

# Sampling Continuous-Time Signals

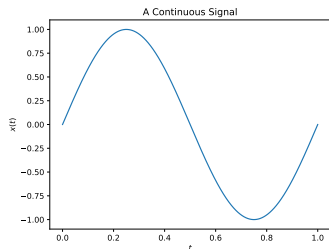
Digital Signal Processing

March 28, 2024

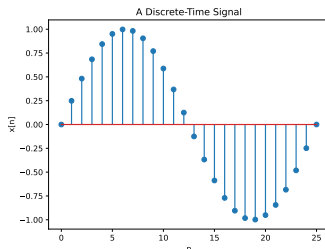


# Sampled Continuous Signals

Discrete-time signals often come from continuous signals:



$$x_c(t)$$

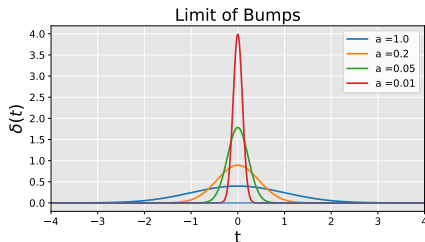
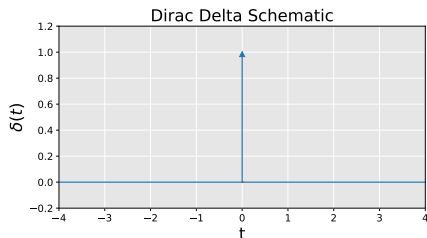


$$x[n] = x_c(nT)$$

Here,  $T \in \mathbb{R}$  is the **sampling period**,  $T = (1/25)\text{s} = 0.04\text{s}$

and  $\frac{1}{T}$  is the **sampling frequency**.  $\frac{1}{T} = 25\text{Hz}$

# Dirac Delta



- Denoted  $\delta(t)$
- Continuous analog to the discrete unit sample function,  $\delta[n]$
- Unlike the discrete case, it is **not a function**
- Can be thought of as a limit of “bump” functions:

$$\delta(t) = \lim_{a \rightarrow 0} \frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{t^2}{2a}\right)$$

# Dirac Delta and Integration

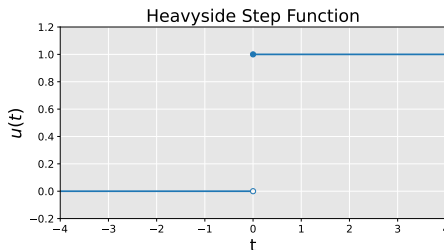
Dirac delta is a *generalized function* (a thing you can integrate):

$$\int_a^b \delta(t) dt = \begin{cases} 1 & \text{if } 0 \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

When we multiply it by a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we get:

$$\int_{-\infty}^{\infty} f(t) \delta(t) dt = f(0).$$

# Integral of the Dirac Delta



Integral of  $\delta(t)$  is the continuous unit step function, a.k.a. the Heavyside step function:

$$u(t) = \int_{-\infty}^t \delta(s) ds = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

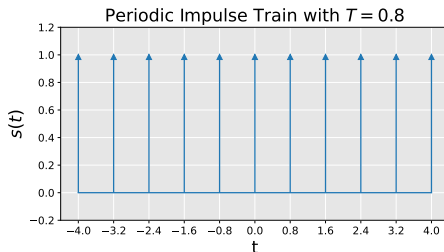
# Shifting the Dirac Delta

Shifting a dirac delta evaluates functions at a different time point:

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0).$$

Mathematical idealization of taking a measurement of some continuous process ( $f$ ) at a particular time ( $t_0$ ).

# Impulse Train



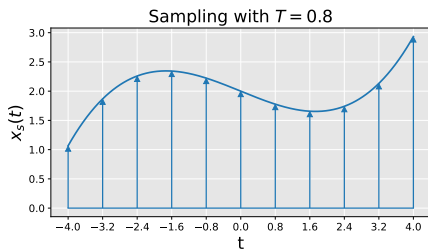
A **periodic impulse train**, a.k.a. a **Dirac comb**, is a sum of dirac deltas, shifted by a sampling period  $T$ :

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

# Sampling: First Step

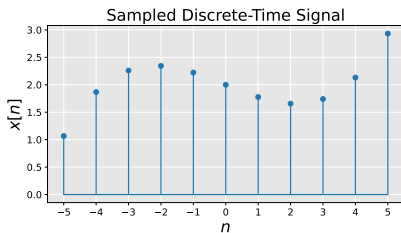
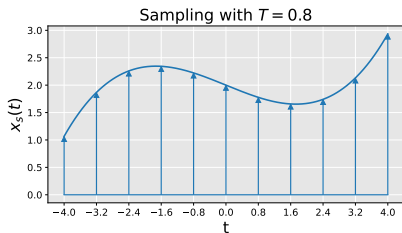
Given a continuous signal,  $x_c(t)$ ,  
define sampled signal,  $x_s(t)$ , as:

$$\begin{aligned}x_s(t) &= x_c(t)s(t) \\&= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\&= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT) \\&= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)\end{aligned}$$





# Sampling: Final Step



Discrete signal  $x[n]$  keeps the sampled values  $x_s(nT)$ .

