Poles and Zeros of the z-Transform

Digital Signal Processing

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UNIVERSITY of VIRGINIA

Review: z-Transform

Definition (z-Transform Analysis)

Given a complex discrete signal $x[n]$, its z-**transform** is given by

$$X(z) = \sum_{n=-\infty}^{\infty} z^{-n} x[n].$$

Review: z-Transform Properties

Let $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, $y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$, and $a, b \in \mathbb{C}$ be constants.

Linearity

$$ax[n] + by[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX(z) + bY(z).$$

Time-Shift

$$x[n-k] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}X(z).$$

Convolution

$$x[n] * y[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)Y(z).$$

Review: Transfer Function

Definition (System Function)

Consider an LTI system:

$$y[n] = x[n] * h[n].$$

Using the convolution property of the z-transform, this means

$$Y(z) = X(z)H(z).$$

$H(z)$ is called the **system function** or **transfer function** for T.

Transfer Function As A Ratio

Again, z-transform of an LTI system looks like:

$$Y(z) = X(z)H(z).$$

Rearranging to solve for $H(z)$, we get

$$\boxed{H(z) = \frac{Y(z)}{X(z)}.}$$

Transfer function is ratio of output to input z-transforms.

Review: Feedforward Comb Filter

Remember the FFCF:

$$y[n] = x[n] + gx[n-k]$$

Its impulse response function is:

$$h[n] = \delta[n] + g\delta[n-k]$$

Using linearity and the time-shift property, we get the transfer function:

$$H(z) = 1 + gz^{-k} = \frac{z^k + g}{z^k}.$$

Review: Feedback Comb Filter

Remember the FBCF:

$$y[n] = x[n] + gy[n-k]$$

To get system function, plug in $x[n] = \delta[n]$:

$$\begin{aligned}
& h[n] = \delta[n] + gh[n-k] & \\
\iff \quad & H(z) = 1 + gz^{-k}H(z) & \text{take } z\text{-transform} \\
\iff \quad & H(z) - gz^{-k}H(z) = 1 & \text{rearrange } H(z) \text{ to left side} \\
\iff \quad & H(z) = \frac{1}{1 - gz^{-k}} & \text{solve for } H(z) \\
\iff \quad & H(z) = \frac{z^k}{z^k - g} & \text{multiply by } \frac{z^k}{z^k}
\end{aligned}$$

Linear Constant-Coefficient Difference Equations

A **linear, constant-coefficient difference equation (LCCDE)** is a system of the form:

$$a_0 y[n]+a_1 y[n-1]+\cdots+a_N y[n-N] = b_0 x[n]+b_1 x[n-1]+\cdots+b_M x[n-M].$$

Or, equivalently:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

z-Transform of LCCDE

$$\text{LCCDE:} \quad \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\Longleftrightarrow \quad \mathcal{Z}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathcal{Z}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\}$$

$$\Longleftrightarrow \quad \sum_{k=0}^{N} a_k \mathcal{Z}\{y[n-k]\} = \sum_{k=0}^{M} b_k \mathcal{Z}\{x[n-k]\} \qquad \text{linearity}$$

$$\Longleftrightarrow \quad \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \qquad \text{time shift}$$

$$\boxed{\text{Transfer function:} \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}}$$

Rational Transfer Functions

General form of LCCDE transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

This is a **rational function**:
Numerator and denominator are both polynomials in z^{-1}

Zeros of Polynomials

Theorem (Fundamental Theorem of Algebra)

A complex polynomial of degree k,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{k-1} z^{k-1} + a_k z^k,$$

can be factored as

$$f(z) = a_k (z - r_1)(z - r_2) \cdots (z - r_k),$$

with complex roots, $r_i \in \mathbb{C}$, for $i = 1, 2, \ldots, k$.

- Roots are the points where $f(r_i) = 0$.
- A root may be repeated multiple times. The number of times is the *multiplicity* of the root.

Poles and Zeros

Again, LCCDE transfer function looks like

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

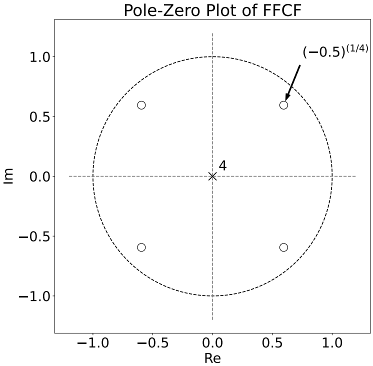
Applying the FTOA, we can factor the numerator and denominator:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=0}^{M}(1 - c_k z^{-1})}{\prod_{k=0}^{N}(1 - d_k z^{-1})}$$

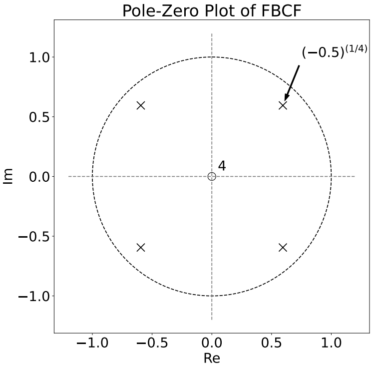
The c_k are **zeros** of $H(z)$ (zeros of the numerator).
The d_k are **poles** of $H(z)$ (zeros of the denominator).

