The Discrete Fourier Transform

Digital Signal Processing

February 8, 2024

Fourier Series Review

Given a real-valued, periodic sequence $x[n]$ with period L, write the fundamental angular frequency as $\omega_0 = \frac{2\pi}{L}$.

Fourier series **synthesis**:

$$x[n] = \frac{a_0}{2} + \sum_{k=1}^{L-1} \left[a_k \cos\left(\omega_0 k n\right) + b_k \sin\left(\omega_0 k n\right) \right].$$

Fourier series **analysis**:

$$a_k = \frac{2}{L} \sum_{n=0}^{L-1} \cos\left(\omega_0 k n\right) x[n],$$

$$b_k = \frac{2}{L} \sum_{n=0}^{L-1} \sin\left(\omega_0 k n\right) x[n].$$

Extending to Complex Signals

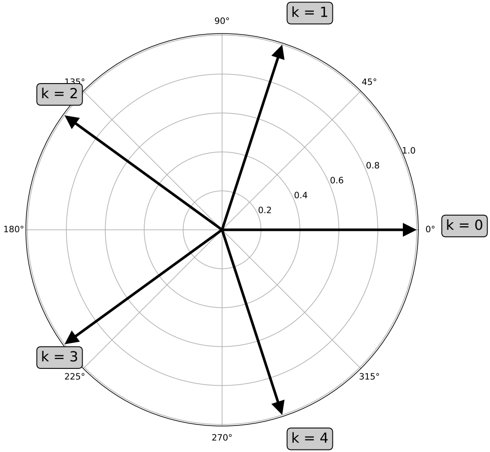
Replace real cosine and sine with a single complex sinusoid:

$$z_k[n] = e^{i\omega_0 kn} = \cos(\omega_0 kn) + i\sin(\omega_0 kn)$$

Here, $z_k[n]$ is a complex exponential sequence with

- **frequency**: $\omega_0 k = \frac{2\pi k}{L}$,
- **amplitude**: $|z_k[n]| = 1$, and
- **phase**: $\phi = 0$.

Example: $L = 5$

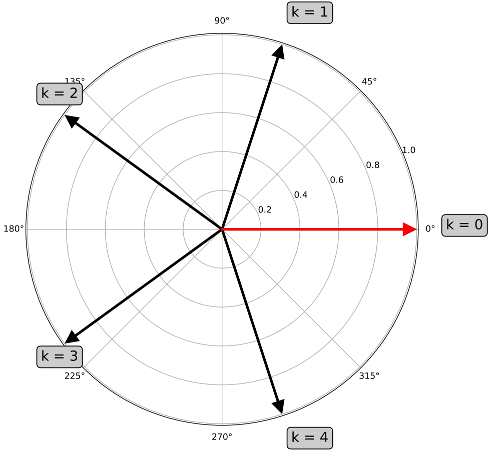


$$\omega_0 = \frac{2\pi}{5} = 72^\circ$$

Note:

