Math 375

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(1) 2.8 Pr 19

- Determine one-to-one
  - Describe its range
  - Find inverse

$$T(x,y,z) = (x, x+y, x+y+z)$$
$$\text{Check one-to-one}$$
$$\text{Let } A = (x_1, y_1, z_1)\ B = (x_2, y_2, z_2)$$
$$\text{If it is one-to-one it must follow } T(A) = T(B) \text{ then } A = B$$

$$T(A) = T(B)$$
$$(x_1, x_1+y_1, x_1+y_1+z_1) = (x_2, x_2+y_2, x_2+y_2+z_2)$$

$$x_1 = x_2$$
$$x_1 + y_1 = x_2 + y_2$$
$$x_1 + y_1 + z_1 = x_2 + y_2 + z_2$$
$$\text{Solving the equation we get}$$
$$x_1 = x_2, y_1 = y_2, z_1 = z_2$$
$$\text{So } A = B$$
$$\text{Therefore it is one-to-one}$$

$$x(1,1,1) + y(0,1,1) + z(0,0,1) = (x, x+y, x+y+z)$$
$$(1,1,1), (0,1,1), (0,0,1) \text{ form a basis for range of } T$$
$$\text{The Range is } \mathbb{R}^3$$

$$T(x,y,z) = (u,v,w)$$
$$(x, x+y, x+y+z) = (u,v,w)$$
$$x = u, y = v-u, z = w-v$$
$$T^{-1}(u,v,w) = (x,y,z)$$
$$T^{-1}(u,v,w) = (u, v-u, w-v)$$

$$\text{Check } TT^{-1}(x,y,z) = (x,y,z)$$
$$T(x, y-x, z-y) = (x, x+y-x, z+y-x+z-y) = (x,y,z)$$
$$\text{Answer) It is one-to-one}$$
$$\text{Spanned by } (1,1,1), (0,1,1), (0,0,1) \text{ which is } \mathbb{R}^3$$
$$T^{-1}(u,v,w) = (u, v-u, w-v)$$

(2) 2.8 Pr 22

$$\text{If S and T commute, prove that} (ST)^n = S^nT^n \text{ for all integers } n \geq 0$$

Before proving for the $(ST)^n = S^nT^n$ we are going to show for $ST^n = T^nS$

It is trivial for $n = 0$ Since $T^0 = I$

for $n = 1$ it is true because $ST^1 = ST = TS$

Assume it is true for n and prove for n+1

$$ST^{n+1} = STT^n$$
$$= TST^n \text{ since we assume it is true for } n$$
$$= TT^nS$$
$$= T^{n+1}S$$

It is true for n+1 therefore it is true

Prove $(ST)^n = S^nT^n$ for all integers $n \geq 0$

It is trivial for $n = 0$ Since $T^0 = I$

for $n = 1$ it is true because $(ST)^1 = S^1T^1$

Assume it is true for n prove for $n + 1$

For $n + 1$ $(ST)^{n+1} = (ST)^n(ST) = S^nT^n(ST)$

Since they commute $S^nT^n(ST) = S^nST^nT = S^{n+1}T^{n+1}$

$(ST)^{n+1} = S^{n+1}T^{n+1}$ is true

Therefore statement $(ST)^n = S^nT^n$ for all integers $n \geq 0$ is true

(3) 2.8 Pr 24

If S and Tare invertible and commute, prove that their inverses also commute.

$ST = TS$ and $S^{-1}\ T^{-1}$ exist

$$ST = TS$$
$$S^{-1}ST = S^{-1}TS$$
$$IT = S^{-1}TS$$
$$T = S^{-1}TS$$
$$T^{-1}T = T^{-1}S^{-1}TS$$
$$I = T^{-1}S^{-1}TS$$
$$IS^{-1} = T^{-1}S^{-1}TSS^{-1}$$
$$S^{-1} = T^{-1}S^{-1}TI$$
$$S^{-1}T^{-1} = T^{-1}S^{-1}TT^{-1}$$
$$S^{-1}T^{-1} = T^{-1}S^{-1}I$$
$$S^{-1}T^{-1} = T^{-1}S^{-1}$$

Therefore inverse is also commute

(4) 2.12 Pr 3

A linear transformation $T : V_2 \to V_2$, maps the basis vectors i andj as follows:

$T(i) = i + j \quad T(j) = 2i - j$

a) Compute $T(3i-4j)$ and $T^2(3i-4j)$ in terms of i and j.

