Part II: Real Analysis

4

Let E be a measurable subset of $\mathbb{R}$ and let $f_n \to f$ a.e., where $f \in L^1(E)$. Show that $\int_E |f_n - f| \; d\mu \to 0$ if and only if $\int_E |f_n| \; d\mu \to \int_E |f| \; d\mu$.

Solution:

Proof. ($\Rightarrow$) Suppose $\int_E |f_n - f| \; d\mu \to 0$. Then

$$\int_E |f_n| \; d\mu - \int_E |f| \; d\mu = \int_E (|f_n| - |f|) \; d\mu \leq \int_E |f_n - f| \; d\mu \to 0.$$

($\Leftarrow$) Suppose $\int_E |f_n| \; d\mu \to \int_E |f| \; d\mu$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\int_E |f_n| \; d\mu < \int_E |f| \; d\mu + 1$. Thus, since $|f_n - f| \leq |f_n| + |f|$,

$$\int_E |f_n - f| \; d\mu \leq \int_E |f_n| \; d\mu + \int_E |f| \; d\mu < 2 \int_E |f| \; d\mu + 1 < \infty,$$

for all $n \geq N$. Hence, by the Dominated Covergence Theorem applied to $\{|f_n - f|\}_{n=N}^{\infty}$,

$$\lim_{n \to \infty} \int_E |f_n - f| \; d\mu = \int_E \lim_{n \to \infty} |f_n - f| \; d\mu = 0.$$

□

5

Construct a function on $[0,1]$ which is monotone increasing and discontinuous precisely at the rationals. Rigorously prove that your function has the desired properties.

Solution: Let $\{\alpha_n\}_{n=0}^{\infty}$ be an enumeration of the rationals in $[0,1]$. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) := \sum_{\{n : \alpha_n \leq x\}} 2^{-n}, \qquad \forall \, x \in [0,1].$$

<u>Claim 1:</u> f is monotone increasing.

<u>Proof of claim 1:</u> Trivial by the construction of f. ■ Claim 1

<u>Claim 2:</u> f is discontinuous at every rational in $[0,1]$ and continuous otherwise.

<u>Proof of claim 2:</u> Let $x \in [0,1]$

Case 1 $x \in [0,1] - \mathbb{Q}$.

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $2^{-N} < \epsilon$. By the denseness of the rationals in $[0, 1]$, we can choose $\delta > 0$ such that $n > N$ for all $\alpha_n \in \mathbb{Q} \cap (x - \delta, x + \delta)$. Now let $y \in (x - \delta, x + \delta)$. Without loss of generality suppose $y > x$. Then

$$|f(x) - f(y)| = \sum_{\{n : x < \alpha_n \leq y_n\}} 2^{-n} \leq \sum_{n=N+1}^{\infty} 2^{-n} = 2^{-N} < \epsilon.$$

Therefore f is continuous at x.

Case 2 $x \in \mathbb{Q}$.

There exists $N \in \mathbb{N}$ such that $\alpha_N = x$. So clearly $f(x) - f(y) > 2^{-N}$ for all $y < x$ (assuming $x \neq 0$ of course). Therefore f is discontinuous at x.

■ Claim 2