

Math 375

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(1) 2.8 Pr 19

- Determine one-to-one
 - Describe its range
 - Find inverse

$$T(x, y, z) = (x, x + y, x + y + z)$$

Check one-to-one

$$\text{Let } A = (x_1, y_1, z_1) \quad B = (x_2, y_2, z_2)$$

If it is one-to-one it must follow $T(A) = T(B)$ then $A = B$

$$T(A) = T(B)$$

$$(x_1, x_1 + y_1, x_1 + y_1 + z_1) = (x_2, x_2 + y_2, x_2 + y_2 + z_2)$$

$$x_1 = x_2$$

$$x_1 + y_1 = x_2 + y_2$$

$$x_1 + y_1 + z_1 = x_2 + y_2 + z_2$$

Solving the equation we get

$$x_1 = x_2, y_1 = y_2, z_1 = z_2$$

$$\text{So } A = B$$

Therefore it is one-to-one

$$x(1, 1, 1) + y(0, 1, 1) + z(0, 0, 1) = (x, x + y, x + y + z)$$

$(1, 1, 1), (0, 1, 1), (0, 0, 1)$ form a basis for range of T

The Range is \mathbb{R}^3

$$T(x, y, z) = (u, v, w)$$

$$(x, x + y, x + y + z) = (u, v, w)$$

$$x = u, y = v - u, z = w - v$$

$$T^{-1}(u, v, w) = (x, y, z)$$

$$T^{-1}(u, v, w) = (u, v - u, w - v)$$

$$\text{Check } TT^{-1}(x, y, z) = (x, y, z)$$

$$T(x, y - x, z - y) = (x, x + y - x, x + y - x + z - y) = (x, y, z)$$

Answer) It is one-to-one

Spanned by $(1, 1, 1), (0, 1, 1), (0, 0, 1)$ which is \mathbb{R}^3

$$T^{-1}(u, v, w) = (u, v - u, w - v)$$

(2) 2.8 Pr 22

If S and T commute, prove that $(ST)^n = S^n T^n$ for all integers $n \geq 0$

Before proving for the $(ST)^n = S^n T^n$ we are going to show for $ST^n = T^n S$

It is trivial for $n = 0$ Since $T^0 = I$

for $n = 1$ it is true because $ST^1 = ST = TS$

Assume it is true for n and prove for $n+1$

$$\begin{aligned} ST^{n+1} &= STT^n \\ &= TST^n \text{ since we assume it is true for } n \\ &= TT^n S \\ &= T^{n+1} S \end{aligned}$$

It is true for $n+1$ therefore it is true

Prove $(ST)^n = S^n T^n$ for all integers $n \geq 0$

It is trivial for $n = 0$ Since $T^0 = I$

for $n = 1$ it is true because $(ST)^1 = S^1 T^1$

Assume it is true for n prove for $n + 1$

$$\text{For } n + 1 \quad (ST)^{n+1} = (ST)^n (ST) = S^n T^n (ST)$$

Since they commute $S^n T^n (ST) = S^n ST^n T = S^{n+1} T^{n+1}$ (use what we proved in previous induction)

$$(ST)^{n+1} = S^{n+1} T^{n+1} \text{ is true}$$

Therefore statement $(ST)^n = S^n T^n$ for all integers $n \geq 0$ is true

(3) 2.8 Pr 24

If S and T are invertible and commute, prove that their inverses also commute.

$$ST = TS \text{ and } S^{-1} T^{-1} \text{ exist}$$

$$\begin{aligned} ST &= TS \\ S^{-1}ST &= S^{-1}TS \\ IT &= S^{-1}TS \\ T &= S^{-1}TS \\ T^{-1}T &= T^{-1}S^{-1}TS \\ I &= T^{-1}S^{-1}TS \\ IS^{-1} &= T^{-1}S^{-1}TSS^{-1} \\ S^{-1} &= T^{-1}S^{-1}TI \\ S^{-1}T^{-1} &= T^{-1}S^{-1}TT^{-1} \\ S^{-1}T^{-1} &= T^{-1}S^{-1}I \\ S^{-1}T^{-1} &= T^{-1}S^{-1} \end{aligned}$$

Therefore inverse is also commute

(4) 2.12 Pr 3

A linear transformation $T : V_2 \rightarrow V_2$, maps the basis vectors i and j as follows:

$$T(i) = i + j \quad T(j) = 2i - j$$

a) Compute $T(3i - 4j)$ and $T^2(3i - 4j)$ in terms of i and j .

