MATH 375

- Prove by contradiction that there are infinitely many primes.

$$\text{Let's assume there are finite primeb numbers.}$$
$$\text{So there are total n primes.}$$
$$p_1 \dots p_n$$
$$\text{Let A be the product of all prime numbers}$$
$$A = p_1 \times p_2 \cdots \times p_n$$
$$\text{and there is number B which is}$$
$$B = A + 1$$
$$\text{Since there are only n prime numbers the B should not be a prime number}$$
$$\text{Therefore the B should be divided by one of the prime nubers in } p_1 \dots p_2$$
$$\text{However there is no prime numbers that can divide B}$$
$$\text{Since A is the product of all prime numbers A can be divided by any prime numbers}$$
$$\text{But 1 cannot be divided by any primenumber from } p_1 \dots p_n$$
$$B \div p_k = \frac{p_1 \times p_2 \dots p_n}{p_k} + \frac{1}{p_k}$$
$$\text{Therefore B can be divided by 1 and itself only and by the definition of prime number}$$
$$\text{B is a prime number.}$$
$$\text{The assumption that there are finite prime number is wrong, therefore there are infinite prime numbers}$$

- Prove the following formula by induction: 1+3+5+ ⋯ +(2n−1)=n^2

$$\text{For } n = 1$$
$$(2 \times 1 - 1) = 1^2 \text{ is correct}$$
$$\text{Let's assume that the equation is true for n}$$
$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$
$$\text{And prove the statement is true for n+1}$$
$$1 + 3 + 5 + \dots (2(n+1) - 1) = (n+1)^2$$
$$1 + 3 + 5 + \dots (2(n+1) - 1)$$
$$= 1 + 3 + 5 + \cdots + (2n - 1) + (2(n+1) - 1)$$
$$= n^2 + (2(n+1) - 1) \text{ (by the assumption)}$$
$$= n^2 + 2n + 1$$
$$n^2 + 2n + 1\text{is equal to } (n+1)^2$$
$$\text{by the induction the formula is correct}$$

- Prove the following formula by induction:

$$1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \cdots + n^3$$

Assume n is a positive integer

For $n = 1$

$$(1-1)^3 < \frac{1^4}{4} < 1^3 \text{ is true}$$

Now we assume that the equation is true for n

$$1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \cdots + n^3$$

And prove the equation is true for n+1

$$1^3 + 2^3 + \cdots + (n-1)^3 + ((n+1)-1)^3 < \frac{(n+1)^4}{4} < 1^3 + 2^3 + \cdots + n^3 + (n+1)^3$$

making the equation simple

$$1^3 + 2^3 + \cdots + (n-1)^3 + ((n+1)-1)^3 = 1^3 + 2^3 + \cdots + (n-1)^3 + n^3$$

$$\frac{(n+1)^4}{4} = \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4}$$

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = 1^3 + 2^3 + \cdots + n^3 + n^3 + 3n^2 + 3n + 1$$

we have to prove this inequality equation

$$1^3 + 2^3 + \cdots + (n-1)^3 + n^3 < \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4} < 1^3 + 2^3 + \cdots + n^3 + n^3 + 3n^2 + 3n + 1$$

first the left part

$1^3 + 2^3 + \cdots + (n-1)^3 + n^3 < \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4}$ we can remove n^3 from the both side

$$1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4} - n^3 = \frac{n^4 + 6n^2 + 4n + 1}{4}$$

$$1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4} + \frac{6n^2 + 4n + 1}{4}$$

$\frac{6n^2 + 4n + 1}{4}$ is positive since n is the positive integer and

by the assumption $1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4}$ is also true

Therefore $1^3 + 2^3 + \cdots + (n-1)^3 + n^3 < \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4}$ is true

Now we are going to prove the right part of the equation

$$\frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4} < 1^3 + 2^3 + \cdots + n^3 + n^3 + 3n^2 + 3n + 1$$

remove $n^3 + 1.5n^2 + n + 0.25$ from the both side

$$\frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4} - (n^3 + 1.5n^2 + n + 0.25) = \frac{n^4}{4}$$

$$1^3 + 2^3 + \cdots + n^3 + n^3 + 3n^2 + 3n + 1 - (n^3 + 1.5n^2 + n + 0.25) = 1^3 + 2^3 + \cdots + n^3 + 1.5n^2 + 2n + 0.75$$

by the assumption $\frac{n^4}{4} < 1^3 + 2^3 + \cdots + n^3$ is true

Since n is positve integer $1.5n^2 + 2n + 0.75$ is also positive

Therefore $\frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4} < 1^3 + 2^3 + \cdots + n^3 + n^3 + 3n^2 + 3n + 1$ is true

