Math 375

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(1) 4.4 Pr 4

$$\text{If } T: V \to V \text{ has the property that } T^2 \text{ has a nonnegative eigenvalue } \lambda^2$$
$$\text{Prove at least one of } \lambda \text{ or } -\lambda \text{ is an eigenvalue for } T.$$
$$T^2(x) = \lambda^2 x$$
$$T^2(x) - \lambda^2 x = 0$$
$$(T^2 - \lambda^2 I)x = 0$$

$$\det\left(T^2 - \lambda^2 I\right) = 0$$
$$\det\left(T^2 - \lambda^2 I\right) = \det\left(T - \lambda I\right)\det\left(T + \lambda I\right)$$

$$\det\left(T - \lambda I\right) \text{ is zero or } \det\left(T + \lambda I\right) \text{ is zero}$$
$$\text{Therefore at least one of } \lambda \text{ or } -\lambda \text{ is an eigenvalue for } T.$$

(2) 4.4 Pr 11

$$\text{Assume that a linear transformation } T \text{ has two eigenvectors } x \text{ and } y \text{ belonging to distinct}$$
$$\text{eigenvalues } i \text{ and } p. \text{ If } ax + by \text{ is an eigenvector of } T \text{ prove that } a = 0 \text{ or } b = 0$$
$$T(x) = ix \quad T(y) = py$$
$$\text{If } ax + by \text{ is an eigenvector } T(ax + by) = \lambda(ax + by)$$
$$aT(x) + bT(y) = \lambda ax + \lambda by$$

$$aT(x) + bT(y) = aix + bpy$$

$$\lambda ax + \lambda by = aix + bpy$$
$$(\lambda - i)ax + (\lambda - p)by = 0$$
$$\text{Since } x \text{ and } y \text{ are linearly independent}$$
$$(\lambda - i)a = 0 \ \ (\lambda - p)b = 0$$
$$\text{if } a \neq 0 \text{ and } b \neq 0$$
$$\lambda = i = p$$
$$\text{However the problem said } i \text{ and } p \text{ are distinct eigenvalues}$$
$$\text{Therefore } a \neq 0 \text{ and } b \neq 0 \text{ is false}$$

$$\text{It proves that } a = 0 \text{ or } b = 0$$

(3) 4.4 Pr 12

$$\text{Let } T: S \to V \text{ be a llinear transformation such that every nonzero element of S is an eigenvector}$$
$$\text{Prove that there exist a scalar } c \text{ such that } T(x) = cx$$
$$\text{In other words the only transformation with this property is a scalar times the identity}$$

$$\text{let } x \text{ is an vector in } S$$
$$y \text{ is also a vector in } S \text{ which is dependent with } x$$
$$y = kx$$
$$T(y) = kT(x) = kcx = cy \text{ therefore } c \text{ exist}$$

$$\text{if } x \text{ and } y \text{ are independent}$$
$$\text{and they are belonging to distinct eigenvalues, the previous problem showed that}$$
$$x + y \text{ can't be eigenvector}$$
$$\text{It means for } ax + by \text{ which } a \neq 0 \text{ and } b \neq 0 \text{ to be eigenvector}$$
$$\text{belonging eigenvalues should be not distinct}$$

$$x \text{ and } y \text{ are independent and having eigenvalues that aren't distinct}$$
$$T(x) = \lambda x$$
$$T(y) = \lambda y$$

$$\text{But also for when } x \text{ and } y \text{ are dependent}$$

$$T(x) = \lambda x$$
$$T(y) = \lambda y$$

$$\text{that means for every single vector } x \in S$$
$$T(x) = \lambda x$$
$$\text{It means } T \text{ must be scalar times identity}$$

$$\text{So } T \text{ is scalar times identity that has every nonzero elements as eigenvector}$$

(4) 4.8 Pr 1

$$\text{Determine the eigenvalues and eigenvectors of each of the matrices}$$
$$\text{for each eigenvalue } \lambda \text{ compute the dimension of the eigenspace } E(\lambda)$$
$$\text{a) } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{let } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\text{eigenvalue is}$$
$$\det(A - \lambda I) = 0$$
$$\det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = 0$$
$$(1-\lambda)^2 = 0$$

$$\text{eigenvalue is } 1, 1$$

$$\text{corresponding eigenvector is}$$
$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 1-1 & 0 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1, x_2 \in \mathbb{R} \text{ except } x_1 = x_2 = 0$$

$$x = (x_1, x_2)$$

$$\text{The dimension of eigenspace is } 2$$

$$\text{b) } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\text{let } A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\text{eigenvalue is}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)^2 = 0$$

$$\text{eigenvalue is } 1, 1$$

$$\text{corresponding eigenvector is}$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-1 & 1 \\ 0 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 \in \mathbb{R}, x_2 = 0$$

$$x = (x_1, 0), x_1 \neq 0$$

$$\text{The dimension of eigenspace is 1}$$

$$\text{c) } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\text{let } A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\text{eigenvalue is}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)^2 = 0$$

