Interpolation and the Sampling Theorem

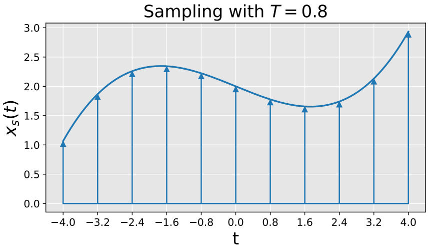
Digital Signal Processing

April 2, 2024

Review: Sampling

Given a continous signal, $x_c(t)$, define sampled signal, $x_s(t)$, as:

$$\begin{aligned} x_s(t) &= x_c(t)s(t) \\ &= x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) \end{aligned}$$



Review: Fourier Transform of Sampling

Remember, a sampled signal is

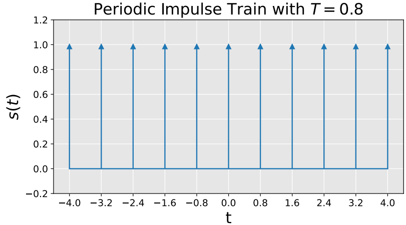
$$x_s(t) = x_c(t)s(t)$$

Taking the Fourier transform of both sides gives:

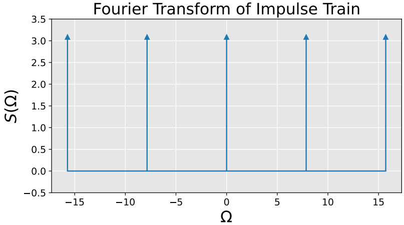
$$X_s(\Omega) = X_c(\Omega) * S(\Omega)$$

So, the Fourier transform of our sampled signal is the convolution of the continuous signal with the Fourier transform of the Dirac comb.

Review: Fourier Transform of Dirac Comb



$$\overset{\mathcal{F}}{\longleftrightarrow}$$

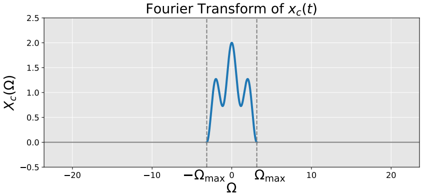


$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$S(\Omega) = \frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

Review: Fourier Transform of Sampling

$$X_s(\Omega) = X_c(\Omega) * S(\Omega)$$



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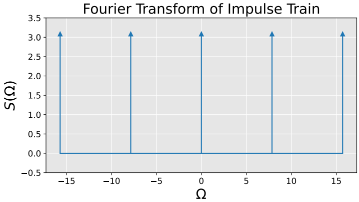

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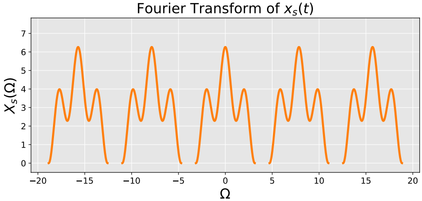
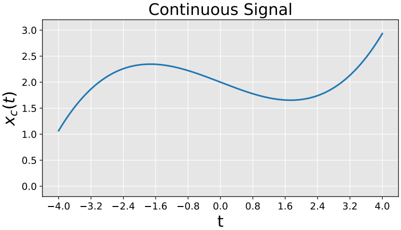


How Do We Reverse the Sampling Process?



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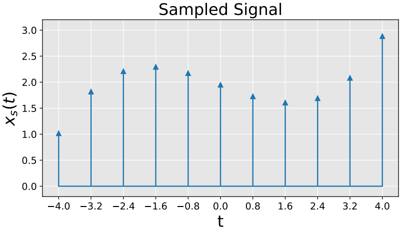
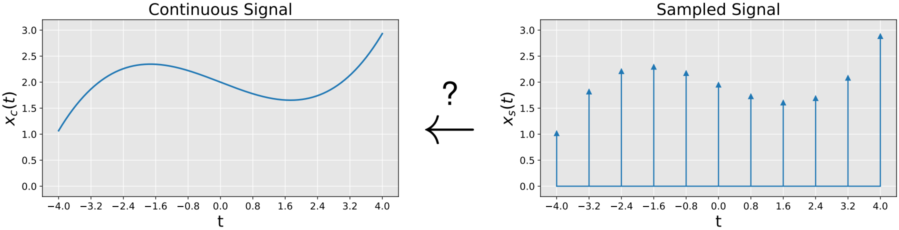
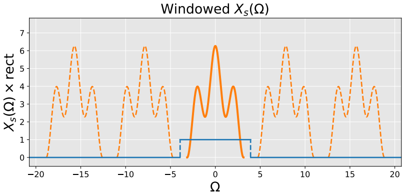


How Do We Reverse the Sampling Process?



