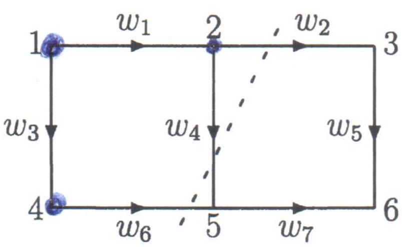
Problem Set 3.3

1 For uniform flow $v=(1,0)=w$, what are the equipotentials and streamlines? For a flow field $w=(0,x)$ what are the streamlines? (Solve for s, there is no u.)

2 Show that this *shear flow* $w=(0,x)$ is not a gradient field. But the streamlines are straight vertical lines, parallel to w. How can there be any rotation when the flow is all upward or downward?

3 *Discrete Divergence Theorem* The flows out of nodes 1, 2, 4 are w_1+w_3 and w_2 w_4 and w_6. The sum of those three "divergences" is the total flow out across the dashed line.



$+\ w_3+w_6-w_4-w_1$ ← left
$w_4+w_7-w_5-w_2$ ← right
$= w_3-w_2-w_1+w_6+w_7$
circulation around the large

4 *Discrete Stokes Theorem* The circulation around the left rectangle is $w_3+w_6-w_4-w_1$. Add to the circulation around the right rectangle to get the circulation around the large rectangle. (The continuous Stokes Theorem is a foundation of modern calculus.)

5 In Stokes' law (8), let $v_1=-y$ and $v_2=0$ to show that the area of S equals the line integral $-\int_C y\,dx$. Find the area of an ellipse ($x=a\cos t$, $y=b\sin t$, $x^2/a^2+y^2/b^2=1$, $0\le t\le 2\pi$).

6 By computing curl v, show that $v=(y^2,x^2)$ is not the gradient of any function u but that $v=(y^2,2xy)$ is such a gradient—and find u.

7 From div w, show that $w=(x^2,y^2)$ does not have the form $(\partial s/\partial y, -\partial s/\partial x)$ for any function s. Show that $w=(y^2,x^2)$ does have that form, and find the stream function s.

8 If $u=x^2$ in the square $S=\{-1<x,y<1\}$, compute both sides when $w=\operatorname{grad} u$:

$$\textbf{Divergence Theorem}\qquad \iint_S \operatorname{div}\operatorname{grad} u\,dx\,dy=\int_C n\cdot\operatorname{grad} u\,ds.$$

9 The curves $u(x,y)=$ constant are orthogonal to the family $s(x,y)=$ constant if grad u is perpendicular to grad s. These gradient vectors are at right angles to the curves, which can be equipotentials and streamlines. Construct $s(x,y)$ and verify $(\operatorname{grad} u)^{\mathrm T}(\operatorname{grad} s)=0$:

(a) $u(x,y)=y$: equipotentials are parallel horizontal lines

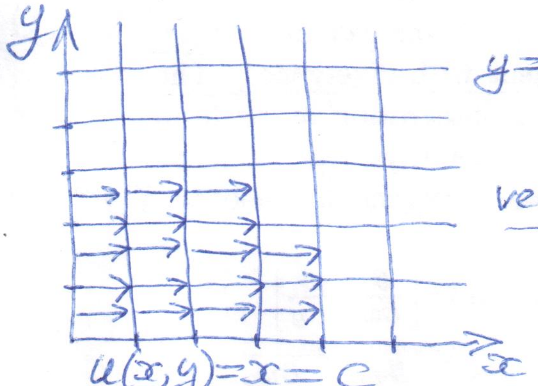
Problem Set 3.3

① $v = (1,0) = w$ need: $\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x} = 0 \iff \underline{v = \text{grad}\, u}$

$u(x,y) = x$, $\nabla u = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, Checked

$\boxed{\text{equipotential lines } u(x,y) = x = \text{const}}$

$\text{div}\, w = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} = 0$; $w_1 = \frac{\partial s}{\partial y} = 1$, $w_2 = -\frac{\partial s}{\partial x} = 0$, $s(x,y) = y = \text{const}$



