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# The Retrieval of Harmonics from a Covariance Function

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*Summary*

A new method for retrieving harmonics from a covariance function is introduced. The method is based on a theorem of Caratheodory about the trigonometrical moment problem. The relation between this method and the 'maximum entropy' spectral estimator is discussed, and the effect of a small addition of a noise component is investigated. A numerical example is discussed.

## Introduction

Several new methods for estimating the spectra of stationary stochastic processes and noise fields have recently been developed:

1. The 'maximum likelihood' estimator (Capon 1969);
2. The 'maximum entropy' or 'autoregression' estimator (which we will refer to in this paper as MEE) due to Burg (Burg 1972; see also Ulrych 1972 and Lacoss 1971). This estimator was treated from the point of view of autoregression processes by Parzen (1969).
3. Spectral estimators based on the eigenvalues and eigenvectors of the covariance matrix, suggested by the present author (Pisarenko 1972).

These methods are particularly useful when good resolution of spectral peaks is desired, and they have been applied in various scientific fields, in geophysics, for example, in studying free oscillations of the Earth, seismic surface waves, spectra of microseisms, etc.

In this paper we introduce yet another method of detecting spectral peaks, based on a theorem of Caratheodory concerning the trigonometrical moment problem, and we show that MEE is in fact a smoothed version of our new method. In the last section of this paper we investigate the effect on the estimator of the addition of a small amount of noise to the record.

## 1. On the restitution of a spectrum from a finite segment of the covariance function

Let $x(t)$ be a complex stationary stochastic process, stationary in the wide sense, which depends on discrete time (see, for example, Doob 1953). We assume the mean value of the process to be zero. The spectral function $F(\lambda)$ of the process $x(t)$ is related to its covariance function $B(\kappa)$ by:

$$B(\kappa) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \exp(i\lambda\kappa)\, dF(\lambda) \quad -\infty \leqslant \kappa \leqslant \infty. \tag{1}$$

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When $x(t)$ is assumed to be a real process we shall say so explicitly.

If $F(\lambda)$ is absolutely continuous its derivative $F'(\lambda)$ is called the spectral density $P(\lambda)$:

$$P(\lambda) = F'(\lambda). \tag{2}$$

If $F(\lambda)$ has a jump $\rho_j$ at frequency $\lambda_j$, i.e. if $F(\lambda_j+0)-F(\lambda_j-0) = \rho_j$, then (2) is still valid if we introduce $\delta$-functions; in this case $P(\lambda)$ contains a component $\rho_j\,\delta(\lambda-\lambda_j)$ and the covariance function $B(\kappa)$ contains a harmonic $\rho_j \exp(i\lambda_j \kappa)$.

Suppose we are given a segment of the covariance function of length $2m$, so that we have

$$B(\kappa) \quad -m \leqslant \kappa \leqslant m. \tag{3}$$

What can we say of the true spectral function $F(\lambda)$? To start with, we assume that the values $B(\lambda)$ are known absolutely, without any noise or error.

*Assertion* 1. Let the segment of $B(\kappa)$ be such that the covariance matrix $B$ (whose order is $m+1$) is non-degenerate:

$$B = \{B(\kappa-j)\}_{\kappa,\, j=0,\,1,\,\ldots m} \tag{4}$$

Then there exist infinitely many spectral functions $F(\lambda)$ which are related by equation (1) to the given function $B(\kappa)$ when $|\kappa| \leqslant m$.

To prove this assertion, consider the MEE $P_E(\lambda)$:

$$P_E(\lambda) = \beta_{00}\left|\sum_{\kappa=0}^{m} \beta_{0\kappa} \exp(-i\lambda\kappa)\right|^{-2} \quad -\pi < \lambda \leqslant \pi \tag{5}$$

where $\{\beta_{\kappa j}\}_{\kappa,\, j\,=\,0,\,1,\,\ldots,\,m}$ is the inverse of the matrix $B$. This estimator has been called the 'maximum entropy' estimator, since it maximizes the entropy integral (see Shannon 1948)

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \log P(\lambda)\, d\lambda$$

among all the spectral density functions which satisfy the relations

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \exp(i\lambda\kappa)\, P(\lambda)\, d\lambda = B(\kappa) \qquad |\kappa| \leqslant m.$$

Note that $P_E(\lambda)$ is just the spectral density of an autoregressive process of order $m$ (see Jenkins & Watts 1970), of which the first $m$ values of the covariance function are equal to the given values of the segment of $B(\kappa)$.

Since $B$ is assumed to be positive definite, the matrix $R(\mu) = B-\mu I$ will also be positive definite for sufficiently small $\mu$ (here $I$ denotes the identity matrix of order $m+1$). Thus for $\mu$ small enough the matrix $B$ can be represented as the sum of two positive definite matrices

$$B = R(\mu)+\mu I. \tag{6}$$

Denoting the inverse of $R(\mu)$ by $\{r_{\kappa j}\}_{\kappa,j\,=\,0,\,1,\,\ldots,\,m}$ we get a family of spectral densities

$$r_{00}\left|\sum_{\kappa=0}^{m} r_{0\kappa} \exp(-i\lambda\kappa)\right|^{-2} + \mu \tag{7}$$

which correspond exactly to the given segment of the covariance function $B(\kappa)$. We notice that the second term in (7) represents the $\delta$-functions $\mu\delta(\kappa)$ in the covariance function. The assertion is proved.

It follows from a theorem of Caratheodory (see Section 2 below) that for any $\lambda_0$ such that $-\pi < \lambda_0 \leqslant \pi$ there exists a spectral function $F(\lambda)$ which corresponds to the given segment of $B(\kappa)$ and has a jump at the frequency $\lambda_0$; moreover, it is possible to choose a function $F(\lambda)$ which has jumps at any prescribed finite set of frequencies $\lambda_0, \lambda_1, \ldots, \lambda_n$ (although the amplitudes of these jumps cannot be chosen arbitrarily, and these amplitudes will generally tend to zero as $n$ increases).

Thus if a spectral estimate of a function of discrete time is based on a finite segment of the covariance function, and the corresponding covariance matrix is non-degenerate, then it must be taken into account that even if the covariance function values are known exactly there exist infinitely many corresponding spectral functions. Also, there is no upper limit for the spectral density at any frequency $\lambda$, since it can always be arranged that a $\delta$-function is present at the frequency $\lambda$. Nevertheless, in such situations some upper and lower limits can be obtained for spectral functions which have been smoothed by some spectral window. In particular, the following theorem is true.

*Theorem.* Let $w(\lambda)$ be a spectral window, that is, a real periodic function of $\lambda$, having period $2\pi$, differentiable $r$ times (where $r \geqslant 1$), and

$$|w^{(r)}(\lambda)| \leqslant K$$

where $K$ is some constant. We denote by $B(\kappa)$ the covariance function related to the spectral function $F(\lambda)$ by equation (1). Then for the smoothed spectrum

$$(2\pi)^{-1} \int_{-\pi}^{\pi} w(\lambda-\lambda_0)\, dF(\lambda)$$

the following upper and lower limits exist, based on the finite segment (3) of the covariance function:

$$\begin{aligned} &\sum_{\kappa=-m}^{m} w_\kappa \exp(-i\lambda_0 \kappa) B^*(\kappa) - KB(0)\, m^{-r}.\, [(4/\pi^2)\log(m+1)+4] \\ &\leqslant (2\pi)^{-1} \int_{-\pi}^{\pi} w(\lambda-\lambda_0)\, dF(\lambda) \\ &\leqslant \sum_{\kappa=-m}^{m} w_\kappa \exp(-i\lambda_0 \kappa) B^*(\kappa) + KB(0)\, m^{-r}.\, [(4/\pi^2)\log(m+1)+4] \end{aligned} \tag{8}$$

where $w_\kappa$ are the Fourier coefficients of the function $w(\lambda)$:

$$w_\kappa = (2\pi)^{-1} \int_{-\pi}^{\pi} \exp(-i\lambda\kappa)\, w(\lambda)\, d\lambda.$$

The proof is given in the Appendix. By this theorem we can estimate the power contribution of both the continuous and the discrete spectral components cut out by the spectral window $w(\lambda-\lambda_0)$.