# Frequency Analysis of Sampling

# Continuous-Time Fourier Transform

Fourier transform of a continuous function,  $f(t)$ :

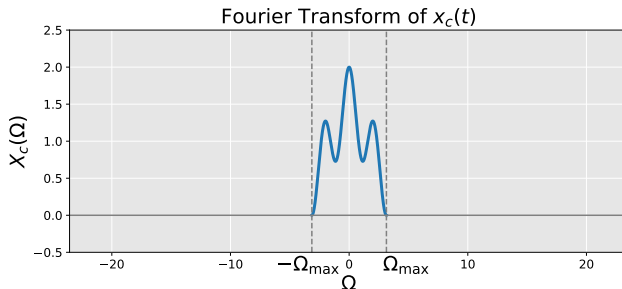
$$F(\Omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\Omega t} f(t) dt$$

Inverse Fourier transform:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\Omega t} F(\Omega) d\Omega$$

Note:  $\Omega$  is angular frequency, in radians per second.

# Bandlimited Continuous Signal



## Definition

A continuous signal  $x_c(t)$  is **bandlimited** if it has a maximum frequency content  $\Omega_{\max}$ , i.e.,

$$X_c(\Omega) = 0 \quad \text{for } |\Omega| > \Omega_{\max}.$$

# Fourier Transform of a Sampled Signal

Remember, a sampled signal is

$$x_s(t) = x_c(t)s(t)$$

Taking the Fourier transform of both sides gives:

$$X_s(\Omega) = X_c(\Omega) * S(\Omega)$$

So, the Fourier transform of our sampled signal is the convolution of the continuous signal with the Fourier transform of the Dirac comb.

# Fourier Transform of a Dirac Comb

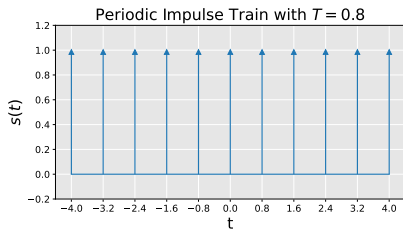
The Fourier transform of a Dirac comb is another Dirac comb:

**Time-Domain Comb:**  $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$

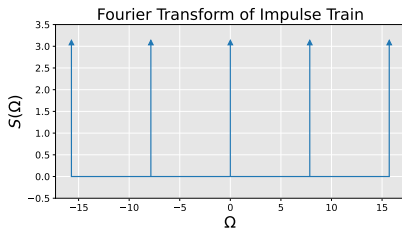
**Fourier transform:**  $S(\Omega) = \frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s),$

where  $\Omega_s = \frac{2\pi}{T}$  is the angular sampling frequency.

# Fourier Transform of a Dirac Comb



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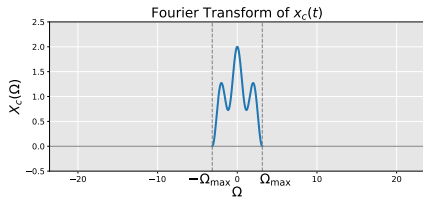


$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

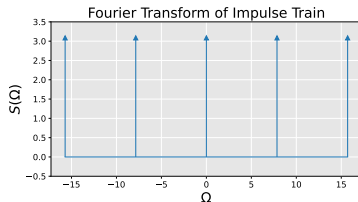
$$S(\Omega) = \frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

# Fourier Transform of a Sampled Signal

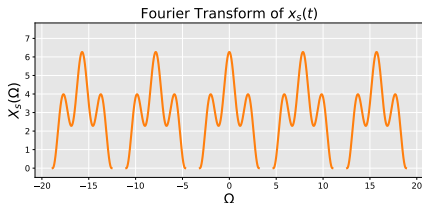
$$X_s(\Omega) = X_c(\Omega) * S(\Omega)$$



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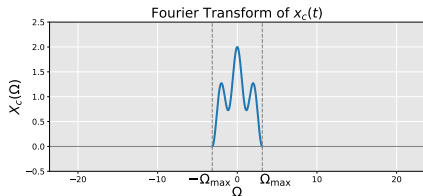
# What if We Decrease the Sampling Rate?

**Increasing** the sampling period from  $T = 0.8$  to  $T = 1.25$

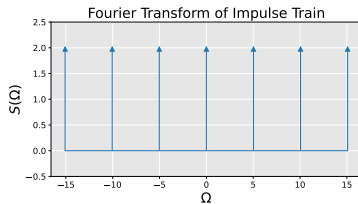
**Decreases** the angular sampling rate from  $\Omega_s = \frac{2\pi}{0.8} \approx 7.85$  to  $\Omega_s = \frac{2\pi}{1.25} \approx 5.03$

# What if We Decrease the Sampling Rate?

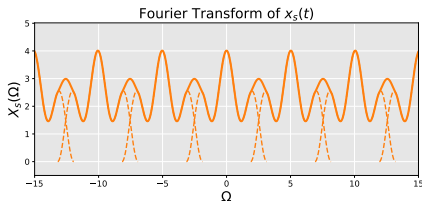
Increasing the sampling period from  $T = 0.8$  to  $T = 1.25$ :



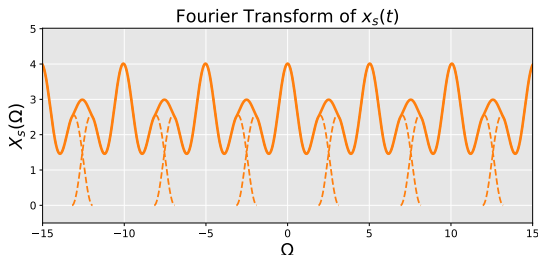
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# Aliasing and the Nyquist Rate



This is **aliasing**: overlapping frequencies are indistinguishable.

To avoid aliasing (no overlap in frequencies), we need to sample at or above the **Nyquist rate**, which is twice the bandwidth of our signal  $= 2\Omega_{\max}$ .

In this example, the bandlimit is  $\Omega_{\max} = \pi$ , and the sampling frequency is  $\Omega_s = \frac{8}{5}\pi$ . Notice,  $\Omega_s < 2\Omega_{\max}$ .