Data-Driven Models: An Alternative Discrete Hedging Strategy

Ke Nian
Supervisor: Prof.Yuying Li

David R. Cheriton School of Computer Science,
University of Waterloo,
Waterloo, Canada

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Agenda

Option Hedging

- Hedging is to take offsetting positions that reduces the risk of existing positions.
- An financial institute that sells derivatives (e.g., options) to an client is faced with the problem of managing its risk.
- The prevailing approach in financial derivative hedging has been to calibrate a pricing model function V and use the various sensitivities (e.g.,Greeks) to hedge the derivative trading risk.
  - The sensitivity of the option value function to the underlying price is used in delta hedging.

Set Up Self-Financing Portfolio

Consider a portfolio P_t which is composed of:

- A short position on option V_t^{mkt}
- A position of δ_t, shares on underlying S_t,
- An amount in the risk-free bank account B_t

The hedging portfolio is rebalanced at discrete times t_i. The hedging position is given by δ_{t_i} Initially, we have

$$P_{t_0} = -V_{t_0}^{mkt} + \delta_{t_0} S_{t_0} + B_{t_0} = 0$$

Thus

$$B_{t_0} = V_{t_0}^{mkt} - \delta_{t_0} S_{t_0}$$

Classical Dynamic Delta Hedging

- Rebalance discretely:

$$B_{t_i} = e^{r\Delta t} B_{t_{i-1}} - S_{t_i}(\delta_{t_i} - \delta_{t_{i-1}})$$

- The market price sensitivity towards underlying asset price $\frac{\partial V^{mkt}}{\partial S}$ is unknown.
  - Assume a parametric model for the underlying asset S.
  - Obtain a risk neutral pricing function V and compute $\frac{\partial V}{\partial S}$ as the hedging position.
  - Delta neutral with respect to the function V, not market option price V^{mkt}.

Parameter Dependence on Underlying Price

- Assume that a pricing model matches the market price $V_{t,T,K}^{mkt}$ exactly:

$$V(S,t,T,K;\theta^*) = V_{t,T,K}^{mkt}. \tag{1}$$

and the calibration (1) holds at any S. Then:

$$\frac{\partial V}{\partial S} + \frac{\partial V}{\partial \theta^*}\frac{\partial \theta^*}{\partial S} = \frac{\partial V^{mkt}}{\partial S} \tag{2}$$

- The calibration (1) only ensures matching in the option values, not matching the change in the market option price.
- It is likely that $\frac{\partial V}{\partial S} - \frac{\partial V^{mkt}}{\partial S} \neq 0 \rightarrow \frac{\partial \theta^*}{\partial S} \neq 0$.

Practitioner Black-Scholes (BS) Delta Hedging

- BS model:

$$\frac{dS}{S} = rdt + \sigma dZ$$

$$\sigma: \text{ Constant}$$

- Implied volatility

$$\sigma_{imp} = V_{BS}^{-1}(V_{mkt}, .)$$

V_{mkt}: market option price
V_{BS}^{-1} : inverse of BS pricing function

- Use BS Delta with implied volatility as hedging position:

$$\delta_{BS} = \frac{\partial V_{BS}}{\partial S}$$

Problem with Black-Scholes Model

Problem with the traditional Black-Scholes model:

- Market violates Black-Scholes assumption
- Dependence of implied volatility on underlying asset price

Improvement over Black-Scholes model:

- Stochastic Volatility (SV) Model
- Local Volatility (LV) Model
- Jump Diffusion Model

Pricing and Hedging Conundrum

- A better fit of a model to option market prices is not a good indicator of its hedging performance nor its ability to describe the underlying dynamics [1].
- Example: SABR delta versus SABR-Bartlett delta.

