Homework 4: Sampling and All-Pass Systems

Instructions: Submit a single Jupyter notebook (.ipynb) of your work to Canvas by 11:59pm on the due date. All code should be written in Python. **Be sure to show all the work involved in deriving your answers! If you just give a final answer without explanation, you may not receive credit for that question.**

You may discuss the concepts with your classmates, but write up the answers entirely on your own. Do not look at another student's answers, do not use answers from the internet or other sources, and do not show your answers to anyone. **Cite any sources you used outside of the class material (webpages, etc.), and list any fellow students with whom you discussed the homework concepts.**

1. Write a function to perform a truncated sinc interpolation of a discrete signal $x[n]$. That is, implement the equation

$$\hat{x}_c(t) = \sum_{n=-N}^{N} x[n] \operatorname{sinc}\left(\frac{t - nT}{T}\right).$$

Now consider three continuous sinusoid signals of the form:

$$x_c(t) = \sin(\Omega t),$$

with angular frequencies $\Omega = 2\pi, 4\pi$, and 8π radians / sec.

(a) What is the Nyquist rate for each of the three sinusoids?

(b) What sampling period (in seconds) do you need to avoid aliasing of these sinusoids?

(c) For each of the three sinusoids, create a discrete signal $x[n]$ by sampling $x_c(t)$ at a rate of 5 Hz. Set N so that $-N \leq n \leq N$ corresponds to a continuous time range of $-2 \leq t \leq 2$ seconds.

(d) Run your truncated sinc interpolation of $x[n]$ to get a reconstructed continuous signal, $\hat{x}_c(t)$. Plot $x_c(t)$ and $\hat{x}_c(t)$ as curves using different colors in the same plot (with time range $-2 \leq t \leq 2$). Still in the same plot, overlay $x[n]$ as a stem plot, making sure to align the discrete samples with their corresponding time points. Repeat this for each of the three sinusoid frequencies.

(e) Describe what you see in the three plots. Which frequencies "work" and which do not? Explain why this happens.

2. Consider the following continuous signal:

$$x_c(t) = \begin{cases} 1 & \text{if floor}(t + 0.5) \text{ is even,} \\ -1 & \text{otherwise.} \end{cases}$$

(a) Plot $x_c(t)$ in the range -2 sec. $\leq t \leq 2$ sec.

(b) What is the Fourier transform of $x_c(t)$? Hint: use the following properties of the Fourier transform:

$$\mathcal{F}\{\mathrm{rect}(t)\} = \frac{1}{\sqrt{2\pi}} \operatorname{sinc}\left(\frac{\Omega}{2\pi}\right) \qquad \text{Fourier of the rectangular function}$$
$$\mathcal{F}\{x(t-t_0)\} = e^{-it_0\omega}\mathcal{F}\{x(t)\} \qquad \text{time-shift property}$$
$$\mathcal{F}\{ax(t)+by(t)\} = a\mathcal{F}\{x(t)\} + b\mathcal{F}\{y(t)\} \qquad \text{linearity}$$

(c) Sample $x_c(t)$ with time period $T = 0.2$ to get a discrete signal $x[n]$ in the range $-10 \leq n \leq 10$. Now try to recover the continuous signal by using your truncated sinc interpolation function. Repeat for $N = 10, 20, 50, 100, 1000$ and with corresponding sampling periods $T = 2/N$ seconds. For each value of N, generate a plot in the same was as in part 1(d) of $x_c(t)$, $x[n]$, and your reconstructed continuous signal, $\hat{x}_c(t)$. What do you notice about the reconstructed signal as you increase N?

(d) Is there some truncation value N and corresponding sampling period $T = 2/N$ seconds where the truncated sinc interpolation of this signal will achieve perfect reconstruction, i.e., $\hat{x}_c(t) = x_c(t)$? Why or why not?

3. Start with the double-zero low-pass filter:

$$H_{\mathrm{LP}}(z) = \frac{1}{4}(1 + 2z^{-1} + z^{-2}).$$

Transform $H(z)$ into a high-pass filter by composing it with an all-pass system:

$$H_{\mathrm{HP}}(z) = H_{\mathrm{LP}}\left(-\frac{z-a}{1-az}\right),$$

where the constant a is given by

$$a = \frac{\cos\left(\frac{\pi}{4} + \frac{\omega_c}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{\omega_c}{2}\right)}.$$

(a) Plot the magnitude and phase of the frequency response for H_{HP} for frequency cutoffs $\omega_c = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$. Verify it is roughly a high-pass filter at those cutoffs. Would you say that the phase plot has close to constant linear slope (except at discontinuities)? What does the shape of the phase plot tell you about how this filter will affect signals it is applied to?

(b) Plot the magnitude and phase of the frequency response for just the all-pass component, $H_{\mathrm{AP}}(z) = -\frac{z-a}{1-az}$. Verify that it is indeed all-pass. From these plots, can you explain how it converted the low-pass filter into a high-pass one?

Grad Only (Extra-Credit for Undergrads)

4. Often we are given a signal that has been blurred (for example, having gone through some sort of low-pass filter), and we want to recover the original unblurred signal. Consider the moving average LTI system we covered in class:

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k].$$

Say you are given an output of this system, $y[n]$, and you want to recover the input, $x[n]$.

(a) One approach is to create an inverse filter through the z-transform. If $H(z)$ is the transfer function of the moving average filter, it's inverse would be:

$$H^{-1}(z) = \frac{1}{H(z)}.$$

What is the equation for $H^{-1}(z)$? What would be the problem with constructing a system from this inverse transfer function?

(b) Another potential approach to reversing the moving average filter is to apply a sequence of systems, $h_1[n]$ followed by $h_2[n]$, where

$$h_1[n] = \sum_{k=0}^{K} \delta[n-kM],$$

$$h_2[n] = \delta[n] - \delta[n-1].$$

What is the impulse response of this system? Explain how this system works and under what conditions applying this system to an output $y[n]$ of the moving average filter will perfectly recover the input, $x[n]$.

(c) Generate a standard Gaussian random signal: $x[n] \sim N(0,1)$ for $n \in [0, 1000]$. (You can use the `np.random.normal` function to do this!) Apply a moving average filter with $M = 10$ to this signal to get a blurred signal $y[n]$. Plot both $x[n]$ and $y[n]$ separately.

(d) Try to recover $x[n]$ by applying the system associated with $H^{-1}(z)$ in part (a) to $y[n]$. What happens?

(e) Try to recover $x[n]$ by applying the two-step system in part (b) to $y[n]$. Plot your result and compare to the plot of $x[n]$. Were you able to reconstruct the original signal?