Data-Driven Models for Discrete Hedging Problem

by

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Abstract

Options hedging is a critical problem in financial risk management. The prevailing approach in financial derivative pricing and hedging has been to first assume a parametric model describing the underlying price dynamics. An option model function V is then calibrated to current market option prices and various sensitivities are computed and used to hedge the option risk. It has been recognized that computing hedging position from the sensitivity of the calibrated model option value function is inadequate in minimizing variance of the option hedge risk, as it fails to capture the model parameter dependence on the underlying price. In this thesis, we demonstrate that this issue can exist generally when determining hedging position from the sensitivity of the option function, either calibrated from a parametric model from spot market option prices or estimated nonparametricaly from historical option prices. Consequently the sensitivity of the estimated model option function typically does not minimize variance of the hedge risk, even instantaneously, unless the parameters dependence is addressed. We propose several data-driven approaches to directly learn a hedging function from the market data by minimizing certain measure of the local hedging risk and total hedging risk. This thesis will focus on answering the following questions: 1) Can we efficiently build direct data-driven models for discrete hedging problem that outperform existing state-of-art parametric hedging models? 2) Can we incorporate feature selection and feature extraction into the data-driven models to further improve the performance of the discrete hedging? 3) Can we build efficient models for both the one-step hedging problem and multi-steps hedging problem based on the state-of-art learning framework such as deep learning framework and kernel learning framework?

Using the S&P 500 index daily option data for more than a decade ending in August 2015, we first propose a direct data-driven approach [**?**] based on kernel learning framework and we demonstrate that the proposed method outperforms the parametric minimum variance hedging method proposed in [**?**], as well as minimum variance hedging corrective techniques based on stochastic volatility or local volatility models. Furthermore, we show that the proposed approach achieves significant gain over the implied Black-Sholes delta hedging for weekly and monthly hedging.

Following the direct data-driven kernel learning approach [**?**], we propose a robust encoder-decoder Gated Recurrent Unit (GRU), GRU_δ, for optimal discrete option hedging. The proposed GRU_δ utilizes the Black-Scholes model as a pre-trained model and incorporates sequential information and feature selection. Using the S&P 500 index European option market data from January 2, 2004 to August 31, 2015, we demonstrate that the weekly and monthly hedging performance of the proposed GRU_δ significantly surpasses that of the data-driven minimum variance (MV) method in [**?**], the regularized kernel data-driven model [**?**], and the SABR-Bartlett method [**?**]. In addition,the daily hedging performance of the proposed GRU_δ also surpasses that of MV methods in [**?**] based on parametric models, the kernel method [**?**] and SABR-Bartlett method [**?**].

Lastly, we design a multi-steps data-driven models $\text{GRU}_{\text{total}}$ based on the GRU_δ to hedge the option discretely until the expiry. We utilize SABR model and Local Volatility Function

(LVF) to augment existing market data and thus alleviate the problem for lack of market option information. The augmented market data is used to train a sufficient multi-steps total hedging model $\text{GRU}_{\text{total}}$ that outperform the one-step hedging model GRU_{δ} when we evaluate the hedging performance on the expiries. We further compare the total hedging model $\text{GRU}_{\text{total}}$ built based on purely market underlying information with the total hedging model $\text{GRU}_{\text{total}}$ built based on both market underlying information and augmented option information to indicate the importance of market option information in determining the data-driven option hedging position.

Acknowledgements

I would like to thank all the little people who made this thesis possible.

Dedication

This is dedicated to the one I love.

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Glossary

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Chapter 1

Introduction

Options hedging is a critical problem in financial risk management. The state-of-art approaches in financial derivative pricing and hedging rely heavily on parametric assumptions describing the dynamics of underlying asset . The common practice is to calibrate an option pricing function based on the specific parametric model and compute various sensitivities to hedge the option risk. For example, the sensitivity of the option value function to the underlying price is used in delta hedging. Ideally, the value of an option written on the underlying asset can be perfectly replicated by a hedging portfolio consisting of the underlying asset and the risk-free asset, when the market is complete [**?**]. In practice, such perfect scenarios does not exist and we have to rebalance the hedging portfolio discretely instead of continuously due to the existence of the transaction cost. The practice of adjusting the hedging portfolio discretely is often referred to as discrete hedging.

