Math 375

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1)

$$\text{Show that if } f(x) = |x| \text{ then for any number } m$$
$$\lim_{h\to 0}(f(h+0) - f(0) - mh) = 0$$
$$\text{but}$$
$$\lim_{h\to 0}\frac{1}{h}(f(h+0) - f(0) - mh) = 0$$
$$\text{never holds, what does this say about } f$$

$$\lim_{h\to 0}(f(h+0) - f(0) - mh) = 0 \text{ since every parts convergence}$$
$$\lim_{h\to 0} f(h+0) - \lim_{h\to 0} f(0) - \lim_{h\to 0} mh$$
$$= 0 + 0 + 0 = 0$$

$$\lim_{h\to 0}\frac{1}{h}(f(h+0) - f(0) - mh) = 0$$
$$\lim_{h\to 0^+}\frac{1}{h}(f(h+0) - f(0) - mh) \neq \lim_{h\to 0^-}\frac{1}{h}(f(h+0) - f(0) - mh)$$
$$\lim_{h\to 0^+}\frac{1}{h}(f(h+0) - f(0) - mh) = 1 - m$$
$$\lim_{h\to 0^-}\frac{1}{h}(f(h+0) - f(0) - mh) = -1 - m$$
$$\lim_{h\to 0}\frac{1}{h}(f(h+0) - f(0) - mh) \text{ cannot be defined}$$
$$\text{therefore it never holds}$$

$$\text{it means } f \text{ is not differentiable at } 0$$

2)

$$\text{Find the Jacobian matrices of the following maps. Show your work.}$$

$$1)\ f(x,y) = e^{x^2+y^3}$$
$$2)\ f(x,y) = (xy, sin(xy))$$

$$1)\ f(x,y) = e^{x^2+y^3}$$

$$Df(x,y) = [\partial f/\partial x, \partial f/\partial y]$$
$$= [\partial f/\partial x, \partial f/\partial y]$$
$$= [2xe^{x^2+y^3}, 3ye^{x^2+y^3}]$$

$$2)\ f(x,y) = (xy, sin(xy))$$

$$Df(x,y) = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}$$
$$= \begin{bmatrix} y & x \\ ycos(xy) & xcos(xy) \end{bmatrix}$$

3)

Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \frac{x^4+y^4}{x^2+y^2} \text{ if } (x,y) \neq 0$$
$$= (0,0) \text{ if } (x,y) = 0$$

show that it is differentiable at $(0,0)$

for $f(x,y)$ to be differentiable it has to satisfy

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - T_{(0,0)}|(x,y) - (0,0)|}{||(x,y) - (0,0)||} = 0$$
$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^4+y^4}{x^2+y^2} - 0 - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2+y^2}}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = h = 0$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = h = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^4+y^4}{x^2+y^2} - 0 - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2+y^2}}$$
$$= \lim_{(x,y)\to(0,0)} \frac{\frac{x^4+y^4}{x^2+y^2}}{\sqrt{x^2+y^2}}$$
$$= \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)\sqrt{x^2+y^2}}$$

along $y = mx$

$$\lim_{x\to 0} \frac{x^4 + m^4x^4}{(x^2+m^2x^2)\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x\to 0} \frac{x^4(1+m^4)}{x^3(1+m^2)\sqrt{1+m^2}} = 0$$

Therefore $f(x,y)$ is differentiable at $(0,0)$

4)

Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$
given by

$$f(x,y) = \frac{x^3}{x^2+y^2} \text{ if } (x,y) \neq 0$$

$$= (0,0) \text{ if } (x,y) = 0$$

show that it has jacobian but not differentiable at $(0,0)$

$$\text{Jacobian}$$
$$[\partial f/\partial x, \partial f/\partial y]$$
$$\text{at } (0,0)$$
$$[\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}, \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}]$$
$$=[1,0]$$
$$\text{so it exist}$$

$$\text{Differentiability}$$
$$\text{it has to satisfy}$$
$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-T_{(0,0)}|(x,y)-(0,0)|}{||(x,y)-(0,0)||}=0$$
$$\lim_{(x,y)\to(0,0)}\frac{\frac{x^3}{x^2+y^2}-0-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}}$$

$$f_x(0,0)=1$$
$$f_y(0,0)=0$$

$$\lim_{(x,y)\to(0,0)}\frac{\frac{x^3}{x^2+y^2}-x}{\sqrt{x^2+y^2}}$$
$$\lim_{(x,y)\to(0,0)}\frac{\frac{-xy^2}{x^2+y^2}}{\sqrt{x^2+y^2}}$$
$$\lim_{(x,y)\to(0,0)}\frac{-xy^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$
$$\text{along } y=mx$$
$$\lim_{x\to 0}\frac{-xm^2x^2}{(x^2+m^2x^2)\sqrt{x^2+m^2x^2}}$$
$$\lim_{x\to 0}\frac{-x^3m^2}{x^2(1+m^2)x\sqrt{1+m^2}}$$
$$\lim_{x\to 0}\frac{-m^2}{(1+m^2)\sqrt{1+m^2}}$$

$$\lim_{x\to 0}\frac{-m^2}{(1+m^2)\sqrt{1+m^2}} \text{ is not always } 0$$

$$\text{therefore it is not differentiable}$$

5. Apostol Section 8.9 Problem 13

compute all first-order partial derivatives

$$
\begin{aligned}
f(x, y) &= \tan\left(x^2/y\right), y \neq 0 \\
\partial f/\partial x &= \frac{2x}{y} \sec^2(x^2/y) \\
\partial f/\partial y &= \frac{-x^2}{y^2} \sec^2(x^2/y)
\end{aligned}
$$

6. Apostol Section 8.14 Problem 2

Evaluate the directional derivatives of the following scalar fields for the points and directions given :

a) $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at $(1, 1, 0)$ in the direction of $i - j + 2k$

$$
\lim_{h \to 0} \frac{f(1, 1, 0 + h(1, -1, 2)) - f(1, 1, 0)}{h}
$$