Data-Driven Models for Discrete Hedging Problem

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Practitioner Black-Scholes (BS) Delta Hedging

- BS model:

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

$$\sigma : \text{ Constant}$$

- Implied Volatility

$$\sigma_{imp} = V_{BS}^{-1}(V_{mkt}, .)$$

$$V_{mkt}\text{: market option price}$$
$$V_{BS}^{-1} : \text{ inverse of BS pricing function}$$

- BS Delta:

$$\delta_{BS} = \frac{\partial V_{BS}}{\partial S}$$

Problem with Black-Scholes Delta

Problem with the traditional Black-Scholes delta:

- Market violates Black-Scholes assumption
- Dependence of implied volatility on underlying asset price

Variants of delta hedging strategy:

- Stochastic Volatility Model
- Local Volatility Model
- Minimum Variance Approach
- Indirect Data-Driven Approach
- **Direct Data-Driven Approach**

Stochastic Volatility Model

Stochastic volatility models:

- Example: Heston Model

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{\upsilon_t} S_t dW_t \\
d\upsilon_t &= \kappa(\overline{\upsilon} - \upsilon_t) dt + \eta \sqrt{\upsilon_t} dZ_t \\
dZ_t dW_t &= \rho dt
\end{aligned}
$$

Minimum Variance Approach

The correction for the dependence of implied volatility on asset price:

- The Minimum Variance (MV) delta:

$$\delta_{MV} = \frac{\partial V_{BS}}{\partial S} + \frac{\partial V_{BS}}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}}{\partial S}$$

- A parametric model [1] learned from market data:

$$\frac{\partial \sigma_{imp}}{\partial S} = \frac{a + b\delta_{BS} + c\delta_{BS}^2}{S\sqrt{T}}$$

a, b and c are the parameters to be fitted using market data.

[1] Hull, J. and White, A., "Optimal delta hedging for options." Journal of Banking and Finance 82 (2017): 180-190.

Local Volatility Model

The local volatility function (LVF) [2]: the volatility is a deterministic function of S and t.

$$\delta_{MV} = \frac{\partial V_{BS}}{\partial S} + \frac{\partial V_{BS}}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}}{\partial S}$$

Local volatility model can also be used to calculate the $\frac{\partial \sigma_{imp}}{\partial S}$.

[2] Coleman, T.F., Kim, Y., Li, Y. and Verma, A., 'Dynamic hedging with a deterministic local volatility function model,' Journal of risk, 4 ,1 (2001):63-89

Problem with Parametric Approach

Parametric approaches:

- Model mis-specification.
- Sub-optimal for discrete hedging problems.

Data-driven approaches:

- Minimum assumptions on S.
- Model is determined by market data.

Indirect Data-driven Approach

The indirect data-driven approach [3]can be summarized as following:

- Let X be the features from market.
  - Asset price S.
  - Strike Price K.
  - Time to expiration $T - t$.
- Determine the data-driven pricing function $V(X)$ using regression model.
- Compute

$$\delta_{ID} = \frac{\partial V(X)}{\partial S}$$

[3]Hutchinson, J.M., Lo, A.W. and Poggio, T., "A nonparametric approach to pricing and hedging derivative securities via learning networks." The Journal of Finance 49.3 (1994): 851-889.

Problem with Indirect Data-driven Approach

Problems with the indirect data-driven approach:

- Unnecessary intermediate procedure.
- Sub-optimal for discrete hedging.
- Model parameters depend on the asset price.

Direct data-driven approach can be more useful in practice.

- Customized hedging position function.
- Directly learn the hedging position.

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Direct Data-driven Approach

The direct data-driven approach is

$$\min_f \left[\frac{1}{N} \sum_{i=1}^{N} (\Delta V_i - \Delta S_i f(X_i))^2 \right]$$

- ΔV_i : the change of option value in data instance i.
- ΔS_i : the change of asset price in data instance i.
- $f(X_i)$: option hedging position function.
- Data-Driven models outperform other delta hedging strategies [4].

[4] Nian, Ke, Thomas F. Coleman, and Yuying Li. "Learning minimum variance discrete hedging directly from the market." Quantitative Finance (2018): 1-14.

