Math 375

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(1) 2.8 Pr 19

- Determine one-to-one
    - Describe its range
    - Find inverse

$$T(x, y, z) = (x, x + y, x + y + z)$$

$$\text{Check one-to-one}$$
$$\text{Let } A = (x_1, y_1, z_1)\ B = (x_2, y_2, z_2)$$
$$\text{If it is one-to-one it must follow } T(A) = T(B) \text{ then } A = B$$

$$T(A) = T(B)$$
$$(x_1, x_1 + y_1, x_1 + y_1 + z_1) = (x_2, x_2 + y_2, x_2 + y_2 + z_2)$$

$$x_1 = x_2$$
$$x_1 + y_1 = x_2 + y_2$$
$$x_1 + y_1 + z_1 = x_2 + y_2 + z_2$$
$$\text{Solving the equation we get}$$
$$x_1 = x_2, y_1 = y_2, z_1 = z_2$$
$$\text{So } A = B$$
$$\text{Therefore it is one-to-one}$$

$$x(1, 1, 1) + y(0, 1, 1) + z(0, 0, 1) = (x, x + y, x + y + z)$$
$$(1, 1, 1), (0, 1, 1), (0, 0, 1) \text{ form a basis for range of } T$$
$$\text{The Range is } \mathbb{R}^3$$

$$T(x, y, z) = (u, v, w)$$
$$(x, x + y, x + y + z) = (u, v, w)$$
$$x = u, y = v - u, z = w - v$$
$$T^{-1}(u, v, w) = (x, y, z)$$
$$T^{-1}(u, v, w) = (u, v - u, w - v)$$

$$\text{Check } TT^{-1}(x, y, z) = (x, y, z)$$
$$T(x, y - x, z - y) = (x, x + y - x, z + y - x + z - y) = (x, y, z)$$
$$\text{Answer) It is one-to-one}$$
$$\text{Spanned by } (1, 1, 1), (0, 1, 1), (0, 0, 1) \text{ which is } \mathbb{R}^3$$
$$T^{-1}(u, v, w) = (u, v - u, w - v)$$

(2) 2.8 Pr 22

$$\text{If S and T commute, prove that} (ST)^n = S^n T^n \text{ for all integers } n \geq 0$$

Before proving for the $(ST)^n = S^nT^n$ we are going to show for $ST^n = T^nS$

It is trivial for $n = 0$ Since $T^0 = I$

for $n = 1$ it is true because $ST^1 = ST = TS$

Assume it is true for n and prove for n+1

$$ST^{n+1} = STT^n$$
$$= TST^n \text{ since we assume it is true for } n$$
$$= TT^nS$$
$$= T^{n+1}S$$

It is true for n+1 therefore it is true

Prove $(ST)^n = S^nT^n$ for all integers $n \geq 0$

It is trivial for $n = 0$ Since $T^0 = I$

for $n = 1$ it is true because $(ST)^1 = S^1T^1$

Assume it is true for n prove for $n + 1$

$$\text{For } n+1 \quad (ST)^{n+1} = (ST)^n(ST) = S^nT^n(ST)$$
$$\text{Since they commute } S^nT^n(ST) = S^nST^nT = S^{n+1}T^{n+1}$$
$$(ST)^{n+1} = S^{n+1}T^{n+1} \text{ is true}$$

Therefore statement $(ST)^n = S^nT^n$ for all integers $n \geq 0$ is true

(3) 2.8 Pr 24

If S and Tare invertible and commute, prove that their inverses also commute.

$ST = TS$ and S^{-1} T^{-1} exist

$$ST = TS$$
$$S^{-1}ST = S^{-1}TS$$
$$IT = S^{-1}TS$$
$$T = S^{-1}TS$$
$$T^{-1}T = T^{-1}S^{-1}TS$$
$$I = T^{-1}S^{-1}TS$$
$$IS^{-1} = T^{-1}S^{-1}TSS^{-1}$$
$$S^{-1} = T^{-1}S^{-1}TI$$
$$S^{-1}T^{-1} = T^{-1}S^{-1}TT^{-1}$$
$$S^{-1}T^{-1} = T^{-1}S^{-1}I$$
$$S^{-1}T^{-1} = T^{-1}S^{-1}$$

Therefore inverse is also commute

(4) 2.12 Pr 3

A linear transformation $T : V_2 \to V_2$, maps the basis vectors i andj as follows:

$T(i) = i + j \quad T(j) = 2i - j$

a) Compute $T(3i - 4j)$ and $T^2(3i - 4j)$ in terms of i and j.

