

The z -Transform

Digital Signal Processing

March 14, 2023



Motivation for the z -Transform

- Until now, we've used the Fourier transform to analyze frequency content of **signals**.
- With the z -transform, we'll analyze **systems** (specifically, LTI systems).
- Helps determine properties of a system, such as stability, causality, frequency response, etc.
- Used to design LTI systems (filters).

Review: Discrete Fourier Transform

Definition

Let $x[n]$ be a complex-valued, periodic signal with period L . The **discrete Fourier transform (DFT)** of $x[n]$ is given by

DFT analysis:

$$X[k] = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n]$$

DFT synthesis:

$$x[n] = \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} e^{i\omega_0 kn} X[k]$$

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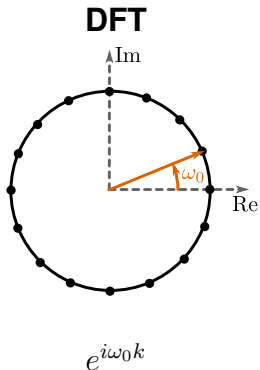
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Transform Domains



Discrete-Time Fourier Transform

Definition (DTFT Analysis)

Let $x[n]$ be a complex signal for $-\infty < n < \infty$. The **discrete-time Fourier transform (DTFT)** of $x[n]$ is given by

$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} e^{-i\omega n} x[n], \quad \text{for } \omega \in [-\pi, \pi).$$

- Note range of n is all of \mathbb{Z} .
- Note $X(e^{i\omega})$ is defined on the **continuous** unit circle in \mathbb{C} .
- Can also think of X as a function of angular frequency, ω .

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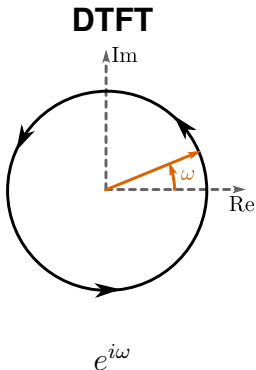
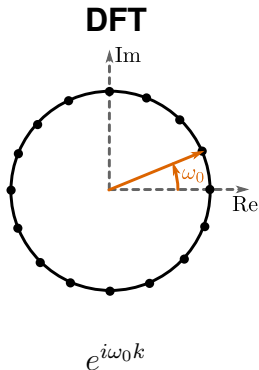
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Transform Domains



Definition (DTFT Synthesis)

Let $X(e^{i\omega})$ be the DTFT of a signal $x[n]$. The **inverse discrete-time Fourier transform** is given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\omega}) e^{i\omega n} d\omega.$$

- Note this is an integral around the unit circle in \mathbb{C} .

The z -Transform

Definition (z -Transform Analysis)

Given a complex discrete signal $x[n]$, its z -**transform** is given by

$$X(z) = \sum_{n=-\infty}^{\infty} z^{-n} x[n].$$

The z -Transform

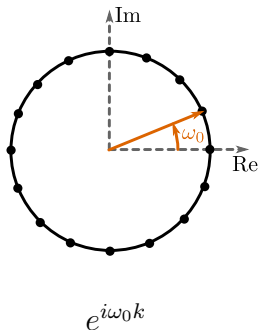
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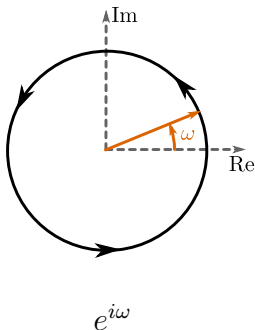
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Transform Domains

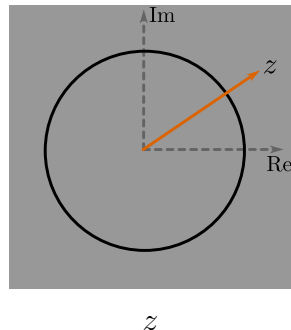
DFT



DTFT



z -Transform



Overloaded Notation!

Notice we reused the notation “ X ” for the DFT, DTFT, and z -transform.

The context is clear from the input to X :

DFT	$X[k]$
DTFT	$X(e^{i\omega})$
z -transform	$X(z)$

Quick Side Note: Geometric Series

A geometric series looks like:

$$\begin{aligned}s &= 1 + r + r^2 + r^3 + \dots \\ &= \sum_{n=0}^{\infty} r^n\end{aligned}$$

The infinite sum evaluates to

$$s = \frac{1}{1-r}, \quad \text{for } |r| < 1.$$

This holds for r real or complex!

