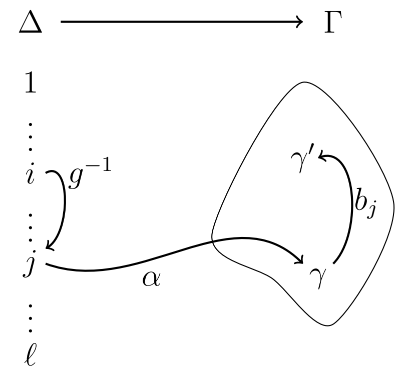

FIGURE 3. The product action.

6. The product action

Wreath products have another 'natural' action which we discuss here. As we shall see this action is often primitive.

Let H and K be groups acting on sets Δ and Γ respectively. Consider the wreath product $K \wr_\Delta H = B \rtimes H$ where $B = K^\Delta$. Let $\Omega := \Gamma^\Delta$, the set of functions from Δ to Γ. Define a function

$$\varphi : (K \wr_\Delta H) \times \Omega \to \Omega, \ \ ((b, g), \alpha) \mapsto \alpha^{(b,g)}$$

where

$$\alpha^{(b,g)} : \Delta \to \Gamma, \ \ \delta \mapsto (\delta^{g^{-1}}\alpha)^{(\delta^{g^{-1}})b}.$$

This definition is rather opaque! So let us consider the stituation where Δ is finite and we can identify it with the set $\{1, \dots, \ell\}$. Now we can think of B as a direct product of ℓ copies of K, and our definition of $\alpha^{(b,g)}$ becomes

$$\alpha^{(b,g)} : \Delta \to \Gamma, \ \ \delta \mapsto (i^{g^{-1}}\alpha)^{b_{ig^{-1}}}.$$

Now Figure 3 demonstrates what is going on – it turns out that the definition is rather natural.

We have still to check that the definition is really an action - to avoid confusion, I will do this only for the case where Δ is finite (so Δ can be taken to be $\{1, \dots, n\}$). Let $(b, g), (b', g') \in K \wr_\Delta H$ and $i \in \Delta$:

- $i\alpha^{(1,\dots,1,1)1} = (i^1\alpha)^1 = i\alpha$ as required.
- Observe that

$$\begin{aligned} i(\alpha^{(a_1,\dots,a_n)g})^{(c_1,\dots,c_n)h} &= (i^{h^{-1}}\alpha^{(a_1,\dots,a_n)g})^{c_{ih^{-1}}} \\ &= (i^{h^{-1}g^{-1}}\alpha)^{a_{ig^{-1}}c_{ih^{-1}}} \\ &= (i^{(gh)^{-1}}\alpha)^{(ac^{g^{-1}})_{i(gh)^{-1}}} \\ &= i\alpha^{(a_1,\dots,a_n)(c_1,\dots,c_n)^{g^{-1}}gh} \\ &= i\alpha^{(a_1,\dots,a_n)g(c_1,\dots,c_n)h}. \end{aligned}$$

Thus $K \wr_\Delta H$ acts on $\Omega = \Gamma^\Delta$, and this action is called the *product action* of the wreath product on Ω.

Example 17. Recall the group $G = \mathrm{Sym}(3) \wr \mathrm{Sym}(2)$ that we studied in Example 16. In that example we examined a subgroup of $\mathrm{Sym}(6)$ that was isomorphic to G and acted imprimitively on $[1, 6]$. In contrast here we will find a subgroup of $\mathrm{Sym}(9)$ that is isomorphic to G.

Recall that $G = B \rtimes \mathrm{Sym}(2)$ where $B \cong \mathrm{Sym}(3) \times \mathrm{Sym}(3)$. Thus we write

$$G = \{(b_1, b_2)h \mid k_1, k_2 \in \mathrm{Sym}(3), h \in \mathrm{Sym}(2)\}$$

and observe that an element $(k_1, k_2)h$ lies in B if and only if $h = 1$. Similarly $(k_1, k_2)h \notin B$ if and only if $h = g$, the unique non-trivial element of $\mathrm{Sym}(2)$.

