Math 375

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- 1.17 Pr 1

  In each case, find an orthonormal basis for the subspace of V_3 spanned by the given vectors

$$\text{(a) } x_1 = (1,1,1), x_2 = (1,0,1), x_3 = (3,2,3)$$
$$\text{(b) } x_1 = (1,1,1), x_2 = (-1,1,-1), x_3 = (1,0,1)$$

$$\text{(a) } x_1 = (1,1,1), x_2 = (1,0,1), x_3 = (3,2,3)$$
$$\text{Since } 2x_1 + x_2 = x_3 \text{ we can ignore } x_3$$

$$\text{Let } v_1 = x_1$$
$$v_2 = x_2 - proj_{v_1}x_2 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{< x_2, v_1 >}{< v_1, v_1 >}v_1$$
$$= \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{bmatrix}$$

$$\text{Orthogonal basis is } \begin{bmatrix}1, \frac{1}{3}\\1, -\frac{2}{3}\\1, \frac{1}{3}\end{bmatrix}$$
$$\text{Change it to orthonormal}$$
$$\begin{bmatrix}\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{6}\\\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\\\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{6}\end{bmatrix}$$

$$\text{(b) } x_1 = (1,1,1), x_2 = (-1,1,-1), x_3 = (1,0,1)$$
$$x_1 - 2x_3 = x_2$$
$$\text{Therefore we can ignore } x_2 \text{ and it is same as previous problem } (a)$$
$$\text{Orthonormal basis for } (a), (b) \text{ is same}$$

$$\begin{bmatrix}\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{6}\\\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\\\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{6}\end{bmatrix}$$

- 1.17 Pr 2a

n each case, find an orthonormal basis for the subspace of V_4 spanned by the given vectors.

$$(a)\ x_1 = (1,1,0,0), x_2 = (0,1,1,0), x_3 = (0,0,1,1), x_4 = (1,0,0,1)$$

$$\text{Since } x_1 + x_3 - x_2 = x_4 \text{ we can ignore } x_4$$

$$\text{Let } v_1 = x_1$$

$$v_2 = x_2 - proj_{v_1} x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{< x_2, v_1 >}{< v_1, v_1 >} v_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = x_3 - \frac{< x_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< x_3, v_2 >}{< v_2, v_2 >} v_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - 0(v_1) - \frac{1}{1.5} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

$$\text{Orthogonal basis is } \begin{bmatrix} 1, -\frac{1}{2}, \frac{1}{3} \\ 1, \frac{1}{2}, -\frac{1}{3} \\ 0, 1, \frac{1}{3} \\ 0, 0, 1 \end{bmatrix}$$

$$\text{Orthonormal basis is } \begin{bmatrix} \frac{\sqrt{2}}{2}, -\frac{\sqrt{6}}{6}, \frac{\sqrt{3}}{6} \\ \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{3}}{6} \\ 0, \frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{6} \\ 0, 0, \frac{\sqrt{3}}{2} \end{bmatrix}$$

- 1.17 Pr 2b

$$(b)\ x_1 = (1,1,0,1), x_2 = (1,0,2,1) x_3 = (1,2,-2,1).$$

$$v_1 = x_1$$

$$v_2 = x_2 - proj_{v_1} x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{< x_2, v_1 >}{< v_1, v_1 >} v_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 2 \\ \frac{1}{3} \end{bmatrix}$$

$$v_3 = x_3 - \frac{< x_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< x_3, v_2 >}{< v_2, v_2 >} v_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 2 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{discard since its zero vector}$$

$$\text{Orthogonal basis is} \begin{bmatrix} 1, \frac{1}{3} \\ 1, -\frac{2}{3} \\ 0, 2 \\ 1, \frac{1}{3} \end{bmatrix}$$

$$\text{Orthonormal basis is} \begin{bmatrix} \frac{\sqrt{3}}{3}, \frac{1}{\sqrt{42}} \\ \frac{\sqrt{3}}{3}, -\frac{2}{\sqrt{42}} \\ 0, \frac{6}{\sqrt{42}} \\ \frac{\sqrt{3}}{3}, \frac{1}{\sqrt{42}} \end{bmatrix}$$

