MATHEMATICAL MODELLING and PROFESSIONAL SKILLS (MA2MMS)

Mathematical Modelling Project 2

Contents

1 General Instruction

- Students work in groups each consisting of 3-4 members. Each group is allocated with one of the listed projects. Members in the same group should work as a team and make evenly distributed contributions to the development of the project.

- The listed projects are based on the mathematical materials covered in Chapters 3-6 of the lecture notes and related problem sheets. Numerical calculations can be done using Matlab, any other programming languages or numerical tools. Please state clearly what numerical methods and/or tools are used and, if applicable, attach your codes as appendix in the report.

- The recommended elements listed in the project description are only for reference. You can choose to address all or some of them, and more importantly consider covering mathematical (analytical and/or numerical) and subject-related contents beyond the recommended scope. Students working individually can decide to cover less elements within the given time and page limit.

- Considering the complexity of real-world problems, if no suitable data sets are available for calibrating the model parameters, you may use parameter values that can reasonably reflect the real-world situation.

- For any downloaded data sets, clearly state their resources;

- For GenAI use, this is a Category 2 assessment, meaning that GenAI use is permitted to support student learning and development, see https://www.reading.ac.uk/cqsd/artificial-intelligence for more details. You must clearly state which parts in your report were developed with the assistance of GenAI.

- The project outputs will be assessed in the form of written report up to 8-11 pages for groups, or 5-8 pages for individual projects. Embedded figures will count towards page limit, but references and computer code (if included) will not.

- The report can be written in either Microsoft Word or LaTex. For group projects, group members should manage to reach consensus on which editor to use. In case that no consensus is reached, you may develop the report using two different editors and merge them into one, say by converting the PDF file generated from LaTex to a Word and then merging it with the Word file produced by other group members. In the latter case, you must state clearly what has been done to the file so that the format conversion will not affect the marking of your report.

- Both the teamwork and individual contributions to the project development will be counted to the final assessment results.

- If having any questions or ideas about the projects, do discuss with the lecturers to avoid any confusions or delays in progress.

- This assessment counts to 45% of the overall module mark. **The submission is due at 12:00pm on Wednesday 14th May.**

2 Project Options

2.1 Project A. Modelling Biological/Ecological Systems

Choose to study a biological or ecological system where two species of populations $x(t)$ and $y(t)$ are either competing for the same resources (such as Lions and Hyenas, Oaks and Maples in a forest, etc) or in mutualistic relation (such as Bees and Flowering Plants, Humans and Gut Bacteria, etc). Develop a mathematical model to describe the population growth of the system, which can take the form of

$$\frac{dx}{dt} = x(r_1 + a_{11}x + a_{12}y) \tag{2.1.1}$$

$$\frac{dy}{dt} = y(r_2 + a_{21}x + a_{22}y) \tag{2.1.2}$$

or its variation. Here the coefficients r_1 and r_2 are positive constants, while a_{11}, a_{12}, a_{21} and a_{22} can have different signs and take different constant values (including zeros), depending on the system to be studied.

Recommended elements

- Describe the selected biological or ecological system, such as the species and the relationship between them and, if available, download data sets about their population changes with time.
- Discuss the choices of the coefficients' signs and values in the selected mathematical model and give reasoning.

- Find the equilibrium points of the system, and discuss the stability and behaviour of trajectories in the vicinity of these points.

- Solve the mathematical model to get $x(t)$ and $y(t)$ by using both analytical (by converting the nonlinear systems into linear systems) and numerical (finite difference algorithm or ODE solvers) methods, or at least one of these two approaches (e.g., for students working individually). If having real-world data, you may use them to guide your model parameter setting.

- Find analytical and/or numerical solutions for the trajectories of the system in the $x - y$ plane.

- Present your analytical and numerical results in plots, and examine your discussions about the equilibrium points and their stability and behaviours of trajectories in the vicinity of these points.

- Make predictions of the populations in later years.

- Discuss the performance of the mathematical model and possible improvements for describing the real-world problem.

- Give advices on how to make the biological or ecological system sustainable based on your model predictions, meaning that the populations of the species are maintained at stable non-zero steady states. (If studying infectious microorganisms such as virus or bacteria, the desired steady-states of their populations would be zero).

- If time allows, consider developing and testing an improved version of the applied model, or testing the performance of a different model on describing the studied system.

2.2 Project B. Modelling Antimicrobial Resistance

Antibiotic is a type of antimicrobial substance active against microorganisms such as bacteria. The efficacy of an antibiotic is typically high at the beginning, but decreases with time as bacteria evolve and develop antimicrobial resistance due to various reasons, such as the appearance of mutants that are insensitive to this antibiotic.

