Problem Set I.1

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[1]:
```
import matplotlib as mpl
import matplotlib.pyplot as plt
```

1.1 # 1

Example: $Ax = 0$ in R^4

$$\begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Dimensions: $(4 \times 3) \cdot (3 \times 1) = (4 \times 1)$

$u + v - (u + v) = 0$

So, any u, v:

$$\begin{bmatrix} u & v & u + v \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$$

In the Example:

$$\begin{bmatrix} u & v & u + 2v \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = u + 2v - (u + 2v) = 0$$

1.2 # 2

Example of $Ax = Ay$

$$A = \begin{bmatrix} u & v & u + v \end{bmatrix}$$

$$A \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = A \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \implies u + v - (u + v) = -u - v + (u + v) = 0$$

$Az = 0$

$$\begin{bmatrix} u & v \end{bmatrix} \cdot z = 0 \implies z_1 \cdot u + z_2 \cdot v = 0$$

So: $z = 0$

Or: $z = \begin{bmatrix} -v \\ u \end{bmatrix}$

Or: $z = \begin{bmatrix} v \\ -u \end{bmatrix}$

1.3 # 3

The vectors $a_1, a_2, \ldots, a_n$ in R^m and $c_1 \cdot a_1 + \cdots + c_n \cdot a_n = 0$

Matrix notation

$$\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

Or

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1 \cdot \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{bmatrix} + \cdots + c_n \cdot \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix} = 0$$

Sigma notation:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} * c_j = 0$$

1.4 # 4

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Two independent vectors

$Ax = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Ay = 0$

$$\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

There is no third independent vector that solve $Az = 0$

1.5 # 5

$v = (1, 1, 0)$ and $w = (0, 1, 1)$ in R^3

a) Find z that $v^T z = w^T z = 0$ or $(cv + dw)^T z = cv^T z + dw^T z = 0$

Check: $z = (1, -1, 1)$

$$v^T z = \begin{bmatrix}1 & 1 & 0\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\1\end{bmatrix} = 0$$

And

$$w^T z = \begin{bmatrix}0 & 1 & 1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\1\end{bmatrix} = 0$$

b) Vector u that not on the plain $u^T z \neq 0$

Check: $u = (1, 0, 1)$

$$u^T z = \begin{bmatrix}1 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\1\end{bmatrix} = 2$$

