5. Primitivity and related notions

In this section we study some key properties of group actions. We will use this material in the next chapter when we discuss the subgroup structure of $\mathrm{Alt}(n)$. Throughout the section G is a finite group acting on a set Ω.

5.1. **Multiple transitivity.** As soon as we have an action of a group G on a set Ω, we can define others. For instance, define an action of G on $\Omega^2 = \Omega \times \Omega$ via

$$(\omega_1, \omega_2)^g := (\omega_1^g, \omega_2^g),$$

for all $g \in G$.

In fact this defines a natural action on the set of distinct pairs,

$$\Omega^{(2)} := \{(\omega_1, \omega_2) \mid \omega_1 \neq \omega_2\}.$$

We say that the *original* action of G on Ω is 2-*transitive* if the induced action of G on $\Omega^{(2)}$ is transitive.

More generally, let k be an integer such that $1 \leq k \leq |\Omega|$. Define

$$\Omega^{(k)} := \{(\omega_1, \omega_2, \dots, \omega_k) \mid \omega_i \in \Omega; \omega_i \neq \omega_j \text{ for all } 1 \leq i < j \leq k\}$$

The action of G on Ω is k-*transitive* if the induced action of G on $\Omega^{(k)}$ is transitive. Note that an action is 1-*transitive* if and only if it is transitive.

(E5.1) *For which values of n is the action of D_{2n} on an n-gon, 2-transitive?*

(E5.2) *Show that, for $k \geq 2$, if an action is k-transitive, then it is $k-1$-transitive.*

(E5.3) *Let $G = S_n$, the symmetric group on n letters. What is the largest value of k for which G is k-transitive? What about $G = A_n$, the alternating group on n letters?*

Lemma 5.1. *The action of G on Ω is k-transitive if and only if the action of G is transitive and, for any $\omega \in \Omega$, the action of the stabilizer of a point G_ω on $\Omega\backslash\{\omega\}$ is $(k-1)$-transitive.*

Proof. Suppose that the action of G on Ω is k-transitive, and let $(\omega_1, \dots, \omega_k)$ and $(\gamma_1, \dots, \gamma_k)$ be elements of $\Omega^{(k)}$. By assumption there exists $g \in G$ such that $\omega_i^g = \gamma_i$ for all $i = 1, \dots, k$. Since (for instance) γ_1 and ω_1 range over all of Ω we conclude that G acts transitively on Ω.

On the other hand let $(\omega_1, \dots, \omega_{k-1})$ and $(\gamma_1, \dots, \gamma_{k-1})$ be elements of $(\Omega\backslash\{\omega\})^{(k-1)}$. Then $(\omega, \omega_1, \dots, \omega_{k-1})$ and $(\gamma_1, \dots, \omega_{k-1})$ are elements of $\Omega^{(k)}$ and so there exists $g \in G$ such that $\omega^g = \omega$ and $\omega_i^g = \gamma_i$ for all $i = 1, \dots, k-1$. Thus $g \in G_\omega$ and we conclude that the action of G_ω on $\Omega\backslash\{\omega\}$ is $(k-1)$-transitive.

For the converse, let $(\omega_1, \dots, \omega_k)$ and $(\gamma_1, \dots, \gamma_k)$ be elements of $\Omega^{(k)}$. Since G acts transitively, there exist $g, h \in G$ such that $\omega_1^g = \omega$ and $\gamma_1^h = \omega$. Observe that $(\omega_2^g, \dots, \omega_k^g)$ and $(\gamma_1^h, \dots, \gamma_k^h)$ are elements of $(\Omega\backslash\{\omega\})^{(k-1)}$. Thus there exists $f \in G_\omega$ such that $(\omega_i^g)f = \gamma_i^h$ for $i = 2, \dots, k$. We conclude that

$$((\omega_i^g)f)^{h^{-1}} = \omega_i^{gfh^{-1}} = \gamma_i$$

for all $i = 1, \dots, k$ and we are done. $\square$

Example 15. Let V be a vector space of dimension at least 2 over a finite field K and consider the action of $G = \mathrm{GL}(V)$ on $V^* := V\backslash\{0\}$ by right multiplication. If $g \in G, v \in V$ and $c \in K$, then we know that $(c \cdot v)^g = c \cdot v^g$. In particular, if $|K| \geq 2$, then we can choose $c \neq 1$ and w a vector linearly independent from v, and we observe that there exists no element $g \in \mathrm{GL}(V)$ which maps the pair (v, cv) to (v, w). We conclude that the action of G on V^* is not 2-transitive.

