Problem Set 1.4

① $-u'' = \delta(x-a)$, $u(0)=2$ and $u(1)=0$

$$u(x)=\begin{cases} Ax+B, & \text{for } 0\le x\le a \\ Cx+D, & \text{for } a\le x\le 1 \end{cases}$$

$u(a)$ → continues function, $(Aa+B=Ca+D)=u(a)$

$u(0)=2=A\cdot 0+B \rightarrow B=2$

$u(1)=0=C+D$

$-u''=\delta(x-a)$, $-u'=+S(x-a)+\text{Const}$

$\int_0^1(-u'')\,dx=\int_0^1\delta(x-a)\,dx=1$, $0\le a\le 1$

$\int_0^1(-u'')\,dx=-[u']_0^1=u'(0)-u'(1)=1$

$u'(0)=A$, $u'(1)=C \rightarrow A-C=1$

$$\boxed{B=2,\ C+D=0,\ A-C=1,\ Aa+B=Ca+D}$$

② free-fixed case $u'(0)=0$ and $u(1)=4$

$u'(0)=\boxed{0=A}$, $\boxed{B=2}$, $\boxed{C=-1}$, $u(1)=4=C+D$

$\boxed{D=5}$

③ $a=\frac{1}{3}$ and $b=\frac{2}{3}$ $\quad -u''=\delta\left(x-\frac{1}{3}\right)+\delta\left(x-\frac{2}{3}\right)$

$$u(x)=\begin{cases} Ax+B, & \text{for } x\le \frac{1}{3} \\ Cx+D, & \text{for } \frac{1}{3}\le x\le\frac{2}{3} \\ Ex+F & \text{for } x\ge\frac{2}{3} \end{cases}$$

General Fixed-Fixed Solution

$$\boxed{u=\begin{cases}(1-a)x, & x\le a\\ (1-x)a, & x\ge a\end{cases}}$$

$-u''=\delta\left(x-\frac{1}{3}\right)$

$u(x)=\begin{cases}(1-1/3)x, & x\le 1/3\\ (1-x)1/3, & x\ge 1/3\end{cases}$, combine them $u(x)=\begin{cases}(1-1/3)x+(1-2/3)x, & x\le 1/3\\ (1-x)1/3+(1-2/3)x, & 1/3\le x\le 2/3\\ (1-x)1/3+(1-x)2/3, & x\ge 2/3\end{cases}$

$-u''=\delta\left(x-\frac{2}{3}\right)$

$u(x)=\begin{cases}(1-2/3)x, & x\le 2/3\\ (1-x)2/3, & x\ge 2/3\end{cases}$

finally → $u(x)=\begin{cases}x, & x\le 1/3\\ 1/3, & 1/3\le x\le 2/3\\ 1-x, & x\ge 2/3\end{cases}$