19. Answers to exercises

Section 1.

(E1.1) Prove that if G is a finite simple abelian group, then $G \cong C_p$, the cyclic subgroup of order p, where p is a prime.

> **Answer.** In an abelian group, any subgroup is normal. Thus a simple abelian group has no non-trivial proper subgroups. Suppose that G is non-cyclic and let $g \in G\backslash\{1\}$. Then, by supposition, $\langle g \rangle$ is proper and non-trivial and we have a contradiction. Suppose that $G = \langle g \rangle$ where $o(g) = st$ where s and t are positive integers greater than 1. Then $\langle g^s \rangle$ is a proper non-trivial subgroup of G and we have a contradiction. The result follows.

Section 2.

(E2.1) Prove that **Set**, **Pfn**, **Grp**, **Top** and $\mathbf{Vect}_K$ are categories.

> **Answer.** Objects in each of these categories are structured sets. Let X be an object in the respective category, with Ω the underlying set. Since function composition is associative, the composition of arrows in these categories is associative and (C1) is satisfied. In addition the identity map $1 : \Omega \to \Omega$ 'carries' an arrow $X \to X$ that we call 1_X, and which satisfies (C2).

(E2.2) Which categories in Example 1 are (full) subcategories of some other category in Example 1?

> **Answer.**
> - **Pfn** is a non-full subcategory of **Set**.
> - **AGrp** is a full subcategory of **Grp**.
> - **Field** is a full subcategory of **Ring**.
> - $\mathbf{Vect}_K$ is a full subcategory of both $\mathbf{Mod} - K$ and $R - \mathbf{Mod}$. (In fact these three categories are equivalent.)

(E2.3) Complete the definition of **Digraph** and prove that it is a category.

> **Answer.** **Objects**: An object is a pair (V, E) where V is a set and E is a set of ordered pairs with entries from V. **Arrows**: An arrow
> $$(V, E) \xrightarrow{f} (V', E')$$
> is just a function $V \to V'$ such that
> $$(e_1, e_2) \in E \implies (f(e_1), f(e_2)) \in E'.$$
> Once again, since objects in **Digraph** are structured sets, (C1) and (C2) follow from the associativity of function composition, and the presence of an identity map on sets.

(E2.4) Give the 'right' definition of the category **Graph** corresponding to graphs that are not necessarily simple, i.e. which may have multiple edges between vertices.

Answer. **Objects**: An object is a triple (V, E, ι) where V and E are sets and $\iota : E \to 2^V$ is a function such that, for all $e \in E$, $\iota(e)$ has cardinality at most 2.

Arrows: An arrow

$$(V, E, \iota) \xrightarrow{f} (V', E', \iota')$$

is a pair of functions, $f_V : V \to V$ and $f_E : E \to E$, such that, for all $einE$,

$$\iota'(f_E(e)) = f_V(\iota(e)).$$

Note that the expression on the right hand side refers to the obvious induced map $f_V : 2^V \to 2^{V'}$.

(When we come to study isomorphisms we shall see why we cannot just extend the definition of **SimpleGraph** to this context.)

(E2.5) Prove that $\mathbf{VectS}_K$ and $\mathbf{IVect}_{\mathbb{R}}$ are categories.

Answer. This is the same as previous answers for categories of structured sets.

(E2.6) Prove that **G-Set** is a category.

Answer. In this category, an arrow is a pair of functions. For a G-set (G, Ω, φ) we take the identity arrow to be $(1_G, 1_\Omega)$.

We need to check that the partial composition has the correct range of definition. Suppose we have two arrows as follows:

$$\begin{tikzcd} G \times \Omega \arrow[r, "\phi"] \arrow[d, "{(\alpha,\beta)}"'] & \Omega \arrow[d, "\beta"] \\ H \times \Gamma \arrow[r, "\psi"] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} H \times \Gamma \arrow[r, "\psi"] \arrow[d, "{(\gamma,\delta)}"'] & \Gamma \arrow[d, "\delta"] \\ J \times \Lambda \arrow[r, "\xi"] & \Lambda \end{tikzcd}$$

By definition, these two diagrams commute, hence if we consider the concatenated diagram -

$$\begin{tikzcd} G \times \Omega \arrow[r, "\phi"] \arrow[d, "{(\alpha,\beta)}"'] & \Omega \arrow[d, "\beta"] \\ H \times \Gamma \arrow[r, "\psi"] \arrow[d, "{(\gamma,\delta)}"'] & \Gamma \arrow[d, "\delta"] \\ J \times \Lambda \arrow[r, "\xi"] & \Lambda \end{tikzcd}$$

- then, since the two small rectangles commute, the large rectangle commutes. Now the pair $(\alpha\gamma, \beta\delta)$ is a well-defined arrow in $\mathbf{G-Set}$, as required. Now (C1) and (C2) follow automatically.

