MA2MMS Project A. Modelling Ecological Systems

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Abstract

Add a brief abstract with a description and conclusion here ...

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Description of the Ecological System

Add some description

Introduction

Add a comprehensive intro

Model Selection

Our aim is modelling *global* populations of red and grey squirrels over time. We begin by considering the *generic Lotka-Volterra* system s.t.

Generic Model

$$\boxed{\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x(r_1 + a_{11}x + a_{12}y),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(r_2 + a_{21}x + a_{22}y).\end{aligned}}$$

where x and y would represent the populations of red and grey squirrels respectively,

- r_1 - idealistic growth rate of red squirrels
- r_2 - idealistic growth rate of grey squirrels
- a_{11} - measure of limitation on red squirrels
- a_{22} - measure of limitation on grey squirrels
- a_{12} - measure of competition towards red squirrels i.e. how much ys hurt xs
- a_{21} - measure of competition towards greys squirrels i.e. how much xs hurt ys

NB a_{11} and a_{22} along with r_1 and r_2 respectively yield the carrying capacities K_R and K_G.

The model in this form is very clear and versatile as it allows easy implementation into numerical algorithms because the equations do not involve division. *NB* This form will prove to be convenient when we will solve the system numerically later on.

For now we introduce a new form of our model with the carrying capacities K_R and K_G explicitly present to aid analysis of stability and picking coefficients below.

Choosing Coefficients

We are looking at ... red and grey squirrel populations over time using the *generic Lotka-Volterra* Competitive model in a *refined form.*

Generic Model - Refined form

$$\boxed{\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}t} &= r_R R\left(1 - \frac{R - G\alpha_{RG}}{K_R}\right)\\ \frac{\mathrm{d}G}{\mathrm{d}t} &= r_G G\left(1 - \frac{G - R\alpha_{GR}}{K_G}\right)\end{aligned}}$$

where $R(t)$ and $G(t)$ represent the population of red and grey squirrels at a given time t, r_R and r_G represent the intrinsic growth rates of red and grey squirrels respectively, K_R and K_G represent the carrying capcity of red and grey squirrels, and finally α_{RG} and α_{GR} represent the competition coefients - the effect of grey squirrels on red squirrels and the effect of red squirrels on grey squirrels respectively.

NB This model is based on the logistic growth model ($\frac{\mathrm{d}x}{\mathrm{d}t} = rx(1 - \frac{x}{K})$), with the addition of competition between the two species as they compete for the same natural resources.

add more reasoning and sources behind coeffs

Stability of the model

We have the coupled differential equations

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}t} &= 0.61R(1 - \frac{R - 0.8G}{K_R})\\ \frac{\mathrm{d}G}{\mathrm{d}t} &= 0.82G(1 - \frac{G - 0.09R}{K_G})\end{aligned}$$

where $K_G = 3\text{Ö}10^6$ and $K_R = 2.5\text{Ö}10^6$

To find the stability of the model, we have to find the equilibria of the system of equations and examine the stability of these points:

The R-nullclines are found to be $R = 0$ or $R = K_R - 0.8G$ by setting $\frac{\mathrm{d}R}{\mathrm{d}t} = 0$.

The G-nullclines are found to be $G = 0$ or $G = K_G - 0.09R$ by setting $\frac{\mathrm{d}G}{\mathrm{d}t} = 0$.

The equilibrium points are found from the intersections of the R and G nullclines, and are given to be: $(0, 0)$, $(K_R, 0)$, $(K_G, 0)$ and $(\frac{K_R - 0.8K_G}{0.928}, \frac{K_G - 0.09K_R}{0.928})$.

The Jacobian matrix of the system is given by:

$$J(R, G) = \begin{bmatrix} 0.61 - \frac{1.22}{K_R}R - \frac{0.488}{K_R}G & -\frac{0.488}{K_R} \\ -\frac{0.0738}{K_G}G & 0.82 - \frac{0.18}{K_G}G - \frac{0.0738}{K_G}R \end{bmatrix}$$

Now looking at the Jacobian at the equilibria points:

At the equilibrium point $(0, 0)$, we have the Jacobian matrix

$$J(0, 0) = \begin{bmatrix} 0.61 & 0 \\ 0 & 0.82 \end{bmatrix}$$

Since $J(0, 0)$ is a diagonal matrix, it has two eigenvalues which are $\lambda_1 = 0.61$ and $\lambda_2 = 0.82$. Since these eigenvalues are non-positive real numbers, the corresponding fixed point is an unstable source. this means....

