Homework 4: Sampling and All-Pass Systems

Instructions: Submit a single Jupyter notebook (.ipynb) of your work to Collab by 11:59pm on the due date. All code should be written in Python. **Be sure to show all the work involved in deriving your answers! If you just give a final answer without explanation, you may not receive credit for that question.**

You may discuss the concepts with your classmates, but write up the answers entirely on your own. Do not look at another student's answers, do not use answers from the internet or other sources, and do not show your answers to anyone. **Cite any sources you used outside of the class material (webpages, etc.), and list any fellow students with whom you discussed the homework concepts.**

1. Consider the following continuous signal:

$$x_c(t) = \begin{cases} 1 & \text{if floor}(t + 0.5) \text{ is even,} \\ -1 & \text{otherwise.} \end{cases}$$

   (a) Plot $x_c(t)$.

   (b) What is the Fourier transform of $x_c(t)$? Hint: use the following properties of the Fourier transform:

$$\mathcal{F}\{\text{rect}(t)\} = \frac{1}{\sqrt{2\pi}} \text{sinc}\left(\frac{\omega}{2\pi}\right) \qquad \text{Fourier of the rectangular function}$$

$$\mathcal{F}\{x(t - t_0)\} = e^{-it_0\omega}\mathcal{F}\{x(t)\} \qquad \text{time-shift property}$$

$$\mathcal{F}\{ax(t) + by(t)\} = a\mathcal{F}\{x(t)\} + b\mathcal{F}\{y(t)\} \qquad \text{linearity}$$

   (c) Sample $x_c(t)$ with time period $T = 0.2$ to get a discrete signal $x[n]$. Now try to recover the continuous signal by a truncated sinc interpolation:

$$\hat{x}_c(t) = \sum_{n=-N}^{N} x[n] \, \text{sinc}\left(\frac{t - nT}{T}\right).$$

   Repeat for $N = 10, 100, 1000$. Each time, plot your reconstructed continuous signal, $\hat{x}_c(t)$. What do you notice about the reconstructed signal as you increase N?

2. Start with the double-zero low-pass filter:

$$H_{\text{LP}}(z) = \frac{1}{4}(1 + 2z^{-1} + z^{-2}).$$

   Transform $H(z)$ into a high-pass filter by composing it with an all-pass system:

$$H_{\text{HP}}(z) = H_{\text{LP}}\left(-\frac{z - a}{1 - az}\right),$$

where the constant a is given by

$$a = \frac{\cos\left(\frac{\pi}{4} + \frac{\omega_c}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{\omega_c}{2}\right)}.$$

(a) Plot the magnitude and phase of the frequency response for H_{HP} for frequency cutoffs $\omega_c = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$. Verify it is roughly a high-pass filter at those cutoffs. Would you say that the phase plot has close to constant linear slope (except at discontinuities)? What does the shape of the phase plot tell you about how this filter will affect signals it is applied to?

(b) Plot the magnitude and phase of the frequency response for just the all-pass component, $H_{\text{AP}}(z) = -\frac{z-a}{1-az}$. Verify that it is indeed all-pass. From these plots, can you explain how it converted the low-pass filter into a high-pass one?