For example, take the Parzen window of width $\Delta$ (see Jenkins & Watts 1969):

$$w(\lambda) = \begin{cases} 1-6(\lambda/\Delta)^2+6|\lambda/\Delta|^3 & |\lambda| \leqslant \Delta/2 \\ 2(1-|\lambda/\Delta|)^3 & \Delta/2 \leqslant |\lambda| \leqslant \Delta \\ 0 & \Delta \leqslant |\lambda| \leqslant \pi. \end{cases} \tag{9}$$

It is easy to verify that the second derivative of $W(\lambda)$ exists and is less than $12/\Delta^2$, and that (Jenkins & Watts 1969)

$$w_\kappa = \frac{3\Delta}{8\pi}\left[\frac{\sin(\kappa\Delta/4)}{\kappa\Delta/4}\right]^4.$$

For $B(\kappa)$ we take the autoregression covariance:

$$B(\kappa) = \alpha^{|\kappa|} \qquad |\alpha| \leqslant 1$$

$$F'(\lambda) = (1-\alpha^2)(1+\alpha^2-2\cos\alpha\lambda)^{-1}$$

and we assume $\Delta = 1$, $\alpha = 0{\cdot}5$, $\lambda_0 = \frac{1}{2}\pi$, and $m = 50$. Then to two significant figures the integral

$$(2\pi)^{-1}\int_{-\pi}^{\pi} w(\lambda-\lambda_0)\,dF(\lambda) = (2\pi)^{-1}\int_{-\pi}^{\pi}\frac{(1-\alpha^2)\,w(\lambda-\lambda_0)}{1+\alpha^2-2\alpha\cos\lambda}\,d\lambda \tag{10}$$

is equal to $0{\cdot}076$. To the same accuracy, the sum

$$\sum_{\kappa=m}^{m} w_\kappa \exp(-i\lambda_0\kappa)\,B^*(\kappa)$$

is $0{\cdot}077$, and the term

$$KB(0)\,m^{-r}[(4/\pi^2)\log(m+1)+4] = 12{\cdot}50^{-2}[(4/\pi^2)\log 51+4]$$

is $0{\cdot}027$. Thus for the true value $0{\cdot}076$ of the integral (10) we find the limits $0{\cdot}077 \pm 0{\cdot}027$. We see that the limits (9) give rather rough estimates for (11) even for comparatively large $m$ and a rather broad width $\Delta$ of the spectral window. Naturally, as $m$ and $\Delta$ decrease the estimators (8) become worse.

Thus when only a finite segment of the covariance function is known, and there is no *a priori* information about the true spectrum, we should not speak of the true spectrum but rather of the whole family of true spectra.

## 2. A new method of retrieving harmonics from a finite segment of the covariance function

In this section we make use of the following theorem of Caratheodory (for a proof due to Szegö, see Grenander & Szegö 1958, Chapter 4).

*Theorem.* Let $c_1, \ldots, c_m$ be complex numbers (not all zero) and assume $m > 1$. Then there exists some integer $r$, $1 \leqslant r \leqslant m$ and some real numbers $\rho_j$ and $\omega_j$ $(j = 1, \ldots, r)$ such that $\rho_j > 0$, $-\pi < \omega_j \leqslant \pi$, and $\omega_j \neq \omega_\kappa$ when $j \neq \kappa$, and such that the following representation of the sequence of numbers $c_\kappa$ is true:

$$c_\kappa = \sum_{j=1}^{r} \rho_j \exp(i\omega_j\kappa) \qquad \kappa = 1, \ldots, m. \tag{11}$$

The constants $\rho_j$ and $\omega_j$ are determined uniquely.

We now consider the segment (3) of the covariance function $B(\kappa)$ and put in Caratheodory's theorem $c_\kappa = B(\kappa)$, $\kappa = 1, \ldots, m$. Then (11) gives the following representation for $B(\kappa)$:

$$B(\kappa) = \sum_{j=1}^{r} \rho_j \exp(i\omega_j\kappa) \qquad \kappa = 1, \ldots, m;\ r \leqslant m. \tag{12}$$

When $\kappa = 0$, then generally

$$B(0) \geqslant \sum_{j=1}^{r} \rho_j. \tag{13}$$

It follows from Szegö's proof that the equality sign in (13) holds when and only when the matrix $B$ is degenerate. If $B$ is non-degenerate, the strict inequality holds. In this case we denote by $\rho_0$ the difference between the left side and the right side of (13):

$$\rho_0 = B(0) - \sum_{j=1}^{r} \rho_j. \tag{14}$$

Since $B(\kappa) = B^*(-\kappa)$, we have the following representation for $B(\kappa)$:

$$B(\kappa) = \rho_0 \delta(\kappa) + \sum_{j=1}^{r} \rho_j \exp(i\omega_j \kappa) \qquad |\kappa| \leqslant m \tag{15}$$

where $\delta(0) = 1$, $\delta(\kappa) = 0$ when $\kappa \neq 0$; $\rho_0 > 0$ for non-degenerate matrices $B$, and $\rho_0 = 0$ for degenerate matrices $B$.

Equation (15) provides a representation for the segment of the covariance function $B(\kappa)$ in terms of a sum of harmonics and possibly a $\delta$-function corresponding to white noise.

It should be noted that if the true covariance function consists of $r$ harmonics (where $r \leqslant m$) and possibly a $\delta$-function as well, then the representation (15) produces the true frequencies $\omega_j$ and amplitudes $\rho_j$, since according to Caratheodory's theorem these quantities are determined uniquely. But if $r > m$ or if $B(\kappa)$ contains some non-white spectral components, then (15) gives numbers $\omega_j$ and $\rho_j$ which may not be the true frequencies and amplitudes. We will comment below on the relation between these numbers and the true spectrum.

If $B(\kappa)$ is real, the complex harmonics in (15) must be even in number and occur in complex conjugate pairs to produce real cosine harmonics:

$$\rho_{j1} = \rho_{j2} \qquad \omega_{j1} = -\omega_{j2} \tag{16}$$

$$\rho_{j1} \exp(i\omega_{j1}\kappa) + \rho_{j2} \exp(i\omega_{j2}\kappa) = 2\rho_{j1} \cos(\omega_{j1}\kappa).$$

Thus for real $B(\kappa)$ the representation (15) can be written in the form:

$$B(\kappa) = \rho_0 \delta(\kappa) + 2 \sum_{j=1}^{p} \rho_j \cos(\omega_j \kappa) \qquad |\kappa| \leqslant m \tag{17}$$

where $p \leqslant m/2$. Thus if the true number of cosine harmonics in a real covariance function is not more than $m/2$, these harmonics can be recovered exactly from a segment of length $2m$ of the covariance function.

We now show how the numbers $r$, $\rho_j$, and $\omega_j$ are derived; the argument follows from Grenander & Szegö (1958).

