Data-Driven Models for Discrete Hedging Problem

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Local Volatility Model

The Dupire formula enables us to deduce the volatility function in a local volatility model from quoted put and call options in the market.

- Under a risk-neutral measure, we assume

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(t, S_t)dZ_t$$

- The forward price for delivery at time T:

$$F_t = F(t, T) = S_t e^{\int_t^T r(s)ds}$$

- We also have:

$$\frac{dF_t}{F_t} = \widetilde{\sigma}(t, F_t)dZ_t$$

$$\widetilde{\sigma}(t, F_t) = \sigma(t, F_t e^{-\int_t^T r(s)ds})$$

Local Volatility Model

- The forward price of a call option for delivery at time T: $C(T, K)$
- The spot price at t is: $C(T, K)e^{-\int_t^T r(s)ds}$.
- It can be shown that, with $\theta(T, K)$ be the normal density function of S_T :

$$C(T, K) = \int_K^\infty (x - K)\theta(T, x)dx$$

Breedon-Litzenberger Formulas

If we differentiate pricing function $C(T, K)$ twice we obtain:

$$\frac{\partial C(T, K)}{\partial K} = \Theta(T, K) - 1$$

$$\frac{\partial^2 C(T, K)}{\partial K^2} = \theta(T, K)$$

Forward Equation

Let h be an arbitrary function and $v(t,x) = E[h(F_T)|F_t = x]$, we can show that:

$$E[h(F_T)] = \int_0^\infty v(t,x)\theta(t,x)dx$$

Differentiate them with regards to t:

$$0 = \int_0^\infty \frac{\partial v(t,x)}{\partial t}\theta(t,x)dx + \int_0^\infty \frac{\partial \theta(t,x)}{\partial t}v(t,x)dx$$

Backward Equation

Similar to the deduction of Black-Scholes equation, using ito's lemma, we can see that.

$$\frac{\partial v(t,x)}{\partial t} + \frac{1}{2}\widetilde{\sigma}^2(t,x)x^2\frac{\partial^2 v(t,x)}{\partial x^2} = 0$$

Note that we are dealing with forward price, the term with interest rate is dropped.

Link Backward Equation and Forward Equation

Plug in the backward equation:

$$\frac{\partial v(t,x)}{\partial t} + \frac{1}{2}\widetilde{\sigma}^2(t,x)x^2\frac{\partial^2 v(t,x)}{\partial x^2} = 0$$

into forward equation:

$$0 = \int_0^{\infty} \frac{\partial v(t,x)}{\partial t}\theta(t,x)dx + \int_0^{\infty} \frac{\partial \theta(t,x)}{\partial t}v(t,x)dx$$

We have:

$$0 = -\int_0^{\infty} \frac{1}{2}\widetilde{\sigma}^2(t,x)x^2\frac{\partial^2 v(t,x)}{\partial x^2}\theta(t,x)dx + \int_0^{\infty} \frac{\partial \theta(t,x)}{\partial t}v(t,x)dx$$

Integration By Parts

Using the rule of Integration By Parts twice:

$$
\begin{aligned}
&\int_0^\infty \widetilde{\sigma}^2(t,x)x^2\frac{\partial^2 v(t,x)}{\partial x^2}\theta(t,x)dx\\
=&\widetilde{\sigma}^2(t,x)x^2\frac{\partial v(t,x)}{\partial x}\theta(t,x)-\int_0^\infty \frac{\partial v(t,x)}{\partial x}\frac{\partial[\widetilde{\sigma}^2(t,x)x^2\theta(t,x)]}{\partial x}dx\\
=&\int_0^\infty \frac{\partial^2[\widetilde{\sigma}^2(t,x)x^2\theta(t,x)]}{\partial x^2}v(t,x)dx
\end{aligned}
$$

Integration By Parts:

$$\int uv'dx = uv - \int u'vdx$$

Forward Equation

Plug in:

$$\int_0^\infty \widetilde{\sigma}^2(t,x)x^2\frac{\partial^2 v(t,x)}{\partial x^2}\theta(t,x)dx$$
$$=\int_0^\infty \frac{\partial^2[\widetilde{\sigma}^2(t,x)x^2\theta(t,x)]}{\partial x^2}v(t,x)dx$$

We finally have:

$$0=\int_0^\infty \frac{1}{2}\frac{\partial^2[\widetilde{\sigma}^2(t,x)x^2\theta(t,x)]}{\partial x^2}v(t,x)dx-\int_0^\infty \frac{\partial\theta(t,x)}{\partial t}v(t,x)dx$$
$$=\int_0^\infty [\frac{1}{2}\frac{\partial^2[\widetilde{\sigma}^2(t,x)x^2\theta(t,x)]}{\partial x^2}-\frac{\partial\theta(t,x)}{\partial t}]v(t,x)dx$$

Forward Equation

Note that, it holds for arbitrary $h(F_t)$ and $v(t,x)$:

$$0 = \int_0^\infty [\frac{1}{2}\frac{\partial^2[\widetilde{\sigma}^2(t,x)x^2\theta(t,x)]}{\partial x^2} - \frac{\partial\theta(t,x)}{\partial t}]v(t,x)dx$$

