

Math 375

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1)

Show that if $f(x) = |x|$ then for any number m

$$\lim_{h \rightarrow 0} (f(h+0) - f(0) - mh) = 0$$

but

$$\lim_{h \rightarrow 0} \frac{1}{h} (f(h+0) - f(0) - mh) = 0$$

never holds, what does this say about f

$\lim_{h \rightarrow 0} (f(h+0) - f(0) - mh) = 0$ since every parts convergence

$$\begin{aligned} \lim_{h \rightarrow 0} f(h+0) - \lim_{h \rightarrow 0} f(0) - \lim_{h \rightarrow 0} mh \\ = 0 + 0 + 0 = 0 \end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{1}{h} (f(h+0) - f(0) - mh) = 0$$

$$\lim_{h \rightarrow 0^+} \frac{1}{h} (f(h+0) - f(0) - mh) \neq \lim_{h \rightarrow 0^-} \frac{1}{h} (f(h+0) - f(0) - mh)$$

$$\lim_{h \rightarrow 0^+} \frac{1}{h} (f(h+0) - f(0) - mh) = 1 - m$$

$$\lim_{h \rightarrow 0^-} \frac{1}{h} (f(h+0) - f(0) - mh) = -1 - m$$

$$\lim_{h \rightarrow 0} \frac{1}{h} (f(h+0) - f(0) - mh) \text{ cannot be defined}$$

therefore it never holds

it means f is not differentiable at 0

2)

Find the Jacobian matrices of the following maps. Show your work.

$$1) f(x, y) = e^{x^2+y^3}$$

$$2) f(x, y) = (xy, \sin(xy))$$

$$1) f(x, y) = e^{x^2+y^3}$$

$$\begin{aligned} Df(x, y) &= [\partial f / \partial x, \partial f / \partial y] \\ &= [\partial f / \partial x, \partial f / \partial y] \\ &= [2xe^{x^2+y^3}, 3ye^{x^2+y^3}] \end{aligned}$$

$$2) f(x, y) = (xy, \sin(xy))$$

$$\begin{aligned} Df(x, y) &= \begin{bmatrix} \partial f_1 / \partial x & \partial f_1 / \partial y \\ \partial f_2 / \partial x & \partial f_2 / \partial y \end{bmatrix} \\ &= \begin{bmatrix} y & x \\ y \cos(xy) & x \cos(xy) \end{bmatrix} \end{aligned}$$

3)

Consider the following function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$
given by

$$\begin{aligned} f(x, y) &= \frac{x^4 + y^4}{x^2 + y^2} \text{ if } (x, y) \neq 0 \\ &= (0, 0) \text{ if } (x, y) = 0 \end{aligned}$$

show that it is differentiable at $(0, 0)$

for $f(x, y)$ to be differentiable

it has to satisfy

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - f(0, 0) - T_{(0,0)}|(x, y) - (0, 0)|}{\|(x, y) - (0, 0)\|} &= 0 \\ \lim_{(x,y) \rightarrow (0,0)} \frac{\frac{x^4+y^4}{x^2+y^2} - 0 - f_x(0, 0)x - f_y(0, 0)y}{\sqrt{x^2 + y^2}} & \end{aligned}$$

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = h = 0$$

$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = h = 0$$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{\frac{x^4+y^4}{x^2+y^2} - 0 - f_x(0, 0)x - f_y(0, 0)y}{\sqrt{x^2 + y^2}} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{\frac{x^4+y^4}{x^2+y^2}}{\sqrt{x^2 + y^2}} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 + y^4}{(x^2 + y^2)\sqrt{x^2 + y^2}} \end{aligned}$$

along $y = mx$

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x^4 + m^4 x^4}{(x^2 + m^2 x^2) \sqrt{x^2 + m^2 x^2}} \\ &= \lim_{x \rightarrow 0} \frac{x^4(1 + m^4)}{x^3(1 + m^2) \sqrt{1 + m^2}} = 0 \end{aligned}$$

Therefore $f(x, y)$ is differentiable at $(0, 0)$

4)

Consider the following function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$
given by

$$\begin{aligned} f(x, y) &= \frac{x^3}{x^2 + y^2} \text{ if } (x, y) \neq 0 \\ &= (0, 0) \text{ if } (x, y) = 0 \end{aligned}$$

show that it has jacobian but not differentiable at $(0, 0)$

Jacobian

$$[\partial f / \partial x, \partial f / \partial y]$$

at $(0, 0)$

$$\left[\lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h}, \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} \right]$$

$$= [1, 0]$$

so it exist

Differentiability

it has to satisfy

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - f(0, 0) - T_{(0,0)}|(x, y) - (0, 0)|}{\|(x, y) - (0, 0)\|} = 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\frac{x^3}{x^2+y^2} - 0 - f_x(0, 0)x - f_y(0, 0)y}{\sqrt{x^2 + y^2}}$$

$$f_x(0, 0) = 1$$

$$f_y(0, 0) = 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\frac{x^3}{x^2+y^2} - x}{\sqrt{x^2 + y^2}}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\frac{-xy^2}{x^2+y^2}}{\sqrt{x^2 + y^2}}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{-xy^2}{(x^2 + y^2)\sqrt{x^2 + y^2}}$$

along $y = mx$

$$\lim_{x \rightarrow 0} \frac{-xm^2x^2}{(x^2 + m^2x^2)\sqrt{x^2 + m^2x^2}}$$

$$\lim_{x \rightarrow 0} \frac{-x^3m^2}{x^2(1 + m^2)x\sqrt{1 + m^2}}$$

$$\lim_{x \rightarrow 0} \frac{-m^2}{(1 + m^2)\sqrt{1 + m^2}}$$

$$\lim_{x \rightarrow 0} \frac{-m^2}{(1 + m^2)\sqrt{1 + m^2}} \text{ is not always } 0$$

therefore it is not differentiable

5. Apostol Section 8.9 Problem 13

compute all first-order partial derivatives

$$f(x, y) = \tan(x^2/y), y \neq 0$$

$$\partial f / \partial x = \frac{2x}{y} \sec^2(x^2/y)$$

$$\partial f / \partial y = \frac{-x^2}{y^2} \sec^2(x^2/y)$$

6. Apostol Section 8.14 Problem 2

Evaluate the directional derivatives of the following scalar fields for the points and directions given :

a) $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at $(1, 1, 0)$ in the direction of $i - j + 2k$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(1, 1, 0 + h(1, -1, 2)) - f(1, 1, 0)}{h} \\ & \lim_{h \rightarrow 0} \frac{f(1 + h, 1 - h, 2h) - f(1, 1, 0)}{h} \\ & \lim_{h \rightarrow 0} \frac{f(1 + h, 1 - h, 2h) - f(1, 1, 0)}{h} \\ & = \lim_{h \rightarrow 0} \frac{15h^2 - 2h}{h} \\ & = -2 \end{aligned}$$

7.