The Discrete Fourier Transform

Digital Signal Processing

February 14, 2023

Fourier Series Review

Given a real-valued, periodic sequence $x[n]$ with period L, write the fundamental angular frequency as $\omega_0 = \frac{2\pi}{L}$.

Fourier series **synthesis**:

$$x[n] = \frac{a_0}{2} + \sum_{k=1}^{L-1} \left[a_k \cos\left(\omega_0 k n\right) + b_k \sin\left(\omega_0 k n\right) \right].$$

Fourier series **analysis**:

$$a_k = \frac{2}{L} \sum_{n=0}^{L-1} \cos\left(\omega_0 k n\right) x[n],$$

$$b_k = \frac{2}{L} \sum_{n=0}^{L-1} \sin\left(\omega_0 k n\right) x[n].$$
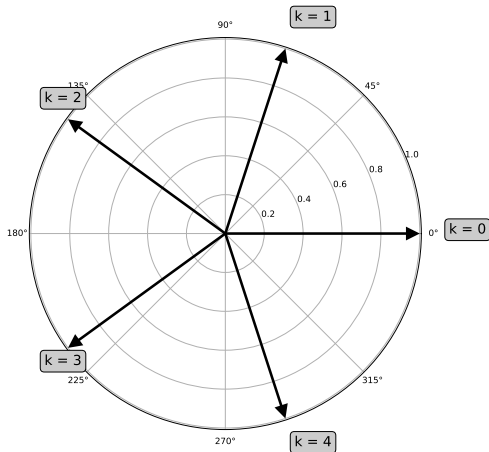
Extending to Complex Signals

Replace real cosine and sine with a single complex sinusoid:

$$z_k[n] = e^{i\omega_0 kn} = \cos(\omega_0 kn) + i\sin(\omega_0 kn)$$

Here, $z_k[n]$ is a complex exponential sequence with

- **frequency**: $\omega_0 k = \frac{2\pi k}{L}$,
- **amplitude**: $|z_k[n]| = 1$, and
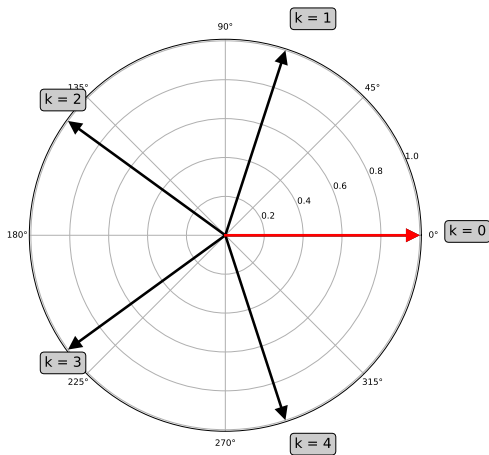- **phase**: $\phi = 0$.

Example: $L = 5$

$$\omega_0 = \frac{2\pi}{5} = 72°$$

Note:

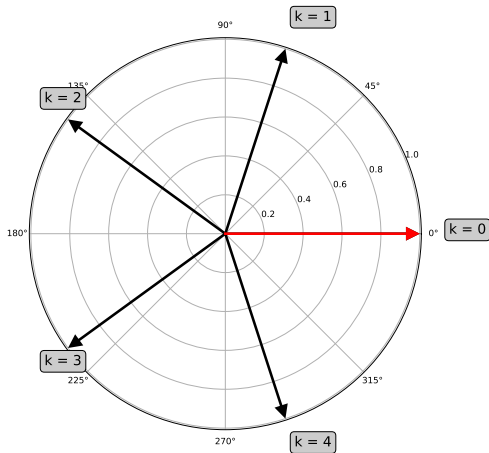$$z_k[n] = e^{i\omega_0 kn} = (e^{i\omega_0})^{kn}$$

Example: $L = 5$
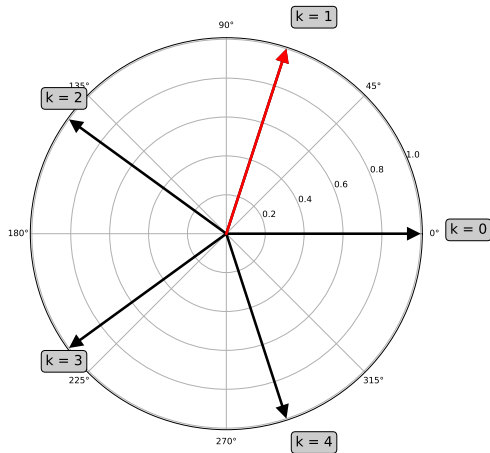

For $k = 0$

$$z_0[n] = e^0 = 1.0$$

Example: $L = 5$

For $k = 1$,

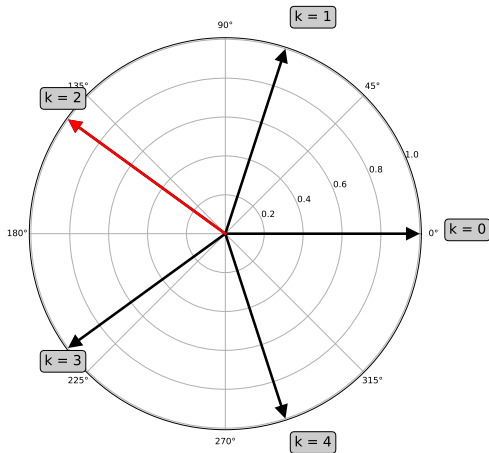$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$

