MATH 375

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- **1.5 Problem#23)** *Determine the set is a real linear space*

$$\text{All vectors } (x, y, z) \text{ in } V_3 \text{ with } x = 0 \text{ or } y = 0$$

$$\begin{gathered} \text{All vector space must satistfy the axiom 1} \\ \text{Let } a, b \in V \\ a = (1, 0, 1) \;\; b = (0, 1, 1) \\ \text{if } V \text{ is an vector space } a + b \in V \\ a + b = (1, 1, 2) \text{ both } x, y \text{ is not } 0 \\ \text{Because it doesn't satisfy axiom 1 it is not a real linear space} \end{gathered}$$

- **1.5 Problem#30a)** (a) Prove that Axiom 10 can be deduced from the other axioms.

$$\text{Axiom 10) For every x in V, we have lx = x.}$$

$$\begin{gathered} x + (-1)x = 0 \\ x + (-1)x + 1x = 0 + 1x \\ x + (1 - 1)x = 1x \end{gathered}$$

Problem #3~#6 In each of Exercises 1 through 10, let S denote the set of all vectors (x, y, z) in V_3 whose components satisfy the condition given. Determine whether S is a subspace of V_3. If S is a subspace, compute dim S.

- **1.10 Problem#3)**

$$S = x + y + z = 0$$

$$\begin{gathered} \text{To prove that } S \text{ is a subspace of } (x, y, z) \text{ in } V_3 \\ \text{We only have to show that it satistfy the closure theorem since it is the subspace of } V_3 \\ \\ \text{Let } w_1 = (a_1, b_1, c_1), \;\; w_2 = (a_2, b_2, c_2) \quad w_1, w_2 \in S \\ \\ \text{For axiom 1} \;\; w_1 + w_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2) \\ (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) = (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) = 0 \\ \text{Therefore, axiom 1 is satisfied} \\ \\ \text{For axiom 2} \;\; \text{let } m \in \mathbb{R} \text{ and let } w_1 = (a_1, b_1, c_1) \in V_3 \\ mw_1 = (ma_1, mb_2, mc_2) \text{ and} \\ ma_1 + mb_2 + mc_2 = m(a_1 + b_1 + c_1) = 0 \;\; \text{Since } a_1 + b_1 + c_1 = 0 \\ \text{Therefore, axiom 2 is satisfied} \end{gathered}$$

$$\text{Since closure axioms are satisfied it is a subspace}$$

$$\text{The dimension of the subspace}$$

$$\begin{gathered} \text{let } (a, b, c) \in V_3 \text{ and } -a = b + c \\ \text{It means } (b + c, b, c) \\ (b + c, b, c) = b(1, 1, 0) + c(1, 0, 1) \text{ two vectors are linearly independent} \\ \text{So that there are two basis and the dimension is 2} \end{gathered}$$

- **1.10 Problem#7)**

$$S = x^2 - y^2 = 0$$

To prove that S is a subspace of (x, y, z) in V_3
We only have to show that it satistfy the closure theorem since it is the subspace of V_3

Let $w_1 = (a_1, b_1)$ and $w_2 = (a_2, b_2)$ $\; w_1, w_2 \in V_3$

For Axiom 1
$$w_1 + w_2 = (a_1 + a_2, b_1 + b_2)$$

Since $S = x + y + z = 0$
$$(a_1)^2 - (b_1)^2 = 0$$
repeating with (a_2, b_2)
$$(a_2)^2 - (b_2)^2 = 0$$

$(a_1 + a_2, b_1 + b_2)$ must satisfy
$$(a_1 + a_2)^2 - (b_1 + b_2)^2 = 0$$
$$(a_1)^2 - (b_1)^2 + (a_2)^2 - (b_2)^2 + 2(a_1a_2 - b_1b_2) = 0$$

Both $(a_1)^2 - (b_1)^2$ $(a_2)^2 - (b_2)^2$ are 0
However $2(a_1a_2 - b_1b_2)$ cannot be 0
Therefore axiom 1 is not satisfied and also S is not a subspace.

- **1.10 Problem#22** *In this exercise, L(S) denotes the subspace spanned by a subset S of a linear space V. Prove each of the statements (a) through (f).*

a)$S \subseteq L(S)$

Since it is a vector space $L(S)$ has every linear combinations of vectors of S
So $span(S)$ has the every elements of S and linear combinations of S
Therefore, $S \subseteq L(S)$

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b) $S \subseteq T \subseteq V$ and if T is a subspace of V $L(S) \subseteq T$
In other words: L(S) is the smallest subspace of V which contains S.

