Problem Set I.3

September 2, 2021

1 Problem Set I.3

[72]:
```
import matplotlib as mpl
import matplotlib.pyplot as plt
import numpy as np
import sympy
import networkx as nx
```

1.1 # 1

$Bx = 0, AB = C \implies ABx = Cx, A(Bx) = Cx, A \cdot 0 = Cx, Cx = 0$ or $ABx = 0$

1.2 # 2

Example of $rank(A^2) < rank(A)$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$rank(A^T A) = rank(A)$

$$A^T A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$rank(AA^T) = rank(A)$

$$AA^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

1.3 # 3

$$C = \begin{bmatrix} A \\ B \end{bmatrix}$$

$$Cx = 0 \implies \begin{bmatrix} A \\ B \end{bmatrix} x = 0, \begin{bmatrix} Ax \\ Bx \end{bmatrix} = 0 \implies Ax = 0, Bx = 0$$

$N(C) = N(A) \cap N(B)$

1.4 # 4

$C(A) = C(A^T)$ and $N(A) = N(A^T)$

$Ax = A^Tx = 0$ because $Ax = 0$ and $A^Ty = 0$ and $A^T = A \implies x = y$

$$\begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} u & v \end{bmatrix}^T, \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} u^* \\ v^* \end{bmatrix}$$

I suppose than $A = A^T \implies S = S^T$ is symmetric matrix

1.5 # 5

1) $r = m = n$, $A_1x = b$ has 1 solution for every b

A_1 is any full-rank matrix

2) $r = m < n$, $A_2x = b$ has 1 or ∞ solutions

$$A_2x = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

A_2 has an extra column

3) $r = n < m, A_3x = b$ has 0 or 1 solutions

$A_3 = A_2^T$

4) $r < m, r < n; A_4x = b$ has 0 or ∞ solutions

$$A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

or dependent columns

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

And etc.

1.6 # 6

$Ax = 0, A^TAx = A^T(Ax) = A^T0 = 0$

$N(A) \subset N(A^TA)$

$A^TAx = 0$, then $x^TA^TAx = x^T0 \implies (Ax)^T(Ax) = 0 \implies ||Ax||^2 = 0$

$N(A^TA) = N(A)$

1.7 # 7

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A and A^2 have different $N(A)$

1.8 # 8

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$C(A) = N(A) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$;

But $C(A) \neq N(A^T)$ because those two subspaces are orthogonal

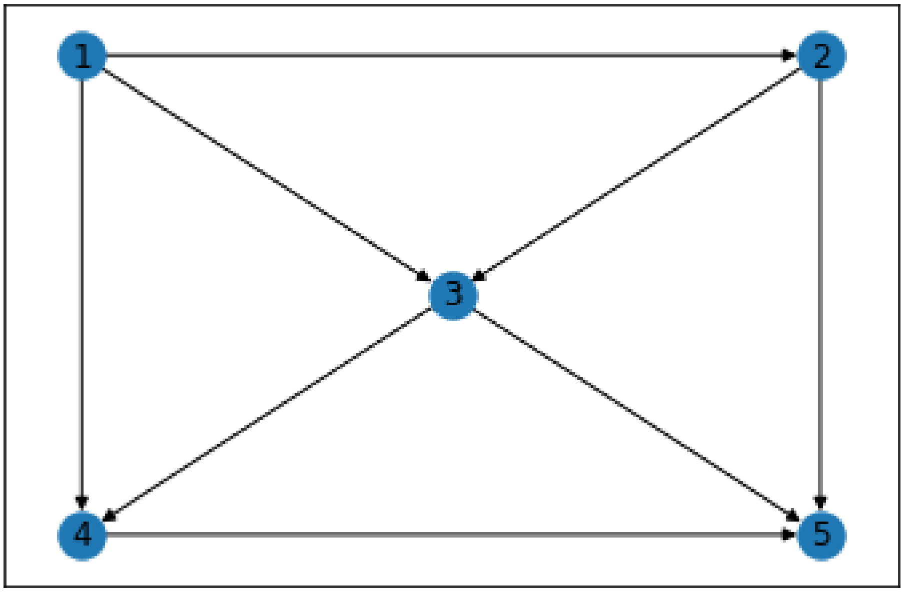
1.9 # 9

[101]:
```
g = nx.DiGraph()
g.add_nodes_from([1,2,3,4,5])
g.add_edge(1,2)
g.add_edge(1,3)
g.add_edge(1,4)
g.add_edge(2,5)
g.add_edge(2,3)
g.add_edge(3,4)
g.add_edge(3,5)
g.add_edge(4,5)

pos = { 1: (0, 20), 2: (20, 20), 3: (10, 10), 4: (0, 0), 5: (20, 0)}

nx.draw_networkx(g, pos=pos, with_labels=True)

plt.show()
```



