Regular maps and the Euler characteristic

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April 15, 2013

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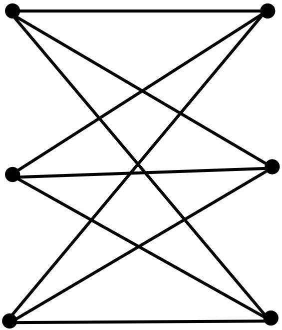
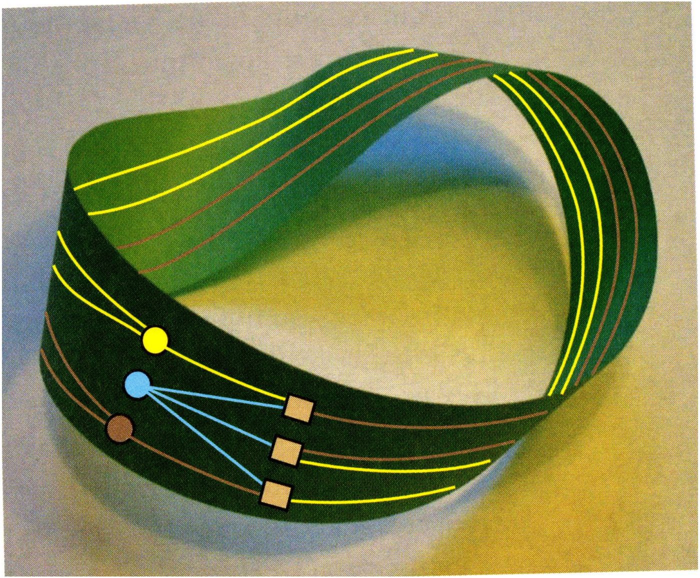
A **map** is a 'nice' embedding of $\mathcal{G}$ in $\mathcal{S}$.

This isn't nice...



... but this is.



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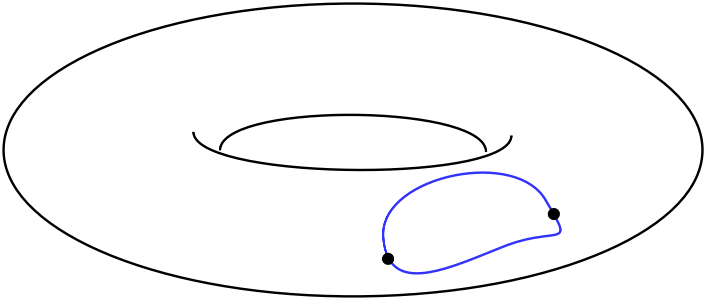

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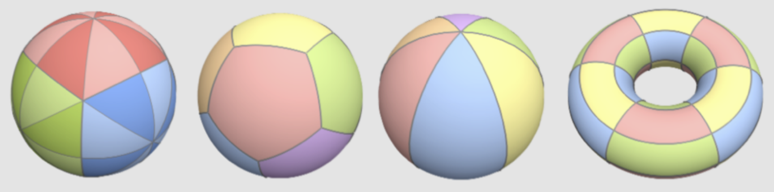

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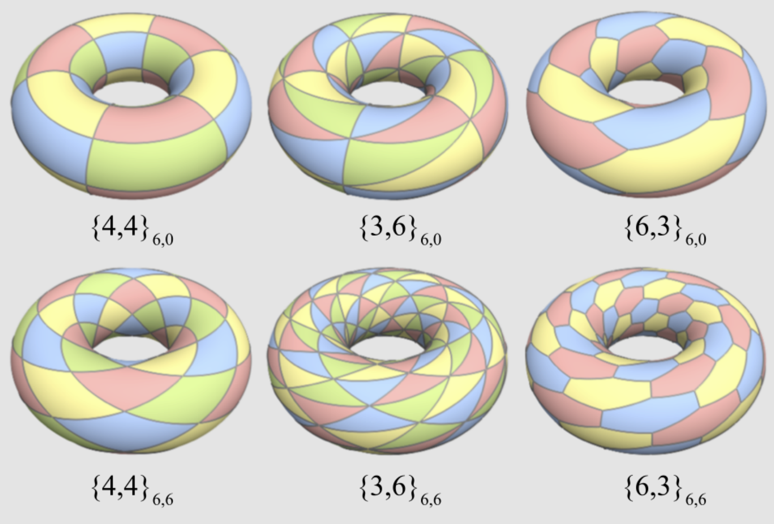