Data-driven Kernel Learning Approach

Kernel Learning Framework:

$$\min_{f \in RKHS} \left[\frac{1}{N} \sum_{i=1}^{N} (\Delta V_i - \Delta S_i f(x_i))^2 + \lambda \|f\|_K^2 \right]$$

Matrix Form:

$$\min_{\alpha} (DK\alpha - \Delta V)^T (DK\alpha - \Delta V) + \lambda \alpha^T K \alpha$$

Where D is the diagonal matrix with ΔS on its diagonal and K is the kernel matrix

Data-Driven Kernel Learning Framework

The data-driven kernel learning framework suffers from several drawbacks:

- Computationally expensive.
- Limited number of variables.
- No feature selection.
- Ignoring the auto-correlation in market data.

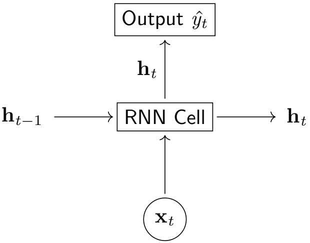
Volatility Clustering and Financial Time Series

Sequential learning framework may further improve the performance:

- Volatility clustering observed in the financial market.
- Autocorrelation between data instances near in time.
- Dependence of option pricing function on the past history of the underlying has been shown in GARCH models [5].

[5] Heston, Steven L., and Saikat Nandi "A closed-form GARCH option valuation model." The review of financial studies 13.3 (2000): 585-625.

Recurrent Neural Network



In each RNN cell:

$$\mathbf{h}_t = f_{act}(\boldsymbol{W}_{hx}\mathbf{x}_t + \boldsymbol{W}_{hh}\mathbf{h}_{t-1} + b_h)$$
$$\hat{y}_t = f_{out}(\boldsymbol{W}_{yh}\mathbf{h}_t + b_y)$$

- Vanishing gradients problem.
- Gated Recurrent Unit (GRU) [6]model is introduced to combat the problem of vanishing gradients.

[6]Cho, Kyunghyun, et al. "Learning phrase representations using RNN encoder-decoder for statistical machine translation." arXiv preprint arXiv:1406.1078 (2014).

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Features

When generating the hedging position, we use local features $\mathbf{x}_L \in \mathbb{R}^d$ and sequential features $\mathbf{X} \in \mathbb{R}^{D \times N}$:

$$\mathbf{X} = [\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_N}] = \begin{bmatrix} (\mathbf{x^1})^\top \\ (\mathbf{x^2})^\top \\ \vdots \\ (\mathbf{x^D})^\top \end{bmatrix} = \begin{bmatrix} x_1^1 & \ldots & x_N^1 \\ \vdots & \ldots & \vdots \\ x_1^D & \ldots & x_N^D \end{bmatrix}$$

Local Features

Local features $\mathbf{x}_L$ for the current day contains:

1. Moneyness S/K.
2. BS delta δ_{BS}.
3. Time to expiry τ.
4. Index close price S .
5. Option bid price V_{bid}.
6. Option offer price V_{offer}.
7. Implied volatility σ_{imp}.
8. BS gamma γ_{BS}.
9. BS vega $vega_{BS}$.
10. Minimum variance delta δ_{MV}

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Sequential Features

Sequential features $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]$ recording the past history contains:

1. Option middle price V_{mid}.
2. Implied volatility σ_{imp}.
3. BS delta δ_{BS}.
4. BS gamma γ.
5. BS vega $vega_{BS}$.
6. Moneyness S/K.

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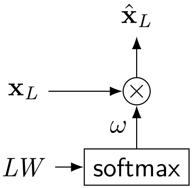
Weighting Local Features

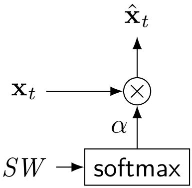


- $LW \in \mathbb{R}^d$: The unnormalized feature weighting vector for the local features $\mathbf{x}_L \in \mathbb{R}^d$.

$$\omega_i = \frac{exp(LW_i)}{\sum_{j=1}^{d} exp(LW_j)}, \; \sum_{i=1}^{d} \omega_i = 1$$

$$\hat{\mathbf{x}}_L = [\omega_1 \mathbf{x}_L^1, \ldots, \omega_d \mathbf{x}_L^d]$$