Consider two types of bacterial mutants coexist and compete in the body of an organism. One type with population $x_1(t)$ is sensitive to an antibiotic, while the other type with population $x_2(t)$ is resistant to this antibiotic. The system can be modelled by

$$\frac{dx_1}{dt} = (d_1 - a_1(q_1x_1 + q_2x_2))\,x + b_2x_2 - cx_1^{\alpha} \tag{2.2.3}$$

$$\frac{dx_2}{dt} = (d_2 - a_2(q_1x_1 + q_2x_2)\,y + b_1x_1 \tag{2.2.4}$$

where the last term on the r.h.s. of Equation (2.2.3) characterises the efficacy of the antibiotic effect on the first type of bacteria. The coefficient d_i ($i = 1$ or 2) is the growth rate of type-i bacteria in the absence of the other type, while a_i is the coefficient characterising the decrease rate of type-i population in proportion to the total number of resource consumed by the two populations. q_i is the resource consumption per microorganism in type-i bacteria. If $c = 0$, there is no antibiotic used. Otherwise c and α are positive coefficients.

Recommended elements

- Search if there is any available data about the populations of two competing bacteria. If not, you may refer to Appendix 5 in Chapter

7 of [1] for some general information which may help choose your own parameter values to reasonably describe the real-world process.

- Consider first studying the population dynamics of the system without antibiotic intervention, i.e. setting $c = 0$. This will allow you to carry out the equilibrium point and stability analysis suggested below:

  - Find the equilibrium points of the system, and discuss the stability and behaviour of trajectories in the vicinity of these points.
  - Solve the mathematical model to get $x_i(t)$ with $i = 1, 2$ by using both analytical (by converting the nonlinear systems into linear systems) and numerical (finite difference algorithm or ODE solvers) methods. or at least one of these two approaches (e.g., for students working individually). If having real-world data sets, find model parameter values by fitting model predictions to these data.
  - Find analytical and/or numerical solutions for the trajectories of the system in the $x_1 - x_2$ plane.
  - Present your analytical and numerical results in plots, and examine your discussions about the equilibrium points and their stability and behaviours of trajectories in the vicinity of these points.
  - Make predictions of the populations in later years.

  - Discuss the performance of the mathematical model and possible improvements for describing the real-world problem.

- Now add the antibiotic effect into the model by taking $c > 0$. Solve the updated mathematical model (possibly numerically) for different values of c and α to obtain results on $x_i(t)$.

- Present the results on $x_i(t)$ and correspondingly trajectories $x_1 - x_2$ plane obtained at different values of $c > 0$ and α in plots, in comparison with those obtained without antibiotic ($c = 0$).

- Based on your model predictions, discuss about the antimicrobial effects of antibiotic on the bacteria populations, and possible ways to enhance the antimicrobial effects of antibiotics and/or reduce the antimicrobial resistance of the microorganism (bacteria).

2.3 Project C. Modelling Economic Competition

Consider two firms produce the same type of products and focus on same group of consumers. In the absence of sales problems, both firms make profits which increase their capitals, x_1 and x_2, and can be invested in growing more products. But their sales and so growth in capitals are limited by the fixed number of potential buyers and their needs. Assuming the decrease in the growth of the firms' income is proportional to the sum of the capitals of both firms the economic competition model is given by

$$\frac{dx_i}{dt} = \left(\epsilon_i - \frac{1}{\beta_i}(x_1 + x_2)\right) x_i,\ i = 1, 2 \tag{2.3.5}$$

where the coefficient ϵ_i is the capital gain of each firm in the absence of the other one. β_i characterises the sales efficiency of products and is associated with the advertising, service quality, etc. A high β_i leads to better capital gain rate. The decisive parameters in the system are $\epsilon_i \beta_i$ which measure the effectiveness of production and marketing of the firms. The firm with lower value of $\epsilon_i \beta_i$ will lose business.

Recommended elements

- Check if there is any available data about the capitals of two competing firms. If so, use these data guide your determination of the model parameter values. Otherwise, you may also refer to Chapter 8.2 in [1] for some general information which can help choose your own parameter values to reasonably describe the real-world problem.

- Find the equilibrium points of the system, and discuss the stability and behaviour of trajectories in the vicinity of these points.

- Solve the mathematical model to get $x(t)$ and $y(t)$ by using both analytical (by converting the nonlinear systems into linear systems) and numerical (finite difference algorithm or ODE solvers) methods, or at least one of these two approaches (e.g., for students working individually). If having real-world data, find model parameter values by fitting model predictions to these data.

- Produce model predictions on $x(t)$ and $y(t)$ for different combinations of $\epsilon_1\beta_1$ and $\epsilon_2\beta_2$ values, present the corresponding trajectories of the system in the $x - y$ plane in different plots, and examine your discussions about the equilibrium points and their stability and behaviours of trajectories in the vicinity of these points.

- Based on your results, discuss what are the possible outcomes of the competition, say win-win, win-lose or lose-lose, and what are the conditions for each of these possibilities in terms of the $\epsilon_i\beta_i$ values.

- Give advice on how to make the competition beneficial to the economics and society, for example, via optimising the total capitals of the whole system.

2.4 Project D. Modelling Bungee Jumping

A theme park is considering to develop a bungee jumping programme. The jump launch spot is going to be set at a height H above the water surface of a lake. The jumper will be connected to the launch spot by a shock cord with elastic constant k. You are invited to develop a mathematical model to help design the bungee programme, which will ensure the programme is exciting and safe.