We then must have:

$$\frac{\partial\theta(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2[\widetilde{\sigma}^2(t,x)x^2\theta(t,x)]}{\partial x^2} \qquad (1)$$

Dupire's equation

$$\frac{\partial C(T,K)}{\partial T} = \int_K^\infty (x-K)\frac{\partial \theta(T,x)}{\partial T}dx$$

$$= \int_K^\infty (x-K)\frac{1}{2}\frac{\partial^2[\widetilde{\sigma}^2(T,x)x^2\theta(T,x)]}{\partial x^2}dx \quad [eq(1)]$$

$$= -\frac{1}{2}\int_K^\infty \frac{\partial[\widetilde{\sigma}^2(T,x)x^2\theta(T,x)]}{\partial x}dx$$

$$= \frac{1}{2}\sigma^2(T,K)K^2\theta(T,K)$$

$$= \frac{1}{2}\sigma^2(T,K)K^2\frac{\partial^2 C(T,K)}{\partial K^2}$$

Volatility Interpolation and Option pricing

logo.png

Given a time of grid of expiries $0 = t_0 < t_1 < \dots$ and a a set of volatility functions, we recursively solve the finite difference discretization of the Dupure forward equation:

$$\frac{C(t_{i+1},k) - C(t_i,k)}{t_{i+1} - t_i} = \frac{1}{2}\sigma(t_i,k)^2 k^2 \frac{\partial^2 C(t_{i+1},k)}{\partial k^2} \qquad (2)$$

Volatility Interpolation and Option pricing

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We can assume $\sigma(t_i, k)$ to be a piecewise constant functions for a given t_i and the initial condition is:

$$C(t_0, k) = max(S(t_0) - k, 0)$$

Given the grid of strike: $k_0 < k_1 < \cdots < k_n$ and a expiry t_i and next expiries t_{i+1}, let $C_{kk}^j = \frac{\partial^2 C(t_{i+1}, k_j)}{\partial k^2}$ and $C^j = C(t_{i+1}, k_j)$. The C_{kk}^j can approximated by finite difference:

$$C_{kk}^j = \frac{\frac{C_{j+1} - C_j}{k_{j+1} - k_j} - \frac{C_j - C_{j-1}}{k_j - k_{j-1}}}{\frac{k_{j+1} - k_{j-1}}{2}}$$

Volatility Interpolation and Option pricing

Let us assume $C^0_{kk} = C^n_{kk} = 0$. Let

$$\alpha_j = \frac{1}{(k_j - k_{j-1})(k_{j+1} - k_{j-1})}$$

$$\beta_j = \frac{1}{(k_{j+1} - k_j)(k_j - k_{j-1})}$$

$$\eta_j = \frac{1}{(k_{j+1} - k_j)(k_{j+1} - k_{j-1})}$$

$$z_j = \sigma(t_i, k_j)^2 k^2 (t_{i+1} - t_i)$$

We have:

$$\begin{aligned} C(t_i, k_1) = &-z_1 \alpha_1 C(t_{i+1}, k_1) \\ &+ (1 + z_1 \beta_1) C(t_{i+1}, k_2) - z_1 \eta_1 C(t_{i+1}, k_2) \end{aligned}$$

Volatility Interpolation and Option pricing

In matrix form let matrix M be::

$$\begin{bmatrix} 1 & & & & & \\ -z_1\alpha_1 & 1+z_1\beta_1 & -z_1\eta_1 & & & \\ & -z_2\alpha_2 & 1+z_2\beta_2 & -z_2\eta_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -z_{n-1}\alpha_{n-1} & 1+z_{n-1}\beta_{n-1} & -z_{n-1}\eta_{n-1} \\ & & & & & 1 \end{bmatrix}$$

$$M^{-1}\begin{bmatrix} C(t_i,k_0) \\ C(t_i,k_1) \\ C(t_i,k_2) \\ \vdots \\ C(t_i,k_{n-1}) \\ C(t_i,k_n) \end{bmatrix} = \begin{bmatrix} C(t_{i+1},k_0) \\ C(t_{i+1},k_1) \\ C(t_{i+1},k_2) \\ \vdots \\ C(t_{i+1},k_{n-1}) \\ C(t_{i+1},k_n) \end{bmatrix}$$

Model Calibration

$$M^{-1}\begin{bmatrix} C(t_i,k_0) \\ C(t_i,k_1) \\ C(t_i,k_2) \\ \vdots \\ C(t_i,k_{n-1}) \\ C(t_i,k_n) \end{bmatrix} = \begin{bmatrix} C(t_{i+1},k_0) \\ C(t_{i+1},k_1) \\ C(t_{i+1},k_2) \\ \vdots \\ C(t_{i+1},k_{n-1}) \\ C(t_{i+1},k_n) \end{bmatrix}$$

We try to find M that so that $C(t_{i+1},k_j) = C_m kt(t_{i+1},k_j)$. This can be done by:

$$\inf_{\sigma(t_i,.)} \sum_j \left(\frac{c(t_{i+1},k_j) - c_{mkt}(t_{i+1},k_j)}{Vega_{bs}^{mkt}(t_{i+1},k_j)}\right)^2$$