1. (10 points) Let $a_1 = \sqrt{2}, a_2 = \sqrt{2+\sqrt{2}}, \dots, a_n = \sqrt{2+\dots\sqrt{2+\sqrt{2+\sqrt{2}}}}$. Prove that $a_n < 2$ for all $n \geq 1$ using mathematical induction.

1. Base step. when $n=1$,

$a_1 = \sqrt{2}$, $\sqrt{2} < 2$ → therefore true when $n=1$

2. ~~Supp~~ Assume $a_k < 2$ and prove $a_{k+1} < 2$.

let $a_k = A$ then $a_{k+1} = \sqrt{2+A}$

$A < 2$ and we have to show $\sqrt{2+A} < 2$

$\sqrt{2+A} < 2$

square

$2+A < 4$

$A < 2$ since we assume $A<2$ the inequality is true.

thus. $a_n < 2$ for $n \geq 1$ is true.

2. (9 points) TRUE or FALSE. You don't have to justify.

False The set of all polynomials of degree 2 is a vector space with the usual operations.

True Let V be an inner product space. Then, any orthogonal set of non-zero vectors is linearly independent.

True The kernel of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ satisfies $3 \geq dim(Ker\,T) \geq 1$.

a) Show that $S = \{(1,1,-3),(2,1,1),(4,-7,-1)\}$ is a basis for $\mathbb{R}^3$. (*Hint*: Is there anything special about this set when you think about $\mathbb{R}^3$ with the dot product?)

$$\begin{matrix} 1 & 2 & 4 \\ 1 & 1 & -7 \\ -3 & 1 & -1 \end{matrix}$$

Any set of independent elements in V is a subset of basis

And every basis for V has same number of elements

therefore any set of n independent elements is basis. //

We need to show S is independent.

for S to be dependent there must be a, b, c which are not all zero.

that satisfy $a\begin{bmatrix}1\\1\\-3\end{bmatrix}+b\begin{bmatrix}2\\1\\1\end{bmatrix}+c\begin{bmatrix}4\\-7\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

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$a+2b+4c=0$

$a+b-7c=0$ → $b+11c=0$

$-3a+b-c=0$ → $4b-22c=0$

$b=-11c$ and $b=\frac{11}{2}c$

~~there does~~ there isn't solution

only $a=b=c=0$ satisfy the equation

so S is independent and also basis for $\mathbb{R}^3$

b) Find the components of the vector $(1,0,1) \in \mathbb{R}^3$ with respect to the basis S. ?

2/1

3 2 -2

2 3 -3

$a(1,1,-3)+b(2,1,1)+c(4,-7,-1) = (1,0,1)$

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$a+2b+4c=1$

$a+b-7c=0$ > $b+11c=1$ > $b=\frac{1}{2}$, $c=\frac{1}{22}$ > $a=-\frac{4}{22}$

$-3a+b-c=1$ > $4b-22c=1$

$-\frac{4}{22}(1,1,-3)+\frac{1}{2}(2,1,1)+\frac{1}{22}(4,-7,-1)=(1,0,1) \in \mathbb{R}^3$

$b+11c=1$

$2b+22c=2$

$4b-22c=1$

$\frac{11}{22}-\frac{7}{22}$

$\frac{4}{22}$

$(-\frac{4}{22}, \frac{1}{2}, \frac{1}{22})_S = (1,0,1)$

$6b=3$

$-\frac{4}{22}+\ \ 7+$

4. (12 points) Prove that if $S = \{x_1, x_2, x_3\}$ is an independent subset of a vector space V and $y \in V$ but $y \notin L(S)$ then $S \cup \{y\}$ is independent.

$S = \{x_1, x_2, x_3\}$

$\sum_{i=1}^{3} c_i x_i = 0 \Rightarrow c_1 = c_2 = c_3 = 0$ for S.

