HOMEWORK 01

PLASENCIA MILTON

1) Y takes values in $\{1..N\}$ and $Pr(Y=i)=\frac{1}{N}$

$$E(Y)=\sum_{i=1}^{N} y_i \cdot Pr(Y=i) = \sum_{i=1}^{N} i \cdot Pr(Y=i) = \sum_{i=1}^{N} i \cdot \frac{1}{N} = \frac{1}{N}\sum_{i=1}^{N} i \overset{\text{Gauss}}{=} \frac{1}{N}\cdot\frac{N(N+1)}{2} = \frac{N+1}{2} \quad //$$

2) $X \sim N(\mu, \sigma^2)$ $\quad E(X)=\int_{-\infty}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

substitution $y=\frac{x-\mu}{\sigma} \Rightarrow dy=\frac{dx}{\sigma} \Rightarrow dx = dy \cdot \sigma$ $\quad$ we have also $y \cdot \sigma = x-\mu$
$\Rightarrow x = \mu + \sigma y$

So $E(X) = \int_{-\infty}^{+\infty} (\mu+\sigma y) \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2}} \cdot \sigma \cdot dy =$

$$= \mu \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{\mathrm{I}} + \sigma \underbrace{\int_{-\infty}^{+\infty} y \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{\mathrm{II}}$$

I $=1$ because $\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ is the density function of a $N(0,1)$ so its integral over $\mathbb{R}$ is equal to 1

II $=0$ because is a odd function $\Rightarrow f(-y) = -y \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(-y)^2}{2}} = -y \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = -f(y)$

$= \mu \cdot 1 + \sigma \cdot 0 = \mu \quad //$

3) $p(x,y) = \begin{cases} \frac{1}{30}(x+y) & x=0,1,2 \\ & y=0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$ $\quad$ X and Y are independent?
they are independent if $p(x,y) = p(x)\cdot p(y)$

$$p(x) = \sum_{y=0}^{3} p(x,y) = \frac{x}{30} + \frac{1+x}{30} + \frac{2+x}{30} + \frac{3+x}{30} = \frac{6+4x}{30} = \boxed{\frac{3+2x}{15}}$$

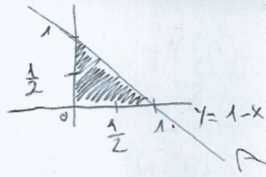
$$p(y) = \sum_{x=0}^{2} p(x,y) = \frac{y}{30} + \frac{1+y}{30} + \frac{2+y}{30} = \frac{3+3y}{30} = \boxed{\frac{1+y}{10}}$$

$p(x,y) \overset{?}{=} p(x)\,p(y) \Rightarrow$ to prove that $\frac{1}{30}(x+y) = \frac{1}{15}(3+2x)\cdot\frac{1}{10}(1+y)$

$\Rightarrow \frac{1}{30}(x+y) = \frac{1}{150}(3+2x)(1+y)$ which is not satisfied in the domain of X ($x=0,1,2$) and Y ($y=0,1,2,3$)

$\Rightarrow$ X and Y are not independent

4) $$f(x,y)=\begin{cases}24xy & 0<x<1,\ 0<y<1,\ x+y<1\\ 0\end{cases}$$



$$f(y)=\int_0^{1-y}24xy\,dx=24y\int_0^{1-y}x\,dx=24y\cdot\left[\frac{x^2}{2}\right]_0^{1-y}=\frac{24}{2}y(1-y)^2=12y(1-y)^2 \;//$$

So $$f(y)=\begin{cases}12y(1-y)^2 & 0<y<1\\ 0 & \text{otherwise}\end{cases}$$

$$f(x|y)=\frac{f(x,y)}{f(y)}=\frac{24xy}{12y(1-y)^2}=\frac{2x}{(1-y)^2}\Rightarrow f(x|y=\tfrac{1}{2})=\frac{2x}{(1-\frac{1}{2})^2}=8x$$

$$\Rightarrow f(x|y=\tfrac{1}{2})=\begin{cases}8x & 0<x<\frac{1}{2}\\ 0 & \text{elsewhere}\end{cases}$$ if $x>\frac{1}{2}$, for $y=\frac{1}{2}$ we are out our domain

5) $$f(x,y)=\frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\cdot x^{\alpha_1-1}y^{\alpha_2-1}(1-x-y)^{\alpha_3-1}$$ with $0<x<1$, $0<y<1$, $x+y<1$

compute $$f(x)=\int_0^{1-x}f(x,y)\,dy=\overbrace{\frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}x^{\alpha_1-1}}^{c}\int_0^{1-x}y^{\alpha_2-1}(1-x-y)^{\alpha_3-1}\,dy=$$ (without c)

substitution $y=(1-x)u$, $dy=(1-x)du$ $$\Rightarrow=\int_0^1(1-x)^{\alpha_2-1}u^{\alpha_2-1}(1-x-(1-x)u)^{\alpha_3-1}(1-x)\,du=$$

$$=(1-x)^{\alpha_2}\int_0^1u^{\alpha_2-1}(1-x-(1-x)u)^{\alpha_3-1}\,du=(1-x)^{\alpha_2+\alpha_3-1}\int_0^1u^{\alpha_2-1}(1-u)^{\alpha_3-1}\,du$$

in which $$\int_0^1u^{\alpha_2-1}(1-u)^{\alpha_3-1}\,du=\frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)}$$

So we put all the ___ together and we have that

$$f(x)=\int_0^{1-x}f(x,y)\,dy=\frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\cancel{\Gamma(\alpha_2)}\cancel{\Gamma(\alpha_3)}}\cdot x^{\alpha_1-1}\cdot(1-x)^{\alpha_2+\alpha_3-1}\cdot\frac{\cancel{\Gamma(\alpha_2)}\cancel{\Gamma(\alpha_3)}}{\Gamma(\alpha_2+\alpha_3)}=$$

$$=\frac{\Gamma(\alpha_1+\alpha_2+\alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2+\alpha_3)}x^{\alpha_1-1}(1-x)^{\alpha_2+\alpha_3-1}$$ which is the density function of a $\beta(\alpha_1,\alpha_2+\alpha_3)$ //