Part I: Real Analysis

1

Let $(X, \mathcal{A}, \mu)$ be an arbitrary measure space with μ a positive measure. Prove that $(X, \mathcal{A}, \mu)$ is σ-finite if and only if there exists a strictly positive function $f \in L^1(\mu)$.

Solution:

Proof. ($\Rightarrow$) Suppose $(X, \mathcal{A}, \mu)$ is σ-finite. Then there exists sets $\{A_n\}_{n=0}^{\infty} \subseteq \mathcal{A}$ such that $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ and $X = \cup_{n=0}^{\infty} A_n$. Without loss of generality assume $\{A_n\}_{n=0}^{\infty}$ is pairwise disjoint and $A_n \neq \emptyset$ for all $n \in \mathbb{N}$. Let $f : X \to \mathbb{R}$ be defined

$$f(x) := \sum_{n=0}^{\infty} \frac{2^{-n}}{\mu(A_n)} \chi_{A_n}(x).$$

Then f is strictly positive and

$$\|f\|_1 = \sum_{n=0}^{\infty} 2^{-n} = 2 < \infty.$$

($\Leftarrow$) Suppose there exists a strictly positive function $f \in L^1(\mu)$. By way of contradiction assume $(X, \mathcal{A}, \mu)$ is not σ-finite. For each $n \in \mathbb{N}$, let $A_n := \{x \in X : f(x) > 2^{-n}\}$. Since f is measurable, A_n is measurable for all n. Also, since $\cup_{n=0}^{\infty} A_n = X$, there must exist some $n_0 \in \mathbb{N}$ such that $\mu(A_{n_0}) = \infty$. Therefore,

$$\|f\|_1 = \int_X f \; d\mu \geq \int_{A_{n_0}} f \; d\mu \geq \int_{A_{n_0}} 2^{-n_0} \; d\mu = \infty.$$

This contradicts the assumption that $f \in L^1(\mu)$. $\square$

2

Give an example of each of the following:

(a) A function f which is unbounded but Lebesgue integrable on $(0, \infty)$.

(b) A function f which is Lipschitz continuous but not differentiable everywhere.

(c) A function f which is absolutely continuous but not Lipschitz continuous on $[0, 1]$.

(d) A sequence $\{f_n\}$ of functions that converges to 0 pointwise on $[0, 1]$ but not in $L^1([0, 1])$.

Solution:

(a) Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) := \sum_{n=0}^{\infty} 2^{-n} \chi_{(n,n+1]}(x).$$

(b) Let $f(x) := |x|$.

(c) Let $f(x) := \sqrt{x}$.

(d) For each $n \in \mathbb{N}$, let $f_n(x) := 2^{-n} \chi_{(0,2^{-n})}(x)$.

3

Let $p, q > 1$ be conjugate and $\Omega \subset \mathbb{R}^N$.

(a) Show that if $f_n \to f$ in $L^p(\Omega)$ and $g_n \to g$ in $L^q(\Omega)$, then $f_n g_n \to fg$ in $L^1(\Omega)$.

(b) Explain carefully what is meant by the statement that $L^q(\Omega)$ is the dual space of $L^p(\Omega)$.

Solution:

(a) *Proof.* Since $f_n \to f$ in $L^p(\Omega)$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\|f_n - f\|_p < 1$. Thus, for $n \geq N$,

$$\Big| \|f_n\|_p - \|f\|_p \Big| \leq \|f_n - f\|_p < 1,$$

so $\{\|f_n\|_p\}_{n \geq N}$ is bounded. But since $\{\|f_n\|_p\}_{n<N}$ is a finite collection of reals, $\{\|f_n\|_p\}$ is bounded. Thus there exists $M > 0$ such that $\|f_n\|_p < M$ for all n. Therefore, by Minkowski's and multiple use of Hölder's Inequality,

$$\begin{aligned} \|f_n g_n - fg\|_1 = \|f_n g_n - f_n g + f_n g - fg\|_1 &\leq \|f_n(g_n - g)\|_1 + \|g(f_n - f)\|_1 \\ &\leq \|f_n\|_p \cdot \|g_n - g\|_q + \|g\|_q \cdot \|f_n - f\|_p \\ &\leq M \cdot \|g_n - g\|_q + \|g\|_q \cdot \|f_n - f\|_p \xrightarrow{n\to\infty} 0. \end{aligned}$$

□

(b) To say that $L^q(\Omega)$ is the dual space of $L^p(\Omega)$ means that for every bounded linear function T on $L^p(\Omega)$, there exists $g \in L^q(\Omega)$ such that

$$T(f) = \int_\Omega f \cdot g \, d\mu,$$

for all $f \in L^p(\Omega)$.

4

If $f \in L^q(\mathbb{R}^N)$ for some $q < \infty$, show that

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

Also show by example that the conclusion may be false without the assumption that $f \in L^q(\mathbb{R}^N)$.

Solution:

Proof. Without loss of generality assume $\|f\|_\infty < \infty$. Let $0 < \delta < \|f\|_\infty$ and let $E := \{x : |f(x)| \geq \|f\|_\infty - \delta\}$. Since $f \in L^q(\mathbb{R}^N)$, $\mu(E) < \infty$. Now for any $0 < p < \infty$,

$$\|f\|_p \geq \left(\int_E (\|f\|_\infty - \delta)^p d\mu\right)^{1/p} = (\|f\|_\infty - \delta) \cdot \mu(E)^{1/p},$$

so $\liminf_{p\to\infty} \|f\|_p \geq \lim_{p\to\infty} \|f\|_\infty - \delta$. Thus,

$$\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty. \tag{1}$$

Further, for any $0 < p < \infty$,

$$\|f\|_p = \left(\int |f|^{p-q}|f|^q \, d\mu\right)^{1/p} \leq \|f\|_\infty^{\frac{p-q}{p}} \|f\|_q^{q/p}.$$

Hence

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty. \tag{2}$$

So by (1) and (2) we have equality. $\square$

For an illustration of how this relationship fails when there does not exist a $q < \infty$ such that $f \in L^q(\mathbb{R}^N)$, consider any constant function $f(x) \equiv c$. Then $\|f\|_\infty = c$ but $\|f\|_p = \infty$ for all $0 < p < \infty$.