Problem Set 4.1

(1) a) $f(x) = \sin^3(x)$
$= \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$, $S(x) = b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \dots$

Odd function

$$= \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} S(x)\sin kx \, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)\right)\sin(nx)dx = \frac{1}{2\pi}\left(\int_0^{\pi} 3\sin(x)\sin(nx)dx - \int_0^{\pi}\sin(3x)\sin(nx)dx\right)$$

Because of Orthogonality
$n = 1, 3$ for any other $n \to b_n = 0$!

$$b_1 = \frac{3}{2\pi}\int_0^{\pi}\sin^2(x)dx = \frac{3}{4}, \quad b_3 = \frac{1}{2\pi}\int_{\pi}^{0}\sin^2(3x)dx = -\frac{1}{4}$$

$$S(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$$

b) $f(x) = |\sin(x)|$, $C(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$

even function

$$a_0 = \frac{1}{\pi}\int_0^{\pi} C(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} C(x)\cos(kx)dx$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi}|\sin(x)|\,dx$$

$$= \frac{1}{\pi}\Big[-\cos(x)\Big]_0^{\pi} = \frac{1}{\pi}\left[-\cos\pi + \cos 0\right] = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}|\sin(x)|\cos(kx)dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos kx\,dx = \frac{1}{\pi}\int_0^{\pi}\left[\sin(k+1)x - \sin(k-1)x\right]dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{k+1}\cos(k+1)x + \frac{1}{k-1}\cos(k-1)x\right]_0^{\pi} =$$

$$= \frac{2}{\pi}\,\frac{(\cos(\pi n) + 1)}{1 - n^2} \longrightarrow \begin{cases} 0, & \text{if } k \text{ is odd} \\ \frac{4}{\pi}\left(\frac{1}{1-n^2}\right), & \text{if } n \text{ is even} \end{cases}$$

$$|\sin(x)| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2\pi n) + 1}{1 - (2n)^2}$$

c) $f(x) = x$, odd function

$$b_k = \frac{2}{\pi}\int_0^{\pi} x\sin(kx)\,dx$$

$$\int x\sin(kx)\,dx = -\frac{x\cos(kx)}{k} + \frac{1}{k}\int\cos(kx)\,dx = -\frac{x\cos(kx)}{k} + \frac{1}{k^2}\sin(kx) + C$$

$$b_k = \frac{+2}{\pi}\left[\frac{-x\cos(kx)}{k} + \frac{1}{k^2}\sin(kx)\right]_0^{\pi} = +\frac{2}{\pi}\left(\frac{-\pi\cos(k\pi)}{k} + \frac{1}{k^2}\sin(k\pi)\right)$$

$$b_k \longrightarrow \begin{cases} \frac{2}{\pi}\,\frac{-\pi(-1)}{k} + \frac{1}{k^2}\cdot 0 = \frac{+2}{k}, \text{ for odd } k \\ \frac{2}{\pi}\,\frac{-\pi(+1)}{k} + \frac{1}{k^2}\cdot 0 = \frac{-2}{k}, \text{ for even } k \end{cases} \longrightarrow -\frac{2(-1)^k}{k} = b_k$$

or $\frac{2(-1)^{k+1}}{k}$

$$f(x) = x = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin(kx)}{k}$$

for $-\pi < x < \pi$

d) $f(x) = e^x$

$$\boxed{F(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)e^{-ikx}\,dx}$$

$$C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx}\,dx$$

$$\int e^x e^{-ikx}\,dx = \int e^{x(1-ik)}\,dx = \frac{1}{(1-ik)}e^{x(1-ik)} + C$$

$$C_k = \frac{1}{2\pi}\left[\frac{1}{(1-ik)}\left[e^{x(1-ik)}\right]_{-\pi}^{\pi}\right] = \frac{1}{2\pi(1-ik)}\left(e^{\pi(1-ik)} - e^{-\pi(1-ik)}\right)$$

$$= \frac{1}{2\pi(1-ik)}\left(e^{\pi}\left(e^{-\pi ik}\right) - e^{-\pi}\left(e^{\pi ik}\right)\right) = \frac{1}{2\pi(1-ik)}\left[e^{\pi}\left(\cos(\pi k) - \cancel{i\sin(\pi k)}\right) - e^{-\pi}\left(\cos(\pi k) + \cancel{i\sin(\pi k)}\right)\right]$$