Method	SABR $\delta_{t,T,K}^{SABR}$	Bartlett $\delta_{t,T,K}^{Bartlett}$
Gain (%)	-4.2	27.1

$$\textsc{Gain} = 1 - \frac{\sum_{i=1}^{m}\left(\Delta V_{t_i,T_i,K_i}^{mkt} - \delta_{t_i,T_i,K_i}\ \Delta S_{t_i}\right)^2}{\sum_{i=1}^{m}\left(\Delta V_{t_i,T_i,K_i}^{mkt} - \delta_{t_i,T_i,K_i}^{BS}\ \Delta S_{t_i}\right)^2}.$$

[1] Nathan Lassance and Frederic Vrins. A comparison of pricing and hedging performances of equity derivatives models. Applied Economics, 50(10):1122–1137, 2018.

Correction For Black-Scholes Delta

The correction for the dependence of implied volatility on asset:

- The Minimum Variance (MV) delta:

$$\delta_{MV} = \frac{\partial V_{BS}}{\partial S} + \frac{\partial V_{BS}}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}}{\partial S}$$

- A parametric model [2]learned from **market** option price time series data can be used to estimate $\frac{\partial \sigma_{imp}}{\partial S}$
- Local volatility model and stochastic volatility model (e.g. SABR) can also be used to estimate the $\frac{\partial \sigma_{imp}}{\partial S}$.

[2] John Hull and Alan White. Optimal delta hedging for options. Journal of Banking &Finance, 82:180–190, 2017.

Problem with Parametric Approach

Parametric approaches:

- Model mis-specification.
- Sub-optimal for discrete hedging problems.

Data-driven approaches:

- Minimum assumptions on the dynamic of S.

The indirect data-driven approach has been proposed:

- Determine the data-driven pricing function $V(\cdot)$ using regression model from historical market data.
- Compute $\frac{\partial V}{\partial S}$ as hedging position

Motivation for Direct Data-Driven Approach

The indirect data-driven approach has the following problems:

- Unnecessary intermediate procedure.
- Sub-optimal for discrete hedging.
- Model parameters depend on the asset price.
- Data-driven pricing model may introduce arbitrage,

Direct data-driven approach: learn hedging function from **market** underlying and option price time series data.

- Directly learn the hedging position $\delta(\cdot)$.
- Flexible objective function: Local hedging risk versus total hedging risk.

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Local Hedging Risk

Assume $r = 0$ for simplicity, the discrete *local hedging risk* measures the changes in the hedging portfolio after a fixed time interval Δt when the hedging position is set to be δ_t.

$$\text{Risk}_t^{local} = \Delta P_t = P_{t+\Delta t} - P_t = \Delta S_t \delta_t - \Delta V_t^{mkt} \tag{3}$$

Kernel Learning Framework

The empirical loss function is chosen to correspond to the square of discrete local hedging risk

$$L\left(\delta(\mathbf{x}_t^{T,K};\widehat{\boldsymbol{\alpha}})\right) = \left(\Delta V_{t,K,T}^{mkt} - \Delta S_t \delta(\mathbf{x}_t^{T,K};\widehat{\boldsymbol{\alpha}})\right)^2. \qquad (4)$$

The kernel hedging position function $\delta(\mathbf{x}_t^{T,K};\widehat{\boldsymbol{\alpha}}^*)$ can be estimated from the regularized optimization below:

$$\min_{\delta \in \mathcal{H}_K} \left\{ \sum_{i=1}^{M} L\left(\delta(\mathbf{x}_{t_i}^{T_i,K_i};\widehat{\boldsymbol{\alpha}})\right)^2 + \lambda_P \|\delta\|_{\mathcal{K}}^2 \right\} \qquad (5)$$

$\delta(\cdot)$: hedging position function

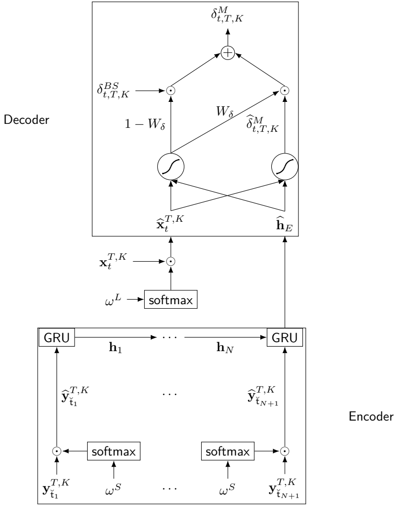
Motivation for Sequential Learning Framework

Recognizing:

- Volatility clustering observed in the financial market.
- Autocorrelation between data instances near in time.
- Dependence of option pricing function on the past history of the underlying has been shown in GARCH models.

we propose a encoder-decoder sequential learning model for hedging function $\delta(\cdot)$ with a robust loss function and the feature weighting mechanism.