1.6 # 6

$v = (4, 2)$, $w = (1, 3)$, $u = (1, 1)$

$z_1 = v + w - u = (4, 4)$

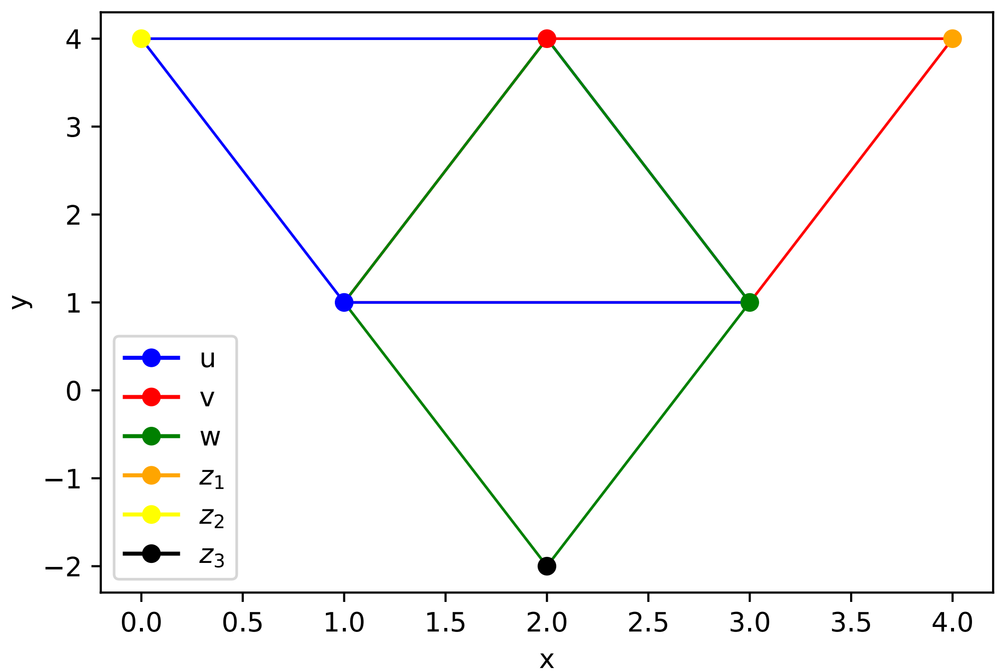
$z_2 = v - w + u = (4, 0)$

$z_3 = -v + w + u = (-2, 2)$

[2]:
```
u = [1, 1]
v = [2, 4]
w = [3, 1]
z_1 = [4, 4]
z_2 = [0, 4]
z_3 = [2, -2]

fig, ax = plt.subplots()
```

```python
ax.plot(u[0], u[1], color="blue", label="u", marker='o')
ax.plot(v[0], v[1], color="red", label="v", marker='o')
ax.plot(w[0], w[1], color="green", label="w", marker='o')
ax.plot(z_1[0], z_1[1], color="orange", label=f'$z_1$', marker='o')
ax.plot(z_2[0], z_2[1], color="yellow", label=f'$z_2$', marker='o')
ax.plot(z_3[0], z_3[1], color="black", label=f'$z_3$', marker='o')

p1 = plt.Polygon([u, v, z_1, w], fill=False, color="red")
p2 = plt.Polygon([u, z_2, v, w], fill=False, color="blue")
p3 = plt.Polygon([z_3, u, v, w], fill=False, color="green")

plt.gca().add_patch(p1)
plt.gca().add_patch(p2)
plt.gca().add_patch(p3)

ax.set_xlabel("x")
ax.set_ylabel("y")

ax.legend()
```

[2]: <matplotlib.legend.Legend at 0x7fb5e43757f0>



1.7 # 7

$$A = \begin{bmatrix} v & w & v + 2w \end{bmatrix}$$

Column space: $C(A) = \begin{bmatrix} v & w \end{bmatrix}$

Plain in R^3 -> $dim(C(A)) = 2$

Null space: $N(A) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = x$

Line in R^3 -> $dim(N(A)) = 1$

$Ax = \begin{bmatrix} v & w & v+2w \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = v + 2w - (v + 2w) = 0$

1.8 # 8

$A = CR$

If $A_{ij} = j^2$ is 3 by 3 $\implies A = \begin{bmatrix} 1 & 4 & 9 \\ 1 & 4 & 9 \\ 1 & 4 & 9 \end{bmatrix}$

$C = (1, 1, 1)$ and $R = (1, 4, 9)$

$$A = CR = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 9 \\ 1 & 4 & 9 \\ 1 & 4 & 9 \end{bmatrix}$$

$rank(A) = 1$

1.9 # 9

If $C(A) = R^3 \implies m = n = rank(A) = 3$

1.10 # 10

$$A_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix}, C_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \implies A_1 = \begin{bmatrix} c_1 & 3c_1 & c_1 - 3c_1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \implies A_2 = \begin{bmatrix} c_1 & c_2 & 2c_2 - c_1 \end{bmatrix}$$

1.11 # 11

$$A_1 = CR = \begin{bmatrix} 1 & 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix}$$

$$A_2 = CR = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

1.12 # 12

$C(A_1) = C_1, dim(C(A_1)) = rank(A_1) = 1$

$C(A_2) = C_2, dim(C(A_2)) = rank(A_2) = 2$

Number of independent rows = rank of a matrix = $dim(C(A)) \implies$ 1 and 2 for A_1 and A_2 respectively

1.13 # 13

$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$

$$A = CR = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}$$

1.14 # 14

$A = \begin{bmatrix} -u & v \end{bmatrix}$

$B = \begin{bmatrix} u & -v & u+v \end{bmatrix}$

$C(A) = C(B) = \begin{bmatrix} u & v \end{bmatrix}$ <= the same col space

$R(A) \neq R(B)$ <= different row space

$R(A) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$R(B) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$rank(A) = rank(B) = 2$

$$A = C(A)R_A = \begin{bmatrix} u & v \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = C(B)R_B = \begin{bmatrix} u & v \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

1.15 # 15

$A = CR$

$A = \begin{bmatrix} u & v \end{bmatrix}$

$$A = CR = \begin{bmatrix} [u_1 & v_1] \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} [u_1 & v_1] \\ u_2 & v_2 \end{bmatrix} \cdot \begin{bmatrix} \ulcorner 1 \urcorner & \ulcorner 0 \urcorner \\ \llcorner 0 \lrcorner & \llcorner 1 \lrcorner \end{bmatrix}$$

