Math 375

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(1) 2.12 Pr 10

$$\text{Let } V \text{ and } W \text{ be linear spaces, each with dimension 2 and each with basis } (e_1, e_2).$$
$$\text{Let } T : V \to W \text{ be a linear transformation such that } T(e_1 + e_2) = 3e_1 + 9e_2,$$
$$T(3e_1 + 2e_2) = 7e_1 + 23e_2.$$
$$\text{a) Compute } T(e_2 - e_1) \text{ and determine the nullity and rank of } T$$

$$T(e_1 + e_2) = 3e_1 + 9e_2$$
$$T(3e_1 + 2e_2) = 7e_1 + 23e_2$$

$$T(e_1) + T(e_2) = 3e_1 + 9e_2$$
$$3T(e_1) + 2T(e_2) = 7e_1 + 23e_2$$
$$T(e_1) = e_1 + 5e_2$$
$$T(e_2) = 2e_1 + 4e_2$$

$$T(e_2 - e_1) = T(e_2) - T(e_1) = 2e_1 + 4e_2 - (e_1 + 5e_2)$$
$$= e_1 - e_2$$

$$T(e_1) = e_1 + 5e_2 = (1)e_1 + (5)e_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$
$$T(e_2) = 2e_1 + 4e_2 = (2)e_1 + (4)e_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

$$\text{Therefore } T(v), v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} v \in V$$
$$T(v) = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 5x_1 + 4x_2 \end{bmatrix}$$

$$\text{When } x_1 = x_2 = 0$$
$$\begin{bmatrix} x_1 + 2x_2 \\ 5x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\text{So Nullity is } 0$$

$$\text{The range is spanned by } \begin{bmatrix} 1 \\ 5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$
$$\text{Therefore Rank is } 2$$

b) Determine the matrix of T relative to given matrix

In a) we get the matrix of T

$$[T]=\begin{bmatrix}1 & 2\\ 5 & 4\end{bmatrix}$$

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c) Use the basis (e_1, e_2) for V and find a new basis of the form $(e_1 + ae_2, 2e_1, +be_2)$ for W, relative to which the matrix of T will be in diagonal form

$$\text{Let } e_1 + ae_2 = e_1 + 5e_2 \text{ and } 2e_1, +be_2 = 2e_1, +4e_2$$
$$T(e_1) = (a_1)e_1 + 5e_2 + (a_2)2e_1, +4e_2$$
$$T(e_2) = (b_1)e_1 + 5e_2 + (b_2)2e_1, +4e_2$$
$$\text{To be diagonal } a_2 = b_1 = 0$$

$$T(e_1) = (a_1)e_1 + 5e_2$$
$$T(e_2) = (b_2)2e_1, +4e_2$$

Since

$$T(e_1) = e_1 + 5e_2 = (1)e_1 + (5)e_2 = \begin{bmatrix}1\\5\end{bmatrix}$$
$$T(e_2) = 2e_1 + 4e_2 = (2)e_1 + (4)e_2 = \begin{bmatrix}2\\4\end{bmatrix}$$
$$a_1 = b_2 = 1$$
$$[T]=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Therefore the basis of W is $(e_1 + 5e_2, 2e_1, +4e_2)$

(2) 2.16 Pr 6

$$\text{Let } A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ Verify that } A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and compute } A^n$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1+0 & 1+1 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\text{Going to prove } A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \text{ for all integers using induction}$$
$$\text{basic step is already showed}$$

$$\text{Assume that it is true when n then prove for n+1}$$
$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$
$$A^{n+1} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1+0 & 1+n \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 1+n \\ 0 & 1 \end{bmatrix}$$

$$\text{Therefore } A^{n+1} = \begin{bmatrix} 1 & 1+n \\ 0 & 1 \end{bmatrix} \text{ So it is true when } n+1$$

$$\text{By induction } A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

(3) 2.16 Pr 9

$$\text{Let } A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \text{ Prove that } A^2 = 2A - Z \text{ and compute } A^{100}$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1+0 & 0+0 \\ -1-1 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$2A - Z = 2\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$
$$\text{Therefore } A^2 = 2A - Z$$

$$\text{Going to prove } A^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \text{ for all integers using induction}$$
basic step is already showed