Set $\Gamma := \{1, 2, 3\}$ and define

$$\Omega := \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \Gamma\}.$$

Observe that Ω is equal to the set of functions $\{1,2\} \to \{1,2,3\}$, a set of cardinality 9. Now the product action of G on Ω is given by

$$(\alpha_1, \alpha_2)^{(k_1,k_2)1} = (\alpha_1^{k_1}, \alpha_2^{k_2}) \text{ and } (\alpha_1, \alpha_2)^{(k_1,k_2)g} = (\alpha_2^{k_2}, \alpha_1^{k_1}).$$

The first of these corresponds to elements of B and it is easy enough to see that B acts transitively on Ω thus, in particular, so does G. Let us consider whether or not G acts primitively or not. Let us calculate the stabilizer of the point $(1,1)$:

$$G_{(1,1)} = \{(k_1,k_2)h \mid k_1,k_2 \in \langle (2,3)\rangle, h \in \mathrm{Sym}(2)\}.$$

Now consider the action of $G_{(1,1)}$ on Ω. It is easy enough to check that the orbits of this action are

$$\begin{aligned}&\{(1,1)\},\\ &\{(1,2),(1,3),(2,1)(3,1)\} \text{ and}\\ &\{(2,2),(2,3),(3,3),(3,2)\}.\end{aligned}$$

Since G is transitive, (E5.5) implies that, if G is imprimitive, then there is only one possible non-trivial G-congruence and it has the property that all blocks have size 3. On the other hand (E5.6) implies that the block containing $(1,1)$ is a union of orbits of the stabilizer $G_{(1,1)}$. We conclude that G acts primitively on Ω.

(E6.1) *Consider the product action of the group* $\mathrm{Sym}(2) \wr \mathrm{Sym}(3)$ *(on a set of size* 8*). Is this action primitive?*

Lemma 6.1. *Let H and K be groups acting on sets Δ and Γ respectively, where $|\Gamma| \geq 2$. Then the product action of $K \wr_\Delta H$ on $\Omega := \Gamma^\Delta$ is faithful if and only if the respective actions of H and K on Δ and Γ are faithful.*

Proof. Suppose that the respective actions of H and K on Δ and Γ are faithful, and suppose that for some $(b,g) \in K \wr_\Delta H$, $\alpha^{(b,g)} = \alpha$ for all $\alpha : \Delta \to \Gamma$. This implies that, for all $\delta \in \Delta$,

$$((\delta^{g^{-1}}\alpha)^{(\delta^{g^{-1}})b} = \delta\alpha.$$

Write γ for $\delta^{g^{-1}}$ and observe that then

$$(\gamma\alpha) = (\delta\alpha)^{((\delta^{g^{-1}})b)^{-1}}.$$

But now if γ and δ are distinct for some δ, then, since α can be any function from $\Delta \to \Gamma$ and $|\Gamma| \geq 2$, we have a contradiction. We conclude that $\gamma = \delta$ for all δ and, since H acts faithfully on Δ, this implies that $g = 1$.

Now since $\delta^{g^{-1}}\alpha$ can be any element of Γ, and K is faithful on Γ, we conclude that $(\delta)b = 1$ for all δ and the result follows.

(E6.2) *Prove the converse.*

$\square$

Lemma 6.2. *Suppose that G is a primitive subgroup of $\mathrm{Sym}(\Omega)$. Then G is regular if and only if, for some (and hence all) $\omega \in \Omega$, G_α is a proper subgroup of $N_G(G_\alpha)$.*

Proof. It is convenient to assume that $|\Omega| > 2$ so that, by Lemma 5.2, G is transitive and G_ω is maximal in G. (When $|\Omega| = 2$ the result is obvious.)

Fix $\omega \in \Omega$ and observe that, since G is transitive, G is regular if and only if G_ω is trivial. Thus if G is regular, then $N_G(G_\alpha) = G$ and G_α is a proper subgroup of $N_G(G_\alpha)$, as required.

On the other hand if G is not regular, then G_ω contains a non-trivial element g and, in particular, G_ω is not normal (since, otherwise, g would fix every element of Ω which is impossible). Thus $G_\Omega \leq N_G(G_\alpha) < G$. Now observe that, since G is primitive, G_α is maximal in G, and we conclude that $G_\Omega = N_G(G_\alpha)$, as required. $\square$

Proposition 6.3. *Suppose that H and K are nontrivial groups acting on the sets Δ and Γ respectively. Then the wreath product $K \wr_\Delta H$ is primitive in the product action on $\Omega := \Gamma^\Delta$ if and only if:*

(1) *K acts primitively but not regularly on Γ; and*
(2) *Δ is finite and H acts transitively on Δ.*

Proof. Suppose that (1) and (2) hold, and, without loss of generality, let $\Delta = \{1, \dots, \ell\}$. It is clear that the base group $B = \underbrace{H \times \cdots \times H}_{\ell}$ acts transitively on Ω, so the same is true of W.