$$\text{eigenvalue is } 1, 1$$

$$\text{corresponding eigenvector is}$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-1 & 0 \\ 1 & 1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0, x_2 \in \mathbb{R}$$

$$x = (0, x_2), x_2 \neq 0$$

$$\text{The dimension of eigenspace is 1}$$

$$d)\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{let } A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\text{eigenvalue is}$$
$$\det(A - \lambda I) = 0$$
$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = 0$$
$$(1-\lambda)^2 - 1 = 0$$
$$-2\lambda + \lambda^2 = 0$$
$$\lambda(\lambda - 2)$$

$$\text{eigenvalue is } 0, 2$$

$$\text{corresponding eigenvector for 0 is}$$
$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 1-0 & 1 \\ 1 & 1-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 = -x_2 x_1 \in \mathbb{R}$$

$$x = (x_1, -x_1), x_1 \neq 0$$

$$\text{The dimension of eigenspace is 1}$$

$$\text{corresponding eigenvector for 2 is}$$
$$(A - \lambda I)x = 0$$
$$\begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 = x_2 x_1 \in \mathbb{R}$$

$$x = (x_1, x_1), x_1 \neq 0$$

$$\text{The dimension of eigenspace is 1}$$

$$\text{Therefore}$$
$$\lambda = 0, \dim(E(\lambda)) = 1$$
$$\lambda = 2, \dim(E(\lambda)) = 1$$

(5) 4.8 Pr 7

$$\text{Calculate the eigenvalues and eigenvectors of each of the following matrices}$$
$$\text{Also, compute the dimension of the eigenspace } E(\lambda) \text{ for each eigenvalue } \lambda$$

$$\text{a)} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix}$$

$$\text{let } A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -7 & 1 \end{bmatrix}$$

eigenvalue is

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 0 & 0 \\ -3 & 1-\lambda & 0 \\ 4 & -7 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)^3 = 0$$

eigenvalue is $1, 1, 1$

corresponding eigenspace is

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ -3 & 1-\lambda & 0 \\ 4 & -7 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 4 & -7 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0, 4x_1 = 7x_2$$

Therefore $x_1 = x_2 = 0$ and $x_3 \in \mathbb{R}$

$$x = (0, 0, x_3), x_3 \neq 0$$

The dimension of eigenspace is 1

$$\text{b) } \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix}$$

$$\text{let } A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix}$$

$$\text{eigenvalue is}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 3 \\ 3 & 3 & 20-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - 24\lambda^2 + 65\lambda - 42 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 21)$$

$$\text{eigenvalue is } 1, 2, 21$$

$$\text{corresponding eigenspace for 1 is}$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 2-1 & 1 & 3 \\ 1 & 2-1 & 3 \\ 3 & 3 & 20-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 = 0, x_1 = -x_2$$

$$x = (x_1, -x_1, 0) x_1 \neq 0$$

$$\text{Therefore dimension of eigenspcae is 1}$$

$$\text{corresponding eigenspace for 2 is}$$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 2-2 & 1 & 3 \\ 1 & 2-2 & 3 \\ 3 & 3 & 20-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 3 & 3 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = x_2, x_1 = -3x_3$$

$$x = (3x_1, 3x_1, x_1) x_1 \neq 0$$

$$\text{Therefore dimension of eigenspcae is 1}$$

corresponding eigenspace for 21 is

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 2-21 & 1 & 3 \\ 1 & 2-21 & 3 \\ 3 & 3 & 20-21 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -19 & 1 & 3 \\ 1 & -19 & 3 \\ 3 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1=x_2, 6x_1=x_3$$

$$x=(x_1,x_1,6x_1)x_1\neq 0$$

Therefore dimension of eigenspcae is 1

Therefore

$$\lambda=1, \dim(E(\lambda))=1$$
$$\lambda=2, \dim(E(\lambda))=1$$
$$\lambda=21, \dim(E(\lambda))=1$$

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c) $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

let $A=\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

eigenvalue is

$$\det(A-\lambda I)=0$$

$$\det\begin{bmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{bmatrix}=0$$

$$\lambda^3-5\lambda^2+8\lambda-4=0$$
$$(\lambda-1)(\lambda-2)(\lambda-2)$$

eigenvalue is $1,2,2$

corresponding eigenspace for 1 is

$$(A-\lambda I)x=0$$

$$\begin{bmatrix} 5-1 & -6 & -6 \\ -1 & 4-1 & 2 \\ 3 & -6 & -4-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3=-3x_2, x_1=-3x_2$$

$$x=(3x_1,-x_1,3x_1)x_1\neq 0$$

Therefore dimension of eigenspcae is 1

corresponding eigenspace for 2 is

$$(A-\lambda I)x=0$$

$$\begin{bmatrix}5-2 & -6 & -6\\ -1 & 4-2 & 2\\ 3 & -6 & -4-2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}3 & -6 & -6\\ -1 & 2 & 2\\ 3 & -6 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1=2x_2+2x_3$$

$$x=(2x_2+2x_3,x_2,x_3)x_1$$
$$=x_2(2,1,0)+x_3(2,0,1)\text{ except for }x_2=x_3=0$$

