Filtering Capabilities and Convergence of the Van-Cittert Deconvolution Technique

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***Abstract*—This paper demonstrates the application of the Van-Cittert technique in iterative frequency-domain deconvolution. The technique is shown to have built-in filtering capabilities which can successfully be used to produce optimum deconvolution estimates. The Bennia-Riad optimization criterion for iterative deconvolution is jointly used with the Van-Cittert technique to optimize the number of iterations required to achieve acceptable (optimum) deconvolution results. An experimental example is provided to illustrate the application of the deconvolution technique and the optimization criterion.**

I. Introduction

THE deconvolution problem is concerned with the separation of two signals combined through convolution. However, the problem is ill-posed as explained by Tikhonov *et al.* [1], and therefore, does not have a unique solution. Only estimates to the deconvolution result are possible. To obtain an acceptable estimate, many deconvolution techniques have been successfully developed to suit a particular application or particular types of signals [2]. A technique that has been widely used and sprung different varieties, and is of particular interest to us in this paper, is the Van-Cittert iterative deconvolution technique.

The Van-Cittert iterative technique has been widely used as a time-domain deconvolution technique. Different forms of the technique have been developed in order to eliminate its noise amplification, which was shown to be increasing linearly with the number of iterations. In this paper, we will show that the technique, as originally proposed, needs no modification when applied in the frequency domain. Also, we will show that the technique can simply be used as a Wiener-type filtering technique. The resulting deconvolution filter is of an adaptive nature which provides band-stop characteristics whenever the signal-to-noise ratio (SNR) is small.

The technique, being iterative, allows for the control of spurious fluctuations by interaction with the solution as it evolves. The technique is somewhat intuitive, and is widely used. The original technique has numerous variants and has engendered a number of constrained nonlinear methods that exhibit significantly better performance [3], [4]. It has been identified with both inverse filtering and the solution of linear equations by relaxation. Some of its variants are concerned with reducing and damping the noise that grows with each iteration when the method is applied to real data [4]. The method, as originally proposed, is thought to have utility where only modest correction is required.

In this paper, the Van-Cittert technique is identified with inverse filtering only and is shown to have capabilities similar to those of adaptive filters that have found wide use in time-domain reflectometry [2]. Except for a needed normalization and a reblurring procedure, no major modifications of the original technique are necessary.

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II. The Van-Cittert Deconvolution Technique

The Van-Cittert technique is an iterative time-domain deconvolution technique based on forming successive approximations of the unknown system impulse response, $h_i(t)$, using the convolution equation. The successive approximations are supposed to converge to the unknown system impulse response, but this is not always possible when the technique is used without any modifications [2]. In addition, the technique is found to be computer time consuming when used in its time-domain form. In the technique, the initial approximation to $h(t)$, $h_o(t)$, is chosen to be $y(t)$, and the $h_i(t)$ term is obtained iteratively by adding an error correction term to the $h_{i-1}(t)$ iteration:

$$h_i(t) = h_{i-1}(t) + [y(t) - h_{i-1}(t) * x(t)]$$

with

$$h_o(t) = y(t). \tag{1}$$

Ideally, the iterative process is ended whenever $h_i(t) * x(t) = y(t)$, which necessarily yields $h_i(t) = h(t)$.

The method requires a very large amount of computation time due to the many numerical convolutions performed in the iterations. In addition, the number of iterations needed to achieve an acceptable estimate is not known. The quality of the obtained estimate can only be assessed through the implementation of qualitative error function criteria. A criterion that achieves a compromise between noise reduction and signal information preservation, has been developed by Bennia and Riad [8]. The criterion has already been successfully used with the op-

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timal compensation deconvolution technique, and will be used in this paper with the Van–Cittert technique.

If the Fourier transform is applied to the equations defining the Van–Cittert technique, the conditions for the convergence of the technique can then be obtained. Using the Fourier transform, we would get:

$$H_i(j\omega) = H_{i-1}(j\omega) + [Y(j\omega) - X(j\omega)H_{i-1}(j\omega)]$$

with

$$H_o(j\omega) = Y(j\omega). \tag{2}$$

After successive substitutions, (2) yields the sequence:

$$H_i(j\omega) = \{1 + [1 - X(j\omega)] + [1 - X(j\omega)]^2 + \cdots + [1 - X(j\omega)]^i\}Y(j\omega). \tag{3}$$

The series in braces { } consists of the first ith terms of the binomial expansion of $\{1 - [1 - X(j\omega)]\}^{-1}$, provided that $|1 - X(j\omega)| < 1$. Hence, the sequence of $H_i(j\omega)$ converges to the ratio $Y(j\omega)/X(j\omega)$ when $|1 - X(j\omega)| < 1$. However, this result is simply what we would obtain when straightforward frequency-domain division is used. It is important to notice that this conclusion is not valid for the frequencies at which $X(j\omega) = 0$.