$$z_k[n] = e^{i\omega_0 kn} = \left(e^{i\omega_0}\right)^{kn}$$

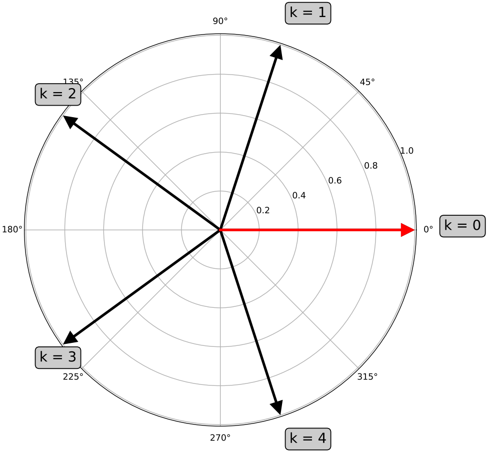
Example: $L = 5$



For $k = 0$

$$z_0[n] = e^0 = 1.0$$

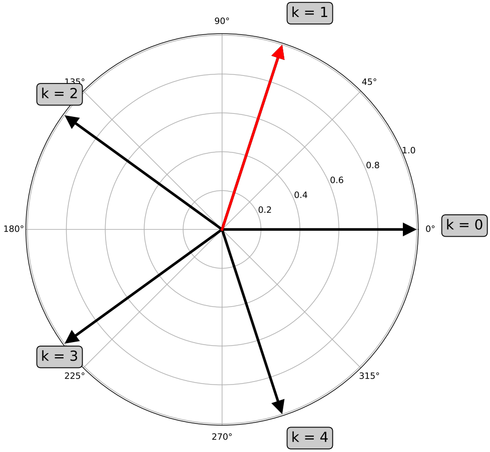
Example: $L = 5$



For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$

Example: $L = 5$



For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$

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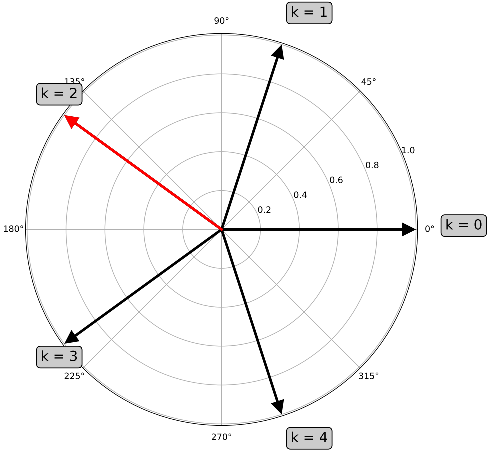

For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$

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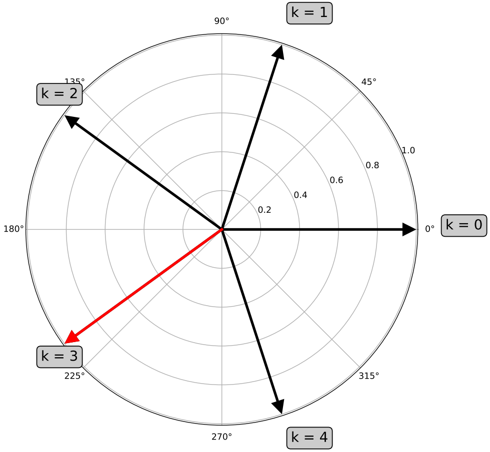

For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$
$$z_1[1] = (e^{i\omega_0})^1$$
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$$z_1[3] = (e^{i\omega_0})^3$$

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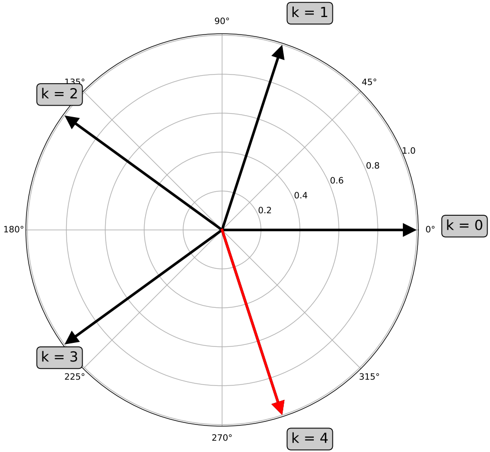

For $k = 1$,

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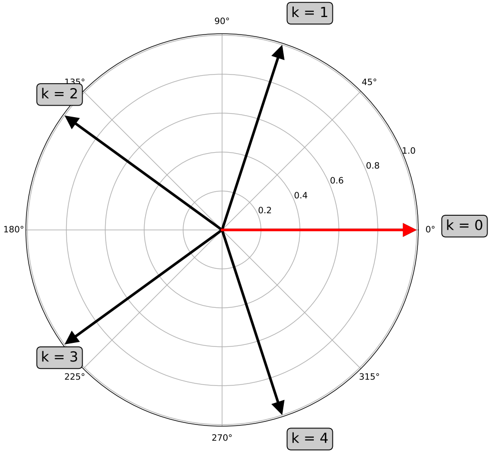
$$z_1[4] = (e^{i\omega_0})^4$$

Example: $L = 5$



For $k = 2$,

$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$

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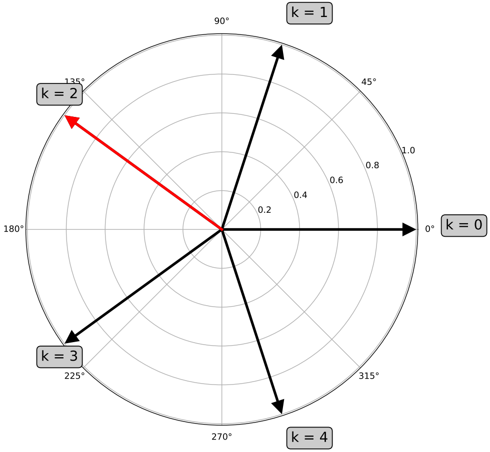