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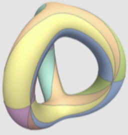
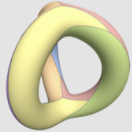
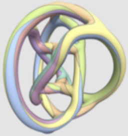


R2.1 $\{3, 8\}$ 16 triangles $\to H_3$

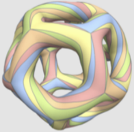


R2.1′ $\{3, 8\}$ 6 octagons $\to H_3$

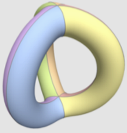
R4.3′ $\{6, 4\}$ 12 hexagons $\to \{6, 3\}_{1,1}$



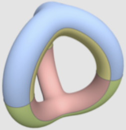
R9.3′ $\{6, 4\}$ 32 hexagons $\to$ R2.1′ $\to H_3$



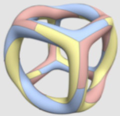
R11.1 $\{4, 6\}$ 60 quads $\to D$



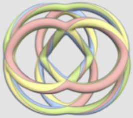
R2.2 $\{4, 6\}$ 6 quads $\to H_3$



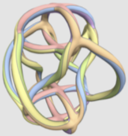
R2.2′ $\{6, 4\}$ 4 hexagons $\to H_3$



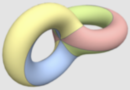
R5.1′ $\{8, 3\}$ 24 octagons $\to C$



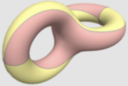
R9.4′ $\{6, 4\}$ 32 hexagons $\to \{4, 4\}_{2,2}$



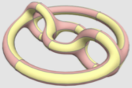
R13.2′ $\{12, 3\}$ 24 faces $\to$ R3.4′ $\to H_4$



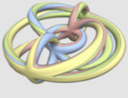
R2.3 $\{4, 8\}$ 4 quads $\to \{4, 4\}_{1,0}$



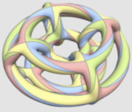
R2.3′ $\{8, 4\}$ 2 octagons $\to \{4, 4\}_{1,0}$



R5.3′ $\{5, 4\}$ 32 pentagons $\to \{4, 4\}_{2,0}$

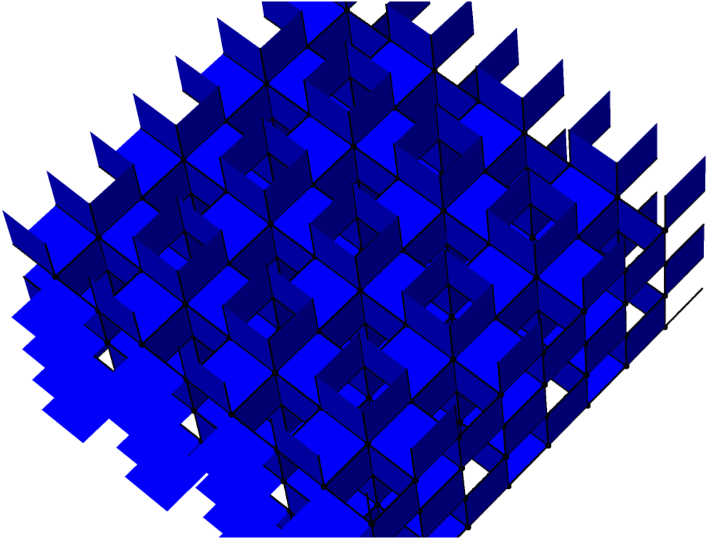


R9.9′ $\{12, 4\}$ 8 faces $\to \{3, 6\}_{2,0}$



R17.3′ $\{6, 4\}$ 64 hexagons $\to \{4, 4\}_{4,0}$

And this one is especially groovy...