*Algorithm for determining $r$, $\rho_j$, and $\omega_j$.* The algorithm consists of four steps:

1. We find the minimal eigenvalue $\mu_0$ of the matrix $B$ of order $m+1$; its multiplicity is denoted by $\nu$. The number $\mu_0$ is equal to the amplitude $\rho_0$ for the $\delta$-function term in (15). If $\rho_0 = 0$, then the $\delta$-function term in (15) vanishes. Two situations are possible: $\nu = 1$ and $\nu > 1$. If $\nu = 1$, we consider the matrix $B - \mu_0 I$ in what follows; if $\nu > 1$, we consider the principal minor of the matrix $B - \mu_0 I$, whose order is $r+1$

(where $r = m-\nu$). We denote this minor by:

$$(B-\mu_0 I)_r = \begin{bmatrix} B(0)-\mu_0 & B(1) & \dots & B(r) \\ B(-1) & B(0)-\mu_0 & \dots & B(r-1) \\ \vdots & \vdots & & \vdots \\ B(-r) & B(-r+1) & \dots & B(0)-\mu_0 \end{bmatrix}. \tag{18}$$

Thus the rank of the matrix $(B-\mu_0 I)_r$ is $r$ and its order is $(r+1)$.

2. We find the eigenvector of $(B-\mu_0 I)_r$ corresponding to its unique zero eigenvalue, and denote its components by $p_0, p_1, \dots, p_r$.

3. We evaluate the roots of the polynomial

$$p_0+p_1 z+\dots+p_r z^r = 0 \tag{19}$$

which roots we denote by $z_1, \dots, z_r$. These roots are all different and of modulus unity, so they can be written uniquely in the form

$$z_j = \exp(i\omega_j) \qquad -\pi < \omega_j \leqslant \pi \qquad j = 1, \dots, r.$$

The numbers $\omega_1, \dots, \omega_r$ are the frequencies of the required harmonics.

4. As shown by Grenander & Szegö (1958), the amplitudes $\rho_j$ can be expressed in terms of the $z_1, \dots, z_r$. We give here another derivation of expressions for $\rho_j$ which may be more convenient for numerical calculation.

The amplitudes $\rho_j$ satisfy the system of linear equations:

$$\sum_{j=1}^{r} \rho_j \exp(i\omega_j \kappa) = B(\kappa)-\mu_0 \delta(\kappa) \qquad |\kappa| \leqslant m. \tag{20}$$

We now show how to derive from (20) a system of real linear equations which determine $\rho_1, \dots, \rho_r$. First suppose that the absolute values $|\omega_1|, \dots, |\omega_r|$ are all different. Then from difference equation theory (Gelfond 1959) it can be shown that:

$$\det\begin{bmatrix} \sin\omega_1 & \sin\omega_2 & \dots & \sin\omega_r \\ \sin 2\omega_1 & \sin 2\omega_2 & \dots & \sin 2\omega_r \\ \vdots & \vdots & & \vdots \\ \sin r\omega_1 & \sin r\omega_2 & \dots & \sin r\omega_r \end{bmatrix} \neq 0 \tag{21}$$

when $\omega_j \neq 0$ and $w_j \neq \pi$.

$$\det\begin{bmatrix} 1 & 1 & & 1 \\ \cos\omega_1 & \cos\omega_2 & \dots & \cos\omega_r \\ \vdots & \vdots & & \vdots \\ \cos(r-1)\omega_1 & \cos(r-1)\omega_2 & \dots & \cos(r-1)\omega_r \end{bmatrix} \neq 0. \tag{22}$$

If none of the $\omega_1, \dots, \omega_r$ is equal to zero or $\pi$ we can take the imaginary part of equations (20) for $\kappa = 0, \dots, r-1$:

$$\sum_{j=1}^{r} \rho_j \sin\omega_j \kappa = \operatorname{Im} B(\kappa) \qquad \kappa = 1, \dots, r. \tag{23}$$

Because of (21), the system (23) determines the amplitudes $\rho_1, \dots, \rho_r$ uniquely. If some $\omega_j = 0$ or $\omega_j = \pi$, the real parts of equations (20) can be taken for $k = 0, 1, \dots, r-1$:

$$\sum_{j=1}^{r} \rho_j \cos \omega_j \kappa = \operatorname{Re} B(\kappa) - \mu_0 \delta(\kappa) \qquad \kappa = 0, \dots, r-1. \tag{24}$$

The determinant of this system is the expression (22), and hence is non-zero.

Now assume that among the $\omega_1, \dots, \omega_r$ there is one pair with identical modulus, say $\omega_1 = -\omega_2$, and the others, $|\omega_3|, \dots, |\omega_r|$ are all different. Taking the terms with $\omega_1$ and $\omega_2$ together in (23) and (24), we have:

$$\rho_1 \sin \omega_1 \kappa + \rho_2 \sin \omega_2 \kappa = (\rho_2 - \rho_1) \sin \omega_2 \kappa$$

$$\rho_1 \cos \omega_1 \kappa + \rho_2 \cos \omega_2 \kappa = (\rho_2 + \rho_1) \cos \omega_2 \kappa.$$

We denote

$$\rho_2 - \rho_1 = \tilde{\rho}_2 \qquad \rho_2 + \rho_1 = \tilde{\tilde{\rho}}_2.$$

Now consider instead of (23) and (24) the truncated system:

$$\tilde{\rho}_2 \sin \omega_2 \kappa + \sum_{j=3}^{r} \rho_j \sin \omega_j \kappa = \operatorname{Im} B(\kappa) \qquad \kappa = 1, \dots, r-1 \tag{25}$$

$$\tilde{\tilde{\rho}}_2 \cos \omega_2 \kappa + \sum_{j=3}^{r} \rho_j \cos \omega_j \kappa = \operatorname{Re} B(\kappa) - \mu_0 \delta(\kappa) \qquad \kappa = 0, 1, \dots, r-2. \tag{26}$$

In these systems all the moduli $|\omega_2|, |\omega_3|, \dots, |\omega_r|$ are different, so that using systems (23) and (24) we can determine $\tilde{\rho}_2, \tilde{\tilde{\rho}}_2, \rho_3, \dots, \rho_r$. From these we get:

$$\rho_1 = \tfrac{1}{2}(\tilde{\tilde{\rho}}_2 - \tilde{\rho}_2) \qquad \rho_2 = \tfrac{1}{2}(\tilde{\tilde{\rho}}_2 + \tilde{\rho}_2). \tag{27}$$

This same method is applicable when there are several pairs of frequencies which differ only in sign. Having grouped together corresponding pairs in (23) and (24), one solves truncated systems and then uses (27) to find amplitudes. The terms with $\omega_j = 0$ or $\omega_j = \pi$ are always determined from (24).

Note that for a real covariance function we have instead of (20) the system of equations (17):

$$2 \sum_{j=1}^{p} \rho_j \cos \omega_j \kappa = B(\kappa) - \mu_0 \delta(\kappa) \qquad |\kappa| \leqslant m. \tag{28}$$

The moduli $|\omega_1|, \dots, |\omega_p|$ are all different, so we can take equations (28) for $\kappa = 0, 1, \dots, p-1$. Because of (22) the determinant of this system is non-zero, so $\rho_1, \dots, \rho_p$ are uniquely determined.

The derivation of the numbers $\omega_j$, $\rho_j$ is thus complete. It is clear that this algorithm can be used in practice on digital computers for values of $m$ up to several dozen.

