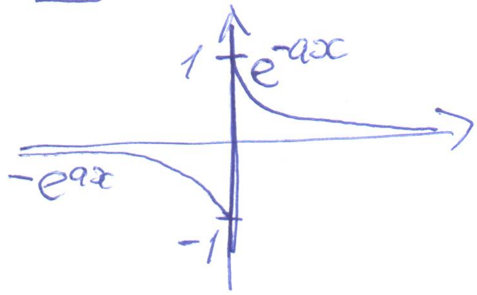
Problem Set 4.5

$$\left[\hat{g}(k)=\int_{-\infty}^{\infty} f(x)e^{-ikx}dx\right]$$

① $g(x)=\begin{cases}-e^{ax} & \text{for } x<0\\ e^{-ax} & \text{for } x>0\end{cases}$



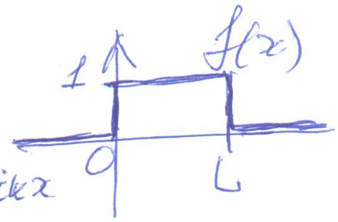
$$\hat{g}(k)=\int_{-\infty}^{0}-e^{ax}e^{-ikx}dx+\int_{0}^{\infty}e^{-ax}e^{-ikx}dx$$

$$=\int_{0}^{\infty}e^{-(a+ik)x}dx-\int_{-\infty}^{0}e^{(a-ik)x}dx$$

$$=\frac{e^{-(a+ik)x}}{-(a+ik)}\Bigg]_0^{\infty}-\frac{e^{(a-ik)x}}{a-ik}\Bigg]_{-\infty}^{0}=0-\frac{1}{-(a+ik)}-\frac{1}{a-ik}-0=-\frac{2ik}{a^2+k^2}$$

decay rate $\hat{g}(k)=\frac{1}{k}$, $g(x)$ has a jump at $x=0$

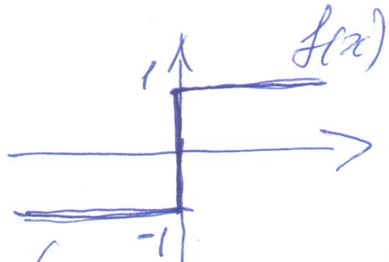
② a) $f(x)=\begin{cases}1 & \text{for } 0<x<L\\ 0 & \text{for } x<0 \text{ and } x>L\end{cases}$



$$\hat{f}(x)=\int_0^L 1\, e^{-ikx}dx=\frac{e^{-ikx}}{-ik}\Bigg]_0^L=\frac{1}{ik}+\frac{e^{-ikx}}{-ik}$$

$$=\frac{1-e^{-ikx}}{ik}$$

b) $f(x)=\begin{cases}1 & \text{for } x>0\\ -1 & \text{for } x<0\end{cases}$



$$\hat{f}(x)=\int_{-\infty}^{0}(-1)e^{-ikx}dx+\int_0^{\infty}1\,e^{-ikx}dx \quad (\text{set up } a=0 \text{ for ①})$$

$$=-\frac{2ik}{k^2}=-\frac{2i}{k}$$

c) $f(x)=\int_0^1 e^{ikx}dk$, $\hat{f}(x)=\int_{-\infty}^{\infty}\left(\int_0^1 e^{ikx}dk\right)e^{-ikx}dx$?

$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(k)e^{ikx}dx$, where $g(k)=\begin{cases}2\pi & \text{for } 0\le k\le 1\\ 0 & \text{otherwise}\end{cases}$

$\hat{f}(k)=g(k)$

d) $f(x) = \sin(x)$ for $0 \le x \le 4\pi$

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_0^{4\pi} \sin(x) e^{-ikx} dx = \frac{e^{-ikx}}{k^2-1}(\cos x + ik\sin x)\Big]_0^{4\pi}$$

$$= \frac{e^{-4\pi ik} - 1}{k^2 - 1}$$

(3) a) $\hat{f}(k) = \delta(k)$, $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(k) e^{ikx} dk = \frac{1}{2\pi}\left[e^0\right]_{-\infty}^{\infty} = \frac{1}{2\pi}$

b) $\hat{f}(k) = e^{-|k|}$, $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|} e^{ikx} dk$

$$\int_{-\infty}^{0} e^{-(-k)} e^{ikx} dk = \int_{-\infty}^{0} e^{k(1+ix)} dk = \frac{e^{k(1+ix)}}{1+ix}\Big]_{-\infty}^{0} = \frac{1}{1+ix}$$

$$\int_0^{\infty} e^{-k} e^{ikx} = \int_0^{\infty} e^{k(-1+ix)} dk = \frac{e^{k(-1+ix)}}{-1+ix}\Big]_0^{\infty}$$

$$\lim_{T\to\infty}\left(\frac{e^{k(-1+ix)}}{-1+ix}\Big]_{k=0}^{k=T}\right) = \lim_{T\to\infty}\left(\frac{e^{-T(1-ix)} - 1}{-1+ix}\right) = \frac{1}{1-ix}$$

$$f(x) = \frac{1}{2\pi}\left(\frac{1}{1-ix} + \frac{1}{1+ix}\right) = \frac{1}{\cancel{2}\pi}\left(\frac{\cancel{2}}{x^2+1}\right) = \frac{1}{\pi(x^2+1)} = f(x) \checkmark$$

(4) Apply Plancherel's formula $2\pi\int_{-\infty}^{\infty}|f(x)|^2 dx = \int_{-\infty}^{\infty}|\hat{f}(k)|^2 dk$

1) $f(x) = \begin{cases} 1 & \text{for } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_{-1}^{1} e^{-ikx} dx = \frac{e^{-ikx}}{ik}\Big]_1^{-1}$$

$$= \frac{1}{ik}\left(e^{ik} - e^{-ik}\right) = \frac{2\sin k}{k} \quad \text{(centered pulse } L=1\text{)}$$

$$\int_{-\infty}^{\infty}\left(\frac{2\sin(k)}{k}\right)^2 dk = 2\pi\int_{-1}^{1} 1^2 dx$$

$$= 2\pi\left[x\right]_{-1}^{1} = 4\pi \checkmark$$

2) $f(x) = e^{-a|x|} = \begin{cases} e^{-ax} & \text{for } x \ge 0 \\ e^{ax} & \text{for } x \le 0 \end{cases}$ $\quad \hat{f}(k) = \frac{2a}{a^2+k^2}$

$$2\pi\int_{-\infty}^{\infty}|e^{-a|x|}|^2 dx = \int_{-\infty}^{\infty}\left|\frac{2a}{a^2+k^2}\right|^2 dk \; ; \; \int_{-\infty}^{\infty}|e^{-a|x|}|^2 dx = \int_{-\infty}^{0} e^{2ax} dx + \int_0^{\infty} e^{-2ax} dx = \frac{1}{2a}\left(\left[e^{2ax}\right]_{-\infty}^{0} - \left[e^{-2ax}\right]_0^{\infty}\right)$$

$$= \frac{1}{2a}(1+1) = \frac{1}{a}$$

$$\frac{2\pi}{a} = 4a^2\int_{-\infty}^{\infty}\frac{dk}{(a^2+k^2)^2}$$

$$\int_{-\infty}^{\infty}\frac{dk}{(a^2+k^2)^2} = \frac{2\pi}{4a^3} = \frac{\pi}{2a^3} \checkmark$$