Exam Review Solutions

Problem 1.

Proof. (a) Suppose $f(x,y) = f(a,b)$, where $x, a > 0$. Then $x^2 - y^2 = a^2 - b^2$ and $2xy = 2ab$. We also know that $||f(x,y)|| = ||f(a,b)||$. Thus

$$\begin{aligned} ||f(x,y)|| &= \sqrt{(x^2-y^2)^2 + (2xy)^2} \\ &= \sqrt{x^4 + 2x^2y^2 + y^4} \\ &= \sqrt{(x^2+y^2)^2} \\ &= x^2 + y^2. \end{aligned}$$

(Note that $\sqrt{(x^2+y^2)^2} = x^2 + y^2$, and not $\pm(x^2+y^2)$, because $x^2 + y^2$ is always positive.) Similarly, we can determine that $||f(a,b)|| = a^2 + b^2$, and hence that $x^2 + y^2 = a^2 + b^2$. Adding this to the equation $x^2 - y^2 = a^2 - b^2$, we obtain

$$2x^2 = 2a^2.$$

Thus $x = \pm a$. As both x and a are positive, it follows that $x = a$.

Using the equation $2xy = 2ab$, we can substitute $a = x$ to obtian

$$2xy = 2xb.$$

Because $x \neq 0$, we can divide both sides by $2x$, and obtain

$$y = b.$$

Therefore $(x,y) = (a,b)$.

(b) We want to find a point (x,y) such that $f(x,y) = (0,1)$ and $x > 0$. Note that $x^2 - y^2$ must equal 0, so $x^2 = y^2$, i.e. $x = \pm y$. We next want $2xy = 1$. Thus $\pm x^2 = \frac{1}{2}$, so y must be positive, and $x = \frac{1}{\sqrt{2}}$. Thus $f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (0,1)$.

By the chain rule, we know that

$$Dg(f(x,y)) = [Df(x,y)]^{-1}.$$

Calculating $Df(x,y)$, we find

$$Df(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}.$$

Thus

$$Df\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}.$$

Therefore

$$Dg(0,1) = Dg\left(f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right) = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}$$

$\square$

Problem 2.

Proof. Let us first calculate Df.

$$Df(x,y) = \begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix}.$$

By the Inverse Function Theorem, we know f is locally invertible everywhere Df is invertible, i.e. all points (x,y) where $\det Df(x,y) \neq 0$. So let us suppose $\det Df(x,y) = 0$. Solving gives

$$0 = \det Df(x,y) = -2y^2 - 2x^2 = -2(x^2+y^2).$$

Clearly the only place $x^2+y^2 = 0$ is the origin, therefore f is locally invertible everywhere except the point $(0,0)$. $\square$

Problem 3.

Proof. (a) Let us calculate Df.

$$Df(x,y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

It follows that

$$\det Df(x,y) = e^{2x}\cos^2 y + e^{2x}\sin^2 y = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}.$$

Because e^{2x} is never equal to 0, the Inverse Function Theorem guarantees that f is locally invertible everywhere.

(b) By the Chain Rule, we know

$$D(g \circ f^{-1}(0,1)) = Dg(f^{-1}(0,1))Df^{-1}(0,1).$$

As $f(0,\pi/2) = (0,1)$, it follows that $f^{-1}(0,1) = (0,\pi/2)$. So let us calculate Dg.

$$Dg(x,y) = \begin{bmatrix} 2x & -2y \\ y & x+2y \\ y^2 & 2xy+1 \end{bmatrix}.$$

Thus

$$Dg(f^{-1}(0,1)) = Dg(0,\pi/2) = \begin{bmatrix} 0 & -\pi \\ \pi/2 & \pi \\ \pi^2/4 & 1 \end{bmatrix}.$$

Now we just need to find $Df^{-1}(0,1)$. By the chain rule, we know

$$Df^{-1}(0,1) = [Df(0,\pi/2)]^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore

$$D(g \circ f^{-1})(0,1) = \begin{bmatrix} 0 & -\pi \\ \pi/2 & \pi \\ \pi^2/4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \pi & 0 \\ -\pi & \pi/2 \\ -1 & \pi^2/4 \end{bmatrix}.$$

$\square$

Problem 4.

