Problem Set 1.1

① $T_5^{-1}T_5 = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I$ ✓

$T_n^{-1} = \begin{bmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & n-1 & n-2 & & 2 & 1 \\ n-2 & n-2 & n-2 & & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & 2 & 1 \\ 2 & 2 & 2 & & \vdots & \vdots \\ 1 & 1 & 1 & & 1 & 1 \end{bmatrix}$

Formula for the entries:

on and below the diagonal:

$T_n^{-1}(i,j) = n-i+1$

on and above the diagonal:

$T_n^{-1}(i,j) = n-j+1$

② $U^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = S$ (sums matrix)

1) $T_3 = U^T U = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$ ✓

2) $U U^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ ✓

3) $T_3^{-1} = (U^T U)^{-1} = U^{-1}(U^{-1})^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ✓

$\underline{(U^T)^{-1} = (U^{-1})^T}$

④ $S_4 S_4^T = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{L} = \underbrace{\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{S\ ?} = T_4^{-1}$

⑤ $K_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\det(K_2) = 3$, $K_2^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\frac{1}{3}$

⑥ $K_4^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 2 & 1 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$, $\overset{i \geq j}{K_n^{-1}}(i,j) = \frac{1}{5}(n-i+1)\,j$! Challenge done!

⑦ $T_4^{-1} - K_4^{-1} = \frac{1}{5}\begin{bmatrix} 16 & 12 & 8 & 4 \\ 12 & 9 & 6 & 3 \\ 8 & 6 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix} = uv^T$

$T_4^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $K_4^{-1} = \frac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

$K_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $T_3 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $K_3 - T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$uv^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = K_3 - T_3$

$T_3^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $K_3^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, $T_3^{-1} - K_3^{-1} = \frac{1}{4}\begin{bmatrix} 9 & 6 & 3 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$uv^T = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$

⑧ a) $T_5^{-1} - K_5^{-1} = \frac{1}{6}\begin{bmatrix} 25 & 20 & 15 & 10 & 5 \\ 20 & 16 & 12 & 8 & 4 \\ 15 & 12 & 9 & 6 & 3 \\ 10 & 8 & 6 & 4 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$

b) $T_5^{-1} = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$, $\cancel{T_5^{-1}} - \cancel{T_5^{-1}} + K_5^{-1} = \frac{1}{6}\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

$= K_5^{-1} =$

⑨ $C_4 = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$, $e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$C_4^T e \overset{!}{=} C_4 e$ (C_4 is symmetric) $C(C_4) = C^T(C_4)$

$C_4^T e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, proved! $Cu = f = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$u = C^+ f = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

Pseudoinverse: Left: $A^+ = (A^TA)^{-1}A^T \rightarrow A^+A = I$

Right: $A^+ = A^T(AA^T)^{-1} \rightarrow AA^+ = I$

(10) $H=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$

$T=\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$ $J=\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}$, $JTJ=H$

$H^{-1}=JT^{-1}J$

$T^{-1}=\begin{bmatrix}3&2&1\\2&2&1\\1&1&1\end{bmatrix}$; $H^{-1}=\underbrace{\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}\begin{bmatrix}3&2&1\\2&2&1\\1&1&1\end{bmatrix}}\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}=$

$\begin{bmatrix}1&1&1\\2&2&1\\3&2&1\end{bmatrix}\begin{bmatrix}&&1\\&1&\\1&&\end{bmatrix}=\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}=H^{-1}$ ✓

Differences matrix

$U=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}$ $H=UU^T=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$ ✓

$U^{-1}=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$, $H^{-1}=(U^{-1})^TU^{-1}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}$ ✓

Sum matrix

(11) $U \to$ upper-triangular, $J=\begin{bmatrix}&&1\\&\dots&\\1&&\end{bmatrix}$, $U=\begin{bmatrix}\text{values}&&\\&\text{vals}&\\&&\text{vals}\end{bmatrix}$

$\begin{bmatrix}&&\text{vals}\\&&\text{values}\\\text{values}&&\end{bmatrix}=JU$

southeast

$UJ=\underbrace{\begin{bmatrix}\text{values}&&\\&\text{values}&\\&&\text{val}\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}&&1\\&\dots&\\1&&\end{bmatrix}}_{J}=\begin{bmatrix}\text{values}&&\\\text{valus}&&\\\text{vals}&&\end{bmatrix}$; $JUJ=\begin{bmatrix}&&\\\text{vals}&&\\\text{values}&&\\\text{values}&&\end{bmatrix}=L$

northwest

$JU \cdot UJ = L$!