(1 row of A)=(1 row of C)$\cdot$(cols of R)

1.16 # 16

Every row of A can be produced from combonation of rows R

1.17 # 17

1)

$$A_1 = \begin{bmatrix} zeros & ones \\ ones & ones \end{bmatrix}_{4\times 4} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$C_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

$R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$$A_1 = C_1 \cdot R_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$(4 \times 2) \cdot (2 \times 4) = (4 \times 4)$

2) $A_2 = \begin{bmatrix} A_1 \\ A_1 \end{bmatrix}_{8\times 4}$

$(8 \times 2) \cdot (2 \times 4) = (8 \times 4)$

$$A_2 = \begin{bmatrix} A_1 \\ A_1 \end{bmatrix}_{8\times 4} = \begin{bmatrix} C_1 \\ C_1 \end{bmatrix}_{8\times 2} \cdot \begin{bmatrix} R_1 \end{bmatrix}_{2\times 4}$$

$$A_2 = C_2 \cdot R_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

3)

$$A_3 = \begin{bmatrix} A_1 & A_1 \\ A_1 & A_1 \end{bmatrix}_{8\times 8} = \begin{bmatrix} C_1 \\ C_1 \end{bmatrix}_{8\times 2} \cdot \begin{bmatrix} R_1 & R_1 \end{bmatrix}_{2\times 8} = \begin{bmatrix} C_1 \cdot R_1 & C_1 \cdot R_1 \\ C_1 \cdot R_1 & C_1 \cdot R_1 \end{bmatrix}_{8\times 8}$$

1.18 # 18

$A = CR$

$$\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix} = \begin{bmatrix} C \\ C \end{bmatrix} \cdot \begin{bmatrix} 0 & R \end{bmatrix} = \begin{bmatrix} 0 & CR \\ 0 & CR \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$$

1.19 # 19

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}_{-(R_1 - R_2)} \rightarrow \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}^{-R_2} \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} -2R_1 \rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = rref(A)$$

1.20 A=CMR

$A = CR = CMR$

$C^T A R^T = C^T C M R R^T \implies M = (C^T C)^{-1} C^T A R^T (R R^T)^{-1}$ (equation *)

1.21 # 20

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

$$CR = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$CMR = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \end{bmatrix}$$

$$C^T C = \begin{bmatrix} 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 4 + 9 = 13$$

$$R R^T = \begin{bmatrix} 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 4 + 16 = 20$$

$$M = \frac{1}{13} \cdot \begin{bmatrix} 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \frac{1}{20} =$$

$$= \frac{1}{260} \begin{bmatrix} 13 & 26 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \frac{1}{260} \begin{bmatrix} 13 & 26 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \frac{1}{260} \cdot 130 = \frac{1}{2}$$

Equation (**):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

1.22 # 21

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = CR$$

$$C^TC = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 14 \end{bmatrix}$$

$$(C^TC)^{-1} = \frac{1}{3} \cdot \begin{bmatrix} 14 & -5 \\ -5 & 2 \end{bmatrix}$$

$$(RR^T)^{-1} = \frac{1}{9} \cdot \begin{bmatrix} 41 & -55 \\ -55 & 74 \end{bmatrix}$$

$$C^TAR^T = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 129 & 96 \\ 351 & 261 \end{bmatrix}$$

$$M = \frac{1}{27} \cdot \begin{bmatrix} 14 & -5 \\ -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 129 & 96 \\ 351 & 261 \end{bmatrix} \cdot \begin{bmatrix} 41 & -55 \\ -55 & 74 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

1.23 # 22

When

$$\begin{bmatrix} b \\ d \end{bmatrix} = m \cdot \begin{bmatrix} a \\ c \end{bmatrix}$$

$$A = \begin{bmatrix} a & ma \\ c & mc \end{bmatrix}$$

$$\frac{1}{ad-bc} = \frac{1}{amc-cma} = \frac{1}{0}$$

Failed! Formula can't be apiplied when dependent columns in a matrix

1.24 # 23

$$A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \\ 2 & 4 \end{bmatrix} = CR = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} = CMR = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \end{bmatrix}$$

1.25 # 24

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} = CR = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = CMR = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$