EXERCISE SHEET 3

(E60) Prove that the left and right radicals are subspaces.

(E61) Prove that if $\dim V < \infty$, then the left and right radicals have the same dimension. Give a counter-example to this assertion when $\dim V = \infty$.

(E62) Check that the following map is a duality.

$\boxed{\texttt{e: perp}}$ (1) $$\mathrm{PG}(V) \to \mathrm{PG}(V), U \mapsto U^{\perp} := \{x \in V \mid \beta(x,y) = 0 \text{ for all } y \in U\}.$$

(E64) Prove that $\lambda \in k \mid kk^{\sigma} = 1\} = \{\epsilon/\epsilon^{\sigma} \mid \epsilon \in k\}$.

(E65) Prove that the following map is a duality.

$$\langle x_1, \ldots, x_n \rangle \longleftrightarrow [x_1, \ldots, x_n].$$

(E69) Fix a basis $\mathcal{B} = \{x_1, \ldots, x_n\}$ for V and let $Q : V \to k$ be a quadratic form. There is a matrix A such that $Q(x) = x^T A x$. Moreover

$$A_{ij} = \begin{cases} \beta_Q(x_i, x_j), & \text{if } i < j, \\ Q(x_i), & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

(E71) Show that the quadratic form Q in the lecture notes polarizes to β.

(E74) Prove that if k is perfect, $\mathrm{char}(k) = 2$ and $Q : V \to k$ is non-degenerate, then $\dim(\mathrm{Rad}(\beta_Q)) \leq 1$.

(E76) Complete the proof of Theorem 34 in lectures.

(E78 Let U_1 and U_2 be subspaces of a vector space V having the same dimension. Show that there is a subspace W of V which is a complement for both U_1 and U_2.

(E80) Let (V, κ) be a formed space. Then the Witt index and the type of a maximal anisotropic subspace are determined.

(E81) Let (V, κ) be a formed space. Any maximal totally isotropic/ totally singular subspaces in V have the same dimension. This dimension is equal to the Witt index.