Proof. Based on the formula, which is xy multiplied by a sin value at all points where $y \neq 0$, we suspect that if the derivative exists, it is equal to $[0\ 0]$. To prove this is the actual derivative, we calculate

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - \begin{bmatrix} 0 & 0 \end{bmatrix}\mathbf{h}}{||\mathbf{h}||} = \lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{h})}{||\mathbf{h}||}$$

Observe that for $y \neq 0$,

$$-1 \leq \sin(x/y) \leq 1 \qquad \text{and} \qquad xy \leq \max\{x^2, y^2\} \leq x^2 + y^2,$$

thus

$$-(x^2+y^2) \leq -xy \leq xy\sin(x/y) \leq xy \leq x^2+y^2.$$

As $f(x,y) = 0$, for $y = 0$, it follows that for any choice of (x,y),

$$-(x^2+y^2) \leq f(x,y) \leq (x^2+y^2).$$

Thus

$$-\sqrt{x^2+y^2} \leq \frac{f(x,y)}{\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2}.$$

As the limits of the left and right functions are both 0, as $(x,y) \to 0$, the Squeeze Theorem implies that $\lim_{(x,y)\to 0} f(x,y)/\sqrt{x^2+y^2}$ is also equal to 0. Therefore the derivatife of f exists at $(0,0)$ and is equal to $\begin{bmatrix} 0 & 0 \end{bmatrix}$. ☐

Problem 5.

Proof. (a) We first calculate

$$D_1 f(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

We next calculate

$$D_2 f(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Thus the Jacobean is $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

(b) (Note: Professor Caglar is not requiring you to make $\mathbf{u}$ a unit vector.) We calculate

$$\lim_{h\to 0} \frac{f(\mathbf{0} + h\mathbf{u}) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,h)}{h} = \lim_{h\to 0} \frac{2h^2/\sqrt{h^2+h^2}}{h} = \sqrt{2}.$$

(c) If f were differentiable at $(0,0)$, then $D_{\mathbf{u}} f$ would equal $Jf(0,0)\cdot\mathbf{u}$. However, for $\mathbf{u} = (1,1)$, we know this isn't case, as $\begin{bmatrix} 0 & 0 \end{bmatrix} \cdot (1,1) = 0 \neq \sqrt{2}$. Therefore f is not differentiable at $(0,0)$. ☐

Problem 6.

Proof. Let us first calculate the eigenvalues by finding the characteristic polynomial:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{bmatrix} = (\lambda - 1)(\lambda - 3) - 8 = (\lambda - 5)(\lambda + 1).$$

Thus the eigenvalues are $\lambda = 5$ and $\lambda = -1$.

To find the eigenvector for $\lambda = 5$, we calculate

$$\ker(5I - A) = \ker \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix} = \operatorname{span}\{(1,1)\}.$$

To find the eigenvector for $\lambda = -1$, we calculate

$$\ker(-I - A) = \ker \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix} = \operatorname{span}\{(2,-1)\}.$$

If we let $B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$, it follows that

$$A = BDB^{-1}.$$

Therefore

$$A^{200} = BD^{200}B^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5^{200} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5^{200} - 2 & 2(5^{200} + 1) \\ 5^{200} + 1 & 2 \cdot 5^{200} - 1 \end{bmatrix}$$

☐

Problem 7.

Proof. We already know the eigenvectors for A: 5 and -1, and $(1,1)$ and $(2,-1)$ respectively. It follows that BAB^{-1} has the same eigenvalues, and as its eigenvectors

$$B\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}10\\12\end{bmatrix}, \qquad \text{and} \qquad B\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}11\\-3\end{bmatrix}.$$

□

Problem 8.

Proof. (a) ($\Rightarrow$) Suppose A is nonsingular. Then the kernel of A is trivial. Thus there is no nonzero vector v such that $Av = 0 = 0 \cdot v$. Therefore 0 is not an eigenvalue of A.

($\Leftarrow$) Suppose 0 is not an eigenvalue of A. Then there is no nonzero vector v such that $Av = 0 \cdot v = 0$. Therefore the kernel of A is trivial. Therefore A is nonsingular.

(b) Suppose A is nonsingular, and let λ be an eigenvalue of A. Then there exists a vector v such that $Av = \lambda v$. Thus

$$v = Iv = A^{-1}Av = A^{-1}(\lambda v).$$

Because part (a) implies that $\lambda \neq o$, we can divide by λ to obtain $A^{-1}v = \frac{1}{\lambda}v$, i.e. $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Now suppose μ is an eigenvalue of A^{-1}. Using the same logic as before, it follows that $1/\mu$ is an eigenvalue of A. Therefore the eigenvalues of A^{-1} are precisely the reciprocals of the eigenvalues of A. □

Problem 9.

Proof. (a) Observe that

$$(-A)A = -A^2 = -(-I) = I,$$

therefore $-A$ is the inverse of A, i.e. A is invertible.

(c) It will be helpful to prove part (c) before part (b). Suppose λ is eigenvalue of A. Then there is a vector v such that $Av = \lambda v$. Thus

$$-v = -Iv = A^2v = A(\lambda v) = \lambda Av = \lambda^2 v.$$

Thus $\lambda^2 = -1$. So λ is not real.