Finally $1^3 + 2^3 + \cdots + (n-1)^3 + ((n+1)-1)^3 < \frac{(n+1)^4}{4} < 1^3 + 2^3 + \cdots + n^3 + (n+1)^3$ is true

by the induction the equation$1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4} < 1^3 + 2^3 + \cdots + n^3$ is true

- $$\text{Let } P(n) \text{ denote the following statement: } 1+2+\cdots+n=\frac{1}{8}(2n+1)^2$$
    - *(a) Prove that if P(k) is true for an integer k then P(k+1)is also true.*

$$\text{Since we are assumming P(k) is true}$$
$$1+2+\cdots+k=\frac{1}{8}(2k+1)^2 \text{ is true}$$

$$P(k+1)\rightarrow 1+2+\cdots+k+k+1=\frac{1}{8}(2(k+1)+1)^2$$
$$\frac{1}{8}(2(k+1)+1)^2=\frac{1}{8}(2k+3)^2=\frac{1}{8}(4k^2+12k+9)=\frac{1}{8}(2k+1)^2+\frac{1}{8}(8k+8)$$
$$\text{by the assumption } \frac{1}{8}(2k+1)^2=1+2+\cdots+k$$

$$\text{Therefore } \frac{1}{8}(2(k+1)+1)^2=1+2+\cdots+k+\frac{1}{8}(8k+8)=1+2+\cdots+k+k+1$$
$$\text{In conclusion } 1+2+\cdots+k+k+1=\frac{1}{8}(2(k+1)+1)^2 \text{ is true}$$

    - *(b) Criticize the statement: "By induction it follows that P(n) is true for all n."*

$$\text{However the statement ”By induction it follows that P(n) is true for all n.” is not true}$$
$$\text{The first step of proving by induction is showing the base case is correct.}$$
$$\text{In this case we have to show that P(1) is true as a base case.}$$
$$\text{However } P(1)\rightarrow 1=\frac{1}{8}(2\times 1+1)^2 \text{ is false.}$$
$$\text{Since the base case is not true we can't say that ”By induction it follows that P(n) is true for all n.”}$$

    - *(c) Amend P(k) by changing equality to an inequality that is true for all positive integer n.*

Changing equality

$$1+2+\cdots+n=\frac{1}{8}(2n+1)^2$$

to inequality

$$1+2+\cdots+n<\frac{1}{8}(2n+1)^2$$

Proving amended P(k) is true for all positive integer n by induction

Step.1

Check base case is true

for $n=1$

$$1<\frac{1}{8}(2\times 1+1)^2=\frac{9}{8}\text{ is true}$$

Step.2

Assume that $n=k$ is true and prove $n=k+1$is also true

By the assumption $1+2+\cdots+k<\frac{1}{8}(2k+1)^2$ is true

For $n=k+1$

$$1+2+\cdots+k+k+1<\frac{1}{8}(2(k+1)+1)^2$$

$$1+2+\cdots+k+k+1<\frac{1}{8}(4k^2+12k+9)=\frac{1}{8}(2k+1)^2+(k+1)$$

we can remove $k+1$ from the both side of the inequality equation

$1+2+\cdots+k<\frac{1}{8}(2k+1)^2$ by the assumption the inequality is true

Therefore by the induction Amended $P(k)$ is true for all positive integer n

- The Fibonacci numbers are given by the recursive formula
  $a_0=1$, $a_1=1$ and $a_{n+1}=a_n+a_{n-1}$ for $n\geq 1$
  Prove that for all $n\geq 1$
  $$a_n<(\frac{1+\sqrt{5}}{2})^n$$

In this case we are going to use strong induction

Step. 1

Showing the base case is true

$n = 1$

$a_1 = 1$ and $(\frac{1+\sqrt{5}}{2})^1 = \frac{1+\sqrt{5}}{2}$

Since $\frac{1+\sqrt{5}}{2}$ is approximately 1.618

Therefore the base case $a_1 < (\frac{1+\sqrt{5}}{2})^1$ is true

Step. 2

$P(n)$ stands for $a_n < (\frac{1+\sqrt{5}}{2})^n$

Assume that the $P(1), P(2), \ldots P(k)$ is true and prove $P(k+1)$ is also true

