Math 452 Homework 3

Due September 20, 2023

1. (5 points) Let $X_1, X_2, \cdots, X_N \sim \mathcal{N}(\mu, \sigma^2)$. Show that the MLE estimations of the mean and variance are given by,

$$\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N} X_n, \quad \hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N} X_n^2 - \hat{\mu}^2.$$

Is $\mathbb{E}[\hat{\mu}] = \mu$? Is $\mathbb{E}[\hat{\sigma}^2] = \sigma^2$? Here $\mathbb{E}$ refers to the expectation with respect to $\mathcal{N}(\mu, \sigma^2)$.

2. (15 points) Let $x \in \{0, 1\}$ denotes the result of a coin toss ($x = 0$ for tails, $x = 1$ for heads). The coin is potentially biased, so that heads occurs with probability θ_1. Suppose that someone else observes the coin flip and reports to you the outcome, $y \in \{0, 1\}$. But this person is unreliable and only reports the result correctly with probability θ_2. i.e., $p(y|x, \theta_2)$ is given by

$$p(y|x,\theta_2) = \begin{cases} \theta_2, & if \quad x = 0, y = 0, \\ 1-\theta_2, & if \quad x = 1, y = 0, \\ 1-\theta_2, & if \quad x = 0, y = 1, \\ \theta_2, & if \quad x = 1, y = 1. \end{cases}$$

i. Assume that θ_2 is independent of θ_1. Show that the joint density is given by $p(x, y|\theta_1, \theta_2) = p(y|x, \theta_2)p(x|\theta_1)$.

ii. Suppose that you have observed the following sequence: $x = (1, 1, 0, 1, 1, 0, 0), y = (1, 0, 0, 0, 1, 0, 1)$. What are the MLEs for θ_1 and θ_2?

iii. Using the prior densities $p(\theta_1) = 1$ and $p(\theta_2) = 5\theta_2^4$, find the MAP estimation of θ_1 and θ_2.

3. Let $Y_1, Y_2, \cdots, Y_N \in \mathbb{B}(\theta)$ be a set of i.i.d. Bernoulli random variables, and let $X = \sum_{n=1}^{N} Y_n$. Assume the prior density is $p(\theta) = 5\theta^4$ ($\theta \in [0, 1]$), and let $\hat{\theta}$ be the MAP estimate of θ. Is $\hat{\theta}$ biased?

4. Following from Problem 3, find the posterior mean and variance of θ when the prior density is $p(\theta) = 5\theta^4$ ($\theta \in [0, 1]$).

5. Following from Problem 3, find the Laplace approximations of the posterior density $p(\theta|x)$.