Problem Set 4.3

(1) $$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \Longleftrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i^2 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{P}$$

$$A \cdot B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i^2 & i & i^3 \\ 1 & 1 & -1 & -1 \\ 1 & i^2 & -i & -i^3 \end{bmatrix}, \quad ABP = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & -1 & 1 & -1 \\ 1 & -i & i^2 & -i^3 \end{bmatrix}$$

$$\underbrace{i^0 = 1,\ i^1 = i,\ i^2 = -1,\ i^3 = -i}_{loop},\ \underbrace{i^4 = 1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i}_{loop}$$

$$[1\ \ -1\ \ 1\ \ -1] \Longleftrightarrow [1\ \ i^2\ \ i^4\ \ i^6];\ [1\ \ -i\ \ i^2\ \ -i^3] \Longleftrightarrow [1\ \ i^3\ \ i^6\ \ i^9]$$

$$F_4 = ABP = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} \begin{bmatrix} even \\ odd \end{bmatrix}$$

$$\omega_N = e^{i2\pi/N} \longrightarrow \omega_2 = e^{i2\pi/2} = e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$$\omega_4 = e^{i2\pi/4} = e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$(\omega_4)^2 = \omega_2 = (i)^2 = -1$$

Let's using (colums)(rows) multiplication $\rightarrow \sum_{i=1}^{i=N} (\text{col } i \text{ of } A)(\text{row } i \text{ of } B)$

$$(\text{col 3 of } A)(\text{row 3 of } B) \cdot P = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} [0\ 0\ 1\ 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{entries for } i^2 = -1 \cdot P} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i^2

$$\left[M = \frac{N}{2} \quad (\omega_N)^M = (\omega_N)^{\frac{N}{2}} \right.$$

$$\omega_N = e^{\frac{2\pi i}{N}} \Rightarrow (\omega_N)^{\frac{N}{2}} = \left(e^{\frac{2\pi i}{N}}\right)^{\frac{N}{2}} = e^{\pi i} = \cos\pi + i\sin\pi = -1$$

Q.E.D.

② row J of $\overline{F_N} \Rightarrow \overline{(\omega_N)^J} = \overline{\exp\left(\frac{i2\pi \cdot J}{N}\right)} = \exp\left(\frac{-i2\pi J}{N}\right)$ ✓

row $(N-J)$ of $F \Rightarrow (\omega_N)^{N-J} = \exp\left(\frac{i2\pi(N-J)}{N}\right) = \underbrace{\exp(i2\pi)}_{1}\exp\left(\frac{-i2\pi J}{N}\right)$

$\exp(i2\pi) = \cos(2\pi) + i\sin(2\pi) = 1$

$\overline{(\omega_N)^J} \overset{\checkmark}{=} (\omega_N)^{N-J}$, Q.E.D.

row J of $\overline{F_N} = [1 \ \overline{\omega^J} \ \overline{\omega^{2J}} \dots \overline{\omega^{(N-1)J}}]$ ✓

the same as $= [1 \ \omega^{(N-J)} \ \omega^{2(N-J)} \ \omega^{(N-1)(N-J)}]$

③ $F = P\overline{F}$, $F_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ \vdots & & & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} \end{bmatrix}$ $\overline{F_N} = \begin{bmatrix} 1 & \frac{1}{\omega} & \frac{1}{\omega^2} & \dots & \frac{1}{\omega^{N-1}} \\ 1 & \overline{\omega} & & & \vdots \\ 1 & & \ddots & & \overline{\omega^{(N-1)^2}} \\ \vdots & & & & \\ 1 & \overline{\omega^{N-1}} & \dots & & \end{bmatrix}$

Or $\overline{F_N} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} \\ 1 & \omega^{N-2} & \omega^{2(N-2)} & \dots & \omega^{(N-1)(N-2)} \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & \omega & \omega^2 & \dots & \omega^{(N-1)} \end{bmatrix}$ $P = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots 1 \dots & \\ \vdots & O & & & O \\ 0 & 1 & & & 0 \end{bmatrix}$

Example

$P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $\overline{F}F = 4I \rightarrow \frac{1}{4}\overline{F}F = I$, $\frac{1}{4}\overline{F}F = PP$ ✓