If $|K| = 2$, then any pair of distinct vectors in V^* is linearly independent. Thus, given (v_1, v_2) and (w_1, w_2) two pairs of distinct vectors in V^* we can extend both to give a basis for V; in other words there exists an element $g \in \mathrm{GL}(V)$ such that $v_1^g = w_1$ and $v_2^g = w_2$, and we conclude that the action of G on V^* is 2-transitive.

We will see a generalization of this situation later on when we study the groups $\mathrm{PGL}_n(q)$; let us briefly foreshadow what is to come: Observe first that, if $g \in \mathrm{GL}(V)$ and $v, w \in V^*$ lie in a 1-dimensional subspace W of V, then v^g and w^g lie in a 1-dimensional subspace of

V that we call W^g. Now, letting Ω be the set of 1-dimensional subspaces of V, we have a well-defined action

$$\mathrm{GL}(V) \times \Omega \to \Omega, (g, W) \mapsto W^g.$$

(Note that if $|K| = 2$, then every element of Ω contains exactly one element of V^*, and the action of $\mathrm{GL}(V)$ on Ω is isomorphic to the action of $\mathrm{GL}(V)$ on V^*.) Now, observe that, given (V_1, V_2) and (W_1, W_2) two pairs of distinct 1-dimensional subspaces of V, there exists an element $g \in \mathrm{GL}(V)$ such that $V_1^g = W_1$ and $V_2^g = W_2$, and we conclude that the action of G on Ω is 2-transitive.

5.2. **Blocks and primitivity.** A *G-congruence* on Ω is an equivalence relation $\sim$ on Ω such that

$$\alpha \sim \beta \implies \alpha^g \sim \beta^g$$

for all $g \in G$. Any action always admits two G-congruences which we call *trivial*, as follows:

- Define $\alpha \sim_1 \beta$ if and only if $\alpha = \beta$;
- Define $\alpha \sim_2 \beta$ always.

The equivalence classes of a G-congruence are called *blocks*. Note that, for $\sim_1$, there are $|\Omega|$ blocks all of cardinality 1 while, for $\sim_2$, there is one block of cardinality $|\Omega|$.

The action of G on Ω is called *primitive* if the only G-congruences on Ω are the trivial ones. We call the action *imprimitive* if it is not primitive.[21]

(E5.4) *Prove that if an action is 2-transitive, then it is primitive.*

Lemma 5.2. *Suppose that G acts primitively on Ω and let $N \trianglelefteq G$ with $N \not\leq G_{(\Omega)}$. Then N acts transitively on Ω.*

Proof. Let $\Lambda_1, \dots, \Lambda_k$ be the orbits of N on Ω. Define an equivalence relation $\sim$ on Ω such that $\alpha \sim \beta$ if and only if there exists i such that $\alpha, \beta \in \Lambda_i$. Now suppose that $\alpha \sim \beta$. By definition $\beta = \alpha^n$ for some $n \in N$. Let $g \in G$ and observe that

$$\beta^g = (\alpha^n)^g = (\alpha^g)^{g^{-1}ng}.$$

Since N is normal, $g^{-1}ng \in N$ and we conclude that $\alpha^g \sim \beta^g$ and hence $\sim$ is a G-congruence on Ω.

Since G is primitive, $\sim$ must be one of the two trivial G-congruences, $\sim_1$ or $\sim_2$. Since $N \not\leq G_{(\Omega)}$ we conclude that $|\Lambda_i| \geq 2$ for some $i = 1, \dots, k$ and so $\sim \neq \sim_1$. We conclude that $\sim = \sim_2$ which implies, in particular that $k = 1$ and N acts transitively on Ω. □

Taking N to equal G in this lemma we observe, in particular, that if an action is primitive, then it is transitive.