$y = c \Rightarrow s(x,y) \leftarrow$ stream lines

vector field $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u(x,y) = x = c$
equipotential lines

$w = (0, x)$, $w_1 = \frac{\partial s}{\partial y} = 0$, $w_2 = -\frac{\partial s}{\partial x} = x$, $s(x,y) = -\frac{1}{2}x^2 \leftarrow$ stream function

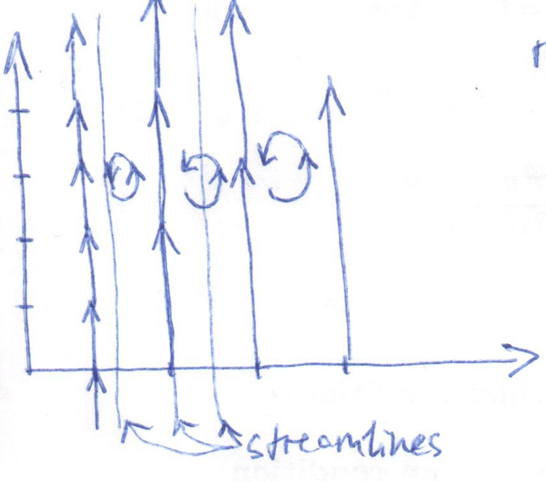
② Shear flow $w = (0, x)$, need: $\frac{\partial w_1}{\partial y} = \frac{\partial w_2}{\partial x} \to 0 \neq 1$ v is not a grad u!

$w_1 = \frac{\partial s}{\partial y}$, $w_2 = -\frac{\partial s}{\partial x}$ $\to s(x,y) = -\frac{1}{2}x^2 = \text{const}$

The rotation field has <u>zero divergence</u>

$\text{div}\, w = 0 + 0 = 0$ $\checkmark \leftarrow w$ is rotation field!

rotation is because of different <u>speed between flow sides</u>!



(5) $V_1 = -y$ and $V_2 = 0$

$$\int_C V_1 dx + V_2 dy = \iint_R \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) dx\, dy$$

Stroke's Theorem

$$\int_C (-y)\, dx = \iint_R -\frac{\partial(-y)}{\partial y}\, dx\, dy$$

$$\left[-\int_C y\, dx = \iint_R dx\, dy \right]$$

$$= \iint_R dS$$

(6) $V_1 = (y^2, x^2)$, $\operatorname{curl} V = \nabla \times V_1 = \frac{\partial(y^2)}{\partial y} - \frac{\partial(x^2)}{\partial x} = 2y - 2x$

$$= 2(y-x) \neq 0$$

V_1 is not a gradient of any function

$V_2 = (y^2, 2xy)$ $\operatorname{curl} V_2 = 2y - 2y = 0$ ✓

V_2 is a gradient of $\quad \frac{du}{dx} = y^2$, $u(x,y) = xy^2 + f_1(y)$

$$\frac{du}{dy} = 2xy, \; u(x,y) = xy^2 \quad + f_2(x)$$

$$\left\lfloor u(x,y) = xy^2 \right\rceil$$

$$f_1(y) = f_2(x) = C \;!$$

(8) $u = x^2$, $S = \{-1 < x, y < 1\}$, $w = \operatorname{grad} u$

$\operatorname{div} \operatorname{grad} u = \nabla \cdot \nabla u = 2$, $\quad w = \begin{bmatrix} 2x \\ 0 \end{bmatrix}$

$$\iint_S \operatorname{div} \operatorname{grad}(u) = \int_{-1}^{1}\int_{-1}^{1} \nabla \cdot \nabla u\, dx\, dy = \int_{-1}^{1}\int_{-1}^{1} 2\, dx\, dy$$

$$= 2\int_{-1}^{1} [x]_{-1}^{1}\, dy = 4[y]_{-1}^{1} = 8$$