Part II: Real Analysis

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Suppose $f \in L^\infty(\mathbb{R})$ and define the Laplace transform $F : (0, \infty) \to \mathbb{R}$ by

$$F(s) := \int_0^\infty f(t)e^{-st} d\mu(t).$$

Prove that F is absolutely continuous on $[a, b]$ for any $b > a > 0$.

Solution:

Proof. Let $\epsilon > 0$ and $b > a > 0$. Since $f \in L^\infty(\mathbb{R})$, there exists $M > 0$ such that $f \leq M$ a.e. Let $\delta := \epsilon a^2/M$ and suppose $\{(x_i, y_i)\}_{i=0}^n \subseteq [a, b]$ are non-overlapping subintervals such that $\sum_{i=0}^n (y_i - x_i) < \delta$. Then,

$$\begin{aligned}
\sum_{i=0}^n |F(x_i) - F(y_i)| &= \sum_{i=0}^\infty \left| \int_0^\infty f(t)[e^{-x_i t} - e^{-y_i t}] d\mu(t) \right| \\
&\leq M \sum_{i=0}^n \left[\int_0^\infty e^{-x_i t} d\mu(t) - \int_0^\infty e^{-y_i t} d\mu(t) \right] \\
&= M \sum_{i=0}^n \left[\frac{1}{x_i} - \frac{1}{y_i} \right] \\
&= M \sum_{i=0}^n \left[\frac{y_i - x_i}{x_i y_i} \right] \leq \frac{M}{a^2} \sum_{i=0}^\infty (y_i - x_i) \leq \frac{M\delta}{a^2} = \epsilon.
\end{aligned}$$

□