## Learning Minimum Variance Discrete Hedging Directly From Market

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# Agenda



### Introduction

### Delta Hedging Variants

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## Data Driven Approach

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# Practitioner Black-Scholes (BS) Delta Hedging



▶ BS model:

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

 $\sigma$ : Constant

► Implied Volatility

$$\sigma_{imp} = V_{BS}^{-1}(V_{mkt},.)$$

 $V_{mkt}$ : market option price  $V_{BS}^{-1}$  : inverse of BS pricing function

▶ BS Delta:

$$\delta_{BS} = \frac{\partial V_{BS}}{\partial S}$$

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## Problem with Black-Scholes Delta



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Problem with the traditional Black-Scholes Delta:

- ► Market violates BS assumption
- ▶ Dependence of volatility on underlying asset price

Variants of Hedging Strategy:

- ► Stochastic Volatility Model
- ► Local Volatility Model
- ► Minimum Variance Approach
- ► Indirect Data-Driven Approach
- ► Direct Data-Driven Approach

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## Stochastic Volatility Model



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## Stochastic volatility models:

- ► Heston Model
- ► SABR Volatility Model
- ► GARCH Model

# Minimum Variance Approach



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Considering the the dependence of imply volatility on asset price:

► The Minimum Variance (MV) delta:

$$\delta_{MV} = \frac{\partial V_{BS}}{\partial S} + \frac{\partial V_{BS}}{\partial \sigma_{imn}} \frac{\partial E(\sigma_{imp})}{\partial S}$$

► The authors <sup>1</sup>propose:

$$\frac{\partial E(\sigma_{imp})}{\partial S} = \frac{a + b\delta_{BS} + c\delta_{BS}^2}{S\sqrt{T}} \tag{1}$$

 $a,\,b$  and  $\,c$  are the parameter to be fitted using market data.

 $<sup>^{1}</sup>$ Hull,J and White,A ,'Optimal Delta Hedging for Options', unpublished manuscript

# Local Volatility Model



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The local volatility function (LVF)  $^2$ : volatility is a deterministic function of S and t.

$$\delta_{MV} = \frac{\partial V_{BS}}{\partial S} + \frac{\partial V_{BS}}{\partial \sigma_{imp}} \frac{\partial E(\sigma_{imp})}{\partial S}$$

Local volatility model can also be used to calculate the  $\frac{\partial E(\sigma_{imp})}{\partial S}$ 

<sup>&</sup>lt;sup>2</sup>Coleman, T, Y. Kim, Y. Li and A. Verma,

<sup>&#</sup>x27;Dynamic hedging with a deterministic local volatility function model,' Journal of risk, 4,1 (2001):63-89

## Problem with Parametric Approach



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Problem with the previous parametric approaches:

► Assumptions do not hold in market

Data-driven approach can be more useful in practice.

- ightharpoonup Minimum assumptions on S
- Model is determined by market data.

## Indirect Data-driven Approach



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The indirect data-driven approach <sup>3</sup>can be summarized as following:

- ▶ Let X the features from market.
  - ► Asset price *S*
  - ▶ Strike Price *K*
  - ▶ Time to expiration T-t
- $\blacktriangleright$  Determine the data driven pricing function  $\,V(X)$  using regression model.
- Compute

$$\delta_{ID} = \frac{\partial V(X)}{\partial S}$$

<sup>&</sup>lt;sup>3</sup>Hutchinson, James M., Andrew W. Lo, and Tomaso Poggio. "A nonparametric approach to pricing and hedging derivative securities via learning networks." The Journal of Finance 49.3 (1994): 851-889.

# Problem with Indirect Data-Driven Approach



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Problem with the Indirect Data-Driven Approach:

- ► Intermediate procedure is not necessary.
- ► Not suitable for weekly and monthly hedging

Direct data-driven approach can be more useful in practice.