There are many parametric models proposed to describe the dynamics of underlying asset. The original and most celebrated parametric Black-Scholes (BS) model uses a constant volatility [**? ?**], which is shown to produce inaccurate option prices for deeply out-of-the-money options and deeply in-the-money options [**?**]. In addition, the BS model is unable to capture the non-zero correlation between the volatility and the underlying asset price [**? ?**]. The practitioner's BS delta hedging approach sets the constant volatility in the BS model to the implied volatility calibrated to the market price at the time of re-balancing. Many alternative parametric models have been proposed to improve the BS model, including the Stochastic Volatility (SV) model [**? ? ? ?**], the Local Volatility Function (LVF) model [**? ? ? ?**], and the jump model [**? ?**]. Unfortunately, all models have been shown to have their limitations in accurately modeling option market prices.

Errors in the option value model have significant implications in hedging. Consider, for example, when the hedging position is computed from the sensitivity of the option value function calibrated at the hedging time, the computed hedging position only depends on the assumed underlying price model and the current market option prices. Unless the assumed model for the underlying price is exact and all assumptions that results in the option pricing function are all valid, the option function calibrated at the hedging time cannot predict the future option market price.

Specifically, assume that $V(S,t,T,K;\theta)$ is the option value function and θ is the vector of model parameters of the assumed option pricing model and, at the hedging time t, option calibration ensures that

$$V(S,t,T,K;\theta) = V^{mkt}_{t,T,K} \quad (1.0.1)$$

where V^{mkt} denotes the actual market option price, t is the current trading time, K is strike price, T is the expiry time, and S is the underlying price input used in the option pricing function. We use $V^{mkt}_{t,T,K}$ to indicate it is the market price at time t for the expiry time T and strike K. The option value function $V(S,t,T,K;\theta)$, calibrated at the hedging time t, does not ensure that $\frac{\partial V}{\partial S}$ equals to $\frac{\partial V^{mkt}}{\partial S}$, which is indeed unknown. This leads to the dependence of the calibrated model parameter on the underlying price [**? ? ?**]. The missing sensitivity $\frac{\partial \theta}{\partial S}$, is difficult to account for and is often ignored, though for some models, corrections have been proposed to account for the dependence [**? ? ?**].

Since machine learning algorithms usually do not impose assumptions on the model to be learned, they have recently been adopted to determine an option value function directly from the market data, with the goal of avoiding the model misspecification issues from the parametric modeling approach e.g.,[**? ? ?**]. Unfortunately, using nonparametric learning, hedging positions still need to be computed from the sensitivity of the model value function. While no assumption is explicitly made for the dynamics of underlying asset, the option value function is determined by data through cross-validation, leading to training errors. Since there is no assurance that the sensitivity of the learned option value function with regards to underlying asset matches that the sensitivity of the market option price, the parameters of the model learned directly from data can similarly exhibit dependence on the underlying price. When the hedging position is computed from the partial derivative of the data-driven option value function, e.g., [**?**], this dependence cannot be accounted for and again is ignored. Hence, option hedging risk remains insufficiently minimized.

Furthermore, using option delta $\frac{\partial V}{\partial S}$ as the hedging position becomes inadequate when discrete hedging is performed in practice, particularly when rebalancing becomes infrequent. Instead, optimal discrete hedging strategy can be determined by determining hedging strategy directly using an appropriate objective in the discrete hedging context, e.g., minimizing the variance of the hedging error, needs to be chosen [**? ? ?**].

In hedging, the ultimate goal is to discover a hedging strategy which minimizes the hedging error which is measured by the market option and underlying prices. With the increasing availability of market option prices, a timely question arises: is it possible to learn optimal hedging positions directly from market option price and underlying price data? Up until now, research in learning the hedging position directly from market data is scarce. Recently, a data-driven approach [**?**] is proposed to learn a parametric model for the minimum variance delta hedging based on the analysis of the BS option greeks and underlying market prices. However, the proposed parametric model focuses on the instantaneous hedging error analysis in a parametric model framework.

