

# Signal Basics

Digital Signal Processing

January 24, 2023



# Review: Discrete-Time Signals

A **discrete-time signal** is a function

$$x : \mathbb{Z} \rightarrow B,$$

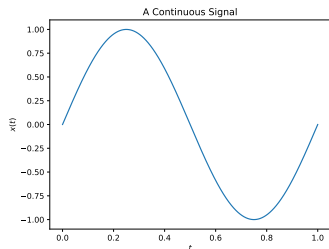
for some output set  $B$  (typically  $B = \mathbb{R}$  or  $B = \mathbb{C}$ ).

Equivalently,  $x$  is a sequence

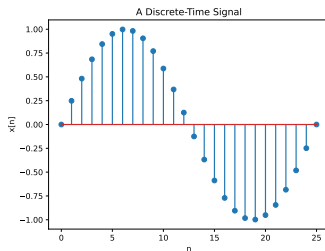
$$x[n] \in B, \quad -\infty \leq n \leq \infty.$$

# Sampled Continuous Signals

Discrete-time signals often come from continuous signals:



$$x_c(t)$$

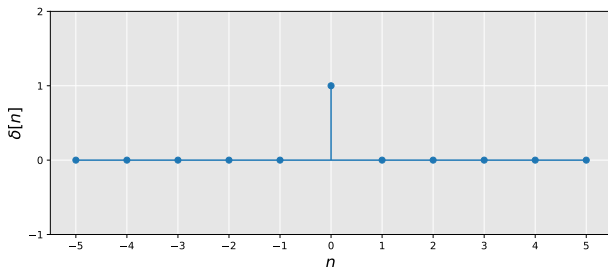


$$x[n] = x_c(nT)$$

Here,  $T \in \mathbb{R}$  is the **sampling period**,  $T = (1/25)\text{s} = 0.04\text{s}$

and  $\frac{1}{T}$  is the **sampling frequency**.  $\frac{1}{T} = 25\text{Hz}$

# Unit Sample Function



$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Also known as the **unit impulse function**.

# Shifting the Unit Impulse

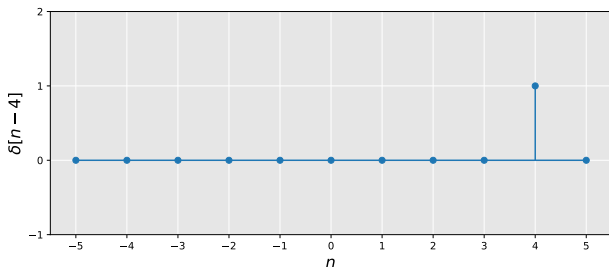
For any integer  $k$ ,

$$\delta[n - k] = \begin{cases} 1, & n - k = 0, \\ 0, & n - k \neq 0. \end{cases}$$

Or, in other words,

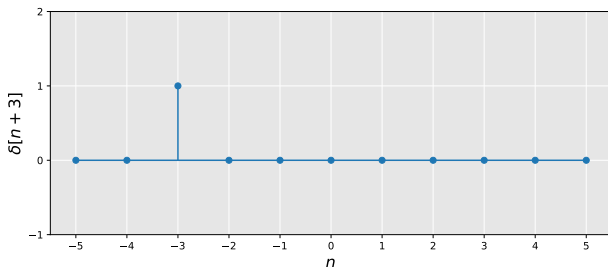
$$\delta[n - k] = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

# Shifting the Unit Impulse



$$\delta[n-4] = \begin{cases} 1, & n = 4, \\ 0, & n \neq 4. \end{cases}$$

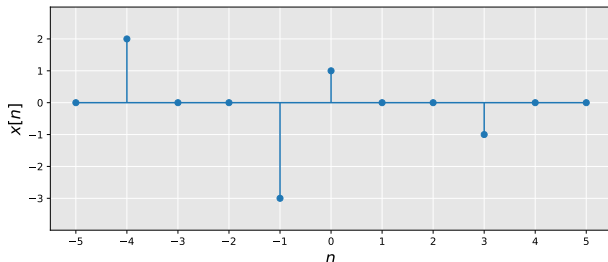
# Shifting the Unit Impulse



$$\delta[n+3] = \begin{cases} 1, & n = -3, \\ 0, & n \neq -3, \end{cases}$$

# Scaling and Adding Shifted Impulses

We can scale and add shifted impulses to construct signals:



$$x[n] = 2\delta[n + 4] - 3\delta[n + 1] + \delta[n] - \delta[n - 3]$$

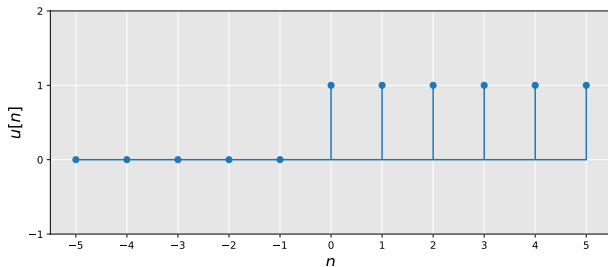


# Scaling and Adding Impulses

In fact, any sequence,  $x[n]$ , can be written as a sum of scaled, shifted impulses:

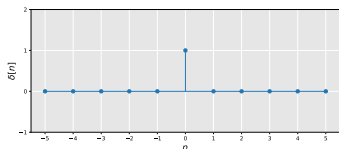
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k].$$

# Unit Step Function

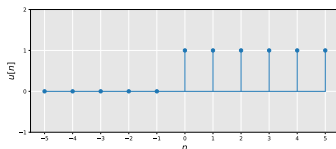


$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

# Relationship Between Step and Impulse



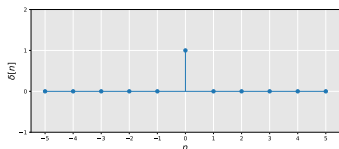
accumulate  
 $\longrightarrow$



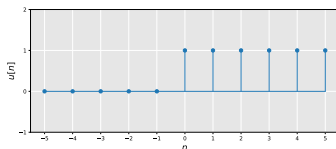
$$u[n] = \sum_{k=-\infty}^n \delta[k]$$

Discrete analogy to integration

# Relationship Between Step and Impulse



difference  
←



$$\delta[n] = u[n] - u[n - 1]$$

Discrete analogy to differentiation

# Real Exponential Function

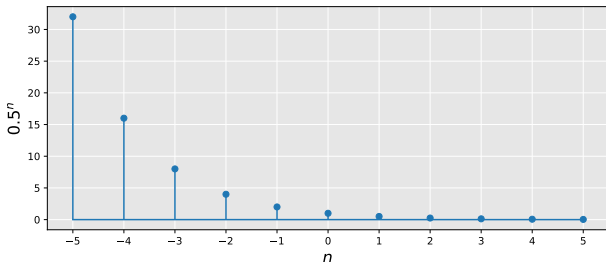
A **real exponential sequence** is of the form

$$x[n] = A\alpha^n,$$

for constants  $A \in \mathbb{R}$  and  $\alpha \in \mathbb{R}$ .

# Real Exponential Function

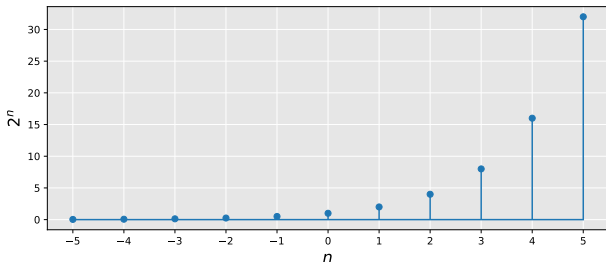
When  $0 < \alpha < 1$ , we get exponential **decay**:



$$x[n] = 0.5^n$$

# Real Exponential Function

When  $\alpha > 1$ , we get exponential **growth**:



$$x[n] = 2^n$$

# Real Exponential Function

Note: taking reciprocal of  $\alpha$  is equivalent to time-reversal:

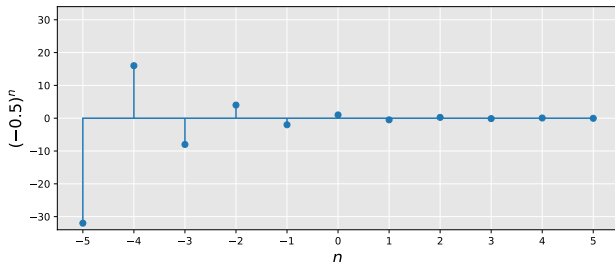
Let  $x[n] = \alpha^n$ , then

$$x[-n] = \alpha^{-n} = (\alpha^{-1})^n = \left(\frac{1}{\alpha}\right)^n$$



# Real Exponential Function

When  $\alpha < 0$ , we get exponential **oscillation**:



$$x[n] = (-0.5)^n$$

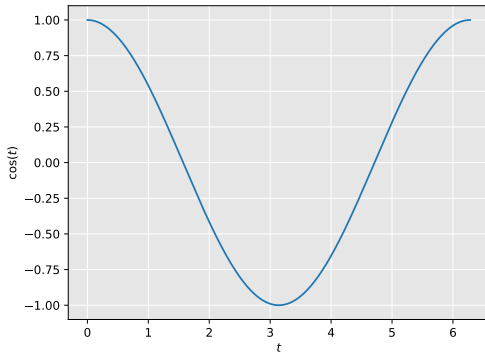
# Real Exponential Function

Note: time shift is equivalent to multiplication:

Let  $x[n] = \alpha^n$ , then

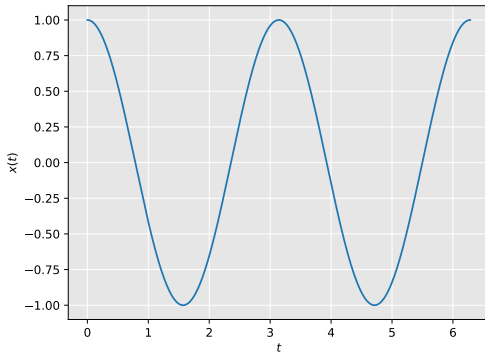
$$x[n - k] = \alpha^{n-k} = \alpha^n \alpha^{-k} = \alpha^{-k} x[n]$$

# Remember the Cosine Function



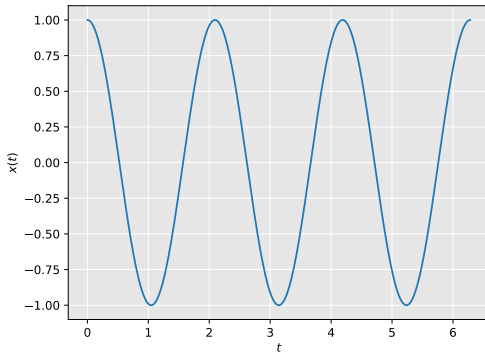
$$x(t) = \cos(t)$$

# Remember the Cosine Function



$$x(t) = \cos(2t)$$

# Remember the Cosine Function



$$x(t) = \cos(3t)$$

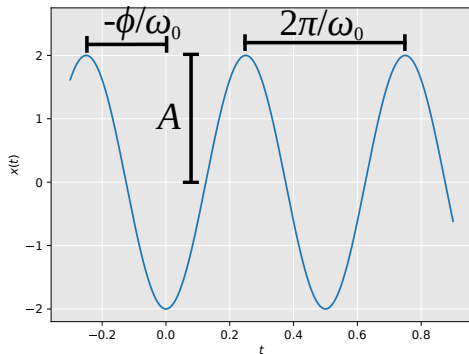
# Frequency

$$x(t) = \cos(\omega_0 t)$$

The **frequency**,  $\omega_0$ , is the number of times the cosine wave repeats in the interval  $[0, 2\pi]$ .

The **period**,  $T_0 = \frac{2\pi}{\omega_0}$ , is the length of time the cosine wave takes to repeat, i.e., the time interval between two consecutive peaks.

# General Continuous Sinusoid



$$A = 2, \quad \omega_0 = 4\pi, \quad \phi = \pi$$

$$x(t) = A \cos(\omega_0 t + \phi)$$

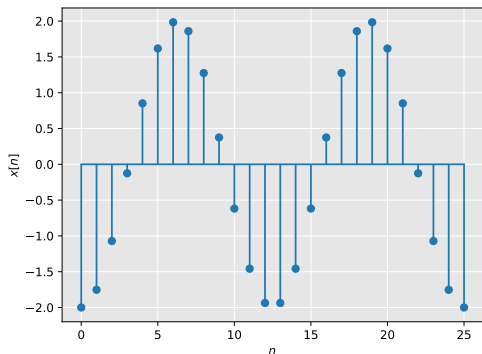
$A$  : amplitude

$\omega_0$  : frequency

$\phi$  : phase

# Discrete-Time Sinusoidal Function

Discrete-time sinusoid is just a sampled continuous sinusoid



$$x[n] = A \cos(\omega_0 n + \phi)$$

$A$  : amplitude

$\omega_0$  : frequency

$\phi$  : phase

$$A = 2, \quad \omega_0 = \frac{4\pi}{25}, \quad \phi = 25\pi$$



# Periodic Discrete-Time Signals

## Definition

A signal,  $x[n]$ , is said to be **periodic** if for some integer  $N > 0$ ,

$$x[n] = x[n + N],$$

for all  $n \in \mathbb{Z}$ .