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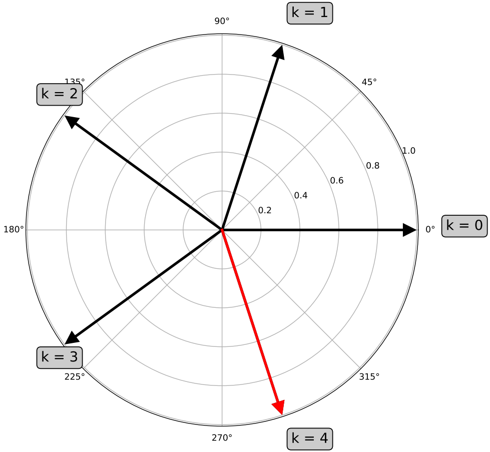

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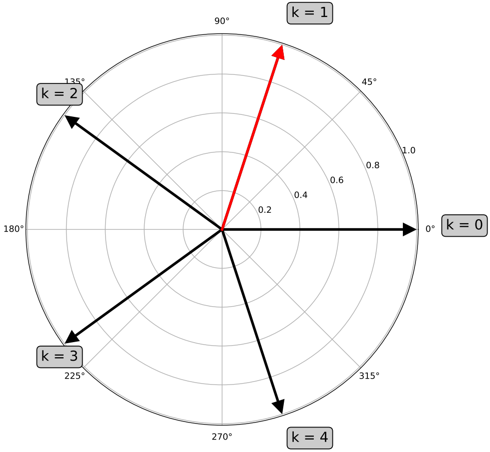

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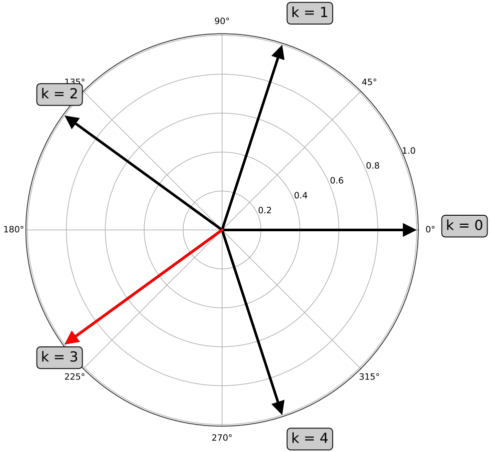

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$$z_2[3] = (e^{i\omega_0})^1$$
$$z_2[4] = (e^{i\omega_0})^3$$

Discrete Fourier Transform (DFT)

Definition

Now let $x[n]$ be a complex-valued, periodic signal with period L. The **discrete Fourier transform (DFT)** of $x[n]$ is given by

DFT **synthesis**:

$$x[n] = \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} e^{i\omega_0 kn} X[k]$$

DFT **analysis**:

$$X[k] = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n]$$

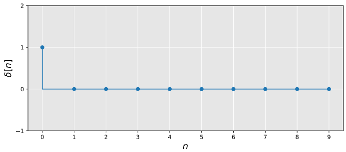
Notation

If $X[k]$ is the DFT of $x[n]$, we'll denote this as

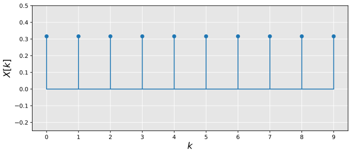
$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k].$$

These are called **DFT pairs**.

