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QUANTIZATION OF LINEAR MAPS ON A TORUS – FRESNEL DIFFRACTION BY A PERIODIC GRATING

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Quantization on a phase space q, p in the form of a torus (or periodized plane) with dimensions Δq, Δp requires that Planck's constant take one of the values $h = \Delta q \Delta p/N$, where N is an integer. Corresponding to a linear classical map T of points q, p is a unitary operator U mapping quantum states that are periodic in q and p; the construction of U involves techniques from number theory. U has eigenvalues $\exp(i\alpha)$. The 'eigenangles' α must be multiples of $2\pi/n(N)$, where $n(N)$ is the lowest common multiple of the lengths of the classical 'cycles' mapped under T by those rational points in q, p which are multiples of $\Delta q/N$ and $\Delta p/N$ (i.e. $n(N)$ is the 'period of T mod N'), at least for odd N. If T is hyperbolic, n is a very erratic function of N, and the classical limit $N \to \infty$ is very different from the 'Bohr–Sommerfeld' behaviour for parabolic maps. The degeneracy structure of the eigenangle spectrum is related to the distribution of cycle lengths. Computation of the quantal Wigner function shows that eigenstates of U do not correspond to individual cycles.

1. Introduction

We are to study a one-dimensional model system whose classical mechanics and quantum mechanics can be solved exactly. The time development of the system can be considered to be governed by a Hamiltonian which is a quadratic form in position and momentum coordinates – a harmonic oscillator with magnetic field. It is to be examined not continuously through time but only 'stroboscopically' at definite equally spaced times. The coefficients in the quadratic form may vary in time provided they do so periodically with this period.

The classical mechanics of this process is described by a repeated canonical (i.e. area preserving) mapping of the phase plane (q, p), and because the Hamiltonian is quadratic the mapping is a linear one. The corresponding quantum mechanics is described by a unitary transformation and one of the many privileges that linear classical processes enjoy is that the associated unitary transformation depends (up to overall phase) only on the resulting linear map and not on the details of the evolution by which it was achieved. It is therefore meaningful to speak of the quantization of a linear map without qualification where it would not be to do so for a general canonical map.

The 'quantization' consists in finding and studying the properties of the unitary transformation, in particular the eigenvalue structure. To do this for a linear map on the *full phase plane* is a simple exercise since it reduces to finding the effective time-independent quadratic Hamiltonian which would achieve the mapping and quantizing that in the standard way. Our problem contains an essential modification. We are to quantize linear maps on a 'torus'. Our phase plane is constrained to admit *only periodic density distributions* with a definite rectangular unit cell – the torus. The maps considered are members of a discrete but infinite set which preserve such periodicity.

We justify the imposition of this apparently rather artificial periodicity constraint on two grounds. Firstly, it enables a new aspect of the relations between classical and quantum mechanics to be studied. The geometry or 'classical mechanics' of linear area preserving maps of the torus has been extensively studied, and it is known that through this very simple prescription of 'dynamics' the points of a torus can be made to exhibit either purely regular or purely irregular motion under successive mappings according to the class of map used. For an 'elliptic' map (one which corresponds to a *stable* oscillator Hamiltonian) all points lie on closed orbits returning to their starting points after at most four mappings. For a 'hyperbolic' map (which corresponds to an unstable oscillator Hamiltonian) on the other hand, a general point exhibits highly irregular behaviour covering the torus ergodically in a motion which has thoroughly stochastic properties (an example is the map commonly known as 'Arnol'd's cat' [1]). It is of interest to discover how this distinction may manifest itself under quantization, particularly since it has not yet proved possible to quantize, analytically, any other system exhibiting irregular classical motion. Our main result is that for irregular maps the quantum-mechanical behaviour is erratic as the classical limit $\hbar \to 0$ is approached, in marked contrast to the smooth 'Bohr–Sommerfeld' or 'WKB' behaviour of regular systems.

Secondly the problem, although artificial from a mechanical point of view, has a direct physical realization in optics in terms of Fresnel diffraction by a periodic diffraction grating. That this is so will become clear in the formal development but we can give a brief account of the analogue system as follows. If an ideal diffraction grating (infinite, with infinitesimal slit widths) is illuminated normally with a monochromatic plane wave, a wavefunction of equally spaced delta functions – a δ comb' – is transmitted. In propagation beyond the grating this wavefunction spreads out (instantly, in principle) to produce a wavefield of uniform intensity with the structure concealed in the phase. However, at definite equal intervals (proportional to the Fresnel distance) an identical image δ comb wavefunction is reproduced. If now the initial delta functions do not all have the same strength but have a periodic (possibly complex) sequence of amplitudes, such as might be produced by a periodically modulated grating then once again image δ combs are produced at the same distances. These images are not now all identical to the original however – only after a certain number of distances is the original reproduced and the sequence repeated – but they all have the same *period* of N spacings, say, as the original. Each set or vector of N amplitudes is obtained from the last by application of an $N \times N$ unitary matrix characteristic of the system. Our problem is to derive and investigate the properties of this propagation matrix for the circumstances just described and for the more general case which corresponds to the insertion of standard ideal lenses into this analogue system. The quantization of linear maps on a torus is thus not as abstract as it sounds.

Some previous studies share features in common with the present one. Schwinger [2, 3] develops the operator formalism appropriate to quantization on a torus, which is then allowed to become infinitely large, as a rigorous approach to quantization in general. No dynamics are applied to the finite torus, however. Rannou [4] on the other hand has investigated some non-linear maps on a torus in a discretized form (such as we will encounter in section 4) from a purely classical standpoint. More recently the semi-classical mechanics ($\hbar \to 0$) of maps on the whole phase plane has been studied by Berry and Balazs [5] and Berry et al., [6] and interesting computations for a quantum map on the cylinder have been made by Casati et al. [7].

Our program in this paper will be firstly in section 2 to derive and discuss the unitary transformation appropriate to quantization of linear maps on a torus. The mathematics involved will be largely that of the integers and many number-theoretical features emerge. One of the most important features to emerge is that the unitary transformation is periodic (i.e. some power of it is the identity) and in section 3 we consider the number-theoretical properties of the period (the power) and its implications for the eigenvalues of the transformation. Finally in section 4 we examine the relationship between the quantum and classical maps through the Wigner function. Much of the analysis requires rather detailed argument and so in section 5 we conclude with a full summary of the paper unencumbered with mathematical details.

2. The unitary propagation matrix

Before the quantization program for linear maps on a torus can begin, two preliminary conditions must be established. The first is of an entirely classical nature and concerns the class of linear maps that are allowed. A linear canonical map of the phase plane (q, p) is described by a 2×2 matrix T:

$$\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} q_1 \\ p_1 \end{pmatrix}, \tag{1}$$

where Liouville's theorem requires that the map be area preserving so that

$$\det T = T_{11}T_{22} - T_{12}T_{21} = 1. \tag{2}$$

A general linear canonical map applied to a periodic (density distribution) pattern with a rectangular unit cell $\Delta q \times \Delta p$ will not yield a pattern which is still periodic with this same unit cell. It will do so if and only if lattice points $(m\Delta q, n\Delta p)$ map to lattice points. In units of q and p in which Δq and Δp are unity, which we will adopt shortly, the required condition on T is simply that it has integer elements.

The second condition is of an entirely quantum nature. The imposition of torus periodicity alone, even before the prescription of the dynamics (i.e. the mapping), greatly restricts the admissable class of quantum wavefunctions. A wavefunction which is periodic in both its position and momentum representations is necessarily a 'comb' of equally spaced delta functions with periodic (complex) amplitudes in both representations. (There is no violation of the uncertainty principle in this, of course, since both combs extend to infinity.) If in the position representation the δ comb has period Δq and spacing $\Delta q/N$ then in the momentum representation it has period $\Delta p = Nh/\Delta q$ and spacing $h/\Delta q$, so from this we see that the period is the same number of spacings, N, in each representation and also that

$$\Delta q \Delta p = Nh. \tag{3}$$

This is our second preliminary condition. It says that the area of the torus must be an integral multiple of h. If we take $\Delta q = \Delta p = 1$ then the only allowed values of h will be the reciprocal integers $1/N$ so that the classical limit $h\to 0$ will correspond to $N\to\infty$ and the (trivial) quantum extreme has $N = 1$.