# Periodicity of Sinusoids

If  $x[n] = x[n + N]$  for a sinusoid, then

$$\begin{aligned} A \cos(\omega_0 n + \phi) &= A \cos(\omega_0(n + N) + \omega_0 N + \phi) \\ &= A \cos(\omega_0 n + \omega_0 N + \phi) \end{aligned}$$

This will only be true when

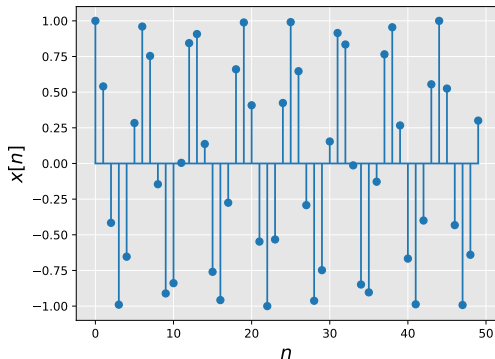
$$\omega_0 N = 2\pi k, \quad \text{for some integer } k.$$

So,  $\omega_0 = 2\pi \frac{k}{N}$  must be  $2\pi$  times a rational number.

# Periodic or Aperiodic?

$$x[n] = \cos(n)$$

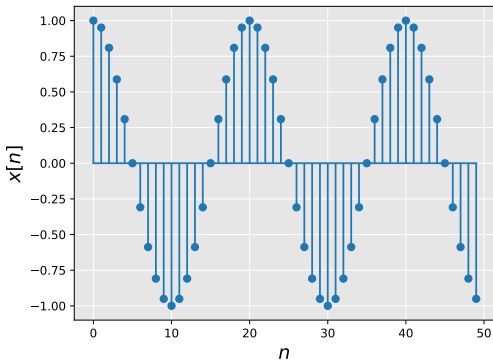
Not periodic!



# Periodic or Aperiodic?

$$x[n] = \cos\left(\frac{n\pi}{10}\right)$$

Periodic!



# Computing the Period of a Sinusoid

To get the period,  $N$ , write the frequency as

$$\omega_0 = 2\pi \frac{k}{N},$$

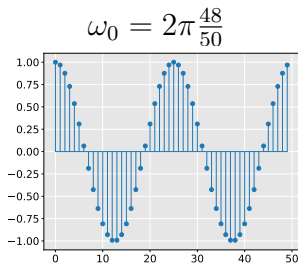
where  $\frac{k}{N}$  is a *simplified* fraction.

For the previous example:

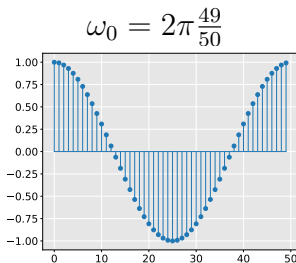
$$\omega_0 = \frac{\pi}{10} = 2\pi \frac{1}{20},$$

so  $N = 20$ .

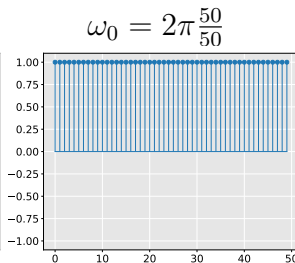
# Weirdness with Discrete-Time Periodicity



Period = 25



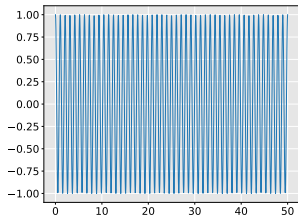
Period = 50



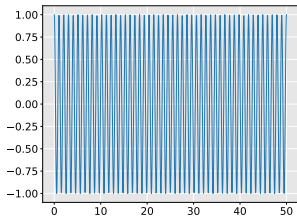
Period = 1

# Continuous Versions of These Cosine Waves

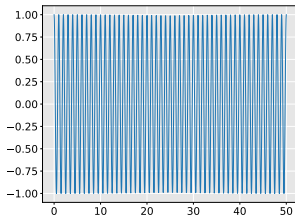
$$\omega_0 = 2\pi \frac{48}{50}$$



$$\omega_0 = 2\pi \frac{49}{50}$$



$$\omega_0 = 2\pi \frac{50}{50}$$



# Energy of a Signal

## Definition

The **energy** of a signal,  $x[n]$ , is defined as

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$



# Power of a Signal

## Definition

The **power** of a signal,  $x[n]$ , is defined as

$$P_x = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2.$$

If  $x[n]$  is periodic, with period  $N$ , then

$$P_x = \frac{1}{N} \sum_{n=0}^N |x[n]|^2.$$