Problem Set 1.2

① $u(x) = \begin{cases} Ax & \text{if } x \le 0 \\ Bx & \text{if } x \geq 0 \end{cases}$ (ramp?) , $u'(x) = \begin{cases} A & \text{if } x < 0 \\ B & \text{if } x > 0 \\ \text{undeterminate} & \text{if } x = 0 \end{cases}$ (step?)

$u''(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \text{undetermitate} & \text{if } x = 0 \end{cases} = \delta(x)$ (spike at $x=0$)

$u_n = \begin{cases} An & \text{if } n \le 0 \\ Bn & \text{if } n \geq 0 \end{cases} = \begin{bmatrix} -2A \\ -A \\ 0 \\ B \\ 2B \end{bmatrix}$, $K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$

$\Delta_- \Delta_+ = \Delta^2 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

$K u_n = \begin{bmatrix} -3A \\ 0 \\ A-B \\ 0 \\ 3B \end{bmatrix}$

$\Delta^2 u_n = \begin{bmatrix} A \\ 0 \\ -A+B \\ 0 \\ B \end{bmatrix}$ or $\begin{bmatrix} \vdots \\ 0 \\ 0 \\ -A+B \\ 0 \\ 0 \\ \vdots \end{bmatrix} \leftarrow$ spike $= \delta(n)$

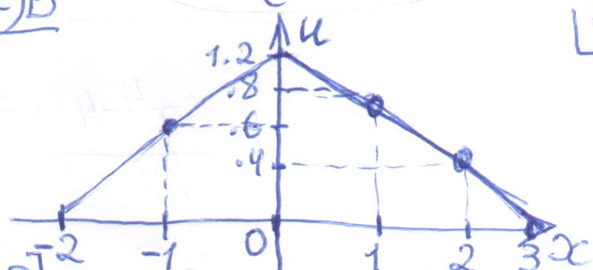
② $-u''(x) = \delta(x)$ with $u(-2) = 0$ and $u(3) = 0$

$\int -u''(x)\,dx = \int \delta(x)\,dx$, $-\left[u'(x)\right]_L^R = 1$, $u_R'(x) - u_L'(x) = -1$

$u = \begin{cases} A(x+2) & x \le 0 \\ B(x-3) & x \geq 0 \end{cases}$ (meet at $x=0$) $A(0+2) = B(0-3)$, $A = (-3/2)B$; $B - A = -1$

$B - \left(\frac{-3}{2}\right)B = -1$, $\frac{5}{2}B = -1$, $B = -\frac{2}{5}$, $A = \frac{3}{5}$

$u = \begin{cases} 0.6(x+2) & x \le 0 \\ -0.4(x-3) & x \geq 0 \end{cases}$



x	-2	-1	0	1	2	3
$u(x)$	0	.6	1.2	.8	.4	0

$U = \begin{bmatrix} u(-1) \\ u(0) \\ u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} .6 \\ 1.2 \\ .8 \\ .4 \end{bmatrix}$, $K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$, $KU = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \delta(x) = F$

Or $U = K^{-1}F = \frac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 1.2 \\ 0.8 \\ 0.4 \end{bmatrix} \Rightarrow U$

③ $\frac{u(x+h)-u(x-h)}{2h}$

for $u(x)=x^3$, $\frac{(x+h)^3-(x-h)^3}{2h} = \frac{\cancel{h^3}+3h^2x+\cancel{3hx^2}+\cancel{x^3}-(\cancel{h^3}+3h^2x-3hx^2+\cancel{x^3})}{\cancel{2h}}$

$= 3x^2$

for $u(x)=x^4$, $\frac{(x+h)^4-(x-h)^4}{2h} = \frac{\cancel{h^4}+4h^3x+\cancel{6h^2x^2}+4hx^3+\cancel{x^4}-\cancel{h^4}+4h^3x-\cancel{6h^2x^2}+4hx^3-\cancel{x^4}}{2h}$

$= \frac{\cancel{8}h^3x+\cancel{8}hx^3}{\cancel{2h}} = 4x^3 + 4h^2x$ e

④ $\Delta_-(\Delta_+) = \begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix} = \begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$

