MAT 2141 Notes

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Contents

1 Linear Maps

1.1 Definiton

Definition 1.1.1: If V and W are vector spaces over the same field F, and $T : V \to W$ is a mapping, we say T is a linear mapping if the following hold:

$$T(u+v) = T(u) + T(v) \;\; \forall u, v \in V$$
$$T(cv) = cT(v) \;\; \forall v \in V, \; \forall c \in F$$

Example 1.1.1: Let $V = C^\infty(\mathbb{R}),\; F = \mathbb{R}$, and define $S : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by $(Sf)(x) = \int_0^x f(t)dt,\; \forall x \in \mathbb{R}$.

Let $f, g \in V$ (i.e. f and g are infinitely differentiable functions) and let $c \in F$. Using the properties of integration, we can show additivity and homogeneity hold:

$$S(f+g)(x) = \int_0^x f(t) + g(t)dt = \int_0^x f(t)dt + \int_0^x g(t)dt = (Sf)(x) + (Sg)(x)$$
$$S(cf)(x) = \int_0^x cf(t)dt = c\int_0^x f(t)dt = c(Sf)(x)$$

Since S satisfies the properties of additivity and homogeneity, we conclude S is a linear map. QED.

1.2 Kernel and Image

Definiton 1.2.1: If $T : V \to W$ is a linear map, then $Ker(T) \subseteq V$ gets mapped to the zero vector under T and $Im(T) \subseteq W$ is where vectors from V can actually get mapped to under T (i.e. range):

$$Ker(T) = T^{-1}(\vec{0}) = \{x \mid Tx = \vec{0}\}$$
$$Im(T) = T(V) = \{Tx \mid x \in V\}$$

Theorem 1.2.1: Suppose $T : V \to W$ is a linear mapping, T is injective iff $Ker(T) = \{\vec{0}\}$.

We prove this in two directions. Assume T is injective and let $u, v \in V$. Since $0v = \vec{0}$ for any vector v and $T(0v) = 0T(v)$ by the property of linear maps, it follows a linear mapping always maps the zero vector to the zero vector. Since for any injective mapping we have $T(u) = T(v) \implies u = v$, it follows if $T(u) = T(v) = \vec{0}$, then $u = v = \vec{0}$ and thus $Ker(T) = \{\vec{0}\}$.

We now prove this in the other direction. Assume $Ker(T) = \vec{0}$. We want to show if $T(u) = T(v)$ for any $u, v \in V$ then $u = v$. If $T(u) = T(v)$, it follows $T(u) - T(v) = \vec{0}$ which simplifies to $T(u - v) = \vec{0}$ by the property of linear maps. Since $Ker(T) = \{\vec{0}\}$ it follows that $T(u - v) = \vec{0} \implies u - v = \vec{0} \implies u = v$. Thus T is injective. Since we have proved both implications this completes the proof. QED.

1.3 Vector Spaces of Linear Maps

Definition 1.3.1: If V, W are vector spaces over a field F, we define the set of linear maps from V to W:

$$\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is linear}\}$$

If $V = W$, we simply write $\mathcal{L}(V)$.

Definition 1.3.2: If A, B, and C are sets with mappings $T : A \to B$ and $S : B \to C$, we can compose the mappings to get $ST : A \to C$ (i.e. $S \circ T$) defined by:

$$(ST)a = S(Ta) \ \ \forall a \in A$$

Theorem 1.3.1: The composition of linear maps is itself a linear map (i.e. $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W) \implies ST \in \mathcal{L}(U, W)$).

Let $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ be linear maps over a field F with $u, v \in U$ and $c \in F$. By the definition of composition we have:

$$(ST)(u + v) = S(T(u + v))$$
$$(ST)(cu) = S(T(cu))$$

By the property of T being a linear map we get:

$$S(T(u + v)) = S(T(u) + T(v))$$
$$S(T(cu)) = S(cT(u))$$

And further, by the property of S being a linear map we get:

$$S(T(u) + T(v)) = S(T(u)) + S(T(v))$$
$$S(cT(u)) = cS(T(u))$$

Thus, ST satisfies additivity and homogeneity and is therefore a linear map. QED.

1.4 Isomorphisms

Definition 1.4.1: If $T : V \to W$ is a bijective linear map, we say T is an isomorphism. We also say that V is isomorphic to W and write $V \cong W$.

Theorem 1.4.1: Suppose V and W are vector spaces. Further, assume $T : V \to W$ is an isomorphism. It follows that $T^{-1} : W \to V$ is also linear and hence an isomorphism.

Assume $T : V \to W$ is an isomorphism. Since it follows that T is bijective, it is invertible. Let $u', v' \in W$. It follows there exists some $u, v \in V$ such that $T(u) = u'$ and $T(v) = v'$. By the property of T being linear, we have:

$$T^{-1}(u' + v') = T^{-1}(T(u) + T(v)) = T^{-1}(T(u + v)) = u + v = T^{-1}(u') + T^{-1}(v')$$
$$T^{-1}(cu') = T^{-1}(cT(u)) = T^{-1}(T(cu)) = cu = cT^{-1}(u')$$

Thus T^{-1} preserves additivity and homogeneity and is therefore a linear map. And moreover, since it is the inverse of a bijective mapping, it is also bijective, and thus an isomorphism. QED.

Theorem 1.4.2: Isomorphisms are an equivalence relation. In other words, if U, V, W are all vector spaces over the same field F, the following properties hold:

$$U \cong U$$
$$U \cong V \implies V \cong U$$
$$U \cong V \text{ and } V \cong W \implies U \cong W$$

This proof is left to the non-existent reader.

