Geometry Exercises

Contents

1 Chapter 1

1.1 Exercise I-1

Show that the function $f : \mathbb{E}^n \to \mathbb{E}^n$ defined by $f(\vec{v}) = 2\vec{v}$ is bijective but not an isometry.

We first show f is bijective. To show injectivity, let $\vec{x}, \vec{y} \in \mathbb{E}^n$ such that $f(\vec{x}) = f(\vec{y})$. Then:

$$2\vec{x} = 2\vec{y} \implies \vec{x} = \vec{y}$$

Since $\vec{x} = \vec{y}$, f is injective. We now show f is surjective, let $\vec{m} \in \mathbb{E}^n$. We aim to find some $\vec{v} \in \mathbb{E}^n$ such that $f(\vec{v}) = \vec{m}$. So:

$$2\vec{v} = \vec{m} \implies \vec{v} = \frac{1}{2}\vec{m}$$

Indeed, checking this:

$$f\left(\frac{1}{2}\vec{m}\right) = 2\left(\frac{1}{2}\vec{m}\right) = \vec{m}$$

This shows f is surjective. Thus, since f is both injective and surjective, f is bijective.

To show f is not an isometry, we proceed by counterexample. Let $(1,5), (2,3) \in \mathbb{E}^n$. For f to be an isometry we must have:

$$\begin{aligned} ||(1,5) - (2,3)|| &= ||f(1,5) - f(2,3)|| \\ ||(-1,2)|| &= ||(-2,4)|| \\ \sqrt{(-1)^2 + (2)^2} &= \sqrt{(-2)^2 + (4)^2} \\ \sqrt{5} &= \sqrt{20} \\ \sqrt{5} &= 2\sqrt{5} \end{aligned}$$

which is clearly false.

Therefore, f is bijective but not an isometry. $\square$

2 Chapter 6