Problems for Foundations of Mathematics

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1 Equivalence Relations

Problem 1.1: Prove that it is impossible to find an equivalence relation $\equiv$ on $\mathbb{R}$ for which $x \equiv y \iff (x - y) \notin \mathbb{Q}$ is true.

Proof: Let $x \in \mathbb{R}$ and assume $\equiv$ is an equivalence relation on the real numbers. Since $\equiv$ is an equivalence relation, it follows $x \equiv x$ as equivalence relations are reflexive. This, however, implies

$$x - x = 0 \notin \mathbb{Q}$$

This is a contradiction. Thus $\equiv$ is not an equivalence relation. $\square$

Problem 1.2: Let $X = \mathbb{R}$. Show that the relation $\equiv$ defined by $x \equiv y \iff \exists_{r \in X\setminus\{0\}}(x = ry)$ is an equivalence relation. How many equivalence classes does $\equiv$ have?

Proof: Let $x, y, z \in X$. To show $x \equiv x$ (reflexivity), it follows

$$x = rx \;\; r \in X\setminus\{0\}$$

For $r = 1$, this is satisfied, and thus $\equiv$ is reflexive. We now show $x \equiv y \implies y \equiv x$ (symmetry). Since $x \equiv y$, it follows $x = ry$ for some $r \in \mathbb{R}\setminus\{0\}$, with some rearrangement:

$$x = ry \implies y = \frac{1}{r}x$$

Since $\mathbb{R}\setminus\{0\}$ is closed under inverses, $\frac{1}{r} \in \mathbb{R}\setminus\{0\}$, thus $\equiv$ is symmetric. We conclude by proving $x \equiv y \wedge y \equiv z \implies x \equiv z$ (transitivity) and stating the number of equivalence classes on $\equiv$. Since $x \equiv y$ and $y \equiv z$, it follows $x = ry$ and $y = sz$ for some $r, s \in \mathbb{R}\setminus\{0\}$. Replacing y in $x = ry$ by sz we obtain:

$$x = r(sz) = (rs)z$$

Since $\mathbb{R}\setminus\{0\}$ is closed under multiplication $rs \in \mathbb{R}\setminus\{0\}$, and so $\equiv$ is transitive. We find the number of equivalence classes ($[a] = \{x \in X : x \equiv a\}$). Let $a \in X$. We consider two cases where $a = 0$ and $a \neq 0$. If $a = 0$, then it follows $x = r0 \implies x = 0$ for any $r \in \mathbb{R}\setminus\{0\}$. So $[0] = \{0\}$. If $a \neq 0$, for nonzero x, we have $x = ra \implies r = \frac{x}{a}$ which is well-defined for some $r \in \mathbb{R}\setminus\{0\}$. Thus $[a] = \mathbb{R}\setminus\{0\}$. $\square$

2 Partial or Transfinite Induction

Problem 2.1: Let $(X, \preceq)$ be a totally ordered set. Prove that if $x_o \in X$ is a minimal element, then it is also a minimum element.

Proof: Recall that x_o is a minimal element if there is no $x \in X$ such that $x \preceq x_o$ and $x \neq x_0$, and it is a minimum element if for any $x \in X$ it follows $x_o \preceq x$. We prove this by contradiction. Assume $x_o \in X$ is minimal but not a minimum element. Since X is totally ordered, it follows $x \preceq x_o$ or $x_o \preceq x$ for any $x \in X$. It follows $x \preceq x_o$ as as x_o is assumed not to be a minimum element. It must be that $x \neq x_o$ as $x = x_o \wedge x \preceq x_o \implies x_o \preceq x$ by reflexivity of $\preceq$. However, if $x \neq x_o$ with $x \preceq x_o$ this is contradictory to our claim that x_o is minimal. Therefore x_o must be a minimum element. $\square$

Problem 2.2: Let $(X, \preceq)$ be a totally ordered set, and $A \subseteq X$. Prove that $\sup(A)$ and $\inf(A)$, if exist, are unique.

Proof: Recall the supremum is defined as an upper bound of A ($a \preceq x$ for all $a \in A$), x, such that for any upper bound of A, x', $x \preceq x'$. And the infimum is defined as a lower bound of A ($y \preceq a$ for all $a \in A$), y, such that for any lower bound of A, y', $y' \preceq y$. We prove this by contradiction. Assume x, x_o are both $\sup(A)$ and y, y_o are both $\inf(A)$ with $x \neq x_o$ and $y \neq y_o$. Since x, x_o are both upper bounds of A, and likewise y, y_o are both lower bounds of A, by the definition of supremum and infimum:

$$x \preceq x_o \text{ and } x_o \preceq x$$
$$y_o \preceq y \text{ and } y \preceq y_o$$

Since $\preceq$ is antisymmetric, the above can only hold if $x = x_o$ and $y = y_o$. This contradicts our assumption that $x \neq x_o$ and $y \neq y_o$. Therefore $\sup(A)$ and $\inf(A)$ must be unique. $\square$

Problem 2.3: Let $\prec_Y$ be a strict partial order on a set Y, and $f : X \to Y$ be a function. Define a relation $\prec_X$ on X as follows. For every $x_1, x_2 \in X$:

$$x_1 \prec_X x_2 \iff f(x_1) \prec_Y f(x_2)$$

Prove that $\prec_X$ is a strict partial order on X.

