Final Exam: Topics and Practice Problems

Exam 3 will focus on the concepts below. This means that you should be able to reproduce any proofs we have done in class, and use theorems in these topics to solve problems or prove new results.

1 Topics

- Eigenvectors and eigenvalues of a linear transformation or a matrix
- Independence of eigenvectors
- Characteristic polynomials
- Trace of a matrix
- Similar matrices
- Eigenvalues of symmetric matrices
- Open sets, interior, exterior, closed sets, boundary
- Limits and continuity of multivariable functions
- Directional derivative, partial derivative
- Total derivative
- Implications of differentiability
- Proving differentiability of a function
- Proving non-differentiability of a function
- Chain Rule for multivariable functions
- Geometric applications of derivative, level curves, direction of most rapid increase
- Inverse function theorem

2 Practice Problems

Once you are clear about all of the concepts above, solve all the homework problems (starting from the LAST HOMEWORK going back earlier). Not just look at their solutions, SOLVE IT AGAIN!

If you are done with all of the above, here are some extra problems:

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be

$$f(x, y) = (x^2 - y^2, 2xy).$$

a) Show that f is 1-1 on the set $A = \{(x, y) \mid x > 0\}$. [Hint: If f(x,y)=f(a,b), then $\|f(x, y)\| = \|f(a, b)\|$.]

b) If g is the inverse function, find $Dg(0,1)$.

2. Where is the function

$$f(x, y) = (xy, x^2 - y^2)$$

guaranteed to be locally invertible by inverse function theorem?

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (e^x \cos y, e^x \sin y)$$

Let $g : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$g(x, y) = (x^2 - y^2, xy + y^2, xy^2 + y)$$

(a) Show that f is locally injective everywhere.

(b) Compute $D(g \circ f^{-1})(0,1)$ where f^{-1} is a local inverse to f near $(0,\pi/2)$.

4. Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x) = \begin{cases} xy \sin \dfrac{x}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Show that the function $f(x, y)$ is differentiable at $(0,0)$.

5. Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{2xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0) \end{cases}$$

a) Find the Jacobian of f at $(0,0)$. **Remark:** You need to use the limit definition of partial derivatives.

b) Find the directional derivative $D_u f(0,0)$ where $u = (1,1)$ using the limit definition.

c) Show that $f(0,0)$ is not differentiable at $(0,0)$.

6. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, compute A^{200}.

7. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 3 \\ 3 & 9 \end{bmatrix}$, and $M = BAB^{-1}$. Find the eigenvalues and eigenvectors of M.

8. Apostol Section 4.10 pr 7

9. Apostol Section 4.10 pr 8

10. Apostol Section 8.3 pr 4a

11. Apostol Section 8.3 pr 8

12. Apostol Section 8.22 pr 15

13. Apostol Section 8.24 pr 3