Option Data Augmentation Using SABR model and Local Volatility Function

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Agenda

Set Up Self-Financing Portfolio

Consider a portfolio P_t which is composed of:

- A short position on option V_t
- Long δ_t (hedging position) shares of S_t
- An amount in a risk-free bank account B_t

The hedging portfolio is rebalanced at discrete times t_i. The hedging position is given by δ_{t_i} Initially, we have

$$P_{t_0} = -V_{t_0} + \delta_{t_0} S_{t_0} + B_{t_0} = 0$$

And

$$B_{t_0} = V_{t_0} - \delta_{t_0} S_{t_0}$$

Rebalance Self-Financing Portfolio

At each rebalancing time t_i, we update our hedging position by change the share we hold from $\delta_{t_{i-1}}$ to δ_{t_i} at t_i, where any required cash is borrowed, and any excess cash is loaned. Assume $\Delta t = t_i - t_{i-1}$ is fixed. The bank account is updated by:

$$B_{t_i} = e^{r\Delta t} B_{t_{i-1}} - S_{t_i}(\delta_{t_i} - \delta_{t_{i-1}})$$

Hedging Objective

Let t_i^+ and t_i^- to be the time immediately after and immediately before t_i. Assume that the performance is measured at the t_N:

$$
\begin{aligned}
P_{t_N^-} &= e^{r\Delta t} B_{t_{N-1}} - V_{t_N} + S_{t_N} \delta_{t_{N-1}} \\
&= \sum_{j=0}^{N-1} \left\{ \left[e^{r(N-j-1)\Delta t} S_{t_{j+1}} - e^{r(N-j)\Delta t} S_{t_j} \right] \delta_{t_i} \right\} \\
&+ e^{rN\Delta t} V_{t_0} - V_{t_N}
\end{aligned}
$$

If we always set $\delta = \frac{\partial V}{\partial S}$ and let $\Delta t \to 0$ (we continuously rebalance the portfolio), then $P_{t_N^-} = 0$. In reality, we can only rebalance discretely and $P_{t_N^-}$ can take positive (profit) and negative value (loss).

Data-Driven Total Hedging Objective

Assume we have M samples of sequences. Each sequence is of length N. Let the hedging position given by a function $\delta_t = f(X_t, y_t)$. The objective of optimizing f is:

$$\min_f \quad \frac{1}{2M} \sum_{i=1}^{M} (P^i_{t_N^-})^2$$

Where $P^i_{t_N^-}$ is the portfolio at t_N^- of sample i:

$$P^i_{t_N^-} = \sum_{j=0}^{N-1} \left\{ \left[e^{r(N-j-1)\Delta t} S^i_{t_{j+1}} - e^{r(N-j)\Delta t} S^i_{t_j} \right] f(\mathbf{X}^i_{t_j}, \mathbf{y}^i_{t_j}) \right\} + e^{rN\Delta t} V^i_{t_0} - V^i_{t_N}$$

Challenges of Obtaining Real Data

The success of building a data-driven model depends on the number of sample paths M. However, the hedging sequence we can gather from real market is limited.

1. Options with specific K and expiry are not traded every day.
   - Interpolation and extrapolation for strikes using SABR Model
2. Options in real market only have fixed expiry dates:
   - Interpolation for expiries using LVF

Arbitrage-Free Surface

Simple interpolation (e.g cubic spline) for strikes and expiries can be problematic:

1. Simple interpolation can create arbitrage.
2. Extrapolation is needed for deep in-money option and deep out-of-money option.

We want to create a price surface that is **arbitrage-free**. In addition, we want to **extrapolate** for deep in-money options and deep out-of-money options.

Static Arbitrage

Let $C(T, K)$ be the continuous call price surface function.

