Math 164 Homework 2

Nick Monozon

3.3 Consider two bases of $\mathbb{R}^3$, $\{e_1, e_2, e_3\}$ and $\{e_1', e_2', e_3'\}$, where $e_1 = 2e_1' + e_2' - e_3'$, $e_2 = 2e_1' - e_2' + 2e_3'$, and $e_3 = 3e_1' + e_3'$. Suppose that a linear transformation has a matrix representation in $\{e_1, e_2, e_3\}$ of the form

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the matrix representation of this linear transformation in the basis $\{e_1', e_2', e_3'\}$.

Solution. Firstly, we have that

$$\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} e_1' & e_2' & e_3' \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}.$$

So the transformation matrix from $\{e_1', e_2', e_3'\}$ to $\{e_1, e_2, e_3\}$ is given by

$$\boldsymbol{T} = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}.$$

Consider the linear transformation $\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $\boldsymbol{A}$ with respect to the basis $\{e_1, e_2, e_3\}$ and $\boldsymbol{B}$ with respect to the basis $\{e_1', e_2', e_3'\}$. Let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ and $\tilde{\boldsymbol{y}} = \boldsymbol{B}\tilde{\boldsymbol{x}}$. Then, we have that

$$\tilde{\boldsymbol{y}} = \boldsymbol{T}\boldsymbol{y} = \boldsymbol{T}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{T}\boldsymbol{A}(\boldsymbol{T}^{-1}\tilde{\boldsymbol{x}}) = (\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1})\tilde{\boldsymbol{x}}.$$

So it follows that

$$\boldsymbol{B} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} 3 & -10 & -8 \\ -1 & 8 & 4 \\ 2 & -13 & -7 \end{bmatrix}$$

is the matrix representation of $\mathcal{T}$ with respect to $\{e_1', e_2', e_3'\}$.

3.4 Consider two bases of $\mathbb{R}^4$, $\{e_1, e_2, e_3, e_4\}$ and $\{e_1', e_2', e_3', e_4'\}$, where $e_1' = e_1$, $e_2' = e_1 + e_2$, $e_3' = e_1 + e_2 + e_3$, and $e_4' = e_1 + e_2 + e_3 + e_4$. Suppose that a linear transformation has a matrix representation in $\{e_1, e_2, e_3, e_4\}$ of the form

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & 3 \end{bmatrix}.$$

Find the matrix representation of this linear transformation in the basis $\{e_1', e_2', e_3', e_4'\}$.

Solution. We have that

$$\begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix} = \begin{bmatrix} e_1' & e_2' & e_3' & e_4' \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So the transformation matrix from $\{e_1', e_2', e_3', e_4'\}$ to $\{e_1, e_2, e_3, e_4\}$ is given by

$$\boldsymbol{T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the linear transformation $\mathcal{T} : \mathbb{R}^4 \to \mathbb{R}^4$ given by the matrix $\boldsymbol{A}$ with respect to the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ and by the matrix $\boldsymbol{B}$ with respect to the basis $\{\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3', \boldsymbol{e}_4'\}$. Let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ and $\tilde{\boldsymbol{y}} = \boldsymbol{B}\tilde{\boldsymbol{x}}$. It follows that

$$\tilde{\boldsymbol{y}} = \boldsymbol{T}\boldsymbol{y} = \boldsymbol{T}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{T}\boldsymbol{A}(\boldsymbol{T}^{-1}\tilde{\boldsymbol{x}}) = (\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1})\tilde{\boldsymbol{x}}.$$

Hence, we have that

$$\boldsymbol{B} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} 5 & 3 & 4 & 3 \\ -3 & -2 & -1 & -2 \\ -1 & 0 & -1 & -2 \\ 1 & 1 & 1 & 4 \end{bmatrix}$$

is the matrix representation of $\mathcal{T}$ with respect to $\{\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3', \boldsymbol{e}_4'\}$.

3.6 Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of the matrix $\boldsymbol{I}_n - \boldsymbol{A}$ are $1 - \lambda_1, \ldots, 1 - \lambda_n$.

