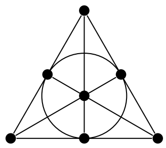

FIGURE 2. The Fano plane.

4. Projective Space

One can approach the study of projective spaces from a number of different angles. Since we will be primarily interested in projective spaces of finite dimension over finite fields, our spaces themselves are finite and so fall most naturally (it seems to me) into the sphere of combinatorics. Our approach is, therefore, combinatorial.

4.1. **Incidence structures.** An *incidence structure*, $(P_1, \dots, P_\ell, I)$, consists of a finite number of sets $P_1, \dots, P_\ell$ along with a set $I \subseteq P_1 \times \dots \times P_\ell$. Elements from set P_i are said to *have type* i. If I contains an element $(p_1, \dots, p_\ell)$ then, for $1 \leq i, j \leq \ell$, we say that p_i *is incident with* p_j. The incidence structure is called *finite* if $P_1, \dots, P_\ell$ are all finite.

This definition is extremely general! Incidence structures are generally only studied subject to extra axioms.

Example 5. An **abstract** *projective plane* is an incidence structure (P, L, I) such that

(1) any two elements of P are incident with a unique element of L;
(2) any two elements of L are incident with a unique element of P.
(3) there are four elements of P such that no element of L is incident with more than two of them.

We call elements of P *points* and elements of L *lines*. The third axiom listed above ('presence of a quadrangle') is there simply to eliminate some degenerate examples.

If P is finite it is easy to see that L is also finite and, in fact, that $|P| = |L|$. The smallest finite projective plane has $|P| = 7$. It is called the *Fano plane* and is represented in Figure 2; in this representation elements of P are drawn as points, and elements of L are drawn as lines (in six cases) or as a circle; each element of L is incident with three elements of P (and vice versa).

f: fano

Let V be a vector space of dimension n over a field k. We define *projective space* $\mathrm{PG}(V)$, or $\mathrm{PG}_{n-1}(k)$, to be the incidence structure $(V_1, \dots, V_{n-1}, I)$ where, for $i = 1, \dots, n-1$, V_i is the set of subspaces of V of dimension i and

$$I := \{(v_1, \dots, v_{n-1}) \in V_1 \times \dots \times V_{n-1} \mid v_1 < v_2 < \dots < v_{n-1}\}.$$

In other words two subspaces are incident if and only if one is contained in the other. This fact allows us to relax language a little: we say things like 'v_1 lies on v_2', or 'v_2 contains v_1' when we really mean that 'v_1 and v_2 are incident'.

A subspace $U \in V_i$ is said to have *projective dimension*, $\mathrm{pdim}(U)$, equal to $i - 1$ and we call elements of V_1 'points', elements of V_2 'lines', and elements of V_{n-1} 'hyperplanes'. If k is finite of order q, then we sometimes write $\mathrm{PG}_{n-1}(q)$ for $\mathrm{PG}_{n-1}(k)$. The incidence structure $\mathrm{PG}_2(q)$ is called the *Desarguesian projective plane of order* q.

e: fano

(E34) *Show that* $\mathrm{PG}_2(2)$ *and the Fano plane are the same incidence structure. (We would do better to write that "*$\mathrm{PG}_2(2)$ *and the Fano plane are isomorphic as incidence structures", but we have not yet defined what we mean by isomorphism.)*

esargues **(E35*)** *Show that, for any prime power q, $\mathrm{PG}_2(q)$ is an abstract projective plane.*

4.2. **Some counting.** We are interested in calculating the order of $V_1, \dots, V_{n-1}$ in $\mathrm{PG}_{n-1}(q)$. To do this it is convenient to introduce *Gaussian coefficients*. Let q be a prime power, m and n positive integers. Then define

e: gco

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{m-1})}{(q^m-1)(q^m-q)\cdots(q^m-q^{m-1})}. \tag{3}$$

Lemma 14. (1) *The number of subspaces of projective dimension $m-1$ in $\mathrm{PG}_{n-1}(q)$ is $\begin{bmatrix} n \\ m \end{bmatrix}_q$. In particular, the number of points in $\mathrm{PG}_{n-1}(q)$ is $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n-1}{q-1}$.*

(2) *The number of subspaces of projective dimension $m-1$ containing a subspace of projective dimension $l-1$ in $\mathrm{PG}_{n-1}(q)$ is $\begin{bmatrix} n-l \\ m-l \end{bmatrix}_q$.*

Proof. Let V be an n-dimensional vector space over $k = \mathbb{F}_q$. To prove (a) we count the number of linearly independent m-tuples of vectors in V. The first entry in the m-tuple can be chosen to be any non-zero vector, there are q^n-1 of these; the second must lie outside the span of the first, so there are q^n-q choices for this, then q^n-q^2 for the third and so on. We conclude that the numerator of the right-hand side of (3) corresponds to the number of linearly independent m-tuples of vectors. Now the result is completed by observing (using the same reasoning) that the denominator of the right-hand side of (3) corresponds to the number of linearly independent m-tuples of vectors all lying inside any given m-dimensional subspace of V.

