Problem Set 3.1

① $f = 1-x$ $\quad w(x) = -\int_0^x f(s)\,ds, \quad u(x) = \int_0^x \frac{w(s)}{c(s)}\,ds$

c – const

$u(0)=0$ with $w(1)=0$ and $u(1)=0$, In other words: ⓐ $u(0) = w(1) = 0$ ⓑ and $u(0) = u(1) = 0$

$$w(x) = -\int_0^x (1-s)\,ds = -\left[s - s^2/2\right]_0^x = \frac{x^2}{2} - x + C_1$$

a) $w(1)=0 \to \frac{1}{2} - 1 + C_1 = 0$, $\boxed{C_1 = \frac{1}{2}}$, $w(x) = \frac{x^2}{2} - x + \frac{1}{2}$

$$u(x) = \frac{1}{c}\int_0^x \left(\frac{s^2}{2} - s + \frac{1}{2}\right) ds = \frac{1}{c}\left[\frac{s^3}{6} - \frac{s^2}{2} + \frac{1}{2}s\right]_0^x = \frac{1}{2c}\left(\frac{x^3}{3} - x^2 + x\right) + C_2$$

$u(0) = 0 \to \frac{1}{2c}\cdot 0 + \boxed{C_2 = 0}$, $\boxed{u(x) = \frac{1}{2c}\left(\frac{x^3}{3} - x^2 + x\right)}$

b) $u(x) = \frac{1}{c}\int_0^x \left(\frac{s^2}{2} - s + C_1\right) ds = \frac{1}{c}\left[\frac{s^3}{6} - \frac{s^2}{2} + C_1 s\right]_0^x = \frac{1}{2c}\left(\frac{x^3}{3} - x^2 + C_1 x\right) + C_2$

$u(0) = 0 \to \boxed{C_2 = 0}$ and $u(1) = 0 \to \frac{1}{2c}\left(\frac{1}{3} - 1 + C_1\right) = 0$, $C_1 - \frac{1}{6} - \frac{1}{2} = 0$

$\boxed{u(x) = \frac{1}{2c}\left(\frac{x^3}{3} - x^2 + \frac{x}{3}\right)}$ $\quad \boxed{C_1 = \frac{1}{3}}$

---

② f – const $\quad w(x) = \int_x^1 f\,ds = (1-x)f + C_1$

$c(x) = (1-x)$

$w(1) = 0 \leftarrow$ conds at the free end $\quad w(1) = 0 = 0\cdot f + C_1 \to C_1 = 0$

$\underline{w(x) = (1-x)f}$

$\underline{w(x) = c\left(\frac{du}{dx}\right) = 0}$ $\quad u(x) = \int_0^x \frac{(1-s)f}{(1-s)}\,ds = fx + C_2$

always non-zero

$\frac{du}{dx} = f \leftarrow$ const speed even where no force

---

⑤ f – const $\quad -\frac{dw}{dx} = f$, $w(1) = 0$, $w(x) = -\int_0^x f\,ds = -fx + C_1$

$c(x) = \begin{cases} 1, & x \le 1/2 \\ 2, & x > 1/2 \end{cases}$ $\quad w(1) = 0 \to -f + C_1 = 0$, $C_1 = f$

$\boxed{w(x) = f - fx = (1-x)f}$

$u(x) = \int_0^x \frac{w(s)}{c(s)}\,ds \to \begin{cases} u(x) = \int_0^{1/2} w(s)\,ds, \text{ for } x \le 1/2 \Rightarrow f\int_0^{1/2}(1-s)\,ds = f\left[s - \frac{s^2}{2}\right]_0^{1/2} \\ u(x) = \int_0^x w(s)\,ds \Rightarrow f\int_0^x (1-s)\,ds \end{cases}$

$= f\left(\frac{1}{2} - \frac{1}{8}\right) = \boxed{f\cdot\frac{3}{8} = u\left(\frac{1}{2}\right)}$

$= f\left(x - \frac{x^2}{2}\right) + C_2$, $\underline{u(0) = 0 = C_2}$

$u(x) = \int_{1/2}^x \frac{w(s)}{2}\,ds$, for $x \geq 1/2 \Rightarrow \frac{1}{2}f\int_{1/2}^x (1-s)\,ds = \frac{1}{2}f\left[s - \frac{s^2}{2}\right]_{1/2}^x = \frac{1}{2}f\left(x - \frac{x^2}{2} \underbrace{- \frac{1}{2} + \frac{1}{8}}_{-3/8}\right)$

for $x \geq 1/2 \to \frac{1}{2}f\int_{1/2}^x (1-s)\,ds + u\left(\frac{1}{2}\right) = \frac{1}{2}f\left(\frac{1}{8} - \frac{1}{2}(x-1)^2\right) = \underline{\frac{1}{16}f - \frac{1}{4}f(x-1)^2}$

$= \frac{7}{16}f - \frac{1}{4}f(x-1)^2$

Answer: $u(x) = \begin{cases} f\left(x - \frac{x^2}{2}\right), & \text{for } 0 \le x \le 1/2 \\ f\left(\frac{7}{16} - \frac{1}{4}(x-1)^2\right), & \text{for } 1/2 \le x \le 1 \end{cases}$