(E5.5) *Prove that if an action is transitive and $\sim$ is a G-congruence, then all of the blocks associated with $\sim$ have the same cardinality.*

(E5.6) *Suppose that G acts imprimitively on Ω. Let Δ be a block associated with the action of G on Ω and let $\omega \in \Delta$. Prove that Δ is a union of orbits of the stabilizer G_ω.*

A *maximal subgroup* of a group G is a subgroup M such that if $M \leq H < G$, then $H = M$. We have the following result.

Lemma 5.3. *Suppose that $|\Omega| > 2$. Then G acts primitively on Ω if and only if G acts transitively and any stabilizer, G_ω, is a maximal subgroup of G.*

Observe that the statement is not true for $|\Omega| = 2$ since the trivial action is intransitive and primitive.

Proof. Suppose first that G acts intransitively on Ω. Since G is primitive, the previous lemma applied with $G = N$ implies that G acts trivially on Ω. But now any equivalence relation on Ω is a G-congruence and so, since $|\Omega| > 2$, non-trivial G-congruences exist. Thus we assume that G acts transitively on Ω. In what follows we let ω be some element of Ω.

[21] I may also write things like "G acts primitively on the set Ω", and will trust you to figure out what I mean.

Suppose first that $\sim$ is a non-trivial G-congruence on Ω. Let Δ be the block associated with $\sim$ that contains ω and observe that (E5.5) implies that $1 < |\Delta| < n$. Consider the setwise stabilizer

$$G_\Delta := \{g \in G \mid \Delta^g = \Delta\}.$$

We know that G_Δ is a subgroup of G. What is more, if $\omega \in \Delta$, then (since Δ is a block) $G_\omega \in G_\Delta$. Indeed (since G is transitive), for any $\gamma \in \Delta \setminus \{\omega\}$, there exists $g \in G$ such that $\omega^g = \gamma$ and (since Δ is a block) $g \in G_\Delta \setminus G_\omega$, i.e. G_ω is a proper subgroup of G_Δ.

On the other hand (since G is transitive) we know that G_Δ is a proper subgroup of G. We conclude that G_ω is not maximal, as required.

The converse is given by the following exercise:

> **(E5.7)** *Show that if G acts transitively on Ω, and if M is a subgroup of G such that $G_\omega < M < G$ for some $\omega \in \Omega$, then the action is imprimitive.*

□

> **Example 16.** Let $G = \mathrm{Sym}(k)$ for some $k \geq 5$. If M is any maximal subgroup of G that **does not** contain $\mathrm{Alt}(k)$, then (E4.28) implies that the natural action of G on the coset space $M\backslash G$ is faithful and transitive. Moreover the stabilizer of the coset M is M itself, a maximal subgroup. Thus, by Lemma 5.3, the action of G on $M\backslash G$ is primitive.
>
> A concrete example is given by taking $M = \mathrm{Sym}(2) \times \mathrm{Sym}(k-2)$, the intransitive subgroup of $\mathrm{Sym}(k)$ that we saw in the previous section. In fact the group M is maximal in $\mathrm{Sym}(k)$ (you may have proved this for (E4.27)) and has index $\frac{1}{2}k(k-1)$ in $\mathrm{Sym}(k)$. Thus Lemma 3.1 implies that the action of $G = \mathrm{Sym}(k)$ on $M\backslash G$ yields an embedding, $\mathrm{Sym}(k) \leq \mathrm{Sym}(\frac{1}{2}k(k-1))$ such that $\mathrm{Sym}(k)$ is a *primitive* subgroup of $\mathrm{Sym}(\frac{1}{2}k(k-1))$.
>
> We shall see later on that primitive subgroups of $\mathrm{Sym}(n)$ that are also almost simple are of considerable interest.

5.3. **Wreath products.** Let H and K be groups and let φ be an action of H on a set Δ. Define $B := K^\Delta = \{b : \Delta \to K\}$, and we define multiplication on B via

$$\delta(bb') := (\delta b)(\delta b')$$

for all $b, b' \in B$ and $\delta \in \Delta$. One can check that this turns B into a group.