Windowing in the Fourier Domain



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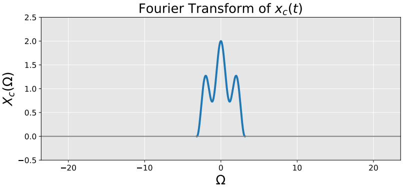


Window the Fourier transform of the sampled signal, $X_s(\Omega)$.

Rectangular Function

The **rectangular function**, or **box window** is defined as

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

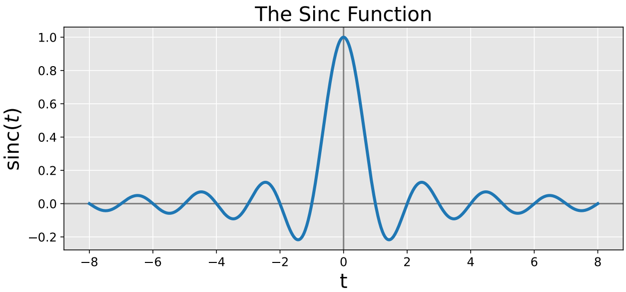
First an Identity

$$
\begin{aligned}
e^{i\theta} - e^{-i\theta} &= \cos(\theta) + i\sin(\theta) - (\cos(-\theta) + i\sin(-\theta)) \\
&= \cos(\theta) + i\sin(\theta) - (\cos(\theta) - i\sin(\theta)) \\
&= 2i\sin(\theta)
\end{aligned}
$$

Inverse Fourier Transform of a Box

$$
\begin{aligned}
\mathcal{F}^{-1}\{\mathrm{rect}(\xi)\} &= \int_{-\infty}^{\infty} \mathrm{rect}(\xi) e^{i2\pi\xi t} d\xi \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi\xi t} d\xi \\
&= \frac{e^{i2\pi\xi t}}{i2\pi t}\bigg|_{\xi=-\frac{1}{2}}^{\xi=\frac{1}{2}} \\
&= \frac{e^{i\pi\xi t}}{i2\pi t} - \frac{e^{-i\pi\xi t}}{i2\pi t} \\
&= \frac{2i\sin(\pi t)}{i2\pi t} = \frac{\sin(\pi t)}{\pi t}
\end{aligned}
$$

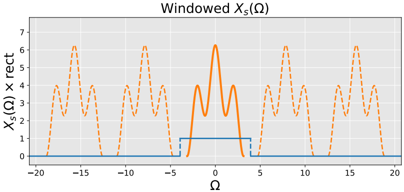
The Sinc Function



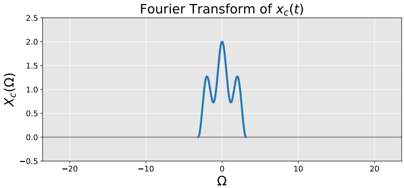
$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Note: $\mathrm{sinc}(0) = 1$

Inverse Fourier Transform of a Box



$$\longrightarrow$$



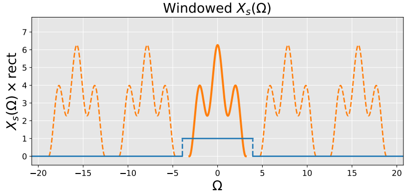
The window we want in the frequency domain is

$$\mathrm{rect}\left(\frac{\Omega}{\Omega_s}\right) = \begin{cases} 1 & \text{if } |\Omega| \leq \frac{\Omega_s}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

The inverse Fourier transform terms of angular frequency:

$$\mathcal{F}^{-1}\left\{\mathrm{rect}\left(\frac{\Omega}{\Omega_s}\right)\right\} = \frac{\sqrt{2\pi}}{T}\,\mathrm{sinc}\left(\frac{t}{T}\right)$$

Windowing in the Fourier Domain



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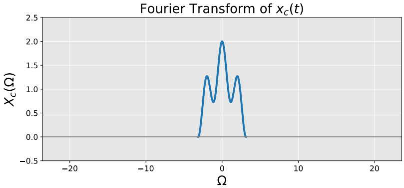

Windowing the Fourier transform of the sampled signal:

$$X_c(\Omega) = \frac{\sqrt{2\pi}}{\Omega_s} X_s(\Omega)\,\mathrm{rect}\left(\frac{\Omega}{\Omega_s}\right)$$

The Sampling Theorem

Theorem (Nyquist-Shannon Sampling Theorem)

Let $x_c(t)$ *be a continuous signal with Fourier transform* $X_c(\Omega)$ *that satisfies* $X_c(\Omega) = 0$ *for* $|\Omega| > \Omega_{\max}$*. Let* $x_s(t)$ *be the sampled signal with sampling period* T *such that*

$$\Omega_s > 2\Omega_{\max},$$

where $\Omega_s = \frac{2\pi}{T}$ *is the angular sampling frequency. Then* $x_c(t)$ *is exactly recoverable from* $x_s(t)$ *as*

$$x_c(t) = \sum_{n=-\infty}^{\infty} x_s(nT)\,\mathrm{sinc}\left(\frac{t - nT}{T}\right).$$