ON REGULARITY OF PARALLEL SETS

Nikolay Pogodaev

October 27, 2015

Abstract

The paper reviews various results concerning regularity of parallel sets.

Keywords: parallel sets, interior sphere condition, tubular neighbourhoods, regularity

1 Introduction

The paper contains a number of facts about regularity of parallel sets. These facts are known, but scattered in the literature. The paper is written mostly for myself and not intended for publication.

Given a closed set $A \subseteq \mathbb{R}^n$, we denote by $d_A(x)$ the distance between $x \in \mathbb{R}^n$ and A:

$$d_A(x) = \inf_{a \in A} |x - a|,$$

and by A_t (with $t > 0$) the t-neighbourhood of A:

$$A_t = \left\{ x : \; d_A(x) \le t \right\}.$$

Definition 1. We say that $E \subseteq \mathbb{R}^n$ is a *parallel set* (of a subset of $\mathbb{R}^n$) if $E = A_t$ for some $A \subset E$ and $t > 0$ [1].

2 Lipschitz manifolds

Definition 2. A map $f \colon X \to Y$ between metric spaces X and Y is called *bi-Lipschitz* if f is a homeomorphism and both f and f^{-1} are Lipschitz.

Lemma 1. *Let $U \subseteq \mathbb{R}^m$ be open and $f \colon U \to \mathbb{R}^n$ be such that*

$$L_1 |x_1 - x_2| \le |f(x_1) - f(x_2)| \le L_2 |x_1 - x_2| \tag{1}$$

for some positive L_1 and L_2. Then, the map $f \colon U \to f(U)$ is bi-Lipschitz.

Proof. We only need to show that f is a homeomorphism between U and $f(U)$. Then the Lipschitz continuity of both f and f^{-1} would follow immediately from (1). The map $f \colon U \to f(U)$ is clearly surjective. But it is also injective, because if there are two different points $x_1, x_2 \in U$ with $f(x_1) = f(x_2)$, then $|f(x_1) - f(x_2)| = 0$, which contradicts (1). □

[1] The synonyms are: sets with interior sphere property (P. Cannarsa, H. Frankowska), sets with interior ball property (O. Alvarez, P. Cardaliaguet, R. Monneau), tubular neighbourhoods (J.H.G. Fu).

Definition 3. A subset $\mathcal{M} \subset \mathbb{R}^n$ is an *m-dimensional Lipschitz manifold* if for every point $x_0 \in \mathcal{M}$ there exist its open neighbourhood $V \subseteq \mathbb{R}^n$, an open set $U \subseteq \mathbb{R}^m$, and a map $\varphi\colon U \to V$ such that $\varphi(U) = \mathcal{M} \cap V$ and $\varphi\colon U \to \varphi(U)$ is bi-Lipschitz.

3 The Implicit Function Theorem

Given a Lipschitz map $f\colon \mathbb{R}^n \to \mathbb{R}$, we define its Clarke subdifferential as follows

$$\partial f(x) = \operatorname{co}\left\{\lim \nabla f(x_k) \;:\; x_k \to x \text{ and all } \nabla f(x_k) \text{ exist}\right\}.$$

All relevant properties of the subdifferential may be found in [1, 8].

Now, as for smooth maps, we may talk about regular and critical points of f.

Definition 4. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. If $0 \in \partial f(x)$, then x is called a *critical point* of f, and the image $f(x)$ is called a critical value. If $x \in \mathbb{R}^n$ fails to be a critical point, it is called a *regular point* of f. Any $t \in \mathbb{R}$ failed to be a critical value is a *regular value* of f.

A version of the Implicit Function Theorem also holds.

Theorem 1 (the Implicit Function Theorem [1])**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and $x_0 \in \mathbb{R}^n$ be a regular point of f. Then there are an open neighbourhood V of x_0, an open subset U of $\mathbb{R}^{n-1}$, and a bi-Lipschitz map $\varphi\colon U \to V \cap \{x \;:\; f(x) = f(x_0)\}$.*

We immediately get the following corollary.

Corollary 1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. If t is a regular value of f, then the level set $\{x \;:\; f(x) = t\}$ is an $(n-1)$-dimensional Lipschitz manifold.*

4 Regularity of parallel sets

Recall that the distance function d_A is Lipschitz, and its Clarke subdifferential can be written as follows

$$\partial d_A(x) = \operatorname{co}\left\{\frac{x-p}{|x-p|} \;:\; p \in \pi_A(x)\right\} \quad \text{whenever} \quad x \notin A, \tag{2}$$

where $\pi_A(x)$ denotes the set of all projections of x onto a closed set $A \subset \mathbb{R}^n$, i.e.,

$$\pi_A(x) = \left\{p \;:\; |x-p| = d_A(x)\right\}.$$

See [4, Lemma 4.2] for details.