(12) $C_4=\begin{bmatrix}2&-1&0&-1\\-1&2&-1&0\\0&-1&2&-1\\-1&0&-1&2\end{bmatrix}\to\begin{bmatrix}2&-1&0&-1\\0&3/2&-1&-1/2\\0&0&4/3&-4/3\\0&-1/2&-1&3/2\end{bmatrix}\to\begin{bmatrix}2&-1&0&-1\\0&3/2&-1&-1/2\\0&0&4/3&-4/3\\0&0&-4/3&4/3\end{bmatrix}\to$

$U(C_4)=\frac{1}{6}\begin{bmatrix}12&-6&0&-6\\0&9&-6&-3\\0&0&8&-8\\0&0&0&0\end{bmatrix}=U_4$ $\leftarrow 0$, because C_4 is singular

(13) $C_4=LU$, $C_4U^+=L$ $\left(\begin{matrix}U^+=U^+\\ \text{Right}\end{matrix}\right)$ $U^+=U^T(UU^T)^{-1}$

$U^+ = U^T (UU^T)^{-1}$

$$UU^T = \frac{1}{36}\begin{bmatrix} 12 & -6 & 0 & -6 \\ 0 & 9 & -6 & -3 \\ 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 12 & 0 & 0 & 0 \\ -6 & 9 & 0 & 0 \\ 0 & -6 & 8 & 0 \\ -6 & -3 & -8 & 0 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 216 & -36 & 48 & 0 \\ -36 & 126 & -24 & 0 \\ 48 & -24 & 128 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(UU^T)^{-1} = \frac{1}{72}\begin{bmatrix} 18 & 12 & 9 \\ 12 & 40 & 30 \\ 9 & 30 & 63 \end{bmatrix}$$

$$U^T (UU^T)^{-1} = \frac{1}{432}\begin{bmatrix} 216 & 144 & 108 \\ 0 & 288 & 216 \\ 0 & 0 & 324 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 6 & 4 & 3 \\ 0 & 8 & 6 \\ 0 & 0 & 9 \end{bmatrix} = U^+$$

$$L = C_4 U^+ = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 4 & 3 & 0 \\ 0 & 8 & 6 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\frac{1}{12}$$

$$= \frac{1}{12}\begin{bmatrix} 12 & 0 & 0 & 0 \\ -6 & 12 & 0 & 0 \\ 0 & -8 & 12 & 0 \\ -6 & -4 & -12 & 0 \end{bmatrix} = L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{3} & -1 & 0 \end{bmatrix}$$

Check $C_4 = LU = \frac{1}{72}\begin{bmatrix} 12 & 0 & 0 & 0 \\ -6 & 12 & 0 & 0 \\ 0 & -8 & 12 & 0 \\ -6 & -4 & -12 & 0 \end{bmatrix}\begin{bmatrix} 12 & -6 & 0 & -6 \\ 0 & 9 & -6 & -3 \\ 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$, Prooved!

(14) $K_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ $\det(K_3) = 4$

$\begin{bmatrix} 1.41421 & -1 & 0 \\ -1 & 1.41421 & -1 \\ 0 & -1 & 1.41421 \end{bmatrix} = M \leftarrow$ singular

diagonal entries $= 1.41421 \approx \sqrt{2}$

Let's solve the equation!

$\left(\begin{bmatrix} 2x & -1 & 0 \\ -1 & 2x & -1 \\ 0 & -1 & 2x \end{bmatrix}\right)$ $\det = 0$, $2x(4x^2-1) - 2x = 0$

$8x^3 - 4x = 0$, $x = 0$ or $8x^2 = 4$

$x = \pm\frac{\sqrt{2}}{2}$!

$2x = \pm\sqrt{2}$

$$M = \begin{bmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{bmatrix}$$

~~det~~ $M = \sqrt{2}\,(\underbrace{\sqrt{2}^2 - 1}_{1}) - \sqrt{2} = \sqrt{2} - \sqrt{2} = 0$!