Therefore dimension of eigenspcae is 2

(6) 4.8 Pr 14

Prove each of the following statements about the trace

let A_{ij} and B_{ij} and $(A+B)_{ij}$ be the i_{th}, j_{th} row and column of each matrices

a) $tr(A+B)=trA+trB$

$$(A+B)_{ij}=A_{ij}+B_{ij}$$
$$\Sigma(A+B)_{ii}=\Sigma A_{ii}+\Sigma B_{ii}$$

$$\Sigma(A+B)_{ii}=tr(A+B)$$
$$\Sigma(A)_{ii}=tr(A)$$
$$\Sigma(B)_{ii}=tr(B)$$

Therefore $tr(A+B)=trA+trB$

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b) $tr(CA)=Ctr(A)$

$$(CA)_{ij}=CA_{ij}$$
$$\Sigma(CA)_{ii}=\Sigma CA_{ij}=C\Sigma A_{ij}$$

Therefore $tr(CA)=Ctr(A)$

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c) $tr(AB)=tr(BA)$

$n\times n$ matrix

$$AB_{ii}=\Sigma_{k=1}^{n}A_{ik}B_{ki}$$
$$tr(AB)=\Sigma(AB)_{ii}=\Sigma_{i=1}^{n}\Sigma_{k=1}^{n}A_{ik}B_{ki}$$
$$=\Sigma_{k=1}^{n}\Sigma_{i=1}^{n}A_{ik}B_{ki}$$
$$=\Sigma_{k=1}^{n}\Sigma_{i=1}^{n}B_{ki}A_{ik}$$
$$=tr(BA)$$

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d) $trA^t = trA$.

$$A^t_{ii} = A_{ii}$$
Therefore $trA^t = trA$.

(7) 8.3 Pr 3

In each of the following, let S be the set of all points (x, y, z) in 3-space satisfying the given inequalities and determine whether or not S is open.

a) $z^2 - x^2 - y^2 - 1 > 0$

This is inside of two-sheeted hyperboloid

From every point $x \in S$ that satifsy the inequality $\exists r,\ B(x,r) \subseteq S$ satisfying the inequality(inside the hyperboloid)

open

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b) $|x| < 1, |y| < 1, |z| < 1$

$|x| < 1, |y| < 1, |z| < 1$ This makes inside of cube

For every point $x \in S$ that satifsy the inequality $\exists r,\ B(x,r) \subseteq S$ satisfying the inequality(inside the cube)

open

---

c) $x + y + z < 1$

$x + y + z = 1$ makes a plane

let d be the distance from (x_1, y_1, z_1) that $x_1 + y_1 + z_1 < 1$ to $x + y + z = 1$

There exist r that $d > r$

For every point $x \in S$ that satifsy the inequality $\exists r,\ B(x,r) \subseteq S$

open

---

d) $|x| \leq 1, |y| < 1, |z| < 1$

$a = (1, 0, 0) \in S$ satisfy the inequalities

but there does not exist r that $B(a,r) \subseteq S$

closed

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e) $x + y + z < 1$ and $x > 0, y > 0, z > 0$.

This is same as c but it is in first octanct

For every point $x \in S$ that satifsy the inequality $\exists r,\ B(x,r) \subseteq S$

open

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f) $x^2 + 4y^2 + 4z^2 - 2x + 16y + 40z + 113 < 0$

This is inside of spheroid

For every point $x \in S$ that satifsy the inequality $\exists r,\ B(x,r) \subseteq S$

open

(8) 8.3 Pr 5

Prove the following properties of open sets in $\mathbb{R}^n$

b) $\mathbb{R}^n$ is open

choosing any points in $x \in \mathbb{R}^n$
$\exists r,$ such that $B(x,r) \subseteq \mathbb{R}^n$

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a) The empty set $\emptyset$ is open

If $\emptyset$ is open its complement is closed
$\mathbb{R}^n - \emptyset = \mathbb{R}^n$
$\emptyset^c = \mathbb{R}^n$ but $\mathbb{R}^n$ is open we proved it in b
Therefore empty set $\emptyset$ is open

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c) The union of any collection of open sets is open.

$\forall x \in$ open set x is interior point
Therefore $\forall x \in$ union of open set we can find at least one open set A
that $x \in A$
Therefore $\forall x\ \exists r > 0$ that $B(x,r) \subseteq A$
and $A \subseteq$ union of open sets

By the definition union of open sets is open

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d) The intersection of a finite collection of open sets is open

let A be the intersection of $A_1 \ldots A_n$

$\forall x \in A$ it also means $x \in A_1 \ldots A_n$ since $A_1 \ldots A_n$ are open
$\exists B(x,r) \subseteq A_1 \ldots A_n$
Therefore $\forall x \in A, \exists r\ \ B(x,r) \subseteq A$

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e) Give an example to show that the intersection of an infinite collection of open sets is not necessarily open

$\cap_{n=1}^{\infty}(0,1/n)$ can be an counter ex