Sequential Learning Model Structure



Call Option Weekly and Monthly Local Risk Hedging Comparison

Delta	Comparing Model(%)											
	Weekly						Monthly					
	MV	Bartlett	$\mathrm{DKL_{SPL}}$	NN_δ	GRU_c	GRU_δ	MV	Bartlett	$\mathrm{DKL_{SPL}}$	NN_δ	GRU_c	GRU_δ
0.1	26.3	-16.9	38.9	35.6	36.6	**47.8**	13.5	-8. 2	22.7	29.7	34.8	**53.9**
0.2	21.6	-5.6	29.0	36.4	39.6	**48.5**	16.4	0.4	23.5	38.4	38.9	**51.7**
0.3	20.1	11.9	23.5	38.6	39.7	**48.5**	17.9	2.1	24.0	40.2	41.7	**50.2**
0.4	18.1	17.3	20.8	38.7	38.9	**45.9**	16.9	2.7	21.0	38.6	42.6	**47.8**
0.5	16.0	21.7	19.9	42.3	37.5	**46.6**	15.2	5.7	13.5	36.3	42.3	**44.5**
0.6	12.1	24.1	17.3	43.4	33.5	**44.8**	12.7	8.4	14.3	36.0	40.7	**44.6**
0.7	8.1	26.3	16.8	**45.6**	31.1	43.9	5.9	7.5	6.1	30.2	26.3	**35.3**
0.8	3.7	25.5	12.5	39.6	31.7	**37.7**	-1.2	4.2	5.3	22.3	26.3	**24.8**
0.9	2.4	21.7	6.2	26.3	**28.7**	16.4	-1.8	9.8	4.1	**21.1**	17.3	10.5
Overall	15.1	18.6	20.2	39.9	33.5	**43.7**	13.4	4.5	16.3	35.4	38.0	**44.5**

Table: S&P 500 call options hedging comparison on traded data, bold entries indicating best Gain. The Gain ratio is a measure for the local hedging performance. The larger the gain ratio is, the better improvement the model achieves over the baseline BS delta hedging method in terms of local hedging risk. The gain ratio is reported on different delta buckets.

Call Option Daily Local Risk Hedging Comparison

Delta	MV (%)	$\mathrm{SABR_{MV}}$(%)	LVF(%)	Bartlett		Data-Driven Model			
						$\mathrm{DKL_{SPL}}$ (%)		GRU_δ (%)	
				Traded	All	Traded	All	Traded	All
0.1	42.1	39.4	42.6	29.0	35.1	**47.1**	**48.6**	32.3	33.8
0.2	35.8	33.4	36.2	28.2	32.3	**37.8**	**40.0**	33.7	36.4
0.3	31.1	29.4	30.3	27.7	28.9	34.1	35.1	**34.1**	**35.5**
0.4	28.5	26.3	26.7	28.7	27.3	32.3	32.0	**33.7**	**34.2**
0.5	27.1	24.9	25.5	26.9	26.7	29.3	29.4	**35.1**	**33.0**
0.6	25.7	25.2	25.2	28.3	26.6	29.9	28.4	**35.6**	**32.1**
0.7	25.4	24.7	25.8	28.5	26.4	29.0	26.8	**31.8**	**29.7**
0.8	24.1	23.5	25.4	23.1	24.9	25.9	24.7	**28.6**	**26.5**
0.9	16.6	17.0	16.9	14.0	15.6	17.7	13.9	**19.3**	**18.9**
Overall	25.7	24.6	25.5	27.1	24.8	31.3	26.0	**32.9**	**28.7**

Table: S&P 500 call option hedging for 1-business day: bold entries indicating best Gain. The Gain ratio is a measure for the local hedging performance. The larger the gain ratio is, the better improvement the model achieves over the baseline BS delta hedging method in terms of local hedging risk. The gain ratio is reported on different delta buckets.