$L(S)$ is a vector space including S also T is a vector space including S
Since both of them are vector space they have linear combinations of S
However L(s) has only linear combinations of S since it is spanned by S
Therefore, every vector in L(s) is included in T which means L(s) is the smallest subspace of V which contains S

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c) A subset S of V is a subspace of V if and only if L(S) = S.

A subspace is a vector space
So for a subset to be a subspace it has to satisfy closure axioms

Closure axioms means that the vector space is closed under linear combination
However span of a set is set of all linear combinations
Therefore if there is a subset S span(S) is a subspace

Therefore for a subset S of V to be a subspace
That means $S = L(S)$

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d) If $S \subseteq T \subseteq V,$ then $L(S) \subseteq L(T)$

In problem a we proved that $S \subseteq L(S)$
In problem b we proved that $L(S) \subseteq T$ if $S \subseteq T \subseteq V$

Applying the two theorems we can say that$S \subseteq L(S) \subseteq T \subseteq L(T) \subseteq V$
Therefore $L(S) \subseteq L(T)$

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e) If S and T are subspaces of V, then so is $S \cap T$

Let $A = S \cap T$
Both S and T are subspaces
Therefore both S and T has $\vec{0}$
Both S and T have zero vector then A has a zero vector too

Since S and T are subspaces $S \cap T$ must be a subset of V
To prove A is a subspace we need to show it satisfy closure axioms

Let B and C a vector in A
Both S and T has B and C too
Since S and T are subspaces They are closed under addition
Therefore $B + C$ is in S and T, also for A
Axiom 1 satisfied

Let B a vector space in A and let c a real number
Since S and T are closed under scalar multiplication cB is in S and T, also for A
Axiom 2 satified

Since A is satisfying closure axioms it is a subspace

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f) If S and T are subsets of V, then $L(S \cap T) \subseteq L(S) \cap L(T)$

Let $A = S \cap T$
$L(S \cap T) = L(A)$
Since $A \subseteq S$ and $A \subseteq T$
As we showed in previous problem (d)
$L(A) \subseteq L(S)$ and $L(A) \subseteq L(T)$
It means $L(A) \subseteq L(S) \cap L(T)$
Therefore, $L(S \cap T) \subseteq L(S) \cap L(T)$

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g) Give an example in which $L(S \cap T) \neq L(S) \cap L(T)$.

When S={x} and T = {2x}

- **1.10 Problem#24** . *Let V be a finite-dimensional linear space, and let S be a subspace of V. Prove each of the following statements.*

a) S is finite dimensional and $dim(S) \leq dim(V)$

Let $(a_1, \dots a_n)$ be the basis of S
and the dimension of S is n
Since the S is subspace of V $(a_1, \dots a_n)$ is an independent set in V and also S is included in V
The basis of V can be written $(a_1, \dots a_n, b_1 \dots b_k)$
$dim(V) = n + k$
k must be $0 \leq k$
Therefore $dim(S) \leq dim(V)$

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b) dim S = dim V if and only if S = V

If $S \neq V$
there is a vector space C
That follows $S + C = V$ and $S \cap C \neq \vec{0}$
Sicne $V = S + C$
$dim(S + C) = dim(S)$
$S \cap C = \emptyset$ therefore $dim(S + C) = dim(S) + dim(C)$
$dim(S) + dim(C) = dim(S)$
$dim(C) = 0$

It means $C = \{0\}$ <Contradiction>
Therefore, $S = V$

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c) Every basis for S is part of a basis for V

First we will show any set of independent elements in V is a subset of basis of V
Let $A = \{x_1, \dots x_k\}$ be any independent set of elements in B
If $L(A)$ is a basis of B then A is a basis
In case it isn't we add some element y: $y \in B$ $y \notin L(A)$ to A
$A' = \{x_1, \dots x_k, y\}$
If A' is dependent there would be some scalars $(c_1, \dots c_{k+1})$
$(\sum_{i=1}^{k} c_i x_i) + c_{k+1} y = 0$ However c_{k+1} cannot be 0
Because $\{x_1, \dots x_k\}$ is independent
Also that means that y can be spanned by A and it is contradiction
Therefore, set A' is independent with $k + 1$ element
If $L(A') = V$ that's it
If it doesn't we can repeat the process with $k + 2$ elements
Since it is finite-dimensional vector space we can get basis with finite steps

Now back to "Every basis for S is part of a basis for V"
basis must be independent
every element of S is also an element of V since S is subset of V

Therefore every basis of S is set of independent elements in V
As we proved any set of independent elements in V is a subset of basis of V
Thus, Every basis for S is part of a basis for V

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d) A basis for V need not contain a basis for S

Let $V = \mathbb{R}^2$
And let $S => y = x$

There can be a basis of V which is $\{(1,0),(0,1)\}$
Subsets of this basis can't be the basis for S

basis of S can be something like (1,1) and it is not contained