Proof. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are eigenvectors of $\boldsymbol{A}$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then for each i such that $1 \le i \le n$, we have that

$$(\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{v}_i = \boldsymbol{v}_i - \boldsymbol{A}\boldsymbol{v}_i = \boldsymbol{v}_i - \lambda_i \boldsymbol{v}_i = (1 - \lambda_i)\boldsymbol{v}_i.$$

As such, we have shown that $1 - \lambda_1, 1 - \lambda_2, \ldots, 1 - \lambda_n$ are the eigenvalues of the matrix $\boldsymbol{I}_n - \boldsymbol{A}$. □

3.11 Let $\mathcal{V}$ be a subspace. Show that $(\mathcal{V}^\perp)^\perp = \mathcal{V}$. *Hint:* Use Exercise 3.10.

Proof. We first show that $\mathcal{V} \subset (\mathcal{V}^\perp)^\perp$. Let $\boldsymbol{x} \in \mathcal{V}$ and $\boldsymbol{y} \in \mathcal{V}^\perp$. By definition, we have that

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0.$$

Since $(\mathcal{V}^\perp)^\perp$ contains all vectors which are orthogonal to every vector in $\mathcal{V}^\perp$, it follows that $\mathcal{V} \subset (\mathcal{V}^\perp)^\perp$. We will now show that $(\mathcal{V}^\perp)^\perp \subset \mathcal{V}$. Let $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_l\}$ be a basis for $\mathcal{V}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k\}$ be a basis for $(\mathcal{V}^\perp)^\perp$. Define the matrix $\boldsymbol{A}$ such that $\boldsymbol{a}_i$ is the ith column of $\boldsymbol{A}$ for each $i = 1, 2, \ldots, l$ and also $\boldsymbol{B}$ such that $\boldsymbol{b}_i$ is the ith column of $\boldsymbol{B}$ for each $i = 1, 2, \ldots, k$. We clearly have that $\mathcal{R}(\boldsymbol{A}) = \mathcal{V}$ and $\mathcal{R}(\boldsymbol{B}) = (\mathcal{V}^\perp)^\perp$. From Exercise 3.10, we want to show that $\mathcal{R}(\boldsymbol{B}) \subset \mathcal{R}(\boldsymbol{A})$, so we must show that $\mathcal{N}(\boldsymbol{A}^\top) \subset \mathcal{N}(\boldsymbol{B}^\top)$. Let $\boldsymbol{x} \in \mathcal{N}(\boldsymbol{A}^\top)$. We know that by definition

$$\boldsymbol{x} \in \mathcal{N}(\boldsymbol{A}^\top) = \mathcal{R}(\boldsymbol{A})^\perp = \mathcal{V}^\perp.$$

From this, for any $\boldsymbol{w} \in (\mathcal{V}^\perp)^\perp$,

$$(\boldsymbol{B}\boldsymbol{w})^\top \boldsymbol{x} = \boldsymbol{w}^\top \boldsymbol{B}^\top \boldsymbol{x} = \boldsymbol{0} \Rightarrow \quad \boldsymbol{B}^\top \boldsymbol{x} = \boldsymbol{0},$$

so $\boldsymbol{x} \in \mathcal{N}(\boldsymbol{B}^\top)$. Hence $(\mathcal{V}^\perp)^\perp \subset \mathcal{V}$. We therefore have shown that $\mathcal{V} \subset (\mathcal{V}^\perp)^\perp \subset \mathcal{V}$, so $\mathcal{V} = (\mathcal{V}^\perp)^\perp$. □

3.13 Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$. Show that there exist matrices $\boldsymbol{V}$ and $\boldsymbol{U}$ such that $\mathcal{V} = \mathcal{R}(\boldsymbol{V}) = \mathcal{N}(\boldsymbol{U})$.

Proof. Let $m = \dim \mathcal{V}$. We fix a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ and let the ith column of the matrix $\boldsymbol{V}$ be $\boldsymbol{v}_i$ for each $i = 1, 2, \ldots, n$. Clearly then $\mathcal{R}(\mathcal{V}) = \mathcal{V}$, as the number of linearly independent columns of $\boldsymbol{V}$ is equal to the dimension of $\mathcal{V}$. Now suppose that we let $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-m}\}$ be a basis for $\mathcal{V}^\perp$. Let the ith row of the matrix $\boldsymbol{U}$ be $\boldsymbol{w}_i^\top$ for each $i = 1, 2, \ldots, n - m$. We then have that $\mathcal{R}(\boldsymbol{U}^\top) = \mathcal{V}^\perp$. Hence, using the results of Exercise 3.11 and Theorem 3.4, it follows that

$$(\mathcal{V}^\perp)^\perp = \mathcal{V} = \mathcal{R}(\boldsymbol{U}^\top)^\perp = \mathcal{N}\big((\boldsymbol{U}^\top)^\top\big) = \mathcal{N}(\boldsymbol{U}).$$

So we have constructed matrices $\boldsymbol{V}$ and $\boldsymbol{U}$ such that $\mathcal{V} = \mathcal{R}(\boldsymbol{V}) = \mathcal{N}(\boldsymbol{U})$. □

3.14 Let $\boldsymbol{P}$ be an orthogonal projector onto a subspace $\mathcal{V}$. Show that

(a) $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathcal{V}$.

