Problem Set 3.2

(1) $u''''(x) = \delta(x-\tfrac{1}{2})$, $u(0) = u'(0) = 0$

2 steps: $M'' = \delta(x-\tfrac{1}{2})$ and $u'' = M(x)$

with boundary cond. $M(1) = M'(1) = 0$

Step 1 $\underline{M'' = \delta(x-\tfrac{1}{2})}$, with $M'(1) = 0 \rightarrow \int_x^1 M''(x)\,dx = \int_x^1 \delta(x-\tfrac{1}{2})\,dx$

$$[M'(x)]_x^1 = [S(x-\tfrac{1}{2})]_x^1 \rightarrow \underbrace{\cancel{M'(1)}}_{0} - M'(x) = \underbrace{\cancel{S(\tfrac{1}{2})}}_{1} - S(x-\tfrac{1}{2})$$

$$\int_x^1 M'(x)\,dx = \int_x^1 \left(S(x-\tfrac{1}{2}) - 1\right)dx \rightarrow \underbrace{\cancel{M(1)}}_{0} - M(x) = R(x-\tfrac{1}{2}) - x\Big]_x^1$$

$$-M(x) = \underbrace{\cancel{R(\tfrac{1}{2})}}_{1/2} - 1 - R(x-\tfrac{1}{2}) + x \rightarrow \boxed{M(x) = R(x-\tfrac{1}{2}) - x + \tfrac{1}{2}}$$

1st step

with indefinite integral

$$\iint M''(x)\,dx\,dx = \iint \delta(x-\tfrac{1}{2})\,dx\,dx \rightarrow M(x) = R(x-\tfrac{1}{2}) + Dx + C$$

$M'(1) = 0$, $\underbrace{\cancel{S(\tfrac{1}{2})}}_{1} + D = 0 \rightarrow D = -1$, $M(1) = 0$, $\underbrace{\cancel{R(\tfrac{1}{2})}}_{\frac{1}{2}} - 1 + C = 0$ $C = \tfrac{1}{2}$

$\boxed{M(x) = R(x-\tfrac{1}{2}) - x + \tfrac{1}{2}}$, Checked!

Step 2 with indefinite integral

$$u(x) = \iint M(x)\,dx\,dx \rightarrow \boxed{u(x) = \overbrace{C(x-\tfrac{1}{2})}^{\text{cubic spline}} - \frac{x^3}{6} + \frac{x^2}{4}}$$

with boundary conditions $u(0) = u'(0) = 0$

$$\int_0^x u''(x)\,dx = \int_0^x \left(R(x-\tfrac{1}{2}) - x + \tfrac{1}{2}\right)dx,\quad u'(x) - \underbrace{\cancel{u'(0)}}_{0} = \overbrace{Q(x-\tfrac{1}{2})}^{\text{quadratic spline}} - \frac{x^2}{2} + \frac{1}{2}x$$

$$\int_0^x u'(x)\,dx = \int_0^x \left(Q(x-\tfrac{1}{2}) - \frac{x^2}{2} + \frac{1}{2}x\right)dx,\quad \boxed{u(x) - \underbrace{\cancel{u(0)}}_{0} = \underbrace{C(x-\tfrac{1}{2})}_{\text{cubic spline}} - \frac{x^3}{6} + \frac{1}{4}x^2}$$

Checked!

$$u(x) = \begin{cases} \dfrac{x^2}{4} - \dfrac{x^3}{6}, & \text{for } 0 \le x \le \tfrac{1}{2} \\ \underbrace{(x-\tfrac{1}{2})^3}_{C(x-\frac{1}{2})} - \dfrac{x^3}{6} + \dfrac{x^2}{4}, & \text{for } \tfrac{1}{2} < x \le 1 \end{cases}$$

② $u'''' = \delta(x)$, $u'''(1) = u''(1) = 0$, $u'(1) = u(-1) = 0$

$u(x) = C(x) + Ax^3 + Bx^2 + Cx + D$, $C(x) = \begin{cases} 0, & x \le 0 \\ x^3/6, & x \ge 0 \end{cases}$