~~there doesn't~~

~~$a_1x_1 + a_2x_2 + a_3x_3$~~ there doesn't exist $a_1x_1 + a_2x_2 + a_3x_3 = y$, $\underbrace{a_1, a_2, a_3}_{\in \mathbb{R}}$ → doesn't exist since $y \notin L(S)$

so we can express $a_1x_1 + a_2x_2 + a_3x_3 + v_1 = y$ and v_1 is $v_1 \notin L(S)$

$S \cup \{y\} = \{x_1, x_2, x_3, (a_1x_1 + a_2x_2 + a_3x_3 + v_1)\}$

for $S \cup \{y\}$ to be dependent

$c_1x_1 + c_2x_2 + c_3x_3 + c_4(a_1x_1 + a_2x_2 + a_3x_3 + v_1) = 0$ except for $c_1 = c_2 = c_3 = c_4 = 0$

not all $= 0$.

$(c_1 + c_4a_1)x_1 + (c_2 + c_4a_2)x_2 + (c_3 + c_4a_3)x_3 + c_4v_1 = 0$

~~$v_1 = \frac{c_1+c_4a_1}{c_4}x_1 + \frac{c_2+c_4a_2}{c_4}x_2 + \frac{c_3+c_4a_3}{c_4}x_3$~~

~~However $v_1 \notin L(S)$ therefore $\frac{c_1+c_4a_1}{c_4} = \frac{c_2+c_4a_2}{c_4} = \frac{c_3+c_4a_3}{c_4} = 0$~~

~~$c_1 + c_4a_1 = c_2 + c_4a_2 = c_3 + c_4a_3 = 0$~~

~~$c_1 = -c_4a_1$~~

~~$c_2 = -c_4a_2$~~

~~$c_3 = -c_4a_3$~~

$(c_1 + c_4a_1) = (c_2 + c_4a_2) = (c_3 + c_4a_3) = c_4 = 0$

since $\{x_1, x_2, x_3\}$ can't span c_4v_1

and since $c_4 = 0$

$c_1x_1 + c_2x_2 + c_3x_3 = 0$ but S is independent

and only $c_1 = c_2 = c_3 = 0$ satisfy the equation.

there fore $c_1 = c_2 = c_3 = c_4 = 0$ to satisfy the equation it means $S \cup \{y\}$ is independent

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$a_1x_1 + a_2x_2 + a_3x_3 + a_4y = 0.$

$a_4 = 0$ $a_4 \neq 0.$

5. (16 points) Let $\mathscr{P}_2$ be the space of polynomials of degree at most two. Start with the basis $B = \{1, x, x^2\}$ for $\mathscr{P}_2$, and obtain an orthogonal basis (using the Gram-Schmidt process) with respect to the following inner product

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$X_1 = 1$ $X_3 = X^2$
$X_2 = X$

$$(p, q) = \int_{-1}^{1} p(x)\,q(x)\,x^2\,dx.$$

$V_1 = 1$ ✓

$V_2 = X_2 - \text{proj}_{V_1} X_2 = X - \frac{\langle X_2, V_1\rangle}{\langle V_1, V_1\rangle} V_1 = X - \frac{0}{\frac{2}{3}} 1 = X$ $\quad V_2 = X$ ✓

$\langle V_1, V_1\rangle = \int_{-1}^{1} x^2 dx = \frac{1}{3}x^3 \Big|_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$

$\langle X_2, V_1\rangle = \int_{-1}^{1} x^3 dx = \frac{1}{4}x^4 \Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$

$V_3 = X_3 - \text{proj}_{V_1} X_3 - \text{proj}_{V_2} X_3 = X^2 - \frac{\langle X_3, V_1\rangle}{\langle V_1, V_1\rangle} V_1 - \frac{\langle X_3, V_2\rangle}{\langle V_2, V_2\rangle} V_2$

$\langle X_3, V_1\rangle = \int_{-1}^{1} x^4 dx = \frac{1}{5}x^5 \Big|_{-1}^{1} = \frac{1}{5} - (-\frac{1}{5}) = \frac{2}{5}$ $\quad = X^2 - \frac{\frac{2}{5}}{\frac{2}{3}} \cdot 1 - \frac{0}{\frac{2}{5}} X = X^2 - \frac{3}{5}$

$V_3 = X^2 - \frac{3}{5}$ ✓

$\langle V_2, V_2\rangle = \int_{-1}^{1} x^4 dx = \frac{2}{5}$

$\langle X_3, V_2\rangle = \int_{-1}^{1} x^2 \cdot x \cdot x^2 dx = \int_{-1}^{1} x^5 dx = 0$

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the problem didn't required to get orthonomal basis.

