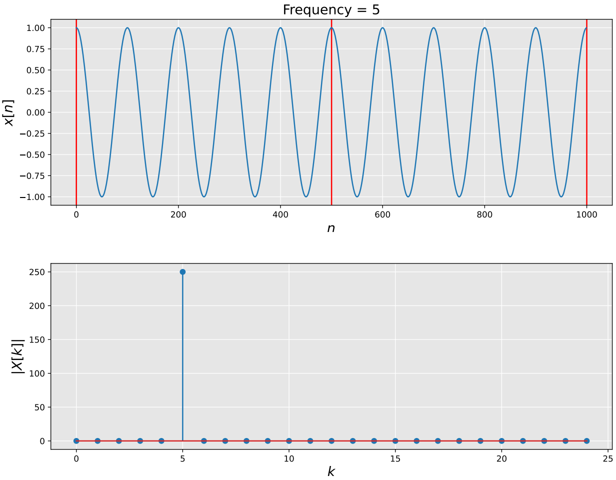
Instantaneous Frequency and Pitch Scaling

Digital Signal Processing

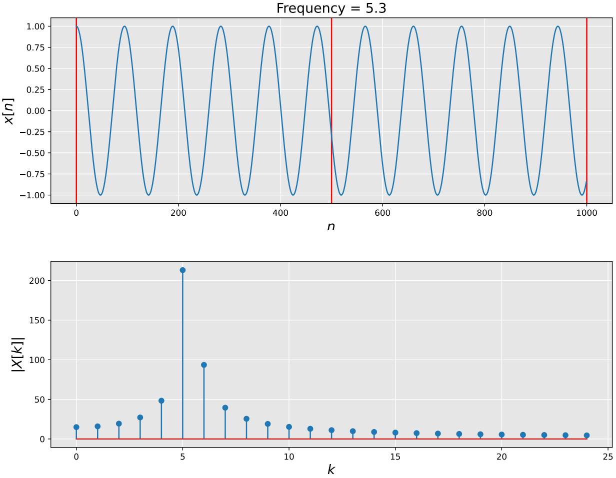
February 20, 2024

Instantaneous Frequency



Frequency is clear in DFT when it is an integer.

Instantaneous Frequency



But it spreads across multiple bins when it is not an integer.

Can we Recover Non-Integer Frequency?

Continuous sinusoid:

$$x(t) = \cos(\omega_0 t + \phi)$$

Its phase angle is:

$$\theta(t) = \omega_0 t + \phi$$

Its derivative is:

$$\frac{d\theta}{dt}(t) = \omega_0$$

Which is the frequency!

Discrete Phase Derivative

Discrete sinusoid:

$$x[n] = \cos(\omega_0 n + \phi)$$

Its phase angle is:

$$\theta[n] = \omega_0 n + \phi$$

Discrete derivative (backward difference):

$$\begin{aligned}\nabla\theta[n] &= \theta[n] - \theta[n-1] \\ &= \omega_0 n + \phi - (\omega_0(n-1) + \phi) \\ &= \omega_0\end{aligned}$$

Again, frequency!

Using the STFT Phase Angle

- The DFT doesn't give us phase angle as a function of time.
- The STFT does give an estimated phase angle as a function of time!
- Given an STFT, $X[k,m]$, we can estimate the instantaneous frequency represented in frequency bin k at time m as:

$$\omega^*[k,m] = \frac{\phi[k,m] - \phi[k,m-1]}{h},$$

  where $\phi[k,m] = \mathrm{Arg}(X[k,m])$, and h is the hop size.
- Note: $\omega^*[k,m]$ is the instantaneous frequency for each bin $0 \leq k < W$ and time $1 \leq m < H$ (no backward difference for $m = 0$)

Units of Frequency

- For STFT of window length W:

$$\omega_0 = \frac{2\pi}{W}$$

- So, kth frequency bin represents frequency

$$\omega_0 k = \frac{2\pi k}{W}$$

- The number k is how many cycles the sinusoid $e^{i\omega_0 kn}$ makes in our window (W time steps)
- If T is the sampling period (time between samples of $x[n]$, in seconds), then the kth frequency bin represents $\frac{k}{WT}$ Hz. (Note: WT is window length in seconds.)