Assume that it is true when n then prove for n+1

$$A^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$$

$$A^{n+1} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ -n-1 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -n-1 & 1 \end{bmatrix}$$

$$\text{Therefore } A^{n+1} = \begin{bmatrix} 1 & 0 \\ -n-1 & 1 \end{bmatrix} \text{ So it is true when } n+1$$

$$\text{By induction } A^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$$

$$\text{Therefore } A^{100} = \begin{bmatrix} 1 & 0 \\ -100 & 1 \end{bmatrix}$$

(4) 2.16 Pr 11

a) Prove that a 2×2 matrix A commutes with every 2×2 matrix if and only if A commutes with each of the four matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

first prove $\rightarrow$ direction

$$\text{let } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ arbitary } 2 \times 2 \text{ matrix}$$

$$\text{then } A \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} A$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A(a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = (a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})A$$

$$(aA \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + bA \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + cA \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + dA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = (a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} A + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} A + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} A + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} A)$$

Therefore A commutes with the four matrices

$$\text{Prove} \leftarrow \text{direction}$$
$$\text{A commutes with the four matrices}$$

$$\text{As we showed above}$$
$$A\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A(a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$$

$$\text{Since } A \text{ commutes with the four matrices}$$
$$A(a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$$
$$= (aA\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + bA\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + cA\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + dA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$$
$$= (a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}A + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}A + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}A + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}A)$$
$$= (a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})A$$
$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}A$$

(5) 2.20 Pr 2

$$\text{Do Gaussian Jordan Elimination and find solution if exist}$$

$$3x + 2y + z = 1$$
$$5x + 3y + 3z = 2$$
$$x + y - z = 1$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 5 & 3 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & 1 \\ 5 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 8 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{2} \\ -2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{-1}{2} \end{bmatrix}$$

$$\text{In the process we found that the last equation } 0x + 0y + 0z = \frac{-1}{2}$$
$$\text{Therefore there is no solution}$$

(6) 2.20 Pr 6

$$
\begin{aligned}
x+y-3z+u&=5\\
2x-y+z-2u&=2\\
7x+y-7z+3u&=3
\end{aligned}
$$

$$
\begin{bmatrix}1&1&-3&1\\2&-1&1&-2\\7&1&-7&3\end{bmatrix}\begin{bmatrix}5\\2\\3\end{bmatrix}
$$

$$
\begin{bmatrix}1&1&-3&1\\0&-3&7&-4\\0&-6&14&-4\end{bmatrix}\begin{bmatrix}5\\-8\\-32\end{bmatrix}
$$

$$
\begin{bmatrix}1&1&-3&1\\0&1&\frac{-7}{3}&\frac{4}{3}\\0&-6&14&-4\end{bmatrix}\begin{bmatrix}5\\\frac{8}{3}\\-32\end{bmatrix}
$$

$$
\begin{bmatrix}1&1&-3&1\\0&1&\frac{-7}{3}&\frac{4}{3}\\0&0&0&1\end{bmatrix}\begin{bmatrix}5\\\frac{8}{3}\\-4\end{bmatrix}
$$

$$
\begin{bmatrix}1&0&\frac{-2}{3}&0\\0&1&\frac{-7}{3}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1\\8\\-4\end{bmatrix}
$$

$$
\begin{aligned}
x-\frac{2}{3}z&=1\\
y-\frac{7}{3}z&=8\\
u&=-4
\end{aligned}
$$

$$
\begin{aligned}
&\text{let } z=a\\
&x=1+\frac{2}{3}a\\
&y=8+\frac{7}{3}a\\
&u=-4
\end{aligned}
$$

$$
\begin{aligned}
(x,y,z,u)&=(1+\frac{2}{3}a,8+\frac{7}{3}a,a,-4)\\
&=a(\frac{2}{3},\frac{7}{3},1,0)+(1,8,0,-4)
\end{aligned}
$$

