The DFT and Convolution

Digital Signal Processing

February 13, 2024

UNIVERSITY of VIRGINIA

Circular Convolution

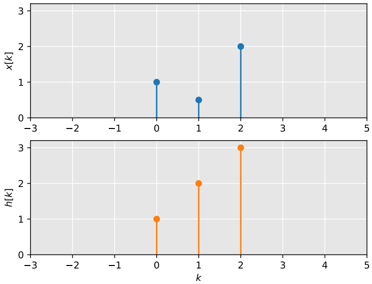
Given two periodic signals, $x[n], h[n]$, both with period L, their **circular convolution** is

$$x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$$

The only thing we've changed is to now "wrap" the index on h.

Circular Convolution Example

Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$



$$x[n] = (1.0, 0.5, 2.0)$$

$$h[n] = (1.0, 2.0, 3.0)$$

Circular Convolution Example

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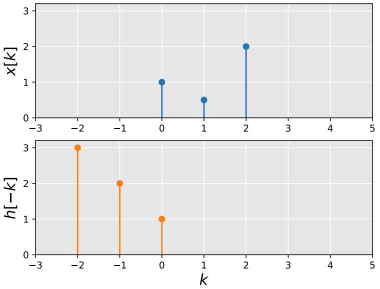

Flip h about 0

Circular Convolution Example

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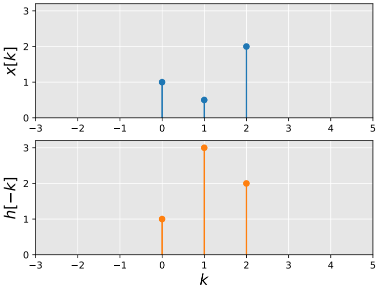
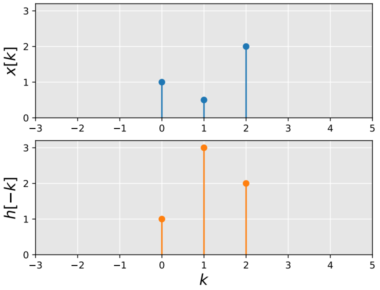


Wrap h

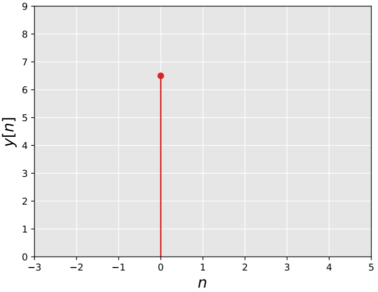
Circular Convolution Example

Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$

For $n = 0$, flip h about 0 to get $h[-k \bmod L]$.



$\Rightarrow$

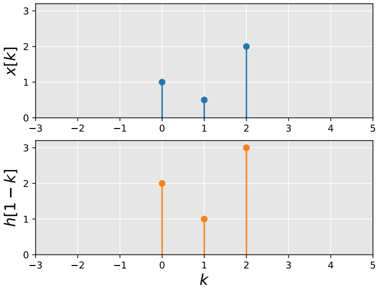


$$\begin{aligned} y[0] &= x[0] \times h[0] + x[1] \times h[2] + x[2] \times h[1] \\ &= 1.0 \times 1.0 + 0.5 \times 3.0 + 2.0 \times 2.0 = 6.5 \end{aligned}$$

Circular Convolution Example

Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k] h[(n-k) \bmod L]$

For $n = 1$, shift h left by one to get $h[(1-k) \bmod L]$.



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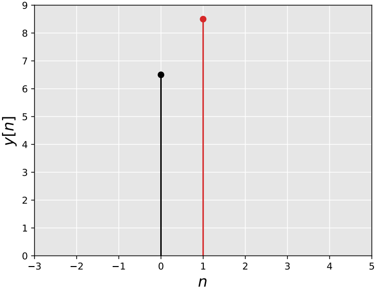
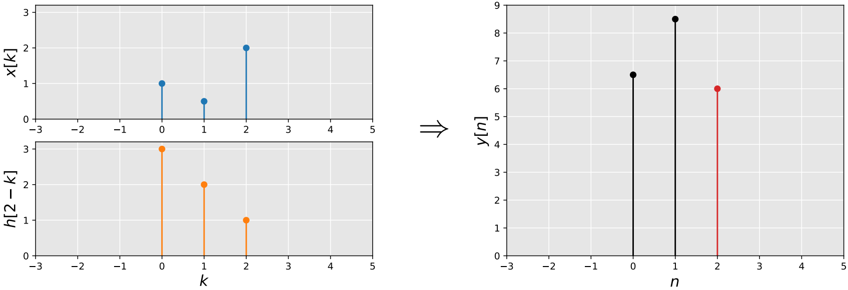


$$
\begin{aligned}
y[1] &= x[0]h[1] + x[1]h[0] + x[2]h[2] \\
&= 1.0 \times 2.0 + 0.5 \times 1.0 + 2.0 \times 3.0 = 8.5
\end{aligned}
$$

Circular Convolution Example

Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$

For $n = 2$, shift h left again to get $h[(2-k) \bmod L]$.



$$y[2] = x[0]h[2] + x[1]h[1] + x[2]h[0]$$
$$= 1.0 \times 3.0 + 0.5 \times 2.0 + 2.0 \times 1.0 = 6.0$$

Convolution and DFT

Theorem (Convolution Theorem)

Given two periodic, complex-valued signals, $x_1[n], x_2[n]$,

$$\mathcal{DFT}\{x_1[n] * x_2[n]\} = \sqrt{L}\left(\mathcal{DFT}\{x_1[n]\} \times \mathcal{DFT}\{x_2[n]\}\right).$$

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Proof on board, also see here:
Convolution Theorem on Wikipedia

Convolution and DFT

Theorem (Convolution Theorem II)

Given two periodic, complex-valued signals, $x_1[n], x_2[n]$,

$$\mathcal{DFT}\{x_1[n] \times x_2[n]\} = \frac{1}{\sqrt{L}} \left(\mathcal{DFT}\{x_1[n]\} * \mathcal{DFT}\{x_2[n]\}\right).$$

Convolution and DFT

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In other words, the **multiplication** in the time domain becomes **convolution** in the frequency domain.