Now define an action of H on the group B:

$$\psi : H \times B \to B, (h, b) \mapsto b^h \text{ where } b^h : \Delta \to K, \delta \mapsto (\delta^{h^{-1}})b. \tag{4}$$

for all $h \in H, b \in B$ and $\delta \in \Delta$.

This definition is a little counter-intuitive, so we should take a moment to check that it really defines an action.

Lemma 5.4. *The function* (4) *defines an action of H on the group B as an object in Group.*

Proof. The given function is clearly well-defined. So now consider $b \in B$ and $h_1, h_2 \in H$ and observe that

$$\begin{aligned}
(b^{h_1})^{h_2} &= \left[\delta \mapsto (\delta^{h_1^{-1}})b\right]^{h_2} \\
&= \left[\delta \mapsto ((\delta^{h_2^{-1}})h_1^{-1})b\right] \\
&= \left[\delta \mapsto (\delta^{h_2^{-1}h_1^{-1}})b\right] \\
&= \left[\delta \mapsto (\delta^{(h_1h_2)^{-1}})b\right] = b^{h_1h_2}.
\end{aligned}$$

Thus the function is indeed an action. Now consider $b_1, b_2 \in B$ and $h \in H$ and observe that

$$\begin{aligned}
(b_1^h)(b_2^h) &= \left[\delta \mapsto (\delta^{h^{-1}})b_1\right] \cdot \left[\delta \mapsto (\delta^{h^{-1}})b_2\right] \\
&= \left[\delta \mapsto (\delta^{h_1^{-1}})(b_1b_2)\right] = (b_1b_2)^h.
\end{aligned}$$

Thus the function defines an action of H on the group B as an object in Group. $\square$

Now the *wreath product* of K and H, written $K \wr_\Delta H$, is defined to be the semidirect product $B \rtimes_\psi H$.[22] We call B the *base group* of the wreath product. Recall that Lemma 4.7 implies that the action by conjugation of the subgroup H in $K \wr_\Delta H$ on the normal subgroup B is isomorphic to the original action of H on B.

For the rest of this subsection we restrict to the situation where $|\Delta| = \ell < \infty$. We identify Δ with the set $\{1, \dots, \ell\}$ and now we can think of the group B in another way:

$$B := \underbrace{K \times \cdots \times K}_{\ell}.$$

(An element of B, then, is an ℓ-tuple of elements from K – we can think of it as a function from Δ to K, just as before, by defining its image at i to equal b_i.) The action ψ of H on the group B now corresponds to permuting coordinates:

$$\psi : H \times B \to B, (h, (a_1, \dots, a_\ell)) \mapsto (a_{1^{h^{-1}}}, \dots, a_{\ell^{h^{-1}}}). \tag{5}$$

(E5.8) *Check that this function is the same as (4) (thereby proving that it defines an action of H on B as an object in Group).*

Example 17. Let us look at a concrete example. We will take $\Delta = \{1, 2\}$ and $K = \mathrm{Sym}(3)$. Then $B = K \times K$ is a group of order 36. Now let $H = \mathrm{Sym}(2) = \langle g \rangle$ where $g = (1, 2)$. Then $H \wr K$ is a group of order 72 and B is a subgroup of index 2. Observe that

$$B = \{((k_1, k_2), 1) \mid k_1, k_2 \in K\} \text{ and}$$
$$(H \wr K) \setminus B = \{((k_1, k_2), g) \mid k_1, k_2 \in K\}.$$

Now set $\Omega := \{1, \dots, 6\}$, and we will find a subgroup of $\mathrm{Sym}(\Omega) = \mathrm{Sym}(6)$ that is isomorphic to $H \wr K$. Let $\Delta = \{1, 2, 3\}$ and define

$$B := \{g \in \mathrm{Sym}(6) \mid \delta \in \Delta \Rightarrow \delta^g \in \Delta\}.$$

It is quite easy to see that $B = \mathrm{Sym}(\{1, 2, 3\}) \times \mathrm{Sym}(\{4, 5, 6\})$, a group of order 72 that is isomorphic to $K \times K$, just as required. Now define

$$h := (1, 4)(2, 5)(3, 6).$$

This is an element of order 2 and so generates $\mathrm{Sym}(2)$, as required. We claim that the group $G = \langle B, h \rangle$ is isomorphic to $H \wr K$.

