8. Polar spaces

This section is something of a diversion, however it seems worthwhile to discuss polar spaces as they prefigure the work of Tits on *buildings* that enabled a uniform geometric understanding of the finite groups of Lie type.

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8.1. **Abstract vs classical polar spaces.** Let (V, β) be a formed space. The polar space associated with V is the incidence structure $\mathcal{I} = (P_1, \dots, P_r, I)$ where P_i is the set of i-dimensional totally isotropic subspaces of V and two such subspaces are incident if and only if one is contained in the other.

Similarly if (V, Q) is a formed space, then the polar space associated with V is the incidence structure $\mathcal{I} = (P_1, \dots, P_r, I)$ where P_i is the set of i-dimensional totally singular subspaces of V and two such subspaces are incident if and only if one is contained in the other.

The objects in these polar spaces are referred to as *flats* and we use the terms points, lines, planes etc as for projective spaces.

Let $k = \mathbb{F}_q$. Let r be a positive integer. In the previous section we encountered a number of different formed spaces with polar rank r. We will use the following labels to refer to the polar space associated with each:

$$\mathrm{Sp}_{2r}, \mathrm{U}_{2r}, \mathrm{U}_{2r+1}, \mathrm{O}^+_{2\mathrm{r}}, \mathrm{O}_{2\mathrm{r}+1}, \mathrm{O}^-_{2\mathrm{r}+2}.$$

These are the *finite classical polar spaces of polar rank r*. Note that the subscript in each case gives the dimension of the formed space with which the polar space is associated.

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Theorem 39. *Let Γ be a finite classical polar space of polar rank r. Then*

(Pol1) *Any flat, together with all the flats that it contains, is isomorphic to $\mathrm{PG}_d(q)$ for some $d \leq r-1$.*
(Pol2) *The intersection of any family of flats is a flat.*
(Pol3) *If U is a maximal flat and p is a point not in U, then the union of all lines joining p to points of U is a maximal flat W and $U \cap W$ is a hyperplane of both U and W.*
(Pol4) *There exist two disjoint maximal flats.*

(E86*) *Prove this.*

Any incidence structure satisfying (Pol1) to (Pol4) is a *thick abstract polar space*. The next theorem is due to Veldkamp and Tits and we will not prove it.

Theorem 40. *(Veldkamp-Tits) A thick abstract projective space with $r \geq 3$ is a classical polar space.*

An abstract polar space with $r = 2$ is called a *generalized quadrangle*. One might justify the terminology by observing that all conditions apart from (Pol4), the 'thickness condition', are satisfied by an ordinary quadrilateral. There are many finite generalized quadrangles other than the classical ones.

The philosophy here is that a thick abstract polar space is some kind of analogue of thick abstract projective space, a concept we encountered in §4.5. The following definition and theorem is included to make this analogue more obvious. Let (P_1, P_2, I) be an incidence structure of points and lines and define the following three properties.

(BS1) Every line has at least 3 points
(BS2) No point is collinear with all the points of S.
(BS3) If x is a point that does not lie on a line L, then either
  (a) exactly one point of L is collinear with x, or
  (b) every point of L is collinear with x.

Theorem 41. *(Buekenhout-Shult) Any finite incidence structure satisfying (BS1) to (BS3) satsifies (Pol1) to (Pol4).*

8.2. **Spherical buildings.** Let us now discuss how these ideas can be pushed further.

Projective and polar spaces. In Section 4.5 we saw the connection between $\mathrm{PG}_n(q)$ and finite thick abstract projective spaces – the former are examples of the latter; indeed they are almost all possible examples of the latter.

In Section 8.1 we saw the connection between classical polar spaces and finite thick abstract polar spaces – the former are examples of the latter; indeed they are all possible examples of the latter except when $r = 2$.

Automorphisms. Recall next that the Fundamental Theorem of Projective Geometry states that, if $\dim(V) \geq 3$, then all collineations of $\mathrm{PG}(V)$ are induced by a semilinear transformation. Combining this with the Veblen-Young theorem we see that, in most cases, the automorphisms of a finite thick projective space are induced by a semilinar transformation of some associated vector space.