- ► Customized hedging position function
- Directly compute the hedging position

## Direct Data-driven Approach



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The direct data-driven approach is

$$\min_{f} \left[ \frac{1}{N} \sum_{i=1}^{N} (\Delta V_i - \Delta S_i f(X_i))^2 \right]$$

 $\Delta\,V_i$ : the change of option value in data instance i  $\Delta S_i$ : the change of asset price in data instance i  $V_i$ : the option price in data instance i

# Regularized Network



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Indirect data-driven approach:

$$\min_{f \in RKHS} \left[ \frac{1}{N} \sum_{i=1}^{N} (V_i - f(x_i))^2 + \lambda ||f||_K^2 \right]$$

Direct data-driven approach:

$$\min_{f \in RKHS} \left[ \frac{1}{N} \sum_{i=1}^{N} (\Delta V_i - \Delta S_i f(x_i))^2 + \lambda ||f||_K^2 \right]$$

# Regularized Network (2)



Given

$$f(x) = \sum_{i=1}^{N} \alpha_i K(x, x_i)$$

Indirect data-driven approach:

$$\min_{\alpha} (K\alpha - V)^{T} (K\alpha - V) + \lambda \alpha^{T} K\alpha$$

Direct data-driven approach:

$$\min_{\alpha} (DK\alpha - \Delta V)^{T} (DK\alpha - \Delta V) + \lambda \alpha^{T} K\alpha$$

Where D is the diagonal matrix with  $\Delta S$  on its diagonal

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# Regularized Network (3)



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Indirect data-driven approach:

$$\alpha = (K + \lambda_P I)^{-1} V$$

Let  $\widetilde{K} = DK$ , direct data-driven approach:

$$\widetilde{K}^{T}(\widetilde{K}\alpha - \Delta V) + \lambda K\alpha = 0$$

$$(\widetilde{K}^{T}\widetilde{K} + \lambda K)\alpha = \widetilde{K}^{T}\Delta V$$

$$\alpha = (\widetilde{K}^{T}\widetilde{K} + \lambda K)^{-1}\widetilde{K}^{T}\Delta V$$

## Cross-Validation



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For the indirect data-driven approach :

$$\alpha = (K + \lambda_P I)^{-1} V$$

We can calculate the eigen-decomposition of  $K = Q \Lambda \, Q^T$  and then

$$\alpha = Q(\Lambda + \lambda I)^{-1} Q^T V$$

Given the eigen-decomposition, finding  $\alpha(\lambda)$  can be done(  $O(N^2)$  ).

## Leave-One-Out Cross-Validation

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- $\triangleright$  For each data point  $x_i$ , building a model using the remaining N-1 data points, and measuring the error for  $x_i$
- ▶ It can be further shown that, given the eigen-decomposition of  $K = Q\Lambda Q^T$ , we can estimate the leave-one-out cross-validation errors in  $O(N^2)$ .

## Modification of the direct data driven model



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► For our direct data-driven approach:

$$\alpha = (\widetilde{K}^T \widetilde{K} + \lambda K)^{-1} \widetilde{K}^T \Delta V$$

▶ If we changed the penalty term from  $\alpha^T K \alpha$  to  $\alpha^T \alpha$ :

$$\alpha = (\widetilde{K}^T \widetilde{K} + \lambda I)^{-1} \widetilde{K}^T \Delta v$$

# Modification of the direct data driven model (2)



Let the SVD of  $\widetilde{K} = U \Sigma V^T$ , the  $\alpha$  can be determined by

$$\begin{split} &(\widetilde{K}^T\widetilde{K} + \lambda I)\alpha = \widetilde{K}^T\Delta V \\ &(V\Sigma^TU^TU\Sigma V^T + \lambda I)\alpha = V\Sigma^TU^T\Delta V \\ &V(\Sigma^T\Sigma + \lambda I)V^T\alpha = V\Sigma^TU^T\Delta V \\ &(\Sigma^T\Sigma + \lambda I)V^T\alpha = \Sigma^TU^T\Delta V \\ &(\Sigma^T\Sigma + \lambda I)V^T\alpha = \Sigma^TU^T\Delta V \end{split}$$

 $(\Sigma^T\Sigma+\lambda I)$  is again a diagonal matrix. Then we can still estimate the leave-one-out cross-validation errors in  $O(N^2)$ 

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## Kernels



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In the following experiments, we use:

The Gaussian kernel

$$K(x,y) = e^{-\frac{\|x-y\|_2^2}{2\sigma_b^2}}$$

► The Spline Kernel:

$$K(x,y) = \int_{lb}^{+\infty} (x-t)_+^d (y-t)_+^d dt + \sum_{k=0}^d x^k y^k$$

## Evaluation Criteria: Local Risk



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The percentage increase in the effectiveness over the BS hedging:

$$Gain = 1 - \frac{SSE[\Delta V_i - \Delta S_i \delta^i]}{SSE[\Delta V_i - \Delta S_i \delta^i_{BS}]}$$