Now we are ready to seek the unitary transformation appropriate to the map T and the quantum parameter N. We shall choose to work in the position representation for its familiarity although we shall require the fact that the momentum representation can be obtained by a mere re-naming of the variables. Because the wavefunctions are discrete and described by N complex amplitudes it is to be expected that our unitary transformation can be expressed as an $N\times N$ *unitary matrix* which acts on an N-component 'vector' of old amplitudes to give a new set,

$$\psi_{Q_2} = \sum_{Q_1=1}^{N} U_{Q_2Q_1}\psi_{Q_1}. \tag{4}$$

We start, however, with the continuous version of the transformation appropriate to the full phase plane. For a linear canonical map the unitary quantum propagator (Green function) is precisely its semiclassical approximation. In terms of the action $S(q_2, q_1)$ along the unique classical path from q_1 to q_2,

$$U(q_2, q_1) = (-1)^r \left(\frac{\mathrm{i}\partial^2 S}{\hbar \partial q_1 \partial q_2}\right)^{1/2} \exp\left[\frac{\mathrm{i}S}{\hbar}(q_2, q_1)\right], \tag{5}$$

where the action is a *quadratic* function of q_1, q_2 plus a trivial constant S_0:

$$S(q_2, q_1) = \tfrac{1}{2}(S_{11}q_1^2 + 2S_{12}q_1q_2 + S_{22}q_2^2) + S_0; \tag{6}$$

The number r is the signature of the map. A given linear classical map has a quantum propagator unique up to its sign. This depends on the history by which the map was obtained. Specifically, if the phase of the square-root in (5) is chosen to satisfy $-\pi/2 < \arg\sqrt{\;} \leqslant +\pi/2$, then r is the number of complete revolutions made by the initial vector $(0, 1)$ in the phase plane. Because the sign is multiplicative under composition of mappings it may be absorbed into the uninteresting constant S_0 which we henceforth omit.

To express the coefficients S_{11} etc. in terms of the elements of the map matrix T_{11} etc. we construct the momenta $p_1 = -\partial S/\partial q_1$, $p_2 = \partial S/\partial q_2$. This gives

$$\begin{pmatrix} -p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{7}$$

and by comparison with the matrix expression (1) we obtain with a little algebra

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix} = \frac{1}{T_{12}} \begin{pmatrix} T_{11} & -1 \\ -1 & T_{22} \end{pmatrix}. \tag{8}$$

With the stated units of q (unit torus) we therefore have

$$U(q_2, q_1) = \left(\frac{\mathrm{i}N}{T_{12}}\right)^{1/2} \exp\left[\frac{\mathrm{i}\pi N}{T_{12}}(T_{11}q_1^2 - 2q_1q_2 + T_{22}q_2^2)\right]. \tag{9}$$

Optically this is the field through the optical system at q_2 due to a point source at q_1.

We are interested in propagation from a source at $q_1 = Q_1/N$, where Q_1 is an integer, together with its equivalent sources at positions

$$q_1 = Q_1/N + m \quad \text{(where } m \text{ is an integer).} \tag{10}$$

Summing the propagator (5) over all such points (fig. 1) (i.e. over all m), yields a function in q_2 which is a periodic δ comb with delta functions at

$$q_2 = Q_2/N + \text{integer}, \tag{11}$$

where Q_2 will later be required to be an integer. The coefficients of the delta functions are the

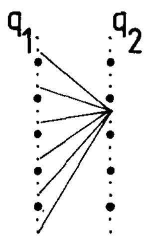

Fig. 1. The propagator is a sum over equivalent source points.

elements of the $N \times N$ unitary quantum propagation matrix U_{Q_2,Q_1}. With all the letters representing integers we have

$$U_{Q_2,Q_1} = \left(\frac{\mathrm{i}T_{12}}{N}\right)^{1/2} \left\langle \exp\left[\frac{\mathrm{i}\pi}{NT_{12}} \{T_{11}(Q_1 + mN)^2 - 2(Q_1 + mN)Q_2 + T_{22}Q_2^2\}\right]\right\rangle_m . \tag{12}$$

Here and henceforth angle brackets $\langle\ \rangle_m$ denote the *average* of their contents over all integer values of m. A sum over m would have given infinity (the height of the delta function) but the average can be shown to extract the finite coefficients in the comb provided the indicated change in the square root prefactor is made. The exponential is an oscillatory Gaussian in m, indeed Gauss himself evaluated such averages and they are discussed in books on number theory [8, 9, 15]. The essential difficulty lies in deciding the sign of the average.

We digress briefly to quote the necessary results; a derivation of them, apart from the sign, by repeated use of the Poisson transformation is given in appendix A. Let a and b be mutually prime positive integers. We need to evaluate averages of the form

$$\left\langle \exp\left[\frac{\mathrm{i}\pi}{b}(am^2 + cm)\right]\right\rangle_m . \tag{13}$$

This is zero unless c is also an integer. In fact it is non-zero if and only if the quantity $(ab + c)$ is an even integer:

$$\left\langle \exp\left[\frac{\mathrm{i}\pi}{b}(am^2 + cm)\right]\right\rangle_m = \begin{cases} \dfrac{1}{\sqrt{b}}\left(\dfrac{a}{b}\right)\exp\left[\dfrac{-\mathrm{i}\pi}{4}(b-1)\right]\exp\left[\dfrac{-\mathrm{i}\pi a}{b}(a\backslash b)^2\left(\dfrac{c}{2}\right)^2\right], & a \text{ even, } b \text{ odd, } c \text{ even,} \\ \dfrac{1}{\sqrt{b}}\left(\dfrac{b}{a}\right)\exp\left[\dfrac{\mathrm{i}\pi a}{4}\right]\exp\left[\dfrac{-\mathrm{i}\pi a}{b}(a\backslash b)^2\left(\dfrac{c}{2}\right)^2\right], & a \text{ odd, } b \text{ even, } c \text{ even,} \\ \dfrac{1}{\sqrt{b}}\left(\dfrac{a}{b}\right)\exp\left[\dfrac{-\mathrm{i}\pi}{4}(b-1)\right]\exp\left[\dfrac{-\mathrm{i}\pi}{b}4a(4a\backslash b)^2c^2\right], & a \text{ odd, } b \text{ odd, } c \text{ odd,} \\ \text{zero} & , \text{ otherwise.} \end{cases} \tag{14}$$

Here $\left(\dfrac{a}{b}\right)$ denotes the Jacobi sign symbol which takes the value ± 1. It represents the irreducible number-theoretical complexity of the Gauss average. Its definition and the properties we need are given in appendix B. The symbol $(a\backslash b)$ is the integer inverse of integer a with respect to a mutually prime integer b – that is the unique integer between 1 and $b-1$ such that

$$a(a\backslash b) \equiv 1 \bmod b. \tag{15}$$

The Gauss average for U in (12) can now be evaluated. Removing the m-independent factors from the average we have

$$U_{Q_2,Q_1} = \left(\frac{\mathrm{i}T_{12}}{N}\right)^{1/2} \exp\left[\frac{\mathrm{i}\pi}{NT_{12}}(T_{11}Q_1^2 - 2Q_1Q_2 + T_{22}Q_2^2)\right]\left\langle \exp\left[\frac{\mathrm{i}\pi}{T_{12}}(NT_{11}m^2 + 2(T_{11}Q_1 - Q_2)m)\right]\right\rangle_m. \tag{16}$$

To use (14) we need to reduce the fraction NT_{11}/T_{12} to its lowest terms. From the unit determinant relation (2), T_{11} and T_{12} are mutually prime but N and T_{12} may not be. We therefore divide each by their highest common factor (which we write $N \cap T_{12}$) and call the results respectively N' and T'_{12}:

$$\begin{aligned} U_{Q_2,Q_1} = {}& \left(\frac{\mathrm{i}T_{12}}{N}\right)^{1/2} \exp\left[\frac{\mathrm{i}\pi}{NT_{12}}(T_{11}Q_1^2 - 2Q_1Q_2 + T_{22}Q_2^2)\right] \\ & \times\left\langle \exp\left[\frac{\mathrm{i}\pi}{T'_{12}}\left(N'T_{11}m^2 + \frac{2}{N\cap T_{12}}(T_{11}Q_1 - Q_2)m\right)\right]\right\rangle_m. \end{aligned} \tag{17}$$

The average here is now ready for substitution into (14). We shall not write out the whole expression but note that it has the following form, non zero if and only if $ab+c$ is even:

$$U_{Q_2,Q_1} = \begin{cases} \left(\frac{\mathrm{i}}{N'}\right)^{1/2}\left(\frac{N'T_{11}}{T'_{12}}\right)\exp\left[\frac{-\mathrm{i}\pi}{4}(T'_{12}-1)\right]\exp\left[\frac{\mathrm{i}\pi}{NT_{12}}(\text{quadratic in } Q_2, Q_1)\right], \\ \qquad \text{for } N'T_{11} \text{ even, } T'_{12} \text{ odd, } \frac{2(T_{11}Q_1 - Q_2)}{N\cap T_{12}} \text{ even,} \\ \quad \text{or for } N'T_{11} \text{ odd, } T'_{12} \text{ odd, } \frac{2(T_{11}Q_1 - Q_2)}{N\cap T_{12}} \text{ odd.} \\ \left(\frac{\mathrm{i}}{N'}\right)^{1/2}\left(\frac{T'_{12}}{N'T_{11}}\right)\exp\left[\frac{\mathrm{i}\pi}{4}N'T_{11}\right]\exp\left[\frac{\mathrm{i}\pi}{NT_{12}}(\text{quadratic in } Q_2, Q_1)\right], \\ \qquad \text{for } N'T_{11} \text{ odd, } T'_{12} \text{ even, } \frac{2(T_{11}Q_1 - Q_2)}{N\cap T_{12}} \text{ even.} \end{cases} \tag{18}$$

Several remarks about these results may now be made.