- 1.17 Pr 3

$$\text{In the real linear space } C(0, \pi) \text{ with inner product } < x, y >= \int_0^\pi x(t) y(t) dt$$

$$\text{let } x(t) = cos(nt) \text{ for } n = 0, 1, 2, 3 \ldots$$

$$\text{Prove } y_0 = \frac{1}{\sqrt{\pi}} \quad y_n = \sqrt{\frac{2}{\pi}} cos(nt) \; n \geq 1$$

$$\text{is forming orthonormal set and spanning same as } \{x_0, x_1 \ldots\}$$

To show they are orthonormal set

First prove they are orthogonal

There can be two cases for choosing elements in y: y_0 and $y_n (n \geq 1), y_a$ and $y_b \;\; a, b \geq 1$

$$y_0 = \frac{1}{\sqrt{\pi}} \quad y_n = \sqrt{\frac{2}{\pi}} cos(nt) \;\; n \geq 1$$

$$< y_0, y_n >= \int_0^{\pi} \frac{1}{\sqrt{\pi}} \sqrt{\frac{2}{\pi}} cos(nt)$$

$$= \int_0^{\pi} \frac{\sqrt{2}}{\pi} cos(nt) = \frac{\sqrt{2}}{\pi} \int_0^{\pi} cos(nt) = \frac{\sqrt{2}}{\pi} \Big| \frac{1}{n} sin(nt) \Big|_0^{\pi} = 0$$

$$y_a = \sqrt{\frac{2}{\pi}} cos(at) \text{ and } y_b = \sqrt{\frac{2}{\pi}} cos(bt)$$

$$< y_a, y_b >= \int_0^{\pi} \sqrt{\frac{2}{\pi}} cos(at) \sqrt{\frac{2}{\pi}} cos(bt)$$

$$= \frac{2}{\pi} \int_0^{\pi} cos(at)cos(bt) = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} [cos(at + bt)cos(at - bt)]$$

$$= \frac{2}{\pi} \frac{1}{2} \int_0^{\pi} [cos(at + bt) + cos(at - bt)]$$

$$\frac{2}{\pi} \frac{1}{2} \int_0^{\pi} [cos((a + b)t) + cos((a - b)t)] = \frac{1}{\pi} \int_0^{\pi} cos((a + b)t) + \frac{1}{\pi} \int_0^{\pi} cos((a - b)t)$$

$$= \frac{1}{\pi} \Big| \frac{1}{a + b} sin((a + b)t) + \frac{1}{a - b} sin((a - b)t) \Big|_0^{\pi} = 0$$

In every cases inner product is 0 therefore it is orthogonal

Next, show the set is orthonormal

When y_0

$$< y_0, y_0 >= \int_0^{\pi} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\pi}} = \int_0^{\pi} \frac{1}{\pi}$$

$$= \Big| \frac{1}{\pi} t \Big|_0^{\pi} = 1$$

therefore y_0 is normalized

when $y_n, n \geq 1$

$$< y_n, y_n >= \int_0^{\pi} \sqrt{\frac{2}{\pi}} cos(nt) \sqrt{\frac{2}{\pi}} cos(nt) = \frac{2}{\pi} \int_0^{\pi} cos^2(nt)$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1 + cos(2nt)}{2} = \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} + \frac{2}{\pi} \int_0^{\pi} \frac{cos(2nt)}{2}$$

$$= \frac{2}{\pi} \Big| \frac{1}{2} t \Big|_0^{\pi} + \frac{2}{\pi} \Big| \frac{sin(2nt)}{4n} \Big|_0^{\pi}$$

$$= \frac{2}{\pi} \frac{\pi}{2} = 1$$

therfore y_n is normalized

The set is orthonormal

And to it is enough to show $\{y_0, y_1 \dots\}$can be obtained by scalar multiplication of $\{x_0, x_1 \dots\}$

Since $x(t) = cos(nt)$ for $n = 0, 1, 2, 3 \dots$

$$y_0 = \frac{1}{\sqrt{\pi}} \quad y_n = \sqrt{\frac{2}{\pi}} cos(nt) \ n \geq 1$$

