

MAT 2141 Notes

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1 Linear Maps

1.1 Definiton

Definition 1.1.1: If V and W are vector spaces over the same field F , and $T : V \rightarrow W$ is a mapping, we say T is a linear mapping if the following hold:

$$\begin{aligned}T(u + v) &= T(u) + T(v) \quad \forall u, v \in V \\T(cv) &= cT(v) \quad \forall v \in V, \forall c \in F\end{aligned}$$

Example 1.1.1: Let $V = C^\infty(\mathbb{R})$, $F = \mathbb{R}$, and define $S : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ by $(Sf)(x) = \int_0^x f(t)dt$, $\forall x \in \mathbb{R}$.

Let $f, g \in V$ (i.e. f and g are infinitely differentiable functions) and let $c \in F$. Using the properties of integration, we can show additivity and homogeneity hold:

$$\begin{aligned}S(f + g)(x) &= \int_0^x f(t) + g(t)dt = \int_0^x f(t)dt + \int_0^x g(t)dt = (Sf)(x) + (Sg)(x) \\S(cf)(x) &= \int_0^x cf(t)dt = c \int_0^x f(t)dt = c(Sf)(x)\end{aligned}$$

Since S satisfies the properties of additivity and homogeneity, we conclude S is a linear map. QED.

1.2 Kernel and Image

Definiton 1.2.1: If $T : V \rightarrow W$ is a linear map, then $Ker(T) \subseteq V$ gets mapped to the zero vector under T and $Im(T) \subseteq W$ is where vectors from V can actually get mapped to under T (i.e. range):

$$\begin{aligned}Ker(T) &= T^{-1}(\vec{0}) = \{x \mid Tx = \vec{0}\} \\Im(T) &= T(V) = \{Tx \mid x \in V\}\end{aligned}$$

Theorem 1.2.1: Suppose $T : V \rightarrow W$ is a linear mapping, T is injective iff $Ker(T) = \{\vec{0}\}$.

We prove this in two directions. Assume T is injective and let $u, v \in V$. Since $0v = \vec{0}$ for any vector v and $T(0v) = 0T(v)$ by the property of linear maps, it follows a linear mapping always maps the zero vector to the zero vector. Since for any injective mapping we have $T(u) = T(v) \implies u = v$, it follows if $T(u) = T(v) = \vec{0}$, then $u = v = \vec{0}$ and thus $Ker(T) = \{\vec{0}\}$.

We now prove this in the other direction. Assume $Ker(T) = \{\vec{0}\}$. We want to show if $T(u) = T(v)$ for any $u, v \in V$ then $u = v$. If $T(u) = T(v)$, it follows $T(u) - T(v) = \vec{0}$ which simplifies to $T(u - v) = \vec{0}$ by the property of linear maps. Since $Ker(T) = \{\vec{0}\}$ it follows that $T(u - v) = \vec{0} \implies u - v = \vec{0} \implies u = v$. Thus T is injective. Since we have proved both implications this completes the proof. QED.

1.3 Vector Spaces of Linear Maps

Definition 1.3.1: If V, W are vector spaces over a field F , we define the set of linear maps from V to W :

$$\mathcal{L}(V, W) = \{T : V \rightarrow W \mid T \text{ is linear}\}$$

If $V = W$, we simply write $\mathcal{L}(V)$.

Definition 1.3.2: If A, B , and C are sets with mappings $T : A \rightarrow B$ and $S : B \rightarrow C$, we can compose the mappings to get $ST : A \rightarrow C$ (i.e. $S \circ T$) defined by:

$$(ST)a = S(Ta) \quad \forall a \in A$$

Theorem 1.3.1: The composition of linear maps is itself a linear map (i.e. $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W) \implies ST \in \mathcal{L}(U, W)$).

Let $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ be linear maps over a field F with $u, v \in U$ and $c \in F$. By the definition of composition we have:

$$\begin{aligned}(ST)(u + v) &= S(T(u + v)) \\ (ST)(cu) &= S(T(cu))\end{aligned}$$

By the property of T being a linear map we get:

$$\begin{aligned}S(T(u + v)) &= S(T(u) + T(v)) \\ S(T(cu)) &= S(cT(u))\end{aligned}$$

And further, by the property of S being a linear map we get:

$$\begin{aligned}S(T(u) + T(v)) &= S(T(u)) + S(T(v)) \\ S(cT(u)) &= cS(T(u))\end{aligned}$$

Thus, ST satisfies additivity and homogeneity and is therefore a linear map. QED.

1.4 Isomorphisms

Definition 1.4.1: If $T : V \rightarrow W$ is a bijective linear map, we say T is an isomorphism. We also say that V is isomorphic to W and write $V \cong W$.

Theorem 1.4.1: Suppose V and W are vector spaces. Further, assume $T : V \rightarrow W$ is an isomorphism. It follows that $T^{-1} : W \rightarrow V$ is also linear and hence an isomorphism.

Assume $T : V \rightarrow W$ is an isomorphism. Since it follows that T is bijective, it is invertible. Let $u', v' \in W$. It follows there exists some $u, v \in V$ such that $T(u) = u'$ and $T(v) = v'$. By the property of T being linear, we have:

$$\begin{aligned}T^{-1}(u' + v') &= T^{-1}(T(u) + T(v)) = T^{-1}(T(u + v)) = u + v = T^{-1}(u') + T^{-1}(v') \\ T^{-1}(cu') &= T^{-1}(cT(u)) = T^{-1}(T(cu)) = cu = cT^{-1}(u')\end{aligned}$$

Thus T^{-1} preserves additivity and homogeneity and is therefore a linear map. And moreover, since it is the inverse of a bijective mapping, it is also bijective, and thus an isomorphism. QED.

Theorem 1.4.2: Isomorphisms are an equivalence relation. In other words, if U, V, W are all vector spaces over the same field F , the following properties hold:

$$\begin{aligned}U &\cong U \\U \cong V &\implies V \cong U \\U \cong V \text{ and } V \cong W &\implies U \cong W\end{aligned}$$

This proof is left to the non-existent reader.

2 Structure of Vector Spaces

2.1 Spans and Generating Sets