M is singular !

diagonal entries $= \pm\sqrt{2}$ or 0

$Mu = 0$, $u = (1, \sqrt{2}, 1)$

(15) $A_{n\times n} \cdot x_{n\times 1}$ ← n Rows with n items times the n items in x

$n \times n = n^2$ individual multiplications

$A_{n\times n} A_{n\times n} = n \times n \times n = n^3$

$A_{m\times n} B_{n\times p} = (AB)_{m\times p} = n \times p \times m$

(16) $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$Ax = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$; $Ax = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot 1 + \begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot 2 = \begin{bmatrix} 2+6 \\ 4+10 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

by rows by cols

(17) $Ax = b$ have a solution vector x exactly when b is a combination of the columns

$\underset{A}{\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}} \underset{x}{x} = \underset{b}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ have a solution, because $C(A) = \mathbb{R}^2$ ← whole dim

In case of less dimensions, we can't solve the equation (dependent cols)

$\underset{A}{\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}} \underset{x}{x} = \underset{b}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ ← not solvable !

(18) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \overset{\text{col1*row1}}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix}} + \overset{\text{col2*row2}}{\begin{bmatrix} 3 \\ 5 \end{bmatrix}\begin{bmatrix} 2 & 4 \end{bmatrix}} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 6 & 12 \\ 10 & 20 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 14 & 28 \end{bmatrix}$ ✓

$\begin{bmatrix} A * \text{col1 of } B & A * \text{col2 of } B \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 14 & 28 \end{bmatrix}$

$\begin{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{bmatrix} =$

(19) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{matrix} 2\begin{bmatrix} 1 & 2 \end{bmatrix} + 3\begin{bmatrix} 2 & 4 \end{bmatrix} \\ 4\begin{bmatrix} 1 & 2 \end{bmatrix} + 5\begin{bmatrix} 2 & 4 \end{bmatrix} \end{matrix} \overset{\checkmark}{=} \begin{bmatrix} 8 & 16 \\ 14 & 28 \end{bmatrix}$

$Bx = 0$, $(AB)x = 0$; $A(Bx) = A \cdot 0 = 0$ ✓

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 8 & 16 \\ 14 & 28 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(20) $\begin{bmatrix}2&3\\4&5\end{bmatrix}\begin{bmatrix}1&2\\2&4\end{bmatrix}=\underbrace{\begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}1&2\end{bmatrix}}_{n}+\underbrace{\begin{bmatrix}3\\5\end{bmatrix}\begin{bmatrix}2&4\end{bmatrix}}_{n}=\begin{bmatrix}2&4\\4&8\end{bmatrix}+\begin{bmatrix}6&12\\10&20\end{bmatrix}$

$=\begin{bmatrix}8&16\\14&28\end{bmatrix}$ ✓

n^2 multiplications + n^2 sums

(21) $AB \neq BA$ for most of cases

$(AB)A = A(BA)$ ✓

$(AB)B \neq B(BA)$ for most of cases

$(AB)^2 \neq A^2B^2$, $\underline{(AB)(AB) \neq A^2B^2}$

(25) $Ku = 0$

$$Ku=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}2u_1-u_2\\-u_1+2u_2-u_3\\-u_2+2u_3\end{bmatrix}$$

means $-u_2+2u_3=0$, $u_2=2u_3$

$-u_1+2u_2-u_3=0$, $\underline{u_2=(u_1+u_3)/2}$

$2u_1-u_2=0$, $u_2=2u_1$, So $\underline{u_1=u_3=u_2\cdot 2}$

possible only for $\underline{u=(0,0,0)}$!

K is invertible

$Tu=\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$ ← $\underline{\text{the same proof}}$

$$Bu=\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}u_1-u_2\\-u_1+2u_2-u_3\\-u_2+u_3\end{bmatrix}, \quad Bu=0$$

means $u_1-u_2=0$, $u_1=u_2$

$-u_1+2u_2-u_3=0$, $2u_2=u_1+u_3$

$-u_2+u_3=0$, $u_2=u_3$, So $u_1=u_2=u_3$

$\underline{u=(1,1,1)\cdot c}$

$$Cu=\begin{bmatrix}2&-1&0&-1\\-1&2&-1&0\\0&-1&2&-1\\-1&0&-1&2\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$C \qquad u = 0$

Prooved!

(26) $v = [2, -1, \text{zeros}(n), \dots, -1]$

(27) $A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$$K = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{A^T} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \checkmark$$

$$T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \checkmark$$

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \checkmark$$