Grade Table.

Question	Points	Score	
1	15	15	
2	10	10	
3	10	10	
4	12	12	
5	6	6	
6	10	8	-2
7	12	6	-6
Total:	75	67	

1. (15 points) Solve the system

$$\begin{bmatrix} 1 & 0 & 1 \\ -4 & 1 & -1 \\ 6 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

by finding A^{-1} where $A = \begin{bmatrix} 1 & 0 & 1 \\ -4 & 1 & -1 \\ 6 & -2 & 1 \end{bmatrix}$.

$A\vec{x} = C$

$A^{-1}A\vec{x} = A^{-1}C$

$IX = A^{-1}C$

find A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -4 & 1 & -1 & 0 & 1 & 0 \\ 6 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 6 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & -2 & -5 & -6 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & -2 & -5 & -3 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & -2 & -1 \\ -2 & -5 & -3 \\ 2 & 2 & 1 \end{bmatrix}$$

$$A^{-1}C = \begin{bmatrix} -1 & -2 & -1 \\ -2 & -5 & -3 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$$

$$= \begin{matrix} -1-14-1 \\ -2-35-3 \\ 2+14+1 \end{matrix} = \begin{matrix} -16 \\ -40 \\ +17 \end{matrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -16 \\ -40 \\ 17 \end{bmatrix}$$

$x = -16$
$y = -40$
$z = 17$

Side work:

1	0	1	-16
-4	1	-1	-40
6	-2	1	17

$-16+17$

$64-33-17$

$64-40-17$

$-96+80+17$

-16

1	0	1	-1	-2	-1
-4	1	-1	-2	-4	-3
6	-2	1	2	2	1

$8-4-2$

2. (10 points) Let A and B be $n \times n$ matrices. Prove that if either $Ker\, A$ or $Ker\, B$ is non-trivial, then $Ker\, AB$ is also non-trivial.

Since $\det AB = \det A \det B$ and ~~if det A = 0 then A is non-trivial~~

if one of $\det A$ or $\det B$ is zero that makes $\det AB$ to zero
since $\det AB = \det A \det B$

if determinant is 0 then it is non-trivial.

3. **(10 points)** Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, with the basis

$$\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

Find the matrix representative with respect to the above basis for the linear transformation T *transposition*. i.e $T: M_{2\times 2}\to M_{2\times 2}$ is given by $T(A)=A^T$. (Hint: M_{2x2} is a 4-dimensional vector space. So, this linear transformation should be represented by a 4×4 matrix.)

$\begin{matrix}1&0\\0&0\end{matrix} \to \begin{matrix}1&0\\0&0\end{matrix}$ $\qquad (1,0)\ (0,1)$ $\qquad \begin{matrix}0&0\\1&0\end{matrix}\ \begin{matrix}a&b\\c&d\end{matrix}$

$\begin{matrix}0&1\\0&0\end{matrix} \to \begin{matrix}0&0\\1&0\end{matrix}$ $\qquad \begin{matrix}0&1\\0&0\end{matrix}$

$\begin{matrix}0&0\\1&0\end{matrix} \to \begin{matrix}0&1\\0&0\end{matrix}$ $\qquad \begin{vmatrix}a\\b\\c\\d\end{vmatrix}$ $\qquad a\ b\ c\ d$

$\begin{matrix}0&0\\0&1\end{matrix} \to \begin{matrix}0&0\\0&0\end{matrix}$

$\cancel{[T]\begin{bmatrix}1&0\\0&0\end{bmatrix}} =$

$T\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\qquad \begin{pmatrix}1&0\\0&0\end{pmatrix}=1\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+0\begin{pmatrix}0&1\\0&0\end{pmatrix}+0\begin{pmatrix}0&0\\1&0\end{pmatrix}+0\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$T\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $\qquad \begin{pmatrix}0&0\\1&0\end{pmatrix}=0\begin{pmatrix}1&0\\0&0\end{pmatrix}+0\begin{pmatrix}0&1\\0&0\end{pmatrix}+1\begin{pmatrix}0&0\\1&0\end{pmatrix}+0\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$T\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\qquad \begin{pmatrix}0&1\\0&0\end{pmatrix}=0\begin{pmatrix}1&0\\0&0\end{pmatrix}+1\begin{pmatrix}0&1\\0&0\end{pmatrix}+0\begin{pmatrix}0&0\\1&0\end{pmatrix}+0\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$T\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ $\qquad \begin{pmatrix}0&0\\0&1\end{pmatrix}=0\begin{pmatrix}1&0\\0&0\end{pmatrix}+0\begin{pmatrix}0&1\\0&0\end{pmatrix}+0\begin{pmatrix}0&0\\1&0\end{pmatrix}+1\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$[T]=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

4. (12 points) If you know that $det \begin{bmatrix} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{bmatrix} = 4$, what is the determinant of

$$B = \begin{bmatrix} \text{3Row 1+2Row 2} \\ \text{Row 2+ 2Row 3} \\ \text{Row 3 +Row 1} \end{bmatrix}$$

let

Row1 = a

Row2 = b

Row3 = c.

