Problem Set 1.3

① $K_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \underset{L_4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}} \underset{U_4}{\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}}$

$\det(K_4) = 5$

$2 \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\boxed{5}}{\cancel{4}} \leftarrow \boxed{n+1}$

$K_4 = \underset{L_4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}} \underset{D_4}{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}} \underset{L_4^T}{\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \cancel{=U}$

② 1. $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2/3 & 1 & 0 & 0 & 1 \end{array}\right] \overset{+\frac{1}{2}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 2/3 & 1 \end{array}\right] +\frac{2}{3}$

$\quad L_3 \quad I_3 \qquad\qquad I_3 \quad \boxed{L_3^{-1}}$

$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4/3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3/4 \end{array}\right]; \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} +\frac{1}{2} \\ +\frac{2}{3} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1/2 & 1/3 \\ 0 & 1 & 0 & 0 & 1 & 2/3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$\quad D_3 \quad I_3 \qquad I_3 \quad \boxed{D_3^{-1}} \qquad L_3^T \quad I_3 \qquad I_3 \quad (L_3^T)^{-1}$

2. $p_K(i) = \frac{(i+1)}{i} \leftarrow$ ith pivot of K

3. $i,j \rightarrow$ [illegible] $\underset{i\ j}{(3,2)} = \boxed{2/3}$

⑤ $A = \begin{bmatrix} \overbrace{2 \quad\ \ }^{B} & & \\ & 7 & \\ & & 6 \end{bmatrix}$, pivots of A $= 2, 7, 6$; pivots of B $= \boxed{2, 7}$

⑥ $K_{(5\times5)} \rightarrow$ 1 diagonal $= 5$
$\quad +2$ upper/lower diag $= 8$ $= \boxed{13 \text{ non-zero items}}$

$D_{(5\times5)} \rightarrow$ 1 diagonal $= \boxed{5 \text{ non-zero items}}$

$L_{(5\times5)} \rightarrow$ 1 diagonal $= 5$
$\quad$ 1 lower diag $= 4$ $= \boxed{9 \text{ non-zero items}}$

⑦ $A_{(m\times n)}, A^T_{(n\times m)}, C_{(m\times m)} \leftarrow$ symmetric

1. $(A^TCA)^T = A^TC^T(A^T)^T = A^TCA \leftarrow$ <u>symmetric</u>

$\underbrace{(n\times m)\cdot(m\times m)}\cdot(m\times n)$

$(n\times m)\cdot(m\times n) = \boxed{n\times n}$

2. On diagonal we have squares of items! <u>can't be negative!</u>

⑧ $A=\begin{bmatrix}1&3\\3&2\end{bmatrix}\rightarrow\underset{\textcircled{u}}{\begin{bmatrix}1&3\\0&-7\end{bmatrix}},\ L=\begin{bmatrix}1&0\\3&1\end{bmatrix},\ D=\begin{bmatrix}1&0\\0&-7\end{bmatrix}$

$\ell_{21}=3$

$$A=\underset{L}{\begin{bmatrix}1&0\\3&1\end{bmatrix}}\underset{D}{\begin{bmatrix}1&0\\0&-7\end{bmatrix}}\underset{L^T}{\begin{bmatrix}1&3\\0&1\end{bmatrix}}\overset{\checkmark}{=}\begin{bmatrix}1&3\\3&2\end{bmatrix}$$

$$A=\begin{bmatrix}1&b\\b&c\end{bmatrix}\rightarrow\underset{\textcircled{u}}{\begin{bmatrix}1&b\\0&c-b^2\end{bmatrix}},\ D=\begin{bmatrix}1&\\&c-b^2\end{bmatrix},\ L=\begin{bmatrix}1&0\\b&1\end{bmatrix}$$

$\ell_{21}=b$

Check $A=LU=\begin{bmatrix}1&0\\b&1\end{bmatrix}\begin{bmatrix}1&b\\0&c-b^2\end{bmatrix}\overset{\checkmark}{=}\begin{bmatrix}1&b\\b&c\end{bmatrix}$

$$A=\underset{L}{\begin{bmatrix}1&0\\b&1\end{bmatrix}}\underset{D}{\begin{bmatrix}1&0\\0&c-b^2\end{bmatrix}}\underset{L^T}{\begin{bmatrix}1&b\\0&1\end{bmatrix}}\overset{\checkmark}{=}\begin{bmatrix}1&b\\b&c\end{bmatrix}$$

$$A=\begin{bmatrix}2&1&0\\1&2&1\\0&1&2\end{bmatrix}\rightarrow\begin{bmatrix}2&1&0\\0&3/2&1\\0&0&4/3\end{bmatrix}=u,\ L=\begin{bmatrix}1&0&0\\1/2&1&0\\0&2/3&1\end{bmatrix}$$

$\ell_{21}=\frac{1}{2}$ $\quad e_{32}=2/3$

$2-\frac{2}{3}=\frac{4}{3}$

$$A=\underset{L}{\begin{bmatrix}1&0&0\\1/2&1&0\\0&2/3&1\end{bmatrix}}\underset{D}{\begin{bmatrix}2&0&0\\0&3/2&0\\0&0&4/3\end{bmatrix}}\underset{L^{-1}}{\begin{bmatrix}1&1/2&0\\0&1&2/3\\0&0&1\end{bmatrix}}\overset{\checkmark}{=}\begin{bmatrix}2&1&0\\1&2&1\\0&1&2\end{bmatrix}$$

⑬ $A=\begin{bmatrix}1&1\\1&0\end{bmatrix}\rightarrow\begin{bmatrix}\textcircled{1}&1\\0&\textcircled{-1}\end{bmatrix}=u\quad L=\begin{bmatrix}1&0\\1&1\end{bmatrix},\ A=\underset{L}{\begin{bmatrix}1&0\\1&1\end{bmatrix}}\underset{D}{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}\underset{L^T}{\begin{bmatrix}1&1\\0&1\end{bmatrix}}\overset{\checkmark}{=}\begin{bmatrix}1&1\\1&0\end{bmatrix}$

$\ell_{21}=1\quad$ pivots $=1,-1$

$$A^{16}\cdot\begin{bmatrix}1\\0\end{bmatrix}\overset{\checkmark}{=}\begin{bmatrix}13\\8\end{bmatrix}$$

⑭ $A=LU,\ Ax=f,\ Lc=f$ and $Ux=c$

$$L=\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\3&1&0&0&1&0\\0&2&1&0&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&-3&1&0\\0&0&1&6&-2&1\end{array}\right]\quad\underset{I\qquad L^{-1}}{}\quad\underbrace{LUx}_{Ax\,=\,f}=\underbrace{Lc}$$

$$U=\left[\begin{array}{ccc|ccc}2&8&0&1&0&0\\0&3&5&0&1&0\\0&0&7&0&0&1\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{4}&-\frac{4}{3}&\frac{20}{21}\\0&1&0&0&\frac{1}{3}&-\frac{5}{21}\\0&0&1&0&0&\frac{1}{7}\end{array}\right]$$

$$c=L^{-1}f=\begin{bmatrix}1&0&0\\-3&1&0\\6&-2&1\end{bmatrix}\begin{bmatrix}0\\3\\6\end{bmatrix}=\begin{bmatrix}0\\3\\0\end{bmatrix};\ x=U^{-1}c=\begin{bmatrix}-4\\1\\0\end{bmatrix}\ \checkmark$$