(i) For simple maps the matrix need not be complicated in form. We introduce three examples for later reference:

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{Elliptic map (a rotation), } U_{Q_2,Q_1} = \left(\frac{\mathrm{i}}{N}\right)^{1/2} \exp\left[\frac{\mathrm{i}2\pi}{N}Q_2Q_1\right]; \tag{19}$$

$$\begin{aligned} T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} & \text{ Parabolic map (a simple shear), } U_{Q_2,Q_1} = \left(\frac{\mathrm{i}}{N}\right)^{1/2}\exp\left[\frac{\mathrm{i}\pi}{2N}(Q_1 - Q_2)^2\right] \\ & \times \begin{cases} \frac{1}{\sqrt{2}}\{1 + (-1)^{Q_1+Q_2+N/2}\}, & \text{for } N \text{ even,} \\ (-1)^{(N^2-1)/8}\exp\left[\frac{\mathrm{i}\pi N}{2}\left(\tfrac{1}{2} - (Q_1-Q_2)^2\right)\right], & \text{for } N \text{ odd;} \end{cases} \end{aligned} \tag{20}$$

$$T=\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}\ \text{Hyperbolic map (a shear with asymptotes)},\ U_{Q_2,Q_1}=\left(\frac{\mathrm{i}}{N}\right)^{1/2}\exp\left[\frac{\mathrm{i}2\pi}{N}(Q_1^2-QQ_2+Q_2^2)\right]. \tag{21}$$

(ii) Not all integer maps can be quantized. The necessary and sufficient condition that T should be quantizable for all N is that it takes one of the two 'checkerboard' forms

$$T=\begin{pmatrix}\text{even} & \text{odd}\\ \text{odd} & \text{even}\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}\text{odd} & \text{even}\\ \text{even} & \text{odd}\end{pmatrix}. \tag{22}$$

The reason is that we must discard as non-quantizable any T and N for which U does not have the correct discreteness and periodicity in Q_2. Although we have considered Q_2 as an integer we must require that the *only* non-zero values of the expressions (18) occur for integral Q_2. We must also require that they are unchanged under the replacement $Q_2\to Q_2+N$. U_{Q_2,Q_1} is non-zero if and only if $ab+c$ is even, that is if

$$T_{12}'N'T_{11}+2(T_{11}Q_1-Q_2)/(N\cap T_{12})=\text{even integer}, \tag{23}$$

so that

$$Q_2=T_{11}Q_1+\tfrac{1}{2}(T_{12}'N'T_{11}-\text{even})(N\cap T_{12}). \tag{24}$$

This is an integer provided that the product

$$T_{12}'N'T_{11}(N\cap T_{12})=N'T_{12}T_{11}=\text{even}. \tag{25}$$

We shall call a map quantizable only if this condition is satisfied for all possible N. So our first condition for quantizability which ensures that U is correctly discrete is

$$T_{12}T_{11}=\text{even}. \tag{26}$$

The second condition is

$$T_{21}T_{22}=\text{even}. \tag{27}$$

This is derived from the analogous requirement that U is correctly *discrete* in its momentum representation (the labels 1 and 2 are merely exchanged) so that it is correctly *periodic* in its position representation. The two conditions (26) and (27) together with the unit determinant relation (2) allow only the checkerboard forms (22) for T. (Note that (22) excludes the Arnol'd 'cat' map whose elements are 1, 1, 1, 2.)

(iii) In any row or column of U_{Q_2,Q_1} the non-zero elements are equally spaced with spacing $N\cap T_{12}$, and are N' in number. They all have square modulus $1/N'$ in accordance with unitarity. The phases differ by multiples of $2\pi/NT_{12}$.

(iv) In the form (12) involving the average over m the m is associated with the Q_1 coordinate so that the expression is manifestly periodic in Q_1 but not obviously in Q_2. However, it follows from the construction of this average (which was not given in detail) that the m can equally well be associated with the Q_2 coordinate instead (i.e. over equivalent images rather than sources in fig. 1), so that

$$U_{Q_2,Q_1}=\left(\frac{\mathrm{i}T_{12}}{N}\right)^{1/2}\left\langle\exp\left[\frac{\mathrm{i}\pi}{NT_{12}}(T_{11}Q_1^2-2Q_1(Q_2+mN)+T_{22}(Q_2+mN)^2)\right]\right\rangle_m. \tag{28}$$

We give an alternative proof of this in the next remark.

(v) The matrix propagator U inherits not only unitarity from its continuous counterpart (5) but also the composition law for propagation under successive maps T_A and T_B. If we label the quantum matrices with their classical matrices as superscripts we have

$$U^{T_B}U^{T_A}=U^{(T_BT_A)}. \tag{29}$$

This is a particularly useful property because it is easier to multiply classical 2×2 matrices than quantum $N\times N$ ones. For example, we can derive the formula (28) for U_{Q_2,Q_1} from the earlier one (12):

$$U^TU^{T^{-1}}=U^{(TT^{-1})}=I\quad(N\times N\text{ identity matrix}). \tag{30}$$

Therefore

$$U^T=(U^{T^{-1}})^{-1}=(U^{T^{-1}})^{\dagger} \tag{31}$$

by unitarity. This is the result we require. If the left hand side stands for the first expression (12) then the right hand side is the same expression with T_{11} and T_{22} interchanged (from the T^{-1}) and Q^1 and Q^2 interchanged (from the $\dagger$) – namely (28).

(vi) It is possible from the results (18) to give conditions on the map matrix T and the number N under which U^T is the identity matrix (apart from an unimportant overall phase factor), i.e.

$$U^T=I\,\mathrm{e}^{\mathrm{i}\phi}. \tag{32}$$

This is of central importance in view of the last remark because the conditions we are about to derive will be satisfied (as will be shown in the next section) by some power T_0^n of any classical map T_0. Then

$$U^{T_0^n}=(U^{T_0})^n=I\,\mathrm{e}^{\mathrm{i}\phi}. \tag{33}$$

In this sense we may call the quantum propagator periodic with a period equal to the lowest such n. The first condition for (32) comes from remark (ii) above. There must be just one non zero element per row or column, so we need $N\cap T_{12}=N$ i.e. N divides T_{12} or $N'=1$. The only other condition is that U_{Q_2,Q_1} should be non-zero for all $Q_1=Q_2$. The case $Q_1=Q_2=0$ evidently excludes the second case of (18) where $T'_{12}\,T_{11}$ is odd. Otherwise the condition is satisfied provided that $(T_{11}-1)$ is divisible by N. To summarize, we need T_{12} and $(T_{11}-1)$ divisible by N and $T_{12}T_{11}/N$ even. But this only ensures that the diagonal terms are non zero. To ensure that they are all identical we must require the equivalent conditions in the momentum representation which simply exchanges the labels 1 and 2. In all then we need

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N \begin{pmatrix} \text{integer} & \text{integer} \\ \text{integer} & \text{integer} \end{pmatrix}, \quad \text{for } N \text{ odd,}$$

or (34)

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N \begin{pmatrix} \text{integer} & \text{even} \\ \text{even} & \text{integer} \end{pmatrix}, \quad \text{for } N \text{ even.}$$

It will sometimes be necessary to restrict attention to the simpler of these cases, N odd, in what follows.

3. Eigenvalues of U

The N eigenvalues of the unitary quantum propagation matrix U lie on the unit circle in the complex plane. We can easily show (below) from the results of the last section that U is periodic with some period $n(N)$ in the sense of (33). This restricts the N eigenvalues to lie on the n possible sites

$$\left\{\exp\left[\frac{i(2m\pi + \phi)}{n}\right]\right\}, \quad 1 \leq m \leq n. \tag{35}$$

Generally $n \neq N$, so that there is no one to one correspondence between eigenvalues and sites. Typically there are both unoccupied and multiply occupied sites. In this section we will examine how the period n depends on N for the different classes of classical map T (elliptic, parabolic, hyperbolic), and how the N eigenvalues are distributed among their n possible values.

According to remark (vi) in the last section, to demonstrate that U is periodic it is sufficient to show that for some power $n(N)$ of the classical map T (previously called T_0)

$$T^{n(N)} = I + N \begin{pmatrix} \text{integer} & \text{integer} \\ \text{integer} & \text{integer} \end{pmatrix}, \quad \text{for } N \text{ odd,}$$

(36)

$$T^{n(N)} = I + N \begin{pmatrix} \text{integer} & \text{even} \\ \text{even} & \text{integer} \end{pmatrix}, \quad \text{for } N \text{ even.}$$

To argue that such a power exists, we consider for simplicity only the case of N odd; the case N even follows in a similar way. For N odd the condition above may be written

$$T^{n(N)} \equiv I \bmod N, \quad \text{(for each element).} \tag{37}$$

But there are only a finite number of such modulo reduced matrices so that the power sequence of matrices $T^k \bmod N$ must eventually contain a repeated matrix. And if the sequence starts with the identity matrix then this must actually be the first matrix to be repeated since each reduced matrix has a unique reduced predecessor.