$$\det \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 4$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \xrightarrow{3\times} \cancel{\begin{matrix} 3a \\ b \\ c \end{matrix}} \rightarrow \cancel{\begin{matrix} 3a+2b \\ b \\ c \end{matrix}} \rightarrow \cancel{\begin{matrix} 3a+2b \\ b+2c \\ c \end{matrix}} \rightarrow \cancel{\begin{matrix} 3a+2b \\ b+2c \end{matrix}} \rightarrow$$

OR

$$\begin{vmatrix} 3a+2b \\ b+2c \\ c+a \end{vmatrix} = \begin{bmatrix} 3a \\ b+2c \\ c+a \end{bmatrix} + \begin{bmatrix} 2b \\ b+2c \\ c+a \end{bmatrix} = \begin{bmatrix} 3a \\ b \\ c+a \end{bmatrix} + \begin{bmatrix} 3a \\ 2c \\ c+a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ c+a \end{bmatrix} + \begin{bmatrix} 2b \\ 2c \\ c+a \end{bmatrix}$$

$$= \begin{bmatrix} 3a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 3a \\ b \\ a \end{bmatrix} + \begin{bmatrix} 3a \\ 2c \\ a \end{bmatrix} + \begin{bmatrix} 3a \\ 2c \\ c \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ c \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ a \end{bmatrix} + \begin{bmatrix} 2b \\ 2c \\ a \end{bmatrix} + \begin{bmatrix} 2b \\ 2c \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 3a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 2b \\ 2c \\ a \end{bmatrix}$$

$$= 4 \times 3 + (-1)^2 \cdot 4 \times 2 \times 2 = 12 + 16$$

$$= 28$$

28

5. (6 points) Show that the set of vectors $\{(1,-3,2),(2,1,-3),(-3,2,1)\}$ are linearly independent using determinants. Justify your answer.

$$\begin{matrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ -3 & 2 & 1 \end{matrix}$$

By the properties of determinants when all 3 rows are independent determinant is not zero, but when dependent it's zero

det $\begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ -3 & 2 & 1 \end{pmatrix}$ = ... = 9

1 -3 2

1(1+6) +3(20 -6)

7 -12 14

2 -6 4

2 -4 +3

~~Since determinant is the rows are dependent~~

Using upper triangular form the det is product of diagonal elements

$$\begin{matrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ -3 & 2 & 1 \end{matrix} \Rightarrow \begin{matrix} 1 & -3 & 2 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{matrix} \Rightarrow \begin{matrix} 1 & -3 & 2 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{matrix}$$

determinant is $1 \times 7 \times 0 = 0$

Since determinant is 0 the rows are dependent

3 -9 6

-7 7

6. (10 points) TRUE or FALSE. You don't have to justify.

a b
c d

~~True~~ If AB and BA are defined then A and B are square matrices.

False If A is a 2×2 matrix, then $det(2A) = 2det(A)$.

True If the rows of a square matrix A are linearly independent, so are the rows of $A^2 = AA$.

False Any system of linear equations has at most one solution.

False If the entries of both A and A^{-1} are integers, it is possible that $detA = 3$.

$\det A \cdot \det A^{-1} = 1$

$\det A^{-1} = \frac{1}{3}$

7. (12 points) Recall that an *isomorphism* between two vector spaces V and W is a linear transformation $T : V \to W$ that is one to one and onto (i.e. $T(V) = W$). In this case we say that V is isomorphic to W. Prove any **TWO** of the following.

   1. Show that V is isomorphic to itself.
   2. Show that if V is isomorphic to W then W is isomorphic to V.
   3. Show that if V is isomorphic to W and W is isomorphic to U then V is isomorphic to U.

iso

1. ~~$T: V \to V$~~

~~$\dim V = \dim(T(V)) + \dim N(T)$~~

2. $T: W \to V$

$T(W) = V$

1. $T(V) \neq V$

$T: V \to V$

$\dim V = \dim(T(V)) + \dim N(T)$

Since

~~let linear transformation $T: V \to V$ is~~

Consider linear transformation that maps every elements of V to themselves then it is one-to-one and onto

2. $T(V) = W$

$\dim V = \dim(T(V)) + \dim N(T)$

Since it is onetoone & onto $\dim V = \dim(T(V)) \Rightarrow \dim V = \dim W$

3

$T(W) = V$

$\dim W = \dim(T(W)) + \dim N(T)$

Since it has to be one-toone and $\dim W = \dim V$

$\dim T(W) = \dim V$. it is one-to-one and having same dimension so W is isomorph

3

3. $T(V) = W$

$T(W) = U$

$T(T(V)) = U$

$TT(V) = U$

So there ~~is~~ exist linear tranformation that satisfy $T(V) = U$