Fix $\gamma \in \Gamma$. We take L to be the stabilizer of the constant element

$$\phi_\gamma : \Delta \to \Gamma, \delta \to \gamma.$$

Observe that

$$L = \{(b, h) \in W \mid b_i \in K_\gamma \text{ for all } i\}.$$

By Lemma 5.2 it is sufficient to show that L is maximal. Thus suppose that $L < M \leq W$; we will show that $M = W$.

Define

$$H_0 := \{(1, h) \mid h \in h\}.$$

Since $W = BH_0 = BL$ we have $M = (M \cap B)L$. Therefore $M \cap B > L \cap B$ and so, for some i_0, there exists $(b, 1) \in M \cap B$ with $b_{i_0} \notin K_\gamma$. Since K is primitive and not regular, Lemma 6.2 implies that $K_\gamma = N_K(K_\gamma)$ and so, for some $u \in K_\gamma$, we have $(b_{i_0})^{-1} u (b_{i_0}) \notin K_\gamma$. Consider the element

$$c := (1, \dots, 1, u, 1, \dots, 1) \in B$$

where the non-identity element is in the i_0-th position.

Then $d := [b, c] \in M \backslash L$ where $d_{i_0} = [b_{i_0}, u] \in K \backslash K_\gamma$ and $d_i = 1$ for all $i \neq i_0$. Since K is primitive, K_γ is maximal, and so $K = \langle K_\gamma, d_{i_0} \rangle$; therefore M contains the subgroup

$$B(i_0) := \{(b, 1) \in B \mid b_i = 1 \text{ for all } i \neq i_0\}.$$

Since $H_0 \leq M$ and H is transitive on Δ we conclude that $B(i) \leq L$ for all $i \in \Delta$. Since Δ is finite we conclude that $B = \prod_{i \in \Delta} B(i) \leq M$. Thus $M = BH_0 = W$ as required.

(E6.3) *Prove the converse.*

∎

(E6.4) *Let p be a prime, $\ell > 1$ any positive integer. Let*

$$C_p = \langle (1, 2, 3, \dots, p) \rangle$$

be a cyclic subgroup of order p in $\mathrm{Sym}(p)$, and consider the wreath product $G = C_p \wr \mathrm{Sym}(\ell)$ in the product action on a set of size p^ℓ. Prove that the action is transitive and imprimitive; calculate the order of the blocks of imprimitivity preserved by G; describe the setwise stabilizer of a block of imprimitivity.

The next result is analogous to Proposition 5.6, and deals with groups 'preserving a product structure'. Specifically a *product structure* on a set Ω is a bijection $\theta : \Omega \to \Gamma^\Delta$ where Γ and Δ are sets. If a group G acts on Ω, then this identification is a *G-product structure* if, for all $g \in G$, there exists $h \in \mathrm{Sym}(\Delta)$ such that,

$$\text{for all } \omega_1, \omega_2 \in \Omega \text{ and all } \delta \in \Delta, \omega_1(\delta^h) = \omega_2(\delta^h) \implies \omega_1^g(\delta) = \omega_2^g(\delta). \tag{7}$$

(To ease notation here and below, I identify Ω and $\theta(\Omega)$, thereby thinking of $\omega \in \Omega$ as a function $\Delta \to \Gamma$.) We will only consider product structures on finite sets Ω. In particular if $|\Omega| = n < \infty$, then we call the product structure *non-trivial* if $1 < |\Gamma|, |\Delta| < n$. If $\theta : \Omega \to \Gamma^\Delta$ is a product structure, and a group G acts on the set Ω, then we say that G *preserves the product structure* θ if θ is a G-product structure.

Proposition 6.4. *Let Ω be a finite set of order n. Suppose that $\theta : \Omega \to \Gamma^\Delta$ is a product structure, with $|\Gamma| = k$ and $|\Delta| = \ell$.*

(1) *θ is a G-product structure for a unique subgroup G of $\mathrm{Sym}(\Omega)$ that is isomorphic to $\mathrm{Sym}(k) \wr_\Delta \mathrm{Sym}(\ell)$;*
(2) *if θ is a H-product structure for some group $H \leq \mathrm{Sym}(\Omega)$, then $H \leq G$.*

Proof. Since $\mathrm{Sym}(\Gamma)$ and $\mathrm{Sym}(\Delta)$ act faithfully on Γ and Δ respectively, Lemma 6.1 implies that $G := \mathrm{Sym}(\Gamma) \wr \mathrm{Sym}(\Delta)$ acts faithfully on Γ^Δ in the product action. This action preserves the product structure associated with Γ^Δ since, for any $g = (f_1, \dots, f_\ell)h$ in G, the definition of the product action implies that

$$\omega_1(\delta^{h^{-1}}) = \omega_2(\delta^{h^{-1}}) \Longrightarrow \omega_1^g(\delta) = \omega_2^g(\delta).$$

We obtain an embedding of $G = \mathrm{Sym}(k) \wr_\Delta \mathrm{Sym}(\ell)$ in $\mathrm{Sym}(\Omega) = \mathrm{Sym}(\Gamma^\Delta)$, as required.

