Problem set 3.4

(2) $u_{xx} + u_{yy} = 4$

$x^2+y^2=1$, $\underline{u(x,y) = x^2+y^2-1}$ $\rightarrow u_{xx}+u_{yy} = 2+2 \overset{\checkmark}{=} 4$

(4) $u = r\cos\theta + r^{-1}\cos\theta$

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$; $\frac{\partial u}{\partial r} = \cos\theta - \frac{1}{r^2}\cos\theta$

$\frac{\partial^2 u}{\partial r^2} = 2\frac{1}{r^3}\cos\theta$

$\frac{\partial^2 u}{\partial \theta^2} = -r\cos\theta - r^{-1}\cos\theta$

$\underbrace{2r^{-3}\cos\theta}_{\frac{\partial^2 u}{\partial r^2}} + \underbrace{\cancel{r^{-1}\cos\theta} - \cancel{r^{-3}\cos\theta}}_{r^{-1}\frac{\partial u}{\partial r}} - \underbrace{\cancel{r^{-3}\cos\theta} - \cancel{r^{-1}\cos\theta}}_{r^{-2}\frac{\partial^2 u}{\partial\theta^2}} \overset{\checkmark}{=} 0$

(5) $u = \log(r) \rightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$, $\frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\left(\frac{1}{r}\right)$, $\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}$

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = -\frac{1}{r^2} + \frac{1}{r^2} + 0 \overset{\checkmark}{=} 0$

$u = \log(r^2) \rightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$, $\frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\left(\frac{2}{r}\right)$, $\frac{\partial^2 u}{\partial r^2} = -\frac{2}{r^2}$

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = -\frac{2}{r^2} + \frac{2}{r^2} + 0 \overset{\checkmark}{=} 0$

$f(x+iy) = u(x,y) + is(x,y)$, $f(z) = \ln(z) = u + is$

$F(z) = \ln(z^2) = U + iS$

where $z = re^{i\theta}$

$f(z) = \ln(r) + i(\theta)$ (14)

$F(z) = \ln(r^2) + i(2\theta)$

Cauchy-Riemann $\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x}$

$U(x,y) = \ln(x^2+y^2) \rightarrow \frac{\partial u}{\partial x} = \frac{2x}{(x^2+y^2)}$ $\frac{\partial u}{\partial y} = \frac{2y}{(x^2+y^2)}$

$S(x,y) = 2\tan^{-1}\left(\frac{y}{x}\right) \rightarrow \frac{\partial S}{\partial y} = 2\left[1+\left(\frac{y}{x}\right)^2\right]^{-1}\left(\frac{1}{x}\right)$; $\frac{\partial S}{\partial x} = 2\left[1+\left(\frac{y}{x}\right)^2\right]\left(-\frac{y}{x^2}\right)$

$= \frac{2x}{(x^2+y^2)}$ $= -\frac{2y}{(x^2+y^2)}$