1. No call spread arbitrage:

$$-1 \leq \frac{\partial C(T, K)}{\partial K} \leq 0$$

2. No butterfly spread arbitrage:

$$\frac{\partial^2 C(T, K)}{\partial K^2} \geq 0$$

3. No calendar spread arbitrage:

$$\frac{\partial C(T, K)}{\partial T} \geq 0$$

SABR Model

In the SABR (Stochastic Alpha, Beta, Rho) model, the asset forward F follows the following stochastic differential equation:

$$
\begin{aligned}
dF_t &= \sigma_t (F_t)^\beta dW_t \\
d\sigma_t &= \alpha \sigma_t (\rho dW_t + \sqrt{1-\rho^2} dZ_t) \\
& dW_t, dZ_t \quad \text{are independent}
\end{aligned}
$$

where $F_t = F(t, T) = S_t e^{(r-q)(T-t)}$ with r be the interest rate and q be the dividend yield. The constant parameters β, α satisfy the conditions $0 \le \beta \le 1, \alpha \ge 0$, α is a volatility-like parameter for the volatility. ρ is the instantaneous correlation between the underlying and its volatility.

Black Model and Implied Volatility

The price of a European call option on F with exercise date T

$$C = D(t, T)\mathbb{E}[(F_T - K)^+]$$

where $\mathbb{E}[\cdot]$ is under the risk-neutral measure. $D(t, T) = e^{-r(T-t)}$. Market convention often quotes the price through Black's formula, with $\tau = T - t$

$$C_B = D(t, T)[F_t N(d_+) - K N(d_-)],\ d_\pm = \frac{\log(F_t/K) \pm \frac{1}{2}\sigma_B^2 \tau}{\sigma_B \sqrt{\tau}}.$$

The remaining problem is how to obtain the implied volatility $\sigma_B = \sigma_B(K, F, \tau)$.

Implied Volatility Approximation Under SABR model

$$\sigma_B(K,F,\tau) \approx \frac{\sigma_0}{(FK)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\log^2(F/K)+\frac{(1-\beta)^4}{1920}\log^4(F/K)+\cdots\right]}\cdot\frac{z}{x(z)}$$

$$\cdot\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\sigma_0^2}{(FK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\sigma_0\alpha}{(FK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\alpha^2\right]\tau+\cdots\right\}$$

where σ_0 is the initial volatility

$$z=\frac{\alpha}{\sigma_0}(FK)^{(1-\beta)/2}\log(F/K)$$

$$x(z)=\log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}$$

Model Calibration

The model calibration of SABR is to match implied volatility approximation $\sigma_B(K, F_t, \tau)$ from SABR model to the market observed $\sigma_B^{MKT}(K, F_t, \tau)$. This can be done by solving a optimization problem with the parameters to be α, ρ, σ_0. β is usually predetermined and not optimized together with other parameters.

Benefit of The SABR model

The benefits of SABR model are:

1. Matching the volatility smile very well.
2. Extrapolation to in-money and out-of-money options.
3. Pricing vanilla options is straightforward.
4. Accurate and efficient implied volatility computation.

Interpolation and Extrapolation for Strikes

The process of interpolation and extrapolation for strikes using SABR Model is

1. Fit $\sigma_B(K, F_t, \tau)$ to the market implied volatility smile for each expiry T.
2. Compute the European call or put price by plugging in the $\sigma_B(K, F_t, \tau)$ for any K into the black pricing formula.

$$C_{SABR}(T, K) = D(t, T)[F_t N(d_+) - KN(d_-)]$$

$$d_\pm = \frac{\log(F_t/K) \pm \frac{1}{2}\sigma_B^2(K, F_t, T-t)\tau}{\sigma_B(K, F_t, T-t)\sqrt{T-t}}$$

Dupire's Equation

The Dupire formula enables us to deduce the volatility function in a local volatility model from quoted put and call options in the market.