2 Structure of Vector Spaces

2.1 Spans and Generating Sets

Theorem 2.1.1: Suppose A is a nonempty subset of a vector space V and that W is a subspace of V. Then $Span(A)$ is the smallest subspace of V containing A. In other words:

$$A \subseteq Span(A)$$
$$A \subseteq W \implies Span(A) \subseteq W$$

Let $\alpha \in A$. Since $Span(A)$ is the set of all linear combinations of vectors in A, it follows the linear combination $1\alpha = \alpha \in Span(A)$. Thus $A \subseteq Span(A)$.

Assume $A \subseteq W$ and $w \in Span(A)$. It follows we can write $w = c_1a_1 + \cdots + c_na_n$ for some $c_1, \ldots, c_n \in F$ and $a_1, \ldots, a_n \in A$. Since W is a vector space, it satisifies closure properties under addition and scalar multiplication. Further, since $A \subseteq W$, it follows that $a_i + a_j \in W$ and $c_ia_i \in W$ for any $a_i, a_j \in A$ and $c_i \in F$. Therefore, it must follow that $w = c_1a_1 + \cdots + c_na_n \in W$ and we can conclude $Span(A) \subseteq W$. QED.
Definition 2.1.1: If V is a vector space with $A \subseteq V$, we say that $Span(A)$ is the subspace of V generated by A, and thus A is the generating set for $Span(A)$. We say that A generates $Span(A)$, so it follows if $Span(A) = V$, we say A generates V.
Theorem 2.1.2: Suppose $T : V \to W$ is a linear map and A is a subset of V. Then $SpanT(A) = T(Span(A))$.

Let $a_1, \ldots, a_n$ be the vectors in A and let $c_1, \ldots, c_n \in F$. We prove this in two directions. First, assume $c_1T(a_1) + \cdots + c_nT(a_n) \in SpanT(A)$. We can use the properties of linear maps to show:

$$\begin{aligned} c_1T(a_1) + \cdots + c_nT(a_n) &= T(c_1a_1) + \cdots + T(c_na_n) \\ &= T(c_1a_1 + \cdots + c_na_n) \in T(Span(A)) \end{aligned}$$

Thus it follows $SpanT(A) \subseteq T(Span(A))$. In the other direction, assume $T(c_1a_1 + \cdots + c_na_n) \in T(Span(A))$. Again, by the properties of linear maps:

$$\begin{aligned} T(c_1a_1 + \cdots + c_na_n) &= T(c_1a_1) + \cdots + T(c_na_n) \\ &= c_1T(a_1) + \cdots + c_nT(a_n) \in SpanT(A) \end{aligned}$$

Thus $T(Span(A)) \subseteq SpanT(A)$. We therefore conclude $SpanT(A) = T(Span(A))$. QED.

2.2 Linear Dependence/Independence

Definition 2.2.1: Suppose V is a vector space and $v_i \in V$ are a list vectors in V and $c_i \in F$ are a list of scalars in F with $i \in \mathbb{N}$. If at least one of the c_i is nonzero and:

$$\sum_{n=1}^{i} c_n v_n = \vec{0}$$

We say that the list of vectors is linearly dependent. If the c_i must all be zero for the equation to hold true, the list is said to be linearly independent.

Definition 2.2.2: If A is any (possibily infinite) set of vectors in a vector space V, if every finite list of distinct vectors in A is linearly independent, then A is linearly independent. However, if there is some finite list of vectors in A that is linearly dependent, then A is linearly dependent.

Theorem 2.2.1: Suppose V is a vector space over a field F with $v_1, \ldots, v_n \in V$. Define the linear map $T : F^n \to V$ by $T(a_1, \ldots, a_n) = a_1v_1 + \cdots + a_nv_n$. The list of vectors $v_1, \ldots, v_n$ are linearly dependent iff T is not injective.

We prove this in two directions. First, assume $v_1, \ldots, v_n$ are linearly dependent. Since the vectors are linearly dependent, there exists some $a_1, \ldots, a_n \in F$, not all zero, such that $a_1v_1 + \cdots + a_iv_i + \cdots + a_nv_n = \vec{0}$. It follows that $(a_1, \ldots, a_i, \ldots a_n) \in Ker(T) \implies Ker(T) \neq \{\vec{0}\}$. Thus T cannot be injective.

Now, assume T is not injective. It follows there is some $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in F^n$ such that $T(a_1, \ldots, a_n) = T(b_1, \ldots, b_n)$ and $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$. We thus have:

$$\begin{aligned} T(a_1, \ldots, a_n) = T(b_1, \ldots, b_n) &\implies a_1v_1 + \cdots + a_nv_n = b_1v_1 + \cdots + b_nv_n \\ &\implies (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n = \vec{0} \end{aligned}$$

Since $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, it follows that $(a_1 - b_1, \ldots, a_n - b_n) \in F^n$ is nonzero, meaning at least one component $a_i - b_i \neq \vec{0}$. Since $(a_1 - b_1, \ldots, a_n - b_n) \in Ker(T)$ and is nonzero, the vectors $v_1, \ldots, v_n$ are linearly dependent. QED.

Theorem 2.2.2: Suppose $T : V \to W$ is an injective linear map and $v_1, \ldots, v_n$ are linearly independent vectors in V, then $T(v_1), \ldots, T(v_n)$ are linearly independent vectors in W.

Let $c_1, \ldots, c_n \in F$. Since T is a linear map, we have:

$$c_1T(v_1) + \cdots + c_nT(v_n) = \vec{0} \implies T(c_1v_1 + \cdots + c_nv_n) = \vec{0}$$

Further, since T is injective and $v_1, \ldots, v_n$ are linearly independent:

$$T(c_1v_1 + \cdots + c_nv_n) = \vec{0} \implies c_1v_1 + \cdots + c_nv_n = \vec{0} \implies c_1 = \cdots = c_n = 0$$

QED.