18.085 Quiz 1 October 5, 2007 Professor Strang

Your PRINTED name is: Nikita

Grading 1
2
3

$A_0=\begin{bmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}$, $A_1=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}$, $A_2=\begin{bmatrix}1&0\\-1&1\\0&-1\end{bmatrix}$

$A_0^TA_0=\begin{bmatrix}-1&0&0\\1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}=\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}=B$! free-free second difference!

1) **(39 pts.)** With $h=\frac{1}{3}$ there are 4 meshpoints $0,\frac{1}{3},\frac{2}{3},1$ and displacements u_0,u_1,u_2,u_3.

$A_1^TA_1=\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}=$ fixed free

a) Write down the matrices A_0,A_1,A_2 with three rows that produce the first differences u_i-u_{i-1}:

$A_2^TA_2=\begin{bmatrix}1&-1&0\\0&1&-1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\\0&-1\end{bmatrix}=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}=K$ fixed-fixed!

A_0 has 0 boundary conditions on u

A_1 has 1 boundary condition $u_0=0$ (left end fixed)

A_2 has 2 boundary conditions $u_0=u_3=0$.

b) Write down all three matrices $A_0^TA_0, A_1^TA_1, A_2^TA_2$.

CROSS OUT IF FALSE / GIVE REASON BASED ON COLUMNS OF A!

$K_0=A_0^TA_0$ is (singular) ~~(invertible)~~ ~~(positive definite)~~

Reason: $K_0\begin{bmatrix}1\\1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ columns are dependent singular

$K_1=A_1^TA_1$ is ~~(singular)~~ (invertible) (positive definite)

Reason: leading det → 2, 3, 1 → positive definite

c) Find all solutions $w=(w_1,w_2,w_3)$ to each of these equations:

$A_0^Tw=0 \qquad A_1^Tw=0 \qquad A_2^Tw=0$

$\underbrace{\begin{bmatrix}-1&0&0\\1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}}_{A_0^T}\underbrace{\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}}_{w}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}=0 \rightarrow w=0$, $\underbrace{\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}}_{A_1^T}\underbrace{\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}}_{w}=0 \rightarrow w=0$

$\underbrace{\begin{bmatrix}1&-1&0\\0&1&-1\end{bmatrix}}_{A_2^T}\underbrace{\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}}_{w}=\begin{bmatrix}0\\0\end{bmatrix}=0 \rightarrow w=(1,1,1)$
$w=c\begin{bmatrix}1\\1\\1\end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0, \quad (1-\lambda)((2-\lambda)(1-\lambda)-1)-(1-\lambda)=0, \quad (1-2\lambda+\lambda^2)(2-\lambda)-1+\lambda-1+\lambda=0$$

$$\cancel{2}-\lambda-4\lambda+2\lambda^2+2\lambda^2-\lambda^3\cancel{-2}+2\lambda=0, \quad -\lambda^3+4\lambda^2-3\lambda=0, \quad \lambda=0$$

$$\lambda^2-4\lambda+3=0, \quad (\lambda-3)(\lambda-1)=0, \quad \lambda=3,1$$

2) **(33 pts.)** a) Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and unit eigenvectors y_1, y_2, y_3 of B.

Hint: one eigenvector is $(1, 0, -1)/\sqrt{2}$.

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

for $\lambda=1$, $x=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

for $\lambda=0$, $x=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

for $\lambda=3$, $x=\frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

for $\lambda=3 \rightarrow (B-3I)x_3=0$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix}\underbrace{\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}}_{x_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b) Factor B into $Q\Lambda Q^{\mathrm{T}}$ with $Q^{-1} = Q^{\mathrm{T}}$. Draw a graph of the energy function $f(u_1, u_2, u_3) = \frac{1}{2}u^{\mathrm{T}}Bu$. This is a surface in 4-dimensional u_1, u_2, u_3, f space so your graph may not be perfect—**OK to describe it in 1 sentence.** Bowl along the eigenvectors

c) What differential equation with what boundary conditions on $y(x)$ at $x = 0$ and 1 is the **continuous analog** of $By = \lambda y$? What are the eigenfunctions $y(x)$ and eigenvalues λ in this differential equation? At which x's would you sample the first three eigenfunctions to get the three eigenvectors in part (a)?

$-y'' = \lambda y$

$y' = 0$ at $x = 0, 1$

$y_k = \cos(k\pi x)$, $k = 0, 1, 2$

$x = \left(\frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$

$$Q\Lambda Q^T = \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 3 & & \\ & 1 & \\ & & 0 \end{bmatrix}}_{\begin{bmatrix} -\frac{3}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{6}{\sqrt{6}} & 0 & 0 \\ -\frac{3}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}}\begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = B$$

3) **(28 pts.)** The fixed-fixed figure shows $n = 2$ masses and $m = 4$ springs. Displacements u_1, u_2.

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

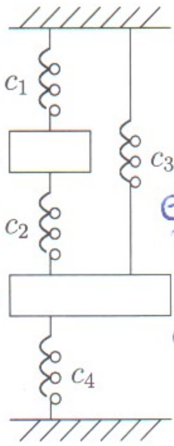
$$\boxed{e = A u}$$

$$K = A^T C A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 + c_4 \end{bmatrix}$$

$e_1 = u_1 \downarrow$

$e_2 = -u_1 + u_2 \downarrow$



$\downarrow u_1$

$e_3 = e_1 + e_2 = u_2$

$e_4 = -u_2$

$\downarrow u_2$

a) Write down the stretching-displacement matrix A in $e = Au$.

b) What is the stiffness matrix $K = A^{\mathrm{T}}CA$ for this system?

c) **Theory question about any $A^{\mathrm{T}}CA$.** C is symmetric positive definite. What condition on A assures that $u^{\mathrm{T}}A^{\mathrm{T}}CAu > 0$ for every vector $u \neq 0$? Explain why this is greater than zero and where you use your condition on A. → Independent Columns!