In this thesis, we study the discrete option hedging problem by explicitly focusing on the issues arising from model specification errors and model parameter dependence. We illustrate that the inability to minimize variance of the hedging error, when determining hedging position from option value function from a parametric model, is also shared by an option model estimated from a nonparametric method. Although a nonparametric modeling approach to option value can potentially lead to smaller mis-specification error, we illustrate that non-parametric model parameters can similarly depend on the underlying. Consequently the sensitivity of the optimally estimated option value function will not lead to minimization of option hedging risk. Furthermore, the estimated pricing function inevitably has errors, due to both model mis-specification, discretization, and numerical roundoff. The error in the value function can potentially be substantially magnified in computing partial derivatives as hedging positions.

We explore several direct market data-driven approaches to bypass challenges mentioned above to achieve effective hedging performance. We first propose a data-driven kernel learning approach [**?**] to learn a local risk minimization hedging model directly from the market data observed at the hedging time t. We learn a hedging function from the market data by minimizing the empirical local hedging risk with a suitable regularization. The local risk corresponds directly to the variance of the hedging error in the discrete rebalancing period. A novel encoder-decoder RNN model GRU_δ [**?**], to extract both sequential and local features at hedging time t from market prices, is later proposed to learn option hedging positions directly from the market. We include a feature weighting procedure to select the most relevant local features at hedging time t and sequential features for the sequential data-driven model GRU_δ. Lastly, in order to deal with multi-steps discrete total hedging scenarios where we hedge until the expiry of the option [**?**], we enhance our sequential local hedging model GRU_δ to be $\mathrm{GRU}_{\mathrm{TOTAL}}$. We compared our data-driven approaches with the parametric approaches and demonstrate the effectiveness of the data-driven hedging models in terms of both local hedging risk and total hedging risk.

1.1 Contribution

The contributions with respect to the data-driven kernel hedging model [**?**] are summarized below:

- We analyze and discuss implications from model mis-specification in the option value function for discrete option hedging. We illustrate challenges in accounting for the dependence of the calibrated model parameters on the underlying, which arises due to model mis-specification.
- We analyze a regularized kernel network for option value estimation and illustrate that the partial derivative of the estimated value function with respect to the underlying similarly does not minimize variance of the hedging risk in general, even infinitesimally.
- We propose a data-driven approach to learn a hedging position function directly by minimizing the variance of the local hedging risk. Specifically we implement a regularized

spline kernel method $\mathrm{DKL}_{\mathrm{SPL}}$ to nonparametrically estimate the hedging function from the market data.

- Using synthetic data sets, we compare daily, weekly, and monthly hedging performance using the kernel direct data-driven hedging approach with the performance of the indirect approach where hedging positions are computed from the sensitivity of the nonparametric option value function. In particular, we present computational results which demonstrate that the direct spline kernel hedging position learning outperforms the hedging position computed from the sensitivity of the spline kernel option value function.
- Using S&P 500 index option market data for more than a decade ending in August 31, 2015, we demonstrate that the daily hedging performance of the direct spline kernel hedging function learning method significantly surpasses that of the minimum variance quadratic hedging formula proposed in [**?**], as well as corrective methods based on LVF and SABR implemented in [**?**].
- We also present weekly and monthly hedging results using the S&P 500 index option market data and demonstrate significant enhanced performance over the BS implied volatility hedging.

The contributions with respect to the data-driven sequential hedging model [**?**] are summarized below:

- We propose a novel encoder-decoder RNN model, to extract both sequential and current features from market prices, to learn option hedging positions directly from the market. We include a feature weighting procedure to select the most relevant local features and sequential time series features for the data-driven model.
- To ensure robust learning, we use the Huber loss function as the learning objective, adaptively setting the error resolution parameter to the BS hedging error, allowing it to varying from data instance to data instance. Furthermore, the proposed GRU_δ can be updated more frequently than the data-driven model in [**?**] to account for the market shifts.
- Using the S&P 500 index option market data from January 2, 2004 to August 31st, 2015, we demonstrate that the weekly and monthly hedging performance of the proposed GRU_δ significantly surpasses that of the data-driven minimum variance (MV) method in [**?**], the regularized kernel data-driven model [**?**], and the SABR-Bartlett method [**?**].
- Using the S&P 500 index option market data from January 2, 2004 to August 31st, 2015, we demonstrate that the daily hedging performance of the proposed GRU_δ surpasses that of the minimum variance quadratic hedging method proposed in [**?**], the corrective methods based on LVF and SABR implemented in [**?**],the SABR-Bartlett method [**?**], as well as the data-driven model in [**?**].