Assume that t is a regular value of d_A. Then, by Corollary 1, the level set $\{x \;:\; d_A(x) = t\}$ is an $(n-1)$-dimensional Lipschitz manifold.

Lemma 2. *If t is a regular value of d_A, then $\partial A_t = \{x \;:\; d_A(x) = t\}$.*

Proof. The inclusion $\partial A_t \subseteq \{x \;:\; d_A(x) = t\}$ is obvious. Now let x_0 lie on the level set. If x_0 is an interior point of A_t, then d_A attains a local maximum at x_0. Thus, by [1, Proposition 2.3.2], $0 \in \partial d_A(x_0)$. But the latter is impossible, since t is regular. □

Thus, ∂A_t is a Lipschitz manifold whenever t is a regular value of d_A. But what happens if t is a critical value? The key observation that will allow us to deal with the case is as follows.

Lemma 3. *Let $A \subset \mathbb{R}^n$ be compact. Then all $t > \operatorname{diam} A$ are regular values of d_A.*

Proof. Formula (2) implies that $x \notin A$ is a critical point if and only if $x \in \operatorname{co} \pi_A(x)$. The points of $\pi_A(x)$ lie on the sphere of radius $d_A(x)$ about x. Fixing a point $p \in \pi_A(x)$, we notice that A is contained in the ball of radius $\operatorname{diam} A$ centered at p. Hence, if $d_A(x) > \operatorname{diam}(A)$, then we can be sure that $x \notin \operatorname{co} \pi_A(x)$. $\square$

Theorem 2. *Let $A \subset \mathbb{R}^n$ be compact.*

(*a*) *If t is a regular value of d_A, then ∂A_t is a compact $(n-1)$-dimensional Lipschitz manifold.*

(*b*) *If t is a critical value of d_A, then ∂A_t belongs to the union of a finite number of compact $(n-1)$-dimensional Lipschitz manifolds.*

Proof. The first statement follows from Corollary 1 and Lemma 2. To prove the second statement, we employ the idea of J. Rataj and S. Winter [7]. We split A into finitely many closed subsets $A^1, \dots, A^m$ of diameters less than t. Now $\partial A_t \subseteq \partial A^1_t \cup \dots \cup \partial A^m_t$ and each A^i_t is a compact $(n-1)$-dimensional Lipschitz manifold by Lemma 3 and (a). $\square$

Critical values are not rare, as it may seem at first glance. In [3] Ferry constructed a compact set $A \subset \mathbb{R}^4$ such that all points of $]0, \frac{1}{100}[$ are critical values of d_A.

5 Parallel sets and sets with positive reach

To define sets with positive reach, we need a little more notations. Let $\operatorname{Unp}(A)$ denote the set of all points x having a unique projection onto A:

$$\operatorname{Unp}(A) = \left\{x : \pi_A(x) \text{ is a singleton}\right\}.$$

Then, the quantity

$$\operatorname{reach}(A) = \sup\left\{r : A_r \subseteq \operatorname{Unp}(A)\right\}$$

is called the reach of A.

Definition 5. We say that $A \subseteq \mathbb{R}^n$ has *positive reach* if $\operatorname{reach}(A) > 0$.

The next theorem by J.H.G. Fu [4] establishes the connection between parallel sets and sets with positive reach.

Theorem 3. *Let $A \subset \mathbb{R}^n$ be closed. If t is a regular value of d_A, then $\overline{(\mathbb{R}^n \setminus A_t)}$ has positive reach.*

The set $\overline{(\mathbb{R}^n \setminus A_t)}$ may not have positive reach when t is a critical value.

Example 1 (from [5]). Suppose that $A \subset \mathbb{R}^3$ is the union of two squares

$$Q^{-1} = [-1,1] \times [-1,1] \times \{-1\} \quad \text{and} \quad Q^1 = [-1,1] \times [-1,1] \times \{1\}.$$

Then $t = 1$ is a critical value of d_A. The square $Q^0 = [-1,1] \times [-1,1] \times \{0\}$, witch is formed by the intersection of $(Q^{-1})_1$ and $(Q^1)_1$, clearly belongs to A_1. It is easy to see that the points lying on the diagonals of Q^0 have multiple projections onto $\overline{\mathbb{R}^3 \setminus A_1}$ (see Fig. 1).