Proof. Note that for any $\boldsymbol{x} \in \mathcal{V}$, we can write $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x} + (\boldsymbol{I} - \boldsymbol{P})\boldsymbol{x}$, which is an orthogonal decomposition of $\boldsymbol{x}$ with respect to $\mathcal{V}$ where $\boldsymbol{P}\boldsymbol{x} \in \mathcal{V}$ and $(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{x} \in \mathcal{V}^\perp$. However, $\boldsymbol{x} = \boldsymbol{x} + \boldsymbol{0}$ is also an orthogonal decomposition. Since orthogonal decompositions are unique, this implies that $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathcal{V}$. □

(b) $\mathcal{R}(\boldsymbol{P}) = \mathcal{V}$.

Proof. Let $\boldsymbol{P}$ be an orthogonal projector onto $\mathcal{V}$. By definition, we have that $\mathcal{R}(\boldsymbol{P}) \subset \mathcal{V}$. From part (a), we showed that $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}$ for all $x \in \mathcal{V}$, so we also have that $\mathcal{V} \subset \mathcal{R}(\boldsymbol{P})$. Combining these, we have that $\mathcal{V} \subset \mathcal{R}(\boldsymbol{P}) \subset \mathcal{V}$, and therefore $\mathcal{R}(\boldsymbol{P}) = \mathcal{V}$. □

3.18 For each of the following quadratic forms, determine if it is positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite.

(a) $f(x_1, x_2, x_3) = x_2^2$
Solution. We have that

$$f(x_1, x_2, x_3) = x_2^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Hence

$$\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where the eigenvalues of $\boldsymbol{Q}$ are $\lambda_1 = 0, \lambda_2 = 0$, and $\lambda_3 = 1$. So this quadratic form is positive semidefinite.

(b) $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - x_1 x_3$
Solution. We have that

$$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - x_1 x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Therefore

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix},$$

where the eigenvalues of $\boldsymbol{Q}$ are $\lambda_1 = 2$, $\lambda_2 = (1-\sqrt{2})/2$, and $\lambda_3 = (1+\sqrt{2})/2$. This quadratic form is indefinite.

(c) $f(x_1, x_2, x_3) = x_1^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$
Solution. We have that

$$f(x_1, x_2, x_3) = x_1^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Therefore

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

where the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1-\sqrt{3}$, and $\lambda_3 = 1+\sqrt{3}$. This quadratic form is therefore indefinite.

3.20 Consider the quadratic form

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 5x_3^2 + 2\xi x_1 x_2 - 2x_1 x_3 + 4x_2 x_3.$$

Find the values of the parameter ξ for which this quadratic form is positive definite.

Solution. This quadratic form can be written as

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x}$$

where we have that

$$Q = \begin{bmatrix} 1 & \xi & -1 \\ \xi & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

for some $\xi \in \mathbb{R}$. The leading principal minors of Q are $\Delta_1 = 1$, $\Delta_2 = 1 - \xi^2$, and $\Delta_3 = -5\xi^2 - 4\xi$. By Sylvester's Criterion, Q is positive definite if and only if all leading principal minors of Q are positive. Hence we want to find the values of ξ such that $1 - \xi^2 > 0$ and $-5\xi^2 - 4\xi > 0$. We know that $1 - \xi^2 > 0$ for $\xi \in (-1, 1)$. Furthermore, we have that

$$\xi = -\frac{4 \pm 4}{10} \Rightarrow \quad \xi = 0, -\frac{4}{5},$$

meaning that since $-5\xi^2 - 4\xi$ has a negative leading coefficient, $-5\xi^2 - 4\xi > 0$ when $\xi \in (-\frac{4}{5}, 0)$. Therefore, since $(-1, 1) \cup (-\frac{4}{5}, 0) = (-\frac{4}{5}, 0)$, Q is positive definite when $\xi \in (-\frac{4}{5}, 0)$.