When one comes to finite thick abstract polar spaces the situation is similar. By the Tits-Veldkamp theorem above we can restrict our attention to the classical polar spaces where it is easy to see that semilinear transformations induce collineations:

> **(E87)** *A semisimilarity of (V, κ) induces a collineation of the associated polar space $\mathcal{I}$. In particular $\mathrm{SemiSim}(\kappa) \leq \mathrm{Aut}(\mathcal{I})$.*

Now deep work of Tits [Tit74] implies that, in most cases, all collineations of a classical polar space are induced in this way – by a semilinear transformation of the associated vector space.

A generalization. In fact finite thick abstract projective spaces and finite thick abstract polar spaces can both be generalized to give the notion, introduced by Jacques Tits, of a *thick spherical building*. That is to say finite thick abstract projective spaces and finite thick abstract polar spaces can be thought of as examples of a finite thick spherical building.

Tits didn't just define these things; he also classified all finite thick spherical buildings, except 'when the rank is 2'.[9] In this pathological case one encounters the *generalized polygons* which include the generalized triangles (or projective planes) and generalized quadrangles, as well as generalized hexagons and octagons.

Automorphisms. The true significance of Tits' classification of the finite spherical buildings lies in their automorphism groups. We saw earlier that $\mathrm{Aut}(\mathrm{PG}_{n-1}(q)) = \mathrm{P\Gamma L}_n(q)$ which, provided $n \geq 2$ or $q \geq 4$, is an almost simple group with simple normal subgroup $\mathrm{PSL}_n(q)$. In the next few sections we will see that the automorphism groups of the classical polar spaces are (generally speaking) almost simple groups with simple normal subgroup equal to a classical group. The beauty of Tits' classification is that the automorphism groups of the spherical buildings are (generally speaking) almost simple groups with simple normal subgroup equal to a finite group of Lie type. Thus the notion of a spherical building gives a uniform geometric description of the finite groups of Lie type.

We briefly considered a related subject – *a group with BN-pair* – in the discussion class. However we will not delve any deeper – those interested should consult Tit's amazing monograph [Tit74] for the full theory.

The remainder of the course will be spent studying the automorphisms of the finite classical polar spaces – the so called finite classical groups.

[9] We haven't defined what we mean by rank here, but we remark that for polar spaces it is precisely the same as the polar rank.

9. Symplectic groups

Throughout this section β is a non-degenerate alternating bilinear form on a $2r$-dimensional vector space V over a field k.

In §7.2.1 we saw that any such pair (V, β) admits a basis $B = \{e_1, f_1, \ldots, e_r, f_r\}$ such that, for all $i, j = 1, \ldots, r$,

$$\begin{aligned} \beta(v_i, w_j) =& \delta_{ij} = -\beta(w_j, v_i), \\ \beta(v_i, v_j) =& 0 = \beta(w, w_j). \end{aligned}$$

This implies the following facts, which we leave as an exercise.

> **(E88*)** *Let β_1 and β_2 be non-degenerate alternating bilinear forms defined on a $2r$-dimensional vector space V over a field k. Then* $\mathrm{Isom}(\beta_1)$ *and* $\mathrm{Isom}(\beta_2)$ *(resp.* $\mathrm{Sim}(\beta_1)$ *and* $\mathrm{Sim}(\beta_2)$*) are conjugate subgroups of* $\mathrm{GL}_{2r}(k)$*. Furthermore* $\mathrm{SemiSim}(\beta_1)$ *and* $\mathrm{SemiSim}(\beta_2)$ *are conjugate subgroups of* $\Gamma\mathrm{L}_{2r}(k)$*.*

These facts allow us to make the following definitions. We write K for the set of invertible scalar matrices over k.

