Math 164 Homework 1

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2.2 Prove that the system $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, has a unique solution if and only if $\operatorname{rank}\boldsymbol{A} = \operatorname{rank}[\boldsymbol{A}, \boldsymbol{b}] = n$.

Proof. $\Rightarrow$: Assume that $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ has a unique solution. By Theorem 2.1, we know that $\operatorname{rank}\boldsymbol{A} = \operatorname{rank}[\boldsymbol{A}, \boldsymbol{b}]$. We must now show that $\operatorname{rank}\boldsymbol{A} = n$. By contradiction, assume that in fact $\operatorname{rank}\boldsymbol{A} < n$. We know that $\operatorname{rank}\boldsymbol{A} \not> n$ because $\boldsymbol{A}$ has n columns. This means that there exists some nonzero vector $\boldsymbol{w} \in \mathbb{R}^n$ such that $\boldsymbol{Aw} = \boldsymbol{0}$. Let $\boldsymbol{x}$ be the unique solution to $\boldsymbol{Ax} = \boldsymbol{b}$. It follows that $\boldsymbol{x} + \boldsymbol{w}$ is also a solution, which contradicts the uniqueness of $\boldsymbol{x}$. Therefore, $\operatorname{rank}\boldsymbol{A} = \operatorname{rank}[\boldsymbol{A}, \boldsymbol{b}]$.

$\Leftarrow$: Now suppose that $\operatorname{rank}\boldsymbol{A} = \operatorname{rank}[\boldsymbol{A}, \boldsymbol{b}] = n$. By Theorem 2.1, a solution to the system exists. We must show that this solution is unique, so let $\boldsymbol{x}$ and $\boldsymbol{w}$ be solutions. This implies that $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Aw} = \boldsymbol{b}$, so $\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{w}) = \boldsymbol{0}$. Since $\operatorname{rank}\boldsymbol{A} = n$ and $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, it follows that $\boldsymbol{x} - \boldsymbol{w} = \boldsymbol{0}$, so $\boldsymbol{x} = \boldsymbol{w}$. Therefore, the system has a unique solution. □

2.9 Show that for any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \le \|\boldsymbol{x} - \boldsymbol{y}\|$. *Hint:* Write $\boldsymbol{x} = (\boldsymbol{x} - \boldsymbol{y}) + \boldsymbol{y}$, and use the triangle inequality. Do the same for $\boldsymbol{y}$.

Proof. By the Triangle Inequality, we have that $\|\boldsymbol{x}\| \le \|\boldsymbol{x} - \boldsymbol{y}\| + \|\boldsymbol{y}\|$. Hence $\|\boldsymbol{x}\| - \|\boldsymbol{y}\| \le \|\boldsymbol{x} - \boldsymbol{y}\|$. Similarly, we have that $\|\boldsymbol{y}\| \le \|\boldsymbol{y} - \boldsymbol{x}\| + \|\boldsymbol{x}\|$, so $\|\boldsymbol{y}\| - \|\boldsymbol{x}\| \le \|\boldsymbol{y} - \boldsymbol{x}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$. Combining these two inequalities, since $\|\boldsymbol{w}\| \ge 0$ for any vector $\boldsymbol{w}$, we find that $|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \le \|\boldsymbol{x} - \boldsymbol{y}\|$. □

3.8 Find the nullspace of

$$\boldsymbol{A} = \begin{bmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix}.$$

Solution. The nullspace of $\boldsymbol{A}$ is the set $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{Ax} = \boldsymbol{0}\}$. We perform elementary row operations on $[\boldsymbol{A}, \boldsymbol{0}]$ to yield a system of equations:

$$[\boldsymbol{A}, \boldsymbol{0}] = \begin{bmatrix} 4 & -2 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 2 & -3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -2 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\implies 4x_1 - 2x_2 = 0, 2x_2 - x_3 = 0$$

We have that x_3 is arbitrary. Furthermore, we have that $x_2 = \frac{1}{2}x_3$, so $x_1 = \frac{1}{2}x_2 = \frac{1}{4}x_3$. It follows that

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} x_3.$$

Hence, this means that

$$\mathcal{N}(\boldsymbol{A}) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} a : a \in \mathbb{R} \right\}.$$

3.21 Consider the function $\langle\cdot,\cdot\rangle_Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by $\langle \boldsymbol{x}, \boldsymbol{y}\rangle_Q = \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Show that $\langle\cdot,\cdot\rangle_Q$ satisfies conditions 1 to 4 for inner products.

Proof. We will show that $\langle\cdot,\cdot\rangle_Q$ is an inner product by showing that it satisfies conditions 1 to 4.

