Part II: Real Analysis

1

Let f be a Borel measurable function which is Lebesgue integrable on $\mathbb{R}$. Suppose for every $a < b$ that $\int_a^b f \; d\mu = 0$. Show that $f = 0$ a.e.

Solution: *Proof.* Let $E := \{x \in \mathbb{R} : f(x) > 0\}$. We will show that $\mu(E) = 0$. The proof that $\mu(\{x : f(x) < 0\}) = 0$ follows by the same argument. Since f is measurable, E is measurable. Let $E_n := E \cap (-n, -n+1] \cup [n-1, n)$ for all $n \geq 1$. Thus $E = \cup_{n=1}^{\infty} E_n$ and each E_n is measurable, and $\{E_n\}_{n=1}^{\infty}$ is pairwise disjoint. Thus, for each n there exists a G_δ set G_n such that $G_n \supseteq E_n$ and

$$\mu(G_n - E_n) = 0. \tag{1}$$

Without loss of generality assume $G_n = \cap_{k=1}^{\infty} O_{n,k}$ where $\{O_{n,k}\}_{k=1}^{\infty}$ is a descending sequence of bounded, open sets. But then $O_{n,k} = \cup_{j \in \mathcal{F}_{n,k}} I_j$, where I_j is a bounded, open interval for every $j \in \mathcal{F}_{n,k}$, $k, n \geq 1$, and therefore

$$\int_{I_j} f \; d\mu = 0$$

by assumption. Thus,

$$\int_{O_{n,k}} f \; d\mu = \sum_{j \in \mathcal{F}_{n,k}} \int_{I_j} f \; d\mu = 0. \tag{2}$$

Hence, using the continuity of integration,

$$\int_E f \; d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \; d\mu \overset{(1)}{=} \sum_{n=1}^{\infty} \int_{G_n} f \; d\mu = \sum_{n=1}^{\infty} \left[\lim_{k \to \infty} \int_{O_{n,k}} f \; d\mu \right] \overset{(2)}{=} 0.$$

Since $\int_E f \; d\mu = 0$ and $f > 0$ on E, $\mu(E) = 0$. $\square$

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This is just the Borel-Cantelli Lemma.

3

Let f be continuous on $\mathbb{R}$ and let

$$F_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right).$$

Prove that F_n converges uniformly on every finite interval $[a, b]$.

Solution:

Proof. Let $-\infty < a < b < \infty$.

<u>**Claim 1:**</u> $F_n(x)$ converges pointwise to $\int_x^{x+1} f \, d\mu$ for all $x \in [a, b]$.

<u>Proof of claim 1:</u> Let $x \in [a, b]$. For each $n \geq 1$ and $0 \leq k < n$, let

$$E_{n,k} := \left\{ t : x + \sum_{i=0}^{k-1} \frac{i}{n} \leq t < x + \sum_{i=0}^{k} \frac{i}{n} \right\}.$$

Now, since f is contiuous on $[x, x+1]$, there exists $M > 0$ such that $|f(t)| < M$ for all $t \in [x, x+1]$. Therefore $\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) \chi_{E_{n,k}}(t) \leq M$ for all $n \in \mathbb{N}, t \in [x, x+1]$. Thus, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) = \lim_{n\to\infty} \int_x^{x+1} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) \chi_{E_{n,k}}(t) \, d\mu(t) = \int_x^{x+1} f \, d\mu.$$

■ Claim 1

Now we will show that the convergence of $\{F_n\}$ is uniform. Let $\epsilon > 0$. Since f is continuous on $[a, b+1]$, f is uniformly contiuous on $[a, b+1]$. Therefore there exists $\delta > 0$ such that $|f(t_0) - f(t_1)| < \epsilon/2$ whenever $|t_0 - t_1| < \delta$. Choose $N \geq 1$ such that $N^{-1} < \delta$. Thus, for $n \geq N$ and $x \in [a, b]$,

$$\begin{aligned}
\left| F_n(x) - \int_x^{x+1} f \, d\mu \right| &= \left| \sum_{k=0}^{n-1} \int_{E_{n,k}} f\left(x + \frac{k}{n}\right) d\mu - \int_x^{x+1} f \, d\mu \right| \\
&= \left| \sum_{k=0}^{n-1} \int_{E_{n,k}} \left[f\left(x + \frac{k}{n}\right) - f(x) \right] d\mu \right| \\
&\leq \sum_{k=0}^{n-1} \int_{E_{n,k}} \left| f\left(x + \frac{k}{n}\right) - f(x) \right| d\mu \\
&\leq \sum_{k=0}^{n-1} \int_{E_{n,k}} \frac{\epsilon}{2} \, d\mu \\
&= \sum_{k=0}^{n-1} \frac{\epsilon}{2n} = \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

□

4

Let $\{f_n\}$, f be measurable functions on $[0,1]$. Show that $f_n \to f$ is measure on $[0,1]$ if and only if

$$\lim_{n\to\infty} \int_0^1 \frac{|f_n - f|}{1 + |f_n - f|}\ d\mu = 0.$$

Solution:

Proof. Let $g(x) = \frac{|x|}{1+|x|}$, $x \geq 0$.

($\Rightarrow$) First suppose $f_n \to f$ in measure on $[0,1]$. Let $\epsilon > 0$. Then

$$\begin{aligned}\int_0^1 g \circ f\ d\mu = \int_{\{|f_n - f| > \epsilon\}} g \circ f\ d\mu + \int_{\{|f_n - f| \leq \epsilon\}} g \circ f\ d\mu \leq \int_{\{|f_n - f| > \epsilon\}} 1\ d\mu + \int_0^1 \epsilon\ d\mu \\ = \mu\left(\{|f_n - f| > \epsilon\}\right) + \epsilon.\end{aligned}$$

Therefore $0 \leq \limsup_{n\to\infty} \int_0^1 g \circ f\ d\mu \leq \epsilon$, so $\lim_{n\to\infty} \int_0^1 g \circ f\ d\mu = 0$.

