Part II: Real Analysis

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Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of increasing, continuously differentiable functions on the interval $[a,b]$ such that, for all $x \in [a,b]$, $s(x) := \sum_{n=0}^{\infty} |f_n(x)| < \infty$. Show that

$$s'(x) := \sum_{n=0}^{\infty} f_n'(x) \quad \text{a.e.}$$

Solution: Note that this is essentially Fubini's theorem on term-by-term differentiation and the assumption that f_n' is continuous is not needed.

Proof. Clearly s is increasing so s' exists a.e. on (a,b). Now suppose let $x \in (a,b)$ such that $s'(x)$ exists. Then

$$\begin{aligned} s'(x) = \lim_{k\to\infty} \mathrm{Diff}_{2^{-k}}(s)(x) = \lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} f_n(x+2^{-k}) - \sum_{n=0}^{\infty} f_n(x)}{2^{-k}} &= \lim_{k\to\infty} \sum_{n=0}^{\infty} \frac{f_n(x+2^{-k}) - f_n(x)}{2^{-k}} \\ &= \lim_{k\to\infty} \sum_{n=0}^{\infty} \mathrm{Diff}_{2^{-k}}(f_n)(x) \\ (\text{Fatou's}) \quad &\geq \sum_{n=0}^{\infty} \liminf_{k\to\infty} \mathrm{Diff}_{2^{-k}}(f_n)(x). \end{aligned}$$

So

$$s'(x) \geq \sum_{n=0}^{\infty} f_n'(x). \tag{1}$$

Now let for each $k \in \mathbb{N}$, let $\alpha_k(x) := \sum_{n=k+1}^{\infty} f_n(x)$. Clearly each α_k is an increasing function of x. Thus

$$0 \leq \int_a^b \alpha_k' \, d\mu \leq \alpha_k(b) - \alpha_k(a) = \sum_{n=k+1}^{\infty} [f_n(b) - f_n(a)],$$

and since $\sum_{n=k+1}^{\infty} [f_n(b) - f_n(a)]$ converges for all $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \int_a^b \alpha_k' \, d\mu = 0. \tag{2}$$

Therefore

$$\int_a^b s' \, d\mu = \int_a^b \left(\sum_{n=0}^{k} f_n \right)' d\mu + \int_a^b \alpha_k' \, d\mu = \int_a^b \sum_{n=0}^{k} f_n' \, d\mu + \int_a^b \alpha_k' \, d\mu \leq \int_a^b \sum_{n=0}^{\infty} f_n' \, d\mu + \int_a^b \alpha_k' \, d\mu.$$

Hence by (2)

$$\int_a^b s' \, d\mu \leq \int_a^b \sum_{n=0}^{\infty} f_n' \, d\mu + \lim_{k\to\infty} \int_a^b \alpha_k' \, d\mu = \int_a^b \sum_{n=0}^{\infty} f_n' \, d\mu. \tag{3}$$

However, the only way that (1) and (3) can be consisted is if $s'(x) = \sum_{n=0}^{\infty} f_n'(x)$ a.e. $\square$