The z-Transform

Digital Signal Processing

February 25, 2025

Motivation for the z-Transform

- Until now, we've used the Fourier transform to analyze frequency content of **signals**.
- With the z-transform, we'll analyze **systems** (specificallly, LTI systems).
- Helps determine properties of a system, such as stability, causality, frequency response, etc.
- Used to design LTI systems (filters).

Review: Discrete Fourier Transform

Definition

Let $x[n]$ be a complex-valued, periodic signal with period L. The **discrete Fourier transform (DFT)** of $x[n]$ is given by

DFT **analysis**:

$$X[k] = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n]$$

DFT **synthesis**:

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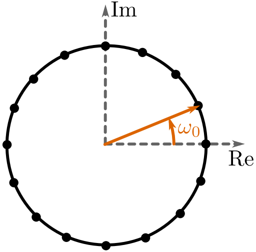
Transform Domains

DFT



$$e^{i\omega_0 k}$$

Discrete-Time Fourier Transform

Definition (DTFT Analysis)

Let $x[n]$ be a complex signal for $-\infty < n < \infty$. The **discrete-time Fourier transform (DTFT)** of $x[n]$ is given by

$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} e^{-i\omega n} x[n], \qquad \text{for } \omega \in [-\pi, \pi).$$

- Note range of n is all of $\mathbb{Z}$.
- Note $X(e^{i\omega})$ is defined on the **continuous** unit circle in $\mathbb{C}$.
- Can also think of X as a function of angular frequency, ω.

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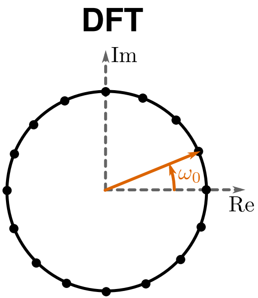
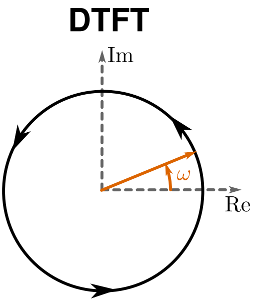


$e^{i\omega_0 k}$



$e^{i\omega}$

Inverse DTFT

Definition (DTFT Synthesis)

Let $X(e^{i\omega})$ be the DTFT of a signal $x[n]$. The **inverse discrete-time Fourier transform** is given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\omega}) e^{i\omega n} d\omega.$$

- Note this is an integral around the unit circle in $\mathbb{C}$.

The z-Transform

Definition (z-Transform Analysis)

Given a complex discrete signal $x[n]$, its z**-transform** is given by

$$X(z) = \sum_{n=-\infty}^{\infty} z^{-n} x[n].$$

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Transform Domains

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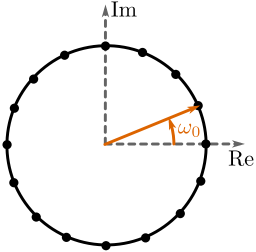
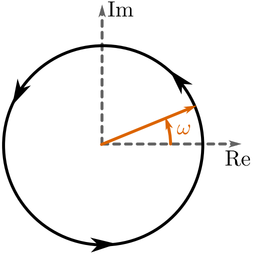
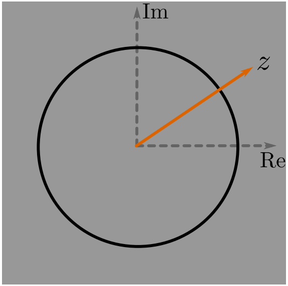
		
$e^{i\omega_0 k}$	$e^{i\omega}$	z

Overloaded Notation!

Notice we reused the notation "X" for the DFT, DTFT, and z-transform.

The context is clear from the input to X:

DFT	$X[k]$
DTFT	$X(e^{i\omega})$
z-transform	$X(z)$

Quick Side Note: Geometric Series

A geometric series looks like:

$$\begin{aligned} s &= 1 + r + r^2 + r^3 + \cdots \\ &= \sum_{n=0}^{\infty} r^n \end{aligned}$$

The infinite sum evaluates to

$$s = \frac{1}{1-r}, \qquad \text{for } |r| < 1.$$

This holds for r real or complex!

DTFT Example: Right-Sided Exponential

Right-sided exponential signal, for some constant $a \in \mathbb{C}$, is:

$$\begin{aligned} x[n] &= \begin{cases} a^n & \text{for } n \geq 0, \\ 0 & \text{for } n < 0, \end{cases} \\ &= a^n u[n]. \end{aligned}$$