DTFT Example: Right-Sided Exponential

Right-sided exponential signal, for some constant $a \in \mathbb{C}$, is:

$$\begin{aligned}x[n] &= \begin{cases} a^n & \text{for } n \geq 0, \\ 0 & \text{for } n < 0, \end{cases} \\ &= a^n u[n].\end{aligned}$$

The DTFT is

$$\begin{aligned}X(e^{i\omega}) &= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-i\omega n} = \sum_{n=0}^{\infty} (ae^{-i\omega})^n \\ &= \frac{1}{1 - ae^{-i\omega}}, \quad \text{for } |ae^{-i\omega}| < 1, \text{ or } |a| < 1.\end{aligned}$$

Diverges for $|a| \geq 1$.

z -Transform of Right-Sided Exponential

The z -transform of $x[n] = a^n u[n]$ is

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \\ &= \frac{1}{1 - az^{-1}}, \quad \text{for } |az^{-1}| < 1, \quad \text{or } |z| > |a| \\ &= \frac{z}{z - a}. \end{aligned}$$

Region of Convergence (ROC)

Definition (ROC)

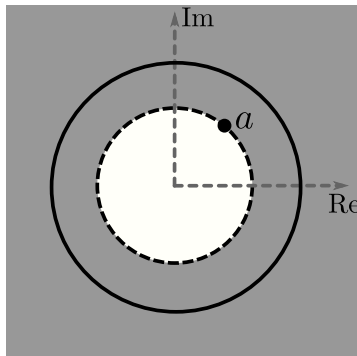
The **region of convergence** for the z -transform of a signal $x[n]$ is defined as all points $z \in \mathbb{C}$ where

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| < \infty.$$

ROC for Right-Sided Exponential

For $x[n] = u[n]a^n$, we had

$$X(z) = \frac{z}{z - a}, \quad \text{for } |z| > |a|.$$



Left-Sided Exponential

Left-sided exponential for a constant $a \in \mathbb{C}$ is

$$\begin{aligned}x[n] &= \begin{cases} -a^n & \text{for } n < 0 \\ 0 & \text{for } n \geq 0 \end{cases} \\ &= -a^n u[-n - 1]\end{aligned}$$

The z -transform is

$$\begin{aligned}X(z) &= \sum_{n=-\infty}^{\infty} -a^n u[-n - 1] z^{-n} = - \sum_{n=-\infty}^{-1} (a z^{-1})^n \\ &= - \sum_{m=1}^{\infty} a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1} z)^m \\ &= 1 - \frac{1}{1 - a^{-1} z}, \quad \text{for } |a^{-1} z| < 1, \quad \text{or } |z| < |a|\end{aligned}$$

Left-Sided Exponential

Further simplifying:

$$\begin{aligned}X(z) &= 1 - \frac{1}{1 - a^{-1}z} \\&= \frac{1 - a^{-1}z}{1 - a^{-1}z} - \frac{1}{1 - a^{-1}z} \\&= \frac{-a^{-1}z}{1 - a^{-1}z} \\&= \frac{z}{z - a}, \quad \text{for } |z| < |a|.\end{aligned}$$

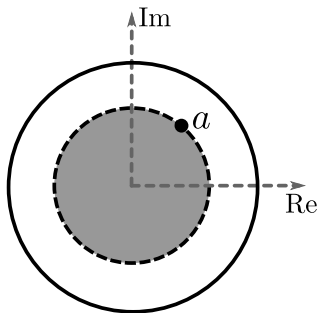
Same as right-sided exponential, but different ROC!

Right-sided ROC: $|z| > |a|$, Left-sided ROC: $|z| < |a|$

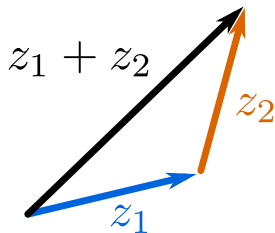
ROC for Left-Sided Exponential

For $x[n] = -a^n u[-n - 1]$, we had

$$X(z) = \frac{z}{z - a}, \quad \text{for } |z| < |a|.$$



Quick Side Note: Triangle Inequality



Complex Addition

Triangle inequality tells us:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Do this iteratively to get:

$$\left| \sum_{n=-\infty}^{\infty} z_n \right| \leq \sum_{n=-\infty}^{\infty} |z_n|.$$

General ROC of z -Transform

In general, we have:

$$\begin{aligned}|X(z)| &= \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| \\ &= \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n}\end{aligned}$$

Note this only depends on the magnitude $r = |z|$, not the angle of z .

General ROC of z -Transform

ROC is an annulus:

$$0 \leq r_R < |z| < r_L \leq \infty$$

r_R : right-side radius

$\sum_{n=0}^{\infty} |x[n]| r^{-n}$ diverges
when $r < r_R$.

r_L : left-side radius

$\sum_{n=-\infty}^{-1} |x[n]| r^{-n}$ diverges
when $r > r_L$.