Weighting Sequential Features

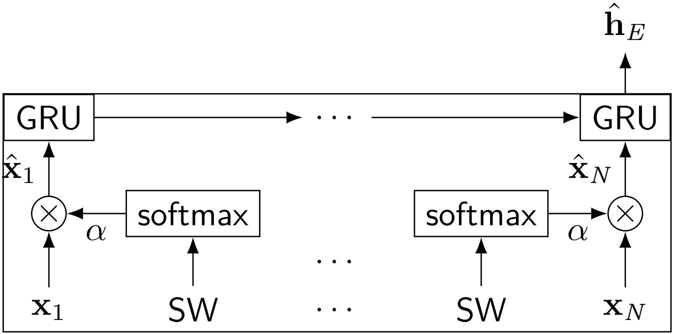


- $SW \in \mathbb{R}^D$:The unnormalized feature weighting vector for the sequential features $\mathbf{X} = [\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_N}]$.
- $\mathbf{x}_t \in \mathbb{R}^D$: The feature vector in $\mathbf{X}$ at time step t

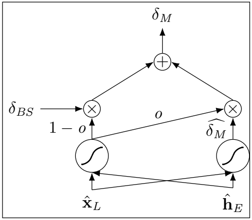
$$\alpha_i = \frac{exp(SW_i)}{\sum_{j=1}^{D} exp(SW_j)}, \ \sum_{i=1}^{D} \alpha_i = 1$$

$$\hat{\mathbf{x}}_t = [\alpha_1 \mathbf{x}_t^1, \ldots, \alpha_D \mathbf{x}_t^D]$$

Encoder Model



Decoder Model



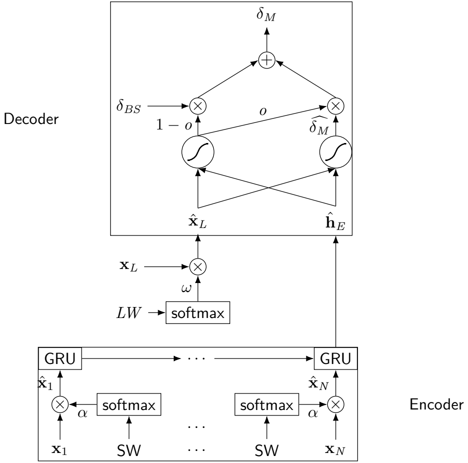
$$\widehat{\delta_M} = sigmoid(\mathbf{v}_{out}^T \ tanh(\boldsymbol{U}^{out}\hat{\mathbf{h}}_E + \boldsymbol{W}^{out}\hat{\mathbf{x}}_L + \mathbf{b}^{out}))$$

$$o = sigmoid(\mathbf{v}_{Gate}^T \ tanh(\boldsymbol{U}^{Gate}\hat{\mathbf{h}}_E + \boldsymbol{W}^{Gate}\hat{\mathbf{x}}_L + \mathbf{b}^{Gate}))$$

The final output is

- Call: $\delta_M = \widehat{\delta_M} \times o + \delta_{BS} \times (1-o)$.
- Put: $\delta_M = -\widehat{\delta_M} \times o + \delta_{BS} \times (1-o)$.

Encoder-Decoder Model



Objective Functions

- Hedging error for BS delta:

$$\eta_i = \Delta V_i - \Delta S_i \delta^i_{BS}$$

- Hedging error for the proposed model:

$$l_i = \Delta V_i - \Delta S_i \delta^i_M$$

- Mean squared loss: $L_S = \frac{1}{2m} \sum_{i=1}^m l_i^2$
- Huber loss:

$$L(l_i, \eta_i) = \begin{cases} \frac{1}{2} l_i^2, & \text{if } |l_i| \leq |\eta_i| \\ |\eta_i|(|l_i| - \frac{1}{2}|\eta_i|), & \text{otherwise} \end{cases}$$

$$L_H = \frac{1}{m} \sum_{i=1}^{m} L(l_i, \eta_i)$$

Training and Regularization

- Optimization technique: trust region method
- Regularization technique: early stopping
- Usage of validation set
- Daily update