- Under a risk-neutral measure, we assume:

$$\frac{dS_t}{S_t} = (r - q)\,dt + \sigma(t, S_t)dZ_t$$

- Let C be the call option pricing function. Dupire's equation states:

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2(T, K)K^2\frac{\partial^2 C}{\partial K^2} - (r - q)K\frac{\partial C}{\partial K} - qC$$

Dupire's Equation Using Forward Price

Define the normalized call price in terms of discount factor $D(t, T)$ and forward price $F(t, T)$ and the moneyness κ as:

$$D(t, T) = e^{-r(T-t)}$$

$$F(t, T) = S_t e^{(r-q)(T-t)}$$

$$\kappa = \frac{K}{F(t, T)}$$

$$\widehat{C}(T, \kappa) = \frac{C(T, \kappa F(t, T))}{D(t, T)F(t, T)} = \frac{C(T, K)}{D(t, T)F(t, T)}$$

Dupire's equation can be rewritten as:

$$\frac{\partial \widehat{C}}{\partial T} = \frac{1}{2}\widehat{\sigma}^2(T, \kappa)\kappa^2 \frac{\partial^2 \widehat{C}}{\partial \kappa^2}, \quad \widehat{\sigma}(T, \kappa) = \sigma(T, K)$$

Interpolation for Expiries using LVF

The purpose of volatility Interpolation [1] is to create a price surface that can be used to obtain option price with unobserved expiries from market. We recursively solve the finite difference discretization of the Dupure forward equation using fully implicit method:

$$\frac{\partial \widehat{C}}{\partial T} = \frac{1}{2}\widehat{\sigma}^2(T,\kappa)\kappa^2 \frac{\partial^2 \widehat{C}}{\partial \kappa^2}$$

with initial condition $\widehat{C}(0,\kappa) = max(1-\kappa, 0)$, lower boundary condition $\widehat{C}(T,0) = 1$, upper boundary condition $\widehat{C}(T,\kappa_{max}) = 0$

[1] Jesper Andreasen and Brian Huge. "Volatility interpolation." Risk, 24(3):76, 2011.

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Volatility Interpolation and Option pricing

Given a grid of expiries $0 = T_0 < T_1 < \cdots < T_{max}$ and the grid of moneyness: $0 = \kappa_0 < \kappa_1 < \cdots < \kappa_{max}$ and a expiry T_i , we assume $\sigma(T_i, K)$ to be a piecewise constant functions for a given T_i. We further assume:

$$\frac{\partial^2 \widehat{C}(T,0)}{\partial \kappa^2} = \frac{\partial^2 \widehat{C}(T,\kappa_{max})}{\partial \kappa^2} = 0$$

Volatility Interpolation Matrix Form

The fully implicit finite difference method in matrix form is, where M is a tri-diagonal matrix parametrized by $\sigma(T_i, .)$:

$$\begin{bmatrix} \widehat{C}(T_i, \kappa_0) \\ \widehat{C}(T_i, \kappa_1) \\ \widehat{C}(T_i, \kappa_2) \\ \vdots \\ \widehat{C}(T_i, \kappa_{max}) \end{bmatrix} = M \begin{bmatrix} \widehat{C}(T_{i+1}, \kappa_0) \\ \widehat{C}(T_{i+1}, \kappa_1) \\ \widehat{C}(T_{i+1}, \kappa_2) \\ \vdots \\ \widehat{C}(T_{i+1}, \kappa_{max}) \end{bmatrix}$$

Model Calibration

$$M^{-1}\begin{bmatrix}\widehat{C}(T_i,\kappa_0)\\ \widehat{C}(T_i,\kappa_1)\\ \widehat{C}(T_i,\kappa_2)\\ \vdots\\ \widehat{C}(T_i,\kappa_{max})\end{bmatrix}=\begin{bmatrix}\widehat{C}(T_{i+1},\kappa_0)\\ \widehat{C}(T_{i+1},\kappa_1)\\ \widehat{C}(T_{i+1},\kappa_2)\\ \vdots\\ \widehat{C}(T_{i+1},\kappa_{max})\end{bmatrix}$$

