④ $-\dfrac{d^2u}{dx^2}=\delta(x-a)$, ~~fixed~~-free
$u(0)=0$ and $\underline{u'(1)=0}$

General Form: $u(x)=-R(x-a)+Ax+B$

assumed $\underline{0\le a\le 1}$

Let's integrate both sides twice: $-\int_x^1 u''(\hat{x})\,dx=\int_x^1 \delta(\hat{x}-a)\,dx$

have no any info about x, so $(x\to 1)$ its an upper bound of $\underline{a}$ and for $u'(1)=0$ must be in a range

$-\left[u'(\hat{x})\right]_x^1=\left[S(\hat{x}-a)\right]_x^1$

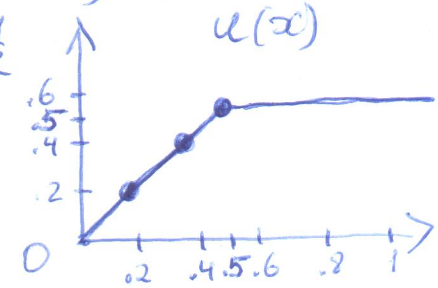
$u'(x)-\cancel{u'(1)}^{\,0}=S(1-a)-S(x-a)$

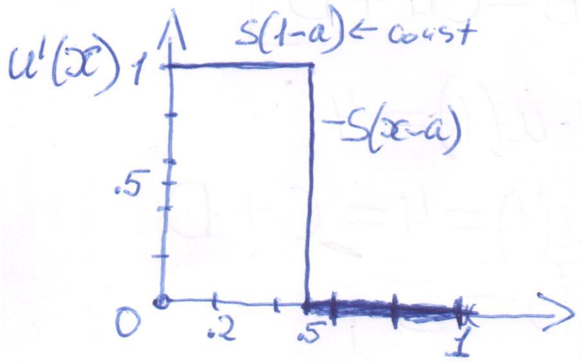
$u(0)=0\to(0\to x)$ $\int_0^x u'(\hat{x})=\int_0^x\big(\underbrace{S(1-a)}_{\text{const } 1 \text{ on the interval}}-S(\hat{x}-a)\big)dx=\underline{\int_0^x\big(1-S(\hat{x}-a)\big)dx}$

$\left[u(\hat{x})\right]_0^x=\left[\hat{x}-R(\hat{x}-a)\right]_0^x$, $u(x)-\cancel{u(0)}^{\,0}=x-R(x-a)+\cancel{R(0-a)}^{\,0}$

$\boxed{u(x)=x-R(x-a)}=\begin{cases}x & \text{for } 0\le x\le a\\ a & \text{for } a\le x\le 1\end{cases}$

$a=\frac{1}{2}$





⑤ $u'(0)=0$ and $u'(1)=0$, free-free
$u(x)=-R(x-a)+Ax+B$, $u'(0)=0\Rightarrow$ [illegible], $A=0$
$u'(1)=0\Rightarrow -1+A=0$, $A=1$) its Not possible
equation $\underline{\text{has no solutions}}$

⑥ $-u''=\delta(x-a)$, $u(0)=u(1)$ and $u'(0)=u'(1)$
$u'(x)=-S(x-a)+A$, $u'(0)=0+A$, $u'(1)=-1+A$) $\underline{\text{can't be equal!}}$

The same for $\underline{u(x)=-R(x-a)+Ax+B}$!

⑦ $f(x) = \delta\left(x-\frac{1}{3}\right) - \delta\left(x-\frac{2}{3}\right) = -u''(x)$, $u'(0)=0$, $u'(1)=0$

find infinitly many solutions

$$u(x) = \begin{cases} Ax+B, & x \le 1/3 \\ Cx+D, & 1/3 \le x \le 2/3 \\ Ex+F, & x \ge 2/3 \end{cases}, \quad u'(0) = \boxed{0 = A}, \quad u'(1) = \boxed{0 = E}$$

$$u\left(\frac{1}{3}\right) = \cancel{A\cdot\frac{1}{3}}^{0} + B = C\cdot\frac{1}{3} + D, \quad 3B = \underline{C+3D}$$

$$u\left(\frac{2}{3}\right) = C\cdot\frac{2}{3} + D = \cancel{E\cdot\frac{2}{3}}^{0} + F, \quad \underline{C+3D} = 3F$$

$\boxed{B = F}$???

no way, dead end...

Integration way

$$-\int_0^x u''(\hat{x})\,dx = \int_0^x \left(\delta\left(\hat{x}-\frac{1}{3}\right) - \delta\left(\hat{x}-\frac{2}{3}\right)\right)dx$$

$$-\left[u'(\hat{x})\right]_0^x = \left[S\left(\hat{x}-\frac{1}{3}\right)\right]_0^x - \left[S\left(\hat{x}-\frac{2}{3}\right)\right]_0^x$$

$$\cancel{u'(0)}^{0} - u'(x) = S\left(x-\frac{1}{3}\right) - \underbrace{\cancel{S\left(-\frac{1}{3}\right)}^{0}}_{\text{out of range}} - S\left(x-\frac{2}{3}\right) - \underbrace{\cancel{S\left(-\frac{2}{3}\right)}^{0}}_{\text{out of range}}$$

$$u'(x) = S\left(x-\frac{2}{3}\right) - S\left(x-\frac{1}{3}\right)$$

$u'(1) = S\left(1-\frac{2}{3}\right) - S\left(1-\frac{1}{3}\right) = 1 - 1 = 0$ ✓ Checked

$$\int_0^x u'(\hat{x})\,dx = \int_0^x S\left(\hat{x}-\frac{2}{3}\right)dx - \int_0^x S\left(\hat{x}-\frac{1}{3}\right)dx$$

$$\left[u(\hat{x})\right]_0^x = \left[R\left(\hat{x}-\frac{2}{3}\right)\right]_0^x - \left[R\left(\hat{x}-\frac{1}{3}\right)\right]_0^x, \quad u(x) - u(0) = R\left(x-\frac{2}{3}\right) - \cancel{R\left(0-\frac{2}{3}\right)}^{0} - R\left(x-\frac{1}{3}\right) + \cancel{R\left(0-\frac{1}{3}\right)}_{0}$$

$$u(x) = R\left(x-\frac{2}{3}\right) - R\left(x-\frac{1}{3}\right) + \underbrace{u(0)}_{\text{const}} = C$$

Ifinitly many solution