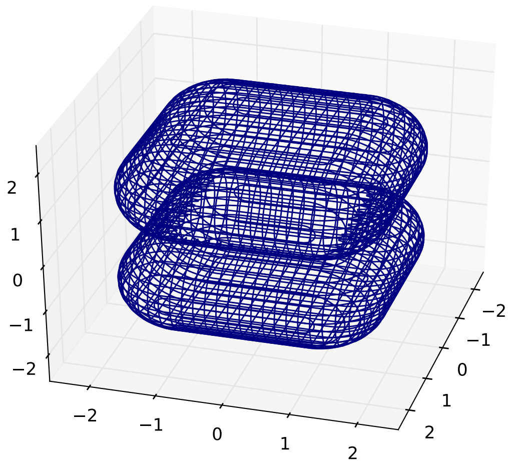

Figure 1: The set A_1. The points lying on the diagonals of the square $Q^0 = [-1,1] \times [-1,1] \times \{0\}$ have multiple projections onto $\overline{\mathbb{R}^3 \setminus A_1}$.

6 Rectifiable sets

Definition 6 ([2])**.** A set E is *m-rectifiable* if there exists a Lipschitz function f mapping a bounded subset $K \subset \mathbb{R}^m$ onto E.

Theorem 4. *A finite union of m-rectifiable sets is m-rectifiable.*

Proof. Let $E_1, \ldots, E_k$ be m-rectifiable subsets of $\mathbb{R}^n$. By $f_j \colon K_j \to E_j$, $j = 1, \ldots, k$, we denote the corresponding maps. Since all f_j are Lipschitz, the numbers

$$a_j = \sup\{|f_j(x)| \,:\, x \in K_j\}, \quad j = 1, \ldots, k,$$

are finite. Without loss of generality, we may assume that

$$|x - y| \geq a_i + a_j \quad \text{whenever} \quad x \in E_i,\ y \in E_j\ , i \neq j.$$

In this case the map f defined by

$$f(x) = f_j(x), \quad x \in K_j, \quad j = 1, \ldots k,$$

is Lipschitz continuous on $K_1 \cup \cdots \cup K_k$ and surjective.

□

It follows from Theorems 2 and 4 that ∂A_t is $(m-1)$-rectifiable whenever A is compact. In particular, by [2, 3.2.39], the set ∂A_t possesses the $(n-1)$-dimensional Minkowski content $\mathcal{M}^{n-1}(\partial A_t)$, which is equal to $\mathcal{H}^{n-1}(\partial A_t)$.

7 Criterion for parallel sets

Given a closed set E, we can use the next theorem to check whether E is a parallel set or not. Below $B_R(y)$ denotes the closed ball of radius R around y.

Theorem 5 ([6]). *Let $E \subset \mathbb{R}^n$ be a nonempty closed set and R_0 be a positive constant. If for any $x \in \partial E$ there exist $y \in E$ and $R \geq R_0$ such that $x \in B_R(y) \subseteq E$ then E is a parallel set.*

Note that Theorem 5 does not say that E is the R_0-neighbourhood of a subset $A \subset E$. It only says that E is the r-neighbourhood of a subset $A \subset E$ for *some* r. Actually, one can safely take any $r \leq \frac{nR_0}{2\sqrt{n^2-1}}$ (see [6] for details).

References

[1] F. H. Clarke. *Optimization and nonsmooth analysis.* 1983.

[2] H. Federer. *Geometric measure theory.* 1969.

[3] S. Ferry. When ε-boundaries are manifolds. *Fundam. Math.*, 90:199–210, 1976.

[4] J. H. G. Fu. Tubular neighborhoods in Euclidean spaces. *Duke Math. J.*, 52:1025–1046, 1985.

[5] C. Nour, R. Stern, and J. Takche. The union of uniform closed balls conjecture. *Control Cybern.*, 38(4):1525–1534, 2009.

[6] C. Nour, R. J. Stern, and J. Takche. Validity of the union of uniform closed balls conjecture. *J. Convex Anal.*, 18(2):589–600, 2011.

[7] J. Rataj and S. Winter. On volume and surface area of parallel sets. *Indiana Univ. Math. J.*, 59(5):1661–1685, 2010.

[8] R. Vinter. *Optimal control.* Boston, MA: Birkhäuser, 2000.