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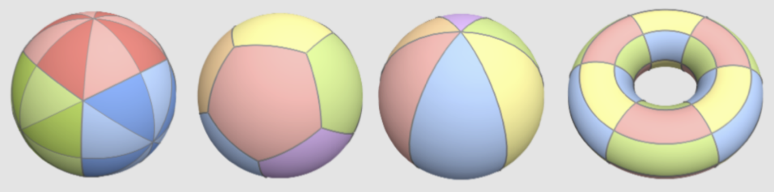
- **Fact:** $\mathrm{Aut}(\mathcal{M})$ acts faithfully and semiregularly on the set of flags.

And a flag is...



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- We would like to classify the regular maps.
- Encouraging fact: For any $g \geq 2$, there are only a finite number of regular maps on a surface of genus g.

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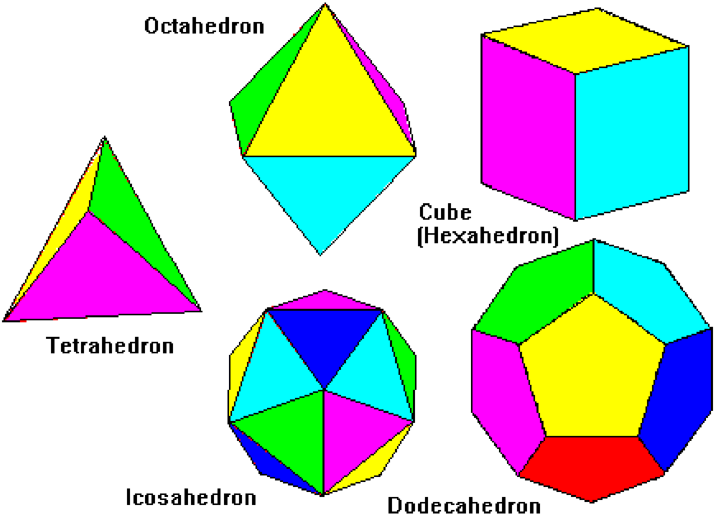

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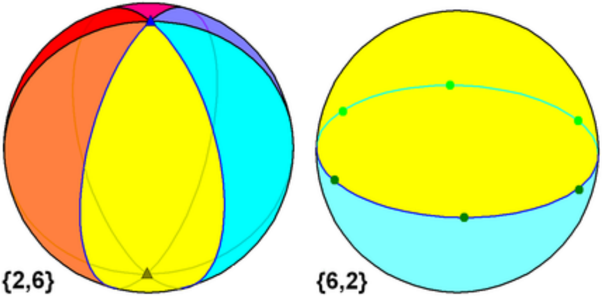

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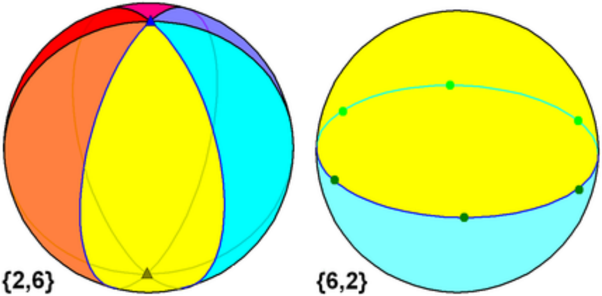

$\mathrm{Aut}(\mathcal{M})$ is solvable except when $\mathcal{M}$ is the dodecahedron or icosahedron, in which case $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Alt}(5)$.

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- If $\mathcal{M}$ is regular with $G = \mathrm{Aut}(\mathcal{M})$, then

$$\chi = V - E + F = |G| \left(\frac{1}{|G_v|} - \frac{1}{|G_e|} + \frac{1}{|G_f|} \right).$$

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χ is complicated $\longleftrightarrow$ The prime factorization of χ has many primes and/or high exponents.

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Theorem (Conder, G., Short, Širáň, 2013)

If T is a non-abelian composition factor of G and y is the maximum exponent in the prime factorization of χ, then T is a simple group of Lie type over a field of order p^a where $a \leq y + 2$.

A word about the proof of Theorem 1

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- Writing m and n for the order of these two cyclic groups we obtain

$$\chi = -|G| \frac{mn - 2m - 2n}{4mn} = -\frac{|G|}{4[m,n]} \left(\frac{mn - 2m - 2n}{(m,n)} \right).$$

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- For large rank groups of Lie type this is impossible. One can establish this formally by studying the prime graph of these groups.

Thanks for coming!