## 3. On the connection between $P_E(\lambda)$ and the harmonics determined from a segment of the covariance function

We now assume that the minimal eigenvalue $\mu_0$ of the matrix $B$ is simple ($\nu = 1$) and that $\mu_0 > 0$. Then $r = m$ and it is easy to show that apart from a constant factor the polynomial (19) can be written as the determinant:

$$\det\begin{bmatrix} 1 & z & z^2 & \ldots & z^m \\ B(-1) & B(0)-\mu_0 & B(1) & \ldots & B(m-1) \\ \vdots & \vdots & \vdots & & \vdots \\ B(-m) & B(-m+1) & B(-m+2) & \ldots & B(0)-\mu_0 \end{bmatrix} = 0. \tag{29}$$

For $\mu < \mu_0$ we denote the maximum entropy estimator (MEE) for the matrix $B-\mu I$ by $P_E^{(\mu)}(\lambda)$: the elements of the first row of the matrix $(B-\mu I)^{-1}$ are denoted by $\beta_{0\kappa}(\mu)$ and the corresponding signed minors by $B_{0\kappa}(\mu)$:

$$\beta_{0\kappa}(\mu) = \frac{B_{0\kappa}(\mu)}{\det(B-\mu I)} \qquad \kappa = 0, 1, \ldots, m.$$

We see that the coefficients of the polynomial (29) are equal to $B_{0\kappa}(\mu)$. But from equation (5), the coefficients $\beta_{0\kappa}$ in $P_E(\lambda)$ can be written

$$\beta_{0\kappa} = \beta_{0\kappa}(0) = \frac{B_{0\kappa}(0)}{\det B} \qquad \kappa = 0, 1, \ldots, m.$$

It is known (Grenander & Szegö 1958, Chapter 2) that if $\det B > 0$, then all the roots of the polynomial

$$\sum_{\kappa=0}^{m} \beta_{0\kappa} z^\kappa = \frac{1}{\det B} \sum_{\kappa=0}^{m} B_{0\kappa}(0) z^\kappa = 0.$$

lie inside the unit circle in the complex $z$-plane. Denoting the roots of the polynomial

$$\sum_{\kappa=0}^{m} \beta_{0\kappa}(\mu) z^\kappa = \frac{1}{\det(B-\mu I)} \sum_{\kappa=0}^{m} B_{0\kappa}(\mu) z^\kappa = 0 \tag{30}$$

by $z_1(\mu), \ldots, z_m(\mu)$, it follows that for $\mu < \mu_0$:

$$|z_j(\mu)| < 1 \qquad j = 1, 2, \ldots, m.$$

From (30) we see that the roots $z_j(\mu)$ tend to the roots of the polynomial (29) as $\mu \uparrow \mu_0$, i.e.

$$z_j(\mu) \to \exp(i\omega_j) \quad \text{when} \quad \mu \uparrow \mu_0, \qquad j = 1, \ldots, m. \tag{31}$$

It follows from the definition (5) of the MEE $P_E(\lambda)$ that

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \exp(i\lambda\kappa) P_E(\lambda)\, d\lambda$$

$$= (2\pi)^{-1} \int_{-\pi}^{\pi} \frac{\beta_{00} \exp(i\lambda\kappa)\, d\lambda}{\left|\sum_{j=0}^{m} \beta_{0j} \exp(-i\lambda j)\right|^2} = B(\kappa) \qquad |\kappa| \leqslant m. \tag{32}$$

Substituting the variable

$$\zeta = \exp(i\lambda)$$

in the integral (32), we find

$$\frac{\beta_{00}}{2\pi}\oint \frac{\zeta^{\kappa-1}\, d\zeta}{\left|\sum_{j=0}^{m} \beta_{0j}\zeta^{j}\right|^{2}} = B(\kappa) \qquad |\kappa| \leqslant m \tag{33}$$

where the path of integration is around the unit circle in the complex $\zeta$-plane. Similarly, for the matrix $B - \mu I$:

$$\frac{\beta_{00}(\mu)}{2\pi}\oint \frac{\zeta^{\kappa-1}\, d\zeta}{\left|\sum_{j=0}^{m} \beta_{0j}(\mu)\,\zeta^{j}\right|^{2}} = B(\mu)-\mu\delta(\kappa) \qquad |\kappa| \leqslant m. \tag{34}$$

Assuming that all the roots $z_j(\mu)$ in the denominator of (34) are distinct, we have from the theory of complex residues:

$$B(\kappa)-\mu\delta(\kappa) = \sum_{j=1}^{m} \rho_j(\mu)\; z_j^{\kappa}(\mu) \qquad \kappa = 0, 1, \ldots, m. \tag{35}$$

For $\kappa < 0$ the equalities (35) are changed to the corresponding complex conjugate expressions. From (31) and (35) we have as $\mu \uparrow \mu_0$:

$$B(\kappa)-\mu_0\,\delta(\kappa) = \sum_{j=1}^{m} \rho_j(\mu_0)\, z_j^{\kappa}(\mu_0) \qquad \kappa = 0, 1, \ldots, m. \tag{36}$$

Comparing (36) and (15) we conclude that $\rho_j(\mu_0) = \rho_j$. Thus

$$\sum_{j=1}^{m} \rho_j(\mu) z_j^{\kappa}(\mu) \to \sum_{j=1}^{m} \rho_j \exp\,(i\omega_j\,\kappa) \quad \kappa \geqslant 0 \quad \text{when } \mu \uparrow \mu_0$$

or in spectral form,

$$P_E^{(\mu)}(\lambda) \to \sum_{j=1}^{m} \rho_j\,\delta(\lambda-\omega_j) \quad -\pi < \lambda \leqslant \pi \qquad \text{when } \mu \uparrow \mu_0. \tag{37}$$

The last equation makes clear some of the amplitude properties of this estimator (compare Lacoss 1971), and shows why the MEE $P_E(\lambda)$ sometimes does well in distinguishing harmonics in a white noise background. It is clear that subtraction of some diagonal matrix $\mu_0 I$ from the matrix $B$ would result in exact determination of the harmonics in Lacoss's examples.

Before taking the limit $\mu \uparrow \mu_0$, the integrand in (34) can be decomposed into simple fractions of the form

$$|e^{i\lambda}-z_j(\mu)|^{-2}$$

which can be considered as ‘smoothed’ versions of the $\delta$-functions $\delta(\lambda-\omega_j)$ in (37). Thus $P_E(\lambda)$ can be considered as a ‘smoothed’ version of the function

$$\mu_0 + \sum_{j=1}^{m} \rho_j\,\delta(\lambda-\omega_j).$$

As $\mu$ increases toward $\mu_0$, the effect of this smoothing diminishes, and in the limit when $\mu = \mu_0$, $P_E^{(\mu)}(\lambda)$ is decomposed into pure harmonics.

If the true spectrum is continuous, then the harmonics $\rho_j \delta(\lambda-\omega_j)$ in its 'linear' spectral representation (37) will 'outline' the true spectral density. The greater $m$ is, the more detailed this outline will be.

Sometimes the linear spectral representation (37) may provide an economic spectral parameterization, which may be useful in practical computations.

## 4. Small random perturbations of the original data

Now we consider the situation when the covariance function is disturbed by additive random noise. This implies that in equation (15) the frequencies $\omega_j$ and amplitudes $\rho_j$ are also random variables. Since they are non-linear functions of the noise and cannot be expressed generally as explicit functions of the noise, it is very difficult to obtain their statistical characteristics. Therefore we consider noise amplitudes small enough that the perturbation method is applicable (see, for example, Gelfand 1966).

