Math 375

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1. *Let x = (x1, . . . , x,) and y = (y1, . . . , yn) be arbitrary vectors in V, . In each case, determine whether (x, y) is an inner product for V,, if (x, y) is defined by the formula given. In case (x, y) is not an inner product, tell which axioms are not satisfied.*

- **1.13 Pr 1a**

$$(x, y) = \sum_{i=1}^{n} x_i|y_i| \text{ Not an inner product}$$

$$\text{Axiom 1) } < x, y >=< y, x >$$

$$\text{To satisfy axiom 1}$$

$$\sum_{i=1}^{n} x_i|y_i| = \sum_{i=1}^{n} y_i|x_i|$$

$$\text{if every elements in x are postive and y are negative}$$

$$\sum_{i=1}^{n} x_i|y_i| > 0 \text{ and } \sum_{i=1}^{n} y_i|x_i| < 0$$

$$\text{Axiom 1 not satisfied}$$

$$\text{Axiom 2) <x,y+z>=<x,y>+<x,z>}$$

$$\text{Let } x = (1), y = (-2), z = (1)$$

$$< x, y + z >= \sum_{i=1}^{n} x_i|y_i + z_i| = 1$$

$$< x, y >= \sum_{i=1}^{n} x_i|y_i| = 2$$

$$< x, z >= \sum_{i=1}^{n} x_i|z_i| = 1$$

$$1 \neq 2 + 1$$

$$\text{Axiom 2 not satisfied}$$

$$\text{Axiom 3) <cx,y>=c<x,y>}$$

$$< cx, y >= \sum_{i=1}^{n} cx_i|y_i| = c\sum_{i=1}^{n} x_i|y_i|$$

$$c < x, y >= c\sum_{i=1}^{n} x_i|y_i|$$

$$c\sum_{i=1}^{n} x_i|y_i| = c\sum_{i=1}^{n} x_i|y_i|$$

$$\text{Therfore, } < cx, y >= c < x, y >$$

$$\text{Axiom 3 satisfied}$$

$$\text{Axiom 4)} \ < x, x >> 0 \text{ if } x \neq \vec{0}$$

$$< x, x >= \sum_{i=1}^{n} x_i |x_i|$$
$$\text{Let } x = (-1)$$
$$< x, x >= \sum_{i=1}^{n} x_i |x_i| = -1$$
$$-1 < 0$$
$$\text{Axiom 4 not satisfied}$$

- **1.13 Pr 1b**

$$(x, y) = \left| \sum_{i=1}^{n} x_i y_i \right| \text{ Not an inner product}$$
$$\text{Axiom 1)} \ < x, y >=< y, x >$$
$$< x, y >= \left| \sum_{i=1}^{n} x_i y_i \right|$$
$$< y, x >= \left| \sum_{i=1}^{n} y_i x_i \right| = \left| \sum_{i=1}^{n} x_i y_i \right|$$
$$< x, y >=< y, x >$$
$$\text{Therefore Axiom 1 satitsfied}$$

$$\text{Axiom 2) <x,y+z>=<x,y>+<x,z>}$$
$$\text{Let } x = (1), y = (-3), z = (2)$$
$$< x, y + z >= \left| \sum_{i=1}^{n} x_i (y_i + z_i) \right| = 1$$
$$< x, y >= \left| \sum_{i=1}^{n} x_i y_i \right| = 3$$
$$< x, z >= \left| \sum_{i=1}^{n} x_i z_i \right| = 2$$
$$1 \neq 3 + 2$$
$$\text{Therefore Axiom 2 not satisfied}$$

$$\text{Axiom 3)} \ < cx, y >= c < x, y >$$

$$\text{Let } c = -1$$
$$< cx, y >= \left| \sum_{i=1}^{n} c x_i y_i \right| = \left| c \sum_{i=1}^{n} x_i y_i \right| = \left| - \sum_{i=1}^{n} x_i y_i \right| \geq 0$$
$$c < x, y >= c \left| \sum_{i=1}^{n} c x_i y_i \right| = - \left| \sum_{i=1}^{n} c x_i y_i \right| \leq 0$$
$$\text{If } \ < x, y > \text{ is not } 0 \ \ < cx, y > \neq c < x, y >$$
$$\text{Therefore Axiom 3 not satisfied}$$

$$\text{Axiom 4)} \ < x, x >> 0 \text{ if } x \neq \vec{0}$$

$$< x, x >= \left| \sum_{i=1}^{n} x_i x_i \right|$$

$$\text{Since it is an absoulute value if } \sum_{i=1}^{n} x_i x_i \neq 0$$

$$< x, x >> 0 \text{ and only when } x = \vec{0}$$

$$< x, x >= 0$$

$$\text{Therefore Axiom 4 satisfied}$$

- **1.13 Pr 1c**

$$(x, y) = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j \text{ Not an inner product}$$