(1) Observe first that h normalizes B: take $g \in B$ and $\delta \in \Delta$. Consider $\delta^{h^{-1}gh}$ - observe that $\delta^{h^{-1}} \notin \Delta$ and so $(\delta^{h^{-1}})^g \notin \Delta$, and so $\delta^{h^{-1}gh} = ((\delta^{h^{-1}})^g)^h \in \Delta$. We conclude that $h^{-1}gh \in B$ and we are done.
(2) Now by (E4.15) we conclude that HB is a group and it must be equal to G. Note that, since $H \cap B$ is clearly trivial, (E4.11) implies that $G \cong B \rtimes H$.
(3) Finally observe that the action by conjugation of $\langle h \rangle$ on B is isomorphic to the action by conjugation of H on K given at (5): Let $(g_1, g_2) \in B$ and observe that, for $i \in \Omega$,

$$i^{h^{-1}(g_1, g_2)h} = (i+3)^{(g_1, g_2)h} = (i+3)^{(g_1, g_2)} + 3 = i^{(g_2, g_1)}$$

as required. (We perform all additions modulo 6 here.)

[22]The definition of wreath product given here is sometimes called the *unrestricted wreath product* to distinguish it from a similar construction known as the *restricted wreath product*. To obtain this second construction, one observes first that the group B is in fact the direct *product* $\prod_{i \in \Omega} K$; now one replaces B by the direct *sum* $\bigoplus_{i \in \Omega} K$. The group H acts on this group just as before, and one can construct a semidirect product just as before. Of course if Ω is finite, then these two definitions yield the same group so there is no ambiguity in this case. (And this is the only case that we shall consider from here on.)

To conclude, let us consider the nature of the action of $G \cong Sym(3) \wr \mathrm{Sym}(2)$ on Ω. It is an easy matter to see that the action is transitive. The action is clearly not primitive, however, since one can define a G-congruence as follows:

(6) $$i \sim j \Longleftrightarrow i, j \leq 3 \text{ or } i, j > 3.$$

The blocks for this G-congruence are $\{1, 2, 3\}$ and $\{4, 5, 6, \}$.

(E5.9) *Prove that the group G in Example 17 is equal to*

$$\{g \in \mathrm{Sym}(6) \mid i \sim j \Longrightarrow i^g \sim j^g\},$$

where $\sim$ is the G-congruence defined at (6).

(E5.10) *Find a copy of $\mathrm{Sym}(2) \wr \mathrm{Sym}(3)$ inside $\mathrm{Sym}(6)$. Describe its action on $[1, 6]$.*

As the previous example suggests, wreath products have a close connection to imprimitivity. Suppose that, in addition to the action of H on $\Delta = \{1, \ldots, n\}$, the group K acts on some set Λ. Consider the following function:

$$\varphi : K \wr H \times (\Lambda \times \Delta) \to \Lambda \times \Delta,$$
$$((\lambda, i), (a_1, \ldots, a_n)h) \mapsto (\lambda, i)^{(a_1, \ldots, a_n)h} := (\lambda^{a_i}, i^h).$$

Let us check that φ is an action:[23]

(1) Clearly $(\lambda, i)^{(1, \ldots, 1)1} = (\lambda, i)$ as required.
(2) Now let $(a_1, \ldots, a_n)g$ and $(c_1, \ldots, c_n)h$ be elements in $K \wr H$. Then

$$\left((\lambda, i)^{(a_1, \ldots, a_n)g}\right)^{(c_1, \ldots, c_n)h} = (\lambda^{a_i}, i^g)^{(c_1, \ldots, c_n)h} = (\lambda^{a_i c_{i^g}}, i^{gh}).$$

On the other hand

$$(\lambda, i)^{(a_1, \ldots, a_n)g(c_1, \ldots, c_n)h} = (\lambda, i)^{(a_1 c_{1^g}, \ldots, a_n c_{n^g})gh} = (\lambda^{a_i c_{i^g}}, i^{gh})$$

as required.