We assume that we have a segment of the disturbed covariance function $\hat{B}(\kappa)$:

$$\hat{B}(\kappa) = B(\kappa)+\varepsilon\xi(\kappa) \qquad |\kappa| \leqslant m$$

where $B(\kappa)$ is the true covariance function, $\xi(\kappa)$ is a random noise such that $\xi(\kappa) = \xi^*(\kappa)$, and $\varepsilon$ is a small real parameter. According to our assumptions, $\hat{B}(\kappa) = \hat{B}^*(-\kappa)$. We further assume that

$$\det B > 0 \tag{38}$$

and that with probability one

$$\det \hat{B} > 0. \tag{39}$$

We assume that the first two moments of the noise $\xi(\kappa)$ exist. We introduce separate notation for the real and imaginary parts of these moments:

$$\left.\begin{aligned} &E\,\mathrm{Re}\,\xi(\kappa) = a_\kappa && \kappa = 0, 1, \ldots, m \\ &E\,\mathrm{Im}\,\xi(\kappa) = a_{m+\kappa} && \kappa = 1, \ldots, m \end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned} &E[\mathrm{Re}\,\xi(\kappa)-a_\kappa][\mathrm{Re}\,\xi(j)-a_j] = \sigma(\kappa,j) && \kappa,j = 0, 1, \ldots, m \\ &E[\mathrm{Re}\,\xi(\kappa)-a_\kappa][\mathrm{Im}\,\xi(j)-a_{m+j}] = \sigma(\kappa, m+j) && \kappa = 0, \ldots, m \\ & && j = 1, \ldots, m \\ &E[\mathrm{Im}\,\xi(\kappa)-a_{m+\kappa}][\mathrm{Im}\,\xi(j)-a_{m+j}] = \sigma(m+\kappa, m+j) && \kappa,j = 1, \ldots, m \end{aligned}\right\}. \tag{41}$$

The column vector of quantities (40) is denoted by $\mathbf{a}$, and the symmetrical covariance matrix $\sigma(\kappa,j)_{\kappa,j=0,\ldots,2m}$ of order $(2m+1)$ is denoted by $\Sigma$.

We assume in the representation (15) for $B(\kappa)$ that $r = m$, i.e.

$$\hat{B}(\kappa) = \rho_0\,\delta(\kappa)+\sum_{j=1}^{m} \rho_j \exp(i\omega_j \kappa) \qquad |\kappa| \leqslant m. \tag{42}$$

Then for sufficiently small $\varepsilon$, the covariance function $\hat{B}(\kappa)$ will be represented with probability arbitrarily close to unity by the similar form:

$$\hat{B}(\kappa) = \rho_0(\varepsilon)\,\delta(\kappa)+\sum_{j=1}^{m} \rho_j(\varepsilon) \exp[i\omega_j(\varepsilon)\,\kappa] \qquad |\kappa| \leqslant m \tag{43}$$

where $\rho_j(\varepsilon)$ and $\omega_j(\varepsilon)$ are random functions of $\varepsilon$, and

$$\left.\begin{array}{ll}\rho_j(\varepsilon) \to \rho_j & j = 0, 1, \ldots, m \\ \omega_j(\varepsilon) \to \omega_j & j = 1, \ldots, m\end{array}\right\} \quad \text{when } \varepsilon \to 0. \tag{44}$$

In the neighbourhood of the point $\varepsilon = 0$ the functions $\rho_j(\varepsilon)$ and $\omega_j(\varepsilon)$ are differentiable and can be represented in the form:

$$\rho_j(\varepsilon) = \rho_j + \varepsilon\tilde{\rho}_j + \varepsilon^2 \tilde{\tilde{\rho}}_j + \ldots \tag{45}$$

$$\omega_j(\varepsilon) = \omega_j + \varepsilon\tilde{\omega}_j + \varepsilon^2 \tilde{\tilde{\omega}}_j + \ldots . \tag{46}$$

We wish to find the principal linear parts of the functions $\rho_j(\varepsilon)$ and $\omega_j(\varepsilon)$: for sufficiently small $\varepsilon$ we have with probability close to one:

$$\begin{aligned}\rho_j(\varepsilon) &\approx \rho_j + \varepsilon\tilde{\rho}_j \\ \omega_j(\varepsilon) &\approx \omega_j + \varepsilon\tilde{\omega}_j.\end{aligned} \tag{47}$$

Inserting (45) and (46) into (43), expanding exponents in powers of $\varepsilon$, and equating coefficients of $\varepsilon$ we get:

$$\tilde{\rho}_0\,\delta(\kappa) + \sum_{j=1}^{m} (\tilde{\rho}_j + \rho_j\,\tilde{\omega}_j\,i\kappa) \exp(i\omega_j\,\kappa) = \xi(\kappa) \qquad |\kappa| \leqslant m. \tag{48}$$

Taking separately the real and imaginary parts of (48) we obtain a system of $2m+1$ linear real equations for the $2m+1$ real unknown parameters $\tilde{\rho}_0, \ldots, \tilde{\rho}_m, \tilde{\omega}_1, \ldots, \tilde{\omega}_m$. For $\kappa = 0$ we get one real equation (48), and for $\kappa = 1, \ldots, m$ we get two real equations from each complex equation (48), making $2m+1$ equations in all. For $\kappa = -1, \ldots, -m$, the equations (48) provide no additional information, since these cases are complex conjugates of those for positive $\kappa$.

We now show that the homogeneous system corresponding to (48), i.e. $\xi(\kappa) \equiv 0$, has only a zero solution. Since the number of equations (48) coincides with the number of unknown parameters, it follows that (48) has a non-degenerate matrix, and its solution is unique.

Thus we have to prove that if

$$\xi(\kappa) \equiv 0 \qquad |\kappa| \leqslant m \tag{49}$$

then

$$\tilde{\rho}_0 = 0 \quad \tilde{\rho}_j = \tilde{\omega}_j = 0 \qquad j = 1, \ldots, m. \tag{50}$$

As shown in Section 3 above, the parameters $\rho_0$ and $\rho_0(\varepsilon)$ are the minimal eigenvalues of the matrices $B$ and $\hat{B}$ respectively. It is known (Gelfand 1966) that

$$\tilde{\rho}_0 = \sum_{\kappa, j=0}^{m} p_\kappa\, p_j^*\, \xi(\kappa - j) \tag{51}$$

where $p_0, \ldots, p_m$ are components of the normalized eigenvector of $B$ corresponding to the eigenvalue $\rho$. Hence if $\xi(\kappa) \equiv 0$ we get from (51) that $\rho_0 = 0$. The system (48) now has the form

$$\sum_{j=1}^{m} \tilde{\rho}_j \exp(i\omega_j\,\kappa) + i\kappa \sum_{j=1}^{m} \tilde{\omega}_j\,\rho_j \exp(i\omega_j\,\kappa) = 0 \qquad |\kappa| \leqslant m \tag{52}$$

and the $2m$ functions

$$\left.\begin{aligned} f_j(\kappa) &= \kappa \exp(i\omega_j \kappa) \\ & \qquad\qquad j = 1, \ldots, m \\ g_j(\kappa) &= \exp(i\omega_j \kappa) \end{aligned}\right. \tag{53}$$

are linearly independent solutions of a linear difference equation of order $2m$ with constant coefficients (see Gelfond 1959):

$$L(S) f(\kappa) = 0 \qquad |\kappa| \leqslant m \tag{54}$$

where $S$ is the shift operator

$$Sf(\kappa) = f(\kappa+1)$$

and $L(S)$ is the difference operator of order $2m$:

$$L(S) = \prod_{j=1}^{m} [S - \exp(i\omega_j)]^2. \tag{55}$$

Since the functions (53) are linearly independent and the number of equations in (52) is larger than the order of the difference equation, it follows that (52) has only the zero solution (50)—which was what we set out to prove.