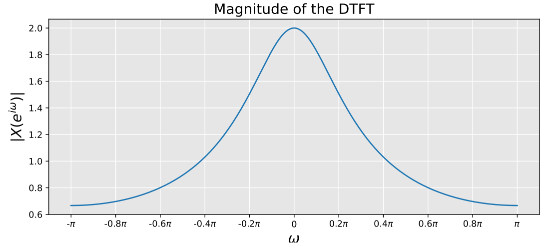
The DTFT is

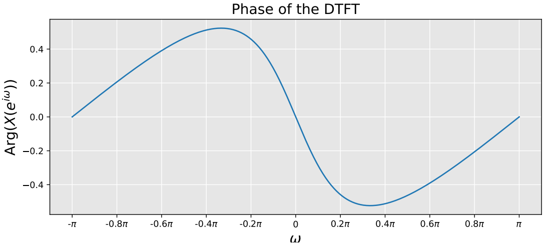
$$\begin{aligned} X(e^{i\omega}) &= \sum_{n=-\infty}^{\infty} a^n u[n] e^{-i\omega n} = \sum_{n=0}^{\infty} (ae^{-i\omega})^n \\ &= \frac{1}{1 - ae^{-i\omega}}, \qquad \text{for } |ae^{-i\omega}| < 1, \quad \text{or} \quad |a| < 1. \end{aligned}$$

Diverges for $|a| \geq 1$.

Plotting the DTFT

$$x[n] = 0.5^n u[n] \qquad \leftrightarrow \qquad X\left(e^{i\omega}\right) = \frac{1}{1-0.5e^{-i\omega}}$$





z-Transform of Right-Sided Exponential

The z-transform of $x[n] = a^n u[n]$ is

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \\
&= \frac{1}{1 - az^{-1}}, \qquad \text{for } |az^{-1}| < 1, \;\; \text{or} \;\; |z| > |a| \\
&= \frac{z}{z - a}.
\end{aligned}
$$

Region of Convergence (ROC)

Definition (ROC)

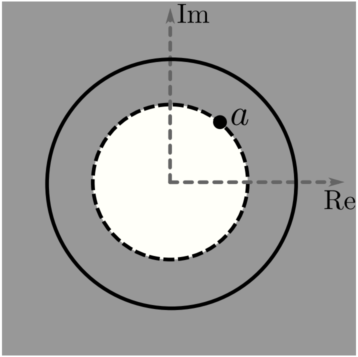
The **region of convergence** for the z-transform of a signal $x[n]$ is defined as all points $z \in \mathbb{C}$ where

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty.$$

ROC for Right-Sided Exponential

For $x[n] = u[n]a^n$, we had

$$X(z) = \frac{z}{z-a}, \qquad \text{for } |z| > |a|.$$



Left-Sided Exponential

Left-sided exponential for a constant $a \in \mathbb{C}$ is

$$x[n] = \begin{cases} -a^n & \text{for } n < 0 \\ 0 & \text{for } n \geq 0 \end{cases}$$
$$= -a^n u[-n-1]$$

The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n$$
$$= -\sum_{m=1}^{\infty} a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m$$
$$= 1 - \frac{1}{1 - a^{-1}z}, \qquad \text{for } |a^{-1}z| < 1, \text{ or } |z| < |a|$$

Left-Sided Exponential

Further simplifying:

$$\begin{aligned} X(z) &= 1 - \frac{1}{1-a^{-1}z} \\ &= \frac{1-a^{-1}z}{1-a^{-1}z} - \frac{1}{1-a^{-1}z} \\ &= \frac{-a^{-1}z}{1-a^{-1}z} \\ &= \frac{z}{z-a}, \qquad \text{for } |z| < |a|. \end{aligned}$$

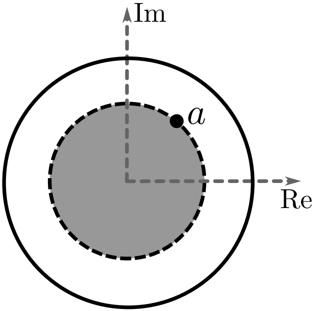
Same as right-sided exponential, but different ROC!

Right-sided ROC: $|z| > |a|$, Left-sided ROC: $|z| < |a|$

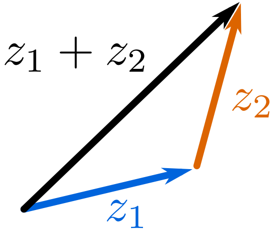
ROC for Left-Sided Exponential

For $x[n] = -a^n u[-n-1]$, we had

$$X(z) = \frac{z}{z-a}, \qquad \text{for } |z| < |a|.$$



Quick Side Note: Triangle Inequality



Complex Addition

Triangle inequality tells us:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Do this iteratively to get:

$$\left| \sum_{n=-\infty}^{\infty} z_n \right| \leq \sum_{n=-\infty}^{\infty} |z_n|.$$

General ROC of z-Transform

In general, we have:

$$\begin{aligned} |X(z)| &= \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| \\ &= \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} \end{aligned}$$

Note this only depends on the magnitude $r = |z|$, not the angle of z.

General ROC of z-Transform

ROC is an annulus:

$$0 \leq r_R < |z| < r_L \leq \infty$$

r_R: right-side radius

$\sum_{n=0}^{\infty} |x[n]| r^{-n}$ diverges when $r < r_R$.

r_L: left-side radius

$\sum_{n=-\infty}^{-1} |x[n]| r^{-n}$ diverges when $r > r_L$.