Since it is a linear transformation it must follow
$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ where $\alpha, \beta \in V_2$ and scalars a, b

$$T(3i - 4j) = 3T(i) - 4T(j) = 3(i + j) - 4(2i - j) = -5i + 7j$$

$$T^2(3i - 4j) = TT(3i - 4j) = T(-5i + 7j)$$
$$= -5T(i) + 7T(j) = -5(i + j) + 7(2i - j) = 9i - 12j$$

b) Determine the matrix of T and T^2

basis is (i, j)

$$T(i) = (1)i + (1)j, \;\; T(j) = (2)i + (-1)j$$

$$T(i) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(j) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\text{So } [T] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

c) Solve b if basis is replaced by (e_1, e_2) $e_1 = i - j, e_2 = 3i + j$

$T(e_1) = T(i - j) = -i + 2j = a(i - j) + b(3i + j)$ using problem a

$$a + 3b = -1$$
$$-a + b = 2$$
$$a = \frac{-7}{4}, b = \frac{1}{4}$$
$$\text{So } T(e_1) = \begin{bmatrix} \frac{-7}{4} \\ \frac{1}{4} \end{bmatrix}$$

$T(e_2) = T(3i + j) = 5i + 2j = a(i - j) + b(3i + j)$ using problem a

$$a + 3b = 5$$
$$-a + b = 2$$
$$a = \frac{-1}{4}, b = \frac{7}{4}$$
$$\text{So } T(e_2) = \begin{bmatrix} \frac{-1}{4} \\ \frac{7}{4} \end{bmatrix}$$

$$[T] = \begin{bmatrix} \frac{-7}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix}$$

$$T^2 = \begin{bmatrix} \frac{-7}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix} \begin{bmatrix} \frac{-7}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

(5) 2.12 Pr 8

A linear transformation $T : V_2 \to V_3$ maps the basis vectors as follows: $T(i) = (1, 0, 1), T(j) = (-1, 0, 1)$

a) Compute $T(2i - 3j)$ and determine the nullity and rank of T.

Since it is a linear transformation it must follow
$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ where $\alpha, \beta \in V_2$ and scalars a, b

$$T(2i - 3j) = 2T(i) - 3T(j) = 2(1,0,1) - 3(-1,0,1) = (5,0,-1)$$

Representative matrix of T

$$T(i) = (1,0,1) = 1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$
$$T(j) = (-1,0,1) = -1(1,0,0) + 0(0,1,0) + 1(0,0,1)$$

$$[T] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

So $T(v), v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} v \in V_2$

$$T(v) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix}$$

The nullity of T

When $x_1 = 0, x_2 = 0$

$$\begin{bmatrix} x_1 - x_2 \\ 0 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore nullity of T is 0

Range of T is spanned by $\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$

Rank is 2

b) Determine the matrix of T

In part a we showed the matrix of T

$$[T] = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

c) Find bases (e_1, e_2) for V_2 and (w_1, w_2, w_3) for V_3 relative to which the matrix of T will be in diagonal form.

$$\text{let } e_1 = i, e_2 = j$$

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(e_1) = a_1 w_1 + a_2 w_2 + a_3 w_3$$
$$T(e_2) = b_1 w_1 + b_2 w_2 + b_3 w_3$$
$$\text{to be a diagonal form } a_2 = a_3 = 0, b_1 = b_3 = 0$$
$$T(e_1) = a_1 w_1$$
$$T(e_2) = b_2 w_2$$

$$T(e_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$T(e_2) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Since } a = b = 1 \text{ matrix of T in a diagonal form is made}$$
$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$e_1 = i, e_2 = j$$
$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(6) 2.16 Pr 1

$$\text{If } A = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 4 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix}, C = \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix}$$
$$\text{Compute } B + C, AB, BA, AC, CA, A(2B - 3C)$$

$$B + C$$
$$= \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 1+2 & 2+2 \\ -1+1 & 3-1 \\ 5+1 & -2-3 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 4 \\ 0 & 2 \\ 6 & -2-3 \end{bmatrix}$$

(7) 2.16 Pr 3

(8) 2.16 Pr 4