We try to find M that so that $\widehat{C}(T_{i+1},\kappa_j)=\widehat{C}_{SABR}(T_{i+1},\kappa_j)$.

$$\widehat{C}_{SABR}(T_{i+1},\kappa_j)=\frac{C_{SABR}(T,K_j)}{D(t,T_{i+1})F(t,T_{i+1})}$$

$$K_j=\kappa_j F(t,T_{i+1})$$

This can be done by:

$$\inf_{\sigma(T_i,.)}\sum_j\left(\frac{\widehat{C}(T_{i+1},\kappa_j)-\widehat{C}_{SABR}(T_{i+1},k_j)}{Vega_B(T_{i+1},\kappa_j)}\right)^2$$

Interpolation For Expiries

After optimization, the local volatility functions are translated into arbitrage-consistent prices for a discrete set of expiries but it does not directly specify the option prices between the expiries. For $T \in [T_i, T_{i+1})$, we can fill in the gaps by:

$$\frac{\widehat{C}(T,\kappa) - \widehat{C}(T_i,\kappa)}{T - T_i} = \frac{1}{2}\widehat{\sigma}^2(T_i,\kappa)\kappa^2 \frac{\partial^2 \widehat{C}(T,\kappa)}{\partial \kappa^2}$$

We then recover the call price by:

$$C(T,K) = \widehat{C}(T,\kappa)D(t,T)F(t,T)$$

Benefits of Interpolation Based On Local Vol Model

Interpolation based on the above procedure can guarantee the option price given by interpolation is arbitrage-free:

1. No call spread arbitrage:

$$-1 \leq \frac{\partial C(T,K)}{\partial K} \leq 0$$

2. No butterfly spread arbitrage:

$$\frac{\partial^2 C(T,K)}{\partial K^2} \geq 0$$

3. No calendar spread arbitrage:

$$\frac{\partial C(T,K)}{\partial T} \geq 0$$

Arbitrage in SABR Model

- Although SABR model is convenient, it is not arbitrage-free. Butterfly and call spread arbitrage can exist for low strike.
- For each expiry, a separate set SABR parameters are calibrated so calender spread arbitrage can also exist.
- Therefore, before using the SABR price as the target for the LVF calibration, we need to fix the potential arbitrage.

Check for Arbitrage

Given the grid of strikes: $0 = K_0 < K_1 < \cdots < K_{max}$, and a grid of expiries $0 = T_0 < T_1 < \cdots < T_{max}$, let $C_{i,j} = C_{SABR}(T_i, K_j)$, we can check and fix

- No call spread arbitrage:

$$0 \leq \frac{C_{i,j-1} - C_{i,j-1}}{K_j - K_{j-1}} \leq 1$$

- No butterfly spread arbitrage:

$$C_{i,j-1} - \frac{K_{j+1} - K_{j-1}}{K_{j+1} - K_j} C_{i,j} + \frac{K_j - K_{j-1}}{K_{j+1} - K_j C_{i,j+1}} \geq 0$$

- No calender spread arbitrage:

$$C_{i+1,j} - C_{i,j} \geq 0$$

SABR Model Calibration Results

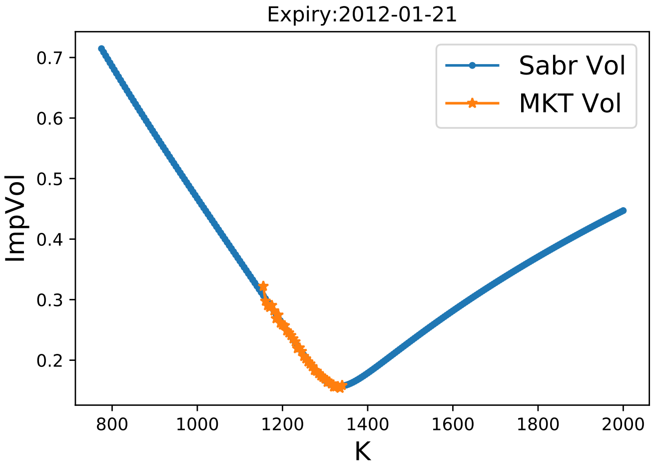
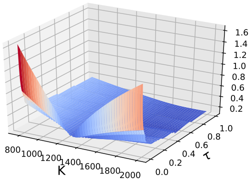
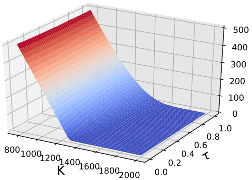