($\Leftarrow$) Now suppose $\lim_{n\to\infty} \int_0^1 g \circ f\ d\mu = 0$. Let $\epsilon > 0$. Then

$$\begin{aligned}\mu\left(\{x \in [0,1] : |f_n(x) - f(x)| > \epsilon\}\right) = \mu\left(\{x \in [0,1] : g(f(x)) > g(\epsilon)\}\right) \leq g(\epsilon) \int_{\{|f_n - f| > \epsilon\}} g \circ f\ d\mu \\ \leq \int_0^1 g \circ f\ d\mu.\end{aligned}$$

Therefore

$$0 \leq \limsup_{n\to\infty} \mu\left(\{|f_n - f| > \epsilon\}\right) \leq \limsup_{n\to\infty} \int_0^1 g \circ f\ d\mu = 0.$$

$\square$

5

Suppose $f : \mathbb{R} \to [0, \infty)$ is measurable and $\int f \, d\mu = c$, where $0 < c < \infty$. Prove

$$\lim_{n\to\infty} \int n \log\left[1 + \left(\frac{f(x)}{n}\right)^{\alpha}\right] d\mu(x) = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ c & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1. \end{cases}$$

Hint: Show when $\alpha \geq 1$ that the integrand is dominated by $\alpha f(x)$.

Solution:

Proof. Let $g \circ f := n \log\left[1 + \left(\frac{f}{n}\right)^{\alpha}\right]$. Then for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} g(f(x)) = \lim_{n\to\infty} \log\left[1 + \left(\frac{f(x)^{\alpha}}{n^{\alpha-1}}\right)\left(\frac{1}{n}\right)\right]^{n} = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ f(x) & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1. \end{cases} \tag{3}$$

Now, for $\varphi(x) \geq 0$ and $\alpha \geq 1$, we have $1 + \varphi(x)^{\alpha} \leq (1 + \varphi(x))^{\alpha}$. Thus

$$\log(1 + \varphi(x))^{\alpha} \leq \alpha \log(1 + \varphi(x)) \leq \alpha\varphi(x).$$

Hence, for $\alpha \geq 1$,

$$g(f(x)) \leq n\alpha \frac{f(x)}{n} = \alpha f(x).$$

So by applying the Dominated Convergence Theorem we have the result for $\alpha = 1$ and $\alpha > 1$. For $0 < \alpha < 1$ we can apply Fatou's Lemma. ∎

6

Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of increasing, continuously differentiable functions on the interval $[a,b]$ such that, for all $x \in [a,b]$, $s(x) := \sum_{n=0}^{\infty} |f_n(x)| < \infty$. Show that

$$s'(x) := \sum_{n=0}^{\infty} f_n'(x) \quad \text{a.e.}$$

Solution: Note that this is essentially Fubini's theorem on term-by-term differentiation and the assumption that f_n' is continuous is not needed.

Proof. Clearly s is increasing so s' exists a.e. on (a,b). Now suppose let $x \in (a,b)$ such that $s'(x)$ exists. Then

$$\begin{aligned} s'(x) = \lim_{k\to\infty} \mathrm{Diff}_{2^{-k}}(s)(x) = \lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} f_n(x+2^{-k}) - \sum_{n=0}^{\infty} f_n(x)}{2^{-k}} &= \lim_{k\to\infty} \sum_{n=0}^{\infty} \frac{f_n(x+2^{-k}) - f_n(x)}{2^{-k}} \\ &= \lim_{k\to\infty} \sum_{n=0}^{\infty} \mathrm{Diff}_{2^{-k}}(f_n)(x) \\ \text{(Fatou's)} \quad &\geq \sum_{n=0}^{\infty} \liminf_{k\to\infty} \mathrm{Diff}_{2^{-k}}(f_n)(x). \end{aligned}$$

So

$$s'(x) \geq \sum_{n=0}^{\infty} f_n'(x). \tag{4}$$

Now let for each $k \in \mathbb{N}$, let $\alpha_k(x) := \sum_{n=k+1}^{\infty} f_n(x)$. Clearly each α_k is an increasing function of x. Thus

$$0 \leq \int_a^b \alpha_k' \, d\mu \leq \alpha_k(b) - \alpha_k(a) = \sum_{n=k+1}^{\infty} [f_n(b) - f_n(a)],$$

and since $\sum_{n=k+1}^{\infty}[f_n(b) - f_n(a)]$ converges for all $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \int_a^b \alpha_k' \, d\mu = 0. \tag{5}$$

Therefore

$$\int_a^b s' \, d\mu = \int_a^b \left(\sum_{n=0}^{k} f_n \right)' d\mu + \int_a^b \alpha_k' \, d\mu = \int_a^b \sum_{n=0}^{k} f_n' \, d\mu + \int_a^b \alpha_k' \, d\mu \leq \int_a^b \sum_{n=0}^{\infty} f_n' \, d\mu + \int_a^b \alpha_k' \, d\mu.$$

Hence by (5)

$$\int_a^b s' \, d\mu \leq \int_a^b \sum_{n=0}^{\infty} f_n' \, d\mu + \lim_{k\to\infty} \int_a^b \alpha_k' \, d\mu = \int_a^b \sum_{n=0}^{\infty} f_n' \, d\mu. \tag{6}$$

However, the only way that (4) and (6) can be consisted is if $s'(x) = \sum_{n=0}^{\infty} f_n'(x)$ a.e. ∎