Lemma 5.5. *If H acts faithfully on Δ and K acts faithfully on Λ, then $K \wr H$ acts faithfully on $\Lambda \times \Delta$.*

Proof. Suppose that $(a_1, \ldots, a_n)g \in A \wr G$ fixes every point $(\lambda, i) \in \Lambda \times \Delta$. Then $i^g = i$ for all $i \in \Delta$ and, since H acts on Δ faithfully, we conclude that $g = 1$. In addition $\lambda^{a_i} = \lambda$ for all $\lambda \in \Lambda$ and, since K acts faithfully on Λ we conclude that $a_i = 1$. We are done. $\square$

(E5.11) *Prove the converse to this lemma: If $A \wr G$ acts faithfully on $\Lambda \times \Omega$, then G acts faithfully on Ω and A acts faithfully on Λ.*

Define an equivalence relation on $\Lambda \times \Delta$ as follows:

$$(\lambda, i) \sim (\lambda', i') \Longleftrightarrow i = i'.$$

Lemma 5.6. *$\sim$ is a $K \wr H$-congruence. In particular, provided $|\Lambda|, |\Omega| > 1$, the action of $K \wr H$ on $\Lambda \times \Delta$ is imprimitive.*

Proof. Take two congruent elements in $\Lambda \times \Delta$: (λ, i) and (λ', i). Then $(\lambda, i)^{(a_1, \ldots, a_n)g} = (\lambda^{a_i}, i^g)$ and $(\lambda', i)^{(a_1, \ldots, a_n)g} = (\lambda'^{a_i}, i^g)$. Since the second entry of both pairs is equal to i^g, we conclude that

$$(\lambda, i)^{(a_1, \ldots, a_n)g} \sim (\lambda', i)^{(a_1, \ldots, a_n)g}$$

as required. $\square$

It turns out that all faithful imprimitive actions lie in a subgroup of the given form. (You should compare this proposition to (E4.25).)

Proposition 5.7. *Let Ω be a finite set of order n, and let $\sim$ be an equivalence relation on Ω with $\ell > 1$ equivalence classes all of size $k := \frac{n}{\ell} > 1$.*

(1) *$\sim$ is a G-congruence for a unique subgroup G of $\mathrm{Sym}(\Omega)$ that is isomorphic to $\mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$;*

[23] Note that in the preceding definition, and in what follows, I have dropped some brackets when writing down an element of the wreath product. I hope this makes things a little clearer.

(2) *if $\sim$ is a J-congruence for some group $J \leq \mathrm{Sym}(\Omega)$, then $J \leq G$.*

Proof. Let Λ be a set of size k, and let Δ be a set of size ℓ. Clearly $\mathrm{Sym}(\Lambda)$ (resp. $\mathrm{Sym}(\Delta)$) act faithfully on Λ (resp. Δ) and so, by the previous two lemmas, $\mathrm{Sym}(\Lambda) \wr \mathrm{Sym}(\Delta)$ acts faithfully and imprimitively on $\Lambda \times \Delta$ with ℓ equivalence classes, each of order k.

We can choose a bijection between Ω and $\Lambda \times \Delta$ that maps $\sim$-equivalence classes to sets of form $\{(\lambda, i) \mid \lambda \in \Lambda\}$ (where i is fixed). This yields an injective homomorphism $\mathrm{Sym}(\Lambda) \wr \mathrm{Sym}(\Delta) \to \mathrm{Sym}(\Omega)$, as required. Write G for the image of this homomorphism.