(7) 2.20 Pr 10

Determine all solutions of the system

$$5x+2y-6z+2u=-1$$
$$x-y+z-u=-2$$

$$\begin{bmatrix}5 & 2 & -6 & 2\\ 1 & -1 & 1 & -1\end{bmatrix}\begin{bmatrix}-1\\ -2\end{bmatrix}$$
$$\begin{bmatrix}1 & -1 & 1 & -1\\ 5 & 2 & -6 & 2\end{bmatrix}\begin{bmatrix}-2\\ -1\end{bmatrix}$$
$$\begin{bmatrix}1 & -1 & 1 & -1\\ 0 & 7 & -11 & 7\end{bmatrix}\begin{bmatrix}-2\\ 9\end{bmatrix}$$
$$\begin{bmatrix}1 & -1 & 1 & -1\\ 0 & 1 & -\frac{11}{7} & 1\end{bmatrix}\begin{bmatrix}-2\\ \frac{9}{7}\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & -\frac{4}{7} & 0\\ 0 & 1 & -\frac{11}{7} & 1\end{bmatrix}\begin{bmatrix}-\frac{5}{7}\\ \frac{9}{7}\end{bmatrix}$$

$$7x-4z=-5$$
$$7y-11z+u=9$$

$$\text{let } z=a, u=b$$
$$x=(-5+4a)/7, y=(9+11a-b)/7$$

$$(x,y,z,u)=((-5+4a)/7,(9+11a-b)/7,a,b)$$
$$a(\frac{4}{7},\frac{11}{7},10)+b(0,-1,0,1)+(-\frac{5}{7},\frac{9}{7},0,0)$$

(8) 2.21 Pr 7

If we interchange the rows and columns of a rectangular matrix A the new matrix so obtained is called the transpose of A and is denoted by A^t

For example, if we have

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}, A^T=\begin{bmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{bmatrix}$$

prove following properties

a) $(A^T)^T=A$

if A is $n\times m$ matrix A^T is $m\times n$ and $(A^T)^T$ is $n\times m$ matrix

$(i,j)^{th}$element of $(A^T)^T$ is $(j,i)^{th}$ element of A^T

$(j,i)^{th}$element of A^T is $(i,j)^{th}$ element of A

therefore $(i,j)^{th}$ element of $(A^T)^T$ is equal to $(i,j)^{th}$ element of A

Therefore $(A^T)^T=A$

b) $(A+B)^t = A^t + B^t$

let $(i,j)^{th}$ element of A is a
let $(i,j)^{th}$ element of B is b

$(j,i)^{th}$ element of $(A+B)^t$ is $(i,j)^{th}$ element of $(A+B)$
$(i,j)^{th}$ element of $(A+B)$ is $a+b$

$(j,i)^{th}$ element of A^t and B^t are a and b
$A^t + B^t = a+b$

therefore $(A+B)^t = A^t + B^t$

c) $(cA)^t = cA^t$

$(j,i)^{th}$ element of $(cA)^t$ is $(i,j)^{th}$ element of cA
Since c is scalar $(i,j)^{th}$ element of cA is $c \times ((i,j)^{th}$ element of $A)$

$(j,i)^{th}$ element of cA^t is $c \times ((i,j)^{th}$ element of $A)$

therefore $(cA)^t = cA^t$

d) $(AB)^t = B^t A^t$

let $(i,j)^{th}$ element of A is a
let $(i,j)^{th}$ element of B is b

$(j,i)^{th}$ element of $(AB)^t$ is $(i,j)^{th}$element of AB
$(i,j)^{th}$element of AB is equal to ab

$(j,i)^{th}$ element of $B^t A^t$ is product of $(j,i)^{th}$ element of B^t and A^t
$(j,i)^{th}$ element of B^t and A^t are equal to a and b
$B^t A^t = ab$

Therefore $(AB)^t = B^t A^t$

e) $(A^t)^{-1} = (A^{-1})^t$ if A is nonsingular

in d) we showed that $(AB)^t = B^t A^t$

$(A^t)^{-1} = (A^{-1})^t$ on both side we are going to multiply A^t
$(A^t)^{-1} A^t = I$

$(A^{-1})^t A^t = (A^{-1}A)^t = (I)t = I$ use what we showed in problem d

Therefore $(A^t)^{-1} = (A^{-1})^t$