For values of ω where $X(j\omega) = 0$, e.g., at frequencies outside the information band of the signal, and in the absence of noise, $Y(j\omega) = 0$. At these frequencies, $H_i(j\omega)$ is undetermined; however, for a finite numerical answer, convergence is possible if and only if:

$$|1 - X(j\omega)| < 1 \qquad \{\omega: X(j\omega) \neq 0\} \tag{4a}$$

$$H(j\omega) = 0 \qquad \{\omega: X(j\omega) = 0\}. \tag{4b}$$

The necessary and sufficient convergence conditions of the Van–Cittert iterative technique, given in (4a) and (4b), were first stated by Bracewell *et al.* [6]. Conditions necessary for the convergence of the Van–Cittert iterative method of deconvolution were also studied and reported [7]. In this study, conditions that can be expressed in the function domain are derived. The restrictions are shown to be apparent from the shape of signals involved in the convolution operation. However, assuming that the convergence conditions are satisfied, the sequence of $H_i(j\omega)$ would converge to $Y(j\omega)/X(j\omega)$, which is simply the straightforward frequency-domain division. Thus, at the poles and zeros of $X(j\omega)$, which are the poles and zeros of $Y(j\omega)$, the ratio becomes indeterminate. Hence applying the technique blindly can result in noise-like errors at and around the poles and zeros. When $H(j\omega)$ is transformed to the time domain, the noise-like errors can cover the entire transform window, swamping the useful information. In addition, it was shown that in the case of the Van–Cittert iterative technique, the noise amplification increases linearly with each iteration [2]. This is seen if the observed response is known with a certain error $n(t)$ (i.e., $y_e(t) = y(t) + n(t)$); then after substituting $y_e(t)$ for $y(t)$, the following relation is obtained:

$$H_i(j\omega) = \{1 + [1 - X(j\omega)] + [1 - X(j\omega)]^2 + \cdots + [1 - X(j\omega)]^i\}Y(j\omega) + N_{H_i}(j\omega),$$

where

$$N_{H_i}(j\omega) = \{1 + [1 - X(j\omega)] + [1 - X(j\omega)]^2 + \cdots + [1 - X(j\omega)]^i\}N(j\omega).$$

Since $N(j\omega)$ is noticeable primarily when $X(j\omega)$ is negligible (at which $[1 - X(j\omega)]$ approaches unity), the above form for $N_{H_i}(j\omega)$ would approximate to $\{1 + i\}N(j\omega)$. This implies that the deconvolution noise content grows linearly with each iteration. Consequently, the technique must be modified to take into account the effects of the additive noise. Some modifications of the Van–Cittert technique were developed to attenuate the noise effects. A method that was very successful used a relaxation parameter that depends on the estimate in the iterative process [2], [3]. The practical application of the new constrained method yielded high-resolution enhancement of spectra.

The conditions for convergence of the Van–Cittert technique given in (4a) and (4b) were first given by Bracewell *et al.* [6]. Another study of the convergence conditions of the technique was also reported [7]. In this study, it was shown that convergence conditions can only be met by a limited number of signals, and the restrictions are shown to be apparent from the shape of the signals involved in the convolution operation.

III. Filtering and Convergence Prediction of the Technique

In the previous section, we showed that for the Van–Cittert technique to converge, certain conditions must be satisfied and that only signals satisfying those conditions are recovered. However, when the technique is considered as a Wiener-type filtering technique, all kinds of physically realizable signals are shown to satisfy the Bracewell convergence conditions.

The frequency-domain equation of the relations given in (3) can be written in the following form:

$$H_i(j\omega) = Y(j\omega) \sum_{k=0}^{i} [1 - X(j\omega)]^k. \tag{5}$$

Now, let us consider the following identity in which we assume that $\eta = 1 - X(j\omega)$:

$$\sum_{k=0}^{i} \eta^k = \frac{1 - \eta^{i+1}}{1 - \eta}. \tag{6}$$

Equation (6) is valid if and only if $|\eta| < 1$, which is the same as Bracewell's convergence condition (4a).

The Van–Cittert technique can be made useful to the other cases where the convergence condition $|1 - X(j\omega)| < 1$ is not satisfied. This is achieved by introducing parameter scaling (normalization) and reblurring procedures [3]. The scaling is achieved by simply replacing $Y(j\omega)$

and $X(j\omega)$ with $CY(j\omega)$ and $CX(j\omega)$ in the convolution equation. Thus, the new condition for the convergence becomes $|1 - CX(j\omega)| < 1$.