$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

[102]:
```
A = sympy.Matrix([
    [-1,  1,  0,  0, 0],
    [-1,  0,  1,  0, 0],
    [-1,  0,  0,  1, 0],
    [ 0, -1,  1,  0, 0],
    [ 0, -1,  0,  0, 1],
    [ 0,  0, -1,  1, 0],
    [ 0,  0, -1,  0, 1],
    [ 0,  0,  0, -1, 1],
])
A
```

[102]:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

[103]:
```
A.nullspace()
```

[103]:
```
[Matrix([
 [1],
 [1],
 [1],
 [1],
 [1]])]
```

[104]:
```
x_1, x_2, x_3, x_4, x_5 = sympy.symbols("x_1, x_2, x_3, x_4, x_5")

eq1 = -x_1 + x_2
eq2 = -x_1 + x_3
eq3 = -x_1 + x_4
eq4 = -x_2 + x_3
eq5 = -x_2 + x_5
eq6 = -x_3 + x_4
eq7 = -x_3 + x_5
eq8 = -x_4 + x_5

sympy.solve([eq1, eq2, eq3, eq4, eq5, eq6, eq7, eq8], [x_1, x_2, x_3, x_4,␣
↪x_5], dict=True)
```

[104]:
```
[{x_1: x_5, x_2: x_5, x_3: x_5, x_4: x_5}]
```

$x_1 = x_2 = x_3 = x_4 = x_5$

[105]:
```
c = sympy.Symbol("c")

x = sympy.Matrix([c, c, c, c, c])

A*x
```

[105]:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$rank(A) = n - 1 = 4, N(A^T) = m - r = 8 - 4 = 4$ also means 4 loops in the graph

[114]:
```
A.T.nullspace()
```

[114]:
```
[Matrix([
 [ 1],
 [-1],
 [ 0],
 [ 1],
 [ 0],
 [ 0],
 [ 0],
 [ 0]]),
 Matrix([
 [ 0],
 [ 1],
 [-1],
 [ 0],
 [ 0],
 [ 1],
 [ 0],
 [ 0]]),
 Matrix([
 [-1],
 [ 1],
 [ 0],
 [ 0],
 [-1],
 [ 0],
 [ 1],
 [ 0]]),
 Matrix([
 [-1],
 [ 0],
 [ 1],
 [ 0],
 [-1],
 [ 0],
 [ 0],
```

```
 [ 1]])]
```

Loop 1:

[106]:
```
y_1 = sympy.Matrix([1, 0, -1, 0, 1, 0, 0, -1])

y_1
```

[106]:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

[107]:
```
A.T*y_1
```

[107]:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Loop 2:

[108]:
```
y_2 = sympy.Matrix([1, -1, 0, 0, 1, 0, -1, 0])

y_2
```

[108]:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

[109]:
```
A.T*y_2
```

[109]:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Loop 3:

[110]:
```
y_3 = sympy.Matrix([1, 0, -1, 1, 0, 1, 0, 0])

y_3
```

[110]:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

[111]:
```
A.T*y_3
```

[111]:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Loop 4:

[112]:
```
y_4 = sympy.Matrix([1, -1, 0, 1, 0, 0, 0, 0])

y_4
```

[112]:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

[113]:
```
A.T*y_4
```

[113]:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1.10 # 10

If $N(A) = x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then what are the nullspace of $B = \begin{bmatrix} A & A & A \end{bmatrix}$

$$Ax = 0; B \cdot \begin{bmatrix} x \\ x \\ x \end{bmatrix} = \begin{bmatrix} A & A & A \end{bmatrix} \cdot \begin{bmatrix} x \\ x \\ x \end{bmatrix} = 0$$

$$N(B) = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$$

1.11 # 11

$dim(S) = 2$ and $dim(T) = 7$ of R^{10}

1) $dim(S \bigcap T) \leq dim(S) \leq 2$
2) $dim(S + T) \leq dim(T) + dim(S)$

Or

$7 \leq dim(S + T) \leq 9$

3) $S^{\perp} = dim(R^{10}) - dim(S^{\perp}) = 10 - 8 = 2$