Next we consider the function $n(N)$ for a given map. An important lowest common multiple relationship exists between periods for different N quite irrespective of the class of map. With the symbol $\cup$ representing lowest common multiple we have

$$n(N_1 \cup N_2) = n(N_1) \cup n(N_2). \tag{38}$$

This means that all the periods can be derived from those of the primes and their powers. If N contains the prime p to the power μ_p,

$$n(N) = \bigcup_p n(p^{\mu_p}). \tag{39}$$

The lowest common multiple rule is proved as follows. Firstly, it is clear from the conditions (36) that if a number N' divides N then $n(N')$ divides $n(N)$. Now N_1 and N_2 both divide $N_1 \cup N_2$; therefore both $n(N_1)$ and $n(N_2)$ divide $n(N_1 \cup N_2)$ and consequently $n(N_1) \cup n(N_2)$ divides $n(N_1 \cup N_2)$. On the other hand, it is easy to see from (36) that any number n that is divisible both by $n(N_1)$ and $n(N_2)$ has

$$T^n = I + (N_1 \cup N_2)\begin{pmatrix} \text{integer} & \text{integer} \\ \text{integer} & \text{integer} \end{pmatrix}, \quad \text{for } N_1,\ N_2 \text{ both odd,}$$

or (40)

$$T^n = I + (N_1 \cup N_2)\begin{pmatrix} \text{integer} & \text{even} \\ \text{even} & \text{integer} \end{pmatrix}, \quad \text{otherwise.}$$

So n satisfies the requirements (36) for $N = N_1 \cup N_2$ and must be a multiple of the period $n(N_1 \cup N_2)$. Since $n(N_1) \cup n(N_2)$ is both a factor of and a multiple of $n(N_1 \cup N_2)$ it must be equal to this number and the result is proved.

Similar and more extensive results are well known in the theory of the Fibonacci sequence [10, 11]:

$$0, 1, 1, 2, 3, 5, 8, \ldots, \quad a_{k+1} = a_k + a_{k-1}. \tag{41}$$

As a 2nd order linear recursion with constant coefficients this can be written in 2×2 matrix form

$$\begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix} = T\begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix}, \tag{42}$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{43}$$

(This differs from our map matrices in that it has determinant minus one.) Conversely, our map matrices generate a 2nd order linear recursion with constant coefficients, as follows: for any integer matrix T with unit determinant,

$$T^2 = (\mathrm{Tr}\ T)T - I, \tag{44}$$

whence

$$T^k = a_k T - a_{k-1} I, \tag{45}$$

with

$$a_{k+1} = (\mathrm{Tr}\ T)a_k - a_{k-1}. \tag{46}$$

This recursion sequence can be useful because if a_k is divisible by N (for N odd) or $2N$ (for N even) and $(a_{k-1}+1)$ is divisible by N then by (36) k is a multiple of the period. We are not sure whether all the results on the Fibonacci sequence listed for example by Robinson [10] can be adapted to our more general matrices with positive unit determinant, but numerical computations do not contradict the following guessed adaptations:

(i) For all primes except 2 and the prime factors of the discriminant of the matrix T, namely $(T_{11}+T_{22})^2-4$, the period $n(N)$ divides $N+1$ or N or $N-1$. Specifically n divides N minus the following Jacobi symbol (in this case a Legendre symbol since N is prime):

$$\left(\frac{(T_{11}+T_{22})^2-4}{N}\right), \tag{47}$$

where this is allowed to take the value zero if the top is a multiple of N.

(ii) For the μth power of any prime p except 2, $n(p^\mu)$ is $n(p)$ multiplied by p raised to a power not greater than $\mu-1$.

We now examine how the period function $n(N)$ depends on the class of map T. The way in which the class influenced the classical mechanics was mentioned in the introduction but we now need to make that classification quantitative. For the three classes, elliptic, parabolic and hyperbolic, points of the full (untorized) phase plane map along curves which are respectively ellipses, straight lines, and hyperbolas. These shapes can be distinguished by the number of straight lines through the origin: 0, 1 or 2, respectively. Each line represents a real eigenvector of T so that the following classification is based on the discriminant $(\mathrm{Tr}\ T)^2-4$ of T. For $|\mathrm{Tr}\ T|<2$ there are no real eigenvectors, the map is elliptic and all orbits on the torus are closed with the same period 2, 4 or 6. For $|\mathrm{Tr}\ T|=2$ there is only one real eigenvector, the map is parabolic and produces regular ('integrable') 'closed straight lines' on the torus. For $|\mathrm{Tr}\ T|>2$ there are two distinct eigenvectors, the map is hyperbolic and the exponential increase in the separation of neighbouring points produces highly irregular motion covering the torus ergodically.

We can now proceed with the proposed examination of $n(N)$ in these three classes. In each case we will use the recurrence sequence (46) to find the periods for prime and prime power values of N and then use the lowest common multiple rule for all other N. The periods of some anomalous low primes and their powers cannot be obtained from the recurrence sequence alone because these primes divide elements of the map matrix T so that the period condition (36) can be satisfied without the sequence divisibility condition (after (46)) being satisfied. Our statements concerning these anomalous numbers are conjectures.

For elliptic maps we have $|\mathrm{Tr}\, T| < 2$ but the checkerboard condition (22) requires $\mathrm{Tr}\, T = \text{even}$; therefore $\mathrm{Tr}\, T = 0$ for any quantizable elliptic map. From the recursion relation (46) the sequence (starting with a_1) runs $(0, 1, 0, -1, 0, 1, \ldots)$ and for all non-anomalous primes and prime powers the period is once again 4 provided N is large enough. To apply the lowest common multiple rule is hardly necessary. Thus, after possible initial fluctuation, $n(N) = 4$.

For parabolic maps $\mathrm{Tr}\, T = \pm 2$. These produce respective recursion sequences $(0, 1, 2, 3, 4, \ldots)$ and $(0, 1, -2, 3, -4, \ldots)$. For nonanomalous primes and prime powers the period is therefore respectively N and $2N$ and by the lowest common multiple rule (38) a similar result holds for all composite numbers composed only of these. For numbers composed of anomalous primes and their powers, $n(N)$ divides N and $2N$ respectively. The complete function $n(N)$ has the property that the ratio $n(N)/N$ is periodic. Thus both elliptic and parabolic quantized maps have very simple period functions.

For hyperbolic maps on the other hand the period function is highly irregular. Even for the non-anomalous primes and their powers there is intrinsic irregularity in the period. The period appears to divide either $N+1$ or $N-1$ in accordance with the guess (i) and other evidence mentioned later. But the factor by which it does so is not known and the periods for composite N consequent upon these are correspondingly random. Computer trials up to $N = 100{,}000$ indicate that the cumulant function of $n(N)$ increases as N^2, so that in an asymptotic sense the period function behaves as

$$n(N) \sim CN, \tag{48}$$

but the limit is approached very slowly and is accompanied by large fluctuations. It follows from the guess (i) that $n(N)$ cannot increase faster than (48), and also that $C < 1$. If the asymptotic form of $n(N)$ were different from (48), then in the semiclassical limit $N \to \infty$ the spectrum of 'eigenangles' in (35) would necessarily have wide gaps (if $n/N \to \infty$) or large degeneracies (if $n/N \to 0$).

A comparison between the period functions $n(N)$ for the three simple maps (19–21) is shown on fig. 2.

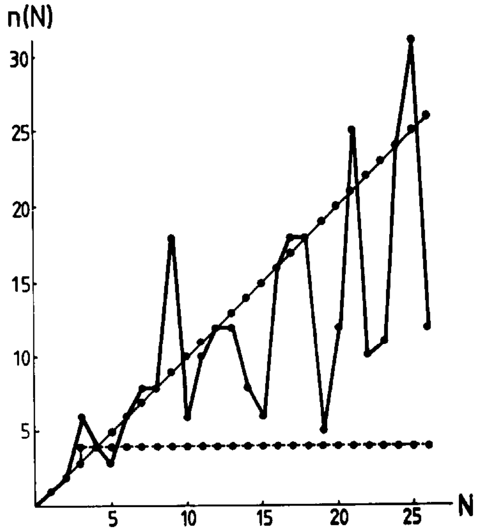

Fig. 2. Period $n(N)$ for the elliptic map (19) (points joined by dashed line), the parabolic map (20) (points joined by thin line), and the hyperbolic map (21) (points joined by thick line).