Your model should be able to describe the dynamics of the bungee jumping process, including an initial free fall followed by bouncing up and down due to the stretching and recoiling of the elastic cord, and address possible risks such as improper choice of the cord's elasticity and/or length that leads the jumper to dive into water, etc.

Recommended elements

1. Identify the forces involved in the bungee jumping process, and find relevant parameter values for calculating these forces, such as typical elastic constant of the cord, air and water viscosities, etc.

2. Find representative values for other model parameters, such as height of the launch spot, length of undeformed cord, and body weight distribution of jumpers, etc.

3. Build the equation of motion of the jumper and solve the equation to obtain model predictions on their position and velocity as a function of time.

4. Produce model predictions under different conditions, such as different combinations of the elasticity and length of the cord, and

different body weights of jumpers.

5. Investigate and discuss the cases where the jumper is not held back by the cord and so dive into water at a non-zero speed; Support your discussions by showing the moving trajectories of the jumper, including the stages diving into water.

6. Find out how to control the length and/or elasticity of the cord to allow the jumper with a given body weight just touch the water surface and then be bounced up. This will make the project more exciting.

7. If the park plans to run projects allowing two people jump together, discuss how to connect them to two or more cords to ensure the safety of jumpers.

2.5 Project E. Modelling Satellite Launching

Consider the process that a rocket carrying a satellite of mass m_s is launched at time $t = 0$ with initial velocity v_0 from the surface of Planet Earth. The initial total mass of the rocket system, including that of the satellite, is $M = M_0$. The rocket is propelled by ejecting mass downwards at a constant rate $\lambda > 0$ in units of kilograms per second and the ejected mass has a constant velocity of magnitude $u > 0$ relative to the centre of the rocket. The rocket flies upwards to reach a Low Earth Orbit (LEO) of altitude H (< 2000 kilometers) from the Earth's surface and projects the satellite into the orbit at an angle θ with respect to the radial vector pointing outwards from the centre of the Earth. The rocket then reduces the rate of mass propelling to allow itself move backwards towards the Earth's surface until landing on the ground with zero velocity. The acceleration of gravity g can be assumed to be constant for the entire process.

For students working in groups, consider modelling both the cases with and without air resistance. For students working individually, consider modelling one of the two cases. If included, the air resistance force can be approximated by the Stokes' law, i.e., being proportional to the product of the moving velocity and the cross-section diameter of the rocket.

Recommended elements

1. Search for model parameter values, such as the masses of the LEO satellites and launch vehicles (rockets), the dimensions of the rockets (such as diameters and/or areas of cross-sections), the altitudes of the LEO, the typical launching velocities of the launch vehicles, the average air density and/or viscosity in the atmosphere, etc.

2. Develop equations of motion to describe the launching and landing processes of the rocket. Choose reasonable model parameter values based on the data collected. Any parameter values within the range of published data and convenient for your model calculations are acceptable. In particular, you should decide on the values of the mass propelling rate during the launching and landing processes to make sure that the rocket can reach the desired LEO altitude and return to the ground safely.

3. Solve the initial value problems for the launching and landing processes separately using suitable coordinate systems to get the travelling distance $x(t)$, velocity $v(t)$ and total mass $M(t)$ of the rocket as a function of time t. Provide analytical (if possible) and numerical solutions under given conditions.

4. Plot the solutions for $x(t)$, $v(t)$ and $M(t)$ in separate figures. If getting both analytical and numerical results, plot them in the same figure for comparison.

5. If taking into account air resistance, consider to produce numerical solutions to the equation of motion, if analytical solutions are difficult to obtain.

6. Discuss if your mathematical models provide a reasonale description of the real launching and landing processes of rockets and what factors could have been included to improve the models.

7. If time allows, modelling the motion of the satellite in its LEO orbit

by referring Chapter 4.2 in [5]. Note that the results depend on the launching angle θ w.r.t. the radial vector.

References

[1] S. Serovajsky, *Mathematical Modelling*, CRC Press, Taylor & Francis Group, LLC, 2022. (Available electronically in university library)

[2] W. J. Duncan, *Physical Similarity and Dimensional Analysis*, Edward Arnold & Co., London, 1953.

[3] Glenn Fulford, Peter Forrester & Arthur Jones, *Modelling with Differential and Difference Equations*, Cambridge University Press, 1996.

[4] Frank R. Giordano, Maurice D. Weir & William P. Fox, *A First Course in Mathematical Modeling*, BrooksCole Publishing Company, 1997.

[5] J. N. Kapur, *Mathematical Modelling*, Mercury Learning and Information, 2023.

[6] B. S. Massey, *Units, Dimensional Analysis and Physics Similarity*, Van Nostrand Reinhold Company, London, 1971.

[7] Douglas D. Mooney & Randall J. Swift, *A Course in Mathematical Modeling*, The Mathematical Association of America, 1999.

[8] Padmanabhan Seshaiyer, *Leading Undergraduate Research Projects in Mathematical Modeling*, PRIMUS (Problems, Resources, and Issues in Mathematics Undergraduate Studies), 27 (2017) 476-493

[9] Books in the category of 511.8 in the library of University of Reading.