18.085 Quiz 3 December 7, 2007 Professor Strang

Your PRINTED name is: ____________________ Grading 1
2
3
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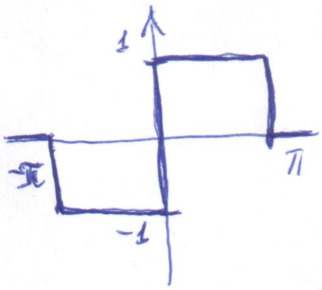
**** NOTE AT NOON A BIG CHEMISTRY CLASS IS COMING !!!**

$u'(x) = \sum b_k \cos(kx)\,k$
$u''(x) = -\sum b_k k^2 \sin(kx)$

1) **(30 pts.)** (a) Solve by a Fourier sine series $u(x) = \sum b_k \sin kx$:



$$-u'' + 4\,u(x) = f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases} \quad \text{with } u(-\pi) = u(\pi) = 0\,.$$

That right side $f(x)$ is the square wave SW(x) on page 318.

(b) What is the decay rate of the coefficients b_k? What is the smoothness of $u(x)$ — which derivative jumps?

$-u'' + 4u(x) = \sum b_k k^2 \sin(kx) + 4 \sum b_k \sin(kx) = \sum (k^2+4) b_k \sin(kx)$

$f(x) = \frac{4}{\pi} \sum_{k=1,\,k \text{ is odd}}^{\infty} \frac{\sin(kx)}{k} \Rightarrow \sum_{k=1}^{\infty} (k^2+4) b_k \sin(kx) = \frac{4}{\pi} \sum_{k=1,\,k \text{ is odd}}^{\infty} \frac{\sin(kx)}{k}$

$\sum_{k \text{ is odd}} b_k = \frac{4}{\pi} \frac{\sum \frac{\cancel{\sin(kx)}}{k}}{\sum (k^2+4) \cancel{\sin(kx)}} = \frac{4}{\pi} \sum \frac{1}{(k^2+4)k}$ | $-u'' + 4u = f$, $u = \frac{f + u''}{4}$

$u''(x) = -\sum \frac{4\cancel{k}^2 \sin(kx)}{\pi(k^2+4)\cancel{k}} = -\frac{4}{\pi} \sum \frac{k \sin(kx)}{(k^2+4)}$

$u(x) = \frac{\cancel{4}}{\pi} \frac{\sum \left(\frac{\sin(kx)}{k} - \frac{k \sin(kx)}{k^2+4} \right)}{\cancel{4}} = \frac{1}{\pi} \sum_{k \text{ is odd}} \frac{\sin(kx)(\cancel{k}^2+4) - \cancel{k}^2 \sin(kx)}{k(k^2+4)} = \frac{4}{\pi} \sum \frac{\sin(kx)}{k(k^2+4)}$

$\frac{1}{k^3}$ is the decay rate

2nd derivative $\frac{1}{6k}$ decay rate → Step function with jumps

2) **(30 pts.)** This problem is about the equation

$$\boxed{\frac{1}{5}x_{n-1} + \frac{3}{5}x_n + \frac{1}{5}x_{n+1} = y_n} \qquad -\infty < n < \infty$$

(a) Suppose the vector $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$ is known. The equation is a non-cyclic convolution $a * x = y$. What is the infinite vector a? Transform the equation into the frequency domain using $X(\omega) = \sum x_n e^{in\omega}$ and $Y(\omega)$ and $A(\omega)$. What is $A(\omega)$ in this problem?

(b) Suppose the vector y is known but the vector x is **not known**. We want to find x. Take two steps:

1. Give a formula for $X(\omega)$ using known things like $Y(\omega)$ and $\frac{1}{5}, \frac{3}{5}, \frac{1}{5}$, or A.
2. Does your formula involve any division by zero or is it safe?

The last step in this deconvolution would recover the Fourier coefficients x_n from your function $X(\omega)$ but this is not on the exam!