Finally, we ask how the N eigenvalues of U are distributed among their n possible values on the unit circle in the complex plane. One result of this type will be proved at the end of section 4. In addition, the lowest common multiple rule for periods suggests that there might be a relationship between the *distribution* of eigenvalues for the number $N = N_1 \cup N_2$ and the distributions for the numbers N_1, N_2 themselves. We believe that the following trace multiplication rule holds for an arbitrary map T

$$\mathrm{Tr}(U^m_{N_1})\,\mathrm{Tr}(U^m_{N_2}) = (\pm)^m\,\mathrm{Tr}(U^m_{N_1\cup N_2})\,\mathrm{Tr}(U^m_1), \tag{49}$$

where the $\pm$ sign depends on the map and on the N's. The basis of this relation is largely empirical – we have not succeeded in proving it or deriving the sign. It leads to the following convolution rule for the 'eigenangle' density. If θ_j are the phase angles of the eigenvalues of a particular map T with a particular N we define the eigen-angle density as

$$\rho_N(\theta) \equiv \sum_{j=1}^{N} \delta(\theta - \theta_j) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{j=1}^{N} \mathrm{e}^{2\pi\, \mathrm{i}m(\theta_j - \theta)} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \mathrm{Tr}(U^m_N)\, \mathrm{e}^{-2\pi\, \mathrm{i}m\theta}. \tag{50}$$

Then

$$\int_0^{2\pi} \mathrm{d}\theta' \rho_{N_1}(\theta')\rho_{N_2}(\theta - \theta') \equiv \rho_{N_1} * \rho_{N_2}(\theta) = \rho_{N_1\cup N_2} * \rho_1(\theta),$$

or (51)

$$\rho_{N_1} * \rho_{N_2}(\theta) = \rho_{N_1\cup N_2} * \rho_1(\theta + \pi),$$

depending on the sign ± 1 in (49). $\rho_1(\theta)$ is a single delta function in θ so that the convolution on the right merely displaces the function $\rho_{N_1\cup N_2}$ in θ. This convolution rule has the lowest common multiple rule (38) as a consequence.

As examples we give in table I the eigenangles for the first few N for the three different maps whose propagation matrices were given in (19–21). It may be remarked with reference to these examples that for prime N all the possible sites except one are equally occupied (i.e. equally degenerate), and the exceptional site has one fewer or one more eigenvalue. For the hyperbolic map (21) most of the prime N eigenangles are nondegenerate; therefore this map can be said to exhibit the 'level repulsion' characteristic of classically nonintegrable systems. For the parabolic map (20) all but the exceptional sites are doubly degenerate, and for the elliptic map (19) there are just four sites and so the degeneracy increases with N; therefore these maps can be said to exhibit the 'level clustering' characteristic of classically integrable systems. The eigenangles in the parabolic map (20) are in fact $2\pi(p^2 \bmod N)/N$ for $1 \le p \le N$, an exact result which can also be obtained by Bohr–Sommerfeld quantization of the classical invariant curves (which for this 'simple shear' map, corresponding to a classical free rotor, are just lines of constant p).

4. Relation with the classical map; the Wigner function

It is interesting to note an intimate connection between the quantized map and the classical behaviour of particular points of the torus. We have described briefly the different sorts of classical

Table I
Eigenangles expressed as a fraction of 2π for the three maps (19–21) with $1 \leqslant N \leqslant 12$

N	1	2	3	4	5	6	7	8	9	10	11	12
	1/8	1/8	−3/8	−3/8	−3/8	−3/8	−3/8	−3/8	−3/8	−3/8	−3/8	−3/8
		−3/8	3/8	3/8	3/8	−3/8	−3/8	−3/8	−3/8	−3/8	−3/8	−3/8
			1/8	1/8	−1/8	−1/8	−1/8	−1/8	−1/8	−3/8	−3/8	−3/8
				1/8	1/8	3/8	3/8	3/8	−1/8	−1/8	−1/8	−1/8
Elliptic map $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$					1/8	1/8	3/8	3/8	3/8	−1/8	−1/8	−1/8
							1/8	1/8	3/8	3/8	3/8	3/8
							1/8	1/8	1/8	3/8	3/8	3/8
								1/8	1/8	1/8	3/8	3/8
									1/8	1/8	1/8	1/8
										1/8	1/8	1/8
											1/8	1/8
												1/8
	0	0	0	0	0	0	0	0	0	0	0	0
		−1/2	−1/3	−1/4	−1/5	−1/6	−1/7	−1/8	−1/9	−1/10	−1/11	−1/12
			−1/3	0	1/5	2/6	3/7	4/8	−4/9	−4/10	−4/11	−4/12
				−1/4	1/5	3/6	−2/7	−1/8	0	1/10	2/11	3/12
					−1/5	2/6	−2/7	0	2/9	4/10	−5/11	−4/12
Parabolic map $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$						−1/6	3/7	−1/8	2/9	5/10	−3/11	−1/12
							−1/7	4/8	0	4/10	−3/11	0
								−1/8	−4/9	1/10	−5/11	−1/12
									−1/9	−4/10	2/11	−4/12
										−1/10	−4/11	3/12
											−1/11	−4/12
												−1/12
	1/8	1/8	13/24	−1/8	−9/24	−7/24	−3/8	−3/8	−19/72	27/24	−13/40	15/24
		−3/8	9/24	3/8	−1/24	−7/24	−2/8	−2/8	41/72	−5/24	−9/40	−5/24
			1/24	3/8	−1/24	−3/24	−1/8	4/8	−7/72	−5/24	23/40	−5/24
				1/8	7/24	9/24	3/8	3/8	33/72	11/24	−5/40	−5/24
Hyperbolic map $\begin{pmatrix}2 & 1\\ 3 & 2\end{pmatrix}$					7/24	5/24	0	0	−3/72	11/24	19/40	13/24
						5/24	2/8	2/8	29/72	−1/24	−1/40	−3/24
							1/8	1/8	−31/72	−1/24	15/40	1/24
								1/8	9/72	7/24	15/40	7/24
									17/22	7/24	3/40	7/24
										3/24	11/40	7/24
											7/40	3/24
												3/24

motion exhibited by a *general* point of the torus under the different classes of map. And we have seen that the quantized map also showed differences for the different classes through its period function $n(N)$ and hence its eigenvalue distribution.

We now show that there is, in fact, a very precise relationship between the distribution of the N eigenvalues of U and the motion of the $(2N)^2$ discrete points which make a $2N \times 2N$ sublattice on the torus (i.e. on the representative unit cell in the phase plane). Thus as N increases quantum mechanics 'experiences' ever more finely the classical motion on the torus.

There is a very useful means of representing a quantum wavefunction (or more generally a density matrix) in phase space, due to Wigner [11, 12, 13]. The 'Wigner function' is a real function in phase space defined in conventional notation (not in our present units) by

$$W(q,p)=\frac{1}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}q'\psi(q+q')\psi^*(q-q')\,\mathrm{e}^{-\mathrm{i}2pq'/\hbar}. \tag{52}$$

It is a 'local spectrum' in the sense that it is the Fourier transform of a correlation function or density matrix of the wavefunction centred on q. It contains all the information in the wavefunction except the unimportant overall phase, and it does so in a manner similar to a classical distribution in phase space in that when integrated through p it gives the q probability $|\psi(q)|^2$ and vice–versa. It has also a crucially important property for our application here. If a quantum wavefunction is propagated by a unitary transformation appropriate to a *linear canonical map* the Wigner function of the propagated wavefunction is simply the Wigner function of the initial wavefunction carried along with the map in a perfectly classical manner.

By straightforward application of the definition above it is found that provided irrelevant infinite normalization difficulties are ignored the Wigner function of an arbitrary periodic δ comb wavefunction is a $2N\times 2N$ periodic 'δ brush' on the phase plane. It has delta function spikes on each of the $(2N)^2$ points whose coordinates have denominator $2N$ in the unit cell. However, not all the strengths of the δ spikes can be different – they are linked in sets of four. The four spikes situated on the corners of a square of side N spacings (i.e. $\frac{1}{2}\times\frac{1}{2}$) have strengths that differ at most in sign:

$$W(q,p)=(-1)^{2Np}W(q+\tfrac{1}{2},p)=(-1)^{2Nq}W(q,p+\tfrac{1}{2})=(-1)^{(2q+2p+1)N}W(q+\tfrac{1}{2},p+\tfrac{1}{2}). \tag{53}$$

Any two on a side differ in sign when the side, if extended to intersect the q or p axis would do so in a 'halfway' coordinate odd/$2N$. In this way the projection integrals of the Wigner function are guaranteed to be zero except on the coordinates integer/N as is required for the position and momentum wavefunctions.

The last preparatory step is to ask how the $2N\times 2N$ sublattice points move under the classical mapping in order to see how the Wigner function changes. The answer is simple – they map amongst themselves – they permute. This is because an integer mapping cannot change the denominator of rational coordinates. The behaviour of such rational points contrasts with that of a general irrational point (at least for parabolic and hyperbolic maps) in that they all lie on closed orbits, which we call cycles. It is now the way in which the $(2N)^2$ points divide up into these (permutation) cycles which distinguishes the different classes of map. The cycles for $N=2$ are shown in fig. 3 for the maps (19–21).

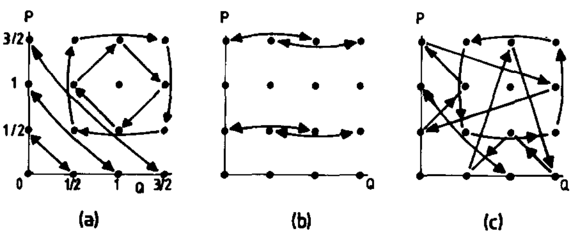

Fig. 3. Wigner point permutation cycles for $N=2$ for: (a) the elliptic map (19); (b) the parabolic map (20); (c) the hyperbolic map (21).

