

The Discrete Fourier Transform

Digital Signal Processing

February 8, 2024



Fourier Series Review

Given a real-valued, periodic sequence $x[n]$ with period L , write the fundamental angular frequency as $\omega_0 = \frac{2\pi}{L}$.

Fourier series **synthesis**:

$$x[n] = \frac{a_0}{2} + \sum_{k=1}^{L-1} [a_k \cos(\omega_0 kn) + b_k \sin(\omega_0 kn)].$$

Fourier series **analysis**:

$$a_k = \frac{2}{L} \sum_{n=0}^{L-1} \cos(\omega_0 kn) x[n],$$
$$b_k = \frac{2}{L} \sum_{n=0}^{L-1} \sin(\omega_0 kn) x[n].$$

Extending to Complex Signals

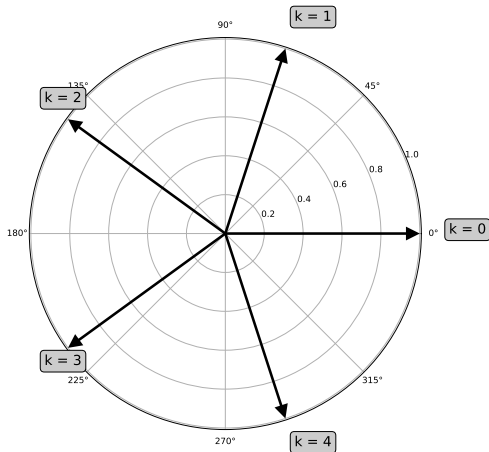
Replace real cosine and sine with a single complex sinusoid:

$$z_k[n] = e^{i\omega_0 kn} = \cos(\omega_0 kn) + i \sin(\omega_0 kn)$$

Here, $z_k[n]$ is a complex exponential sequence with

- **frequency:** $\omega_0 k = \frac{2\pi k}{L}$,
- **amplitude:** $|z_k[n]| = 1$, and
- **phase:** $\phi = 0$.

Example: $L = 5$

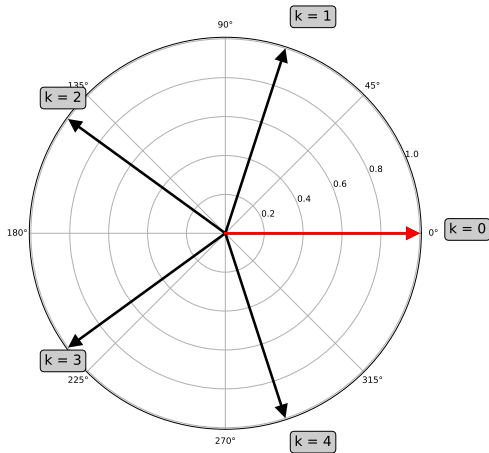


$$\omega_0 = \frac{2\pi}{5} = 72^\circ$$

Note:

$$z_k[n] = e^{i\omega_0 kn} = (e^{i\omega_0})^{kn}$$

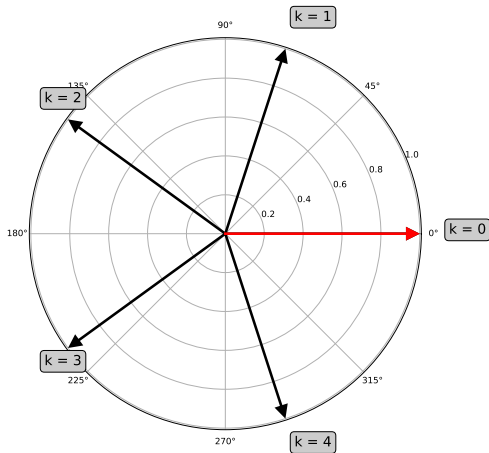
Example: $L = 5$



For $k = 0$

$$z_0[n] = e^0 = 1.0$$

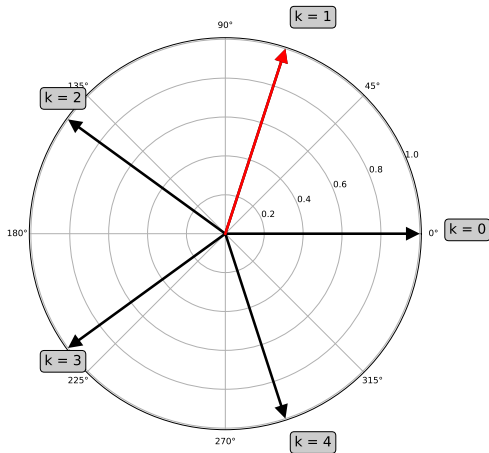
Example: $L = 5$



For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$

Example: $L = 5$

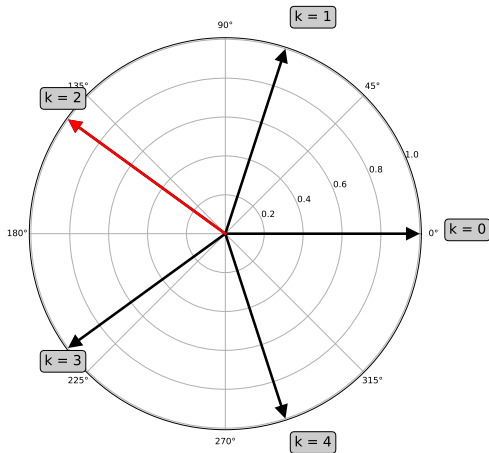


For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$

$$z_1[1] = (e^{i\omega_0})^1$$

Example: $L = 5$



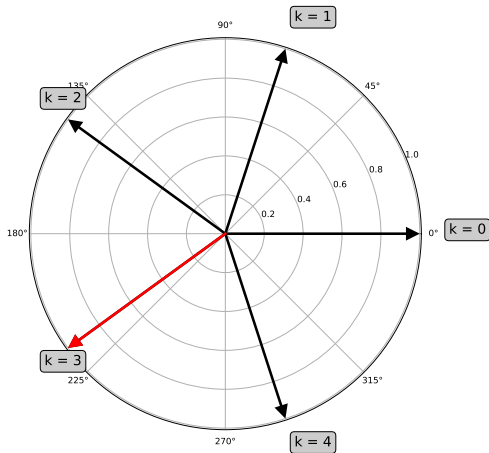
For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$

$$z_1[1] = (e^{i\omega_0})^1$$

$$z_1[2] = (e^{i\omega_0})^2$$

Example: $L = 5$



For $k = 1$,

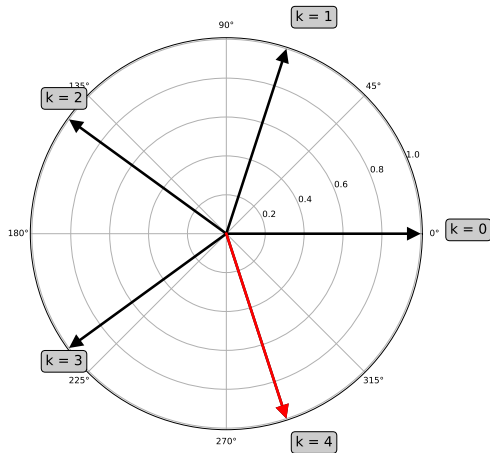
$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$

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$$z_1[3] = (e^{i\omega_0})^3$$

Example: $L = 5$



For $k = 1$,

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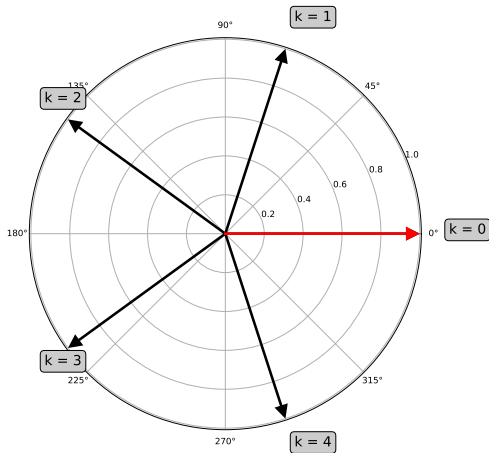
$$z_1[1] = (e^{i\omega_0})^1$$