$a * x = y$

$A(\omega)X(\omega) = Y(\omega)$

$$\begin{array}{ccccc}
 & & a_{n+1} & a_n & a_{n-1} \\
 & & x_{n+1} & x_n & x_{n-1} \\
\hline
 & & a_{n+1}x_{n-1} & a_n x_{n-1} & a_{n-1}x_{n-1} \\
 & a_{n+1}x_n & a_n x_n & a_{n-1}x_n & \\
a_{n+1}x_{n+1} & a_n x_{n+1} & a_{n-1}x_{n+1} & &
\end{array}$$

y_n

If $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$

$a = (\ldots, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, \ldots)$

$X(\omega) = \sum_{n=-\infty}^{\infty} x_n e^{in\omega}$

$Y(\omega) = \sum_{n=-\infty}^{\infty} y_n e^{in\omega}$

$A(\omega) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega}$

Substitute to $A(\omega) = \frac{1}{5}e^{-i\omega} + \frac{3}{5}e^{0} + \frac{1}{5}e^{i\omega}$

$A(\omega) = \frac{1}{5}\underbrace{(e^{-i\omega} + e^{i\omega})}_{\substack{\cos\omega - i\sin\omega \\ + \cos\omega + i\sin\omega}} + \frac{3}{5} = \frac{2}{5}\cos\omega + \frac{3}{5}$

$$\boxed{X(\omega) = \frac{Y(\omega)}{A(\omega)} = \frac{Y(\omega)}{\frac{3}{5} + \frac{2}{5}\cos(\omega)}}$$

$-1 < \cos\omega < 1$

$\frac{3}{5} + \frac{2}{5}\cos(\omega) \neq 0$

formula is save

3) **(40 pts.)** This circulant equation $Cd = b$ is a cyclic convolution:

$$Cd = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = b \qquad \text{is} \quad c \circledast d = b$$

(a) The eigenvectors of that matrix C are the four columns e_0, e_1, e_2, e_3 of the Fourier matrix F (this F is on page 347). Multiply F times the e's to find the four eigenvalues. Check that their sum is correct.

(b) Write the right side $b = (1, 0, 0, 0)$ as a combination of those four eigenvectors (columns of F). Using the eigenvalues, the solution d is what combination of the four eigenvectors? Find the vector d.

(c) A direct way to solve $c \circledast d = b$ would be to take the 4-point discrete transform of both sides. What are the transforms of b and c in this problem? What is the transform of the solution d? Isn't this just the same method in different words (yes or no).

Thank you for taking 18.085 !

a)

$Ce = \lambda e$, $CF = F\Lambda$, $\underline{C = F\Lambda F^{-1}}$, $CF = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & 5 & 5 & 5 \\ 1 & 5i & -5 & -5i \\ 1 & -5 & 5 & -5 \\ 1 & -5i & -5 & 5i \end{bmatrix}$,

$F\Lambda = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ or $[1\ 5\ 5\ 5] \leftarrow$ 1st row of CF

$\underline{\text{trace } C = 16 = \sum \lambda_i}$ ✓ $\quad [4\ {-1}\ {-1}\ {-1}] \cdot F = [1\ 5\ 5\ 5]$!

b) $b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \delta_4$, $b = \frac{1}{4} \sum_{i=1}^{4} e_i$, $Cd = b$, $d = C^{-1}b = (F\Lambda^{-1}F^{-1})b$; $C^{-1}b = C_0^{-1}$

$F^{-1}b = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$; $\Lambda^{-1}F^{-1}b = \frac{1}{20}\begin{bmatrix} 5 \\ 1 \\ 1 \\ 1 \end{bmatrix}$; $(F\Lambda^{-1}F^{-1})b = \frac{1}{20} F \begin{bmatrix} 5 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 8 \\ 4 \\ 4 \\ 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = d$ ✓

c) $c \circledast d = F^{-1}((Fc) .* (Fd))$; $Fb = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $FC = \begin{bmatrix} 1 & 5 & 5 & 5 \\ 1 & 5i & -5 & -5i \\ 1 & -5 & 5 & -5 \\ 1 & -5i & -5 & 5i \end{bmatrix}$

$Fd = \frac{1}{5}\begin{bmatrix} 5 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad Fc = \begin{bmatrix} 1 \\ 5 \\ 5 \\ 5 \end{bmatrix}$