$\boxed{F = P\overline{F}, \overline{F} = PF}$ $\frac{1}{4}\underbrace{P\overline{F}}_{F}F = P \rightarrow \frac{1}{4}F^2 = P$

$P^2 = \frac{1}{16}F^4 = I$ ✓ $F^4 = N^2 I$

④ $F^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 & 0 \\ 0 & -i/2 & 0 & i/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$= \frac{1}{4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}$

(5) $F_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \begin{matrix} \}\text{even} \\ \\ \}\text{odd} \end{matrix}$

$F_4 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} \begin{bmatrix} \text{even} \\ \text{odd} \end{bmatrix} \rightarrow F_N^T = \left(\begin{bmatrix} I & D_{N/2} \\ I & -D_{N/2} \end{bmatrix} \begin{bmatrix} F_{N/2} & \\ & F_{N/2} \end{bmatrix} \begin{bmatrix} \text{even} \\ \text{odd} \end{bmatrix} \right)^T$

$F_N^T = \begin{bmatrix} \text{even} \\ \text{odd} \end{bmatrix} \begin{bmatrix} F_{N/2} & \\ & F_{N/2} \end{bmatrix} \begin{bmatrix} I & I \\ D_{N/2} & -D_{N/2} \end{bmatrix}$, ✓

More then the many another ~~pr~~ examples I like this one

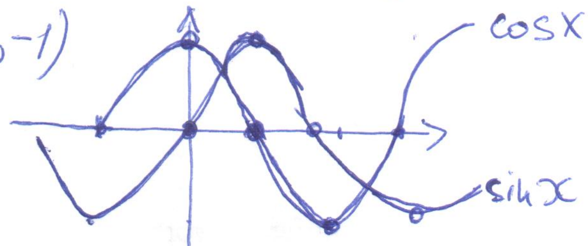
$F_N^T = \left(\begin{bmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & -D_{N/2} \end{bmatrix} \begin{bmatrix} F_{N/2} & \\ & F_{N/2} \end{bmatrix} \begin{bmatrix} \text{even}_{N/2} & \text{even}_{N/2} \\ \text{odd}_{N/2} & \text{odd}_{N/2} \end{bmatrix} \right)^T$

$F_N = F_N^T = \begin{bmatrix} \text{even}_{N/2} & \text{even}_{N/2} \\ \text{odd}_{N/2} & \text{odd}_{N/2} \end{bmatrix} \begin{bmatrix} F_{N/2} & \\ & F_{N/2} \end{bmatrix} \begin{bmatrix} I_{N/2} & I_{N/2} \\ D_{N/2} & -D_{N/2} \end{bmatrix} = \underline{FFT\,2}\,!$

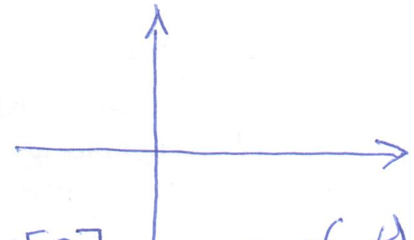
(symmetrick)

(6) $F_6 = \begin{bmatrix} 1 & & & 1 & & \\ & 1 & & & \omega & \\ & & 1 & & & \omega^2 \\ 1 & & & -1 & & \\ & 1 & & & -\omega & \\ & & 1 & & & -\omega^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & & & \\ 1 & \omega^2 & \omega^4 & & & \\ 1 & \omega^4 & \omega^2 & & & \\ & & & 1 & 1 & 1 \\ & & & 1 & \omega^2 & \omega^4 \\ & & & 1 & \omega^4 & \omega^2 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & & 1 & & & \\ & & & & 1 & \\ & 1 & & & & \\ & & & 1 & & \\ & & & & & 1 \end{bmatrix}$

(7) discrete sine $(0, +1, 0, -1)$



discrete cos $(+1, 0, -1, 0)$



$\cos(\omega t) = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$ ✓

$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(10) $\omega = \exp\left(\frac{2\pi i}{64}\right)$; $\omega^2 = \exp\left(\frac{2\pi i}{32}\right)$, $\sqrt{\omega} = \omega^{\frac{1}{2}} = \exp\left(\frac{2\pi i}{128}\right)$

among the 32^{th} and 128^{th} roots of 1