$u'''(1) = 0 = \underbrace{S(1)}_{1} + A \cdot 6$, $\boxed{A = -\frac{1}{6}}$; $u''(1) = 0 = \underbrace{R(1)}_{1} - \frac{1}{6} \cdot 6 + 2 \cdot B = 0$ $\boxed{B = 0}$

$u'(1) = 0 = \underbrace{Q(1)}_{1/2} + \overbrace{\left(-\frac{1}{6}\right) 3}^{-1/2} + \boxed{C = 0}$; $u(-1) = 0 =$ [illegible] $+ \left(-\frac{1}{6}\right)(-1) + D$ $\boxed{D = -\frac{1}{6}}$

$\boxed{u(x) = C(x) - \frac{1}{6}x^3 - \frac{1}{6}}$

Matrix way

$u(x) = \frac{x^3}{6} + Ax^3 + Bx^2 + Cx + D$, $u(-1) = 0 =$ [illegible] $- A + B - C + D$

$u'(x) = \frac{x^2}{2} + 3Ax^2 + 2Bx + C$, $u'(1) = 0 = \frac{1}{2} + 3A + 2B + C$

$u''(x) = x + 6Ax + 2B$, $u''(1) = 0 = 1 + 6A + 2B$

$u'''(x) = 1 + 6A$, $u'''(1) = 0 = 1 + 6A$

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 3 & 2 & 1 & 0 \\ 6 & 2 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \\ -1 \\ -1 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} D \\ C \\ B \\ A \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ -1/2 \\ -1 \\ -1 \end{bmatrix}}_{b}$$

$x = A^{-1}b = \begin{bmatrix} -1/6 \\ 0 \\ 0 \\ -1/6 \end{bmatrix} = \begin{bmatrix} D \\ C \\ B \\ A \end{bmatrix}$ ✓, Checked!

③ $u'''' = \delta(x)$,

$$u(x) = \begin{cases} Ax^3 + Bx^2 + Cx + D, & \text{for } x \leq 0 \\ \frac{x^3}{6} + Ax^3 + Bx^2 + Cx + D, & \text{for } x \geq 0 \end{cases}$$

$u(\pm 1) = u'(\pm 1) = 0$

$$u'(x) = \begin{cases} A \cdot 3x^2 + B \cdot 2x + C, & \text{for } x \leq 0 \\ \frac{x^2}{2} + A \cdot 3x^2 + B \cdot 2x + C, & \text{for } x \geq 0 \end{cases}$$

$u(1) = \frac{1}{6} + A + B + C + D = 0$, $u(-1) = -A + B - C + D = 0$

$u'(1) = \frac{1}{2} + A \cdot 3 + B \cdot 2 + C = 0$, $u'(-1) = A \cdot 3 + B \cdot (-2) + C = 0$

$$\begin{array}{c} u(1) \\ u(-1) \\ u'(1) \\ u'(-1) \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 3 & 2 & 1 & 0 \\ 3 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -1/6 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -1/12 \\ -1/8 \\ 0 \\ 1/24 \end{bmatrix} \checkmark$$

⑤ $\varphi_0^d(x), \varphi_0^s(x), \dots, \varphi_3^d(x), \varphi_3^s(x)$ based at the meshpoints $x = 0, \frac{1}{3}, \frac{1}{2}, 1$

$-u'' = f$

ⓐ Fixed-fixed: $u(0) = u(1) = 0 \rightarrow \varphi_0^d = \varphi_3^d = 0$

ⓑ Fixed-free: $u(0) = u'(1) = 0 \rightarrow \varphi_0^d = 0$

but φ_3^s won't be essential, so $\neq 0$!

⑥ $u'''' = f$

ⓐ $u = u' = 0$ at both ends $\rightarrow \varphi_0^d, \varphi_0^s, \varphi_3^d, \varphi_3^s$ would be dropped

ⓑ $u = u'' = 0$ at both ends

only $u(0) = u(1) = 0$ are essential cond. $\rightarrow \varphi_0^d$ and φ_3^d dropped

ⓒ $u(0) = u'(0) = u''(1) = u'''(1) = 0$

only $u(0) = u'(0) = 0$ are essential $\rightarrow \varphi_0^d$ and φ_0^s dropped