Now we divide equations (48) into real and imaginary parts for $\kappa = 0, 1, \ldots, m$:

$$\tilde{\rho}_0 \delta(\kappa) + \sum_{j=1}^{m} \tilde{\rho}_j \cos(\omega_j \kappa) - \kappa \sum_{j=1}^{m} \tilde{\omega}_j \rho_j \sin(\omega_j \kappa) = \operatorname{Re} \xi(\kappa) \qquad \kappa = 0, 1, \ldots, m$$

$$\sum_{j=1}^{m} \tilde{\rho}_j \sin(\omega_j \kappa) + \kappa \sum_{j=1}^{m} \tilde{\omega}_j \rho_j \cos(\omega_j \kappa) = \operatorname{Im} \xi(\kappa) \qquad \kappa = 1, \ldots, m.$$

In order to write this in vector form, we introduce a column vector $\mathbf{x}$ of the unknown parameters. The components of $\mathbf{x}$ are:

$$\left.\begin{aligned} x_0 &= \tilde{\rho}_0, \ldots, x_m = \tilde{\rho}_m \\ x_{m+1} &= \tilde{\omega}_1, \ldots, x_{2m} = \tilde{\omega}_m. \end{aligned}\right\} \tag{57}$$

Similarly, we introduce the column vector $\boldsymbol{\xi}$ of the right-hand parts of (56), having components

$$\left.\begin{aligned} \xi_0 &= \xi(0), \xi_1 = \operatorname{Re} \xi(1), \ldots, \xi_m = \operatorname{Re} \xi(m) \\ \xi_{m+1} &= \operatorname{Im} \xi(1), \ldots, \xi_{2m} = \operatorname{Im} \xi(m) \end{aligned}\right\} \tag{58}$$

The matrix of the system (58) is denoted by $D$:

$$D = \left[\begin{array}{c|ccc|ccc} 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\ \hline 0 & \cos\omega_1 & \ldots & \cos\omega_m & -\rho_1 \sin\omega_1 & \ldots & -\rho_m \sin\omega_m \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cos m\omega_1 & \ldots & \cos m\omega_m & -m\rho_1 \sin m\omega_1 & \ldots & -m\rho_m \sin m\omega_m \\ \hline 0 & \sin\omega_1 & \ldots & \sin\omega_m & \rho_1 \cos\omega_1 & \ldots & \rho_m \cos\omega_m \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \sin m\omega_1 & \ldots & \sin m\omega_m & m\rho_1 \cos m\omega_1 & \ldots & m\rho_m \cos m\omega_m \end{array}\right]. \tag{59}$$

The system (56) can thus be written:

$$Dx = \xi. \tag{60}$$

From what has been already proved, det $D \neq 0$. The solutions of equations (60) can thus be written:

$$\mathbf{x} = D^{-1}\,\xi \tag{61}$$

where $D^{-1}$ is the inverse of $D$. From (61) we can find the statistical moments of the components of $\mathbf{x}$, in particular their mean value vector $\mathbf{b}$ and their covariance matrix $W$:

$$\mathbf{b} = E\mathbf{x} = D^{-1}\,E\xi = D^{-1}\,\mathbf{a},$$

where

$$b_0 = E\rho_0 \qquad \begin{matrix} b_j = E\tilde{\rho}_j \\ b_{m+j} = E\tilde{\omega}_j \end{matrix} \qquad j = 1, \ldots, m \tag{62}$$

$$W = E(\mathbf{x}-\mathbf{b})(\mathbf{x}-\mathbf{b})^* = ED^{-1}(\xi-\mathbf{a})(\xi-\mathbf{a})^*\,(D^{-1})^* = D^{-1}\,\Sigma(D^{-1})^* \tag{63}$$

where the asterisk denotes the complex conjugate matrix or vector, and $\mathbf{a}$ and $\Sigma$ were defined in (40) and (41). Thus we have the following approximate mean values of the disturbed parameters:

$$\left.\begin{aligned} E\rho_j(\varepsilon) &\approx \rho_j+\varepsilon E\tilde{\rho}_j = \rho_j+\varepsilon b_j \qquad & j = 0, \ldots, m \\ E\omega_j(\varepsilon) &\approx \omega_j+\varepsilon E\tilde{\omega}_j = \omega_j+\varepsilon b_{m+j} \qquad & j = 1, \ldots, m. \end{aligned}\right\} \tag{64}$$

The covariance matrix of the disturbed parameters is $\varepsilon^2 W$.

For practical use it is important to know how large the neglected terms of order $\varepsilon^2$ were in equations (45) and (46). Substituting (45) and (46) into (43) and equating coefficients of $\varepsilon^2$, we have the system of equations for the quadratic corrections $\tilde{\tilde{\rho}}_j$ and $\tilde{\tilde{\omega}}_j$:

$$\begin{aligned} &\tilde{\tilde{\rho}}_0\,\delta(\kappa)+\sum_{j=1}^{m}\tilde{\tilde{\rho}}_j\exp(i\omega_j\kappa)+i\kappa\sum_{j=1}^{m}\tilde{\tilde{\omega}}_j\,\rho_j\exp(i\omega_j\kappa) \\ &\qquad = \sum_{j=1}^{m}[\tfrac{1}{2}\rho_j(\tilde{\omega}_j\kappa)^2-i\kappa\tilde{\rho}_j\,\tilde{\omega}_j]\exp(i\omega_j\kappa) \qquad |\kappa| \leqslant m. \end{aligned} \tag{65}$$

This differs from (48) by the right-hand parts only, so that the system (65) always has a unique solution. We see that $\tilde{\tilde{\rho}}_j$ and $\tilde{\tilde{\omega}}_j$ are quadratic functionals of the noise $\xi(\kappa)$. If the third and fourth moments of $\xi(\kappa)$ are known, then along the same lines as in the derivation of (63) and (64) it is possible to find $E(\tilde{\tilde{\rho}}_j)^2$ and $E(\tilde{\tilde{\omega}}_j)^2$. If it happens that

$$\left.\begin{aligned} E(\varepsilon^2\,\tilde{\tilde{\rho}}_j)^2 &\ll E(\varepsilon\tilde{\rho}_j)^2 \qquad & j = 0, 1, \ldots, m \\ E(\varepsilon^2\,\tilde{\tilde{\omega}}_j)^2 &\ll E(\varepsilon\tilde{\omega}_j)^2 \qquad & j = 1, \ldots, m \end{aligned}\right\} \tag{66}$$

then in (45) and (46) the quadratic corrections are negligible and the linear approximation (47) is justified.

When $r < m$ in the representation (15) of $B(\kappa)$, the corresponding representation for the disturbed function $\hat{B}(\kappa)$ will nevertheless generally contain $m$ harmonics; i.e. $m-r$ false harmonics will appear with fictitious amplitudes and frequencies, the amplitudes being of order $\varepsilon$. Statistical analysis of the disturbances is much more difficult in this case, since even for the $\tilde{\rho}_j$ and $\tilde{\omega}_j$ the system of equations is non-linear.

For this situation a rough simplified procedure has merit: if we find in (15) that $m-r$ amplitudes $\rho_j(\varepsilon)$ are of order of magnitude $\varepsilon$, then it is possible to take a segment of the disturbed covariance function of length $2r$ rather than the original length $2m$ and carry out the analysis of the disturbances as above.

We note that in order to use (63) and (64) in practice, we have to know the matrix $D$, which depends on the true unknown parameters $\rho_j$ and $\omega_j$. In practice we usually put into $D$ the disturbed parameters $\rho_j(\varepsilon)$ and $\omega_j(\varepsilon)$ obtained from data. The justification of this procedure lies in the fact that the disturbed matrix $D(\varepsilon)$, i.e. the matrix that contains $\rho_j(\varepsilon)$ and $\omega_j(\varepsilon)$ in place of $\rho_j$ and $\omega_j$, differs from the true matrix $D$ only by a matrix whose elements are of magnitude $\varepsilon$ (for sufficiently small $\varepsilon$). Of course, care must be taken, particularly when some frequencies $\omega_i$, $\omega_j$ are close together or when the noise is not small. It would be possible to estimate the term linear in $\varepsilon$ for $D(\varepsilon)$,

$$D(\varepsilon) = D+\varepsilon\tilde{D}+\varepsilon^2\,\tilde{\tilde{D}}+\ldots$$

and check that this term is negligible, but we omit such considerations here.

Now we write down the equations analogous to (48), (59), and (65) for the case of real covariance functions. These equations differ slightly from the complex expressions, but all the proofs and discussion are almost the same as in the more general case.