(E2.7) Prove that Example 5 yields a category.

Answer. Clearly the partial composition is well-defined. For an object A, we define the identity arrow as follows:

$$1_A : A \times A \to \mathbb{R}, (a, b) \mapsto \begin{cases} 1, & \text{if } a = b; \\ 0, & \text{otherwise.} \end{cases}$$

To check (C1), suppose that the following arrows,

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D.$$

Now observe that

$$\begin{aligned}(fg)h : A \times D &\to \mathbb{R}\\ (a,d) &\mapsto \sum_{c\in C} fg(a,c)h(c,d)\\ &= \sum_{c\in C}\left((\sum_{b\in B} f(a,b)g(b,c)\right) h(c,d)\\ &= \sum_{b\in B} f(a,b)\left(\sum_{c\in C} g(b,c)h(c,d)\right)\\ &= \sum_{b\in B} f(a,b)gh(b,d).\end{aligned}$$

We conclude that $(fg)h = f(gh)$ as required. For (C2) consider the following arrows,

$$A \xrightarrow{1_A} A \xrightarrow{f} B \xrightarrow{1_B} B.$$

Now observe that

$$1_A f : A \times B \to \mathbb{R},\ (a,b) \mapsto \sum_{a'\in A} 1_A(a,a')f(a',b) = f(a,b).$$

Thus $1_A f = f$ and, similarly, $f1_B = f$ and we are done.

(E2.8) What is $\mathrm{Aut}(X)$ when X is an object in **Top**?

Answer. The set of homeomorphisms of X.

(E2.9) Show that

(1) an isomorphism is monic and epic;
(2) if $\mathbf{C}$ is a category of structured sets (so that each arrow is carried by a total function between the carriers of the two objects), then

$$\text{injective} \Longrightarrow \text{monic, and surjective} \Longrightarrow \text{epic}.$$

(3) Show that epic does not imply surjective in **Ring**.
(4) Show that bijective does not imply isomorphism in **Top**.

Answer.

(1) Let $A \longrightarrow^f B$ be an isomorphism and suppose that $X \underset{h}{\overset{g}{\rightrightarrows}} A$ are arrows such that $gf = hf$. Since f is an isomorphism, there exists $e : B \to A$ such that $fe = 1_B : B \to B$. Then

$$\begin{aligned} gf = hf &\Longrightarrow (gf)e = (hf)e\\ &\Longrightarrow g(fe) = h(fe)\\ &\Longrightarrow g1_B = h1_B\\ &\Longrightarrow g = h.\end{aligned}$$

We conclude that f is monic. The proof that f is epic is similar.

(2) In this answer we abuse notation by conflating the objects of **C** with their underlying sets. Suppose

$$A \xrightarrow{f} B$$

is injective, with A, B objects in **C**. Suppose that $X \underset{h}{\overset{g}{\rightrightarrows}} A$ are arrows such that $gf = hf$. Thus, for all $x \in X$,

$$\begin{aligned} x(gf) = x(hf) &\Longrightarrow (xg)f = (xh)f \\ &\Longrightarrow xg = xh \text{ (by injectivity)} \\ &\Longrightarrow g = h. \end{aligned}$$

We conclude that f is monic. The other implication is similar.

(3) It is enough to show that the injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic. The proof is left for the reader.

(4) This is equivalent to showing the existence of a topological space X that admits continuous bijections that do not have continuous inverse. This is an exercise in analysis, a solution to which can be found here:
`http://math.stackexchange.com/questions/68800/functions-which-are-continuous-but-not-bicontinuous`

(E2.10) What are automorphisms in **Graph**? Can you see why one needs a different definition in this context?

Answer. We gave the definition of an arrow in **Graph** for (E2.4). One immediately obtains that an isomorphism

$$(V, E, \iota) \xrightarrow{f} (V', E', \iota')$$

is a pair of bijections, $f_V : V \to V$ and $f_E : E \to E$, such that, for all $e \in E$,

$$\iota'(f_E(e)) = f_V(\iota(e)).$$

If one tries to use the definition of arrow given for **SimpleGraph** in the category **Graph**, then one obtains (for instance) that the following two graphs are isomorphic (and it is clear that this is not what we want!):