- To motivate the roles of each major component of the proposed GRU$_\delta$, we demonstrate performance sensitivity through computational experiments. In addition, we illustrate and analyze the relative importance of selected features.

The contributions with respect to the data-driven total hedging model [**?**] are summarized below:

- We enhance the sequential data-driven local hedging model GRU$_\delta$ [**?**] to cope with total hedging scenarios where we rebalance multiple times until the expiries of the options.
- We augment the market data using SABR model and Local Volatility Function to cope with the challenges of lacking market option data.
- Using the S&P 500 index option market data from January 2, 1996 to August 31st, 2015, we demonstrate that the weekly, bi-weekly and monthly hedging performance of the proposed total hedging model GRU$_{\text{TOTAL}}$ surpasses that of sequential data-driven local hedging model GRU$_\delta$ [**?**] and the SABR-Bartlett method [**?**], when the hedging performance is evaluated on the expiries of the option.
- We compare GRU$_{\text{TOTAL}}$ based on market option information with GRU$_{\text{TOTAL}}$ based on purely underlying asset information and indicate the importance of market option information in determining the total hedging position.

1.2 Outline

The remainder of the thesis is organized as follows. Chapter 2 reviews basic concept of derivative pricing models, discrete hedging problems, local and total hedging risk and various existing parametric approaches to hedge options. Chapter 3 discusses the kernel learning framework and introduces the data-driven kernel local hedging model DKL$_{\text{SPL}}$. Empirical results from data-driven kernel local hedging model DKL$_{\text{SPL}}$ are also discussed in Chapter 3. Chapter 4 discusses the Recurrent Neural Network(RNN) framework and introduces the data-driven sequential local hedging model GRU$_\delta$. Empirical results from the data-driven sequential local hedging model GRU$_\delta$ are also discussed in Chapter 4. Chapter 5 discusses the challenges of using market data to build data-driven total hedging models and the data augmentation procedure to cope with the challenges. Chapter 6 introduces the data-driven sequential total hedging model GRU$_{\text{TOTAL}}$ and presents the empirical comparisons between local hedging model GRU$_\delta$ and total hedging model GRU$_{\text{TOTAL}}$. We conclude in Chapter 7 with summary remarks and potential extensions.

Chapter 2

Background

2.1 Option Pricing Model

In this chapter, we review the basic concept of option pricing models and discuss the problem of pricing model parameters dependence on underlying asset. In addition, we specify the discrete hedging problem and define the total and local hedging risk. Most of the discussion in this chapter are drawn from [**? ? ? ? ?**].

2.1.1 Black-Scholes Model

A European style call or put option gives its buyer the right to buy the underlying asset on the option expiry with a strike price. Let the strike price be K and the S_T be the underlying price at expiry T. The payoff of call options C_T is :

$$C_T = \max(S_T - K, 0)$$

The payoff of put options P_T is:

$$P_T = \max(K - S_T, 0)$$

Black and Scholes [**?**] drive the famous closed-form pricing formula for European options. They show that one can construct a riskless portfolio consisting of one option and shares of the underlying asset. The riskless portfolio needs to be continuously adjusted so that the number of shares always equal to the partial derivative of the option pricing function with regards to the underlying asset. No-arbitrage condition implies that the the return of the riskless portfolio must be equal to the risk-free interest rate. This leads to the renowned Black-Scholes (BS) partial differential equation and the closed-form pricing formula.

