(11) $\dfrac{r u(x)+s u(x-\Delta x)+t u(x-2\Delta x)}{\Delta x}=\dfrac{du}{dx}$ for $u=1, x, x^2$

$u=1$, $\dfrac{du}{dx}=0$, $\dfrac{r+s(1-\Delta x)+t(1-2\Delta x)}{\Delta x}=0$, $\left(\dfrac{r+s-s\Delta x+t-2\Delta x t}{\Delta x}=0\right)$

$u=x$, $\dfrac{du}{dx}=1$, $\dfrac{rx+s(x-\Delta x)+t(x-2\Delta x)}{\Delta x}=1$, $\left(\dfrac{rx+sx-s\Delta x+tx-2\Delta x t}{\Delta x}=1\right)$

$u=x^2$, $\dfrac{du}{dx}=2x$, $\dfrac{rx^2+s(x-\Delta x)^2+t(x-2\Delta x)^2}{\Delta x}=2x$, $\dfrac{\left(rx^2+sx^2+tx^2-2sx\Delta x-4tx\Delta x+s(\Delta x)^2+4(\Delta x)^2 t\right)}{\Delta x}=2x$

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$u=1$, $\dfrac{r+s-s\Delta x+t-2\Delta x t}{\Delta x}=0$, $r+s+t=s\Delta x+2\Delta x t$, $\begin{bmatrix} r\\ s\\ t\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$u=x$, $\dfrac{rx+sx-s\Delta x+tx-2\Delta x t}{\Delta x}=1$, ~~[illegible]~~
$rx+sx-s\Delta x+tx-2\Delta x t=\Delta x$
$(r,s,t)=(1,-1,0)$, $\Delta x=\Delta x$ ✓

$u=x^2$, $\begin{cases} rx^2+sx^2+tx^2=0\\ -2sx\cancel{\Delta x}-4tx\cancel{\Delta x}=2x\\ s+4t=0,\ s=-4t\\ \quad s=-2\end{cases}$ ; $-2(-4t)\cancel{x}-4t\cancel{x}=2\cancel{x}$
$4t-2t=1$, $t=\frac{1}{2}$

$r\cancel{x^2}-2\cancel{x^2}+\frac{1}{2}\cancel{x^2}=0$, $r=\frac{3}{2}$, $\begin{bmatrix} r\\ s\\ t\end{bmatrix}=\begin{bmatrix}3/2\\-2\\1/2\end{bmatrix}$

$\begin{bmatrix}0&0&0\\1&-1&0\\3/2&-2&1/2\end{bmatrix}\begin{bmatrix}u_n\\u_{n-1}\\u_{n-2}\end{bmatrix}$

lower-triangular

(12) Difference of difference (7) ← there is no any stretching so, its centered around point $i$

(14) a) $-u''=12x^2$, $u'(0)=0$ and $u(1)=0$

$u(x)=-x^4+Cx+D$, ~~[illegible]~~

$u_p(x)=-x^4+1$ ✓

$u'(0)=0=-4(0)^3+C=0$, $u(1)=-(1)^4+0\cdot 1+D=0$
$D=1$

b) $\dfrac{u_{i+1}-2u_i+u_{i-1}}{h^2}=12(ih)^2$ ? Can't find the way without matrix...

16) $-u'' = \cos(4\pi x)$, $u(0) = u(1) = 0$

$u' = -\int \cos(4\pi x)\,dx = -\frac{1}{4\pi}\sin(4\pi x) + C$

$u'' = -\frac{1}{4\pi}\int(\sin(4\pi x) + C)\,dx = +\frac{1}{(4\pi)^2}\cos(4\pi x) + Cx + D$

$u(0) = 0 = \frac{1}{(4\pi)^2}\cos(0) + D$, $D = -\frac{1}{(4\pi)^2}$

$u(1) = 0 = \cancel{\frac{1}{(4\pi)^2}\cos(2\pi)} + C - \cancel{\frac{1}{(4\pi)^2}}$, $C = 0$

$\boxed{u_p(x) = \frac{1}{(4\pi)^2}\cos(4\pi x) - \frac{1}{(4\pi)^2}}$ ← analog solution

17) $u(x) = e^{ax}$, $\Delta_0 u = (u_{i+1} - u_{i-1})$, $\Delta^2 u = u_{i+1} - 2u_i + u_{i-1}$

$\Delta_0 u = (u(x+\Delta x) - u(x-\Delta x)) = e^{a(x+\Delta x)} - e^{a(x-\Delta x)} = e^{ax}(e^{a\Delta x} - e^{-a\Delta x})$

$\Delta^2 u = e^{a(x+\Delta x)} - 2e^{ax} + e^{a(x-h)} = e^{ax}(e^{a\Delta x} - 2 + e^{-a\Delta x})$

$$\begin{cases} e^{a\Delta x} = 1 + a\Delta x + \frac{(a\Delta x)^2}{2} + \frac{(a\Delta x)^3}{6} + \frac{(a\Delta x)^4}{24} + \frac{(a\Delta x)^5}{120} + \dots \\ e^{-a\Delta x} = 1 - a\Delta x + \frac{(a\Delta x)^2}{2} - \frac{(a\Delta x)^3}{6} + \frac{(a\Delta x)^4}{24} - \frac{(a\Delta x)^5}{120} + \dots \end{cases}$$

