More About Filter Design

Digital Signal Processing

April 3, 2024

Zero-Phase Filters

Say we want to design a filter with **zero phase delay**.

This is equivalent to the condition:

$$\mathrm{Arg}(H(e^{i\omega})) = 0.$$

Or, in other words, or frequency response is a **real-valued function**:

$$H(e^{i\omega}) = A(\omega).$$

Inverse DTFT of Zero-Phase Condition

Because $h[n]$ is real, we have an even function: $A(\omega) = A(-\omega)$.

The DTFT is:

$$
\begin{aligned}
h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)e^{i\omega n}d\omega && \text{definition of DTFT}\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)(\cos(\omega n) + i\sin(\omega n))d\omega && \text{expanding } e^{i\omega}\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)\cos(\omega n)d\omega && A(\omega)\sin(\omega n) \text{ is odd}\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\omega)\cos(-\omega n)d\omega && \text{cosine is even}\\
&= h[-n]
\end{aligned}
$$

Impulse Response of Zero-Phase Filter

Fact

An LTI system has zero phase only if it has an even impulse response function, i.e.,

$$h[n] = h[-n]$$

Note: This means that the only interesting zero-phase filters are **not causal**.

Also, note: All zero-phase filters have an **odd** length (of non-zero coefficients).

Not All Even $h[n]$ Are Zero-Phase

If $h[n]$ is zero-phase, then it is even.

The impulse response $-h[n]$ is also even. It has frequency response:

$$-H(e^{-i\omega}) = -A(\omega) = A(\omega)e^{i\pi}.$$

So, it has phase $\mathrm{Arg}(-H(e^{-i\omega})) = \pi$.

Linear-Phase Causal Filters

Let's say that we want a causal filter. Zero-phase is out of the question, so the next best thing is a linear phase response.

This implies:

$$\mathrm{Arg}(H(e^{i\omega})) = -\omega\alpha,$$

for some positive constant α. Note, both the **phase delay** and **group delay** are equal to α.

In other words,

$$H(e^{i\omega}) = A(\omega)e^{-i\omega\alpha},$$

where $A(\omega) = |H(e^{i\omega})|$.

Linear-Phase as Shifted Zero-Phase

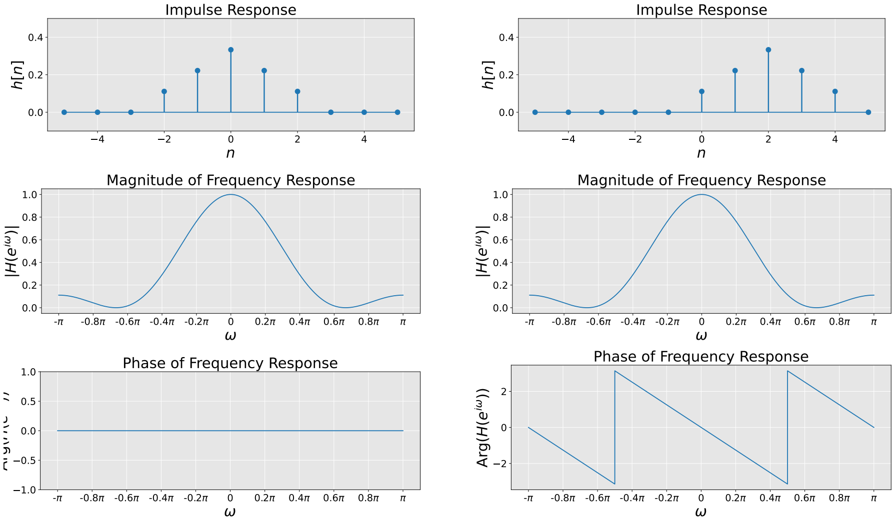
We can get a linear-phase causal filter by shifting the impulse response of a zero-phase filter.

Let $h_{\mathrm{ZP}}[n]$ be a zero-phase filter with L non-zero coefficients. These will be at time points $-\frac{L-1}{2} \leq n \leq \frac{L-1}{2}$.