SSE: sum of squared errors  $\delta$ : hedging position computed from different models  $\delta_{BS}$ : BS delta

# Synthetic Data: Experimental Setting



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▶ Dynamic for asset: Heston Model

Training data: a 2 year stock path

▶ Testing Data: 100 independent 6-months stock path

Methods considered:

 $lackbox{}{f\delta}^{
m BS}$ : implied volatility BS delta

► HESTON: analytic Heston delta

 $\blacktriangleright \ \mathrm{DKL}_{\mathsf{SPL}}$  : direct learning a spline kernel hedging function

ightharpoonup DKL<sub>RBF</sub> : directly learning a RBF kernel hedging function

▶ IKL<sub>SPL</sub>: Indirectly computing  $\frac{\partial f}{\partial S}$  using a spline kernel

 $ightharpoonup IKL_{\mathsf{RBF}}$ : Indirectly computing  $rac{\partial f}{\partial S}$  using a RBF kernel

# Daily Hedging Performance



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| Method               | Gain (%) | $\mathbf{E}( \Delta V - \Delta Sf(\mathbf{x}) )$ | Std   | VaR   | CVaR  |
|----------------------|----------|--|-------|-------|-------|
| $\delta^{ m BS}$     | 0.0      | 0.185  | 0.286 | 0.380 | 0.574 |
| IKL <sub>RBF</sub>   | -3.3     | 0.171  | 0.291 | 0.356 | 0.566 |
| $IKL_{SPL}$          | -183.3   | 0.291  | 0.482 | 0.669 | 1.105 |
| DKL <sub>RBF</sub>   | 63.1     | 0.120  | 0.174 | 0.251 | 0.352 |
| $\mathrm{DKL}_{SPL}$ | 64.9     | 0.121  | 0.170 | 0.255 | 0.345 |
| HESTON               | 63.6     | 0.121  | 0.173 | 0.266 | 0.360 |

Table: Daily Hedging Comparison

<sup>&</sup>lt;sup>1</sup> FS #1:  $X = \{\text{MONEYNESS}, \text{TIME-TO-EXPIRY}\}$ 

<sup>&</sup>lt;sup>2</sup> Bold entry indicating best Gain

## Real Data Hedging Experiments



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▶ Data: S&P 500 index option from Jan 2007 and Aug 2015

Model Calibration:

► SABR: daily calibration

LVF:  $\frac{\partial E(\sigma_{imp})}{\partial S}$  from implied volatility surface

MV: Use a 36 months time window to train

▶ DKL<sub>SPL</sub>: Use a 36 months time window to train. Models are separately calibrated for different Black-Sholes delta range.

## S&P 500 Call Options

|         |          |         |        | DKL <sub>SPL</sub> (%)<br>Leave-One-Out <sup>1</sup> |      |
|---------|----------|---------|--------|--|------|
| Delta   | SABR (%) | LVF (%) | MV (%) |  |      |
|         |          |         |        | Traded   | All  |
| 0.1     | 42.1     | 39.4    | 42.6   | 44.1   | 44.4 |
| 0.2     | 35.8     | 33.4    | 36.2   | 37.8   | 38.1 |
| 0.3     | 31.1     | 29.4    | 30.3   | 33.1   | 33.6 |
| 0.4     | 28.5     | 26.3    | 26.7   | 30.9   | 31.3 |
| 0.5     | 27.1     | 24.9    | 25.5   | 30.0   | 30.4 |
| 0.6     | 25.7     | 25.2    | 25.2   | 29.3   | 29.8 |
| 0.7     | 25.4     | 24.7    | 25.8   | 28.4   | 30.2 |
| 8.0     | 24.1     | 23.5    | 25.4   | 22.5   | 28.0 |
| 0.9     | 16.6     | 17.0    | 16.9   | 8.1  | 12.7 |
| Overall | 25.7     | 24.6    | 25.5   | 31.3   | 26.8 |

Table: S&P 500 Call Option Daily Hedging: bold entry indicating best Gain



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<sup>&</sup>lt;sup>1</sup> For each month, the penalties for models are determined by leaveone-out cross validation.

## Conclusion

▶ loosing assumption on the market dynamic is a good practise.

▶ Data driven approach can leads to better performance.



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Thank you very much!

Any Questions?