MAT 2362 Notes

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1 Lecture 4

Definition 4.1.1: Let $\{A_i : i \in \mathbb{N}\}$ be a set with sets in it. We can union and intersect sets with specified indices (like summations):

$$\bigcup_{i=1}^{n \in \mathbb{N}} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i=1}^{n \in \mathbb{N}} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Example 4.1.1: Prove that $\bigcup_{n=2}^{\infty}(0, 1 - \frac{1}{n}) = (0, 1)$.
Proof: We first prove $(0, 1) \subseteq \bigcup_{n=2}^{\infty}(0, 1 - \frac{1}{n})$. Let $x \in (0, 1)$. Since x is strictly less than 1, it follows that there will exist some $n \in \mathbb{N}$ where $1 - \frac{1}{n} > x$ as $1 - \frac{1}{n} = 1$ for $n \to \infty$. So for any x in $(0, 1)$ we can find an n that is large enough such that $x \in (0, 1 - \frac{1}{n})$.

To prove the converse in $\bigcup_{n=2}^{\infty}(0, 1-\frac{1}{n}) \subseteq (0, 1)$ we can use the same reasoning. Let $x \in (0, 1-\frac{1}{n})$. Since $1 - \frac{1}{n} = 1$ as $n \to \infty$ and since any element in $(0, 1)$ is strictly less than 1, it follows there is some large enough n such that $x \in (0, 1)$.

Since $(0, 1) \subseteq \bigcup_{n=2}^{\infty}(0, 1 - \frac{1}{n})$ and $\bigcup_{n=2}^{\infty}(0, 1 - \frac{1}{n}) \subseteq (0, 1)$, they are both equal. QED.

Definition 4.1.2: If $f : X \to Y$ and $A \subseteq X$, we can "restrict" the domain of X to A using the notation $f|_A$ where $f|_A(a) = f(a)$ for all $a \in A$. This is called the restriction of f to A.

Remark 4.1.1: It is not really clear why restricting f to A is useful, and tbh I have no idea. But, (maybe interestingly?) given the mapping $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ it is clear this is not injective as $f(-1) = f(1) = 1$ and so on with every number and its negative counterpart. However, the restriction of f to $[0, \infty)$ where $f|_{[0,\infty)}(x) = x^2$ is injective! Also, restricting the codomain to $[0, \infty)$ makes f surjective.

Lemma 4.1.1: If $f : X \to Y$ and $g : Y \to Z$ are both surjective mappings, then so is $g \circ f : X \to Z$.
Proof: Assume f, g are both surjective mappings and let $z \in Z$ such that $(g \circ f) = z$. Since f is surjective, for any $y \in Y$ there is a corresponding $x \in X$ such that $f(x) = y$. So we have $g(f(x)) = g(y)$. Further, since g is surjective, for any $z \in Z$ there is a corresponding $y \in Y$ such that $g(y) = z$. So we have $g(f(x)) = g(y) = z$. Therefore $g \circ f$ is surjective. QED.

Theorem 4.1.1: For a mapping $f : X \to Y$, f is an injection iff for all $A \subseteq X(f^{-1}(f(A)) = A)$.
Proof: We prove this statement in two directions. We will first show if $f : X \to Y$ is an injection, then $f^{-1}(f(A)) = A$. Assume that f is injective.
Prove $f^{-1}(f(A)) \subseteq A$: Let $x \in f^{-1}(f(A))$. This implies we have:

$$f(x) \in f(f^{-1}(f(A))) \implies f(x) \in f(A) \implies f(x) = f(a) \text{ for some } a \in A$$

Since f is injective, it must hold that $x = a$ and so $x \in A$. Thus $f^{-1}(f(A)) \subseteq A$.
Prove $A \subseteq f^{-1}(f(A))$: Let $x \in A$. This implies we have:

$$f(x) \in f(A) \implies f^{-1}(f(x)) \in A \implies x \in f^{-1}(f(A))$$

Thus $A \subseteq f^{-1}(f(A))$. We can therefore conclude if $f : X \to Y$ is an injection, then $f^{-1}(f(A)) = A$.

We now prove the statement that if $f^{-1}(f(A)) = A$, then f is injective. Assume $f^{-1}(f(A)) = A$ and that $a, b \in X$ such that $f(a) = f(b)$. Consider the set $A = \{a\} \subseteq X$, by our assumption, it follows that $f^{-1}(f(A)) = \{a\}$. Since $f(a) = f(b)$, it follows $b \in f^{-1}(f(A)) = \{a\}$ and thus $a = b$. This concludes the proof in both directions. QED.

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2 Lecture 5

Definition 5.1.1: If $R \subseteq X \times X$, that is, R is a set of relations on X, we can define the "reflexive closure" of R. This is the "smallest" reflexive relation that contains R:

$$S = R \cup \{(x, x) \mid x \in X\} \tag{1}$$

The symmetric and transitive closures are defined similarily.

Example 5.1.1: Let $R \subseteq \mathbb{Z} \times \mathbb{Z} = \{(1,1), (1,2), (4,5), (3,3), (4,4), (2,2), (5,5)\}$ be a reflexive set. The reflexive closure of R would be the set $S \subseteq R = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$.

Example 5.1.2: If n is a positive integer, then the realtion $\equiv_n$ on $\mathbb{Z}$ that is defined by $x \equiv_n y \iff (x - y)$ is divisible by n is an equivalence relation.
Proof: We must prove this relation is reflexive, symmetric, and transitive.
Reflexivity: Let $x \in \mathbb{Z}$. $(x - x)$ is divisible by n implies $x - x = nm$ for some $n \in \mathbb{Z}$. Since this simplifies to $0 = nm$, it is clear this is true for $m = 0$. Thus $x \equiv_n x$.
Symmetry: Let $x, y \in \mathbb{Z}$ and assume $x \equiv_n y$. This implies that $x - y = nm$ for some $m \in \mathbb{Z}$. We can manipulate this a bit to get $y - x = -nm \implies y - x = n(-m)$. Since $(-m) \in \mathbb{Z}$, $(y - x)$ is divisble by n using our assumption and therefore $x \equiv_n y \implies y \equiv_n x$.
Transitivity: Let $x, y, z \in \mathbb{Z}$ and assume $x \equiv_n y$ and $y \equiv_n z$. Using our assumption it follows that $y - z = nm \implies y = nm + z$ for some $m \in \mathbb{Z}$. Since we have $x - y = nv$ for some $v \in \mathbb{Z}$, we substitute our y to get $x - (nm + z) = nv \implies x - z = nv + nm \implies x - z = n(v + m)$. Since $v, m \in \mathbb{Z}$ and adding two integers gives you another integer, $x - z$ is divisible by n. Thus $x \equiv_n y \wedge y \equiv_n z \implies x \equiv_n z$.
Since $\equiv_n$ is reflexive, transitive, and symmetric, it is an equivalence relation. QED.

Definition 5.1.2: If $\equiv$ is an equivalence relation on a set X, for every $a \in X$ we define the equivalence class of a (denoted $\overline{a}$ or $[a]$) to be the set:

$$[a] = \{x \in X \mid x \equiv a\} \tag{2}$$