Part II: Real Analysis

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Let $1 < p < \infty$ and q be conjugate to p. Fix a real-valued function $g \in L^q(\mathbb{R})$ and define the functional $F : L^p(\mathbb{R}) \to \mathbb{R}$ by

$$F(f) := \int_{\mathbb{R}} f \cdot g \; d\mu,$$

for all $f \in L^p(\mathbb{R})$. Show that

(i) F is in fact a linear functional on $L^p(\mathbb{R})$.

(ii) $||F|| = ||g||_q$ where $||F|| := \sup\{|F(f)| : ||f||_p = 1\}$.

Solution:

Proof. Let $g \in L^q(\mathbb{R})$.

Claim 1: F is a linear functional.

Proof of claim 1: Let $f, h \in L^p(\mathbb{R}), \alpha, \beta \in \mathbb{R}$. Then

$$F(\alpha f + \beta h) = \int_{\mathbb{R}} (\alpha f + \beta h) g \; d\mu = \alpha \int_{\mathbb{R}} f \cdot g \; d\mu + \beta \int_{\mathbb{R}} h \cdot g \; d\mu = \alpha F(f) + \beta F(h).$$

■ Claim 1

Now note that for any $f \in L^p(\mathbb{R})$,

$$|F(f)| = \left| \int_{\mathbb{R}} f \cdot g \; d\mu \right| \leq \int_{\mathbb{R}} |f \cdot g| \; d\mu \leq ||f||_p \cdot ||g||_q = ||g||_q, \tag{1}$$

by Hölder's Inequality. Now define $f^* := ||g||_q^{1-q} \text{sign}(g) |g|^{q-1}$.

Claim 2: $f^* \in L^p(\mathbb{R})$.

Proof of claim 2: We have

$$||f^*||_p^p = \int_{\mathbb{R}} |f^*|^p \; d\mu = ||g||_q^{1-q} \int_{\mathbb{R}} |g|^{qp-p} \; d\mu = ||g||_q < \infty.$$

■ Claim 2

Claim 3: $||F|| = ||g||_q$.

Proof of claim 3: Note that

$$|F(f^*)| = \left| \int_{\mathbb{R}} ||g||_q^{1-q} |g|^{q-1} |g| \; d\mu \right| = ||g||_q^{1-q} \cdot ||g||_q^q = ||g||_q. \tag{2}$$

Thus, by (1) and (2) we are done. ■ Claim 3

□

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Let $\{f_n\}_{n=0}^{\infty}$ be a sequence in $L^2(\mathbb{R})$ such that $\sum_{n=0}^{\infty} ||f_n||_2 < \infty$ and so that $\sum_{n=0}^{\infty} f_n(x) = 0$ for almost all $x \in \mathbb{R}$. Prove that for each $g \in L^2(\mathbb{R})$,

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} f_n(x)g(x)\ d\mu(x) = 0.$$

Solution:

Proof. Let $g \in L^2(\mathbb{R})$. For each $k \in \mathbb{N}$, define $h_k := \sum_{n=0}^{\infty} f_n \cdot g$. Clearly

$$|h_k| = \left|\sum_{n=0}^{k} f_n \cdot g\right| \leq \sum_{n=0}^{k} |f_n \cdot g| \leq \sum_{n=0}^{\infty} |f_n \cdot g|, \tag{3}$$

and

$$\int_{\mathbb{R}} \sum_{n=0}^{\infty} |f_n \cdot g|\ d\mu = \sum_{n=0}^{\infty} \int_{\mathbb{R}} |f_n \cdot g|\ d\mu \leq ||g||_2 \cdot \sum_{n=0}^{\infty} ||f_n||_2 < \infty, \tag{4}$$

by the Cauchy-Schwarz Inequality. So by (3) and (4) we can apply the Dominated Convergence Theorem to $\{h_k\}_{k=0}^{\infty}$. Thus,

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} f_n \cdot g\ d\mu = \lim_{k\to\infty} \int_{\mathbb{R}} h_k\ d\mu = \int_{\mathbb{R}} \lim_{k\to\infty} h_k\ d\mu = \int_{\mathbb{R}} g \sum_{n=0}^{\infty} f_n\ d\mu = 0.$$

$\square$

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Suppose $f : [0,1] \to \mathbb{R}$ is absolutely continuous.

(i) Prove that f is Lipschitz continuous on $[0,1]$ if and only if $\sup_{x\in[0,1]} |f'(x)| < \infty$.

(ii) Does (i) still hold if we assume f has bounded variation instead?

Solution:

(i) *Proof.*

($\Rightarrow$) Suppose f is Lipschitz continuous on $[0,1]$. Then there exists $c > 0$ such that $|f(a) - f(b)| < c|a - b|$ whenever $a, b \in [0,1]$. Therefore

$$|\text{Diff}_h(f)(x)| = \left|\frac{f(x+h) - f(x)}{(x+h) - x}\right| < c,$$

for all $h > 0, x \in [0,1]$. Hence $|f'(x)| \leq c$ for all $x \in [0,1]$, and so $\sup_{x\in[0,1]} |f'(x)| \leq c$.

($\Leftarrow$) Now suppose $\sup_{x\in[0,1]}|f'(x)| < \infty$. Then there exists $c > 0$ such that $|f'| < c$. Now let $a, b \in [0,1]$. Without loss of generality, suppose $a < b$. Since f is absolutely continuous, $\int_a^b f' d\mu = f(b) - f(a)$. Thus,

$$|f(b) - f(a)| = \left|\int_a^b f' d\mu\right| \leq \int_a^b |f'| d\mu \leq \int_a^b c d\mu = c(b-a).$$

Hence f is Lipschitz continuous. $\square$

(ii) No, (i) does not remain true. For example, consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 0 & : x < 1/2, \\ 1 & : x \geq 1/2. \end{cases}$$