Evaluation Criteria: Local Risk

The percentage increase in the effectiveness over the BS hedging:

$$Gain = 1 - \frac{SSE[\Delta V_i - \Delta S_i \delta^i]}{SSE[\Delta V_i - \Delta S_i \delta^i_{BS}]}$$

- SSE: sum of squared errors
- δ: hedging position computed from different models
- δ_{BS}: BS delta

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Experimental Setting

- Data: S&P 500 index option from Jan 2007 to Aug 2015
- The models to be compared:
  - $\mathrm{DKL}_{\mathsf{SPL}}$: Direct data-driven kernel learning model.
  - MV: Minimum variance hedging formula.
  - LVF: Local volatility function model.
  - SABR: SABR stochastic volatility model.
  - DRNN: The proposed encoder-decoder model

Call Option Daily Hedging

Delta	MV (%)	SABR(%)	LVF(%)	Data-Driven Model DKL$_{\text{SPL}}$ (%) Traded	DKL$_{\text{SPL}}$ (%) All	DRNN (%) Traded	DRNN (%) All
0.1	42.1	39.4	42.6	47.1	48.6	32.3	33.8
0.2	35.8	33.4	36.2	37.8	40.0	33.7	36.4
0.3	31.1	29.4	30.3	34.1	35.1	34.1	**35.5**
0.4	28.5	26.3	26.7	32.3	32.0	**33.7**	**34.2**
0.5	27.1	24.9	25.5	29.3	29.4	**35.1**	**33.0**
0.6	25.7	25.2	25.2	29.9	28.4	**35.6**	**32.1**
0.7	25.4	24.7	25.8	29.0	26.8	**31.8**	**29.7**
0.8	24.1	23.5	25.4	25.9	24.7	**28.6**	**26.5**
0.9	16.6	17.0	16.9	17.7	13.9	**19.3**	**18.9**
Overall	25.7	24.6	25.5	31.3	26.0	**32.9**	**28.7**

- Performance will be slighted better than $\mathrm{DKL}_{\mathrm{SPL}}$.

Put Option Daily Hedging

Delta	MV (%)	SABR(%)	LVF(%)	Data-Driven Model $\mathrm{DKL}_{\mathrm{SPL}}$ (%) Traded	$\mathrm{DKL}_{\mathrm{SPL}}$ (%) All	DRNN (%) Traded	DRNN (%) All
-0.9	15.1	11.2	-7.4	8.6	13.6	15.1	**17.2**
-0.8	18.7	19.6	6.8	6.5	16.7	**23.2**	**28.5**
-0.7	20.3	17.7	9.1	10.6	19.8	**28.5**	**32.8**
-0.6	20.4	16.7	9.2	14.9	21.0	**28.3**	**33.9**
-0.5	22.1	16.7	10.8	22.5	23.1	**29.2**	**34.5**
-0.4	23.8	17.7	12.0	24.2	25.2	**29.9**	**34.7**
-0.4	27.1	21.7	16.8	27.7	28.3	**30.6**	**33.6**
-0.2	29.6	25.8	20.6	30.1	30.8	25.4	29.9
-0.1	27.5	26.9	17.7	29.1	31.2	18.7	21.4
Overall	22.5	19.0	10.2	23.4	23.2	**26.2**	**29.7**

- Performance will be slighted better than $\mathrm{DKL}_{\mathrm{SPL}}$.

Call Option Weekly Hedging and Monthly Hedging

Delta	Data-Driven Model			
	$\mathrm{DKL}_{\mathsf{SPL}}$(%)		DRNN(%)	
	Traded	All	Traded	All
0.1	38.9	38.3	**47.8**	**45.6**
0.2	29.0	26.9	**48.5**	**46.0**
0.3	23.5	25.3	**48.5**	**46.6**
0.4	20.8	24.3	**45.9**	**45.4**
0.5	19.9	22.8	**46.6**	**45.0**
0.6	17.3	19.5	**44.8**	**43.1**
0.7	16.8	17.7	**43.9**	**42.4**
0.8	12.5	12.3	**37.7**	**39.0**
0.9	6.2	5.1	**16.4**	**29.1**
Overall	20.2	17.1	**43.7**	**40.5**