$$\text{Axiom 1)} \ < x, y >=< y, x >$$

$$< x, y >= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j$$

$$< y, x >= \sum_{i=1}^{n} y_i \sum_{j=1}^{n} x_j$$

$$< x, y >=< y, x >$$

$$\text{Therefore Axiom 1 satisfied}$$

$$\text{Axiom 2)} \ < x, y + z >=< x, y > + < x, z >$$

$$\text{Let } X = \sum_{i=1}^{n} x_i, \ Y = \sum_{i=1}^{n} y_i, \ Z = \sum_{i=1}^{n} z_i$$

$$< x, y + z >= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j + z_j = X(Y + Z)$$

$$< x, y >= \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i = XY$$

$$< x, z >= \sum_{i=1}^{n} x_i \sum_{i=1}^{n} z_i = XZ$$

$$X(Y + Z) = XY + XZ$$

$$\text{Therefor Axiom 2 satisfied}$$

$$\text{Axiom 3)} \;\; < cx, y >= c < x, y >$$

$$< cx, y >= \sum_{i=1}^{n} cx_i \sum_{j=1}^{n} y_j = c \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j$$

$$c < x, y >= c(\sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j) = c \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j$$

$$c \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j = c \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j$$

$$\text{Therefore Axiom 3 satisfied}$$

$$\text{Axiom 4)} \;\; < x, x >> 0 \text{ if } x \neq \vec{0}$$

$$< x, x >= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j$$

$$\text{Let } x = (1, -1)$$

$$\text{Then } \sum_{i=1}^{n} x_i = \sum_{j=1}^{n} x_j = 0$$

$$\text{Even } x \neq \vec{0} < x, x >= 0$$

$$\text{Therefore Axiom 4 not satisfied}$$

- **1.13 Pr 8**

$$\text{In the real linear space C(l, e), define an inner product by the equation}$$

$$(f, g) = \int_1^e (\log x) f(x) g(x) dx,$$

$$\text{a) If } f(x) = \sqrt{x}, \text{ compute } ||f||$$

$$(f, f) = ||f||^2$$

$$\text{Therefore we have to get the value of } (f, f) \text{ and compute the value of square root}$$

$$(f, f) = \int_1^e (\log x)) f(x) f(x) dx$$

$$\text{Since } f(x) = \sqrt{x}$$

$$\int_1^e (\log x) f(x) f(x) dx = \int_1^e (x \log x) dx$$

$$\int_1^e (x \log x) dx$$

$$\text{Using partial integration}$$

$$\int_1^e (x \log x) dx = \frac{1}{2} x^2 \log x \Big|_1^e - \frac{1}{2} \int_1^e x dx = \frac{1}{4}(e^2 + 1)$$

$$||f||^2 = \frac{1}{4}(e^2 + 1)$$

$$||f|| = \sqrt{\frac{1}{4}(e^2 + 1)} = \frac{1}{2}\sqrt{(e^2 + 1)}$$

b) Find a linear polynomial $g(x) = a + bx$ that is orthogonal to the constant function $f(x) = 1$

$$\text{To be orthogonal } (g, f) = 0$$

$$\int_1^e (\log x) f(x) g(x) dx = 0$$

$$f(x) = 1 \text{ and } g(x) = a + bx$$

$$\int_1^e (\log x) f(x) g(x) dx = \int_1^e (\log x)(1)(a + bx)$$

$$= \int_1^e (\log x)(a + bx)$$

$$\text{Using partial integration}$$

$$\int_1^e (\log x)(a + bx) = \log x(ax + \frac{1}{2}bx^2)\Big|_1^e - \int_1^e a + \frac{1}{2}bx$$

$$= ae + \frac{1}{2}be^2 - ae - \frac{1}{4}be^2 + a + \frac{1}{4}b$$

$$= \frac{1}{4}be^2 + a + \frac{1}{4}b$$

$$\text{Therefore } a = -\frac{1}{4}be^2 - \frac{1}{4}b$$

$$g(x) = bx - \frac{1}{4}b(e^2 + 1)$$

- **1.13 Pr 9**

$$\text{In the real linear space } C(-1, 1)$$

$$\text{Let } (f, g) = \int_{-1}^1 f(t) g(t) dt$$

$$\text{Consider the three function } u_1, u_2, u_3$$

$$u_1(t) = 1, u_2(t) = t, u_3(t) = 1 + t$$

$$\text{Prove that two of them are orthogonal, two make an angle pi/3 with each other,}$$
$$\text{and two make an angle pi/6 with each other. } pi = \pi$$

$$\text{So there are 3 combinations } (u_1, u_2), (u_2, u_3), (u_1, u_3)$$

$$\text{The angle can be calculated by using formula } cos\theta = \frac{v_1 v_2}{\|v_1\| \|v_2\|}$$

