# Part II: Real Analysis

## 1

Let $f$ be an unbounded real valued function defined on the open interval $(0,1)$. Is it possible that $f$ is uniformly continuous on $(0,1)$?

**Solution:** No.

*Proof.* By way of contradiction, suppose $f:(0,1)\to\mathbb{R}$ is uniformly continuous yet unbounded. Let $\epsilon>0$. Then there exists $\delta>0$ such that $|f(x)-f(y)|<\epsilon$ whenever $x,y\in(0,1)$ such that $|x-y|<\delta$. Now let $0<a<b<1$ such that $a<\delta$ and $1-b<\delta$. Then there exists $M>0$ such that $|f(x)|<M$ for all $x\in[a,b]$ and therefore $|f(y)|<M+\epsilon$ for all $y\in(0,1)-[a,b]$. So $|f(x)|<M+\epsilon$ for all $x\in(0,1)$. This is a contradiction. □

## 3

Let $p,q,r$ be positive real numbers with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, and let $f\in L^p, g\in L^q, h\in L^r$. Prove that $fgh\in L^1$ and

$$\|fgh\|_1\le\|f\|_p\cdot\|g\|_q\cdot\|h\|_r.$$

**Solution:**

*Proof.* Let $s:=qr/(q+r)$.

**Claim 1:** $gh\in L^s$ and $\|gh\|_s\le\|g\|_q\cdot\|h\|_r$.

Proof of claim 1: Let $q':=(q+r)/r$ and $r':=(q+r)/q$. Clearly $q'$ and $r'$ are conjugate and $|g|^s\in L^{q'}$ and $|h|^s\in L^{r'}$. Thus, by Hölder's Inequality,

$$\|gh\|_s^s=\int|g|^s\cdot|h|^s\,d\mu\le\left(\int|g|^q\,d\mu\right)^{\frac{r}{q+r}}\left(\int|h|^r\,d\mu\right)^{\frac{q}{q+r}}=\|g\|_q^s\cdot\|h\|_r^s.$$

■ Claim 1

Now note that $p$ and $s$ are conjugate. Thus, by claim 1 and Hölder's applied to $f$ and $gh$,

$$\|fgh\|_1\le\|f\|_p\cdot\|gh\|_s\le\|f\|_p\cdot\|g\|_q\cdot\|h\|_r.$$

□