There are $c_k \in \mathbb{R}$ which satisfy $c_k x_k = y_k$

So they are spanning same space

- 1.17 Pr 6

In the real number space $C(1,3)$ with inner product $< f.g >= \int_1^3 f(x)g(x)dx$

let $f(x) = \frac{1}{x}$ and show that the constant polynomial g nearest to f is $g = \frac{1}{2} log(3)$

Compute $\|g - f\|^2$ for this g

1 is a basis for g

$\frac{\sqrt{2}}{2}$is orthonormal basis

$$\text{let } w = \{\frac{\sqrt{2}}{2}\}$$

$$proj_w f(x) = proj_w \frac{1}{x}$$

$$= \frac{< f, w >}{< w, w >} w = \frac{\frac{\sqrt{2} ln(3)}{2}}{1} \frac{\sqrt{2}}{2}$$

$$= \frac{ln(3)}{2}$$

Therefore the constant polynomial g nearest to f is $\frac{ln(3)}{2}$

Compute $\|g - f\|^2, g = \frac{ln(3)}{2}$

$$g - f = \frac{ln(3)}{2} - \frac{1}{x} = \frac{ln(3)x - 2}{2x}$$

$$\|g - f\|^2 =< g - f, g - f >= \int_1^3 \frac{ln(3)x - 2}{2x} \frac{ln(3)x - 2}{2x} dx$$

$$= \int_1^3 \frac{ln^2(3)x^2 - 4ln(3)x + 4}{4x^2} dx = \int_1^3 \frac{ln^2(3)x^2}{4x^2} dx + \int_1^3 \frac{-4ln(3)x}{4x^2} dx + \int_1^3 \frac{4}{4x^2} dx$$

$$= \frac{ln^2(3)}{2} + (-1)ln^2(3) + \frac{2}{3} = -\frac{ln^2(3)}{2} + \frac{2}{3}$$

- 1.17 Pr 9

In the real number space $C(0,2\pi)$ with inner product $< f.g >= \int_0^{2\pi} f(x)g(x)dx$
let $f(x) = x$ In the subspace spanned by $u_0(x) = 1, u_1(x) = cos(x), u_2(x) = sin(x)$,
Find the trigometric polynomial nearest to f

Get projection of $f(x)$ onto u_0, u_1, u_2

$$proj_{u_0} f(x) = \frac{< f, u_0 >}{< u_0, u_0 >} u_0 = \frac{2\pi^2}{2\pi} = \pi$$

$$proj_{u_1} f(x) = \frac{< f, u_1 >}{< u_1, u_1 >} u_1 = \frac{\int_0^{2\pi} xcos(x)dx}{\int_0^{2\pi} 1 + cos(2x)dx} cos(x)$$

$$= \frac{\Big|xsin(x)\Big|_0^{2\pi} - \int_0^{2\pi} sin(x)dx}{\frac{1}{2}\int_0^{2\pi} 1 + cos(2x)dx} cos(x)$$

$$= \frac{-\Big| - cos(x)\Big|_0^{2\pi}}{\frac{1}{2}\int_0^{2\pi} 1 + cos(2x)dx} cos(x), \quad \text{Since } -\Big| - cos(x)\Big|_0^{2\pi} = 0$$

$$\frac{-\Big| - cos(x)\Big|_0^{2\pi}}{\frac{1}{2}\int_0^{2\pi} 1 + cos(2x)dx} cos(x) = 0$$

$$proj_{u_2} f(x) = \frac{< f, u_2 >}{< u_2, u_2 >} u_2 = \frac{\int_0^{2\pi} xsin(x)dx}{\int_0^{2\pi} 1 - cos(2x)dx} sin(x)$$

$$= \frac{\Big| - xcos(x)\Big|_0^{2\pi} + \int_0^{2\pi} cos(x)dx}{\frac{1}{2}\int_0^{2\pi} 1 + cos(2x)dx} sin(x)$$

$$= \frac{-2\pi + 0}{\frac{1}{2}\Big|x + \frac{1}{2}sin(2x)\Big|_0^{2\pi}} sin(x)$$

$$= -2sin(x)$$

Therefore $g(x) = \pi - 2sin(x)$