$$z_1[2] = (e^{i\omega_0})^2$$

$$z_1[3] = (e^{i\omega_0})^3$$

$$z_1[4] = (e^{i\omega_0})^4$$

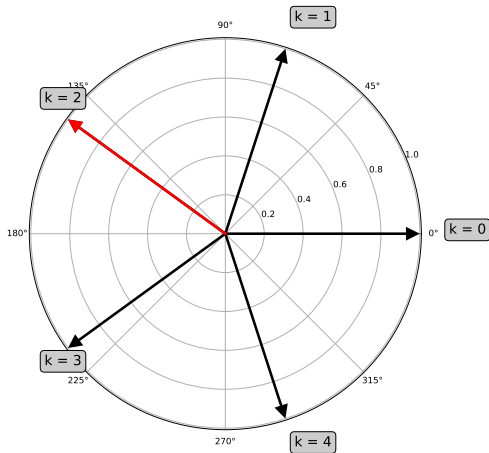
Example: $L = 5$



For $k = 2$,

$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$

Example: $L = 5$

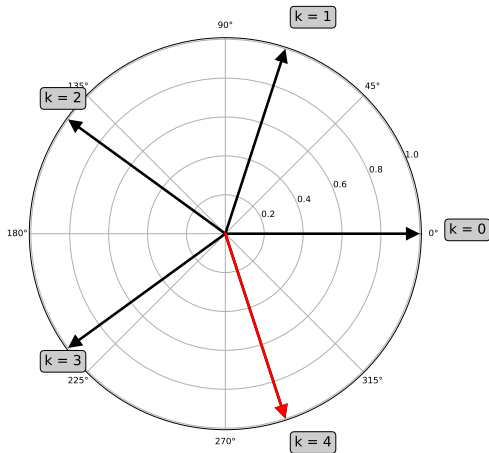


For $k = 2$,

$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$

$$z_2[1] = (e^{i\omega_0})^2$$

Example: $L = 5$



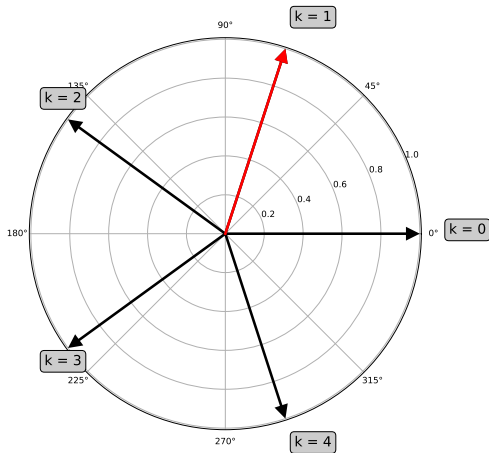
For $k = 2$,

$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$

$$z_2[1] = (e^{i\omega_0})^2$$

$$z_2[2] = (e^{i\omega_0})^4$$

Example: $L = 5$



For $k = 2$,

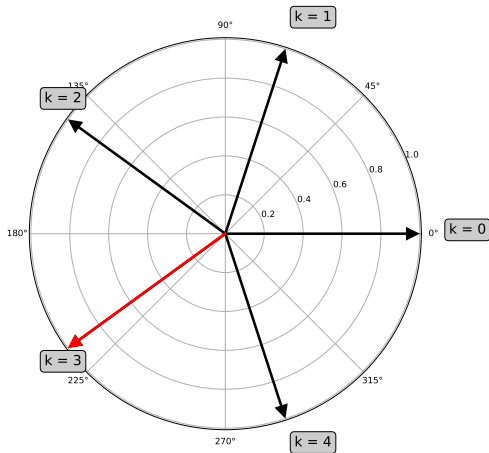
$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$

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$$z_2[3] = (e^{i\omega_0})^1$$

Example: $L = 5$



For $k = 2$,

$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$

$$z_2[1] = (e^{i\omega_0})^2$$

$$z_2[2] = (e^{i\omega_0})^4$$

$$z_2[3] = (e^{i\omega_0})^1$$

$$z_2[4] = (e^{i\omega_0})^3$$

Discrete Fourier Transform (DFT)

Definition

Now let $x[n]$ be a complex-valued, periodic signal with period L . The **discrete Fourier transform (DFT)** of $x[n]$ is given by

DFT synthesis:

$$x[n] = \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} e^{i\omega_0 kn} X[k]$$

DFT analysis:

$$X[k] = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n]$$

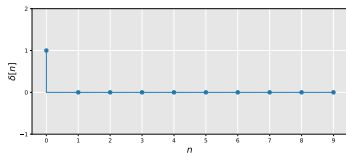
Notation

If $X[k]$ is the DFT of $x[n]$, we'll denote this as

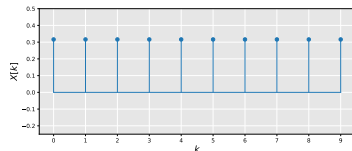
$$x[n] \xleftrightarrow{\mathcal{DFT}} X[k].$$

These are called **DFT pairs**.