Figure: The Price Surface and Implied Vol Surface for 2012-01-04

LVF Model Calibration Results



(a) ImpVol



(b) Price

Figure: The Price Surface and Implied Vol Surface for 2012-01-04 SP500 index option

Hedging in SABR model

Based on SARR model, a correction formula for BS delta hedging is given by:

$$\delta_{SABR} = \frac{\partial C_B}{\partial F} + \frac{\partial C_B}{\partial \sigma_B} \frac{\partial \sigma_B}{\partial F}$$

The first term is the regular Black delta. The second term is the SABR model's correction to the risk-Black vega times the predicted change in the implied volatility σ_B caused by the change in the forward F.

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Problem with δ_{SABR}

Let the SABR implied volatility after calibration be

$$\sigma_B(K, F, \tau; \alpha, \beta, \rho, \sigma_0)$$

δ_{SABR} assume σ_0 is fixed. However, the σ and F are correlated, whenever F changes, σ_0 changes.

Bartlett Corrective Formula

Assume:

$$dF_t = \sigma_t (F_t)^\beta dW_t$$

$$d\sigma_t = \alpha\sigma_t(\rho dW_t + \sqrt{1-\rho^2}\, dZ_t)$$

$$dW_t, dZ_t \quad \text{are independent}$$

We can easily show that:

$$d\sigma_t = \frac{\rho\alpha}{(F_t)^\beta} dF_t + \alpha\sigma_t\sqrt{1-\rho^2}\, dZ_t$$

Therefore, a better Bartlett formula is given by:

$$\delta_{Bartlett} = \frac{\partial C_B}{\partial F} + \frac{\partial C_B}{\partial \sigma_B}\left(\frac{\partial \sigma_B}{\partial F} + \frac{\partial \sigma_B}{\partial \sigma_0}\frac{\rho\alpha}{F^\beta}\right)$$

Comparison between δ_{SABR} and $\delta_{Bartlett}$

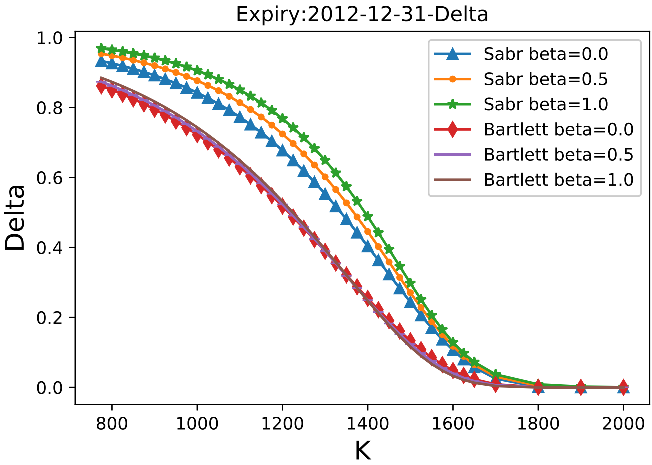

Figure: Comparison based on 2012-01-04 SP500 index option

Connection with Our Research

- The dependence exists generally for all parametric pricing models.
- Accounting for the dependence is not always straightforward as it is under SABR models.
- This motivates us to directly learn a hedging position from the market data bypassing the calibration process.

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Discussion

- In this talk, we discuss an efficient way of creating arbitrage-free surface and we use the surface to augment option data.
- We also discuss the how to adjust the delta hedging position using SABR model.