- $\mathrm{Sp}_{2r}(k)$ is the isometry group of β;
- $\mathrm{GSp}_{2r}(k)$ is the similarity group of β;
- $\Gamma\mathrm{Sp}_{2r}(k)$ is the semi-similarity group of β;
- $\mathrm{PSp}_{2r}(k) = \mathrm{Sp}_{2r}(k)/(K \cap \mathrm{Sp}_{2r}(k))$;
- $\mathrm{PGSp}_{2r}(k) = \mathrm{GSp}_{2r}(k)/K$;
- $\mathrm{P}\Gamma\mathrm{Sp}_{2r}(k) = \Gamma\mathrm{Sp}_{2r}(k)/K$

If $k = \mathbb{F}_q$ we may write $\mathrm{Sp}_{2r}(q)$ for $\mathrm{Sp}_{2r}(k)$ and likewise for the other groups.

> **(E89)** $|\mathrm{Sp}_{2r}(k) \cap K| = \begin{cases} 2, & \textit{if } \mathrm{char}(k) \neq 2; \\ 1, & \textit{otherwise.} \end{cases}$

We can write $\mathrm{Sp}_{2r}(k)$ in terms of matrices:

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$$\mathrm{Sp}_{2r}(k) = \{X \in \mathrm{GL}_{2r}(k) \mid XAX^T = X\} \tag{11}$$

where A can be written in one of the following ways (each is obtained from the others by permutating the basis appropriately):

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{pmatrix}, \quad \begin{pmatrix} 0_r & I_r \\ -I_r & 0_r \end{pmatrix}, \quad \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}.$$

In what follows we will study the action of $\mathrm{Sp}_{2r}(k)$ on the points of its associated polar space, with a view to applying Iwasawa's criterion to this action. Note that, since β is alternating, all 1-dimensional subspaces of V are totally isotropic, and thus they all arise as points of the associated polar space.

1: pip

Lemma 42. $\mathrm{Sp}_2(k) \cong \mathrm{SL}_2(k)$.

Proof. Write elements of $V = k^2$ as row vectors and define

$$\beta : V \times V \to k, (x, y) \mapsto \det \begin{pmatrix} x \\ y \end{pmatrix}.$$

It is clear that β is a non-degenerate bilinear form. Now, if $X \in GL_2(k)$, then

$$\beta(xX, yX) = \det \begin{pmatrix} xX \\ yX \end{pmatrix} = \det \left(\begin{pmatrix} x \\ y \end{pmatrix} X \right) = \det \begin{pmatrix} x \\ y \end{pmatrix} \det(X).$$

Thus $\beta(xX, yX) = \beta(x, y)$ if and only if $\det(X) = 1$. $\square$

(E90) *Give an alternative proof of Lemma 42 by showing that*

$$X \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Longleftrightarrow \det(X) = 1.$$

Lemma 43. $|\mathrm{Sp}_{2r}(q)| = q^{r^2} \prod_{i=1}^{r} (q^{2i} - 1)$.

Proof. Clearly $G = \mathrm{Sp}_{2r}(q)$ acts transitively on the set of r-tuples of hyperbolic pairs that span the space. On the other hand if $g \in G$ fixes such an r-tuple, then $g = 1$. Thus the action is regular and $|G|$ is equal to the number of r-tuples of hyperbolic pairs. Let us count these r-tuples.

If (v, w) is a hyperbolic pair, then the number of choices for v is $q^{2r} - 1$; the number of vectors w in $V \backslash \langle v \rangle^\perp$ is $q^{2r} - q^{2r-1}$ and, of these $\frac{1}{q-1}(q^{2r} - q^{2r-1})$ satisfy $\beta(v, w) = 1$.

If we fix (v, w) and continue in $\langle v, w \rangle^\perp$, which is a symplectic non-degenerate space of dimension $2r - 2$ over k, then the order formula follows by induction. $\square$

Suppose that a group G acts transitively on a set Ω and let $\omega \in \Omega$. The *permutation rank* of G is the number of orbits of G_ω on Ω.

