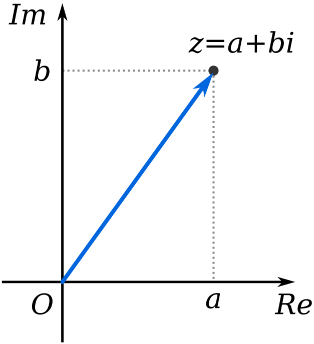
Complex-Valued Signals

Digital Signal Processing

February 1, 2024

Complex Numbers



The **standard form** for $z \in \mathbb{C}$:

$$z = a + bi,$$

where $i = \sqrt{-1}$.

Notation for real and imaginary parts:

$$\mathrm{Re}(z) = a, \quad \mathrm{Im}(z) = b.$$

Complex Addition

Adding two complex numbers:

$$z_1 = a_1 + b_1 i,$$

$$z_2 = a_2 + b_2 i.$$

Just add the real parts and imaginary parts, respectively:

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i.$$

Complex Multiplication

Multiplying two complex numbers:

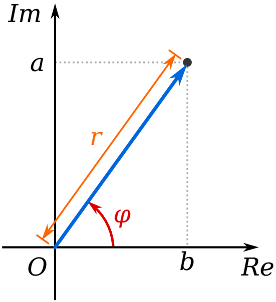
$$z_1 = a_1 + b_1 i,$$

$$z_2 = a_2 + b_2 i.$$

Use the rule that $i^2 = -1$:

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i \end{aligned}$$

Polar Form



The polar form for $z \in \mathbb{C}$:

$$z = r\cos\phi + ir\sin\phi$$

r is the **modulus** or **magnitude**
ϕ is the **argument**

Equations:

$$r = |z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$$
$$\phi = \mathrm{atan2}\left(\mathrm{Im}(z), \mathrm{Re}(z)\right)$$

Euler Notation

Any complex number $z \in \mathbb{C}$ can be written as an exponential:

$$z = re^{i\phi} = r\cos\phi + ir\sin\phi$$

Why? Power series definition of exponential:

$$\begin{aligned} e^{i\phi} &= 1 + \frac{i\phi}{1!} + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^3}{3!} + \cdots \\ &= 1 + \frac{i\phi}{1!} - \frac{\phi^2}{2!} - \frac{i\phi^3}{3!} + \cdots \\ &= \left(1 - \frac{\phi^2}{2!} + \cdots\right) + i\left(\frac{\phi}{1!} - \frac{\phi^3}{3!} + \cdots\right) \\ &= \cos\phi + i\sin\phi \end{aligned}$$

Multiplication in Euler Form

Let's multiply again:

$$z_1 = re^{i\phi}, \quad z_2 = se^{i\theta}$$

$$\begin{aligned} z_1 z_2 &= \left(re^{i\phi}\right)\left(se^{i\theta}\right) \\ &= (rs)\left(e^{i\phi}e^{i\theta}\right) \\ &= (rs)e^{i\phi+i\theta} \\ &= (rs)e^{i(\phi+\theta)} \end{aligned}$$

Notice:

$$|z_1 z_2| = rs = |z_1||z_2|,$$ **Moduli multiply**

$$\mathrm{Arg}(z_1 z_2) = \phi + \theta = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$$ **Arguments add**

Exponentiation in Euler Form

Take $z = re^{i\phi}$ to the power $x \in \mathbb{R}$:

$$\begin{aligned} z^x &= \left(re^{i\phi}\right)^x \\ &= r^x e^{i\phi x} \end{aligned}$$

Inversion in Euler Form

The inverse of $z \in \mathbb{C}$ is just exponentiation by -1:

$$z^{-1} = \left(re^{i\phi}\right)^{-1} = \frac{1}{r}e^{-i\phi}.$$

We can verify $zz^{-1} = z^{-1}z = 1$.

Conjugation

The conjugate of $z = a + bi$ is

$$\bar{z} = a - bi.$$

Useful tricks with conjugate:

- Magnitude is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$
- Inverse is $z^{-1} = \frac{\bar{z}}{|z|^2}$
- Argument is $\mathrm{Arg}(\bar{z}) = -\mathrm{Arg}(z)$

Roots in Euler Form

The kth root of $z \in \mathbb{C}$ is just exponentiation by $\frac{1}{k}$:

$$\sqrt[k]{z} = z^{\frac{1}{k}} = \left(re^{i\phi}\right)^{\frac{1}{k}} = r^{\frac{1}{k}} e^{\frac{i\phi}{k}}.$$

Again, we can verify $(\sqrt[k]{z})^k = z$.

Complex Exponential Signals

Remember our exponential signal:

$$x[n] = A\alpha^n$$

Now let A and α both be **complex numbers**.

Complex Exponential Signals

Let's write that using Euler notation:

$$A = |A|e^{i\phi}, \quad \alpha = |\alpha|e^{i\omega_0}$$

$$\begin{aligned} x[n] &= A\alpha^n \\ &= |A|e^{i\phi}|\alpha|^n e^{i\omega_0 n} \\ &= |A||\alpha|^n e^{i(\omega_0 n + \phi)} \\ &= |A||\alpha|^n (\cos(\omega_0 n + \phi) + i\sin(\omega_0 n + \phi)) \end{aligned}$$

This is a sinusoid, with **frequency** ω_0, **phase** ϕ, and **exponential weighting** by $|\alpha|^n$.