More specifically, under BS model, it is assumed that the underlying asset price follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a standard Brownian motion, μ is the constant drift rate of the asset and σ is the constant volatility of the asset. We can easily show that:

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

Let $C(t,S)$ be the option value function for call option. Follows Ito's lemma [**?**],we have:

$$dC(t,S_t) = \left(\frac{\partial C}{\partial t}(t,S_t) + \mu S_t \frac{\partial C}{\partial S}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}(t,S_t) \right) dt + \sigma S_t \frac{\partial C}{\partial S}(t,S_t) dW_t$$

Considering a self-financing trading strategy, where any required cash is borrowed, and any excess cash is loaned, we continously adjust the shares in underlying asset to always hold $-\frac{\partial C}{\partial S}(t,S_t)$ shares at time t. The total value of the replicating portfolio V_{rep} is:

$$V_{rep}(t,S_t) = C(t,S_t) - S_t \frac{\partial C}{\partial S}(t,S_t)$$

The instantaneous profit or loss is:

$$dV_{rep}(t,S_t) = \left(\frac{\partial C}{\partial t}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S^2}(t,S_t) \right)$$

Assume there is a riskless asset with constant rate of return r, which is also known as risk-free interest rate. We can see that the replicating portfolio V_{rep} is also riskless because the diffusion term associated with dW_t is dropped. Under no-arbitrage condition, two riskless investment must earn the same rate of return so we must have:

$$dV_{rep}(t,S_t) = rV_{rep}(t,S_t)dt$$

This leads to the Black-Scholes partial differential equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{2.1.1}$$

The solution with European call option is the well-known Black-Scholes pricing formula:

$$C(t,S) = S\,CDF(d_1) - e^{-r(T-t)}K\,CDF(d_2) \tag{2.1.2}$$

where CDF is the cumulative density function of standard normal distributon

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \;\; d_2 = d_1 - \sigma\sqrt{T-t}$$

Similarly, we can derive the Black-Scholes pricing formula for European put option to be:

$$P(t,S) = e^{-r(T-t)}K\,CDF(-d_2) - S\,CDF(-d_1) \tag{2.1.3}$$

Alternatively, we can derive the Black-Scholes formula under the risk-neutral pricing framework. As the name suggests, under a risk-neural measure Q, all agents in the economy are neutral to risk, so that they are indifferent between investments with different risk as long as these investments have the same expected return. Under a risk-neutral measure, all tradable assets should have the same expected rate of return as the risk-free asset, which is the risk free interest rate r. The derivative price can thus derived from the expected payoff, discounted back to the current time at the risk-free rate r. It can been shown that [**?**], following the Black-Scholes assumption, there is a unique risk-neutral probability measure Q that is equivalent to the actual physical probability measure. Under risk-neutral pricing framework, we have:

$$C(t,S) = e^{-r(T-t)}E^{Q}[\max(S_T - K, 0)] \tag{2.1.4}$$

$$P(t,S) = e^{-r(T-t)}E^{Q}[\max(K - S_T, 0)] \tag{2.1.5}$$

where $E^{Q}[\cdot]$ is the expectation under the risk-neutral measure Q. More specifically, define Θ_1 to be the market price of risk:

$$\Theta_1 = \frac{\mu - r}{\sigma} \tag{2.1.6}$$

We change the original Brownian motion dW_t to $d\hat{W}_t$ with

$$d\hat{W}_t = dW_t + \Theta_1 dt \tag{2.1.7}$$

The underlying dynamic will have the drift to be the risk-free interest rate r.

$$dS_t = rS_t dt + \sigma S_t d\hat{W}_t$$

It can be shown that $d\hat{W}_t$ is the Brownian motion under the risk-neutral measure Q defined through Radon-Nikodym derivative via Girsanov's theorem [**?**]. Using the fact that the drift rate of underlying asset dynamic under risk-neutral measure Q is r, following (**??**) and (**??**), we can arrive at the same pricing formula as (**??**) and (**??**). Interest reader can refer to [**?**] for more details about risk-neutral pricing and change from physical measure to risk-neutral measure Q.