DFT of the Unit Sample Function



$$\xleftrightarrow{\mathcal{DFT}}$$



$$x[n] = \delta[n] = \begin{cases} 1 & n = 0, \\ 0 & 1 \leq n < L. \end{cases}$$

$$X[k] = \frac{1}{\sqrt{L}}, \quad \forall k.$$

DFT of the Unit Sample Function

$$
\begin{aligned}
X[k] &= \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n] \\
&= \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} \delta[n] \\
&= \frac{1}{\sqrt{L}} e^{0} \\
&= \frac{1}{\sqrt{L}}
\end{aligned}
$$

Properties of DFT

1. Duality
2. Conjugation
3. Linearity
4. Time Shift
5. Modulation

1. Duality

Duality Property

The inverse DFT is the same as the forward DFT composed with time-reversal:

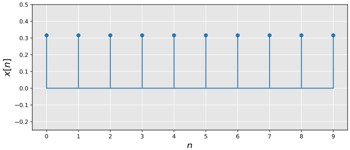
If

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k],$$

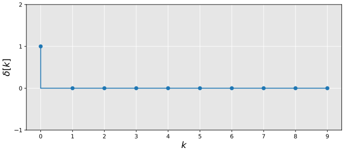
then

$$X[n] \overset{\mathcal{DFT}}{\longleftrightarrow} x[L-k].$$

DFT of a Constant Signal



$$\xleftrightarrow{\mathcal{DFT}}$$



$$x[n] = \frac{1}{\sqrt{L}}, \quad \forall n.$$

$$X[k] = \delta[k] = \begin{cases} 1 & k = 0, \\ 0 & 1 \leq k < L. \end{cases}$$

Using duality

2. Conjugation

Conjugation Property

Conjugating a signal results in its DFT being conjugated and time-reversed:

$$\bar{x}[n] \xleftrightarrow{\mathcal{DFT}} \bar{X}[L-k]$$

3. Linearity

Linearity Property

The DFT is a linear operator:

If $x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]$, and $y[n] \overset{\mathcal{DFT}}{\longleftrightarrow} Y[k]$, then

$$ax[n] + by[n] \overset{\mathcal{DFT}}{\longleftrightarrow} aX[k] + bY[k],$$

for all complex constants $a, b \in \mathbb{C}$.

4. Time Shift

Time Shift Property

Shifting a signal in time results in a modulation of its DFT:

$$x[n-m] \overset{\mathcal{DFT}}{\longleftrightarrow} e^{-i\omega_0 km} X[k].$$

Derivation of Time Shift Property

$$
\begin{aligned}
\mathcal{DFT}\{x[n-m]\} &= \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n-m] \\
&= \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} e^{-i\omega_0 k(j+m)} x[j] \qquad \text{defining } j = n-m \\
&= \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} e^{-i\omega_0 kj} e^{-i\omega_0 km} x[j] \\
&= e^{-i\omega_0 km} \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} e^{-i\omega_0 kj} x[j] \\
&= e^{-i\omega_0 km} \mathcal{DFT}\{x[n]\}
\end{aligned}
$$

5. Modulation

Modulation Property

Modulation of a signal results in a time shift of its DFT:

$$e^{i\omega_0 nm}x[n] \xleftrightarrow{\mathcal{DFT}} X[k-m].$$

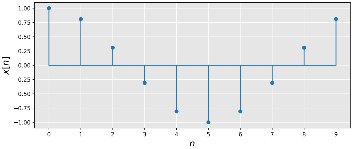
DFT of Real-Valued Signals

Let $x[n]$ be a real-valued signal. In other words, $\mathrm{Im}(x[n]) = 0$.

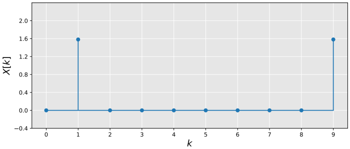
Then the following symmetry in the DFT holds:

$$X[k] = \bar{X}[L - k].$$

DFT of a Cosine

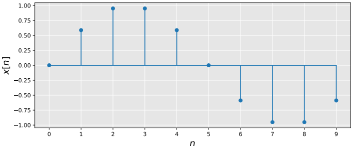


$$\overset{\mathcal{DFT}}{\longleftrightarrow}$$

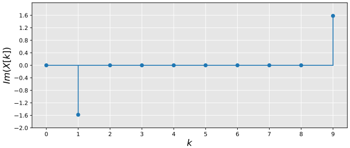


$$x[n] = \cos(\omega_0 m n)$$

$$X[k] = \frac{\sqrt{L}}{2}(\delta[k-m] + \delta[k-L+m]).$$



$$\xleftrightarrow{\mathcal{DFT}}$$



$$x[n] = \sin(\omega_0 m n)$$

$$X[k] = \frac{i\sqrt{L}}{2}(-\delta[k-m] + \delta[k-L+m])$$