Divisibility of Central Stirling Numbers

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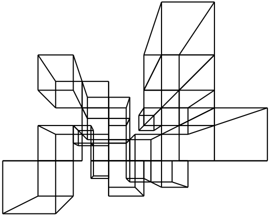

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Stirling numbers

$$\{1, 2, 3, 4\}$$

Stirling numbers

$$\{1, 2\}\ \{3, 4\}$$

Stirling numbers

$$\{1, 3\}\ \{2, 4\}$$

Stirling numbers

$$\{2\}\ \{1,3,4\}$$

Stirling Numbers of the Second Kind

Definition

A stirling number of the second kind $S(n, k)$ counts the number of ways to partition a set of n objects into k non-empty sets.

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$$\left\{ n \atop k \right\}$$

Stirling Numbers of the Second Kind

Definition

A stirling number of the second kind $S(n, k)$ counts the number of ways to partition a set of n objects into k non-empty sets.

For example $S(4, 2)$ counts the partitions of $\{1, 2, 3, 4\}$ into 2 sets

$$\{1,2\}\ \{3,4\}$$
$$\{1,3\}\ \{2,4\}$$
$$\{1,4\}\ \{2,3\}$$
$$\{1\}\ \{2,3,4\}$$
$$\{2\}\ \{1,3,4\}$$
$$\{3\}\ \{1,2,4\}$$
$$\{4\}\ \{1,2,3\}$$

so $S(4, 2) = 7$

Stirling Numbers of the Second Kind

They enjoy the following recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

and can be explicitly calculated with the following alternating sum

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n$$

What are the Central Stirling Numbers?

The central stirling numbers are numbers of the form $S(2n, n)$

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$\underline{n}$													
0							$\left\{ {0 \atop 0} \right\}$						
1						$\left\{ {1 \atop 0} \right\}$		$\left\{ {1 \atop 1} \right\}$					
2					$\left\{ {2 \atop 0} \right\}$		$\left\{ {2 \atop 1} \right\}$		$\left\{ {2 \atop 2} \right\}$				
3				$\left\{ {3 \atop 0} \right\}$		$\left\{ {3 \atop 1} \right\}$		$\left\{ {3 \atop 2} \right\}$		$\left\{ {3 \atop 3} \right\}$			
4			$\left\{ {4 \atop 0} \right\}$		$\left\{ {4 \atop 1} \right\}$		$\left\{ {4 \atop 2} \right\}$		$\left\{ {4 \atop 3} \right\}$		$\left\{ {4 \atop 4} \right\}$		
5		$\left\{ {5 \atop 0} \right\}$		$\left\{ {5 \atop 1} \right\}$		$\left\{ {5 \atop 2} \right\}$		$\left\{ {5 \atop 3} \right\}$		$\left\{ {5 \atop 4} \right\}$		$\left\{ {5 \atop 5} \right\}$	
6	$\left\{ {6 \atop 0} \right\}$		$\left\{ {6 \atop 1} \right\}$		$\left\{ {6 \atop 2} \right\}$		$\left\{ {6 \atop 3} \right\}$		$\left\{ {6 \atop 4} \right\}$		$\left\{ {6 \atop 5} \right\}$		$\left\{ {6 \atop 6} \right\}$

What are the Central Stirling Numbers?

The central stirling numbers are numbers of the form $S(2n, n)$

n													
0							1						
1						0		1					
2					0		1		1				
3				0		1		3		1			
4			0		1		7		6		1		
5		0		1		15		25		10		1	
6	0		1		31		90		65		15		1

Odd Central Stirling Numbers

```
In[1]=  Table[
            If[OddQ@StirlingS2[2n, n], n, Nothing],
            {n, 1, 50}
        ]

Out[1]:= {1, 2, 4, 5, 8, 9, 10, 16, 17, 18, 20,
         21, 32, 33, 34, 36, 37, 40, 41, 42}
```

Odd Central Stirling Numbers

```
In[2]:= IntegerString[%, 2]

Out[2]= {1, 10, 100, 101, 1000, 1001, 1010,
10000, 10001, 10010, 10100, 10101,
100000, 100001, 100010, 100100,
100101, 101000, 101001, 101010}
```

Odd Central Stirling Numbers

Definition (Fibbinary Number)

A number whose binary representation does not contain two consecutive ones

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A number whose binary representation does not contain two consecutive ones

- 😄 10000101010
- 🤠 10010101001
- 😊 10100000010
- 😥 10001101010

Odd Central Stirling Numbers

```
In[2]:= IntegerString[%, 2]

Out[2]= {1, 10, 100, 101, 1000, 1001, 1010,
10000, 10001, 10010, 10100, 10101,
100000, 100001, 100010, 100100,
100101, 101000, 101001, 101010}
```

Odd Central Stirling Numbers

Theorem (Chan & Manna, 2010)

$S(2n, n)$ is odd if and only if n is a Fibbinary number.

