4. (12 points) Prove that if $S = \{x_1, x_2, x_3\}$ is an independent subset of a vector space V and $y \in V$ but $y \notin L(S)$ then $S \cup \{y\}$ is independent.

$$\boxed{\begin{array}{l} S = \{x_1, x_2, x_3\} \\ \sum_{i=1}^{3} c_i x_i = 0 \Rightarrow c_1 = c_2 = c_3 = 0 \end{array}}$$ for S.

~~there doesn't~~

$\cancel{a_1x_1 + a_2x_2 + a_3x_3}$ there doesn't exist $a_1x_1 + a_2x_2 + a_3x_3 = y$, $\underbrace{a_1, a_2, a_3}_{\in \mathbb{R}} \rightarrow$ doesn't exist

since $y \notin L(S)$

so we can express $a_1x_1 + a_2x_2 + a_3x_3 + v_1 = y$ and v_1 is $v_1 \notin L(S)$

$S \cup \{y\} = \{x_1, x_2, x_3, (a_1x_1 + a_2x_2 + a_3x_3 + v_1)\}$

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for $S \cup \{y\}$ to be dependent

$c_1x_1 + c_2x_2 + c_3x_3 + c_4(a_1x_1 + a_2x_2 + a_3x_3 + v_1) = 0$ except for $c_1 = c_2 = c_3 = c_4 = 0$

not all $= 0$.

$(c_1 + c_4a_1)x_1 + (c_2 + c_4a_2)x_2 + (c_3 + c_4a_3)x_3 + c_4v_1 = 0$

~~However $v_1 \notin L(S)$ therefore~~ $\cancel{\frac{c_1 + c_4a_1}{c_4} = \frac{c_2 + c_4a_2}{c_4} = \frac{c_3 + c_4a_3}{c_4} = 0}$

$\cancel{c_1 + c_4a_1 = c_2 + c_4a_2 = c_3 + c_4a_3 = 0}$

$\cancel{c_1 = -c_4a_1}$

$\cancel{c_2 = -c_4a_2}$

$\cancel{c_3 = -c_4a_3}$

$(c_1 + c_4a_1) = (c_2 + c_4a_2) = (c_3 + c_4a_3) = c_4 = 0$

since $\{x_1, x_2, x_3\}$ can't span c_4v_1

and since $c_4 = 0$

$c_1x_1 + c_2x_2 + c_3x_3 = 0$ but S is independent

and only $c_1 = c_2 = c_3 = 0$ satisfy the equation.

therefore $c_1 = c_2 = c_3 = c_4 = 0$ to satisfy the equation it means $S \cup \{y\}$ is independent

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4y = 0.$$

$a_4 = 0$ $\qquad$ $a_4 \neq 0$.