(E91*) *Prove that the permutation rank is 2 if and only if G acts 2-transitively on Ω.*

(E92) *Prove that the permutation rank of G is equal to the number of orbits of G in the induced action on Ω^2.*

Lemma 44. $\mathrm{Sp}_{2r}(k)$ *acts primitively on the set of points of its polar space. If $r \geq 2$, then the permutation rank is* 3.

Proof. Witt's Lemma implies that $G = \mathrm{Sp}_{2r}(k)$ acts transitively on points. Any pair of distinct points either spans a totally isotropic 2-space, or a hyperbolic plane. By Witt's lemma, $\mathrm{Sp}_{2r}(k)$ is transitive on the pairs of each type. Thus G has three orbits in the induced action on Ω^2 (the other being on the diagonal $\{\omega, \omega) \mid \omega \in \Omega\}$), i.e. the permutation rank is 3.

We can think of a non-trivial G-congruence on Ω as a subset of Ω^2, in which case such a congruence must be a union of the diagonal and one of the other two orbits on Ω^2. We must prove that neither of these two possibilities yields an equivalence relation. The following exercises do this by showing that, whichever union we consider, the consequent relation is not transitive.

e: cc **(E93*)** *Prove that if $\beta(x, y) = 0$, then there exists z with $\beta(x, z), \beta(y, z) \neq 0$.*

(E94*) *Prove that if $\beta(x, y) \neq 0$, then there exists z with $\beta(x, z) = \beta(y, z) = 0$.*

$\square$

l: iii **Lemma 45.** *Let $G = \mathrm{Sp}_{2r}(k)$ and $\omega \in \Omega$, the set of points of its polar space. Then*

$$G_\omega \cong Q \rtimes (\mathrm{Sp}_{2r-2}(k) \times GL_1(k))$$

where Q is an abelian group isomorphic to the additive group $(k^{2r-1}, +)$.

Proof. We assume $\mathrm{Sp}_{2r}(k)$ is defined via (11) and

$$A = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}. \tag{12}$$

Thus the associated basis is $\{v_1, \dots, v_r, w_r, \dots, w_1\}$ and, since $\mathrm{Sp}_{2r}(k)$ acts transitively on the set of points of its polar space, we can take $\omega = \langle w_1 \rangle$.

Now it is easy enough to see that

$$G_{\langle w_1 \rangle} = \left\{ g := \left(\begin{array}{c|ccc|c} a & a_1 & \cdots & a_{2r-2} & a_{2r-1} \\ \hline 0 & & & & b_{2r-2} \\ \vdots & & A & & \vdots \\ 0 & & & & b_1 \\ \hline 0 & 0 & \cdots & 0 & a^{-1} \end{array}\right) \;\middle|\; \begin{array}{l} a_1, \dots, a_{2r-1}, b_1, \dots, b_{2r-2} \in k, \\ b_i = \begin{cases} -a_i, & \text{if } i \leq r-1; \\ a_i, & \text{otherwise}; \end{cases} \\ a \in k^*, A \in \mathrm{Sp}_{2r-2}(k) \end{array} \right\}. \tag{13}$$

Now there is a natural epimorphism

$$G_{\langle w_1 \rangle} \to \mathrm{Sp}_{2r-2}(k) \times \mathrm{GL}_1(k), g \mapsto (A, a)$$

and the kernel of this map is the group

$$Q := \left\{ g := \left(\begin{array}{c|ccc|c} 1 & a_1 & \cdots & a_{2r-2} & a_{2r-1} \\ \hline 0 & & & & b_{2r-2} \\ \vdots & & I & & \vdots \\ 0 & & & & b_1 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right) \;\middle|\; \begin{array}{l} a_1, \dots, a_{2r-1}, b_1, \dots, b_{2r-2} \in k, \\ b_i = \begin{cases} -a_i, & \text{if } i \leq r-1; \\ a_i, & \text{otherwise}; \end{cases} \end{array} \right\} \tag{14}$$

and one can check that Q is indeed isomorphic to the additive group $(k^{2r-1}, +)$.

(E95) *Prove that this extension is split.*

□

9.1. Symplectic transvections. Recall that a *transvection* on V is an element $t \in \mathrm{GL}(V)$ such that

- $\mathrm{r}k(t - I) = 1$;
- $(t - I)^2 = 0$.