$(\Delta_-^{-1}) = \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-1&1&0\\0&0&1&0&-1&1\end{array}\right] \begin{matrix}\\ +1\\ +1\end{matrix} \rightarrow \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\1&1&0&0&1&0\\1&1&1&0&0&1\end{array}\right]$

$S=(\Delta_-^{-1})$ $\quad I$

$\Delta_0 = \frac{1}{2}\left(\underbrace{\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}}_{\Delta_-} + \underbrace{\begin{bmatrix}-1&+1&0\\0&-1&+1\\0&0&-1\end{bmatrix}}_{\Delta_+}\right) = \frac{1}{2}\begin{bmatrix}0&1&0\\-1&0&1\\0&-1&0\end{bmatrix}$ = centered difference matrix

$n=3$, $\Delta_0 u = 0 \Rightarrow \frac{1}{2}\begin{bmatrix}0&1&0\\-1&0&1\\0&-1&0\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $u=(c,0,c)$

$n=5$, $\Delta_0 u = 0 \rightarrow \frac{1}{2}\underbrace{\begin{bmatrix}0&1&0&0&0\\-1&0&1&0&0\\0&-1&0&1&0\\0&0&-1&0&1\\0&0&0&1&0\end{bmatrix}}_{\text{pattern detected}}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$, $u=(c,0,c,0,c)$

pattern detected too !

⑤ $u(x+h) = u(x)+hu'(x)+\frac{1}{2}h^2u''(x)+\frac{1}{6}h^3u'''(x)+\frac{1}{24}h^4u''''(x)$

$u(x)=x^4$, $u'=4x^3$, $u''=12x^2$, $u'''=24x$, $u''''=24$

$ah^4u^{(4)}(x) = \frac{1}{\cancel{24}}h^4\cdot\cancel{24} = h^4$

⑥ $u(x)=x^4$, $u''(x)=12x^2$, $\Delta^2u/(\Delta x)^2 \approx \frac{d^2[u(x)]}{dx^2}$

$\frac{\Delta^2 u}{(\Delta x)^2} = \frac{u(x+\Delta x)-2u(x)+u(x-\Delta x)}{(\Delta x)^2} = \frac{(x+h)^4-2x^4+(x-h)^4}{h^2}$

$= \frac{2h^{\cancel{4}2}+12\cancel{h^2}x^2+\cancel{2x^4}-\cancel{2x^4}}{\cancel{h^2}} = 12x^2+2h^2$ e

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1) $$\frac{-u_2+8u_1-8u_{-1}+u_{-2}}{12h}=\frac{du}{dx}+bh^4\frac{d^5u}{dx^5}+\ldots$$

fourth-order accuracy

$u(x)=1$ (constant):

$$\frac{-1+8-8+1}{12h}=0=\frac{d}{dx}[1]$$

$u(x)=x^2$:

$$\frac{-(x+2h)^2+8(x+h)^2-8(x-h)^2+(x-2h)^2}{12h}$$

$$=\frac{-(\cancel{x^2}+4xh+\cancel{4h^2})+\cancel{8x^2}+16xh+\cancel{8h^2}-\cancel{8x^2}+16xh-\cancel{8h^2}+\cancel{x^2}-4xh+\cancel{4h^2}}{12h}$$

$$=\frac{32x\cancel{h}-8x\cancel{h}}{12\cancel{h}}=2x=\frac{d}{dx}[x^2]$$

$u(x)=x^4$:

$$\frac{(x-2h)^4-(x+2h)^4+8(x+h)^4-8(x-h)^4}{12h}=\left[\begin{array}{l}\left((x-2h)^2-(x+2h)^2\right)\left((x-2h)^2+(x+2h)^2\right)\\+8\left(\left((x+h)^2-(x-h)^2\right)\left((x+h)^2+(x-h)^2\right)\right)\end{array}\right]\frac{1}{12h}$$

$$=\frac{-8xh(2x^2+8h^2)+8(4xh(2x^2+2h^2))}{12h}=\frac{8x\cancel{h}(-2x^2\cancel{-8h^2}+8x^2\cancel{+8h^2})}{12\cancel{h}}$$

$$=\frac{48x^3}{12}=4x^3=\frac{d}{dx}[x^4]$$

2) $u_2=u(x+2h)=u(x)+2hu'(x)+\frac{1}{2}(2h)^2u''(x)+\frac{1}{6}(2h)^3u'''(x)+\frac{1}{24}(2h)^4u''''(x)$

