Math 375

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1)

$$\text{Show that if } f(x) = |x| \text{ then for any number } m$$
$$\lim_{h\to 0}(f(h+0) - f(0) - mh) = 0$$
$$\text{but}$$
$$\lim_{h\to 0}\frac{1}{h}(f(h+0) - f(0) - mh) = 0$$
$$\text{never holds, what does this say about } f$$

$$\lim_{h\to 0}(f(h+0) - f(0) - mh) = 0 \text{ since every parts convergence}$$
$$\lim_{h\to 0} f(h+0) - \lim_{h\to 0} f(0) - \lim_{h\to 0} mh$$
$$= 0 + 0 + 0 = 0$$

$$\lim_{h\to 0}\frac{1}{h}(f(h+0) - f(0) - mh) = 0$$
$$\lim_{h\to 0^+}\frac{1}{h}(f(h+0) - f(0) - mh) \neq \lim_{h\to 0^-}\frac{1}{h}(f(h+0) - f(0) - mh)$$
$$\lim_{h\to 0^+}\frac{1}{h}(f(h+0) - f(0) - mh) = 1 - m$$
$$\lim_{h\to 0^-}\frac{1}{h}(f(h+0) - f(0) - mh) = -1 - m$$
$$\lim_{h\to 0}\frac{1}{h}(f(h+0) - f(0) - mh) \text{ cannot be defined}$$
$$\text{therefore it never holds}$$

$$\text{it means } f \text{ is not differentiable at } 0$$

2)

$$\text{Find the Jacobian matrices of the following maps. Show your work.}$$

$$1)\ f(x,y) = e^{x^2+y^3}$$
$$2)\ f(x,y) = (xy, sin(xy))$$

$$1)\ f(x,y)=e^{x^2+y^3}$$

$$\begin{aligned}Df(x,y)&=[\partial f/\partial x,\partial f/\partial y]\\&=[\partial f/\partial x,\partial f/\partial y]\\&=[2xe^{x^2+y^3},3ye^{x^2+y^3}]\end{aligned}$$

$$2)\ f(x,y)=(xy,sin(xy))$$

$$\begin{aligned}Df(x,y)&=\begin{bmatrix}\partial f_1/\partial x & \partial f_1/\partial y\\ \partial f_2/\partial x & \partial f_2/\partial y\end{bmatrix}\\&=\begin{bmatrix}y & x\\ ycos(xy) & xcos(xy)\end{bmatrix}\end{aligned}$$

3)

Consider the following function $f:\mathbb{R}^2\to\mathbb{R}$ given by

$$\begin{aligned}f(x,y)&=\frac{x^4+y^4}{x^2+y^2}\text{ if }(x,y)\neq 0\\&=(0,0)\text{ if }(x,y)=0\end{aligned}$$

show that it is differentiable at $(0,0)$

for $f(x,y)$ to be differentiable it has to satisfy

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-T_{(0,0)}|(x,y)-(0,0)|}{||(x,y)-(0,0)||}=0$$

$$\lim_{(x,y)\to(0,0)}\frac{\frac{x^4+y^4}{x^2+y^2}-0-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}}$$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=h=0$$

$$f_y(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=h=0$$

$$\begin{aligned}&\lim_{(x,y)\to(0,0)}\frac{\frac{x^4+y^4}{x^2+y^2}-0-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}}\\&=\lim_{(x,y)\to(0,0)}\frac{\frac{x^4+y^4}{x^2+y^2}}{\sqrt{x^2+y^2}}\\&=\lim_{(x,y)\to(0,0)}\frac{x^4+y^4}{(x^2+y^2)\sqrt{x^2+y^2}}\end{aligned}$$

using squeeze theorem

$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)\sqrt{x^2+y^2}} \le \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$
$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)\sqrt{x^2+y^2}} \le \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$$
$$0 \le \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)\sqrt{x^2+y^2}} \le 0$$
$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)} = 0$$

Therefore $f(x, y)$ is differentiable at $(0, 0)$

4)

Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$
given by

$$f(x,y) = \frac{x^3}{x^2+y^2} \text{ if } (x,y) \neq 0$$
$$= (0,0) \text{ if } (x,y) = 0$$

show that it has jacobian but not differentiable at $(0, 0)$

$$\text{Jacobian}$$
$$[\partial f/\partial x, \partial f/\partial y]$$
$$\text{at } (0,0)$$
$$[\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}, \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}]$$
$$=[1,0]$$
$$\text{so it exist}$$

$$\text{Differentiability}$$
$$\text{it has to satisfy}$$
$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-T_{(0,0)}|(x,y)-(0,0)|}{||(x,y)-(0,0)||}=0$$
$$\lim_{(x,y)\to(0,0)}\frac{\frac{x^3}{x^2+y^2}-0-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}}$$

$$f_x(0,0)=1$$
$$f_y(0,0)=0$$

$$\lim_{(x,y)\to(0,0)}\frac{\frac{x^3}{x^2+y^2}-x}{\sqrt{x^2+y^2}}$$
$$\lim_{(x,y)\to(0,0)}\frac{\frac{-xy^2}{x^2+y^2}}{\sqrt{x^2+y^2}}$$
$$\lim_{(x,y)\to(0,0)}\frac{-xy^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$
$$\text{along } y=mx$$
$$\lim_{x\to 0}\frac{-xm^2x^2}{(x^2+m^2x^2)\sqrt{x^2+m^2x^2}}$$
$$\lim_{x\to 0}\frac{-x^3m^2}{x^2(1+m^2)x\sqrt{1+m^2}}$$
$$\lim_{x\to 0}\frac{-m^2}{(1+m^2)\sqrt{1+m^2}}$$

$$\lim_{x\to 0}\frac{-m^2}{(1+m^2)\sqrt{1+m^2}} \text{ is not always } 0$$

$$\text{therefore it is not differentiable}$$

5. Apostol Section 8.9 Problem 13

compute all first-order partial derivatives

$$f(x,y) = \tan\left(x^2/y\right), y \neq 0$$
$$\partial f/\partial x = \frac{2x}{y}\sec^2(x^2/y)$$
$$\partial f/\partial y = \frac{-x^2}{y^2}\sec^2(x^2/y)$$