$\{1, X, X^2 - \frac{3}{5}\}$ is a orthogonal basis.

$x^5 - \frac{3}{5}x^3$

$\int_{-1}^{1} x^4 - \frac{3}{5}x^2$

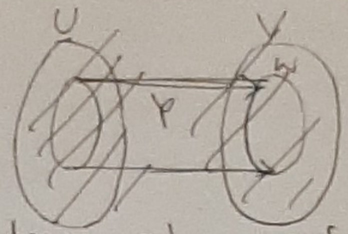
$\frac{1}{5}x^5 - \frac{1}{5}x^3$

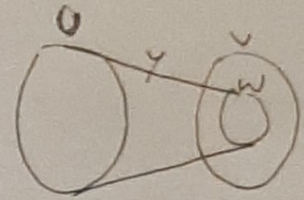
Let W be a subspace of V. Let Y be the subset of $\mathrm{Hom}(U, V)$ defined as follows:

$$Y = \{T : U \to V \mid T(U) \subset W\}.$$

Show that Y is a subspace.

$U \to V$





For a ~~subsp~~ subset to be subspace, it has to satisfy closure axioms.
Have to show Y st satisfy ~~clos~~ closure axioms

$S, T \in Y$

so $S + T \in Y$
and
$cS \in Y$ $\quad c = \mathbb{R}$ $\quad x \in U$

using axiom of linear transformat

• first $cS(x) = S(cx)$ since $S \in Y$
satisfied

need to show $cSx \in W$.

• Second ~~$(S+T)(x) = S(T(x))$ $(ST)x = S(T(x))$~~
~~satisfied~~

~~Therefore Y is a subspace.~~

for any S and T, $S(U) + T(U)$ since $S(U) \subset W$ $T(U) \subset W$ and W is a subspace
W is ~~closed~~ satisfying closure axioms. So it is closed under addition
therefore $S(U) + T(U) \subset W$ so $S+T$ also satisfy $S+T \in Y$

Theefore Y is a subspace

W

(scrap)

Correction//

Q.4

4) $S = \{X_1, X_2, X_3\}$ ⇒ independent subset of vector space V $y \in V$ but $y \notin L(S)$

prove $S \cup y$ is independent.//

$S \cup y = \{X_1, X_2, X_3, y\}$ to prove it is independent $a_n \in \mathbb{R}$

have to show that $a_1X_1 + a_2X_2 + a_3X_3 + a_4y = 0$ and $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$

$a_1X_1 + a_2X_2 + a_3X_3 + a_4y = 0$

there can be two cases $a_4 = 0$ or $a_4 \neq 0$

i) $a_4 = 0$

$a_1X_1 + a_2X_2 + a_3X_3 = 0$ since S is independent.

a_1, a_2, a_3 must be zero

ii) $a_4 \neq 0$

let $a_4 = c$ which isn't zero

$a_1X_1 + a_2X_2 + a_3X_3 = -cy$

$(-\frac{a_1}{c})X_1 + (-\frac{a_2}{c})X_2 + (-\frac{a_3}{c})X_3 = y$ However the equation can't be true since $y \notin L(S)$

which means y can't be made by linear combination of S.

Therefore a_1, a_2, a_3, a_4 are all zero and $y \cup S$ is independent

For a subset to be a subspace it must satisfy closure axioms

so $S, T \in Y$ and $c \in \mathbb{R}$ then $S+T \in Y$ and $cS \in Y$

first for any S and T, ~~since~~ and for any x that $x \in U$ $S(x) \in W$ $T(x) \in W$
and W is a subspace therefore it is closed under ~~a~~ addition
$S(x)+T(x) \in W$ therefore $S+T \in Y$

Second. for any scalar c, and any x that $x \in U$ $S(x) \in W$
and W is a subspace which is closed under scalar multiplication
that means $cS(x) \in W$ therefore $cS \in Y$

Q. 6

Since two axioms are satisfied Y is a subspace.