$u_1=u(x+h)=u(x)+hu'(x)+\frac{1}{2}h^2u''(x)+\frac{1}{6}h^3u'''(x)+\frac{1}{24}h^4u''''(x)+\frac{1}{120}h^5u^{(5)}(x)$

$u_{-1}=u(x-h)=u(x)-hu'(x)+\frac{1}{2}h^2u''(x)-\frac{1}{6}h^3u'''(x)+\frac{1}{24}h^4u''''(x)-\frac{1}{120}h^5u^{(5)}(x)$

$u_{-2}=u(x-2h)=u(x)-2hu'(x)+\frac{1}{2}(2h)^2u''(x)-\frac{1}{6}(2h)^3u'''(x)+\frac{1}{24}(2h)^4u''''(x)-\frac{1}{120}(2h)^5u^{(5)}(x)$

$$\frac{-u_2+8u_1-8u_{-1}+u_{-2}}{12h}=\left[\begin{array}{l}0\cdot u(x)+12hu'(x)+0\cdot u''(x)+0\cdot u'''(x)\\+\underbrace{\frac{16}{120}h^5u^{(5)}(x)-\frac{64}{120}h^5u^{(5)}(x)}_{-48/120}\end{array}\right]\frac{1}{12h}$$

$$=u'(x)-\frac{1}{30}h^4u^{(5)}(x)$$

$\Rightarrow b=-1/30$

8) $\frac{\Delta_0}{2h}\frac{\Delta_0}{2h}u_n = \frac{u_{n+2}-2u_n+u_{n-2}}{(2h)^2} =$

$$\Delta_0\Delta_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix};$$

$$(\Delta_0\Delta_0)\begin{bmatrix} u_{n-2} \\ u_{n-1} \\ u_n \\ u_{n+1} \\ u_{n+2} \end{bmatrix} = \begin{bmatrix} \vdots \\ u_{n-2}-2u_n+u_{n+2} \\ \vdots \end{bmatrix} \leftarrow$$ center is fine, change the latest and first

$$(\Delta_0\Delta_0)\begin{bmatrix} u_{n-1} \\ u_{n-1/2} \\ u_n \\ u_{n+1/2} \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} \vdots \\ u_{n-1}-2u_n+u_{n+1} \\ \vdots \end{bmatrix} \leftarrow \checkmark$$

9) $\frac{\Delta^4 u}{(\Delta x)^4} = \frac{u_2-4u_1+6u_0-4u_{-1}+u_{-2}}{(\Delta x)^4} = \frac{\partial^4 u}{\partial x^4} + e$

$u=x$, $\frac{\cancel{x+2h}-4(x+h)+6x-4(x-\cancel{h})+\cancel{x-2h}}{h^4} = \frac{8x-8x}{h^4} \overset{\checkmark}{=} 0$

$u=x^2$, $\frac{(x+2h)^2-4(x+h)^2+6x^2-4(x-h)^2+(x-2h)^2}{h^4} = \frac{\cancel{2x^2}+\cancel{8h^2}-\cancel{8x^2}-\cancel{8h^2}+\cancel{6x^2}}{h^4} \overset{\checkmark}{=} 0$

$u=x^3$, $\frac{(x+2h)^3-4(x+h)^3+6x^3-4(x-h)^3+(x-2h)^3}{h^4}$

$= \frac{\cancel{24h^2x}+\cancel{2x^3}-\cancel{24h^2x}-\cancel{8x^3}+\cancel{6x^3}}{h^4} \overset{\checkmark}{=} 0$

$u=x^4$, $\frac{(x+2h)^4-4(x+h)^4+6x^4-4(x-h)^4+(x-2h)^4}{h^4} = \frac{24\cancel{h^4}}{\cancel{h^4}} \overset{\checkmark}{=} 24$

10) $$\Delta_+\Delta_- = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix}\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & -1 & 1 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ & & & 1 & -1 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow u=0 \\ \\ \\ \\ \leftarrow u'=0 \end{array}$$