Since it is a linear transformation it must follow
$T(a\alpha+b\beta)=aT(\alpha)+bT(\beta)$ where $\alpha,\beta\in V_2$ and scalars a,b

$$T(3i-4j)=3T(i)-4T(j)=3(i+j)-4(2i-j)=-5i+7j$$

$$T^2(3i-4j)=TT(3i-4j)=T(-5i+7j)$$
$$=-5T(i)+7T(j)=-5(i+j)+7(2i-j)=9i-12j$$

b) Determine the matrix of T and T^2

basis is (i,j)

$$T(i)=(1)i+(1)j,\;\; T(j)=(2)i+(-1)j$$
$$T(i)=\begin{bmatrix}1\\1\end{bmatrix}, T(j)=\begin{bmatrix}2\\-1\end{bmatrix}$$
$$\text{So } [T]=\begin{bmatrix}1&2\\1&-1\end{bmatrix}$$

$$T^2=\begin{bmatrix}1&2\\1&-1\end{bmatrix}\begin{bmatrix}1&2\\1&-1\end{bmatrix}=\begin{bmatrix}3&0\\0&3\end{bmatrix}$$

c) Solve b if basis is replaced by (e_1,e_2) $e_1=i-j, e_2=3i+j$

$T(e_1)=T(i-j)=-i+2j=a(i-j)+b(3i+j)$ using problem a

$$a+3b=-1$$
$$-a+b=2$$
$$a=\frac{-7}{4}, b=\frac{1}{4}$$
$$\text{So } T(e_1)=\begin{bmatrix}\frac{-7}{4}\\\frac{1}{4}\end{bmatrix}$$

$T(e_2)=T(3i+j)=5i+2j=a(i-j)+b(3i+j)$ using problem a

$$a+3b=5$$
$$-a+b=2$$
$$a=\frac{-1}{4}, b=\frac{7}{4}$$
$$\text{So } T(e_2)=\begin{bmatrix}\frac{-1}{4}\\\frac{7}{4}\end{bmatrix}$$

$$[T]=\begin{bmatrix}\frac{-7}{4}&\frac{-1}{4}\\\frac{1}{4}&\frac{7}{4}\end{bmatrix}$$

$$T^2=\begin{bmatrix}\frac{-7}{4}&\frac{-1}{4}\\\frac{1}{4}&\frac{7}{4}\end{bmatrix}\begin{bmatrix}\frac{-7}{4}&\frac{-1}{4}\\\frac{1}{4}&\frac{7}{4}\end{bmatrix}=\begin{bmatrix}3&0\\0&3\end{bmatrix}$$

(5) 2.12 Pr 8

A linear transformation $T:V_2\to V_3$ maps the basis vectors as follows: $T(i)=(1,0,1), T(j)=(-1,0,1)$

a) Compute $T(2i-3j)$ and determine the nullity and rank of T.

Since it is a linear transformation it must follow
$T(a\alpha+b\beta)=aT(\alpha)+bT(\beta)$ where $\alpha,\beta\in V_2$ and scalars a,b

$$T(2i-3j)=2T(i)-3T(j)=2(1,0,1)-3(-1,0,1)=(5,0,-1)$$

Representative matrix of T
$$T(i)=(1,0,1)=1(1,0,0)+0(0,1,0)+1(0,0,1)$$
$$T(j)=(-1,0,1)=-1(1,0,0)+0(0,1,0)+1(0,0,1)$$

$$[T]=\begin{bmatrix}1 & -1\\ 0 & 0\\ 1 & 1\end{bmatrix}$$

So $T(v), v=\begin{bmatrix}x_1\\ x_2\end{bmatrix} v\in V_2$

$$T(v)=\begin{bmatrix}1 & -1\\ 0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1-x_2\\ 0\\ x_1+x_2\end{bmatrix}$$

The nullity of T
When $x_1=0, x_2=0$
$$\begin{bmatrix}x_1-x_2\\ 0\\ x_1+x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$
Therefore nullity of T is 0

Range of T is spanned by $\begin{bmatrix}1 & -1\\ 0 & 0\\ 1 & 1\end{bmatrix}$

Rank is 2

b) Determine the matrix of T

In part a we showed the matrix of T
$$[T]=\begin{bmatrix}1 & -1\\ 0 & 0\\ 1 & 1\end{bmatrix}$$

c) Find bases (e_1, e_2) for V_2 and (w_1, w_2, w_3) for V_3 relative to which the matrix of T will be in diagonal form.

$$T(e_1) = a_1 w_1 + a_2 w_2 + a_3 w_3$$
$$T(e_2) = b_1 w_1 + b_2 w_2 + b_3 w_3$$
$$\text{to be a diagonal form } a_2 = a_3 = 0, b_1 = b_3 = 0$$
$$T(e_1) = a_1 w_1$$
$$T(e_2) = b_2 w_2$$

$$T(v) = \begin{bmatrix} x_1 - x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e_2 = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$
$$T(e_1) = \begin{bmatrix} c_1 - c_2 \\ 0 \\ c_1 + c_2 \end{bmatrix}$$
$$T(e_2) = \begin{bmatrix} d_1 - d_2 \\ 0 \\ d_1 + d_2 \end{bmatrix}$$

(6) 2.16 Pr 1

(7) 2.16 Pr 3

(8) 2.16 Pr 4