Since actual drift μ is irrelevant in determining the option price under Black-Scholes framework, in this thesis, we assume we are dealing with risk-neutral measure Q and the drift of underlying dynamic is always the risk-free interested r. Also, in this thesis, we use $V_{BS}(S,t,T,K,r;\sigma)$ to denote the European Black-Scholes pricing function regardless of the call or put nature.

Although, Black-Scholes framework provides a nice close-form formula, the real markets are never as ideal as the assumption of Black-Scholes model. Empirical evidence indicates markets often violate the assumption of Black-Scholes model. The two major aspects that has been criticized about Black-Scholes model are:

1. The constant volatility does not hold in real market. In practice, the implied volatility σ_{imp}, which equates the Black-Scholes option price $V_{BS}(S,t,T,K,r;\sigma)$ to market option price $V_{t,T,K}^{mkt}$, is often used to make sure that Black-Scholes price match the market observation.

However, one can often find that the implied volatility σ_{imp} tends to differ across different strikes and expiries. This breaks down the assumption of a constant volatility

2. Transaction cost exist in real market. Due the existence of transaction cost, continuously adjusting the shares of underlying is prohibitively expensive. Therefore, perfect replication of the option is impossible and the argument of the Black-Scholes theory falls apart.

The market deviations from the assumption of Black-Scholes model motivates people to propose various approaches for relaxing the assumptions of Black and Scholes. These attempts include, but not limited to, local volatility models [**? ? ? ?**], stochastic volatility models [**? ? ? ?**], jump diffusion models [**? ?**] and nonparametric pricing models based on regression [**? ? ? ? ?**]. In the following section, we discuss two stochastic volatility models, Heston model and SABR model, which provide efficient closed-form solutions for the option price similar as Black-Scholes model.

2.1.2 Heston Model

Heston [**?**] proposed a version of the stochastic volatility model, which has become quite popular to model the volatility smiles. One of the key reason for its popularity is that European call and put option under Heston model have closed-form solution which makes the calibration of the model computationally efficient and accurate. The Heston model assumes that the underlying, S_t follows a Black-Scholes type stochastic process, but with a stochastic variance υ_t that follows a Cox, Ingersoll, Ross (CIR) process [**?**].

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{\upsilon} S_t dW_t \\ d\upsilon_t &= \kappa(\overline{\upsilon} - \upsilon_t)dt + \eta\sqrt{\upsilon_t} dZ_t \\ E[dZ_t dW_t] &= \rho dt \end{aligned}$$

These parameters are described as follows:

- μ is the drift coefficient of the underlying asset
- $\overline{\upsilon}$ is the long term mean of variance
- κ is the rate of mean reversion
- η is the volatility of volatility
- S_t is the underlying asset price
- υ_t is the instantaneous variance
- W_t and Z_t are correlated Wiener process with correlation coefficient ρ

Similarly, the Heston dynamics can be transformed to be under a risk-neutral measure Q. Heston [**?**] assumes that the market price of volatility risk is proportional to the volatility $\sqrt{\upsilon_t}$:

$$\Theta_2 = \frac{\lambda}{\eta}\sqrt{\upsilon_t} \tag{2.1.8}$$

λ is a parameter used to generate the market price of volatility risk. Recall that market price of risk is:

$$\Theta_1 = \frac{\mu - r}{\sqrt{\upsilon_t}}$$

It can be shown that a risk-neutral measure Q can be defined through Radon-Nikodym derivative via Multi-dimensional Girsanov's theorem [**?**] using the Θ_1 and Θ_2.

Therefore, Heston model under a risk-neutral measure Q is:

$$\begin{aligned} dS &= rSdt + \sqrt{\upsilon}Sd\hat{W} \\ d\upsilon &= \kappa^*(\overline{\upsilon}^* - \upsilon)dt + \eta\sqrt{\upsilon}d\hat{Z} \\ E[d\hat{Z}d\hat{W}] &= \rho dt \end{aligned} \tag{2.1.9}$$

where

$$\kappa^* = \kappa + \lambda, \overline{\upsilon}^* = \frac{\kappa\overline{\upsilon}}{\kappa + \lambda}$$

$$d\hat{W} = dW + \Theta_1 dt$$

$$d\hat{Z} = dZ + \Theta_2 dt$$

Similar to Black-Scholes model, Heston model has closed-form solution. The closed formed solution for European call option under the risk-neutral measure Q is

$$C(S,t,T,K,r;\upsilon,\kappa^*,\overline{\upsilon}^*,\eta,\rho) = S\,N_1 - Ke^{-r(T-t)}\,N_2$$