(E36) *Prove (b).*

$\square$

Gaussian coefficients have properties resembling those of binomial coefficients, to which they tends as $q \to 1$.

(E37) *Prove that* $\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix}_q = \binom{n}{m}$.

(E38) *Prove that*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q + q^{n-m+1} \begin{bmatrix} n \\ m-1 \end{bmatrix}_q = \begin{bmatrix} n+1 \\ m \end{bmatrix}_q.$$

4.3. **Collineations and the fundamental theorem.** Let V be an n-dimensional vector space over a field k.

Let $\mathcal{I}_1 = (P_1^1, \dots, P_\ell^1, I^1), \mathcal{I}_2 = (P_1^2, \dots, P_\ell^2, I^2)$ be incidence structures. An *isomorphism* $\mathcal{F} : \mathcal{I}_1 \to \mathcal{I}_2$ is a set of ℓ bijections $\phi_i : P_i^1 \to P_i^2$ such that

$$(p_1, \dots, p_\ell) \in I^1 \iff (\phi(p_1), \dots, \phi(p_\ell)) \in I^2.$$

Note that we only define isomorphisms between incidence structures having the same number of types. An isomorphism $\mathcal{F} : \mathcal{I}_1 \to \mathcal{I}_1$ is called a *collineation* (such a thing would usually be called an *automorphism*, but the terminology is classical and well-established). The set of all collineations of an incidence structure $\mathcal{I}$ is a group under composition and is denoted $\mathrm{Aut}(\mathcal{I})$.

Example 6. The action of $\Gamma L(V)$ on the set V extends naturally to an action on the set of subspaces of V: if $v_1, \dots, v_k \in V$ and $g \in \Gamma\mathrm{L}(V)$, then we write $U = \langle v_1, \dots, v_n \rangle$ and define

$$U^g = \langle v_1^g, \dots, v_k^g \rangle.$$

(E39*) *Prove that this action is well-defined, and that the action preserves the incidence relation for $\mathrm{PG}(V)$.*

This exercise implies that we have a homomorphism $\phi : \Gamma\mathrm{L}(V) \to \mathrm{Aut}(\mathrm{PG}(V))$. The next result asserts that this homomorphism is also surjective, i.e. all collineations of $\mathrm{PG}(V)$ are induced by a semilinear transformation.

Theorem 15. *(Fundamental theorem of projective geometry) If* $\dim V \geq 3$*, then* $\mathrm{Im}(\phi) = \mathrm{Aut}(\mathrm{PG}(V))$.

Proof. This is omitted. See [Cam, Chapter 1] or [Tay92]. □

The first isomorphism theorem of group theory implies, then, that

$$\mathrm{Aut}(\mathrm{PG}(V)) \cong \Gamma\mathrm{L}(V)/\ker(\phi).$$

e: K **(E40)** *Prove that* $\ker(\phi) = \{\alpha I \in \mathrm{GL}(V) \mid \alpha \in k\}$.

Let us write K for $\ker(\phi)$ and observe that K is just the group of scalar transformations in $\Gamma L(V)$. Note that K is actually a subgroup of $\mathrm{GL}(V)$.

We now define three new groups in terms of this subgroup K.

(1) $\mathrm{P}\Gamma\mathrm{L}(V) := \Gamma\mathrm{L}(V)/K$;
(2) $\mathrm{PGL}(V) := \mathrm{GL}(V)/K$;
(3) $\mathrm{PSL}(V) := \mathrm{SL}(V)/(K \cap \mathrm{SL}(V))$.

For $X \in \{\Gamma, G, S\}$ we write $\mathrm{P}XL_n(k)$ as a synonym for $\mathrm{P}XL(V)$.

Observe that the Fundamental theorem of projective geometry could be expressed as follows: If $\dim V \geq 3$, then

$$\mathrm{P}\Gamma\mathrm{L}(k) = \mathrm{Aut}(\mathrm{PG}(V)).$$

In particular the three groups just defined all act faithfully on $\mathrm{PG}(V)$.

