18.085 Quiz 2 November 2, 2007 Professor Strang

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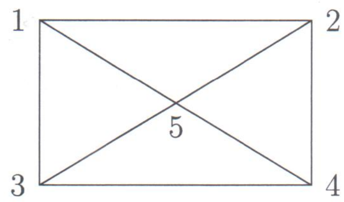
Grading 1
2
3

1) **(40 pts.)** This problem is based on a 5-node graph.

$A=\begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$ $A^TA=\begin{bmatrix} 12 & -5 & -5 & 3 & -5 \\ -5 & 12 & 3 & -5 & -5 \\ -5 & 3 & 12 & -5 & -5 \\ 3 & -5 & -5 & 12 & -5 \\ -5 & -5 & -5 & -5 & 20 \end{bmatrix}$



I have not included edge numbers and arrows. Add them if you want to: not needed.

√ (a) Find $A^{\mathrm{T}}A$ for this graph. A is the incidence matrix.

$\det(A^TA)=0$, $\prod \lambda_i = 0$ → √ (b) The sum of the eigenvalues of $A^{\mathrm{T}}A$ is $\underline{\text{trace}(A^TA)}$. $= 68 = \sum \lambda_i$
The product of those eigenvalues is $\underline{\text{rank}(A^TA)}$. $=4$, $A^TA\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=0$, $\det(A^TA)=0$

$A^TA=\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ → √ (c) What is $A^{\mathrm{T}}A$ for a graph with **only one edge**? How can that small $A^{\mathrm{T}}A$ be used in constructing $A^{\mathrm{T}}A$ for a large graph?

√ (d) Suppose I want to solve $Au = \text{ones}(8,1) = b$ by least squares. What equation gives a best $\widehat{u}$? For the incidence matrix A, is there exactly one best $\widehat{u}$ solving that equation? (If your equation has more than one best $\widehat{u}$, describe the difference between any two solutions.)

$Au=b$, $(A^TA)\hat{u}=A^Tb$, $\hat{u}=(A^TA)^{-1}A^Tb$

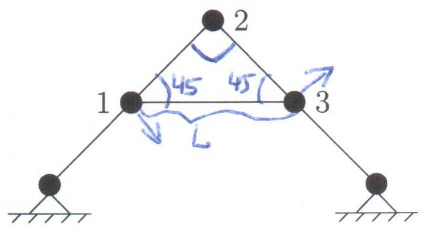
There is no best solutions A can't be invertable!

$Au=0 \to \underline{n-r}$ solution; $A^T w=0 \to \underline{m-r}$ solutions

2) **(30 pts.)** (a) Suppose A is an m by n matrix of rank r (so it has r independent columns). How many independent solutions to $Au = 0$ and $A^{\mathrm{T}} w = 0$?

(b) Draw a full set of mechanisms (solutions to $e = Au = 0$ with no stretching) for this truss with unit length bars and 45° angles.

only one mechanism $= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$



(c) Suppose a mechanism has $u_1^{\mathrm{H}} = .01$. What are u_1^{V} and u_3^{H} and u_3^{V}? What is the actual new length of the bar between joints 1 and 3?

$$u_1^H = .01 \to \begin{bmatrix} u_1^H \\ u_1^V \\ u_2^H \\ u_2^V \\ u_3^H \\ u_3^V \end{bmatrix} = \begin{bmatrix} .01 \\ -.01 \\ 0 \\ 0 \\ .01 \\ .01 \end{bmatrix} \qquad L_{New} = \sqrt{L_{old} + (.02)^2}$$

3) **(30 pts.)** This problem is about the equation

$$\underline{-u''(x)+u(x)=1} \qquad \text{with } \overset{\text{fixed}}{u(0)=0} \text{ and } \overset{\text{fixed}}{u(1)=0}.$$

√ (a) Multiply by a test function $v(x)$. Find the weak form of the equation, after an integration by parts.

√ (b) With $h=\Delta x=\frac{1}{3}$ draw the admissible piecewise linear trial functions $\phi_1(x),\ldots,\phi_n(x)$. What is n? With test functions = trial functions, give a *formula* for the entry K_{12} in the finite element equation $KU=F$.

√ (c) Find all the numbers in K and F.

Weak form: $\int_0^1 c(x)\frac{du}{dx}\frac{dv}{dx}dx=\int_0^1 f(x)v(x)dx$ <u>for all $v(x)$</u>

$-u''(x)v(x)+u(x)v(x)=v(x) \to -\int_0^1 u''(x)v(x)dx+\int_0^1 u(x)v(x)dx=\int_0^1 v(x)dx$

$\int_0^1 u''(x)v(x)dx=\left[u'(x)v(x)\right]_0^1-\underbrace{\int_0^1 u'(x)v'(x)dx}_{\text{non-zero}} \to \left[u'(x)v(x)\right]_0^1=u'(1)v(1)-u'(0)v(0)$

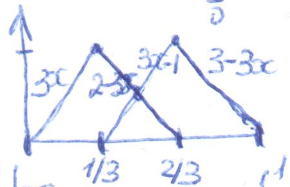
Integration by parts:

Required $\underline{v(1)=v(0)=0}$

$\int u(x)v'(x)dx=u(x)v(x)-\int u'(x)v(x)dx$

Weak form: $\int_0^1 u'(x)v'(x)dx+\int_0^1 u(x)v(x)dx=\int_0^1 v(x)dx$

(b) $u(0)=0,\ u(1)=0,\ h=\Delta x=\frac{1}{3}$



$\underline{n=2}$

$K_{ij}=\int_0^1 \phi_i'(x)\phi_j'(x)dx+\int_0^1 \phi_i(x)\phi_j(x)dx$, $F_i=\int_0^1 \phi_i(x)dx$, $\underline{\phi_1(x)=3x}$

$K_{12}=\int_0^1 \phi_1'(x)\phi_2'(x)dx+\int_0^1 \phi_1(x)\phi_2(x)dx$

(c) $K_{11}=\int_0^{1/3}3\cdot 3\,dx+\int_{1/3}^{2/3}(-3)(-3)\,dx+\int_0^{1/3}(3x)(3x)\,dx+\int_{1/3}^{2/3}(2-3x)(2-3x)\,dx$

$=3+3+\frac{1}{9}+\frac{1}{2}=\frac{56}{9}$

$K_{12}=\int_{1/3}^{2/3}(-3)(3)\,dx+\int_{1/3}^{2/3}(2-3x)(3x-1)\,dx=-\frac{53}{18}$

$$K=\begin{bmatrix}56/9 & -53/18\\ -53/18 & 56/9\end{bmatrix} \qquad F=\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}$$