Since it is a linear transformation it must follow
 $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ where $\alpha, \beta \in V_2$ and scalars a, b

$$T(3i - 4j) = 3T(i) - 4T(j) = 3(i + j) - 4(2i - j) = -5i + 7j$$

$$\begin{aligned} T^2(3i - 4j) &= TT(3i - 4j) = T(-5i + 7j) \\ &= -5T(i) + 7T(j) = -5(i + j) + 7(2i - j) = 9i - 12j \end{aligned}$$

b) Determine the matrix of T and T^2

basis is (i, j)

$$T(i) = (1)i + (1)j, \quad T(j) = (2)i + (-1)j$$

$$T(i) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(j) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\text{So } [T] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

c) Solve b if basis is replaced by (e_1, e_2) $e_1 = i - j, e_2 = 3i + j$

$$T(e_1) = T(i - j) = -i + 2j = a(i - j) + b(3i + j) \quad \text{using problem a}$$

$$a + 3b = -1$$

$$-a + b = 2$$

$$a = \frac{-7}{4}, b = \frac{1}{4}$$

$$\text{So } T(e_1) = \begin{bmatrix} \frac{-7}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$T(e_2) = T(3i + j) = 5i + 2j = a(i - j) + b(3i + j) \quad \text{using problem a}$$

$$a + 3b = 5$$

$$-a + b = 2$$

$$a = \frac{-1}{4}, b = \frac{7}{4}$$

$$\text{So } T(e_2) = \begin{bmatrix} \frac{-1}{4} \\ \frac{7}{4} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \frac{-7}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix}$$

$$T^2 = \begin{bmatrix} \frac{-7}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} \frac{-7}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

a) Compute $T(2i - 3j)$ and determine the nullity and rank of T .

Since it is a linear transformation it must follow

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \text{ where } \alpha, \beta \in V_2 \text{ and scalars } a, b$$

$$T(2i - 3j) = 2T(i) - 3T(j) = 2(1, 0, 1) - 3(-1, 0, 1) = (5, 0, -1)$$

Representative matrix of T

$$T(i) = (1, 0, 1) = 1(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1)$$

$$T(j) = (-1, 0, 1) = -1(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1)$$

$$[T] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\text{So } T(v), v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} v \in V_2$$

$$T(v) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix}$$

The nullity of T

$$\text{When } x_1 = 0, x_2 = 0$$

$$\begin{bmatrix} x_1 - x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore nullity of T is 0

$$\text{Range of } T \text{ is spanned by } \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Rank is 2

b) Determine the matrix of T

In part a we showed the matrix of T

$$[T] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

c) Find bases (e_1, e_2) for V_2 and (w_1, w_2, w_3) for V_3 relative to which the matrix of T will be in diagonal form.

$$\text{let } e_1 = i, e_2 = j$$

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(e_1) = a_1 w_1 + a_2 w_2 + a_3 w_3$$

$$T(e_2) = b_1 w_1 + b_2 w_2 + b_3 w_3$$

to be a diagonal form $a_2 = a_3 = 0, b_1 = b_3 = 0$

$$T(e_1) = a_1 w_1$$

$$T(e_2) = b_2 w_2$$

$$T(e_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Since $a = b = 1$ matrix of T in a diagonal form is made

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e_1 = i, e_2 = j$$

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(6) 2.16 Pr 1

$$\text{If } A = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix}, C = \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix}$$

Compute $B + C, AB, BA, AC, CA, A(2B - 3C)$

$$B + C$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & 2+2 \\ -1+1 & 3-1 \\ 5+1 & -2-3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 \\ 0 & 2 \\ 6 & -5 \end{bmatrix}$$

