Math 375

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(1) 2.8 Pr 19

- Determine one-to-one
    - Describe its range
    - Find inverse

$$T(x, y, z) = (x, x + y, x + y + z)$$

$$\text{Check one-to-one}$$
$$\text{Let } A = (x_1, y_1, z_1)\ B = (x_2, y_2, z_2)$$
$$\text{If it is one-to-one it must follow } T(A) = T(B) \text{ then } A = B$$

$$T(A) = T(B)$$
$$(x_1, x_1 + y_1, x_1 + y_1 + z_1) = (x_2, x_2 + y_2, x_2 + y_2 + z_2)$$

$$x_1 = x_2$$
$$x_1 + y_1 = x_2 + y_2$$
$$x_1 + y_1 + z_1 = x_2 + y_2 + z_2$$
$$\text{Solving the equation we get}$$
$$x_1 = x_2, y_1 = y_2, z_1 = z_2$$
$$\text{So } A = B$$
$$\text{Therefore it is one-to-one}$$

$$x(1, 1, 1) + y(0, 1, 1) + z(0, 0, 1) = (x, x + y, x + y + z)$$
$$(1, 1, 1), (0, 1, 1), (0, 0, 1) \text{ form a basis for range of } T$$
$$\text{The Range is } \mathbb{R}^3$$

$$T(x, y, z) = (u, v, w)$$
$$(x, x + y, x + y + z) = (u, v, w)$$
$$x = u, y = v - u, z = w - v$$
$$T^{-1}(u, v, w) = (x, y, z)$$
$$T^{-1}(u, v, w) = (u, v - u, w - v)$$

$$\text{Check } TT^{-1}(x, y, z) = (x, y, z)$$
$$T(x, y - x, z - y) = (x, x + y - x, z + y - x + z - y) = (x, y, z)$$
$$\text{Answer) It is one-to-one}$$
$$\text{Spanned by } (1, 1, 1), (0, 1, 1), (0, 0, 1) \text{ which is } \mathbb{R}^3$$
$$T^{-1}(u, v, w) = (u, v - u, w - v)$$

(2) 2.8 Pr 22

If S and T commute, prove that$(ST)^n = S^n T^n$ for all integers $n \geq 0$

It is trivial for $n = 0$ Since $T^0 = I$

for $n = 1$ it is true because $(ST)^1 = S^1 T^1$

Assume it is true for n prove for $n + 1$

For $n + 1$ $(ST)^{n+1} = (ST)^n (ST) = S^n T^n (ST)$

Since they commute $S^n T^n (ST) = S^n S T^n T = S^{n+1} T^{n+1}$

$(ST)^{n+1} = S^{n+1} T^{n+1}$ is true

Therefore statement $(ST)^n = S^n T^n$ for all integers $n \geq 0$ is true

(3) 2.8 Pr 24

If S and Tare invertible and commute, prove that their inverses also commute.

$ST = TS$ and S^{-1} T^{-1} exist

$$ST = TS$$
$$S^{-1}ST = S^{-1}TS$$
$$IT = S^{-1}TS$$
$$T = S^{-1}TS$$
$$T^{-1}T = T^{-1}S^{-1}TS$$
$$I = T^{-1}S^{-1}TS$$
$$IS^{-1} = T^{-1}S^{-1}TSS^{-1}$$
$$S^{-1} = T^{-1}S^{-1}TI$$
$$S^{-1}T^{-1} = T^{-1}S^{-1}TT^{-1}$$
$$S^{-1}T^{-1} = T^{-1}S^{-1}I$$
$$S^{-1}T^{-1} = T^{-1}S^{-1}$$

Therefore inverse is also commute

(4) 2.12 Pr 3

A linear transformation $T : V_2 \to V_2$ is defined as follows: Each vector (x, y) is reflected in the y-axis and then doubled in length to yield $T(x, y)$

Determine the matrix of T and T^2

(5) 2.12 Pr 8

(6) 2.16 Pr 1

(7) 2.16 Pr 3

(8) 2.16 Pr 4