(b) Suppose n is odd. Then the characteristic polymomial of A is of odd degree. As every odd degree polynomial has at least one real root, the characteristic polynomial of A has at least one real root, meaning it has at least one real eigenvalue. However (c) shows that this is a contradiction, meaning n must be even.

(d) Note that

$$\det(A)^2 = \det(A^2) = \det(-I) = (-1)^n.$$

Because n is even, we know $(-1)^n = 1$. □

Problem 10.

Proof. Let us first prove that $\mathbb{R}^n - \{x\}$ is open. Suppose $y \in \mathbb{R}^n - \{x\}$. Then $y \neq x$, so $||y-x|| > 0$. Let $d = ||y-x||$, and it follows that the open ball around y of radius $\frac{1}{2}d$ does not have x in it, i.e. it is contained in $\mathbb{R}^n - \{x\}$. Therefore $\mathbb{R}^n$ is open.

Note that $A - \{x\} = A \cap (\mathbb{R}^n - \{x\})$. As the intersection of finitely many open sets is open, it follows that $A - \{x\}$ is open. □

Problem 11.

Proof. (a) By 5(b), $\mathbb{R}^n$ is open. As $\mathbb{R}^n$ is the complement of $\emptyset$, it follows that $\emptyset$ is closed.

(b) By 5(a), $\emptyset$ is open. As $\emptyset$ is the complement of $\mathbb{R}^n$, it follows that $\mathbb{R}^n$ is closed.

(c) The complement of an intersection of sets is the union of the complements of those sets. Therefore the complement of the intersection of closed sets is the union of open sets. By 5(c), this is also open. Therefore the intersection of closed sets has an open complement, i.e. it is closed.

(d) The complement of the union of sets is the intersection of the complements. Therefore the complement of the union of finitely many closed sets is an intersection of finitely many open sets. by 5(d), the intersection of finitely many open sets is also open. Therefore the union of finitely many closed sets has an open complement, i.e. it is closed.

(e) Define an infinitely family of sets in $\mathbb{R}^1$ by

$$\begin{aligned} I_1 &= [0, 1/2] \\ I_2 &= [0, 2/3] \\ I_3 &= [0, 3/4] \\ \vdots \ &\ \ \vdots \\ I_n &= [0, n/(n+1)] \\ \vdots \ &\ \ \vdots \end{aligned}$$

Each of these intervals is closed. However, the union of this infinite family of sets is $[0, 1)$, which is not closed. $\square$

Problem 12.

Proof. (a) Observe that

$$Df(x,y,z) = \begin{bmatrix} 2x & 1 & 1 \\ 2 & 1 & 2z \end{bmatrix} \qquad Dg(u,v,w) = \begin{bmatrix} v^2w^2 & 2uvw^2 & 2uv^2w \\ 0 & w^2\cos v & 2w\sin v \\ 2ue^v & u^2e^v & 0 \end{bmatrix}.$$

(b) We calculate

$$f(g(u,v,w)) = f(uv^2w^2, w^2\sin v, u^2e^v) = \begin{bmatrix} u^2v^4w^4 + w^2\sin v + u^2e^v \\ 2uv^2w^2 + w^2\sin v + u^4e^{2v} \end{bmatrix}.$$

(c) The Jacobean of the composition is

$$\begin{bmatrix} 2uv^2w^2 + 2ue^v & 4u^2v^3w^4 + w^2\cos v + u^2e^v & 4u^2v^4w^3 + 2w\sin v \\ 2v^2w^2 + 4u^3e^{2v} & 4uvw^2 + w^2\cos v + 2u^4e^{2v} & 4uv^2w + 2w\sin v \end{bmatrix}$$

$\square$

Problem 13.

Proof. (a) Let $\mathbf{a} = (a, b)$. If $a \neq 0$, we calculate

$$D_{\mathbf{a}}f(0,0) = \lim_{h\to 0} \frac{f(\mathbf{0} + h\mathbf{a}) - f(\mathbf{0})}{h} = \lim_{h\to 0} \frac{f(ha, hb)}{h} = \lim_{h\to 0} \frac{(h^4ab^3)/(h^3a^3 + h^6b^6)}{h} = \lim_{h\to 0} \frac{ab^3}{a^3 + h^3b^6} = \frac{b^3}{a^2}.$$

If $a = 0$, then

$$D_{\mathbf{a}}f(0,0) = \lim_{h\to 0} \frac{f(0, hb) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Thus $D_{\mathbf{a}}f$ is defined everywhere.

(b) Consider the path $x = y^2$. The limit as $y \to 0$ along this path is

$$\lim_{y\to 0} \frac{y^5}{2y^6} = \lim_{y\to 0} \frac{1}{2y}.$$

As the above limit is ∞ from the right and $-\infty$ from the left, the limit does not exist. Therefore $\lim_{(x,y)\to 0} f$ does not exist, meaning f is not continuous. $\square$