(E41) *Prove that K is central in $\mathrm{GL}(V)$. Can you characterize those fields k and those vector spaces V for which K is central in $\Gamma\mathrm{L}(V)$?*

(E42*) *Prove that*

$$|\mathrm{PGL}_n(\mathbb{R}) : \mathrm{PSL}_n(\mathbb{R})| = \begin{cases} 1, & \text{if } n \text{ is odd;} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

4.4. **Duality.** Let $\mathcal{I}_1 = (P_1^1, \ldots, P_\ell^1, I^1), \mathcal{I}_2 = (P_1^2, \ldots, P_\ell^2, I^2)$ be incidence structures. A *weak isomorphism* $\mathcal{F} : \mathcal{I}_1 \to \mathcal{I}_2$ is a permutation $\pi \in S_k$, and set of ℓ bijections $\phi_i : P_i^1 \to P_{i\pi}^2$ such that

$$(p_1, \ldots, p_\ell) \in I^1 \iff (\phi(p_{1\pi^{-1}}), \ldots, \phi(p_{\ell\pi^{-1}})) \in I^2.$$

In other words, with this weaker definition, we allow the types of objects to be jumbled. Of course an isomorphism is a weak isomorphism. The set of all weak collineations of an incidence structure $\mathcal{I}$ is a group under composition and is denoted $\mathrm{WAut}(\mathcal{I})$.

Let V be a vector space over a field k of dimension $n \geq 3$. Then we claim that there are weak isomorphisms of $\mathrm{PG}(V)$ that are not isomorphisms. To see this, let $V^* = \mathrm{Hom}(V, k)$, the dual space of V.

(E43) *V^* is a vector space over k of dimension n.*

For $U \leq V$, define the *annihilator of* U, given by

$$U^\dagger := \{f \in V^* \mid uf = 0 \text{ for all } u \in U\}.$$

(E44) *$U \mapsto U^\dagger$ is a bijection between the subspaces of V and the subspaces of V^*.*

e: ff **(E45)** *$U_1 \leq U_2$ if and only if $U_1^\dagger \geq U_2^\dagger$.*

(E46) *If $U \leq V$, then* $\dim(U^\dagger) = n - \dim(U)$ *and* $\mathrm{pdim}(U^\dagger) = n - 2 - \mathrm{pdim}(U)$

Since all vector spaces over k of dimension n are mutually isomorphic, it follows that $V^* \cong V$ and so the function $V \to V^*$, $U \mapsto U^\dagger$ can be thought of as a map from V to V; indeed by (E45) we have a function $\mathrm{PG}(V) \to \mathrm{PG}(V)$.

(E47) *Prove that $U \to U^\dagger$ is a weak automorphism of* $\mathrm{PG}(V)$*.*

A weak automorphism $\mathrm{PG}(V) \to \mathrm{PG}(V)$ with the property that a subspace of dimension d is mapped to a subspace of dimension $n-d$ is called a *duality*. Clearly $U \to U^\dagger$ is a duality. In particular, if $n \geq 3$, the function $U \to U^\dagger$ cannot be an automorphism of $\mathrm{PG}(V)$.

und dual

Proposition 16. *If $n \geq 3$, and Δ is a duality of $\mathrm{PG}(V)$, then $\Delta = st^{-1}$ where s is induced by a semilinear automorphism $V \to V^*$ and t is the annihilator map, $U \to U^\dagger$.*

Proof. Since Δ is a duality, the map Δt is an isomorphism $\mathrm{PG}(V) \to \mathrm{PG}(V*)$. Since $V \cong V^*$, the Fundamental Theorem of Projective Geometry implies that it is induced by a semilinear isomorphism $V \to V^*$. □

e: waut

s: aps

(E48*) *Prove that, for $n \geq 3$,* $\mathrm{WAut}(\mathrm{PG}_n(q))$ *contains* $\mathrm{Aut}(\mathrm{PG}_n(q))$ *as an index 2 subgroup. Can you say any more about the structure of* $\mathrm{WAut}(\mathrm{PG}_n(q))$*?*

4.5. **Abstract projective space.** It turns out that one can characterize the geometric properties of $\mathrm{PG}_{n-1}(q)$ rather straightforwardly. For $\mathcal{I} = (P_1, P_2, I)$ a finite incidence structure of points and lines we define the following three properties.

