Higher Order Markov Chains and Spacey Walks

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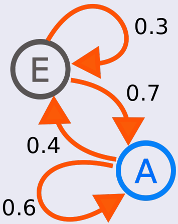
Markov Chains

Markov Chains are stochastic models describing a sequence of states with a probability of transitioning between each.

- Defined as matrix: $\mathbf{P} \in \mathbb{R}^{n \times n}$, $\mathbf{P}_{ij}$ gives probability of moving to state i from j, for chain with n states.
- Given distribution/state vector $\mathbf{x}^{(n)}$, one step of a random walk is represented by matrix-vector product $\mathbf{x}^{(n+1)} = \mathbf{P}\mathbf{x}^{(n)} \;\Leftrightarrow\; x_i^{(n+1)} = \sum_j P_{ij} x_j^{(n)}$.



Transition matrix:

$$\mathbf{P} = \begin{array}{cc} & \begin{array}{cc} A & E \end{array} \\ \begin{array}{c} A \\ E \end{array} & \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \end{array}$$

Figure from https://commons.wikimedia.org/wiki/File:Markovkate_01.svg

Definitions

Time Homogeneous

Each transition probability is independent of the time step t.

Irreducible

Every state can be reached from every other state.

Periodic

Periodic markov chain:

$$\exists i \in S \quad s.t. \quad P_{ii} = 0$$

Periodicity implies a state may only return to itself after some multiple d "hops". If every state has self loops the chain is **aperiodic**.

Canonical Examples

Google PageRank

- Markov chain represents the "idealized" web surfer. States are webpages and transition probability is given by number of outgoing links between pages.
- Ranking is determined by probability of being on certain page.

Board Games

- Each spot on game board is a state.
- Probability of moving between spots is given by dice, spinner, etc.

Weather

- States are possible weather conditions.
- Tomorrow's weather is randomly chosen based on today's conditions.

Steady-State Distribution

Steady state vector $\boldsymbol{\pi} \in \mathbb{R}^n$ does not change when multiplied by Markov chain:

$$\boldsymbol{\pi} = \mathbf{P}\boldsymbol{\pi}$$

- Average distribution as random walk tends to length infinity, $\lim_{k \to \infty} \mathbf{P}^k \mathbf{x}$
- Is equivalent to **Perron vector** of matrix, dominant eigenvector with nonnegative components.
- Compute iteratively with power method: $\mathbf{x}^{(k+1)} = \mathbf{P}\mathbf{x}^{(k)}$.

Unique steady states are guaranteed for **irreducible**, **aperiodic** Markov chains.

Markov Chains with Memory

What if we want our chain to **remember** where it has been? Introduce ***m*-order Markov chain**, where transition probability depends on previous *m* states.

- Probability is now stored as $\mathbf{P} \in \mathbb{R}^{n \times n^m}$, where columns are a *m*-tuple of each state permutation.
- Random step becomes slightly more complicated than just a *mat-vec*, will see a better way of doing this.

(Example indexing for 2$^{\text{nd}}$ order chain with $n = 2$ states.)

Higher Order Markov Chains

We can **fold the last dimension** of the higher-order Markov matrix to obtain a higher dimensional **tensor**.

Order m Markov chain with n states is order $m+1$ tensor

$$\mathcal{P} \in \mathbb{R}^{\overbrace{n \times n \times \cdots \times n}^{m+1 \text{ dimensions}}}$$

Transition probability indexed by $P_{i_{(n+1)}, i_{(n)}, i_{(n-1)}, \dots, i_{(n-m+1)}}$

- State/distribution is stored as order m tensor.
- Example random step for order 2 Markov chain: $X_{ij}^{(n+1)} = \sum_k P_{ijk} X_{jk}^{(n+1)}$
- Tensor form gives nice analysis properties — will see shortly.

Higher Order Steady-State Distribution

Steady state tensor $\mathbf{X} \in \mathbb{R}^{n \times n}$ does not change when multiplied along first mode.

$$X_{ij} = \sum_k P_{ijk} X_{jk}$$

- Conceptually simple, but requires $\mathcal{O}(n^m)$ space to store state tensor.

Replace state tensor with **low rank approximation** $\mathbf{x} \in \mathbb{R}^n$:

$$x_i = \sum_{jk} P_{ijk} x_j x_k$$

Actually just the ***z*-eigenvector** of $\mathcal{P}$!

Spacey Walks

Applications, Examples

References

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