At the equilibrium point $(K_R, 0)$, we have the Jacobian matrix

$$J(K_R, 0) = \begin{bmatrix} -0.61 & -0.488 \\ 0 & 0.82 - \frac{0.073K_R}{K_G} \end{bmatrix}$$

Since $J(K_R, 0)$ is an upper triangular matrix, it has two eigenvalues which are $\lambda_1 = -0.61$ and $\lambda_2 = 0.82 - \frac{0.073K_R}{K_G}$. Since these eignevalues are real numbers with $\lambda_1 < 0 < \lambda_2$, the corresponding fixed point is an unstable saddlepoint. This means...

At the equilibrium point $(0, K_G)$, we have the Jacobian matrix

$$J(0, K_G) = \begin{bmatrix} 0.61 - \frac{0.488K_G}{K_R} & 0 \\ -0.0738 & -0.82 \end{bmatrix}$$

Since $J(0, K_G)$ is a lower triangular matrix, it has two eigenvalues which are $\lambda_1 = 0.61 - \frac{0.488K_G}{K_R}$ and $\lambda_2 = -0.82$. Since these eignevalues are real numbers with $\lambda_2 < 0 < \lambda_1$, the corresponding fixed point is an unstable saddlepoint. This means...

Finally at the equilibrium point $(\frac{K_R - 0.8K_G}{0.928}, \frac{K_G - 0.09K_R}{0.928})$, we have the Jacobian matrix

$$J(\frac{K_R - 0.8K_G}{0.928}, \frac{K_G - 0.09K_R}{0.928}) = \begin{bmatrix} \frac{0.488K_G - 0.61K_R}{0.928K_R} & \frac{0.3904K_G - 0.488K_R}{0.928K_R} \\ \frac{0.006642K_R - 0.0738K_G}{0.928K_G} & \frac{0.738K_R - 0.82K_G}{0.928K_G} \end{bmatrix}$$

$J(\frac{K_R - 0.8K_G}{0.928}, \frac{K_G - 0.09K_R}{0.928})$ has two eigenvalues which are

$$\lambda_{1,2} = \frac{1}{2}\left[\frac{0.488K_G^2 + 0.738K_R^2 - 1.43K_RK_G}{0.928K_RK_G} \pm \sqrt{\left(\frac{\text{num1}}{0.928K_RK_G}\right)^2 - 4\frac{\text{num2}}{(0.928)^2K_RK_G}}\right]$$

NB See *Notes on Stability* for precise calculations of the numerators inside the square root.

Since these eigenvalues are real, negative numbers, ($\lambda_1 \approx -0.0186$ and $\lambda_2 \approx -0.2287$) the corresponding fixed point is an asymptotically stable sink point. This means...

Numerical Solution

For the purposes of constructing our numerical algorithm, we return to our generic model in its original form. So we must calculate the coefficients $r_1, r_2, a_{11}, a_{12}, a_{21}, a_{22}$ for our populations x and y of red and grey squirrels respectively.

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}t} &= r_R R\left(1 - \frac{R - G\alpha_{RG}}{K_R}\right) \\ \frac{\mathrm{d}G}{\mathrm{d}t} &= r_G G\left(1 - \frac{G - R\alpha_{GR}}{K_G}\right) \end{aligned} \quad \longrightarrow \quad \begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x(r_1 + a_{11}x + a_{12}y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(r_2 + a_{21}x + a_{22}y). \end{aligned}$$

We expand the equations and compare coefficients for $x = R$ and $y = R$ s.t.