It is also important to recognize that although the nonnegativity of $X(j\omega)$ does not ensure the convergence of the technique, it represents a necessary condition to achieve convergence. To obtain a nonnegative transfer function for use with this technique, the reblurring concept is introduced [3]. The reblurring concept is based on convolving both sides of the convolution equation with a reversed version of the input waveform, i.e., $x(-t)$ or, in other words, multiplying both sides of the frequency-domain form of the convolution equation by $X^*(j\omega)$, the complex conjugate of $X(j\omega)$. The result as shown in the following is to replace $X(j\omega)$ in the Van-Cittert development above by $|X(j\omega)|^2$, which is nonnegative. This requires that $Y(j\omega)$ be simultaneously replaced by $X^*(j\omega)$ $Y(j\omega)$:

$$X^*(j\omega) \cdot [Y(j\omega)] = X^*(j\omega) \cdot [H(j\omega) \cdot X(j\omega)] = H(j\omega) \cdot |X(j\omega)|^2. \tag{7}$$

In cases where the convergence conditions are satisfied, we can observe that (3) takes the form of a filter $B(j\omega)$ applied to the ratio transfer function:

$$H_i(j\omega) = H_r(j\omega) \cdot B(j\omega) \tag{8}$$

where

$$H_r(j\omega) = \{Y(j\omega)/X(j\omega)\}, \quad \text{and}$$

$$B(j\omega) = 1 - [1 - X(j\omega)]^{i+1}.$$

To examine the filtering nature of $B(j\omega)$, we discuss the effect of the iteration order i on this function. For $i \to \infty$, the term $B(j\omega)$ approaches unity at all frequencies (an all-pass filter), provided that condition (4), $|1 - X(j\omega)| < 1$, is satisfied. In this case, $H_i(j\omega)$ approaches the transfer function ratio $\{H_r(j\omega) = Y(j\omega/X(j\omega)\}$. For a finite i, $H_r(j\omega)$ is modified by a factor that suppresses its values at the frequencies for which $X(j\omega)$ is small (i.e., regions with low SNR). In other words, adaptive filtering of noise regions takes place. The suppression factor is determined by the values of $X(j\omega)$ and i. The suppression is very high for small i and affects the entire frequency response of the signal, i.e., for a few number of iterations, most of the signal is attenuated, and the filter acts as a stop-band over most frequencies.

From the above discussion it is clear that an optimum filter performance can be obtained with a large, but not infinite, number of iterations. For the large values of i, the adaptive filter performance is illustrated as follows: 1) for signal regions with useful signal information, $X(j\omega) > 0$, we have $B(j\omega) \to 1$ to provide a pass-band for the signal in those regions; and 2) for noise regions, $X(j\omega) \to 0$ (where the signal is absent), we have $B(j\omega) \to 0$ to provide a stop-band within those regions containing no useful signal information.

Another noteworthy observation is the contrast between the filter performance of the Van-Cittert technique and that of a typical frequency-domain iterative deconvolution filter. The typical frequency-domain deconvolution method starts off with the accurate but noisy ratio estimate, $H_r(j\omega) = Y(j\omega)/X(j\omega)$, then uses iterative filtering methods to reduce the noise while maintaining the deconvolution accuracy. The Van-Cittert method as discussed here starts with a low-noise but inaccurate estimate, $H_o(j\omega) = Y(j\omega)$, then uses an iterative correction process to improve the accuracy while maintaining the noise content at a low level. The ratio estimate is approached as the number of iterations approaches infinity.

The convergence prediction (optimization), which means the determination of the number of iterations required to obtain an acceptable estimate of the deconvolved result, is achieved through the use of a frequency-domain criterion. The optimization method selected for use in this work was developed and used successfully by Bennia and Riad [8] in the deconvolution of noisy time-domain waveforms. This method has been successfully used with other adaptive deconvolution filters widely used in time-domain measurement techniques [9], [10]. This optimization method has the advantage of being performed in the frequency domain and does not require the examination of the time-domain waveforms at each iteration; therefore the use of the inverse Fourier transformation at each iteration is eliminated.

The Bennia and Riad optimization method involves the examination of the effect of the iteration parameter on the deconvolution result in the two frequency-domain regions: the information region, where $X(j\omega) > 0$, and the noise region, where $X(j\omega)$ is negligible. For each frequency region, the standard deviation of the difference $\{H_i(j\omega) - H_{r(\text{same as } i=\infty)}(j\omega)\}$ is normalized to a unity maximum value and plotted versus the number of iterations i. For optimum results, the value of this standard deviation should be lowest (zero) for the information region and highest (unity) for the noise region. The optimum estimate corresponds to the deconvolution result with the lowest noise content and the least noticeable degradation in useful information. The illustration in the following section is to provide further insight into the method and its application.