The implications of the Wigner construction just described are now evident. If the map is repeated a number of times equal to the lowest common multiple of lengths (number of points) of the cycles then each point has returned to its starting point and the Wigner function of the final state is identical to that of the (arbitrary) initial state. It follows that the final state is the initial state (apart from an overall phase factor) so that the number of mappings must be the period n of the quantized map or a multiple of it. The reason that it may be a multiple of the period is that the Wigner delta function strengths are linked by (53) and it can be that the Wigner function (of an arbitrary initial state) is reproduced before each point has returned. A short analysis confirms the previously desired conditions for the period. Also the checkerboard quantizability condition (22) can be rederived – it is in fact the condition that sublattice points whose coordinates are (odd/$2N$, odd/$2N$) map amongst themselves for all N. This is necessary and sufficient to ensure that the projections onto the q and p axes at halfway coordinates odd/$2N$ remain zero after the mapping.

An eigenstate of the quantized map has a Wigner function in which members of the same cycle share the same delta function strength, so that the Wigner function as a whole is unchanged under the map. Different cycles may have different strengths except in so far as they are linked by the relationships (53). Fig. 4 shows the Wigner functions for the five eigenstates of the hyperbolic map (21) for $N = 5$, and fig. 5 shows the nine (unlinked) cycles in this case. The pictures show that the Wigner functions of eigenstates 'cover' (i.e. have nonzero values on) nearly all the cycles – there is no particular association between an eigenstate and a group of cycles (as might have been expected since an eigenstate is the minimal quantal invariant object and a cycle is the minimal classical invariant object).

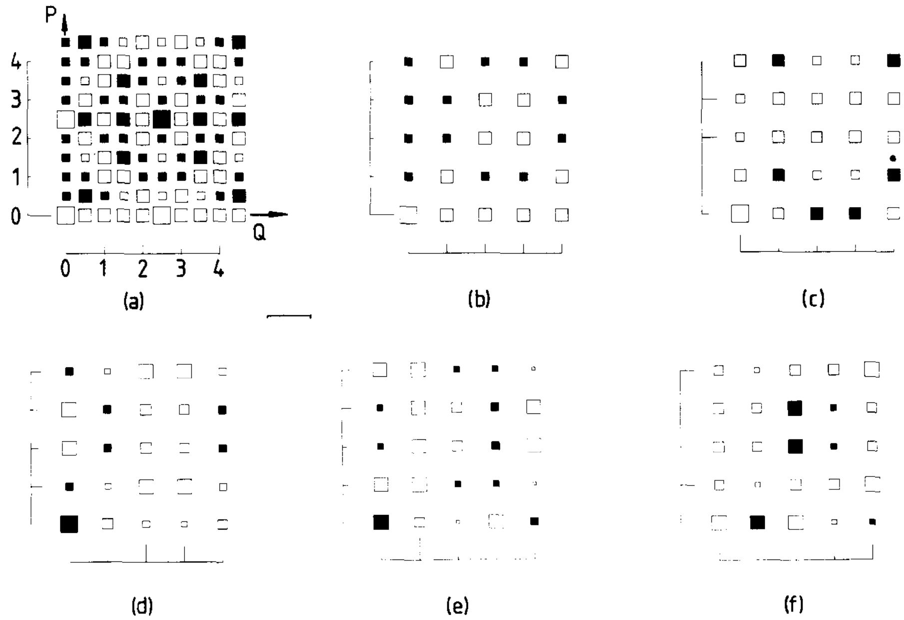

Fig. 4. Wigner functions W for the five eigenstates of the hyperbolic map (21) for $N = 5$. (a) shows W on the full $2N \times 2N$ sublattice for one eigenstate, and (b–f) show W on the $N \times N$ representative lattice for states with eigenangles $-135°$, $-15°$, $-15°$, $105°$ and $105°$, respectively (cf. table I). The values of W are proportional to the sizes of the squares, and negative values are represented by black squares. Position and momentum probability densities (projections of W) are indicated below and to the left of each picture, respectively.

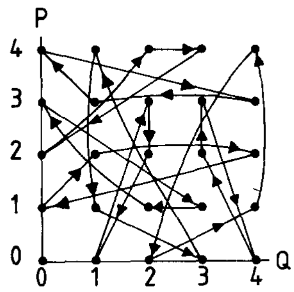

Fig. 5. The nine representative classical cycles executed by points of the $N \times N$ lattice under the hyperbolic map (21) when $N = 5$.

It is not only the Wigner functions for the N pure eigenstates that are invariant under the map but also those of statical mixtures of pure states, which are merely weighted sums of the constituent pure state Wigner functions. From this observation a detailed relationship can be derived between the eigenvalue degeneracies d_m of the quantized map and the number $\mathcal{N}$ of independent (unlinked) classical cycles. Of course, by summing over the n possible sites for eigenvalues we have in any case

$$\sum_{m=1}^{n} d_m = N. \tag{54}$$

A more informative relation derives from the argument that the number of parameters specifying the density matrix for any mixture of eigenstates should be the same as the number of parameters in the corresponding Wigner function, namely the number of independent cycles. The number of parameters specifying a density matrix is just N if the eigenvalues are distinct, but if there are degeneracies it is greater. In the eigen-representation the matrix can be arranged into n separate $d_m \times d_m$ blocks on the diagonal, and zero elsewhere. Because the matrix is hermitian, each block requires d_m^2 parameters to specify it. Equality of parameters and cycles now gives the desired relation, namely

$$\sum_{m=1}^{n} d_m^2 = \mathcal{N}. \tag{55}$$

In the case of N odd there is a considerable simplification because each linked set of four Wigner points has exactly one representative point on the $N \times N$ lattice. The period of the map is the lowest common multiple of the cycle lengths (not merely a factor of it) and the sum of the squares of the eigenvalue degeneracies is the number of representative cycles. By applying this rule to powers of the map it is possible to find, from the eigenvalue degeneracies, the number of cycles of each length. For example, for hyperbolic maps the eigenvalues for N prime (non-anomalous) are (we believe) all equally degenerate except for one site which has one extra or one fewer eigenvalue. This rule implies that all cycles on the $N \times N$ lattice have equal length n (the period) except for the fixed point at the origin. Numerical computation supports this conclusion.

5. Concluding summary

We have considered a one-dimensional model whose classical and quantum mechanics can be solved exactly. The development of the system was examined not continuously through all time but only 'stroboscopically' at equal intervals. The classical mechanics is therefore described by an area-preserving map. Now the quantum mechanics of linear maps on the full phase plane is trivial. (It amounts to quantization of a Harmonic oscillator with a magnetic field). But if the phase plane (q, p) is constrained to admit only probability distributions which are periodic in q and p so that the phase space is effectively a torus, the problem is more interesting. It is known that under such mappings on a torus, a point can exhibit highly regular motion (for an elliptic map) or highly irregular motion (for a hyperbolic map) and we have shown how this irregularity manifests itself in the quantum mechanics through the period function $n(N)$ (described below). Also while the system is rather artificial from the point of view of mechanics, its quantized form has a direct physical realization in optics in terms of diffraction by a periodic grating.

The imposition of torus periodicity alone, before the prescription of the dynamics (i.e. the mapping), greatly restricts the admissible class of quantum wavefunctions. A wavefunction which is periodic both in its position and momentum representations is necessarily a period 'comb' of equally spaced delta functions, period N spacings in both representations. (Such a wavefunction is produced by an idea diffraction grating with periodic infinitesimal slit widths.) If the position and momentum periods Δq and Δp are considered fixed then $\hbar$ can only take discrete values $\Delta q \Delta p = 2N\pi\hbar$. As $\hbar$ is reduced N increases so that the spacing between the delta functions decreases and the comb becomes a continuum in the classical limit.

The linear canonical mapping which constitutes the classical dynamics is described by a 2×2 matrix. This is constrained by the toroidal periodicity to be a matrix of integers (in units where the periods Δq, Δp are unity), and to be quantizable for all N the elements have to satisfy the additional restriction of being alternately even and odd. Associated with a given such mapping and a given value of N there is an $N\times N$ unitary quantum propagation matrix. Under the map an initial periodic δ comb wavefunction is carried into another one with identical spacings and periods but with a different set of N amplitudes given in terms of the initial set by the unitary matrix. Its elements can be written as a Gauss sum which on evaluation give expressions involving the Legendre and Jacobi sign symbols of number theory. The matrix obeys an important composition law: the matrix for the product of two classical maps is the product of the two matrices for the individual maps.

This composition law leads to an essential feature of the quantized map – it is periodic in the sense that when raised to some power the unitary matrix equals the identity (apart from an unimportant phase factor). The power $n(N)$ to which it has to be raised – the 'period' of the quantized map – constrains the N eigenvalues of the matrix to lie on some or all of n equally spaced sites around the unit circle (to which they are confined by unitarity). As a function of N the period $n(N)$ behaves differently for different classes of map.

