CLASSICAL GROUPS: DISCUSSION CLASS

The aim of this class is to discuss the (B, N)-structure of $\mathrm{GL}_n(k)$.[1] Let V be an n-dimensional vector space over a field k. Let $\{e_1, \ldots, e_n\}$ be a basis for V and let $G = \mathrm{GL}_n(k)$.

(D1) The chain of subspaces

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \cdots \subset \langle e_1, \ldots, e_{n-1} \rangle$$

is called a *chamber*.

Let B be the stabilizer in G of this chamber. What is B?

(D2) Given a basis $\{e_1, \ldots, e_n\}$, the corresponding *frame* is the set

$$\mathcal{F} = \{\langle e_1 \rangle, \langle e_2 \rangle, \ldots, \langle e_n \rangle\}.$$

Let N be the stabilizer in G of the given frame. What is N?

(D3) Show that $G = \langle N, B \rangle$.

(D4) Let $H = B \cap N$. Show that H is a normal subgroup of N.

(D5) The group $W := N/H$ is called the *Weyl group* of G. What well-known group is N/H isomorphic to?

[1] This discussion class is taken from a course given by Michael Giudici.
My thanks to him for letting me use it.