(APS1) Two points lie on a unique line.
(APS2) A line meeting two sides of a triangle, not at a vertex, meets the third also.[4]
(APS3) A line contains at least two points.
(APS4) A line contains at least three points.

An incidence structure satisfying (APS1) to (APS3) is called an *abstract projective space*. If it satisfes (APS4) then it is a *thick abstract projective space*.

(E49) $\mathrm{PG}_{n-1}(q)$ *is a thick abstract projective space.*

The key theorem here is the following which we will not prove.

Theorem 17. (Veblen-Young) *A finite thick abstract projective space $\mathcal{I} = (P_1, P_2, I)$ with $1 < |P_1|, |P_2|$ is either*

- *a projective plane, or*
- *isomorphic to* $\mathrm{PG}_{n-1}(q)$ *for some n and q.*

Note that the condition $1 < |P_1|, |P_2|$ is only present to eliminate some obvious and uninteresting degeneracies. The Veblen-Young theorem reduces the question of classifying the finite thick abstract projective spaces to that of classifying the finite abstract projective planes. Unfortunately the latter is a very difficult project! For instance there are many finite abstract projective planes other than $\mathrm{PG}_2(q)$.

We remark finally that the ordinary triangle is an example of an abstract projective space that is not thick. For this reason projective planes are sometimes thought of as 'generalized triangles'. This terminology will assume more significance when we come to consider polar spaces.

5. Linear groups acting on projective space

In this section V is an n-dimensional vector space over a field k. In this section we investigate the action of $\mathrm{PGL}(V)$ (or, equivalently, of $\mathrm{GL}(V)$) on $\mathrm{PG}(V)$. You will see that we gradually put together the pieces to apply Iwasawa's criterion to $\mathrm{PSL}(V)$.

[4] We are using descriptive language here to save pain. It is hopefully clear what we mean by a 'triangle' and 'meeting'...

5.1. **Transitivity properties.**

l: pgl **Lemma 18.** *If $n \geq 2$, then* $\mathrm{PGL}(V)$ *acts 2-transitively on the set of points of* $\mathrm{PG}(V)$*.*

Proof. Let (v_1, v_2) and (w_1, w_2) be pairs of linearly independent vectors in V. Extend both pairs to bases: $B_V := \{v_1, \dots, v_n\}$ and $B_W\{w_1, \dots, w_n\}$. For a vector $v \in V$ we write $(v)_V$ for the coordinate vector of v with respect to the basis B_V.

This means that $(v_i)_v = e_i = (\underbrace{0 \cdots 0}_{i-1} 1 \underbrace{0 \cdots 0}_{n-i})$. Now define the following matrix

$$A = \begin{bmatrix} (w_1)_V \\ (w_2)_V \\ \vdots \\ (w_n)_v \end{bmatrix}$$

and note that $(v_i)_V \cdot A = (w_i)_V$. The result follows. □

In fact Lemma 18 can be deduced from a stronger result which we leave as an exercise. Recall that if a group acts transitively on a set with trivial stabilizers, then the action is called *regular* or *sharply transitive.*

e: reg **(E50*)** *A tuple of $n+1$ points in $\mathrm{PG}_{n-1}(F)$ (i.e. a tuple of $n+1$ lines in V) is said to be* special *if no n of its entries lie in a hyperplane. Write Σ_V for the set of special tuples. Prove that the action of $\mathrm{PGL}(V)$ on Σ_V is regular.*

e: 2t **(E51*)** *Prove that $\mathrm{PSL}_n(k)$ is 2-transitive on the points of $\mathrm{PG}_{n-1}(k)$. Prove, furthermore, that $\mathrm{PSL}_n(k)$ is 3-transitive if and only if $n = 2$ and every element of k is a square.*

abilizer **Lemma 19.** *Let $G = \mathrm{SL}_n(k)$ and $\omega \in \Omega$, the set of points of $\mathrm{PG}(V)$. Then*

$$G_\omega \cong Q \rtimes GL_{n-1}(k)$$

where Q is an abelian group isomorphic to the additive group $(k^{n-1}, +)$.

The proof below sheds more light on the groups Q and G_ω.