Proof: Recall that $\prec_Y$ is a strict partial order if it is transitive and irreflexive. Let $x_1, x_2, x_3 \in X$. We first prove transitivity. Assume $x_1 \prec_X x_2$ and $x_2 \prec_X x_3$. It follows that

$$f(x_1) \prec_Y f(x_2) \text{ and } f(x_2) \prec_Y f(x_3)$$

This implies $f(x_1) \prec_Y f(x_3)$ by transitivity of $\prec_Y$ being a strict partial order. So it also follows $x_1 \prec_X x_3$, and hence $\prec_X$ is transitive. We now show $\prec_X$ is irreflexive by contradiction. Assume $x_1 \prec_X x_1$. It follows by the definition of $\prec_X$

$$f(x_1) \prec_Y f(x_1)$$

This is, however, a contradiction as $\prec_Y$ is a strict partial order and so is irreflexive. Thus $x_1 \not\prec_X x_1$, and since $\prec_X$ is both transitive and irreflexive it is a strict partial order. $\square$

3 Cardinality

Problem 3.1: Prove the following statements by giving explicit bijections:

a) $2\mathbb{Z} \sim \mathbb{N}$

Proof: We split $2\mathbb{Z}$ into two disjoint sets $2\mathbb{Z}_{<0} = \{\dots, -6, -4, -2\}$ and $2\mathbb{Z}_{\geq 0} = \{0, 2, 4, \dots\}$ that satisfy $2\mathbb{Z}_{<0} \cup 2\mathbb{Z}_{\geq 0} = 2\mathbb{Z}$. We can define the bijection

$$f_1 : 2\mathbb{Z}_{<0} \to O \subseteq \mathbb{N}, \;\; x \mapsto -x - 1$$

where $O = \{1, 3, 5, 7, \dots\}$, as the mapping of negative even integers onto odd natural numbers. Now, we can define the bijection

$$f_2 : 2\mathbb{Z}_{\geq 0} \to E \subseteq \mathbb{N}, \;\; x \mapsto x + 2$$

where $E = \{2, 4, 6, 8, \dots\}$, as the mapping of positive even integers onto even natural numbers. Since O and E are disjoint and $O \cup E = \mathbb{N}$, we define

$$f : 2\mathbb{Z} \to \mathbb{N}, \quad f(x) = \begin{cases} f_1(x) \text{ if } x < 0 \\ f_2(x) \text{ if } x \geq 0 \end{cases}$$

which is a bijection as both f_1 and f_2 are bijective. Since there exists a bijection from $2\mathbb{Z}$ to $\mathbb{N}$, the set of even integers is equinumerous with the natural numbers. $\square$

Problem 3.2: Find an explicit surjection $f : \mathbb{R} \to (0, 1)$.

Proof: To obtain a surjective function, it suffices to map each real number to its corresponding decimal expansion and each integer to a fraction (since f need not be injective). For example:

$$10.4353 \mapsto 0.4353 \text{ and } -100000.4353 \mapsto 0.4353$$
$$0 \mapsto \frac{1}{2} \text{ and } 1 \mapsto \frac{1}{3}$$

We thus define the surjective function f as:

$$f(x) = \begin{cases} x - \lfloor x \rfloor & \text{if } x \notin \mathbb{Z} \\ \frac{1}{|x|+2} & \text{if } x \in \mathbb{Z} \end{cases}$$

To prove surjectivity on f let $y \in (0, 1)$. Consider the first case if $y \in (0, 1) \backslash \{\frac{1}{n} : n \in \mathbb{N}\}$. Since the decimal expansion of numbers in $\mathbb{R}$ densely fills $(0, 1)$, it follows if $x = a + y$ for some $a \in \mathbb{N}$, then $f(x) = x - \lfloor x \rfloor = (a + y) - a = y$. Further, if $y = \frac{1}{n}$ for some $n \in \mathbb{N}_{\geq 2}$, choose any $(n - 2) \in \mathbb{Z}$, we get: $f(n - 2) = \frac{1}{|n-2|+2} = \frac{1}{n} = y$. Therefore f is surjective. $\square$