Fourier Series

Digital Signal Processing

February 6, 2024

Sum of Cosine and Sine

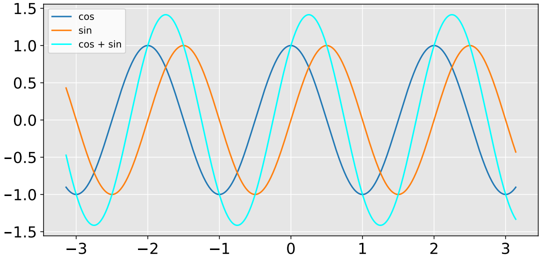
Adding a cosine and sine with same frequency, results in another cosine with the same frequency, but **different amplitude** and **phase**.

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \phi),$$

where

$$C = \sqrt{A^2 + B^2}, \quad \text{and} \quad \phi = \operatorname{atan2}(B, A).$$

Sum of Cosine and Sine



$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \phi),$$

$$A = 1, \quad B = 1, \quad \omega_0 = \pi.$$

So,

$$C = \sqrt{A^2 + B^2} = \sqrt{2}, \quad \text{and} \quad \phi = \operatorname{atan2}(B, A) = \frac{\pi}{4}.$$

Derivation (Optional)

Consider the sum of complex exponentials:

$$\begin{aligned} Ae^{i\omega_0 t} + Be^{i(\omega_0 t - \frac{\pi}{2})} &= e^{i\omega_0 t}(A + e^{-i\frac{\pi}{2}}B) \\ &= e^{i\omega_0 t}(A - iB) \\ &= e^{i\omega_0 t}\sqrt{A^2 + B^2}e^{-i\phi} \\ &= \sqrt{A^2 + B^2}e^{i(\omega_0 t - \phi)} \end{aligned}$$

Taking real part of left side gives:

$$A\cos(\omega_0 t) + B\cos(\omega_0 t - \frac{\pi}{2}) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

Taking real part of right side gives:

$$\sqrt{A^2 + B^2}\cos(\omega_0 t - \phi)$$

Can we build other functions as the sum of cosines and sines?

Fourier Series

We can write (almost) any continuous periodic function, $x(t)$, with period L, as a linear combination of cosines and sines:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi k t}{L}\right) + b_k \sin\left(\frac{2\pi k t}{L}\right) \right],$$

where a_k, b_k are constants.

Fourier Series

The constants are given by

$$a_k = \frac{2}{L}\int_0^L \cos\left(\frac{2\pi kt}{L}\right) x(t)dt,$$
$$b_k = \frac{2}{L}\int_0^L \sin\left(\frac{2\pi kt}{L}\right) x(t)dt,$$

Frequencies in the Fourier Series

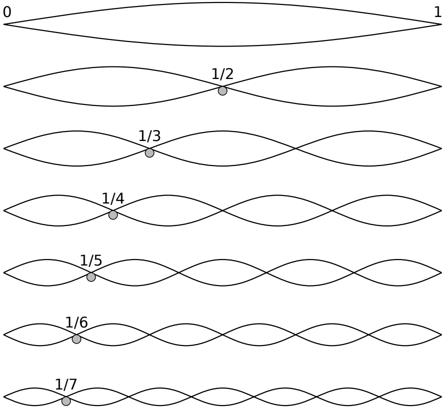
If $x(t)$ has period L, it's frequency is $\omega_0 = \frac{2\pi}{L}$.

Notice in the Fourier series for $x(t)$, the frequencies are integer multiples, $k\omega_0 = \frac{2\pi k}{L}$:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi k t}{L}\right) + b_k \sin\left(\frac{2\pi k t}{L}\right) \right].$$

ω_0 is called the **fundamental frequency**, and the $k\omega_0$ are called **harmonics**.

Harmonics of a Vibrating String



A vibrating string, with fixed endpoints, will move at a fundamental frequency and at integer multiples of that frequency.

These are called **harmonics** because they are in harmony with the fundamental frequency.

Discrete-Time Fourier Series

Now consider a discrete, periodic signal, $x[n]$, with period L.

The discrete-time Fourier series just replaces the continuous variable, $t \in \mathbb{R}$, with the discrete variable, $n \in \mathbb{Z}$!

$$x[n] = \frac{a_0}{2} + \sum_{k=1}^{L-1} \left[a_k \cos\left(\frac{2\pi k n}{L}\right) + b_k \sin\left(\frac{2\pi k n}{L}\right) \right].$$

Discrete-Time Fourier Series

Replace integration with summation to get the constants:

$$a_k = \frac{2}{L} \sum_{n=0}^{L-1} \cos\left(\frac{2\pi kn}{L}\right) x[n],$$

$$b_k = \frac{2}{L} \sum_{n=0}^{L-1} \sin\left(\frac{2\pi kn}{L}\right) x[n].$$