Proof. Since G acts transitively on the set of points of $\mathrm{PG}(V)$, all stabilizers are isomorphic. We set $\{e_1, \dots, e_n\}$ to be the basis of elementary vectors and observe that the stabilizer of $\langle e_n \rangle$ is

e: gln (4)
$$G_{\langle e_n \rangle} = \left\{ g := \left(\begin{array}{ccc:c} & & & a_1 \\ & A & & \vdots \\ & & & a_{n-1} \\ \hdashline 0 & \cdots & 0 & a^{-1} \end{array} \right) \;\middle|\; \begin{array}{l} a_1, \dots, a_{n-1} \in k, a \in k^*, \\ A \in \mathrm{GL}_{n-1}(k), \\ a = \det(A) \end{array} \right\}.$$

Now there is a natural epimomorphism

$$G_{\langle e_n \rangle} \to \mathrm{GL}_{n-1}(q), g \mapsto A,$$

and the kernel of this map is the group

e: Q (5)
$$Q := \left\{ g := \left(\begin{array}{ccc:c} & & & a_1 \\ & I & & \vdots \\ & & & a_{n-1} \\ \hdashline 0 & \cdots & 0 & 1 \end{array} \right) \mid a_1, \dots, a_{n-1} \in k \right\}$$

which is clearly isomorphic to $(k^{n-1}, +)$. Thus $G_{\langle e_n \rangle}$ is an extension of Q by $\mathrm{GL}_{n-1}(q)$.

(E52*) *Prove that this extension is split.*

□

5.2. **Transvections and generation.** The motivation for this subsection is to establish that $\mathrm{SL}(V) = \langle Q^g \mid g \in G \rangle$ where Q is the subgroup of $\mathrm{GL}(V)$ defined in Lemma 19.[5]

A *transvection* on V is an element $t \in \mathrm{GL}(V)$ such that

- $\mathrm{r}k(t - I) = 1$;
- $(t - I)^2 = 0$.

We define

- the *axis* of t to be $\ker(t - I)$;
- the *centre* of t to be $\mathrm{Im}(t - I)$.

Notice that the axis of t is a hyperplane in V, while the centre is a 1-dimensional subspace of that hyperplane.

ans conj **Lemma 20.** *All transvections lie in* $\mathrm{SL}_n(q)$ *and are conjugate in* $\mathrm{GL}_n(q)$.

Proof. We choose a basis for V as follows:

- $v_n \in \mathrm{Im}(t - I)$;
- $v_2, \dots, v_{n-1}$ are chosen so that $\langle v_2, \dots, v_n \rangle$ is the axis of t;
- v_1 such that $v_n = v_1^{t-I}$.

Then it is easy to see that the matrix of t with respect to this matrix is

e: T (6)
$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

(Recall that we are acting on the right.) The result follows. □

By comparing (5) and (6) it is clear that the group Q in Lemma 19 contains a transvection.

nj trans **(E53*)** *Prove that if $n \geq 3$, then $\mathrm{SL}_n(k)$ contains a unique conjugacy class of transvections. Prove that if $n = 2$, then $\mathrm{SL}_n(k)$ contains one or two conjugacy classes of transvections. Can you characterise when $\mathrm{SL}_n(k)$ contains two conjugacy classes, and describe how the subgroup Q intersects each class? (In particular you should show that each class has non-empty intersection with Q.)*

generate **Lemma 21.** *If $n \geq 2$, then $\mathrm{SL}_n(k)$ is generated by the set of all transvections.*

Proof. Let $G = \mathrm{SL}_n(k)$ and define

$$D := \langle t \mid t \text{ is a transvection} \rangle.$$

We proceed by induction on n.

Let $n = 2$ and consider the group

$$Q := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in k \right\}.$$

(This is the same group Q which was defined in the proof of Lemma 19, and is normal in $G_{\langle e_2 \rangle}$.) Observe that all non-identity elements of Q are transvections. Furthermore it is easy to see that Q is transitive on the set of 1-subspaces of V that are distinct from $\langle e_2 \rangle$. Since we can easily find a transvection that does not fix $\langle e_2 \rangle$, we conclude that D is 2-transitive on the points of $\mathrm{PG}_{n-1}(k)$.

[5]Refer to Lemma 3, Iwasawa's criterion, to see why we would want to know this.

Thus we will be done if we can show that the stabilizer in G of a pair of distinct points of $\mathrm{PG}_{n-1}(k)$ is equal to the stabilizer in D of that pair. One can calculate directly that

$$G_{(\langle e_1\rangle,\langle e_2\rangle)} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in k^* \right\}.$$

Now we can write elements of this group as products of transvections as follows:

e: psl2

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a-a^2 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}. \tag{7}$$

We conclude that $\mathrm{SL}_2(k)$ is generated by transvections.