For elliptic and parabolic maps $n(N)$ behaves in a regular way but for hyperbolic maps it is highly irregular. For all maps, though, the period obeys a lowest common multiple rule (39) from which the period $n(N)$ can be obtained from the $n(p^\mu)$ where p are the distinct prime factors of N and μ is the multiplicity of them in N. The cumulant $\Sigma_{N'=1}^{N} n(N')$ goes asymptotically like const. $\times N$ for elliptic maps and const. $\times N^2$ for parabolic and hyperbolic maps.

The way in which the N eigenvalues of the unitary matrix are distributed among their n possible sites appears to be governed by a convolution rule (51) for the 'eigenangle' density $\Sigma_j\, \delta(\theta-\theta_j)$ where the eigenvalues are $\exp[\mathrm{i}\theta_j]$. Its essential content is that the eigenangle density for N is the multiple convolution of those for p^μ, the prime powers in N as above. Thus the eigenvalues for general N are governed by those of the primes and powers of primes. For the primes the eigenvalues appear to be very regularly distributed with an equal number on each of the n possible sites except one which has one less or one more. For composite numbers this regularity is preserved for elliptic and parabolic maps but not for hyperbolic ones.

The quantized map is intimately connected with the classical map. Classically every point on the torus (dimensions 1×1) which has rational coordinates is a member of a closed orbit trajectory or 'cycle' – it is a fixed point of some power of the mapping since the 2×2 classical map matrix with its integer elements cannot change the (common) denominator of the coordinates. (For elliptic maps the irrational points are also closed orbits.) Under quantization a subset of rational points, namely the lattice of those with a fixed denominator $2N$, take on special significance in the following sense. If the Wigner function of the periodic delta comb is constructed it is a double comb (a 'brush') in phase space with delta functions sited on this $2N \times 2N$ sub-lattice. Now for a linear map the Wigner function (which is fully quantum-mechanical) is simply carried along with the classical map. The $2N \times 2N$ points map among themselves as a permutation carrying the Wigner values with them. The permutation divides up into permutation cycles of different lengths (numbers of points) and the lowest common multiple of these is the period of the map (or a multiple of it) because the Wigner function has then necessarily returned to its original configuration.

The situation is rather simple for N odd when the mapping of the $2N \times 2N$ lattice is 'governed' by that of an $N \times N$ lattice whose coordinates have denominator N (like the delta functions of the position and momentum wavefunctions). In this case, N odd, there is an easily stated relationship between the eigenvalues of the quantum matrix and the cycles on this $N \times N$ lattice: the number of cycles is the sum of the squares of the degeneracies of the eigenvalues. This rule can be used on powers of the map to obtain from the eigenvalues the number of cycles of each length. It derives from an argument that the number of free parameters in the density matrix of an eigenstate of the mapping should be the same as the number of free parameters in the prescription of the Wigner function of any eigenstate.

To conclude the summary we return to the optical analogy to mention a possible application. A finite diffraction grating is fitted across the mouth of a waveguide consisting of two parallel mirrors and illuminated with a monochromatic 'picture'. The picture is discretized by the grating and the mirrors create virtual images of it on each side out to infinity. Thus the conditions of our problem are realized and at equal intervals along the guide real discretized 'images' are produced. These 'images' may bear no recognizable relation to the initial picture until the nth image, where n is the period of the map, when a faithful reproduction is obtained.

Appendix A

Evaluation of the Gauss average

We require

$$G(a, b, c) = \left\langle \exp\left[\frac{i\pi}{b}(am^2 + cm)\right]\right\rangle_m . \tag{A.1}$$

The average here is over all integer values of m with a and b mutually prime positive integers. It will turn out that this average is zero unless c is also an integer. In this case the exponential is evidently periodic in m and the average can be expressed as a sum over a period divided by the period. This sum is the 'Gauss sum' of number theory. We shall continue to work in terms of the average to avoid having to specify the period.

We shall show below that $G(a, b, c)$ is non zero if $(ab + c)$ is an even integer and zero otherwise. Firstly, however, we show that if indeed $ab + c$ is even the average may be simplified to a form in which c is zero (which is that usually found in texts on number theory [8, 9, 15]). It is easy to check that this condition implies that the exponential is periodic in m with period b (or a factor of b). Essential use is to be made of this fact.

Case (ab) *even,* c *even*

$$G(a,b,c) = \left\langle \exp\left[\frac{\mathrm{i}\pi a}{b}\left(m + \frac{c}{2}(a\backslash b)\right)^2\right]\right\rangle_m \exp\left[\frac{-\mathrm{i}\pi a}{b}\left(\frac{c}{2}\right)^2(a\backslash b)^2\right]$$

$$= \left\langle \exp\left[\frac{\mathrm{i}\pi a}{b} m^2\right]\right\rangle_m \exp\left[\frac{-\mathrm{i}\pi a}{b}\left(\frac{c}{2}\right)^2(a\backslash b)^2\right], \tag{A.2}$$

where $(a\backslash b)$ is the integer inverse of a with respect to the mutually prime integer b, namely that unique integer in the range of 1 to $b-1$ such that

$$a(a\backslash b) \equiv 1 \bmod b. \tag{A.3}$$

Case (ab) *odd,* c *odd*

Here we use the following rule. For any periodic function $f_n(m)$ with an integer period n and any integer l mutually prime with n

$$\langle f_n(m)\rangle_m = \langle f_n(lm)\rangle_m, \tag{A.4}$$

since the sequence $(lm) \bmod n$ for $1 \leq m \leq n$ is then simply a permutation of the ordered integers from 1 to n. We take $l = 2$ which is mutually prime with b and therefore with the period of the exponential. This reduces the average to the previous case. Thus

$$G(a,b,c) = \left\langle \exp\left[\frac{\mathrm{i}\pi}{b}(4am^2 + 2cm)\right]\right\rangle_m$$

$$= \left\langle \exp\left[\frac{\mathrm{i}\pi}{b} 4am^2\right]\right\rangle_m \exp\left[\frac{-\mathrm{i}\pi}{b} 4a(4a\backslash b)^2 c^2\right]. \tag{A.5}$$

It remains to reduce the simplified averages to some standard form. We take that given by Lang [5] who proves that for b odd and a even (and mutually prime)

$$\left\langle \exp\left[\frac{\mathrm{i}\pi a m^2}{b}\right]\right\rangle_m = \frac{1}{\sqrt{b}}\left(\frac{a/2}{b}\right)\exp\left[\frac{\mathrm{i}\pi}{8}(b-1)^2\right] = \frac{1}{\sqrt{b}}\left(\frac{a}{b}\right)\left(\frac{2}{b}\right)\exp\left[\frac{\mathrm{i}\pi}{8}(b-1)^2\right]$$

$$= \frac{1}{\sqrt{b}}\left(\frac{a}{b}\right)\exp\left[\frac{\mathrm{i}\pi}{8}((b-1)^2 - (b^2-1))\right] = \frac{1}{\sqrt{b}}\left(\frac{a}{b}\right)\exp\left[\frac{-\mathrm{i}\pi}{4}(b-1)\right], \tag{A.6}$$

where we have used some standard manipulations of the Jacobi symbol to produce a simplified expression in our case. Now the average in the second case (A.5) above already has the standard form (A.6). That in the first case also has this form if a is even and b is odd. However, if a is odd and b is even the following reciprocity law derived from a Poisson transformation discussed below converts the average to standard form:

$$\left\langle \exp\left[\frac{\mathrm{i}\pi}{b}am^2\right]\right\rangle_m = \sqrt{\frac{a}{b}}\exp[-\mathrm{i}\pi/4]\left\langle \exp\left[-\frac{\mathrm{i}\pi b}{a}m^2\right]\right\rangle. \tag{A.7}$$

The results of (14) are thus obtained.

Finally we return to the beginning to prove that $G(a, b, c)$ is zero unless $ab + c$ is an even integer. It is in fact possible to pursue this analysis to derive the results of (14) up to their sign. The procedure we use is to reduce the coefficients of $\mathrm{i}\pi m^2$ (i.e. the ratio a/b) in G to an integer by a repeated pair of operations: 'truncation' and 'Poisson transformation'. Within G only the numbers a and b are changed in each operation.