Pole-Zero Plots

Place ‘o’ at zeros and ‘x’ at poles:



$$H(z) = \frac{z^4+0.5}{z^4}$$



$$H(z) = \frac{z^4}{z^4+0.5}$$

Review: Region of Convergence (ROC)

ROC is an annulus:

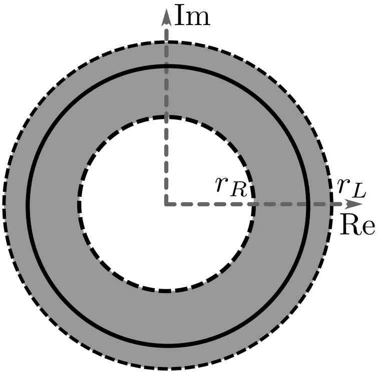
$$0 \leq r_R < |z| < r_L \leq \infty$$

r_R: right-side radius

$\sum_{n=0}^{\infty} |x[n]| r^{-n}$ diverges when $r < r_R$.

r_L: left-side radius

$\sum_{n=-\infty}^{-1} |x[n]| r^{-n}$ diverges when $r > r_L$.



ROC and Poles

Rules for the ROC and Poles:

1. The ROC cannot contain any poles.
2. A left-sided sequence will satisfy $|z| < r_L$, where r_L is the smallest magnitude of a pole.
3. A right-sided sequence will satisfy $|z| > r_R$, where r_R is the largest magnitude of a pole.
4. A sequence that is neither left- or right-sided will be an annulus satisfying $r_R < |z| < r_L$, where r_R and r_L are magnitudes of two poles.

Review: Causal Systems

Definition

A system is said to be **causal** if, for any $n_0 \in \mathbb{Z}$, $T\{x[n_0]\}$ depends only on previous values of $x[n]$, for $n \leq n_0$

A causal system cannot "look into the future."

If $x[n] = y[n]$ for all $n < n_0$, then $T\{x[n]\} = T\{y[n]\}$ for all $n < n_0$.

Review: Causality of LTI Systems

Theorem

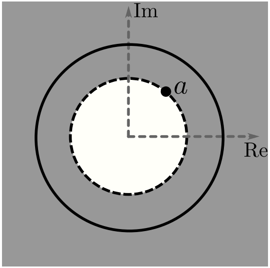
An LTI system is causal if and only if its impulse response function, $h[n]$, satisfies $h[n] = 0$ for all $n < 0$.

Sketchy proof.
Our LTI system output evaluated for some n_0 is:

$$(h * x)[n_0] = \sum_{k=-\infty}^{\infty} h[k]x[n_0 - k]$$

This will avoid using $x[n]$ for $n > n_0$ if and only if $h[k] = 0$ when $n_0 - k > n_0$. That is, when $k < 0$.

Causality from the z-Transform



An LTI system is causal if and only if its impulse response, $h[n]$, is right-sided. So, we have:

Theorem (Causal ROC)

A causal LTI system will have ROC $|z| > r_R$, where r_R is the largest magnitude of a pole.

Review: BIBO Stability

Definition

A signal, $x[n]$, is **bounded** if $|x[n]| \leq B$ for some $B < \infty$ and for all $n \in \mathbb{Z}$

Definition

A system, $T\{\cdot\}$, is said to be **bounded-input, bounded-output (BIBO) stable** if for every bounded input $x[n]$, the resulting output $T\{x[n]\}$ is also bounded.

Review: BIBO Stability of LTI Systems

Theorem

An LTI system is BIBO stable if and only if its impulse response, $h[n]$, is absolutely summable:

$$\sum_{k=-\infty}^{\infty} |h[n]| < \infty.$$

Stability from z-Transform

Theorem

An LTI system is BIBO stable if and only if the ROC of its z-transform contains the unit circle.

Proof:
ROC condition on unit circle ($|z| = 1$) is same as BIBO:

$$\text{ROC:} \quad \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$$

$$\text{BIBO:} \quad \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Stable and Causal LTI Systems

Theorem

For an LTI system to be both causal and stable, all of its poles must lie inside the unit circle, and the ROC is right-sided.