1. Positivity: By the properties of symmetric positive definite matrices, $Q = Q^\top > 0$. Furthermore, $Q = Q^{1/2}Q^{1/2}$ and $Q^{1/2} = (Q^{1/2})^\top > 0$. Therefore, we have that for any vector $x \in \mathbb{R}^n$,

$$\langle x, x\rangle_Q = (Q^{1/2}x)^\top (Q^{1/2}x) = (Q^{1/2}x)(Q^{1/2}x) = \|Q^{1/2}x\|^2 \geq 0.$$

We now want to show that $\langle x, x\rangle = \mathbf{0}$ if and only if $x = \mathbf{0}$. Therefore, since $Q^{1/2}$ is invertible,

$$\begin{aligned}\langle x, x\rangle &\iff \|Q^{1/2}x\|^2 = 0\\ &\iff Q^{1/2}x = 0\\ &\iff x = \mathbf{0}.\end{aligned}$$

2. Symmetry: Let $x, y \in \mathbb{R}^n$ be arbitrary vectors. We have that

$$\langle x, y\rangle_Q = x^\top Q y = y^\top Q^\top x = y^\top Q x = \langle y, x\rangle_Q.$$

3. Additivity: Let x, y, and z be arbitrary vectors in $\mathbb{R}^n$. We have that

$$\begin{aligned}\langle x + y, z\rangle_Q &= (x+y)^\top Q z\\ &= x^\top Q z + y^\top Q z\\ &= \langle x, z\rangle_Q + \langle y, z\rangle_Q.\end{aligned}$$

4. Homogeneity: Let $x, y \in \mathbb{R}^n$ be arbitrary vectors and $c \in \mathbb{R}$ be an arbitrary scalar. We have that

$$\begin{aligned}\langle cx, y\rangle_Q &= (cx)^\top Q y\\ &= cx^\top Q y\\ &= c\langle x, y\rangle_Q.\end{aligned}$$

Since $\langle\cdot,\cdot\rangle_Q$ satisfies conditions 1 to 4, it is an inner product. □

3.23 Consider the vector norm $\|\cdot\|_1$ on $\mathbb{R}^n$ given by $\|x\|_1 = \sum_{i=1}^n |x_i|$, where $x = \begin{bmatrix}x_1, \ldots, x_n\end{bmatrix}^\top$. Define the norm $\|\cdot\|_1$ on $\mathbb{R}^m$ similarly. Show that the matrix norm induced by these vector norms is given by

$$\|A\|_1 = \max_k \sum_{i=1}^m |a_{ik}|$$

where a_{ij} is the (i, j)th element of $A \in \mathbb{R}^{m\times n}$.

Proof. This induced matrix norm is defined by

$$\|A\|_1 = \max_{\|x\|_1 = 1} \{\|Ax\|_1\}.$$

For every x where $\|x\|_1 = 1$, we have by definition that

$$\begin{aligned}\|Ax\|_1 &= \sum_{i=1}^m \left|\sum_{k=1}^n a_{ik}x_k\right|\\ &\leq \sum_{i=1}^m \sum_{k=1}^n |a_{ik}||x_k|\\ &\leq \sum_{k=1}^m |x_k| \sum_{i=1}^m |a_{ik}|\\ &\leq \left(\max_k \sum_{i=1}^m |a_{ik}|\right) \sum_{k=1}^m |x_k|\\ &= \max_k \sum_{i=1}^m |a_{ik}|\end{aligned}$$

because $\sum_{k=1}^{m} |x_k| = \|\boldsymbol{x}\|_1 = 1$. So we have that

$$\|\boldsymbol{A}\|_1 \leq \max_k \sum_{i=1}^{m} |a_{ik}|.$$

We must now show that there exists some $\boldsymbol{y} \in \mathbb{R}^m$ where $\|\boldsymbol{y}\|_1 = 1$ such that $\|\boldsymbol{A}\boldsymbol{y}\|_1 = \max_k \sum_{i=1}^{m} |a_{ik}|$. We choose j such that

$$\sum_{i=1}^{m} |a_{ij}| = \max_k \sum_{i=1}^{m} |a_{ik}|$$

where we define $\boldsymbol{y}$ to be $\boldsymbol{y}_k = 1$ if $k = j$ and $\boldsymbol{y}_k = 0$ otherwise. Hence, it follows that

$$\|\boldsymbol{A}\boldsymbol{y}\|_1 = \sum_{i=1}^{m} \left| \sum_{k=1}^{n} a_{ik} y_k \right| = \sum_{i=1}^{m} |a_{ij}| = \max_k \sum_{i=1}^{m} |a_{ik}|.$$

Hence, we have shown that $\|\boldsymbol{A}\| = \max_k \sum_{i=1}^{m} a_{ik}$. $\square$