Expected Phase Shift

If frequency was exactly the kth bin frequency, $\omega^* = \omega_0 k$, then

$$\omega^*[k,m] = \frac{\phi[k,m] - \phi[k,m-1]}{h} = \omega_0 k$$

Rearranging, we get

$$\phi[k,m] - \phi[k,m-1] = \omega_0 k h$$

This is the **expected phase shift in one hop**.

Instantaneous Frequency

The remainder between actual and expected phase shifts:

$$\phi_r[k,m] = \underbrace{\phi[k,m] - \phi[k,m-1]}_{\text{actual phase shift}} - \underbrace{\omega_0 kh}_{\text{expected}} .$$

Now, we wrap this remainder to be in $[-\pi, \pi)$, and calculate our final instantaneous frequency:

$$\omega^*[k,m] = \frac{\mathrm{wrap}(\phi_r[k,m])}{h} + \omega_0 k.$$

Or, in terms of integer frequency bins:

$$\kappa[k,n] = \frac{\omega^*[k,m]}{\omega_0} = \frac{\mathrm{wrap}(\phi_r[k,m])}{\omega_0 h} + k.$$

Wrapping to Principal Angle

To wrap an angle between $[-\pi, \pi)$:

$$\mathrm{wrap}(\phi) = (\phi + \pi) \bmod (2\pi) - \pi.$$

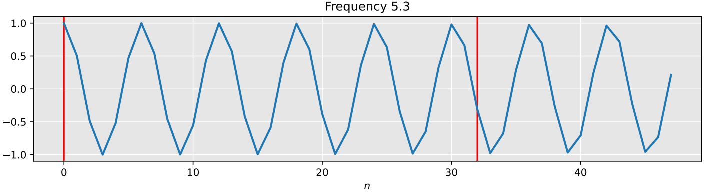
Here, the notation $x \bmod y$ means the remainder from floating point division.

The `%` operator in Python will do this (or the function `np.mod`)

Example

Cosine with frequency $f = 5.3$ and window $W = 32$:

$$x[n] = \cos(2\pi f n / 32)$$



Example

Cosine with frequency $f = 5.3$ and window $W = 32$:

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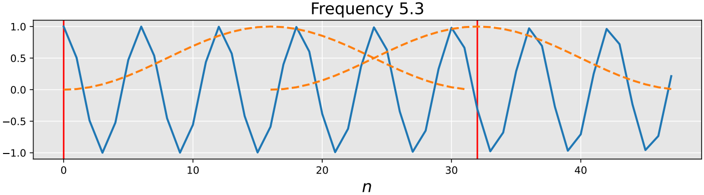
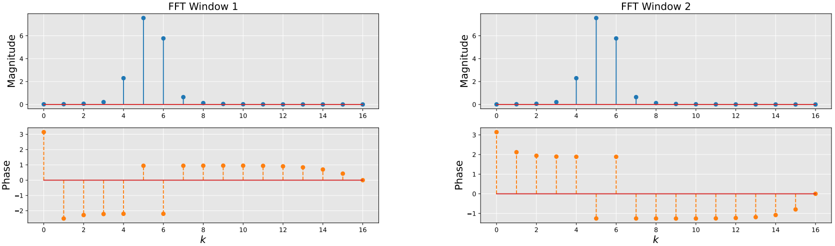
Estimate STFT with just two windows ($h = 16$):



Example

STFT:



Example

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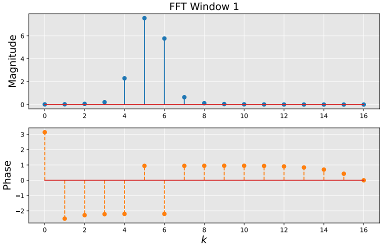
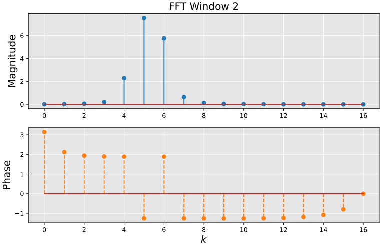
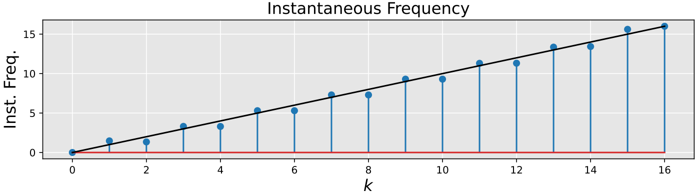


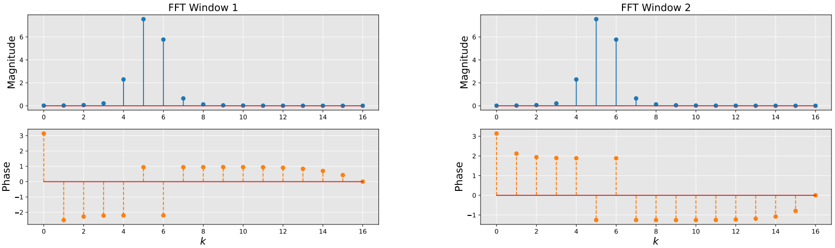


Estimated Frequencies $\omega^*[k,n]$:



Example

STFT:



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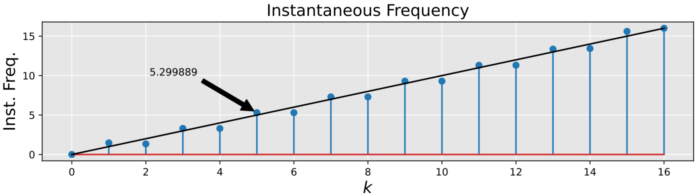
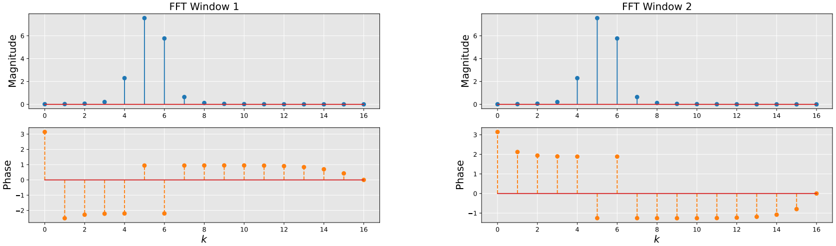


Example

STFT:



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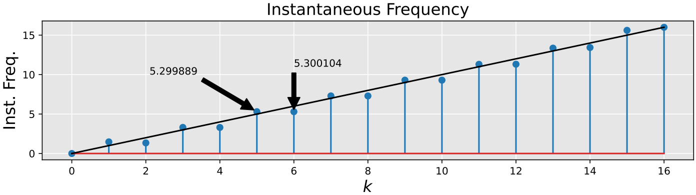
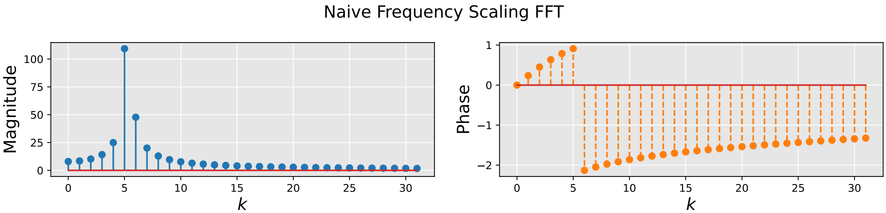


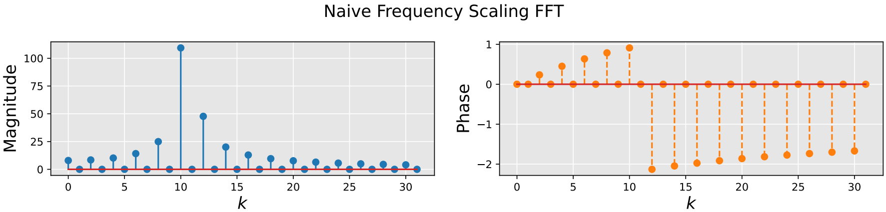
Frequency (Pitch) Scaling

How To Change Pitch?