DFT of the Unit Sample Function



\xleftrightarrow{DFT}



$$x[n] = \delta[n] = \begin{cases} 1 & n = 0, \\ 0 & 1 \leq n < L. \end{cases}$$

$$X[k] = \frac{1}{\sqrt{L}}, \quad \forall k.$$

DFT of the Unit Sample Function

$$\begin{aligned}X[k] &= \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n] \\&= \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} \delta[n] \\&= \frac{1}{\sqrt{L}} e^0 \\&= \frac{1}{\sqrt{L}}\end{aligned}$$

Properties of DFT

- ① Duality
- ② Conjugation
- ③ Linearity
- ④ Time Shift
- ⑤ Modulation

1. Duality

Duality Property

The inverse DFT is the same as the forward DFT composed with time-reversal:

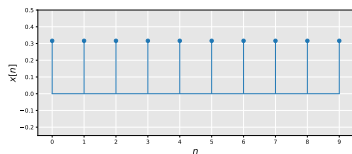
If

$$x[n] \xleftrightarrow{\mathcal{DFT}} X[k],$$

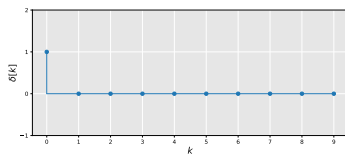
then

$$X[n] \xleftrightarrow{\mathcal{DFT}} x[L - k].$$

DFT of a Constant Signal



$\longleftrightarrow \mathcal{DFT}$



$$x[n] = \frac{1}{\sqrt{L}}, \quad \forall n.$$

$$X[k] = \delta[k] = \begin{cases} 1 & k = 0, \\ 0 & 1 \leq k < L. \end{cases}$$

Using duality

2. Conjugation

Conjugation Property

Conjugating a signal results in its DFT being conjugated and time-reversed:

$$\bar{x}[n] \xleftrightarrow{\mathcal{DFT}} \bar{X}[L - k]$$

3. Linearity

Linearity Property

The DFT is a linear operator:

If $x[n] \xleftrightarrow{\mathcal{DFT}} X[k]$, and $y[n] \xleftrightarrow{\mathcal{DFT}} Y[k]$, then

$$ax[n] + by[n] \xleftrightarrow{\mathcal{DFT}} aX[k] + bY[k],$$

for all complex constants $a, b \in \mathbb{C}$.

4. Time Shift

Time Shift Property

Shifting a signal in time results in a modulation of its DFT:

$$x[n - m] \xleftrightarrow{\mathcal{DFT}} e^{-i\omega_0 km} X[k].$$

Derivation of Time Shift Property

$$\begin{aligned}\mathcal{DFT}\{x[n - m]\} &= \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n - m] \\&= \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} e^{-i\omega_0 k(j+m)} x[j] && \text{defining } j = n - m \\&= \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} e^{-i\omega_0 kj} e^{-i\omega_0 km} x[j] \\&= e^{-i\omega_0 km} \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} e^{-i\omega_0 kj} x[j] \\&= e^{-i\omega_0 km} \mathcal{DFT}\{x[n]\}\end{aligned}$$

5. Modulation

Modulation Property

Modulation of a signal results in a time shift of its DFT:

$$e^{i\omega_0 nm} x[n] \xleftrightarrow{\mathcal{DFT}} X[k - m].$$

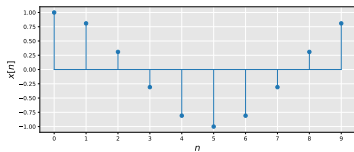
DFT of Real-Valued Signals

Let $x[n]$ be a real-valued signal. In other words, $\text{Im}(x[n]) = 0$.

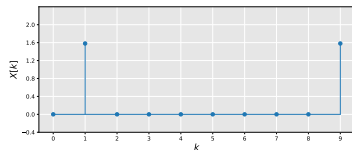
Then the following symmetry in the DFT holds:

$$X[k] = \bar{X}[L - k].$$

DFT of a Cosine



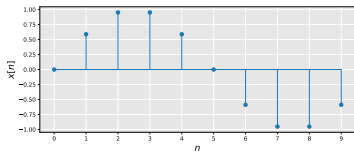
$\xleftrightarrow{\mathcal{DFT}}$



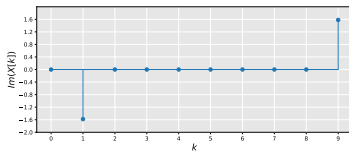
$$x[n] = \cos(\omega_0 mn)$$

$$X[k] = \frac{\sqrt{L}}{2}(\delta[k - m] + \delta[k - L + m]).$$

DFT of a Sine



$\xleftrightarrow{\mathcal{DFT}}$



$$x[n] = \sin(\omega_0 mn)$$

$$X[k] = \frac{i\sqrt{L}}{2}(\delta[k - m] - \delta[k - L + m]).$$