$$\frac{\mathrm{d}R}{\mathrm{d}t} = r_R R\left(1 - \frac{R}{K_R} + \frac{G\alpha_{RG}}{K_R}\right) = \underbrace{r_R}_{r_1} R + \underbrace{\left(-\frac{r_R}{K_R}\right)}_{a_{11}} R^2 + \underbrace{\left(\frac{r_R\alpha_{RG}}{K_R}\right)}_{a_{12}} RG$$

$$\frac{\mathrm{d}G}{\mathrm{d}t} = r_G G\left(1 - \frac{G}{K_G} + \frac{R\alpha_{GR}}{K_G}\right) = \underbrace{r_G}_{r_2} G + \underbrace{\left(\frac{r_G\alpha_{GR}}{K_G}\right)}_{a_{21}} RG + \underbrace{\left(-\frac{r_G}{K_G}\right)}_{a_{22}} G^2$$

NB We use coefficients found above, $K_G = 3 \cdot 10^6$, $K_R = 2.5 \cdot 10^6$, $\alpha_{RG} = 0.8$ and $\alpha_{GR} = 0.09$.

By comparison we have
$$\boxed{\begin{aligned} r_1 &= r_R \\ r_2 &= r_G \\ a_{11} &= -\frac{r_R}{K_R} \\ a_{12} &= \frac{r_R \alpha_{RG}}{K_R} \\ a_{21} &= \frac{r_G \alpha_{GR}}{K_G} \\ a_{22} &= -\frac{r_G}{K_G} \end{aligned}} \longrightarrow \boxed{\begin{aligned} r_1 &= 0.61 \\ r_2 &= 0.82 \\ a_{11} &= -2.44 \cdot 10^{-7} \\ a_{12} &= 1.952 \cdot 10^{-6} \\ a_{21} &= 2.46 \cdot 10^{-8} \\ a_{22} &= -2.7\dot{3} \cdot 10^{-7} \end{aligned}}$$

! The a_{22} is a recurring decimal marked $\dot{3}$. In the actual implementation, 8 decimal points will be used to minimise the *roundoff error*.

Local Truncation Error

add discussion on error inherited by the numerical method

Potential Bifurcation

add some on bifurcation [1]

Appendix

Add any ideas/manuscripts, links and references below, treat this as a draft for now

Notes on Stability

$$\text{num1} = 0.488K_G^2 + 0.738K_R^2 - 1.43K_R K_G$$

$$\begin{aligned} \text{num2} = &(0.488K_G - 0.61K_R)(0.738K_R - 0.82K_G) \\ &- (0.3904K_G - 0.488K_R)(0.006642K_R - 0.0738K_G) \end{aligned}$$

Jacobian formula

$$J(R,G) = \begin{bmatrix} \frac{\partial F}{\partial R} & \frac{\partial F}{\partial G} \\ \frac{\partial E}{\partial R} & \frac{\partial E}{\partial G} \end{bmatrix}$$

Where $F(R,G) = \frac{\mathrm{d}R}{\mathrm{d}t}$ and $E(R,G) = \frac{\mathrm{d}G}{\mathrm{d}t}$

Notes on Runge-Kutta

An s-stage **Runge–Kutta method** approximates u by specifying constants a_{ij}, b_j and c_i for $i, j = 1, 2, \ldots, s$ where $s \in \mathbb{N}$.

Compute the s intermediary steps

$$\boxed{k_i = f\big(t_{n-1} + c_i\, h,\ U^{n-1} + h \sum_{j=1}^{s} a_{ij}\, k_j\big) \quad \text{for} \quad i = 1, 2, \ldots, s}$$

Compute the nth approximation

$$\boxed{U^n = U^{n-1} + h \sum_{j=1}^{s} b_j\, k_j}$$

[1] Bifurcation theory: https://en.wikipedia.org/wiki/Bifurcation_theory

Python code

4th Order Runge-Kutta method

```python
import numpy as np
import matplotlib.pyplot as plt
```

to be added

Bibliography

Draft (to be removed)

$$\lambda_{1,2} = \frac{1}{2}\left[\frac{0.488K_G^2 + 0.738K_R^2 - 1.43K_RK_G}{0.928K_RK_G} \pm \sqrt{\left(\frac{0.488K_G^2 + 0.738K_R^2 - 1.43K_RK_G}{0.928K_RK_G}\right)^2 - 4\frac{(0.488K_G - 0.61K_R)(0.738K_R - 0}{}}\right.$$