.

(E96*) *Given a transvection t, there exists $f \in V^*$ and $a \in \ker(f)$ such that*

$$vt = v + (vf)a \text{ for all } v \in V.$$

Let β be an alternating bilinear non-degenerate form on V. A *symplectic transvection for β* is a transvection t that lies in $\mathrm{Isom}(\beta) \cong \mathrm{Sp}_{2r}(q)$. If a and f are as in the previous exercise, we have

$$\begin{aligned} \beta(vt, wt) &= \beta(v + (vf)a, w + (wf)a) \\ &= \beta(v, w) + (wf)\beta(v, a) + (vf)\beta(a, w). \end{aligned}$$

Thus t is symplectic if and only if $(wf)\beta(v,a) = (vf)\beta(w,a)$ for all $v, w \in V$. Take w such that $\beta(w,a) = 1$ and let $\lambda = wf$, then we require that $vf = \lambda\beta(v,a)$ and so a symplectic transvection for β is given by

$$v \mapsto v + \lambda\beta(v,a)a.$$

Conversely, one can check that a transvection of this form does indeed lie in $\mathrm{Sp}_{2r}(k)$.

l: sss **Lemma 46.** *The symplectic transvections generate the symplectic group $G = \mathrm{Sp}_{2r}(k)$.*

Proof. Lemma 42 implies that the result is true for $n = 2$. Now we induct on n. Define

$$D := \langle t \mid t \text{ is a transvection in } G\rangle.$$

Claim: D is transitive on $V\backslash\{0\}$.
Proof of claim: Let $u, v \in V\backslash\{0\}$. If $\beta(u,v) \neq 0$, then the symplectic transvection

e: st
$$x \mapsto x + \frac{\beta(x, v-u)}{\beta(u,v)}(v-u) \tag{15}$$

carries u to v. If $\beta(u,v) = 0$, then (E93) implies that we can choose w such that $\beta(u,w), \beta(v,w) \neq 0$, and so we can map u to w to v.

Claim: D is transitive on the set of hyperbolic pairs in V.
Proof of claim: By the previous claim it is sufficient to prove that if (v, w_1) and (v, w_2) are hyperbolic pairs, then there exists a transvection $t \in G_v$ such that $w_1^g = w_2$. If $\beta(w_1, w_2) \neq 0$, then the following will do:

$$x \mapsto x + \frac{\beta(x, w_1 - w_2)}{\beta(w_1, w_2)}(w_1 - w_2).$$

If $\beta(w_1, w_2) = 0$, then go via $v + w_1$ as before.

Now it is sufficient to prove that any symplectic transformation g fixing a hyperbolic pair (u,v) is a product of symplectic transvections. It is easy to see that the stabilizer of (u,v) is the isometry group of $\langle u,v\rangle^\perp$ (cf. (11)), a symplectic polar space of dimension $2r-2$. Induction now allows us to assume that g is a product of transvections in $\mathrm{Sp}_{2r-2}(q)$ and hence in $\mathrm{Sp}(q)$. $\square$

Corollary 47. $\mathrm{Sp}_{2r}(k) \leq SL_{2r}(k)$.

In the next lemma we will use the fact, found in the proof of Lemma 46, that $\mathrm{Sp}_{2r}(k)$ is transitive on hyperbolic pairs.

ery symp **Lemma 48.** *Every symplectic transvection is contained in a conjugate of the group Q defined in Lemma 45.*

Proof. We can use the definition for Q given by (14) provided we are careful to define $\mathrm{Sp}_{2r}(k)$ with respect to the matrix (12).