Suppose that in (17), $p = m/2$, $\omega_j \neq 0$, and $|\omega_j| < \pi$. For (48) we have:

$$\tilde{\rho}_0\,\delta(\kappa)+2\sum_{j=1}^{m}\tilde{\rho}_j\cos(\omega_j\kappa)-2\kappa\sum_{j=1}^{p}\tilde{\omega}_j\rho_j\sin(\omega_j\kappa) = \xi(\kappa) \qquad |\kappa| \leqslant m = 2p. \tag{67}$$

The matrix $D$ has the form:

(Shown on p. 361)

For (58) we have

$$\xi_\kappa = \xi(\kappa) \qquad \kappa = 0, 1, \ldots, 2p. \tag{69}$$

For (40) and (41) we have

$$E\xi(\kappa) = a_\kappa \qquad \kappa = 0, 1, \ldots, 2p \tag{70}$$

$$E[\xi(\kappa)-a_\kappa][\xi(j)-a_j] = \sigma(\kappa, j) \qquad \kappa, j = 0, 1, \ldots, 2p. \tag{71}$$

The matrix $\Sigma$ has elements $\sigma(\kappa, j)$, $\kappa, j = 0, \ldots, 2p$. With these changes, the equations (60)–(64) are valid. For (65) we have the system:

$$\sum_{j=1}^{p}[\tilde{\rho}_j\cos(\omega_j\kappa)-\kappa\tilde{\tilde{\omega}}_j\rho_j\sin(\omega_j\kappa)] = \sum_{j=1}^{p}[\tfrac{1}{2}\rho_j(\kappa\tilde{\omega}_j)^2\cos(\omega_j\kappa)+\tilde{\rho}_j\tilde{\omega}_j\kappa\sin(\omega_j\kappa)]$$
$$\kappa = 0, 1, \ldots, 2p. \tag{72}$$

*Example of application of the perturbation method.* We suppose that:

$$B(\kappa) = \rho_0\,\delta(\kappa)+\rho_1\exp(i\omega_1\kappa) \qquad |\kappa| \leqslant 1. \tag{73}$$

$$D = \begin{bmatrix} 1 & 2 & \cdots & 2 & 0 & \cdots & 0 \\ 0 & 2\cos\omega_1 & \cdots & 2\cos\omega_p & -2\rho_1\sin\omega_1 & \cdots & -2\rho_p\sin\omega_p \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 2\cos p\omega_1 & \cdots & 2\cos p\omega_p & -2p\rho_1\sin p\omega_1 & \cdots & -2p\rho_1\sin p\omega_p \\ 0 & 2\cos(p+1)\omega_1 & \cdots & 2\cos(p+1)\omega_p & -2(p+1)\rho_1\sin(p+1)\omega_1 & \cdots & -2(p+1)\rho_p\sin(p+1)\omega_p \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 2\cos 2p\omega_1 & \cdots & 2\cos 2p\omega_p & -4p\rho_1\sin 2p\omega_1 & \cdots & -4p\rho_p\sin 2p\omega_p \end{bmatrix} \quad (68)$$

The matrices $D$ and $D^{-1}$ are:

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \cos\omega_1 & -\rho_1 \sin\omega_1 \\ 0 & \sin\omega_1 & \rho_1 \cos\omega_1 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1 & -\cos\omega_1 & -\sin\omega_1 \\ 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & -\rho_1^{-1}\sin\omega_1 & \rho_1^{-1}\cos\omega_1 \end{bmatrix} \tag{74}$$

We further assume that $\xi(\kappa)$ is a complex Gaussian process, and:

$$E\xi(0) = E\xi(1) = 0$$

$$E\xi^2(0) = \sigma^2 \qquad E\xi(0)\,\xi^*(1) = 0$$

$$E[\mathrm{Re}\,\xi(1)]^2 = E[\mathrm{Im}\,\xi(1)]^2 = \sigma^2$$

$$\Sigma = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}.$$

From (62) we have

$$E\tilde{\rho}_0 = E\tilde{\rho}_1 = E\tilde{\omega}_1 = 0$$

and from (63) we have

$$\varepsilon^2 W = \varepsilon^2 D^{-1}\,\Sigma(D^{-1})^* = \varepsilon^2\sigma^2 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \rho_1^{-2} \end{bmatrix}. \tag{75}$$

We now put in the following numerical values for the parameters $\rho_0$, $\rho_1$, and $\omega_1$ : $\rho_0 = 1$, $\rho_1 = 10$, $\omega_1 = \pi/4$, $B(0) = 11$, $B(1) = 10(1+i)/2^{\frac{1}{2}} = 7{\cdot}07 + i.7{\cdot}07$. We assume that $\varepsilon^2\sigma^2 = 0{\cdot}01$ and take the noise values

$$\varepsilon\xi(0) = -0{\cdot}1 \qquad \varepsilon\xi(1) = 0{\cdot}03 - 0{\cdot}07i.$$

Then the observed disturbed covariances $\hat{B}(\kappa)$ are:

$$\hat{B}(0) = 10{\cdot}9 \qquad \hat{B}(1) = 7{\cdot}1 + 7i$$

and using the algorithm of Section 2 above we find that

$$\rho_0(\varepsilon) = 10{\cdot}9 - \surd(99{\cdot}41) \approx 0{\cdot}9295$$

$$\exp[i\omega_1(\varepsilon)] = \frac{7{\cdot}1 + 7i}{99{\cdot}41} \qquad \omega_1(\varepsilon) \approx \tfrac{1}{4}\pi - 0{\cdot}007$$

$$\rho_1(\varepsilon) = \surd(99{\cdot}41) \approx 9{\cdot}97.$$

From (75) we have:

$$E[\rho_0(\varepsilon) - \rho_0]^2 = 2\varepsilon^2\sigma^2 = 0{\cdot}02$$

$$E[\rho_1(\varepsilon) - \rho_1]^2 = \varepsilon^2\sigma^2 = 0{\cdot}01$$

$$E[\omega_1(\varepsilon) - \omega_1]^2 = \frac{\varepsilon^2\sigma^2}{\rho_1^2} \approx \frac{\varepsilon^2\sigma^2}{\rho_1^2(\varepsilon)} = 10^{-4}.$$

We now estimate the quadratic corrections $\varepsilon^2\tilde{\tilde{\rho}}_0$, $\varepsilon^2\tilde{\tilde{\rho}}_1$, and $\varepsilon^2\tilde{\tilde{\omega}}_1$. Equations (65) have the form

$$\tilde{\tilde{\rho}}_0+\tilde{\tilde{\rho}}_1 = 0$$
$$\tilde{\tilde{\rho}}_1\cos\omega_1-\tilde{\tilde{\omega}}_1\rho_1\sin\omega_1 = \tfrac{1}{2}\rho_1\tilde{\omega}_1^{\,2}\cos\omega_1+\tilde{\rho}_1\tilde{\omega}_1\sin\omega_1$$
$$\tilde{\tilde{\rho}}_1\sin\omega_1+\tilde{\tilde{\omega}}_1\rho_1\cos\omega_1 = \tfrac{1}{2}\rho_1\tilde{\omega}_1^{\,2}\sin\omega_1-\tilde{\rho}_1\tilde{\omega}_1\cos\omega_1$$

and hence we find

$$\tilde{\tilde{\rho}}_0 = -\tfrac{1}{2}\rho_1\tilde{\omega}_1^{\,2} \qquad \tilde{\tilde{\rho}}_1 = \tfrac{1}{2}\rho_1\tilde{\omega}_1^{\,2} \qquad \tilde{\tilde{\omega}}_1 = \frac{-\tilde{\rho}_1\tilde{\omega}_1}{\rho_1}.$$