Central Stirling Divisibility

```
In[3]:= Table[StirlingS2[2 n, n]/n, {n, 1, 10}]

Out[3]= {1, 7/2, 30, 1701/4, 8505, 661826/3,
7047040, 2141764053/8, 11797266195,
1183516992931/2}
```

Central Stirling Divisibility

```
In[4]:= Table[
            Mod[StirlingS2[2 n, n], n],
            {n, 1, 20}
        ]

Out[4]= {0, 1, 0, 1, 0, 4, 0, 5, 0, 5,
        0, 0, 0, 0, 0, 5, 0, 13, 0, 1}
```

Central Stirling Divisibility

```
In[5]:= Table[
            Mod[StirlingS2[2 n, n], n],
            {n, 1, 1000, 2}
        ]

Out[5]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
        0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
        0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
        0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
        0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
        0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
        0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, ... , 0}
```

Divisibility Properties

Conjecture

if n is odd, n divides $S(2n, n)$

Grouping Set Partitions

We can categorize every partition by its shape!

Grouping Set Partitions

In our previous example, $S(4,2)$, groups $\{1,2,3,4\}$ into 2 sets

$\{1,2\}\ \{3,4\}$	$\{1\}\ \{2,3,4\}$
$\{1,3\}\ \{2,4\}$	$\{2\}\ \{1,3,4\}$
$\{1,4\}\ \{2,3\}$	$\{3\}\ \{1,2,4\}$
	$\{4\}\ \{1,2,3\}$

Grouping Set Partitions

In our previous example, $S(4,2)$, groups $\{1,2,3,4\}$ into 2 sets

$$\begin{array}{ll} \{1,2\}\ \{3,4\} & \{1\}\ \{2,3,4\} \\ \{1,3\}\ \{2,4\} & \{2\}\ \{1,3,4\} \\ \{1,4\}\ \{2,3\} & \{3\}\ \{1,2,4\} \\ & \{4\}\ \{1,2,3\} \end{array}$$

The first partitions share the shape $\{\_,\_\}\ \{\_,\_\}$ while the last share $\{\_\}\ \{\_,\_,\_\}$.

Grouping Set Partitions

In our previous example, $S(4,2)$, groups $\{1,2,3,4\}$ into 2 sets

$$\underbrace{\begin{matrix}\{1,2\}\ \{3,4\}\\ \{1,3\}\ \{2,4\}\\ \{1,4\}\ \{2,3\}\\ \phantom{\{4\}}\end{matrix}}_{2:2} \qquad \underbrace{\begin{matrix}\{1\}\ \{2,3,4\}\\ \{2\}\ \{1,3,4\}\\ \{3\}\ \{1,2,4\}\\ \{4\}\ \{1,2,3\}\end{matrix}}_{1:3}$$

We will label these shapes by the number of elements, so $2:2$ and $1:3$ respectively.

Shape as an Integer Partition

For a central stirling number, its shape is an integer partition of $2n$ of length n.

Previous example:

$$2 + 2 = 4$$

$$\underbrace{1 + 3}_{\text{2 elements}} = 4$$

Shape as an Integer Partition

For a central stirling number, its shape is an integer partition of $2n$ of length n.

$$1+1+1+1+6=10$$

$$1+1+1+2+5=10$$

$$1+1+1+3+4=10$$

$$1+1+2+2+4=10$$

$$1+1+2+3+3=10$$

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$$1+1+2+3+3=10$$

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$$2+2+2+2+2=10$$

if $\lambda = (1:2:2:2:3)$, then $\lambda \vdash 10$
note that $\text{len}(\lambda) = 5$

Divisibility of Groups

[1]:
```
sn2_central_group_counts(5)
```

```
210 partitions in 1:1:1:1:6
2520 partitions in 1:1:1:2:5
4200 partitions in 1:1:1:3:4
9450 partitions in 1:1:2:2:4
12600 partitions in 1:1:2:3:3
12600 partitions in 1:2:2:2:3
945 partitions in 2:2:2:2:2
```

