Exam 1: Topics and Practice Problems

Exam 1 will focus on the concepts below. This means that you should be able to reproduce any proofs we have done in class, and use theorems in these topics to solve problems or prove new results.

1 Topics

- Proof by induction
- Proof by contradiction
- Definition and properties of a vector space (only ($\mathbb{R}$) ones)
- Subspaces, and showing a subset is a subspace
- Dependent and independent sets, various different definitions
- Basis and dimension, showing that a set is a basis, and finding the dimension of a vector space
- Components of vectors with respect to a given basis
- Definition of an inner product, and inner product space. (Only real ($\mathbb{R}$) ones)
- Cauchy-Schwartz inequality (not the proof)
- Angles and norms for inner product spaces
- Orthogonality, orthonormality
- Using Gram-Schmidt to obtain orthonormal bases
- Consequences of orthogonality in terms of computations
- Orthogonal projections and orthogonal complements
- Orthogonal decomposition and Pythagorean formula
- Best approximation in subspaces
- Definition of a linear transformation, finding the Kernel and Range
- Rank-Nullity Theorem
- Algebraic Operations on linear transformations, vector space of linear transformations

These topics are contained through Section 2.5 in the book.

2 Practice Problems

Once you are clear about all of the concepts above, solve all the homework problems. Not just look at their solutions, solve it again!

Here are some extra problems:

1. Prove that $5^{2n+1}+2^{2n+1}$ is divisible by 7 for all integers $n \geq 0$.

2. Prove that $1^3+2^3+\ldots+n^3 = \frac{1}{4}n^2(n+1)^2$

3. Is it possible that vectors v_1, v_2, v_3 are linearly dependent, but the vectors $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$ and $w_3 = v_3 + v_1$ are linearly independent?

4. Let $W \subset \mathbb{R}^4$ be the set of all vectors of the form $\begin{pmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{pmatrix}$

Show that W is a subspace of $\mathbb{R}^4$, and find a basis for W.

5. Determine whether the following subsets of $\mathbb{R}^3$ are linearly independent or not.

a)

$$\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\}$$

b)

$$\{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -7 \\ 5 \\ 4 \end{pmatrix}\}$$

c)

$$\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$$

6. Let $B = \{v_, \ldots, v_n\}$ be a linearly independent set in $\mathbb{R}^n$. Explain why B must be a basis for $\mathbb{R}^n$.

7. Let V, W be vector spaces, and $T : V \to W$ be a linear transformation. Let $S = \{v_1, \ldots, v_k\}$ be linearly dependent set of vectors. Show that $S' = \{T(v_1), \ldots, T(v_k)\}$ is dependent in W.