Let us define the imaginary unit $\mathscr{I}^2 = -1$. Then the N_1 and N_2 are defined as:

$$N_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-\mathscr{I}\varphi\ln K} f_j(S,\upsilon,t,T;\varphi)}{\mathscr{I}\varphi}\right]d\varphi; \quad j = 1,2$$

with $Re[\cdot]$ denoting the real part. The characteristic function $f_j(S,\upsilon,t,T;\varphi)$ is :

$$f_j(S,\upsilon,t,T;\varphi) = e^{A_j(t,T,\varphi) + B_j(t,T,\varphi)\upsilon + \mathscr{I}\varphi\ln S} \quad j = 1,2$$

Where:

$$
\begin{aligned}
A_j(t,T,\varphi) &= r\varphi(T-t) + \frac{\kappa^*\overline{\upsilon}^*}{\eta^2}\left\{(b_j - \rho\eta\varphi\mathscr{I} + d_j)(T-t) - 2\ln\left[\frac{1-g_je_j^d(T-t)}{1-g_j}\right]\right\} \\
B_j(t,T,\varphi) &= g_j\left[\frac{1-e_j^d(T-t)}{1-g_je^{d_j(T-t)}}\right] \\
g_j &= \frac{b_j - \rho\varphi\mathscr{I} + d_j}{b_j - \rho\varphi\mathscr{I} - d_j} \\
d_j &= \sqrt{(\rho\eta\varphi\mathscr{I} - b_j)^2 - \eta^2(2u_j\varphi - \varphi^2)} \\
u_1 &= 0.5, u_2 = -0.5, b_1 = \kappa^* - \rho\upsilon, b_2 = \kappa^*
\end{aligned}
$$

European put option price can be derived from the call-put parity:

$$
P(S,t,T,K,r;\upsilon,\kappa^*,\overline{\upsilon}^*,\eta,\rho) = C(S,t,T,K,r;\upsilon,\kappa^*,\overline{\upsilon}^*,\eta,\rho) - S + Ke^{-r(T-t)}
$$

The parameters to be calibrated from the market option prices are $\{\upsilon,\kappa^*,\overline{\upsilon}^*,\eta,\rho\}$ where υ is the initial instantaneous variance, $\overline{\upsilon}^*$ is the long term mean of variance under risk-neutral measurement Q, κ^* is the rate of mean reversion under risk-neutral measurement Q, η is the volatility of volatility, and ρ is correlation.

Under Black-Scholes model, an option is dependent on tradable asset S_t. The randomness in option value is solely determined by the randomness of the asset S_t. Such uncertainly can be hedged by continuously adjusting the shares of underlying asset as we have discussed in section **??**. This makes the market complete (i.e, we can construct the replicating portfolio to replicate the option value). Under a stochastic volatility model such as Heston model, the uncertainty of option value comes from both the underlying asset S_t and the volatility (or variance υ_t as in Heston model). The volatility itself is not tradable which makes the market incomplete under stochastic volatility model. Incompleteness implies the risk-neutral measure is not unique. In other words, λ is not unique. Different choices of λ will lead to different risk-neutral measurements. However, one can assume a risk-neutral measure Q exists and calibrate the Heston model to match the the market option prices directly using the dynamics in (**??**) without specifying the λ. In this way, λ has been implied and embedded into the calibrated model parameters κ^* and $\overline{\upsilon}^*$. Interested reader can refer to [**? ?**] for more details of risk-neutral pricing under Heston model. In this thesis, we deal with Heston model under the risk-neural measurement Q. For simplicity, in this thesis, we use $V_{Heston}(S,t,T,K,r;\upsilon,\kappa^*,\overline{\upsilon}^*,\eta,\rho)$ to denote the European Heston pricing function regardless of the call or put nature.