To complete the proof, we must show that if θ is a J-product structure for some group $J \leq \mathrm{Sym}[\Omega)$, then J is a subgroup of G (this will yield (ii) as well as the uniqueness part of (i)). Suppose that $j \in J$ and let h^{-1} be the associated permutation of $\mathrm{Sym}(\Omega)$ satisfying (7).

Then, for each $\delta \in \Delta$, (7) implies that we have an associated element $g_\delta \in \mathrm{Sym}(\Gamma)$ such that, for any $\omega \in \Omega$ and $\delta \in \Delta$,

$$\omega^j(\delta) = (\omega(\delta^{h^{-1}})^{g_\delta}.$$

In other words, for all $\omega \in \Omega$,

$$\omega^j = \omega^{(g_1, \dots, g_\ell)h}$$

where $(g_1, \dots, g_\ell)h \in G$ and we use the product action of G on Ω. We are done. $\square$

As usual we have a categorical restatement, as follows.

(E6.5) *Our category is called* ***ProductStruct***

Objects*: An object is a pair (Ω, θ) where Ω is a finite set and $\theta : \Omega \to \Gamma^\Delta$ is a product structure. Equivalently an object is a direct product $\underbrace{\Gamma \times \cdots \times \Gamma}_{\ell}$ where Γ is a finite set of size k.*

Arrows*: An arrow is a pair (g, h) where $g : \Omega \to \Omega$ and $h : \Delta \to \Delta$ are functions, and we require that (7) holds.*

1. *Prove that* ***ProductStruct*** *is a category.*
2. *Prove that if X is an object in* ***ProductStruct****, then $\mathrm{Aut}(X) \cong \mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$.*
3. *Prove that if G acts on $X = \Gamma^\ell$ as an object from* ***ProductStruct****, then $\sim$ is a G-product structure, and conversely.*

The next proposition is a refinement of Proposition 5.7, making use of the previous two propositions.

Proposition 6.5. *Let $H \leq \mathrm{Sym}(\Omega)$ where $|\Omega| < \infty$. One of the following holds:*

1. *H is intransitive and $H \leq \mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$ for some $1 < k < n$;*
2. *H is transitive and imprimitive and $H \leq \mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ for some $1 < k, l < n$ with $n = kl$;*
3. *H is primitive, preserves a non-trivial product structure, and $H \leq \mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ for some $1 < l < n$, $2 < k < n$ with $n = k^l$;*
4. *H is primitive and does not preserve a non-trivial product structure.*[25]

Proof. We apply Proposition 5.7 and are able to assume that H is primitive. If $\theta : \Omega \to \Gamma\Delta$ is a H-product structure, then Proposition 6.4 implies that H is a subgroup of a group $\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ inside $\mathrm{Sym}(n)$, with $n = k^l$; moreover, since the product structure is non-trivial, we have $1 < l < n$, $1 < k < n$ with $n = k^l$. If $k = 2$, then $\mathrm{Sym}(2)$ acts regularly on the associated set of order 2 and Proposition 6.3 implies that $\mathrm{Sym}(2) \wr \mathrm{Sym}(l)$ is imprimitive, which is a contradiction. The result follows. $\square$

(E6.6) *Let Ω be a finite set of order n and let $X = (\Omega, \theta)$ (resp. $Y = (\Omega, \theta')$) be an object from* ***ProductStruct****. Let $H = \mathrm{Aut}(X)$ (resp. $K = \mathrm{Aut}(Y)$) be subgroups of $\mathrm{Sym}(n)$. When is H maximal? Are H and K conjugate? How many conjugacy classes of subgroups isomorphic to H does $\mathrm{Sym}(n)$ contain? Describe the intersection of H and $\mathrm{Alt}(n)$.*

To classify the subgroups of $\mathrm{Sym}(\Omega)$, then, we need to study those primitive groups that do not preserve a product structure. To do this we change our approach slightly, and turn our attention to the *socle* of a permutation group.

[25] Peter Cameron uses the notation *basic primitive group* to refer to a permutation group that is primitive and does not preserve a non-trivial product structure.