To complete the proof we must show that if $\sim$ is a J-congruence for some group $J \leq \mathrm{Sym}(\Omega)$, then J is a subgroup of G (this will yield (ii) as well as the uniqueness part of (i)). We maintain the identification of Ω with $\Lambda \times \Delta$ and we let $J \leq \mathrm{Sym}(\Omega)$ be a group for which $\sim$ is a J-congruence. Suppose that $j \in J$, let $(\lambda, i) \in \Omega$ and write $(\lambda, i)^j = (\lambda', i')$. Since $\sim$ is a J-congruence, the choice of λ does not affect the value of i', in other words h defines a permutation $g \in \mathrm{Sym}(\Delta)$ such that $i' = i^g$ for all $i \in \Delta$. Now, for fixed i, j defines a permutation $a_i \in \mathrm{Sym}(\Lambda)$ for which $\lambda' = \lambda^{a_i}$. In other words

$$(\lambda, i)^j = (\lambda^{a_i}, i^g) = (\lambda, i)^{(a_1, \ldots, a_n)g}$$

where $(a_1, \ldots, a_n)g \in K \wr H$. Thus $J \leq K \wr H$ and the result follows. $\square$

The previous lemma can be expressed in categorical language – the following exercise explores this idea.

(E5.12) *Our category is called* ***ImprimTrans***

Objects*: Our objects are pairs $(\Omega, \sim)$ where Ω is a finite set and $\sim$ is an equivalence relation for which all equivalence classes have the same size.*

Arrows*: An arrow $(\Omega, \sim) \to (\Omega', \sim')$ is a function $f : \Omega \to \Omega'$ such that $x \sim y \Longrightarrow f(x) \sim' f(y)$.*

(1) *Prove that* ***ImprimTrans*** *is a category.*
(2) *Prove that if X is an object in* ***ImprimTrans****, then $\mathrm{Aut}(X) \cong \mathrm{Sym}(\Lambda) \wr \mathrm{Sym}(\Delta)$ for some finite sets Λ and Δ.*
(3) *Prove that if G acts on $X = (\Omega, \sim)$ as an object from* ***ImprimTrans****, then $\sim$ is a G-congruence, and conversely.*
(4) *Consider an alternative category – called* ***Imprim*** *in which we drop the condition that equivalence classes all have the same size. What would $\mathrm{Aut}(X)$ look like in this case?*

The next proposition is a refinement of Proposition 4.12, making use of Proposition 5.7.

Proposition 5.8. *Let $H \leq \mathrm{Sym}(\Omega)$ where $|\Omega| < \infty$. One of the following holds:*

(1) *H is intransitive and $H \leq \mathrm{Sym}(k) \times \mathrm{Sym}(n-k)$ for some $1 < k < n$;*
(2) *H is transitive and imprimitive and $H \leq \mathrm{Sym}(k) \wr \mathrm{Sym}(\ell)$ for some $1 < k, l < n$ with $n = kl$;*
(3) *H is primitive.*

Proof. We apply Proposition 4.12 to H. If H satisfies (1) of that proposition, then (1) of this proposition holds and we are done. Assume that H satisfies (2) of that proposition, i.e. that H acts transitively on Ω.

If H is imprimitive, then (E5.5) implies that the associated blocks all have equal size, and we can apply Proposition 5.7, and the result follows immediately. $\square$

Proposition 5.8 implies that, to understand the subgroup structure of $\mathrm{Sym}(n)$, we need to understand the finite primitive actions.

(E5.13) *Let H be a primitive subgroup of $\mathrm{Sym}(n)$. Prove that*

(1) *if H contains a transposition, then $H = \mathrm{Sym}(n)$.*
(2) *if H contains a 3-cycle, then H contains $\mathrm{Alt}(n)$.*[24]

(E5.14) *Let Ω be a finite set of order n, and let $X = (\Omega, \sim)$ (resp. $Y = (\Omega, \sim')$ be an object from* ***ImprimTrans****. Assume that neither $\sim$ nor $\sim'$ are trivial. Let $H = \mathrm{Aut}(X)$ (resp. $K = \mathrm{Aut}(Y)$) be subgroups of $\mathrm{Sym}(n)$.*

(1) *Use the result of the previous exercise to prove that H is maximal.*
(2) *Are H and K conjugate? How many conjugacy classes of subgroups isomorphic to H does $\mathrm{Sym}(n)$ contain?*
(3) *Describe the intersection of H and $\mathrm{Alt}(n)$.*

[24]This is a famous result of Jordan. Its proof is a little tricky.