2.1.3 SABR Model

The SABR model [**? ?**] is another stochastic volatility model, which attempts to capture the volatility smile in derivatives markets. The name stands for "Stochastic Alpha, Beta, Rho", referring to the parameters of the model. The SABR model is another popular stochastic volatility model widely used in financial risk management. Its popularity is due to the fact that it can reproduce comparatively well the market-observed volatility smile and that it provides a closed-form formula for the volatility. Given the risk-free interest rate r, the forward F_t with expiry T is:

$$F_t = S_t e^{r(T-t)}$$

In the SABR stochastic volatility model, the forward F_t price follows the following stochastic differential equation:

$$\begin{aligned} dF_t &= \alpha_t (F_t)^{\beta} dW_t \\ d\alpha_t &= \nu \alpha_t dZ_t \\ E[dW_t dZ_t] &= \rho dt \end{aligned}$$

These parameters are described as follows:

- α_t is the instantaneous volatility of the Forward F_t.
- ν is the volatility of instantaneous volatility α_t .
- W_t and Z_t are correlated Wiener process with correlation coefficient ρ

A variant of the Black–Scholes option pricing model, Black model [**?**], is often used together with SABR model. Under Black model, the forward F_t price follows the following stochastic differential equation:

$$dF_t = \sigma_B F_t dW_t$$

where σ_B is the volatility. We use $V_B(F,K,r,t,T;\sigma_B)$ to denote the Black pricing function. For European call option:

$$V_B(F,t,T,K,r;\sigma_B) = e^{-r(T-t)}[F\,CDF(d3) - K\,CDF(d4)]$$

For European put option:

$$V_B(F,t,T,K,r;\sigma_B) = e^{-r(T-t)}[K\,CDF(-d4) - F\,CDF(-d3)]$$

where CDF is the cumulative density function of standard normal distributon

$$d_3 = \frac{\ln(F/K) + \sigma^2/2(T-t)}{\sigma\sqrt{T-t}}, \;\; d_4 = d_3 - \sigma\sqrt{T-t}$$

Consider an option on the forward F with expiry T and strike K at time t. If we force the SABR model price of the option into the form of the Black model valuation formula. Then the implied volatility, which is the value of the σ_B in Black's model that forces it to match the SABR price, is approximately given by:

$$
\begin{aligned}
\sigma_B(F,t,T,K;\alpha,\beta,\nu,\rho) \;\approx\; & \frac{\alpha}{(FK)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\log^2(F/K)+\frac{(1-\beta)^4}{1920}\log^4(F/K)+\cdots\right]}\cdot\frac{z}{x(z)} \\
& \cdot\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(FK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(FK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right](T-t)+\cdots\right\}
\end{aligned}
$$

where

$$
z=\frac{\nu}{\alpha}(FK)^{(1-\beta)/2}\log(F/K),\; x(z)=\log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}
$$

For the special case of at-the-money options, options struck at $K=F$, this formula reduces to

$$
\begin{aligned}
\sigma_{ATM} &= \sigma_B(F,t,T,F;\alpha,\beta,\nu,\rho) \\
&\approx \frac{\alpha}{F^{(1-\beta)}}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{F^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha\nu}{F^{(1-\beta)}}+\frac{2-3\rho^2}{24}\nu^2\right](T-t)+\cdots\right\}
\end{aligned}
$$

Therefore, the European option value under SABR model is given by:

$$
V_{SABR}=V_B(F,t,T,K,r;\sigma_B(F,t,T,K;\alpha,\beta,\nu,\rho))
$$

The $\sigma_B(F,t,T,K;\alpha,\beta,\nu,\rho)$ under SABR model depends on the forward F, the strike K, the time to expiry $T-t$, the initial SABR volatility α, the power of forward β, the volatility of volatility ν, and the correlation ρ.

2.2 Discrete Hedging Problem

2.2.1 Total and Local Hedging Risk

2.3 Parameter Dependence on Underlying Asset

2.3.1 Option Hedging Using the Sensitivity from Pricing Model

2.3.2 Hull-White Correction

2.3.3 SABR and Bartlett Correction

APPENDICES