IV. Illustrative Example

An example of deconvolution is considered in this section to illustrate the applicability of the deconvolution technique. It is also to demonstrate the built-in filter performance of the Van-Cittert technique as well as the implementation of the Bennia-Riad criterion for optimum choice of the iteration parameter.

The technique is applied to an experimental case where no *a priori* knowledge of the system's impulse response is available. All the waveforms used were acquired using the Hypres PSP-1000 TDR system. A time-domain reflectometry, TDR, measurement was performed on a wideband coaxial device. The reference waveform, shown

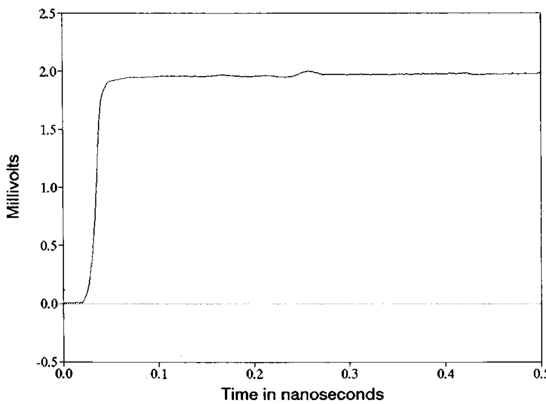

Fig. 1. TDR reference waveform $x(t)$.

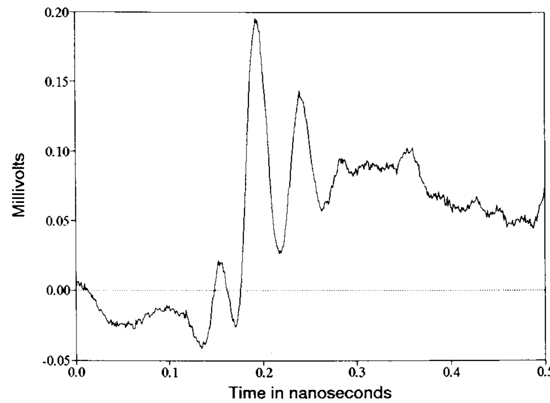

Fig. 2. TDR response waveform $y(t)$ of the DUT.

in Fig. 1, was acquired using a precision short circuit at the end of a reference transmission line. Next, the short circuit was replaced by the device under test, and the corresponding TDR response waveform was acquired; this waveform is shown in Fig. 2.

The computed transfer function of the device under test is shown in Fig. 3 for different numbers of iterations. In the figure, three cases are shown: the case of 25 000 iterations, the case of 250 000 iterations, and the case of no filter (straightforward ratio, this would also be the case of an infinite number of iterations). It can be seen from the figure that increasing the number of iterations increases the bandwidth of the filter. This results in a simultaneous increase in both the information content of the deconvolution result as well as its noise content.

To apply the Bennia–Riad optimization criterion, the frequency band of interest is divided into two major intervals: 0–80, and 80–512 GHz. The choice of the 80 GHz break point is to make the first interval as the primary information one, while the second interval is primarily the noise interval. This is the case since the bandwidth of this step-like excitation is about 64 GHz, and the information bandwidth of $H(j\omega)$ of the coaxial device appears to be about 40 GHz. The standard deviation parameter of the difference $\{H_i(j\omega) - H_r(j\omega)\}$ for each frequency region is then computed for various values of the number of iterations. The results were then normalized to a unity maximum for each frequency region independently, then plotted versus the number of iterations i. The results of these computations are shown in Fig. 4.

In Fig. 4, the decline of curve 1 is an indication of the continued improvement in the deconvolution accuracy as the number of iterations increased. The improvement is clearly observed to about 250 000 iterations, after which accuracy improvements continue slowly then vanish at about 1 000 000 iterations. On the other hand, examining curve 2, the unity standard deviation clearly indicates a low noise content in the deconvolution result. The degradation in this measure of the quality of the SNR starts at about 1 000 000 iterations and becomes more visible when the number of iterations exceeds about 1 400 000.