Further from (75) we have

$$\left.\begin{aligned} E(\varepsilon^2\tilde{\tilde{\rho}})^2 &= \tfrac{1}{4}\varepsilon^4 E\tilde{\omega}_1^{\,4} = \frac{3\varepsilon^4\sigma^4}{4}\rho_1^{-4} \approx 7{\cdot}5.10^{-9} \\ E(\varepsilon^2\tilde{\tilde{\rho}}_1)^2 &= 7{\cdot}5.10^{-9} \\ E(\varepsilon^2\tilde{\tilde{\omega}}_1)^2 &= \frac{\varepsilon^4\sigma^4}{\rho_1^{\,4}} = 10^{-8} \end{aligned}\right\} \tag{76}$$

and

$$\left.\begin{aligned} E(\varepsilon\tilde{\rho}_0)^2 &= 2\varepsilon^2\sigma^2 = 0{\cdot}02 \\ E(\varepsilon\tilde{\rho}_1)^2 &= \varepsilon^2\sigma^2 = 0{\cdot}01 \\ E(\varepsilon\tilde{\omega}_1)^2 &= \frac{\varepsilon^2\sigma^2}{\rho_1^{\,2}} = 10^{-4}. \end{aligned}\right\} \tag{77}$$

Comparing (77) to (76) we see that in this case the quadratic corrections $\varepsilon^2\tilde{\tilde{\rho}}_0$, $\varepsilon^2\tilde{\tilde{\rho}}_1$, and $\varepsilon^2\tilde{\tilde{\omega}}_1$ are much smaller than the linear corrections $\varepsilon\tilde{\rho}_0$, $\varepsilon\tilde{\rho}_1$, and $\varepsilon\tilde{\omega}_1$, and hence are negligible.

## Conclusions

The method we have described for retrieving harmonics from a finite segment of the covariance function is applicable to the case of non-ergodic processes whose spectral functions contain jumps or discontinuities. The amplitudes $\rho_j$ of the harmonics in the covariance functions are the *ensemble-averaged* squared amplitudes of the harmonics of the process itself. However, the method is also applicable to a different situation which is often encountered in practice, namely, the case when the observed time series $x(\kappa)$ contains some deterministic harmonics with unknown (but not random) amplitudes and frequencies, in addition to a random component having a continuous spectrum. In this case the sample covariance function based on a single realization of the process will contain the same harmonics but with some additions caused by the finite length of the realization. These additions can be regarded as noise, since they decrease as the length of the realization increases.

We are thus led to suggest the use of the method described here in the problem of 'hidden periodicities' (see, for example, Whittle 1952 or Serebrennikov & Pervozvansky 1965). This problem is the following: a time series $x(\kappa)$ is observed:

$$x(\kappa) = \sum_{j=1}^{m}\gamma_j\exp[i\omega_j\kappa+i\phi_j]+n(\kappa) \qquad |\kappa| \leqslant T$$

where $n(\kappa)$ is a noise component. The problem is to estimate the amplitudes $\gamma_j$ and frequencies $\omega_j$, and possibly also the phases $\phi_j$ of the harmonics. The application of the present method to this problem would be to compute a sample covariance function:

$$\hat{B}(\kappa) = (2T+1)^{-1} \sum_{j=-T+\kappa}^{T} x(j)\, x^*(j-\kappa) \qquad |\kappa| \leqslant m, m \ll T$$

and then use a sample of $\hat{B}(\kappa)$ of length $2m$ as described above. In using $\hat{B}(\kappa)$ rather than $x(\kappa)$ we of course lose the phase information, but the phase $\phi_j$ can easily be found from $x(\kappa)$ by means of standard least-squares methods after determining the frequencies $\omega_j$.

The problem of hidden periodicities can, of course, be treated by itself without regard to covariance function analysis (for discussion, see Pisarenko 1973). However, the power of the method proposed here is that it allows the determination of several tens of harmonics at once, using high-speed digital computers.

The procedure described here may also be useful for resolving peaks in continuous spectra; for that problem it is not necessary to go to the limit of discrete harmonics, but our new procedure does offer a way to decrease the ‘ smoothing ’ which is caused by the maximum entropy spectral estimator.

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## Appendix

Here we prove the theorem stated in Section 1 above. Consider the function $\phi_m(\lambda-\lambda_0)$:

$$\phi_m(\lambda-\lambda_0) = \sum_{\kappa, j=0}^{m} g_{\kappa-j} \exp\left[i(\lambda-\lambda_0)(\kappa-j)\right]$$

$$= \sum_{\kappa=-m}^{m} g_\kappa (m+1-|\kappa|) \exp\left[i(\lambda-\lambda_0)\kappa\right] \tag{78}$$

where $g_\kappa = g^*_{-\kappa}$ are some complex numbers specified below. We have:

$$(2\pi)^{-1} \int_{-\pi}^{\pi} w(\lambda-\lambda_0)\, dF(\lambda) = (2\pi)^{-1} \int_{-\pi}^{\pi} [w(\lambda-\lambda_0)-\phi_m(\lambda-\lambda_0)]\, dF(\lambda)$$

$$+ (2\pi)^{-1} \int_{-\pi}^{\pi} \phi_m(\lambda-\lambda_0)\, dF(\lambda). \tag{79}$$

First we evaluate the first term in the right-hand side of (79); to do this we set:

$$g_\kappa = \frac{w_\kappa}{m+1-|\kappa|} \qquad |\kappa| \leqslant m \tag{80}$$

where $w_\kappa$ are the Fourier coefficients of the function $w(\lambda)$:

$$w_\kappa = (2\pi)^{-1} \int_{-\pi}^{\pi} w(\kappa\lambda) \exp(-i\kappa\lambda)\, d\lambda. \tag{81}$$

It is seen from (78) that $\phi_m(\lambda-\lambda_0)$ represents a finite Fourier sum of order $m$ for the function $w(\lambda-\lambda_0)$. Since we assumed that $r$ derivatives of $w(\lambda)$ exist (where $r \geqslant 1$), and that $|w^{(r)}(\lambda)| \leqslant K$, we have the following estimate for the deviation of the Fourier sum $\phi_m(\lambda-\lambda_0)$ from its generating function $w(\lambda-\lambda_0)$:

$$\max_\lambda |w(\lambda-\lambda_0)-\phi_m(\lambda-\lambda_0)| \leqslant Km^{-r}[(4/\pi^2)\log(m+1)+4] \tag{82}$$

(see Achiezer 1965). Now if $m \to \infty$ the first term on the right-hand side of (79) tends to zero. Hence the second term tends to the integral on the left side, i.e. as $m \to \infty$,

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \phi_m(\lambda-\lambda_0)\, dF(\lambda) \to (2\pi)^{-1} \int_{-\pi}^{\pi} w(\lambda-\lambda_0)\, dF(\lambda) \quad \text{as} \quad m \to \infty. \tag{83}$$

The integral on the left-hand side of (83) can be written:

$$(2\pi)^{-1}\int_{-\pi}^{\pi}\phi_m(\lambda-\lambda_0)\,dF(\lambda) = (2\pi)^{-1}\sum_{\kappa=-m}^{m} w_\kappa \int_{-\pi}^{\pi}\exp\left[i(\lambda-\lambda_0)\kappa\right]dF(\lambda)$$

$$= \sum_{\kappa=-m}^{m} w_\kappa \exp\left(-i\lambda_0\kappa\right)B^*(\kappa). \tag{84}$$

Substituting (84) into (79) and using (82) we get finally:

$$\left|(2\pi)^{-1}\int_{-\pi}^{\pi} w(\lambda-\lambda_0)\,dF(\lambda) - \sum_{\kappa=-m}^{m} w_\kappa \exp\left(-i\lambda_0\kappa\right)B^*(\kappa)\right|$$

$$\leqslant KB(0)\,m^{-r}\left[(4/\pi^2)\log(m+1)+4\right]$$

which was to be proved.