$$\begin{aligned}
& AB \\
&= \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 1+4+10 & 2-12-4 \\ -1-4-10 & -2+12+4 \end{bmatrix} = \begin{bmatrix} 15 & -14 \\ -15 & 14 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& BA \\
&= \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 1-2 & -4+8 & 2-4 \\ -1-3 & 4+12 & -2-6 \\ 5+2 & -20-8 & 10+4 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -2 \\ -4 & 16 & -8 \\ 7 & -28 & 14 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& AC \\
&= \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix} \\
&= \begin{bmatrix} 2-4+2 & 2+4-6 \\ -2+4-2 & -2-4+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& CA \\
&= \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 2-2 & -8+8 & 4-4 \\ 1+1 & -4-4 & 2+2 \\ 1+3 & -4-12 & 2+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -8 & 4 \\ 4 & -16 & 8 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& A(2B-3C) \\
&= \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix} \left(2 \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix} - 3 \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix} \right) \\
&= \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} 2-6 & 4-6 \\ -2-3 & 6+3 \\ 10-3 & -4+9 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} -4 & -2 \\ -5 & 9 \\ 7 & 5 \end{bmatrix} \\
&= \begin{bmatrix} -4+20+14 & -2-36+10 \\ 4-20-14 & 2+36-10 \end{bmatrix} \\
&= \begin{bmatrix} 30 & -28 \\ -30 & 28 \end{bmatrix}
\end{aligned}$$

(7) 2.16 Pr 3

find a, b, c, d

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 6 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} c \\ a \\ b \\ d \end{bmatrix}$$

$$\begin{bmatrix} c \\ a \\ b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 6 \\ 5 \end{bmatrix}$$

$$a = 9, b = 6, c = 1, d = 5$$

$$\begin{bmatrix} a & b & c & d \\ 1 & 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 & 6 \\ 1 & 9 & 8 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c & d \\ 1 & 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a & c & 2a + b + d & b \\ 1 & 9 & 2 + 4 + 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a & c & 2a + b + d & b \\ 1 & 9 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 6 & 6 \\ 1 & 9 & 8 & 4 \end{bmatrix}$$

$$a = 1, c = 0, 2a + b + d = 6, b = 6$$

Solving the equation

$$a = 1, b = 6, c = 0, d = -2$$

(8) 2.16 Pr 4

Calculate $AB - BA$ in each case

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} & AB \\ &= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4-8+2 & 1+4+4 & 1+0+2 \\ 8-4+2 & 2+2+4 & 2+0+2 \\ 4-8+3 & 1+4+6 & 1+0+3 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 9 & 3 \\ 6 & 8 & 4 \\ -1 & 11 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} & BA \\ &= \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 4+2+1 & 8+1+2 & 8+2+3 \\ -4+4+0 & -8+2+0 & -8+4+0 \\ 1+4+1 & 2+2+2 & 2+4+3 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 11 & 13 \\ 0 & -6 & -4 \\ 6 & 6 & 9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AB - BA &= \begin{bmatrix} -2 & 9 & 3 \\ 6 & 8 & 4 \\ -1 & 11 & 4 \end{bmatrix} - \begin{bmatrix} 7 & 11 & 13 \\ 0 & -6 & -4 \\ 6 & 6 & 9 \end{bmatrix} \\ &= \begin{bmatrix} -2-7 & 9-11 & 3-13 \\ 6-0 & 8-(-6) & 4-(-4) \\ -1-6 & 11-6 & 4-9 \end{bmatrix} \\ &= \begin{bmatrix} -9 & -2 & -10 \\ 6 & 14 & 8 \\ -7 & 5 & -5 \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & 11 \end{bmatrix}$$

$$\begin{aligned} & AB \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & 11 \end{bmatrix} \\ &= \begin{bmatrix} 6+0+0 & 2+0+0 & -4+0+0 \\ 3+3-6 & 1-2+10 & -2+4+22 \\ -3+6-3 & -1-4+5 & 2+8+11 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 2 & -4 \\ 0 & 9 & 24 \\ 0 & 0 & 21 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} & BA \\ &= \begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & 11 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6+1+2 & 0+1-4 & 0+2-2 \\ 6-2-4 & 0-2+8 & 0-4+4 \\ -6+5-11 & 0+5+22 & 0+10+11 \end{bmatrix} \\ &= \begin{bmatrix} 9 & -3 & 0 \\ 0 & 6 & 0 \\ -12 & 27 & 21 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AB - BA &= \begin{bmatrix} 6 & 2 & -4 \\ 0 & 9 & 24 \\ 0 & 0 & 21 \end{bmatrix} - \begin{bmatrix} 9 & -3 & 0 \\ 0 & 6 & 0 \\ -12 & 27 & 21 \end{bmatrix} \\ &= \begin{bmatrix} 6-9 & 2-(-3) & -4-0 \\ 0-0 & 9-6 & 24-0 \\ 0-(-12) & 0-27 & 21-21 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 5 & -4 \\ 0 & 3 & 24 \\ 12 & -27 & 0 \end{bmatrix} \end{aligned}$$