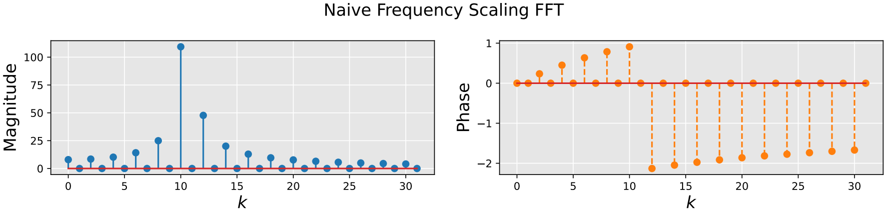
Why not just move everything to double the frequency?

$$Y[2k] = X[k]$$

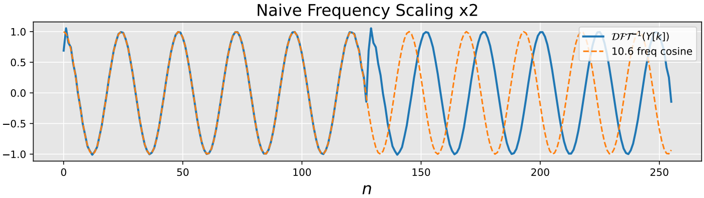


Naive Frequency Scaling Result

$$Y[2k] = X[k]$$

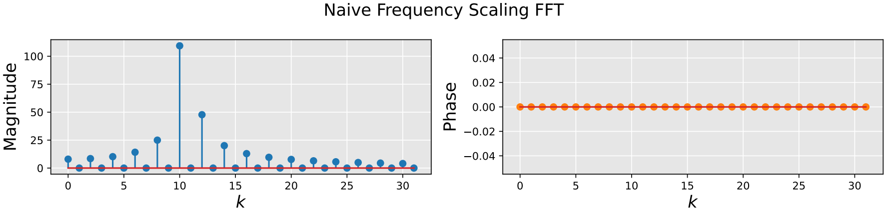


$$\mathcal{DFT}^{-1}(Y[k])$$



How About Zeroing-Out the Phase?

$$|Y[2k]| = |X[k]|, \quad \mathrm{Arg}(Y[2k]) = 0$$



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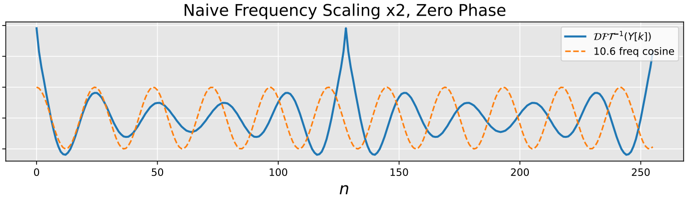

Smart Frequency Scaling Algorithm

1. Use STFT to compute instantaneous frequency, $\kappa[k, m]$
2. Scale by some factor: $\kappa_s[k, m] = R\kappa[k, m]$
3. Find new bins: $k_s = \mathrm{round}(Rk)$
4. Find new phase shifts: $\Delta\phi[k_s, m] = \omega_0 h(\kappa_s[k, m] - k_s)$
5. Accumulate with phase from previous time hop:

$$\phi_s[k_s, m] = \mathrm{wrap}\left(\phi_s[k_s, m-1] + \Delta\phi[k_s, m] + \omega_0 k_s h\right).$$

6. Set $Y[k_s, m] = |X[k, m]|e^{i\phi_s[k_s, m]}$.
7. Synthesize output signal: $y[n] = \mathrm{OLA}(Y[k, m])$

Smart Frequency Scaling Result