a) $v_1 = u_1, v_2 = u_2$

$$(v_1, v_2) = \int_{-1}^{1} u_1 u_2 dt = \int_{-1}^{1} (1) t dt = 0$$

$$\|v_1\| = \sqrt{(v_1, v_1)} = \sqrt{\int_{-1}^{1} u_1 u_1 dt} = \sqrt{\int_{-1}^{1} 1 dt}$$
$$= \sqrt{2}$$
$$\|v_2\| = \sqrt{(v_2, v_2)} = \sqrt{\int_{-1}^{1} u_2 u_2 dt} = \sqrt{\int_{-1}^{1} t^2 dt}$$
$$= \sqrt{\frac{2}{3}}$$

$$cos\theta = \frac{0}{\sqrt{2}\sqrt{\frac{2}{3}}} = 0$$

$\theta = \pi/2$ Which means orthogonal

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b) $v_1 = u_2, v_2 = u_3$

$$(v_1, v_2) = \int_{-1}^{1} u_2 u_3 dt = \int_{-1}^{1} t(1+t) dt = \int_{-1}^{1} t + t^2 dt = \frac{2}{3}$$
$$\|v_1\|^2 = \int_{-1}^{1} u_2 u_2 dt = \int_{-1}^{1} t^2 dt = \frac{2}{3}$$
$$\|v_2\|^2 = \int_{-1}^{1} u_3 u_3 dt = \int_{-1}^{1} (1+t)^2 dt = \int_{-1}^{1} 1 + 2t + t^2 dt = \frac{8}{3}$$
$$\|v_1\| = \sqrt{\frac{2}{3}}$$
$$\|v_2\| = \sqrt{\frac{8}{3}}$$

$$cos\theta = \frac{\frac{2}{3}}{\sqrt{\frac{2}{3}}\sqrt{\frac{8}{3}}} = \frac{1}{2}$$
$$\theta = \pi/3$$

$$c)\ v_1 = u_1, v_2 = u_3$$

$$(v_1, v_2) = \int_{-1}^{1} u_1 u_3 dt = \int_{-1}^{1} (1)(1+t)dt = \int_{-1}^{1} (1+t)dt = 2$$

$$\|v_1\| = \sqrt{2},\ \|v_2\| = \sqrt{\frac{8}{3}} \text{ Using the value we already calculated before}$$

$$cos\theta = \frac{2}{\sqrt{2}\sqrt{\frac{8}{3}}} = \frac{\sqrt{3}}{2}$$

$$\theta = \pi/6$$

$$\text{All proved}$$

- **1.13 Pr 16**

$$\text{Prove that the following identities are valid in every Euclidean space}$$

$$a)\|x+y\|^2 = \|x\|^2 + \|y\|^2 + (x,y) + (x,y)$$

$$b)\|x+y\|^2 - |x-y\|^2 = 2(x,y) + 2(y,x)$$

$$c)\|x+y\|^2 - |x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

$$a)\|x+y\|^2 = \|x\|^2 + \|y\|^2 + (x,y) + (x,y)$$

$$\|x+y\|^2 = (x+y, x+y) \text{ use def of norm}$$

$$(x+y, x+y) = (x+y, x) + (x+y, y) = (x,x) + (x,y) + (x,y) + (y,y) \text{ use distributivity axiom}$$

$$= \|x\|^2 + \|y\|^2 + (x,y) + (x,y)$$

$$b)\|x+y\|^2 - |x-y\|^2 = 2(x,y) + 2(y,x)$$

$$\text{Using the result from } a$$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + (x,y) + (x,y)$$

$$\|x+(-y)\|^2 = \|x\|^2 + \|(-y)\|^2 + (x,-y) + (x,-y)$$

$$\|(-y)\|^2 = (-y,-y) = (-1)(-1)(y,y) = \|y\|^2 \text{ use associativity axiom}$$

$$(x,-y) + (x,-y) = -1(x,y) - 1(x,y) \text{ use associativity axiom}$$

$$\|x+(-y)\|^2 = \|x\|^2 + \|y\|^2 - 1(x,y) - 1(x,y) = \|x\|^2 + \|y\|^2 - 2(x,y)$$

$$\|x+y\|^2 - |x-y\|^2 = \|x\|^2 + \|y\|^2 + 2(x,y) - (\|x\|^2 + \|y\|^2 - 2(x,y)) = 2(x,y) + 2(x,y)$$

$$(x,y) = (y,x) \text{ commutativity axiom}$$

$$\text{Therefore } |x+y\|^2 - |x-y\|^2 = 2(x,y) + 2(y,x)$$

$$c) \|x+y\|^2 + |x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Just using the results from previous problem

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2(x,y)$$
$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2(x,y)$$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$