Delta	Data-Driven Model			
	$\mathrm{DKL}_{\mathsf{SPL}}$ (%)		DRNN (%)	
	Traded	All	Traded	All
0.1	22.7	24.8	**53.9**	**39.4**
0.2	23.5	25.5	**51.7**	**48.3**
0.3	24.0	24.6	**50.2**	**49.1**
0.4	21.0	20.7	**47.8**	**48.3**
0.5	13.5	12.7	**44.5**	**47.6**
0.6	14.3	13.5	**44.6**	**47.4**
0.7	6.1	7.0	**35.3**	**42.9**
0.8	5.3	4.1	**24.8**	**34.1**
0.9	4.1	2.3	**10.5**	**19.9**
Overall	16.3	12.5	**44.5**	**42.3**

Table: Weekly(Left) and Monthly(Right)

- Performance will be significantly better than $\mathrm{DKL}_{\mathsf{SPL}}$.

Put Option Weekly Hedging and Monthly Hedging

Delta	Data-Driven Model $\mathrm{DKL}_{\mathrm{SPL}}$(%) Traded	All	DRNN(%) Traded	All
-0.9	10.1	7.3	**34.7**	**35.7**
-0.8	18.3	11.5	**44.2**	**45.1**
-0.7	20.2	16.3	**49.6**	**47.3**
-0.6	20.8	18.4	**51.3**	**49.6**
-0.5	22.4	21.2	**53.5**	**51.0**
-0.4	21.0	23.9	**53.2**	**51.2**
-0.3	22.2	26.1	**51.1**	**51.7**
-0.2	20.8	29.7	**46.3**	**51.8**
-0.1	19.2	29.1	**37.2**	**47.6**
Overall	20.4	20.3	**49.1**	**49.4**

Delta	Data-Driven Model $\mathrm{DKL}_{\mathrm{SPL}}$ (%) Traded	All	DRNN (%) Traded	All
-0.9	6.5	5.8	**32.6**	**33.1**
-0.8	6.1	7.8	**49.5**	**45.3**
-0.7	7.3	11.9	**52.4**	**46.3**
-0.6	10.3	9.5	**51.6**	**47.0**
-0.5	13.9	12.8	**51.4**	**46.7**
-0.4	15.6	16.7	**53.4**	**45.1**
-0.3	19.5	13.4	**48.4**	**40.7**
-0.2	20.6	18.4	**44.7**	**35.1**
-0.1	13.0	19.9	**26.8**	**25.3**
Overall	13.5	12.7	**49.5**	**41.2**

Table: Weekly(Left) and Monthly(Right)

- Performance will be significantly better than $\mathrm{DKL}_{\mathrm{SPL}}$.

Importance of Sequential Learning

- The RNN part is removed.

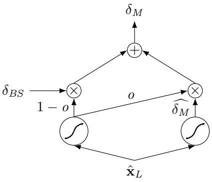


Figure: DNN

Delta	Data-Driven Model(%) Weekly DNN	Weekly DRNN	Monthly DNN	Monthly DRNN
0.1	35.6	**47.8**	29.7	**53.9**
0.2	36.4	**48.5**	38.4	**51.7**
0.3	38.6	**48.5**	40.2	**50.2**
0.4	38.7	**45.9**	38.6	**47.8**
0.5	42.3	**46.6**	36.3	**44.5**
0.6	43.4	**44.8**	36.0	**44.6**
0.7	**45.6**	43.9	30.2	**35.3**
0.8	39.6	**37.7**	22.3	**24.8**
0.9	**26.3**	16.4	**21.1**	10.5
Overall	39.9	**43.7**	35.4	**44.5**

- Performance will be decreased if we remove the RNN part.

A Simplified Model for Comparison

- Directly outputting δ_M without combining with δ_{BS}:

$$\delta_M = sigmoid(\mathbf{v}_{out}^T\ tanh(\boldsymbol{U}^{out}\hat{\mathbf{h}}_E + \boldsymbol{W}^{out}\hat{\mathbf{x}}_L + \mathbf{b}^{out}))$$

- The model is trained without early stopping.
- The objective function is fixed to be the mean squared loss.