We are assuming $a_1 < (\frac{1+\sqrt{5}}{2})^1 \ldots a_k < (\frac{1+\sqrt{5}}{2})^k$ are true

So $P(k+1)$ is

$$a_{k+1} < (\frac{1+\sqrt{5}}{2})^{k+1}$$

$$a_{k+1} = a_k + a_{k-1} \quad (a_{k+1} = a_k + a_{k-1})$$

$$(\frac{1+\sqrt{5}}{2})^{k+1} = (\frac{1+\sqrt{5}}{2})^k \times (\frac{1+\sqrt{5}}{2})$$

$$P(k+1) \rightarrow a_k + a_{k-1} < (\frac{1+\sqrt{5}}{2})^k \times (\frac{1+\sqrt{5}}{2})$$

By the assumption

$a_k < (\frac{1+\sqrt{5}}{2})^k$ and $a_{k-1} < (\frac{1+\sqrt{5}}{2})^{k-1}$

Therefore $a_k + a_{k-1} < (\frac{1+\sqrt{5}}{2})^k + (\frac{1+\sqrt{5}}{2})^{k-1}$

If the inequality $(\frac{1+\sqrt{5}}{2})^k + (\frac{1+\sqrt{5}}{2})^{k-1} \leq (\frac{1+\sqrt{5}}{2})^k \times (\frac{1+\sqrt{5}}{2})$ is true

$a_k + a_{k-1} < (\frac{1+\sqrt{5}}{2})^k \times (\frac{1+\sqrt{5}}{2})$ is true

$$(\frac{1+\sqrt{5}}{2})^k + (\frac{1+\sqrt{5}}{2})^{k-1} \leq (\frac{1+\sqrt{5}}{2})^k \times (\frac{1+\sqrt{5}}{2}) \text{ divide the both side by } (\frac{1+\sqrt{5}}{2})^k$$

$$1 + (\frac{1+\sqrt{5}}{2})^{-1} \leq (\frac{1+\sqrt{5}}{2})$$

$$1 + (\frac{2}{1+\sqrt{5}}) = (\frac{3+\sqrt{5}}{1+\sqrt{5}}) \leq (\frac{1+\sqrt{5}}{2})$$

$$(\frac{3+\sqrt{5}}{1+\sqrt{5}}) \leq (\frac{1+\sqrt{5}}{2})$$

$$(\frac{6+2\sqrt{5}}{2+2\sqrt{5}}) \leq (\frac{(1+\sqrt{5})^2}{2+2\sqrt{5}})$$

$$6+2\sqrt{5} < (1+\sqrt{5})^2 \leq 1+2\sqrt{5}+5$$

Sicne the both sides are equal the inequality equation is true

$$a_{k+1} < (\frac{1+\sqrt{5}}{2})^{k+1} \text{ is true}$$

Thus for all $n \geq 1$

$$a_n < (\frac{1+\sqrt{5}}{2})^n$$

- (A curios vector space) Let $V = (0, \infty)$ be the set of positive real numbers. Define "addition ◈" on V as follows: x ◈ $y = x \times y$

  where $\times$ is the usual multiplication of real numbers.

  Define "Scalar multiplication ◉" as c ◉ $x = x^c$ where $x \in V$ and $c \in R$

  Prove that V is a real vector space with respect to addition and scalar multiplication defined above

To prove that the V is a vector space we have to prove that V satisfy 8 axioms

1) $x \diamond (y \diamond z) = (x \diamond y) \diamond z$

$$x \diamond (y \diamond z) = xyz = (x \diamond y) \diamond z = xyz$$

2) $x \diamond y = y \diamond x$

$$x \diamond y = xy = y \diamond x = yz$$

3) $x \diamond \vec{0} = \vec{0} \diamond x = x$

$$1 \times x = x \times 1 = x$$

4) Existence of negative vector $x \diamond y = \vec{0}$

$$y = \frac{1}{x},\ y \diamond x = 1$$

5) $(a \odot b) \odot x = a \odot (b \odot x)$

$$(x^b)^a = (x^b)^a$$

6) $(a + b) \odot x = a \odot x \diamond b \odot x$

$$x^{a+b} = x^a \times x^b = x^{a+b}$$

7) $a \odot (x \diamond y) = a \odot x \diamond a \odot y$

$$(xy)^a = x^a \times y^a = (xy)^a = x^a$$

8) $1 \odot x = x$

$$x^1 = x$$

Since all of the 8 axioms are satisfied V is a real vector space