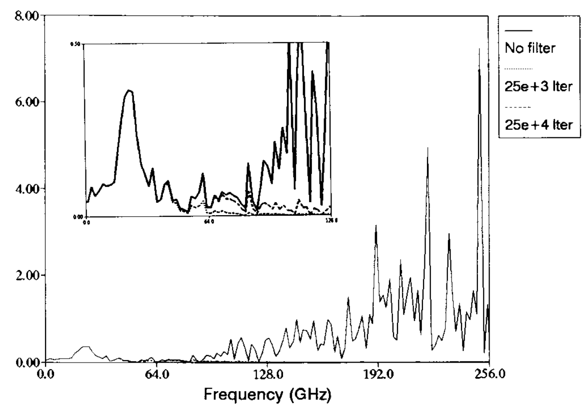

Fig. 3. Magnitude of the transfer function using the direct division and the deconvolution filter with different filtering levels (numbers of iterations).

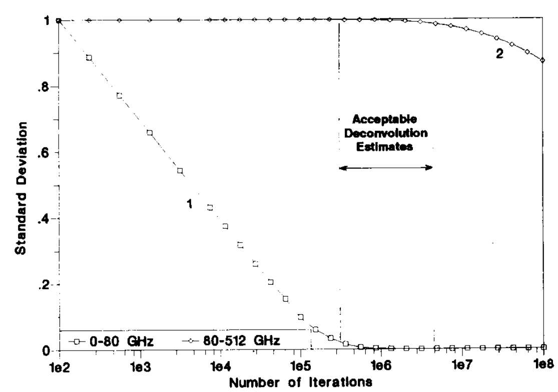

Fig. 4. Results of the frequency domain optimization criteria.

Based on these observations, it is safe to state that acceptable deconvolution estimates are obtained for a number of iterations between 250 000 and 1 400 000. The value of 250 000 was chosen in this case as a balance between the slow improvement in accuracy and the linear increase in computation time. The corresponding decon-

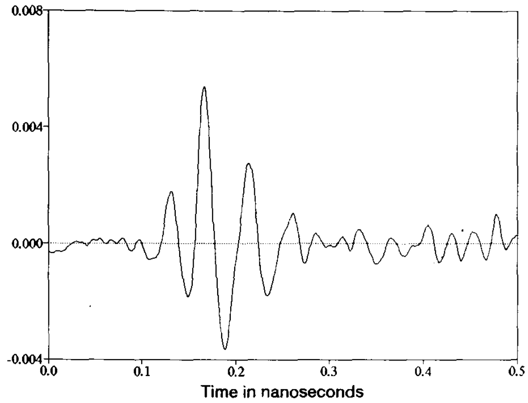

Fig. 5. Impulse response $h(t)$ obtained using an optimal deconvolution filter.

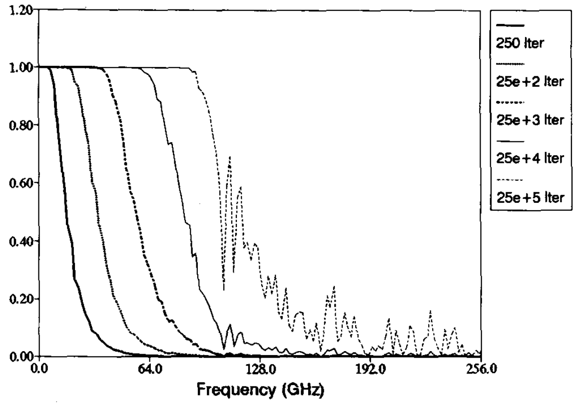

Fig. 6. Magnitude of the deconvolution filter using different numbers of iterations.

volution result, $h(t)$, for this optimized number of iterations is shown in Fig. 5.

The adaptive nature of the Van–Cittert deconvolution filter is demonstrated in Fig. 6. The figure plots the magnitude of the filter response $|B(j\omega)|$ of (7) versus frequency for two iteration counts. The figure demonstrates the adaptive nature of this filter with its pass-band coinciding with the information frequency band and its stop-band affecting the noise interval. The number of iterations is clear to affect the filter's bandwidth. The larger the number of iterations the wider the filter's bandwidth, and consequently, the more accurate the result is.

V. Conclusion

In this paper, we showed that the time-domain Van–Cittert iterative deconvolution technique can be used as a Wiener-type filtering (frequency-domain) technique. Using scaling (normalization) and reblurring, the convergence restrictions are alleviated. Consequently, the convergence of the technique is made possible regardless of the type of excitation signal used. We also showed that the number of iterations needed for its convergence can be predicted using a previously developed frequency-domain optimization criterion. The determination of the number of iterations required to obtain an acceptable result is achieved through the application of the Bennia–Riad optimization criterion. The criterion is successfully applied to single out a range for the number of iterations that would yield an appropriate result. An experimental example of deconvolution in time-domain measurements was used to illustrate the findings of this paper. Successful results were obtained as reported.

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