5. (16 points) Let $\mathscr{P}_2$ be the space of polynomials of degree at most two. Start with the basis $B = \{1, x, x^2\}$ for $\mathscr{P}_2$, and obtain an orthogonal basis (using the Gram-Schmidt process) with respect to the following inner product

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$X_1 = 1$ $X_3 = x^2$
$X_2 = x$

$$(p, q) = \int_{-1}^{1} p(x)\,q(x)\,x^2\,dx.$$

$V_1 = 1$ ✓

$V_2 = X_2 - \text{proj}_{v_1} X_2 = X - \frac{\langle X_2, V_1\rangle}{\langle V_1, V_1\rangle} V_1 = X - \frac{0}{\frac{2}{3}} 1 = X$ $\quad V_2 = X$ ✓

$\langle V_1, V_1\rangle = \int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$

$\langle X_2, V_1\rangle = \int_{-1}^{1} x^3\,dx = \frac{1}{4}x^4\Big|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$

$V_3 = X_3 - \text{proj}_{v_1} X_3 - \text{proj}_{v_2} X_3 = X^2 - \frac{\langle X_3, V_1\rangle}{\langle V_1, V_1\rangle} V_1 - \frac{\langle X_3, V_2\rangle}{\langle V_2, V_2\rangle} V_2$

$\langle X_3, V_1\rangle = \int_{-1}^{1} x^4\,dx = \frac{1}{5}x^5\Big|_{-1}^{1} = \frac{1}{5} - (-\frac{1}{5}) = \frac{2}{5}$

$= X^2 - \frac{\frac{2}{5}}{\frac{2}{3}} \cdot 1 - \frac{0}{\frac{2}{5}} X = X^2 - \frac{3}{5}$

$V_3 = X^2 - \frac{3}{5}$ ✓

$\langle V_2, V_2\rangle = \int_{-1}^{1} x^4\,dx = \frac{2}{5}$

$\langle X_3, V_2\rangle = \int_{-1}^{1} x^2 \cdot x \cdot x^2\,dx = \int_{-1}^{1} x^5\,dx = 0$

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the problem did not required to get orthonormal basis.

$\{1, X, X^2 - \frac{3}{5}\}$ is a orthogonal basis.

$x^5 - \frac{3}{5}x^3$

$\int_{-1}^{1} x^4 - \frac{3}{5}x^2$

$\frac{1}{5}x^5 - \frac{1}{5}x^3$