Then, $h[n] = h_{\mathrm{ZP}}\left[n - \frac{L-1}{2}\right]$ is a causal filter with phase delay $\frac{L-1}{2}$. In other words, it has linear phase:

$$\mathrm{Arg}(H(e^{i\omega})) = -\omega\,\frac{L-1}{2}.$$

Linear-Phase Causal Filter Example



Even-Length Linear-Phase Filters

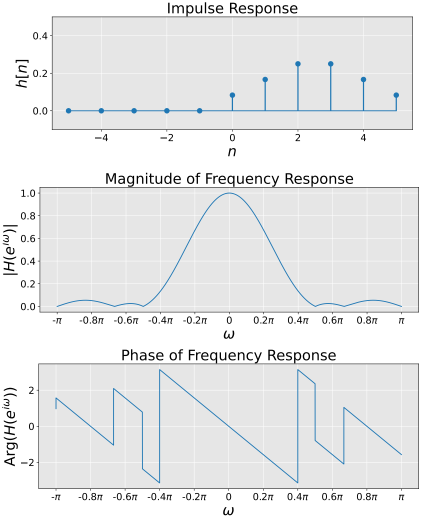
Shifting zero-phase filters can only give us odd-length filters.

These filters have symmetry:

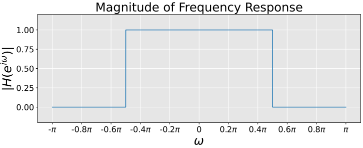
$$h[n] = h[L - n - 1].$$

We can also get an **even-length**, linear-phase, causal filter if we have a similar symmetry.

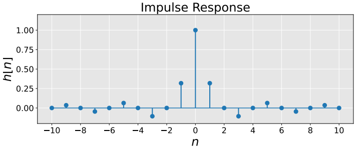
Even-Length Linear-Phase Example



Review: Ideal Low-Pass Filter



$$\xrightarrow{\mathcal{F}^{-1}}$$



$$H_{\mathrm{LP}}(e^{i\omega}) = \begin{cases} 1 & \text{for } |\omega| < \omega_c, \\ 0 & \text{otherwise.} \end{cases}$$

$$h_{\mathrm{LP}}[n] = \frac{\sin(\omega_c n)}{\pi n}$$

- The ideal low-pass filter is a box in frequency domain.
- Inverse DTFT gives us a sinc impulse response.
- Non-causal and can't be implemented (infinite extent).

The Window Method

Construct a real-valued window function $w[n]$, such that:

$$w[n] = 0 \quad \text{for } n < -M \text{ and } n > M$$

$$w[n] = w[-n] \quad \text{(even function)}$$

1. Shift $h_{\mathrm{LP}}[n]$ to $h_{\mathrm{LP}}[n-M]$.
2. Multiply by shifted window function $w[n-M]$ to get:

$$h[n] = w[n-M]h_{\mathrm{LP}}[n-M]$$

The Window Method

Resulting frequency response of windowed sinc:

$$\begin{aligned} H(e^{i\omega}) &= \mathcal{F}\left\{w[n-M]h_{\mathrm{LP}}[n-M]\right\} && \text{taking DTFT} \\ &= e^{-iM\omega}\mathcal{F}\left\{w[n]h_{\mathrm{LP}}[n]\right\} && \text{shift property} \\ &= e^{-iM\omega}\left(W(e^{i\omega}) * H_{\mathrm{LP}}(e^{i\omega})\right) && \text{convolution property} \end{aligned}$$

See Jupyter notebook: WindowFunctions.ipynb.

The Window Method

Note $W(e^{i\omega})$ and $H_{\mathrm{LP}}(e^{i\omega})$ are both zero-phase.

Why? Because $w[n]$ and $h_{\mathrm{LP}}[n]$ are even, real-valued functions.

So, $W(e^{i\omega}) * H_{\mathrm{LP}}(e^{i\omega})$ has phase 0 or π everywhere.

Therefore $H(e^{i\omega}) = e^{-iM\omega}\left(W(e^{i\omega}) * H_{\mathrm{LP}}(e^{i\omega})\right)$ is linear phase with group/phase delay of M.