1. Abstract groups

Throughout this section G is a group.

1.1. **Simple groups.** The group G is called *simple* if the only normal subgroups of G are $\{1\}$ and G.

> **(E1)** *Prove that if G is a finite simple abelian group, then $G \cong C_p$, the cyclic subgroup of order p, where p is a prime.*

1.2. **Composition series and abelian series.** Let $H \leq G$. A *series from H to G* is a finite sequence $(G_i)_{0 \leq i \leq k}$ of subgroups of G, such that

$$H = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_k = G. \tag{1}$$

If H is unspecified, then you should assume that $H = \{1\}$. The sequence $(G_i)_{0 \leq i \leq k}$ is called

- a *composition series* if, for $i = 1, \dots, k$, G_k/G_{k-1} is non-trivial and simple. The abstract group G_k/G_{k-1} is called a *composition factor* of G.
- an *abelian series* if for $i = 1, \dots, k$, G_k/G_{k-1} is abelian.
- a *normal series* if, for $i = 0, \dots, k$, $G_i \trianglelefteq G$.

Suppose that we have two series from H to G, the first given by (1), the second by:

$$H = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_l = G. \tag{2}$$

Series (1) and (2) are called *equivalent* if $k = l$ and there exists a permutation $\pi \in S_k$ such that, for $i = 1, \dots, k$,

$$G_i/G_{i-1} \cong H_{i\pi}/H_{i\pi-1}.$$

The series (2) is said to be a *refinement* of series (1) if $k \leq l$ and there are non-negative integers $j_0 < j_1 < \cdots < j_k \leq l$ such that $G_i = H_{j_i}$ for $i = 0, \dots, k$.

Now the key result concerning series is due to Schreier [Ros94, 7.7]:

Lemma 1. *Any two series have equivalent refinements.*

> **(E2)** *Prove this.*

One important consequence of Lemma 1 is that if G is a group admitting a composition series, then the multiset of composition factors associated with any composition series of G is an invariant of the group G. In §2.4 we will briefly examine how, given M a finite multiset of simple groups, one might construct a group G for which M is the multiset of composition factors.

1.3. **Derived series.** For $g, h \in G$, define the *commutator* of g and h,

$$[g,h] := g^{-1}h^{-1}gh.$$

The *commutator subgroup*, or *derived subgroup* of G, written G' or $[G,G]$ or $G^{(1)}$, is the group

$$\langle [g,h] \mid g,h \in G \rangle.$$

Warning. G' is the group *generated* by all commutators of the group G, i.e. the smallest subgroup of G that contains all commutators. The set of all commutators in G is not necessarily a group.

> **(E3*)** *Prove that, for N a normal subgroup of G, the quotient G/N is abelian if and only if $G' \leq N$.*
>
> **(E4)** *Find an example of a group G such that G' is not equal to the set of all commutators.*

We can generalize this construction as follows:

$$G^{(0)} := G;$$
$$G^{(n)} := [G^{(n-1)}, G^{(n-1)}] \text{ for } n \in \mathbb{N}.$$

We obtain a descending sequence of groups

$$\cdots \trianglelefteq G^{(2)} \trianglelefteq G^{(1)} \trianglelefteq G$$

which is called the *derived series* of G. If, for some k, $G^{(k)} = G^{(k+1)}$ then, clearly, $G^{(k)} = G^{(l)}$ for every $l \geq k$ and we say that the derived series *terminates* at $G^{(k)}$. Note that if the derived series does not terminate for any k then it is not strictly speaking a series. (Of course the derived series of a finite group always terminates.)

(E5) *Prove that (provided it terminates) the derived series is a normal series.*

We call G *perfect* if $G = [G, G]$. If G is finite, then the derived series terminates after k steps at a perfect group.

1.4. **Solvable groups.** We say that G is *soluble* or *solvable* if G has an abelian series.

(E6) *Prove that, if G is finite, then G is solvable if and only if all composition factors of G are cyclic of prime order. Give an example of a solvable group that does not have a composition series.*

(E7*) *Prove that a finite group G is solvable if and only if the derived series of G terminates at $\{1\}$.*

2. Permutation groups

n groups

Throughout this section, assume that G is a group that acts (on the right) on some set Ω. Equivalently, there exists a group homomorphism $\phi : G \to \mathrm{Sym}(\Omega)$, the set of permutations on the set Ω. Recall that

- for $\omega \in \Omega$, $G_\omega := \{g \in g \mid \omega^g = \omega\}$, is the *stabilizer* of ω;
- $G_{(\Omega)} := \bigcap\limits_{\omega\in\Omega} G_\omega$ is the *kernel* of the action;
- for $\omega \in \Omega$, $\omega^G := \{\omega^g \mid g \in G\}$ is the *orbit* of ω.

Note that $G_{(\Omega)}$ is precisely the kernel of ϕ.

We say that the action of G on Ω is

- *faithful*, if $G_{\Omega)} = \{1\}$; equivalently, ϕ is a monomorphism and we think of G as a subgroup of $\mathrm{Sym}(\Omega)$;
- *transitive*, if $\omega^G = \Omega$ for some (and hence all) $\omega \in \Omega$.

Remark. When a group theorist speaks of a 'permutation group', they mean an abstract group G accompanied by some fixed embedding of G in $\mathrm{Sym}(\Omega)$, for some set Ω. Equivalently, they mean an abstract group G accompanied by some faithful action. Indeed for a long time this was the only context in which groups were studied, in the immediate aftermath of the work of Galois.

Example 1. Let H be any subgroup of G. The group G acts transitively on $H\backslash G$, the set of right cosets of H via right multiplication.

o action

(E8*) *Prove that any transitive action is isomorphic to an action of this kind, i.e. given a transitive action of G on Ω, there exists a subgroup $H \leq G$ such that the action of G on Ω is isomorphic to the action of G on $H\backslash G$. You may need to recall what it means for two group actions to be isomorphic.*

Recall that when G is finite the Orbit-Stabilizer Theorem asserts that, for all $\omega \in \Omega$,

$$|G| = |G_\omega| \cdot |\omega^G|.$$

(E9) *Use (E8) to prove the orbit-stabilizer theorem.*

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(E10) *Prove that if G acts transitively on Ω and G_ω is a stabilizer, then the set of all stabilizers equals the set of all conjugates of G_ω. Under what conditions is the action of G by conjugation on this set of conjugates is isomorphic to the action of G on Ω?*

(E11) *What conditions on H result in the action of G on $H\backslash G$ being faithful?*

(E12*) *Let G be a finite group acting transitively on a set Ω. Show that the average number of fixed points of the elements of G is 1, i.e.*

$$\frac{1}{|G|}\sum_{g\in G}|\{\omega \in \Omega \mid \omega^g = \omega\}| = 1.$$

Example 2. Let $3 \leq n \in \mathbb{Z}^+$ and let $G := D_{2n}$, the dihedral group of order $2n$. In other words

$$G := \langle g, h \mid g^n = h^2 = 1, h^{-1}gh = g^{-1}\rangle.$$

Define Ω to be the corners of an n-gon which we might as well label $1, \dots, n$. We can define g to act like the permutation $(1, 2, \dots, n)$ and h to reflect the polygon through a line passing through 1; see Figure 1 for an example when $n = 5$. Thus

$$h := (2, n-1)(3, n-2)\dots\left(\lfloor\frac{n+2}{2}\rfloor, \lceil\frac{n+2}{2}\rceil\right).$$