Now assume that $n > 2$ and the result holds for $n-1$. Let v_1 and v_n be linearly independent vectors in V. We can extend to a basis $\{v_1, v_2, \ldots, v_n\}$ and now observe that the matrix (6) is a transvection which maps $\langle v_n\rangle$ to $\langle v_1\rangle$. Thus D is transitive on the set of points of $\mathrm{PG}_{n-1}(k)$ (that is, on the set of 1-spaces of V).

It is sufficient, then, to show that $G_{\langle v_n\rangle} = D_{\langle v_n\rangle}$. (Recall that the structure of $G_{\langle v_n\rangle}$ is given in Lemma 19.) Now one can check that every tranvection on the vector space $V/\langle v_n\rangle$ is induced by a transvection on V that fixes v_n. Thus, by induction, the set of transvection on V that fix v_n generate the group $\mathrm{SL}_{n-1}(k)$ on this quotient space.

Thus if $g \in G_{\langle v_n\rangle}$ has the from (4) then, by multiplying g by transvections we obtain an element

$$h := \begin{pmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 0 \\ a_1 & a_2 & \cdots & a_{n-1} & a \end{pmatrix}$$

where $a, a_1, \ldots, a_{n-1} \in k$ with $a \neq 0$. Further multiplication by transvections allows us to assume that $a_1 = \cdots a_{n-1} = 0$, and now the identity (7) allows us to multiply by more transvections to assume that $a = 1$. Thus we have written g as a product of transvections and the result follows. □

mutators

Lemma 22. *Let t be a transvection in $\mathrm{SL}_n(k)$. Then t is a commutator except when $n = 2$ and $|k| \leq 3$.*

Proof. If $n = 1$, $\mathrm{SL}_n(k)$ is trivial and the result is immediate, so assume that $n \geq 2$.

By (E53) all transvections are conjugate to a non-trivial element from Q, thus we need only show that all non-trivial element of Q are commutators. If $n = 2$ and $|k| > 3$ one can do this by taking $a, x \in k$ with $a^2 \neq 0, 1$ and observing that

e: comm

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & (a^2-1)x \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Clearly, as x varies across k, we cover all non-trivial elements of Q.

If $n > 2$, then all transvections are conjugate, and we need only show that the transvection (6) is a commutator. If $|k| > 3$, then this is achieved using (8) by taking $x = \frac{1}{a^2-1}$, and enlarging each matrix to size n by n, by appending blocks equal to the identity of rank $n-2$.

We are left with the case $n > 2$ and $|k| = 2$ or 3.

(E54*) *Prove the remaining case.*

□

The two previous lemmas imply the following corollary.

perfect

Corollary 23. $\mathrm{SL}_n(k)$ *is perfect except when $n = 2$ and $q \leq 3$.*

Remark. The projective image of a transvection in $\mathrm{PSL}_n(k)$ is called an *elation*. Now it is easy to see that the three previous results, Lemmas 21 and 22 and Corollary 23 all remain true if one replaces all instances of the word 'transvection' by 'elation' and all instances of '$\mathrm{SL}_n(k)$' by '$\mathrm{PSL}_n(k)$'.

5.3. **Finite groups.** In this section $k = \mathbb{F}_q$. In this situation, for

$$\mathrm{X} \in \{\Gamma, \mathrm{G}, \mathrm{S}, \mathrm{P}\Gamma, \mathrm{PG}, \mathrm{PS}\},$$

we write $\mathrm{XL}_n(q)$ as a synonym for $\mathrm{XL}_n(k)$.

q simple **Theorem 24.** *The group* $\mathrm{PSL}_n(q)$ *is simple except when* $n = 2$ *and* $q \leq 3$.

Proof. If $n = 1$, then $\mathrm{PSL}_n(q)$ is trivial and the result is immediate. Assume that $n \geq 2$ and that $q > 3$. Observe that (E51) implies that $\mathrm{PSL}_n(q)$ acts faithfully and primitively on the set of points of $\mathrm{PG}_{n-1}(q)$. Let Q_0 be the subgroup of $\mathrm{PSL}_n(q)$ equal to the projective image of the subgroup Q in Lemma 19. Now (E53) implies that $\bigcup\limits_{g \in G} Q^g$ contains all transvections of $\mathrm{SL}_n(q)$ and Lemma 21 implies, therefore, that

$$\langle Q_0^g \mid g \in G \rangle = \mathrm{PSL}_n(q).$$

Finally Corollary 23 implies that $\mathrm{PSL}_n(q)$ is perfect. Now Lemma 3 (Iwasawa's Criterion) implies that $\mathrm{PSL}_n(q)$ is simple. □