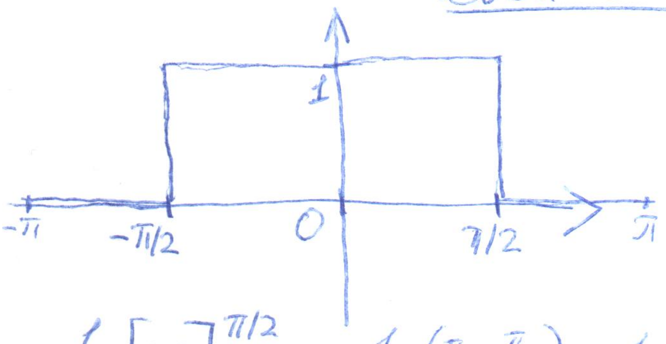
for any k

$$C_k \longrightarrow \frac{(-1)^k\left(e^{\pi} - e^{-\pi}\right)}{2\pi(1-ik)}$$

$$f(x) = e^x = \frac{e^{\pi} - e^{-\pi}}{2\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{inx}}{1-ik}$$

3 $f(x)=\begin{cases}1, & \text{for } |x|<\frac{\pi}{2}\\ 0, & \text{for } \frac{\pi}{2}<|x|<\pi\end{cases}$

even fuc



Because $f(x)$ is even

$b_n = 0$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\,dx = \frac{1}{2\pi}\Big[x\Big]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{1}{2} = a_0$$

average

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cdot\cos(nx)\,dx = \frac{1}{n\pi}\Big[\sin(nx)\Big]_{-\pi/2}^{\pi/2} = \frac{1}{n\pi}\left(\sin\left(\frac{\pi n}{2}\right)+\sin\left(\frac{\pi n}{2}\right)\right)$$

$$a_n = \frac{2}{n\pi}\sin\left(\frac{\pi n}{2}\right) \qquad C(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{\pi n}{2}\right)\cos(nx)$$

5 $x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{27}+\frac{\sin 5x}{125}+\dots\right], \quad 0<x<\pi$

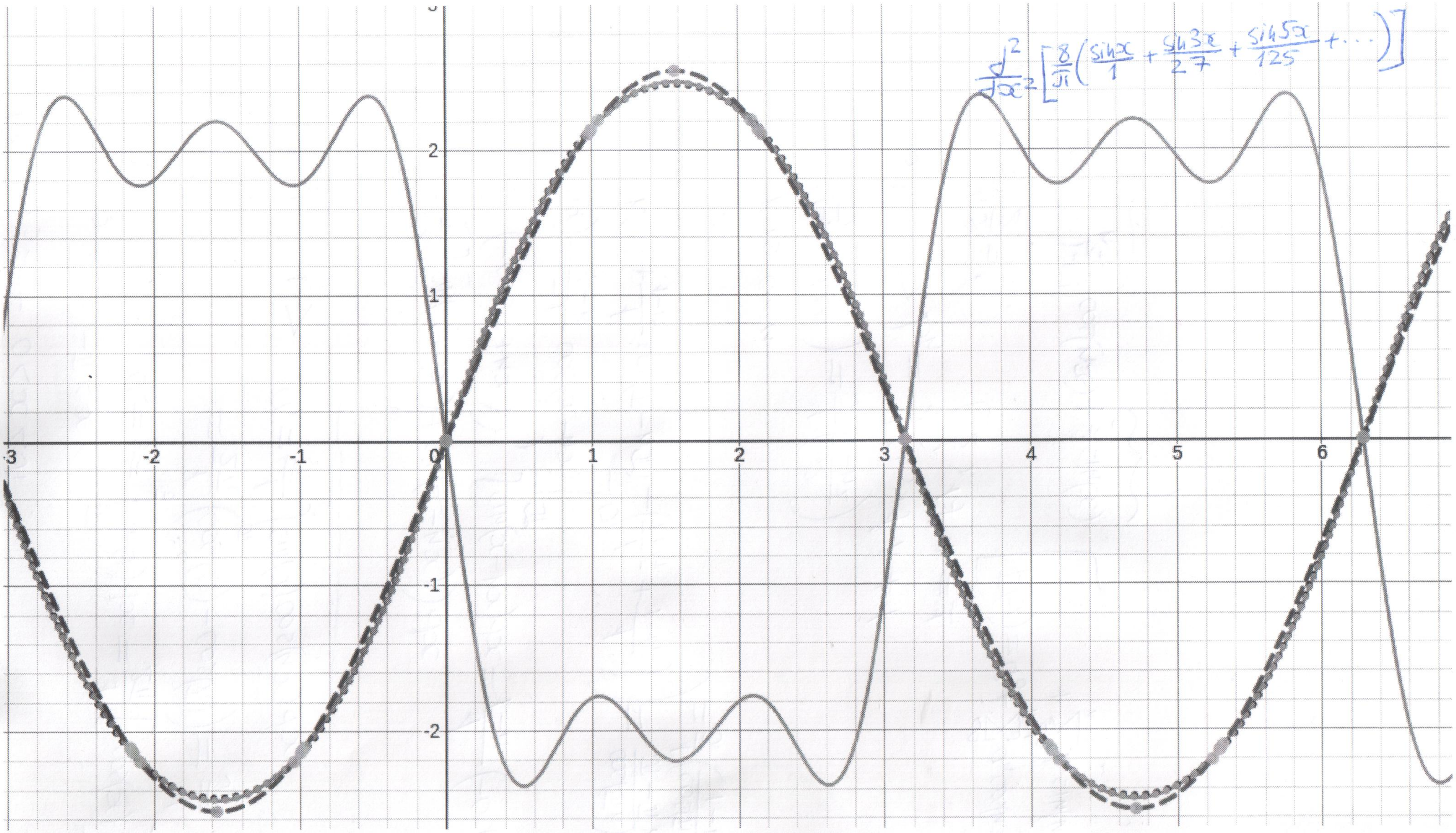
$$\frac{d}{dx}\left[\frac{8}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{27}+\frac{\sin 5x}{125}\right)\right] = \frac{8}{\pi}\left(\frac{\cos x}{1}+\frac{\cos 3x}{9}+\frac{\cos 5x}{25}+\dots\right)$$