We have first the truncation step:

$$G(a, b, c) = G(a', b, c), \tag{A.8}$$

where a' is a with an appropriate *even* number of b added or subtracted so that

$$-b \leqslant a' \leqslant b \tag{A.9}$$

(we are now admitting arguments of G with negative sign). The second operation is the Poisson transformation. With the square root sign meaning the root with $-\pi/2 < \arg\sqrt{} \leqslant \pi/2$, we have

$$G(a, b, c) = G(-b, a, c)\sqrt{\left(\frac{\mathrm{i}a}{b}\right)}\exp\left[\frac{-\mathrm{i}\pi c^2}{4ab}\right]. \tag{A.10}$$

This derives from representing G as an integral over a periodic δ comb and then using Parseval's theorem as follows. Writing the uniform δ comb with delta functions on the integers as $\delta_c(x)$, the average of any function f can be expressed as

$$\langle f(m)\rangle_m = \lim_{\alpha\to 0}\int_{-\infty}^{\infty}\sqrt{\alpha}\,\delta_c(x)f(x)\,\mathrm{e}^{-\pi\alpha x^2}\,\mathrm{d}x. \tag{A.11}$$

By expressing our average in this way and then using Parseval's theorem we obtain

$$\begin{aligned} G(a, b, c) &= \lim_{\alpha\to 0}\int_{-\infty}^{\infty}\sqrt{\alpha}\,\mathrm{e}^{-\pi\alpha x^2}\,\delta_c(x)\exp\left[\frac{\mathrm{i}\pi}{b}(ax^2 + cx)\right]\mathrm{d}x \\ &= \lim_{\alpha\to 0}\int_{-\infty}^{\infty}\sqrt{\alpha}\,\delta_c(y)\exp\left[\frac{-\mathrm{i}\pi}{b}(a+\mathrm{i}\alpha b)^{-1}\left(\frac{c}{2}+by\right)^2\right]\left(\frac{b}{-\mathrm{i}a+\alpha b}\right)^{1/2}\mathrm{d}y \\ &= \lim_{\alpha\to 0}\int_{-\infty}^{\infty}\sqrt{\alpha}\,\delta_c(y)\exp\left[\frac{-\mathrm{i}\pi}{b}\left(\frac{-\mathrm{i}\alpha b}{a^2}\right)\right]\exp\left[\frac{-\mathrm{i}\pi}{ab}\left(\frac{c}{2}+by\right)^2\right]\left(\frac{b}{-\mathrm{i}a}\right)^{1/2}\mathrm{d}y \\ &= \left(\frac{\mathrm{i}a}{b}\right)^{1/2}\exp\left[\frac{-\mathrm{i}\pi c^2}{4ab}\right]G(-b, a, c). \end{aligned} \tag{A.12}$$

It is clear that the effect of first truncation and then Poisson transformation is to decrease (the absolute value of) the second argument of G (the denominator in the exponent) unless its value is ± 1. If we repeat the pair of operations and list the first two arguments of G together with the coefficient of $\mathrm{i}\pi c^2/4$ in the exponent outside G and the prefactor we obtain the sequence

$$\begin{array}{llllllllllll}
 & T & P & T & P & T & P & T & P & & \\
 & a_0 & a_1 & -b_0 & -b_1 & a_1 & a_2 & -b_1 & -b_2 & a_2\ldots & \\
 & b_0 & b_0 & a_1 & a_1 & b_1 & b_1 & a_2 & a_2 & b_2 & \\[1em]
\text{coefficient of } \mathrm{i}\pi c^2/4 & & \dfrac{-1}{b_0 a_1} & & \dfrac{+1}{a_1 b_1} & & \dfrac{-1}{b_1 a_2} & & \dfrac{+1}{a_2 b_2} & \text{(cumulative)} \\[1em]
\text{prefactor} & & \left(\dfrac{\mathrm{i}a_1}{b_0}\right)^{1/2} & & \left(\dfrac{-\mathrm{i}b_1}{a_1}\right)^{1/2} & & \left(\dfrac{\mathrm{i}a_2}{b_1}\right)^{1/2} & & \left(\dfrac{-\mathrm{i}b_2}{a_2}\right)^{1/2} & \text{(multiplicatively cumulative).}
\end{array} \tag{A.13}$$

At each stage the first two arguments of G remain mutually prime so that eventually the second argument must become ± 1 and here the sequence is terminated. The Gauss average then has the form

$$\langle \exp[\mathrm{i}\pi(Km^2+cm)]\rangle_m = \langle \exp[\mathrm{i}\pi(K+c)m]\rangle_m = \begin{cases} 1, & \text{for } K+c \text{ even,} \\ 0, & \text{for } K+c \text{ odd.} \end{cases} \tag{A.14}$$

But it is straightforward to check that the product of the first arguments of G remains even or odd through the sequence so that K is even or odd like $a_0 b_0$ and the assertion is proved.

To conclude we remark that the coefficients in the square root prefactors evidently cancel each other out to give the correct absolute value for G. It is also possible to keep cumulative track of the factors i here and to obtain straightforwardly the phase of the result modulo π. The overall sign, however, depends on the sign of the individual square roots in the sequence and gives rise to the Jacobi sign symbol in the results (14). As far as the coefficients of $\mathrm{i}\pi c^2/4$ are concerned they can be accumulated correctly using the identity

$$\frac{1}{ab} = \frac{(a\backslash b)}{a} + \frac{(b\backslash a)}{b} - 1. \tag{A.15}$$

Appendix B

Legendre and Jacobi sign symbols

The definitions and required properties of these [8, 9, 15] are as follows. Consider the squares of the integers reduced modulo p where p is any prime except 2. The Legendre symbol $\left(\frac{n}{p}\right)$ is defined in terms of this sequence. For example, for $p = 5$ we have

$$\begin{array}{lllllllll} 0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64\ldots \\ 0 & 1 & 4 & 4 & 1 & 0 & 1 & 4 & 4\ldots \text{ periodic} \end{array} \tag{B.1}$$

1 and 4 are *present* in the reduced sequence. Therefore

$$\left(\frac{1}{5}\right) = \left(\frac{4}{5}\right) = +1. \tag{B.2}$$

2 and 3 are *absent* so

$$\left(\frac{2}{5}\right) = \left(\frac{3}{5}\right) = -1. \tag{B.3}$$

Generally the definition of the Legendre symbol (for n not divisible by p) is

$$\left(\frac{n}{p}\right) = \begin{cases} +1, & \text{if there is an integer } m \text{ such that } m^2 \equiv n \bmod p, \\ -1, & \text{if there is no integer } m \text{ such that } m^2 \equiv n \bmod p. \end{cases} \tag{B.4}$$

(If n is divisible by p it is sometimes useful to extend the definition by taking $\left(\frac{n}{p}\right) = 0$ in this case.) All numbers which are equivalent modulo p have the same Legendre symbol.

The Jacobi symbol $\left(\frac{n}{p}\right)$ is a generalization of the Legendre symbol. The prime p is replaced by any odd number m. It is defined as the product of the Legendre symbols for *all* the prime factors p of m (irrespective of whether these are distinct):

$$\left(\frac{n}{m}\right) = \prod_{p \in m} \left(\frac{n}{p}\right), \tag{B.5}$$

whence the denominator product rule

$$\left(\frac{n}{m_1 m_2}\right) = \left(\frac{n}{m_1}\right)\left(\frac{n}{m_2}\right). \tag{B.6}$$

But the Legendre symbols also obey a numerator product rule

$$\left(\frac{n_1 n_2}{p}\right) = \left(\frac{n_1}{p}\right)\left(\frac{n_2}{p}\right), \tag{B.7}$$

which therefore the Jacobi symbols inherit:

$$\left(\frac{n_1 n_2}{m}\right) = \left(\frac{n_1}{m}\right)\left(\frac{n_2}{m}\right). \tag{B.8}$$

The remaining properties required are the Gauss reciprocity rule for the Jacobi symbol with m and n both odd:

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{(m-1)(n-1)/4}, \tag{B.9}$$

and the value of the Jacobi symbol

$$\left(\frac{2}{m}\right) = (-1)^{(m^2-1)/8}. \tag{B.10}$$

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References

[1] V.I. Arnold and A. Avez, Ergodic Problems of Classical Mechanics, (Benjamin New York, 1968).
[2] J. Schwinger, Quantum Kinematics and Dynamics (Benjamin, New York, 1970).
[3] J. Schwinger, Proc. National Academy of Sciences 46 (1960) 570.
[4] F. Rannou, Astron. and Astrophys. 31 (1974) 289.
[5] M.V. Berry and N.L. Balazs J. Phys. A 12 (1979) 625.
[6] M.V. Berry, N.L. Balazs, M. Tabor and A. Voros, Ann. Phys. 122 (1979) 26.
[7] G. Casati, B.V. Chirikov, F.M. Izraelev and J. Ford, in Stochastic Behaviour in Classical and Quantum Hamiltonian Systems, Casati and Ford, eds. Springer Lecture notes in Physics No. 93, 334.
[8] H. Rademacher, Lectures on Elementary Number Theory (Blaisdell, Waltham, 1964).
(9) W. Sierpinski, Elementary Theory of Numbers, Math. Non. Pol. Acad. No. 42 (Warsaw, 1964).
[10] D.W. Robinson, Fibonacci Quarterly, April 1963, 29.
[11] D.D. Wall, Amer. Math. Monthly 67 (1960) 525.
[12] E.P. Wigner, Phys. Rev. 40 (1932) 749.
[13] G.A. Baker Jr., Phys. Rev. 108 (1958) 2198.
[14] M.V. Berry, Phil. Trans. 287 (1977) 237.
[15] S. Lang, Algebraic Number Theory (Addison-Wesley, New York, 1970).