6. Apostol Section 8.14 Problem 2

Evaluate the directional derivatives of the following scalar fields for the points and directions given :

a) $f(x,y,z) = x^2 + 2y^2 + 3z^2$ at $(1,1,0)$ in the direction of $i - j + 2k$

$$\lim_{h\to 0} \frac{f(1,1,0+h(1,-1,2)) - f(1,1,0)}{h}$$
$$\lim_{h\to 0} \frac{f(1+h,1-h,2h) - f(1,1,0)}{h}$$
$$\lim_{h\to 0} \frac{f(1+h,1-h,2h) - f(1,1,0)}{h}$$
$$= \lim_{h\to 0} \frac{15h^2 - 2h}{h}$$
$$= -2$$

divide by norm

$$\frac{-2}{\sqrt{6}}$$

b) $f(x,y,z) = (x/y)^z$ at (1,1,1) in direction of $(2,1,-1)$

$$D_{(2,1,-1)}f_{(1,1,1)} = f_x(1,1,1)2 + f_y(1,1,1)1 + f_z(1,1,1)(-1)$$
$$f_x = zy^{-z}x^{z-1}, f_y = -zx^z y^{-z-1}, f_z = (x/y)^z \ln(x/y)$$
$$f_x(1,1,1)2 + f_y(1,1,1)1 + f_z(1,1,1)(-1) = 1$$
$$\text{divide by norm } \frac{1}{\sqrt{6}}$$

Since the quality of pdf is too bad i am not sure $f(x,y,z) = (x/y)^z$ or $f(x,y,z) = (x/y)^2$

If $f(x,y,z) = (x/y)^2$

$$D_{(2,1,-1)}f_{(1,1,1)} = f_x(1,1,1)2 + f_y(1,1,1)1 + f_z(1,1,1)(-1)$$
$$f_x = 2y^{-2}x, f_y = -2x^2y^{-3}, f_z = 0$$
$$f_x(1,1,1)2 + f_y(1,1,1)1 + f_z(1,1,1)(-1) = 2$$
$$\text{divide by norm } \frac{2}{\sqrt{6}}$$

7.

$$\text{Let } f:\mathbb{R}^3\to\mathbb{R}^2 \text{ satisfy the condition } f(\vec{0})=(1,2)$$
$$Df(\vec{0})=\begin{bmatrix}1&2&3\\0&0&1\end{bmatrix}$$
$$\text{let } g:\mathbb{R}^2\to\mathbb{R}^2 \text{ be } g(x,y)=(x+2y+1,3xy)$$
$$\text{find } D(g\circ f)(\vec{0})$$
$$\text{Using chainrule}$$
$$D(g\circ f)(\vec{0})=D(g)(f(\vec{0}))D(f)(\vec{0})$$

$$D(f)(\vec{0})=\begin{bmatrix}1&2&3\\0&0&1\end{bmatrix}$$
$$D(g)(f(\vec{0}))$$
$$=\begin{bmatrix}1&2\\3y&3x\end{bmatrix}_{(1,2)}$$
$$=\begin{bmatrix}1&2\\6&3\end{bmatrix}$$

$$\begin{bmatrix}1&2\\6&3\end{bmatrix}\begin{bmatrix}1&2&3\\0&0&1\end{bmatrix}$$
$$=\begin{bmatrix}1&2&5\\6&12&21\end{bmatrix}$$

$$D(g\circ f)(\vec{0})=\begin{bmatrix}1&2&5\\6&12&21\end{bmatrix}$$

8.

$$\text{Let } f:\mathbb{R}^3\to\mathbb{R}^2 \text{ Let } g:\mathbb{R}^2\to\mathbb{R} \text{ be differentiable}$$
$$F:\mathbb{R}^2\to\mathbb{R} \text{ be defined by}$$
$$F:(x,y)=f(x,y,g(x,y))$$

a) find DF in term of partials of f and g
b) if F(x,y)=0 for all (x,y), find D_1g and D_2g in terms of the partials of f

$$D_1f(x,y,g(x,y)),D_3f(x,y,g(x,y)),D_1g(x,y),D_2f(x,y,g(x,y)),dotD_2g(x,y)$$
$$\text{briefly write}$$
$$D_1f,D_2f,D_3f,D_1g,D_2g$$

a) find DF in term of partials of f and g

$$F(x,y)=f(x,y,g(x,y)) \text{ so } F=f\circ h$$

$$DF=(D_1F,D_2F)$$
$$=(D_1f+D_3f\cdot D_1g,D_2f+D_3f\cdot D_2g)$$

b) if F(x,y)=0 for all (x,y), find $D_1 g$ and $D_2 g$ in terms of the partials of f

$$\text{if F(x,y)=0 for all (x,y) that means } DF = 0$$
$$(D_1 f + D_3 f \cdot D_1 g, D_2 f + D_3 f \cdot D_2 g) = (0,0)$$
$$D_1 f + D_3 f \cdot D_1 g = 0$$
$$D_1 g = -D_1 f / D_3 f$$

$$D_2 f + D_3 f \cdot D_2 g = 0$$
$$D_2 g = -D_2 f / D_3 f$$