In the next result and hereafter, for integers k, l, we write (k, l) for their greatest common divisor.

r orders **Proposition 25.**

$$\begin{aligned}
|\mathrm{GL}_n(q)| &= (q^n - 1)(q^n - q) \cdots (q^n - q^{n-2})(q^n - q^{n-1}) \\
&= q^{\frac{1}{2}n(n-1)}(q^n - 1)(q^{n-1} - 1) \cdots (q^2 - 1)(q - 1). \\
|\mathrm{PGL}_n(q)| &= |\mathrm{SL}_n(q)| = |\mathrm{GL}_n(q)|/(q - 1). \\
|\mathrm{PSL}_n(q)| &= |\mathrm{SL}_n(q)|/(q - 1, n).
\end{aligned}$$

Proof. We write down elements of $\mathrm{GL}_n(q)$ with respect to a fixed basis for $V = \mathbb{F}_q^n$. There are $q^n - 1$ choices for the first column, then $q^n - q$ choices for the second column (since we cannot choose vectors that are in the span of the first), then $q^n - q^2$ choices for the third column, and so on. The identity for $|\mathrm{GL}_n(q)|$ follows.

Now $|\mathrm{PGL}_n(q)| = |\mathrm{GL}_n(q)|/|K|$ where, by (E40),

$$K = \{\alpha I \in \mathrm{GL}(V) \mid \alpha \in k\}.$$

Since $|K| = q - 1$, the identity for $|\mathrm{PGL}_n(q)|$ follows. On the other hand $|\mathrm{SL}_n(q)|$ is the kernel of the determinant map $\mathrm{GL}_n(q) \to k*$. Since this map is surjective, the first isomorphism theorem for groups yields the identity for $\mathrm{SL}_n(q)$.

Finally observe that $|\mathrm{PSL}_n(q)| = |\mathrm{SL}_n(q)|/|K \cap \mathrm{SL}_n(q)|$. Using the fact that k^* is cyclic of order $q - 1$ we conclude immediately that $|K \cap \mathrm{SL}_n(q)| = (n, q - 1)$ and we are done. □

(E55*) *Show that the set of upper-triangular matrices with 1's on the diagonal is a Sylow* p*-subgroup of* $\mathrm{GL}_n(q)$*.*

Isomorphisms between 'different' simple groups turn out to be very significant in the group theory universe. The next result discusses some of these.

order 1 **Proposition 26.** (1) $\mathrm{SL}_2(2) \cong S_3$;
(2) $\mathrm{PSL}_2(3) \cong A_4$;
(3) $\mathrm{SL}_2(4) \cong PSL_2(5) \cong A_5$;

(4) $\mathrm{PSL}_2(7) \cong \mathrm{SL}_3(2)$;
(5) $\mathrm{PSL}_2(9) \cong A_6$;
(6) $\mathrm{SL}_4(2) \cong A_8$.

Note that we write SL rather than PSL in cases where $(n, q-1) = 1$, since in these cases $\mathrm{PSL}_n(q) \cong \mathrm{SL}_n(q)$.

Proof of (1) to (4) only. A 2-dimensional vector space over $\mathbb{F}_2$ has 3 lines through the origin, on which $\mathrm{SL}_2(2)$ acts faithfully. Thus $\mathrm{SL}_2(2)$ embeds into S_3; comparing orders we conclude that $\mathrm{SL}_2(2) = S_3$.

A 2-dimensional vector space over $\mathbb{F}_3$ has 4 lines through the origin, on which $\mathrm{PSL}_2(3)$ acts faithfully. Thus $\mathrm{PSL}_2(3)$ embeds into S_4 as a subgroup of index 2. Now S_4 has a unique subgroup of index 2, namely A_4, thus $\mathrm{PSL}_2(3) \cong A_4$.

A 2-dimensional vector space over $\mathbb{F}_4$ has 5 lines through the origin, on which $\mathrm{SL}_2(4)$ acts faithfully. Thus $\mathrm{SL}_2(4)$ embeds into S_5 as a subgroup of index 2. Either $\mathrm{SL}_2(4) = A_5$ or $|A_5 : \mathrm{SL}_2(4) \cap A_5| = 2$. But, since A_5 is simple and index 2 subgroups are normal, the latter possibility cannot occur. Thus $\mathrm{SL}_2(4) = A_5$.