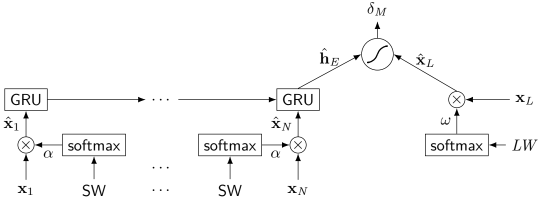


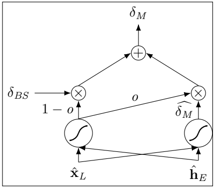
Figure: $DRNN_C$

Importance of Robust Model Design and Regularization

Delta	Data-Driven Model (%)			
	$DRNN_C$		DRNN	
	Weekly	Monthly	Weekly	Monthly
0.1	36.6	34.8	**47.8**	**53.9**
0.2	39.6	38.9	**48.5**	**51.7**
0.3	39.7	41.7	**48.5**	**50.2**
0.4	38.9	42.6	**45.9**	**47.8**
0.5	37.5	42.3	**46.6**	**44.5**
0.6	33.5	40.7	**44.8**	**44.6**
0.7	31.1	33.0	**43.9**	**35.3**
0.8	31.7	**26.3**	**37.7**	24.8
0.9	**28.7**	**17.3**	16.4	10.5
Overall	33.5	38.0	**43.7**	**44.5**

- Performance will be decreased if we use the simplified model $DRNN_C$.

Importance of Output Gate



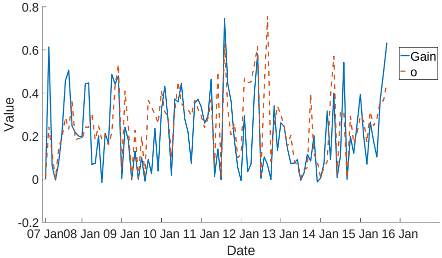


Figure: Call Option Monthly Hedging

- When o is close to 0, what the model output is close to δ_{BS}.

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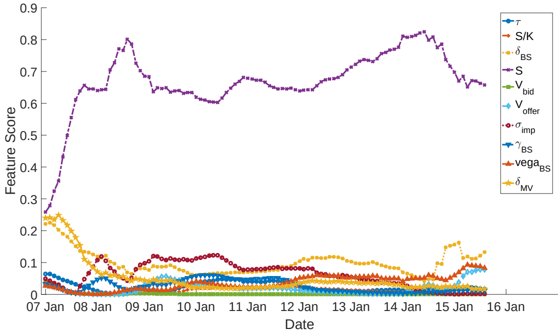
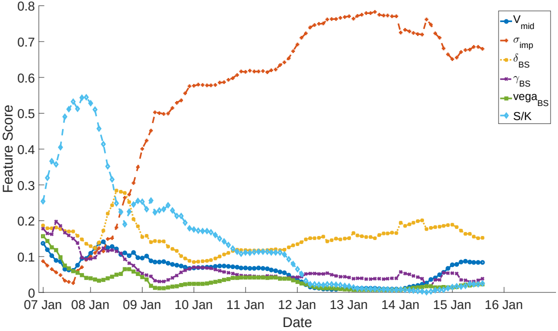
Local Feature Score (ω) for Monthly Hedging



(a) Local Features

Figure: Feature Score of S&P500 Call Option

Sequential Feature Score (α) for Monthly Hedging



(a) Sequential Features

Figure: Feature Score of S&P500 Call Option

Summary

- Loosing assumption on the market dynamic is a good practise
- Data-driven approach can lead to better performance.
- Incorporating the information about the past history can further improve the hedging performance.
- Robust model design is also beneficiary.

What to do next?

- Extend the learning framework to multi-step total hedging problems.
- Use convolution neural network to extract features from volatility surface.

Q & A

Thank you very much!

Any Questions?