Now let t be a symplectic transvection and write

$$t : V \to V, v \mapsto v + \lambda\beta(v,a)a$$

where $\lambda \in k^*$ and $a \in V$. Let $w \in V$ be such that (w,a) is a hyperbolic pair. Now extending this to a symplectic basis (with w as the first element of the basis and a the last which, in particular, is consistent with (12)) and invoking Witt's lemma, we know that we can conjugate by an element g of $\mathrm{Sp}_{2r}(k)$ so that

$$t^g = \begin{pmatrix} 1 & 0 & \cdots & 0 & \lambda\beta(v,a) \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$

Now $t \in Q^{g^{-1}}$ as required. □

Corollary 49. *Let Q be the group defined in Lemma 45. Then*

$$\mathrm{Sp}_{2r}(q) = \langle t \mid t \text{ is a transvection}\rangle = \langle Q^g \mid g \in \mathrm{Sp}_{2r}(q)\rangle.$$

Lemma 50. *Symplectic transvections in $\mathrm{Sp}_{2r}(k)$ are commutators in $\mathrm{Sp}_{2r}(k)$ except if*

$$(2r, |k|) \in \{(2,2), (2,3), (4,2)\}.$$

Proof. Let t be a transvection, and let $v \in V$ such that vt is linearly independent of v. It is easy to see that $U = \langle v, vt\rangle$ is a hyperbolic plane, and that t acts trivially on $U^\perp$.

Then t acts as a transvection on U and, by Lemma 22, t is a commutator in $SL(U)$ provided $|k| \geq 3$. Now Lemma 42 implies that t is a commutator in $\mathrm{Sp}(U)$ and hence in G, as required.

To complete the proof we must deal with $|k| \leq 3$. The following exercise does that.

(E97*) *Prove that symplectic transvections in $\mathrm{Sp}_6(2)$ and $\mathrm{Sp}(4,3)$ are commutators.*

□

Corollary 51. *$\mathrm{Sp}_{2r}(k)$ is perfect.*

Iwasawa's criterion and the preceding results allow us to conclude our big result.

c simple **Theorem 52.** *$\mathrm{PSp}_{2r}(q)$ is simple unless*

$$(2r, q) \in \{(2,2), (2,3), (4,2)\}.$$

We mentioned earlier that coincidences in order between simple groups, and isomorphisms between 'different' simple groups are important. The following theorem deals with all such coincidences, and isomorphisms, between $\mathrm{PSp}_{2r}(q)$ and the other simple groups that we have encountered thus far. In light of Lemma 42 we restrict to $r > 1$.

Proposition 53. *Let K and L be simple with $K = \mathrm{PSp}_{2r}(q)$ (with $r > 1$) and $L \cong \mathrm{PSL}_m(q')$ or A_m. Then $K \not\cong L$.*

It is worth dealing with the three cases listed in Theorem 52 for which $\mathrm{Sp}_{2r}(q)$ is not simple. Lemma 42 and our results for $\mathrm{SL}_2(q)$ immediately imply that

$$\mathrm{Sp}_2(2) \cong \mathrm{SL}_2(2) \cong S_3;$$
$$\mathrm{Sp}_2(3) \cong \mathrm{SL}_2(3) \cong A_4.$$

Our final lemma deals with the remaining case.

Lemma 54. $\mathrm{Sp}_4(2) \cong S_6$.

Proof. Let V be a 6-dimensional vector space over $\mathbb{F}_2$. After fixing a basis for V, observe that S_6 acts on V by permuting coordinates.

Define the form $\beta(x,y) = \sum_{i=1}^{6} x_i y_i$ and write $j = (1,1,1,1,1,1)$. Then $\langle j\rangle^\perp$ is of dimension 5 and contains j; we define $W := \langle j\rangle^\perp/\langle j\rangle$, a vector space of dimension 4.

Observe that the action of S_6 on V induces, by restriction, a faithful action on W. Furthermore the form β induces a form β_W on W, since $\beta(x,j) = 0$ for $x \in \langle j\rangle^\perp$. Since $\beta(x,x) = 0$ for $x \in \langle j\rangle^\perp$, the form β_W is alternating and one can check that it is non-degenerate.

Since S_6 preserves β_W we obtain an embedding $S_6 \leq \mathrm{Sp}_4(2)$. Since the two groups have the same order, the result follows. □