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Total Hedging Risk

The discrete *total hedging risk* measures the hedging portfolio profit and loss at the expiry T for the entire hedging period $[t_0, T]$:

$$\text{Risk}_{t_0}^{total} = \sum_{j=0}^{N_{rb}-1} \left\{ \Delta S_{t_j} \delta_{t_j} - \Delta V_{t_j}^{mkt} \right\} = \sum_{j=0}^{N_{rb}-1} \text{Risk}_{t_j}^{local}$$

where $N_{rb} = \frac{T-t_0}{\Delta t}$.

Data Augmentation For Total Risk Hedging

- Total risk hedging requires the time series over $[t_0, T]$.
- Scarcity of market option time series:
  - Option market only offers options with fixed expiry dates.
  - Option with specific $\{T, K\}$ are not traded every day.
- Calibrate arbitrage-free surfaces to augment option time series:

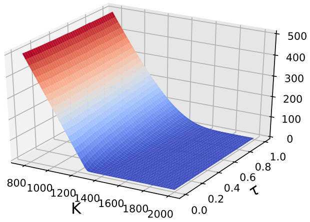


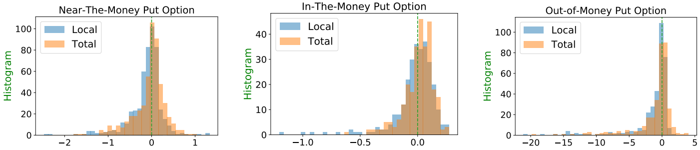
Figure: An Arbitrage-Free Surface

Local Versus Total Risk Hedging

The *discrete total hedging risk* is the summation of the *discrete local hedging risk* evaluated at discrete rebalancing time $\{t_0, t_1, \ldots, t_{N_{rb}-1}\}$.

$$\mathsf{Risk}_{t_0}^{total} = \sum_{j=0}^{N_{rb}-1} \left\{ \Delta S_{t_j} \delta_{t_j} - \Delta V_{t_j}^{mkt} \right\} = \sum_{j=0}^{N_{rb}-1} \mathsf{Risk}_{t_j}^{local}$$

As a consequence, building a model reducing the discrete local hedging risk will reduce the discrete total hedging risk as well.



(a) (b) (c)

Figure: Weekly Hedging Put Option

Conclusion

Data-driven hedging model shows potential for improved performance over classical methods:

- The kernel model $\mathrm{DKL}_{\mathsf{SPL}}$ improves local risk hedging performance over the classical parametric hedging strategy.
- The sequential model GRU_{δ} significantly outperforms MV, SABR-Bartlett, and kernel model $\mathrm{DKL}_{\mathsf{SPL}}$ in local risk hedging.
- The extension to multi-step total risk hedging scenarios often outperforms SABR-Bartlett, and Black-Scholes in total risk hedging measurement.
- The data-driven local risk hedging model remains competitive in terms of total risk measurements.

Future Works

- Generate hedging risk scenarios (e.g., using GAN) to build more complex model for hedging.
- Incorporate transaction cost, allow flexible rebalancing frequency, extend to hedge more complex derivatives, etc.
- Extension to calculate implicit exposure across different assets.

Contribution

- Ke Nian, Thomas F Coleman, and Yuying Li. Learning minimum variance discrete hedg- ing directly from the market. Quantitative Finance, 18(7): 1115–1128, 2018.
- Ke Nian, Thomas F Coleman, and Yuying Li. Learning sequential option hedging models from market data. Journal of Banking & Finance, 133:106-277, 2021.
- Ke Nian, Thomas F Coleman, and Yuying Li. Learning sequential total hedging models from market data. In Preparation, 2023.

Q & A

Thank you very much!

Any Questions?