Divisibility of Groups

[1]:
```
sn2_central_group_counts(5)
```

```
210 partitions in 1:1:1:1:6     mod 5 = 0
2520 partitions in 1:1:1:2:5    mod 5 = 0
4200 partitions in 1:1:1:3:4    mod 5 = 0
9450 partitions in 1:1:2:2:4    mod 5 = 0
12600 partitions in 1:1:2:3:3   mod 5 = 0
12600 partitions in 1:2:2:2:3   mod 5 = 0
945 partitions in 2:2:2:2:2     mod 5 = 0
```

Divisibility of Groups

```
[2]: sn2_central_group_counts(4)
```

```
56 partitions in 1:1:1:5     mod 4 = 0
420 partitions in 1:1:2:4    mod 4 = 0
280 partitions in 1:1:3:3    mod 4 = 0
840 partitions in 1:2:2:3    mod 4 = 0
105 partitions in 2:2:2:2    mod 4 = 1
```

Counting Groups

Denote the # of partitions of $S(2n, n)$ w.r.t a shape λ as $S_\lambda(2n, n)$

Counting Groups

Denote the # of partitions of $S(2n, n)$ w.r.t a shape λ as $S_\lambda(2n, n)$

How can we count $S_\lambda(2n, n)$?

Counting Groups

Denote the # of partitions of $S(2n, n)$ w.r.t a shape λ as $S_\lambda(2n, n)$

The number of partitions in a grouping can be counted combinatorially with multinomial coefficients

Counting Groups

Denote the # of partitions of $S(2n,n)$ w.r.t a shape λ as $S_\lambda(2n,n)$

The number of partitions in a grouping can be counted combinatorially with multinomial coefficients

$$\binom{n}{\lambda_1,...,\lambda_n} = \frac{n!}{\lambda_1! \cdot ... \cdot \lambda_n!}$$

Counting Groups

For example, consider the $\lambda = 1:2:2:2:3$ group when $n = 5$.

$$S_\lambda(10,5) = \binom{10}{1,2,2,2,3} = \frac{10!}{1!\cdot 2!\cdot 2!\cdot 2!\cdot 3!}$$

Counting Groups

For example, consider the $\lambda = 1 : 2 : 2 : 2 : 3$ group when $n = 5$.

$$S_\lambda(10, 5) = \binom{10}{1, 2, 2, 2, 3} = \frac{10!}{1! \cdot 2! \cdot 2! \cdot 2! \cdot 3!}$$

... almost

Counting Groups

For example, consider the $\lambda = 1:2:2:2:3$ group when $n = 5$.

$$S_\lambda(10,5) = \binom{10}{1,2,2,2,3} = \frac{10!}{1!\cdot 2!\cdot 2!\cdot 2!\cdot 3!}$$

We need to account for the fact that there are 3 sets of length 2, so we divide by $3!$

Counting Groups

For example, consider the $\lambda = 1:2:2:2:3$ group when $n = 5$.

$$S_\lambda(10,5) = \binom{10}{1,2,2,2,3} \cdot \frac{1}{3!} = \frac{10!}{1!\cdot 2!\cdot 2!\cdot 2!\cdot 3!} \cdot \frac{1}{3!} = 12600$$

Counting Groups

For example, consider the $\lambda = 1:2:2:2:3$ group when $n = 5$.

$$S_\lambda(10,5) = \binom{10}{1,2,2,2,3} \cdot \frac{1}{3!} = \frac{10!}{1! \cdot 2! \cdot 2! \cdot 2! \cdot 3!} \cdot \frac{1}{3!} = 12600$$

```
12600 partitions in 1:2:2:2:3
```

A Sum Over Groups of Shapes

$$S(2n, n) = \sum_{\substack{\lambda \vdash 2n \\ \mathsf{len}(\lambda) = n}} S_\lambda(2n, n)$$

A Sum Over Groups of Shapes

$$S(2n,n) = \sum_{\substack{\lambda \vdash 2n \\ \mathsf{len}(\lambda)=n}} \binom{2n}{\lambda_1, \lambda_2, ..., \lambda_n} \prod_{\mathsf{unique}\ \lambda_i} \frac{1}{\mathsf{count}(\lambda_i)!}$$

Questions?!