

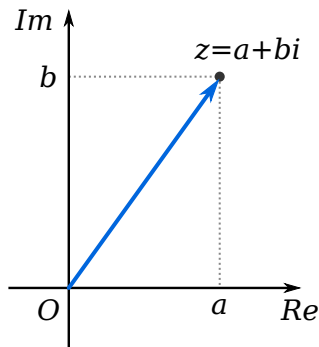
# Complex-Valued Signals

Digital Signal Processing

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# Complex Numbers



The **standard form** for  $z \in \mathbb{C}$ :

$$z = a + bi,$$

where  $i = \sqrt{-1}$ .

Notation for real and imaginary parts:

$$\operatorname{Re}(z) = a, \quad \operatorname{Im}(z) = b.$$

# Complex Addition

Adding two complex numbers:

$$z_1 = a_1 + b_1i,$$

$$z_2 = a_2 + b_2i.$$

Just add the real parts and imaginary parts, respectively:

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i.$$

# Complex Multiplication

Multiplying two complex numbers:

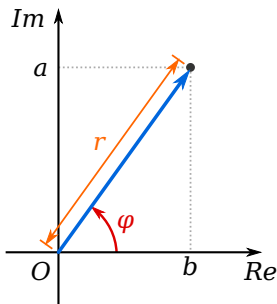
$$z_1 = a_1 + b_1i,$$

$$z_2 = a_2 + b_2i.$$

Use the rule that  $i^2 = -1$ :

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1i)(a_2 + b_2i) \\ &= a_1a_2 + a_1b_2i + b_1a_2i + b_1b_2i^2 \\ &= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i \end{aligned}$$

# Polar Form



The polar form for  $z \in \mathbb{C}$ :

$$z = r \cos \phi + ir \sin \phi$$

$r$  is the **modulus** or **magnitude**  
 $\phi$  is the **argument**

Equations:

$$r = |z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$$

$$\phi = \operatorname{atan2}(\operatorname{Im}(z), \operatorname{Re}(z))$$

# Euler Notation

Any complex number  $z \in \mathbb{C}$  can be written as an exponential:

$$z = re^{i\phi} = r \cos \phi + ir \sin \phi$$

Why? Power series definition of exponential:

$$\begin{aligned} e^{i\phi} &= 1 + \frac{i\phi}{1!} + \frac{(i\phi)^2}{2!} + \frac{(i\phi)^3}{3!} + \dots \\ &= 1 + \frac{i\phi}{1!} - \frac{\phi^2}{2!} - \frac{i\phi^3}{3!} + \dots \\ &= \left(1 - \frac{\phi^2}{2!} + \dots\right) + i \left(\frac{\phi}{1!} - \frac{\phi^3}{3!} + \dots\right) \\ &= \cos \phi + i \sin \phi \end{aligned}$$

# Multiplication in Euler Form

Let's multiply again:

$$z_1 = re^{i\phi}, \quad z_2 = se^{i\theta}$$

$$\begin{aligned} z_1 z_2 &= (re^{i\phi}) (se^{i\theta}) \\ &= (rs) (e^{i\phi} e^{i\theta}) \\ &= (rs) e^{i\phi + i\theta} \\ &= (rs) e^{i(\phi + \theta)} \end{aligned}$$

Notice:

$$\begin{aligned} |z_1 z_2| &= rs = |z_1| |z_2|, \\ \text{Arg}(z_1 z_2) &= \phi + \theta = \text{Arg}(z_1) + \text{Arg}(z_2) \end{aligned}$$

**Moduli multiply**  
**Arguments add**

# Exponentiation in Euler Form

Take  $z = re^{i\phi}$  to the power  $x \in \mathbb{R}$ :

$$\begin{aligned} z^x &= (re^{i\phi})^x \\ &= r^x e^{i\phi x} \end{aligned}$$



# Inversion in Euler Form

The inverse of  $z \in \mathbb{C}$  is just exponentiation by  $-1$ :

$$z^{-1} = (re^{i\phi})^{-1} = \frac{1}{r}e^{-i\phi}.$$

We can verify  $zz^{-1} = z^{-1}z = 1$ .

# Conjugation

The conjugate of  $z = a + bi$  is

$$\bar{z} = a - bi.$$

Useful tricks with conjugate:

- Magnitude is  $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$
- Inverse is  $z^{-1} = \frac{\bar{z}}{|z|^2}$
- Argument is  $\text{Arg}(\bar{z}) = -\text{Arg}(z)$

# Roots in Euler Form

The  $k$ th root of  $z \in \mathbb{C}$  is just exponentiation by  $\frac{1}{k}$ :

$$\sqrt[k]{z} = z^{\frac{1}{k}} = (re^{i\phi})^{\frac{1}{k}} = r^{\frac{1}{k}} e^{\frac{i\phi}{k}}.$$

Again, we can verify  $(\sqrt[k]{z})^k = z$ .

# Complex Exponential Signals

Remember our exponential signal:

$$x[n] = A\alpha^n$$

Now let  $A$  and  $\alpha$  both be **complex numbers**.

# Complex Exponential Signals

Let's write that using Euler notation:

$$A = |A|e^{i\phi}, \quad \alpha = |\alpha|e^{i\omega_0}$$

$$\begin{aligned}x[n] &= A\alpha^n \\&= |A|e^{i\phi}|\alpha|^n e^{i\omega_0 n} \\&= |A||\alpha|^n e^{i(\omega_0 n + \phi)} \\&= |A||\alpha|^n (\cos(\omega_0 n + \phi) + i \sin(\omega_0 n + \phi))\end{aligned}$$

This is a sinusoid, with **frequency**  $\omega_0$ , **phase**  $\phi$ , and **exponential weighting** by  $|\alpha|^n$ .