CAAM 402/502 Spring 2013
Homework 4
Solutions

1. Let $f : V \to \mathbb{R}^n$ be a continuous function defined on the open set $V \subset \mathbb{R}^n$. Suppose that f is injective on V, has all the first order partial derivatives, and its Jacobian $J_f(x)$ satisfies $\det J_f(x) \neq 0$ for all $x \in V$. Prove that the inverse of f, call it $f^{-1} : f(V) \to V$ is continuous.

Proof. We will rely on the following theorem proved in class. Assuming the hypothesis above. Let $a \in V$ and $r > 0$ be sufficiently small such that $\overline{B}_a(r) \subset V$. If $f(a) \notin f(\partial B_a(r))$ then there exists $\delta > 0$ such that $B_\delta(f(a)) \subset f(B_r(a))$.

Now we are ready to prove that $f^{-1} : f(V) \to V$ is continuous. Let $\epsilon > 0$ and $b \in f(V)$ be arbitrary. Since f is injective there exists unique $a \in V$ such that $f(a) = b$. We assume that $\epsilon > 0$ is sufficiently small so that $\overline{B}_\epsilon(a) \subset V$. Now since f is injective, we have that $f(a) \notin f(\partial B_\epsilon(a))$. From the theorem in class, then there exists $\delta > 0$ such that $B_\delta(b) \subset f(B_\epsilon(f^{-1}(b)))$. In other words, if $y \in B_\delta(b)$ then $f^{-1}(y) \in B_\epsilon(f^{-1}(b))$, which is the definition for the continuity of f^{-1} at $b \in f(V)$. Since b is arbitrary, then we conclude that $f^{-1} : f(V) \to V$ is continuous. $\square$

2. Suppose that $f : V \to W$ is a bijection from $V = B_r(a) \subset \mathbb{R}^n$ to $W \subset \mathbb{R}^n$, and it is differentiable at a. Suppose also that the inverse $f^{-1} : W \to V$ is differentiable at $f(a)$. Prove that $\det J_f(a) \neq 0$.

Proof. We have that $f^{-1} \circ f = id$ on the set V. Since f is differentiable at a and f^{-1} is differentiable at $f(a)$, we can use the chain rule (pp. 471-472 in Lang) to obtain,

$$I = J_{id}(a) = J_{f^{-1}\circ f}(a) = J_{f^{-1}}(f(a))\, J_f(a).$$

Therefore $J_f(a)$ is invertible and hence $\det J_f(a) \neq 0$. $\square$

3. XVIII, 3.1 (Lang page 520).

Proof. The solution is from the book 'Problems and Solutions for Undergraduate Analysis' by Rami Shakarchi. Let $y \in f(U)$ and select x such that $f(x) = y$. By the inverse mapping theorem we know that f is locally C^1-invertible at x, so by definition there exists an open set U_1 such that $x \in U_1$ and $f(U_1)$ is open. Since $y \in f(U_1)$ we conclude that $f(U)$ is open. $\square$

4. XVIII, 3.2 (Lang page 520).

Proof. The solution is from the book 'Problems and Solutions for Undergraduate Analysis' by Rami Shakarchi. The Jacobian of f at a point (x, y) is

$$J_f(x, y) = \begin{pmatrix} e^x & e^y \\ e^x & -e^y \end{pmatrix}$$

whose determinant is $-2e^x e^y \neq 0$, so f is locally invertible around every point of R^2. Note that f is injective because if $f(x_1, y_1) = f(x_2, y_2)$, then

$$\begin{cases} e^{x_1} + e^{y_1} = e^{x_2} + e^{y_2}, \\ e^{x_1} - e^{y_1} = e^{x_2} - e^{y_2}, \end{cases}$$

so adding the two equations we get $x_1 = x_2$ and subtracting the two equations we see that $y_1 = y_2$. This shows that

$$f : R^2 \to f(R^2)$$

has a set inverse. Note that

$$f(R^2) = \{(x, y) \in R^2 : x > y\} = V$$

because $e^x + e^y > e^x - e^y$ and if $(a, b) \in V$, then

$$f(\log \frac{a+b}{2}, \log \frac{a-b}{2}) = (a, b)$$

From this analysis, we also see that the map $g : V \to R^2$ defined by

$$g(x, y) = (\log \frac{x+y}{2}, \log \frac{x-y}{2})$$

is a C^1-inverse for f because $g(f(x, y)) = (x, y)$ and all the partial derivatives of g exist and are continuous on V. $\square$

5. XVIII, 3.3 (Lang page 520). Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that $Df(x, y)$ is invertible for all $(x, y) \in \mathbb{R}^2$, that f is locally invertible at every point, but does not have an inverse defined on all of $\mathbb{R}^2$.

Proof. Given $f(x, y) = (f_1, f_2) = (e^x \cos y, e^x \sin y)$, Jacobian J_f can be computed as,

$$J_f(x, y) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \tag{1}$$

and $\text{Det}J_f(x, y) = e^{2x} \neq 0$ for any $(x, y) \in \mathbb{R}^2$. Hence J_f is invertible at every point, so that f is invertible at every point $(x, y) \in \mathbb{R}^2$ from inverse mapping theorem.

Observe that $f(x, y) = f(x, y+2\pi)$. i.e., f is not one to one. Hence f does not have an inverse defined on all of $\mathbb{R}^2$ $\square$

6. XVIII, 3.4 (Lang page 520). Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x^2 - y^2, 2xy)$. Determine the points of $\mathbb{R}^2$ at which f is locally invertible, and determine whether f has an inverse defined on all of $\mathbb{R}^2$.

Proof. Given $f(x, y) = (f_1, f_2) = (x^2 - y^2, 2xy)$, Jacobian J_f can be computed as,

$$J_f(x, y) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} \tag{2}$$

and $\text{Det}J_f(x, y) = 4(x^2 + y^2) \neq 0$ for any $(x, y) \neq (0, 0)$. Hence J_f is invertible at every point $(x, y) \in \mathbb{R}^2 \backslash \{0\}$, so that f is invertible at every point $(x, y) \in \mathbb{R}^2 \backslash \{0\}$ from inverse mapping theorem.

At $(0, 0)$ in any neighborhood $B_r((0, 0))$, if $(x, y) \in B_r((0, 0))$, then $(-x, -y) \in B_r((0, 0))$ and $f(x, y) = f(-x, -y)$. Hence f is not locally invertible at $(0, 0)$ and thus does not have a global inverse on all of $\mathbb{R}^2$. $\square$

Note: Converse of inverse mapping theorem is not true. i.e., jacobian is not invertible does not imply that local inverse does not exist. For example consider $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^3$. We can observe that $f'(x) = 3x^2$ and $f'(0) = 0$, that is derivative is not invertible at $x = 0$ but this function has a global inverse defined as $f^{-1}(x) = x^{1/3}$.