We use the same reasoning on the simplicity of A_5 to see that if $\mathrm{PSL}_2(5)$ acts on a set of size 5, then $\mathrm{PSL}_2(5)$ is isomorphic to A_5. We claim that $\mathrm{PSL}_2(5)$ has 5 Sylow 2-subgroups. One can compute these directly or else observe that the possible number of Sylow 2-subgroups is 1,3,5 or 15. Since $\mathrm{PSL}_2(5)$ is simple the first two are ruled out (why?). Now observe that, since a Sylow 2-subgroup of $\mathrm{PSL}_2(5)$ is equal to the centralizer of all of its non-trivial elements, we conclude that these elements lie in a unique Sylow 2-subgroup. If there were 15 Sylow 2-subgroups, then we would have 45 elements of order 2 in $\mathrm{PSL}_2(5)$ which is impossible - there are, for instance, 24 elements of order 5.

We know that $\mathrm{Aut}(\mathrm{PG}_2(2)) = \mathrm{SL}_3(2)$ and, by (E34), we know that $\mathrm{PG}_2(2)$ is equal to the incidence structure represented in Figure 2, the Fano plane. Thus it is sufficient to show that $\mathrm{PSL}_2(7)$ acts non-trivially on the Fano plane – since $\mathrm{PSL}_2(7)$ is simple, this action will therefore be faithful, inducing an embedding of $\mathrm{PSL}_2(7)$ into $\mathrm{SL}_3(2)$ which must be an isomorphism by comparison of orders.

Now define an incidence structure $\mathcal{I}$ as follows: Let S be a Sylow 2-subgroup of $G = \mathrm{PSL}_2(7)$ - it is dihedral of order 8 and contains two K_4-subgroups, U and V. One can check that $N_G(U) \cong N_G(W) \cong S_4$, thus there are 7 conjugates of U and 7 conjugates of V; what is more these conjugates are distinct. We set the conjugates of U to be the points of our incidence structure, the conjugates of V to be the lines, and say that a point and a line are incident if they are contained in the same Sylow 2-subgroup of G. Now one must check that this incidence structure is isomorphic to the Fano plane, and that the natural conjugation action of G on the conjugates of U and V respectively, induces an action on $\mathcal{I}$.

(E56*) *Check the details of the last paragraph.*

□

It turns out that the above list is a complete list of all isomorphisms between groups of form $\mathrm{PSL}_n(q)$ and A_n and S_n (a hardish fact that we won't prove). In fact this list contains almost all instances of a coincidence of cardinality between groups of form $\mathrm{PSL}_n(q)$ and groups of form A_n - there is one more such coincidence which is considered in the next exercise.

(E57*) *Prove that* $\mathrm{PSL}_3(4) \not\cong \mathrm{SL}_4(2) \cong A_8$, *despite the fact that these groups have the same orders.*

We have seen, in Lemma 5, that $\mathrm{PSL}_n(q)$ embeds into its own automorphism group. In fact, as the next proposition makes clear, we have (kind of) already seen the automorphism group of

$\mathrm{PSL}_n(q)$. To state the proposition we need one definition: fix a basis of $V = \mathbb{F}_q^n$ and define

$$\iota : \mathrm{PSL}_n(q) \to \mathrm{PSL}_n(q), x \mapsto x^{-T}.$$

To be clear: given $x \in \mathrm{PSL}_n(q)$, let X be an element in $\mathrm{SL}_n(q)$ that projects onto x, then define x^ι to be the projective image of X^{-T}, the inverse transpose of the matrix X with respect to the fixed basis.

(E58) *Check that this is a well-defined automorphism of* $\mathrm{PSL}_n(q)$.

psln aut **Proposition 27.** $\mathrm{Aut}(\mathrm{PSL}_n(q)) = \begin{cases} \mathrm{P\Gamma L}_2(q), & \text{if } n = 2; \\ \mathrm{P\Gamma L}_2(q) \rtimes \langle \iota \rangle, & \text{if } n \neq 3. \end{cases}$

The proof is omitted, although one inclusion is covered in the following exercise. You should compare the statement of the proposition to the statement of (E48).

(E59*) *Prove that*

$$\mathrm{Aut}(\mathrm{PSL}_n(q)) \geq \begin{cases} \mathrm{P\Gamma L}_2(q), & \text{if } n = 2; \\ \mathrm{P\Gamma L}_2(q) \rtimes \langle \iota \rangle, & \text{if } n \neq 3. \end{cases}$$

Hint: you need to study the natural action of, say, $\mathrm{P\Gamma L}_n(q)$ *on its normal subgroup* $\mathrm{PSL}_n(q)$.