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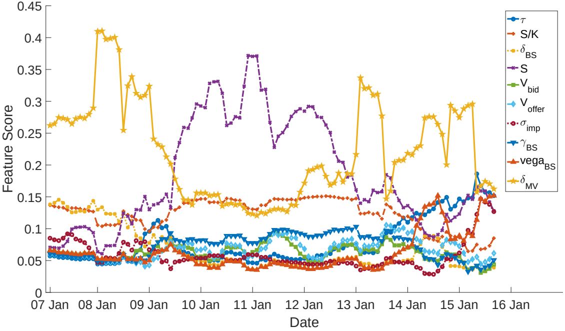
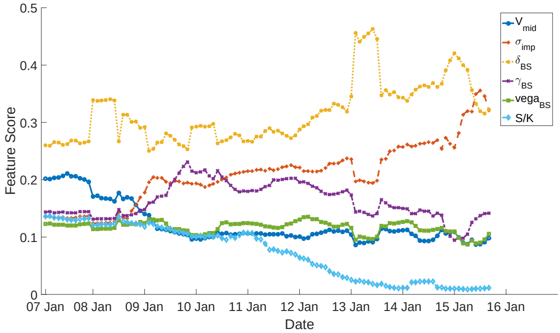
Local Feature Score (ω) for Daily Hedging



(a) Local Features

Figure: Feature Score of S&P500 Call Option

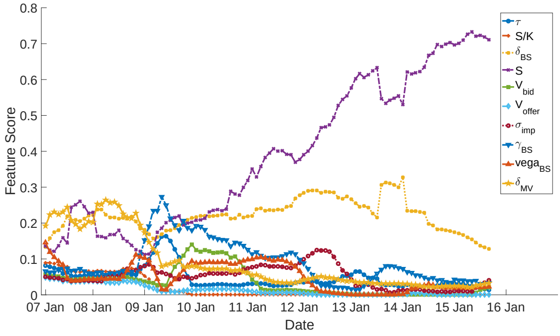
Sequential Feature Score (α) for Daily Hedging



(a) Sequential Features

Figure: Feature Score of S&P500 Call Option

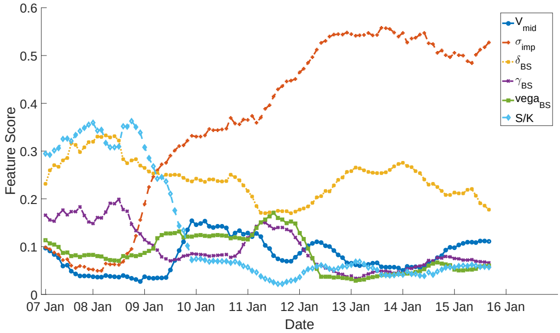
Local Feature Score (ω) for Weekly Hedging



(a) Local Features

Figure: Feature Score of S&P500 Call Option

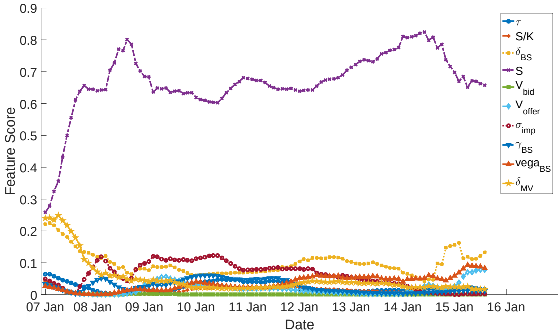
Sequential Feature Score (α) for Weekly Hedging



(a) Sequential Features

Figure: Feature Score of S&P500 Call Option

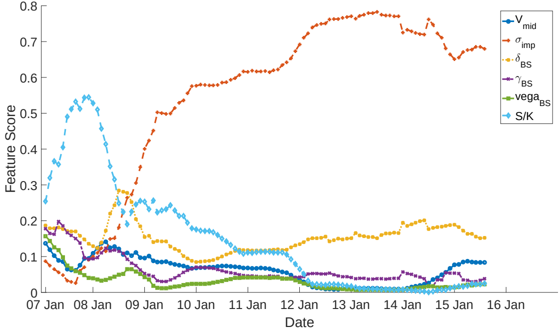
Local Feature Score (ω) for Monthly Hedging



(a) Local Features

Figure: Feature Score of S&P500 Call Option

Sequential Feature Score (α) for Monthly Hedging



(a) Sequential Features

Figure: Feature Score of S&P500 Call Option