$e^{a\Delta x} - e^{-a\Delta x} = 2\left(a\Delta x + \frac{(a\Delta x)^3}{6} + \dots\right) = 2a\Delta x\left(1 + \frac{(a\Delta x)^2}{6} + \frac{(a\Delta x)^4}{120} + \dots\right)$

$= 2a\Delta x + \frac{(a\Delta x)^3}{3} + \frac{(a\Delta x)^5}{60} + \dots$

where is 2nd order accuracy? ?

can't recognize ...

error found ??? NO

$e^{a\Delta x} + e^{-a\Delta x} = 1 + (a\Delta x)^2 + \frac{(a\Delta x)^4}{12} + \dots$

↑error $= a^2$?

18) $\frac{d^2u}{dx^2} = x$, $u(0) = u(1) = 0$

$u(x) = \frac{x^3}{6} + Cx + D$, $u(0) = 0 = D$, $u(1) = 0 = \frac{1}{6} + C$

$C = -1/6$

$u_p(x) = \frac{x^3}{6} - \frac{x}{6} = \frac{x}{6}(x^2 - 1)$

discret solution, $n = 4$, $h = 1/(n+1) = 1/5$, $h^2 = 1/25$, $h^{-2} = 25$

$x \to$ linear $= (1,2,3,4) = f$

$h^{-2}Ku = f \Rightarrow u = h^2K^{-1}f$, $u = h^2(-K^{-1})f$ ← solution $u$

for $-\frac{d^2u}{dx}$, for $\frac{d^2u}{dx} = -K$

(18) $u = \frac{1}{25}\left(-\frac{1}{5}\right)\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{meshpoints}} = -\frac{1}{125}\left(\frac{1}{5}\right)\begin{bmatrix} 20 \\ 35 \\ 40 \\ 30 \end{bmatrix} = -\frac{1}{125}\begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$

$x \in (0, 1)$

(19) $-\frac{d^2u}{dx^2} + \frac{du}{dx} = 1$ with $u(0) = u(1) = 0$, $K$

$u = u_h + u_p$, $-u''(x) + u'(x) = 0$, $u = e^{sx}$

$-s^2 + s = 0$, $s = 1, 0$, $u_h = c_1 + c_2 e^{x}$

$u_p = x$, $u(x) = u_h + u_p = c_1 + c_2 e^{x} + x$

$u(0) = 0 = c_1 + c_2 e^0 + 0$, $c_1 = -c_2$; $u(1) = 0 = c_1 + c_2 e^1 + 1$

$-c_2 + c_2 e + 1 = 0$

$\boxed{u(x) = \underbrace{\frac{1}{e-1}}_{A} - \underbrace{\frac{1}{e-1}}_{B} e^{x} + x}$, $c_1 = \frac{1}{e-1}$, $c_2 = -\frac{1}{e-1}$

$\frac{du}{dx} = \Delta_+ U / h$, $\Delta_0 = (\Delta_+ + \Delta_-)/2h$, $h = 1/5$, $n = 4$

$\frac{1}{h^2} K u + \frac{1}{2h}\Delta_0 u = f$, $\underbrace{\left(\frac{1}{h^2}K + \frac{1}{2h}\Delta_0\right)}_{D} u = f$

$u = D^{-1} f$, $u = \begin{bmatrix} 0.0714 \\ 0.1141 \\ 0.1220 \\ 0.0821 \end{bmatrix}$

Discret forward difference (uncentered)

$\left(\frac{1}{h^2}K + \frac{1}{h}\Delta_+\right)^{-1} f = u$, $u = \begin{bmatrix} 0.0782 \\ 0.1258 \\ 0.1355 \\ 0.0975 \end{bmatrix}$

(20) $\sum_{-\infty}^{\infty}(f_i g_{i+1} - f_i g_{i-1}) = -\sum_{-\infty}^{\infty}(f_{i+1} g_i - f_{i-1} g_i)$

$\sum f_i g_{i+1}$, $\begin{bmatrix} i+1 = i \\ i = i-1 \end{bmatrix}$, $= \sum f_{i-1} g_i$ and $\sum f_i g_{i-1}$, $\begin{bmatrix} i-1 = i \\ i = i+1 \end{bmatrix}$ $= \sum f_{i+1} g_i$

$\sum_{-\infty}^{\infty}(f_{i-1} g_i - f_{i+1} g_i) = -\sum_{-\infty}^{\infty}(f_{i+1} g_i - f_{i-1} g_i)$

proved Q.E.D.

(21) $u(h) = u(0) + h u'(0) + \frac{1}{2} h^2 u''(0) + \dots$

$u'(0) = 0$ and $-u'' = f(x)$

$$\frac{u(h) - u(0)}{h} = \cancelto{0}{u'(0)} + \frac{1}{2} h \cancelto{-f(x)}{u''(0)} + \dots$$

$$= -\frac{1}{2} h f(x) + \dots$$

$u(h) - u(0) = -\frac{1}{2} h^2 f(x)$, $u(0) - u(h) = \frac{1}{2} h^2 f(x)$

$u(0) - u(1) = \frac{1}{2} h^2 f(0)$