$$\frac{d^2}{dx^2}\left[\frac{8}{\pi}\left(\frac{\cos x}{1}+\frac{\cos 3x}{9}+\frac{\cos 5x}{25}+\dots\right)\right] = \frac{8}{\pi}\underbrace{\left(\frac{-\sin x}{1}+\frac{-\sin 3x}{3}+\frac{-\sin 5x}{5}+\dots\right)}_{\text{square wave!}}$$

Step function has decay rate $1/k$

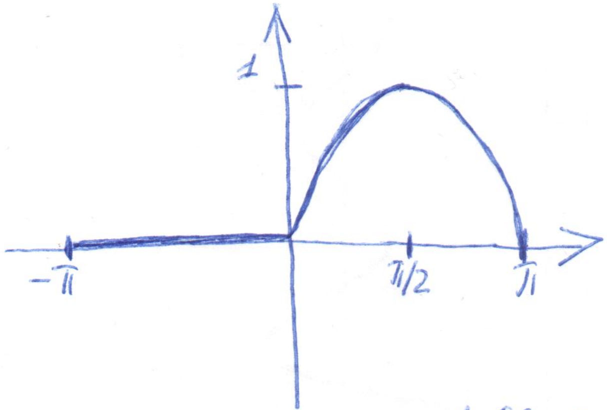
Each ~~[illegible]~~ step integration ~~[illegible]~~ dacay rate increase ~~[illegible]~~ $1/k$ times faster $\rightarrow 1/k^3$ after 2 step

5


$\frac{d^2}{dx^2}\left[\frac{8}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{27}+\frac{\sin 5x}{125}+\dots\right)\right]$
-3
-2
-1
0
1
2
3
4
5
6
2
1
-1
-2

7

$$f(x) = \begin{cases} \sin(x), & \text{for } 0 < x < \pi \\ 0, & \text{for } -\pi < x < 0 \end{cases}$$



$$a_0 = \frac{1}{2\pi}\int_0^\pi \sin(x)\,dx = \frac{1}{2\pi}\Big[-\cos(x)\Big]_0^\pi$$

$$= \frac{1}{2\pi}\big(\cos(0) - \cos(\pi)\big) = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^\pi \sin(x)\cos(nx)\,dx$$

$$\int \sin(x)\cos(nx)\,dx = \frac{1}{2}\int\big(\sin(x+nx) + \sin(x-nx)\big)\,dx$$

$$\int \sin(x+nx)\,dx = \frac{1}{1+n}\big(-\cos(x+nx)\big); \quad \int \sin(x-nx)\,dx = \frac{1}{1-n}\big(-\cos(x-nx)\big)$$

$$a_n = \frac{1}{2\pi}\left[\frac{-1}{1+n}\cos(x+nx) + \frac{-1}{1-n}\cos(x-nx)\right]_0^\pi$$

$$= \frac{-1}{2\pi}\left(\frac{1}{1+n}\cos(\pi + n\pi) + \frac{1}{1-n}\cos(\pi - n\pi) - \left(\frac{1}{1+n} + \frac{1}{1-n}\right)\right) = \frac{\cos(\pi n) + 1}{\pi - \pi n^2}$$

$$= \frac{1}{\pi}\left(\frac{\cos(\pi n) + 1}{1 - n^2}\right)$$

$$a_n = \begin{cases} 0, & \text{for } n \text{ is odd} \\ \frac{2}{\pi}\left(\frac{1}{1-n^2}\right), & \text{for } n \text{ is even} \end{cases}$$

$$-\frac{1}{2\pi}\left(\frac{-2}{1+n} + \frac{-2}{1-n}\right) = \frac{1}{\pi}\left(\frac{1-\cancel{n}+1+\cancel{n}}{1-n^2}\right) = \frac{1}{\pi}\left(\frac{2}{1-n^2}\right)$$

$$b_n = \frac{1}{\pi}\int_0^\pi \sin(x)\sin(nx)\,dx \Rightarrow b_n = \begin{cases} 0, & \text{if } n \neq 1 \\ \frac{1}{2}, & \text{if } n = 1 \end{cases}$$

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin^2(x)\,dx = \frac{1}{\cancel{\pi}}\frac{\cancel{\pi}}{2} = \frac{1}{2}$$

$$\left[F(x) = a_0 + \sum_{n=1}^\infty a_n\cos(nx) + \sum_{n=1}^\infty b_n\sin(nx)\right]$$

$$f(x) = \underbrace{\frac{1}{\pi}}_{a_0} + \underbrace{\frac{1}{\pi}\sum_{n=1}^\infty \frac{\cos(\pi n)+1}{1-n^2}\cos(nx)}_{C(x)} + \underbrace{\frac{1}{2}\sin(x)}_{S(x)}$$

$a_0 + C(x)$

(13) $f(x) = \begin{cases} 1, \text{ for } |x| < \frac{\pi}{2} \\ 0, \text{ for } \frac{\pi}{2} < |x| < \pi \end{cases}$

a) $\int_{-\pi}^{\pi} |F(x)|^2 dx = \int_{-\pi/2}^{\pi/2} 1^2 dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

b) $C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ikx} dx = \frac{1}{2\pi(-ik)} \left[e^{-ikx} \right]_{-\pi/2}^{\pi/2}$

$= -\frac{1}{2\pi(ik)} \left(\exp\left(-\frac{ik\pi}{2}\right) - \exp\left(\frac{ik\pi}{2}\right) \right) = -\frac{1}{2\pi k i} \left(\cancel{\cos\left(\frac{k\pi}{2}\right)} - i\sin\left(\frac{k\pi}{2}\right) \cancel{-\cos\left(\frac{k\pi}{2}\right)} - i\sin\left(\frac{k\pi}{2}\right) \right)$

$= \frac{1}{\pi k} \sin\left(\frac{k\pi}{2}\right) = C_k$

c) $C_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^0 dx = \frac{1}{2\cancel{\pi}} \cancel{\pi} = \frac{1}{2}$

$\int_{-\pi}^{\pi} |F(x)|^2 dx = 2\pi \left(|C_0|^2 + |C_1|^2 + |C_{-1}|^2 + \dots \right) \Rightarrow$

$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 = 2\pi |C_0|^2 + 2\pi \left(2 \sum_{n=1}^{\infty} |C_n|^2 \right)$

$= \frac{\pi}{2} + 4\pi \sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{2}\right)}{\pi n} \right|^2 = \frac{\pi}{2} + \frac{4\cancel{\pi}}{\pi^{\cancel{2}}} \underbrace{\sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{n\pi}{2}\right)}{n^2}}_{\text{wolfram} = \frac{\pi^2}{8}}$

$= \frac{\pi}{2} + \frac{\cancel{4}^{1}}{\cancel{\pi}} \left(\frac{\pi^{\cancel{2}}}{\cancel{8}\,2} \right) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ ✓