Example: $L = 5$

For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$
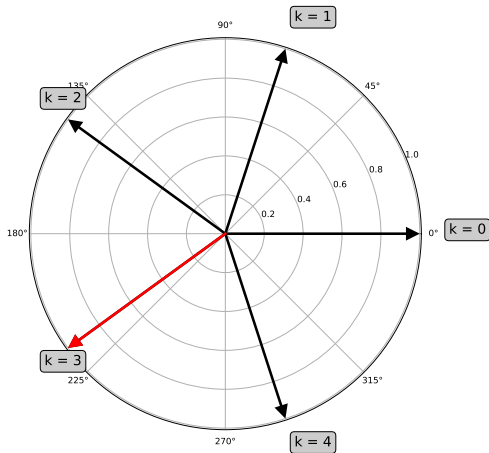$$z_1[1] = (e^{i\omega_0})^1$$

Example: $L = 5$

For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$
$$z_1[1] = (e^{i\omega_0})^1$$
$$z_1[2] = (e^{i\omega_0})^2$$
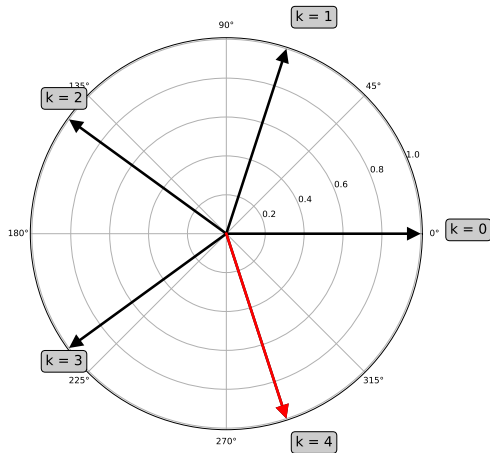
Example: $L = 5$

For $k = 1$,

$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$
$$z_1[1] = (e^{i\omega_0})^1$$
$$z_1[2] = (e^{i\omega_0})^2$$
$$z_1[3] = (e^{i\omega_0})^3$$

For $k = 1$,

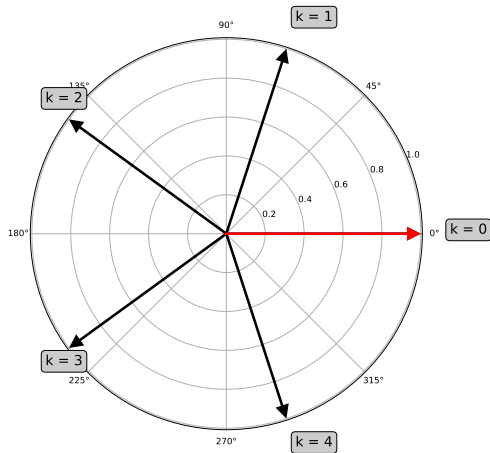$$z_1[0] = (e^{i\omega_0})^0 = 1.0$$
$$z_1[1] = (e^{i\omega_0})^1$$
$$z_1[2] = (e^{i\omega_0})^2$$
$$z_1[3] = (e^{i\omega_0})^3$$
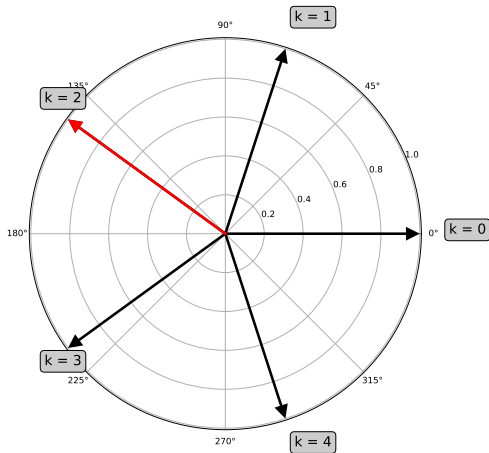$$z_1[4] = (e^{i\omega_0})^4$$

Example: $L = 5$

For $k = 2$,

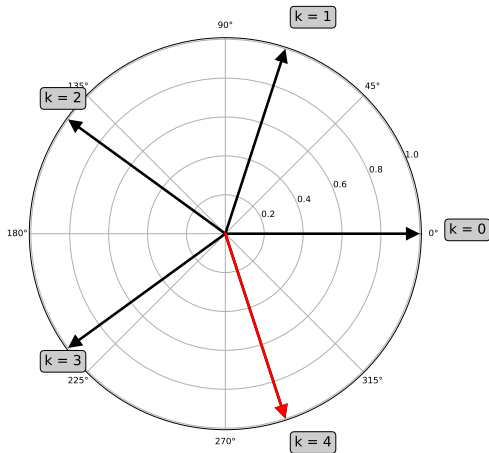$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$

Example: $L = 5$

For $k = 2$,

$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$
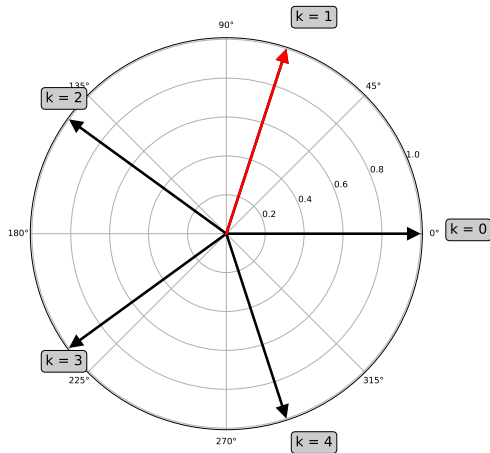$$z_2[1] = (e^{i\omega_0})^2$$

Example: $L = 5$

For $k = 2$,

$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$
$$z_2[1] = (e^{i\omega_0})^2$$
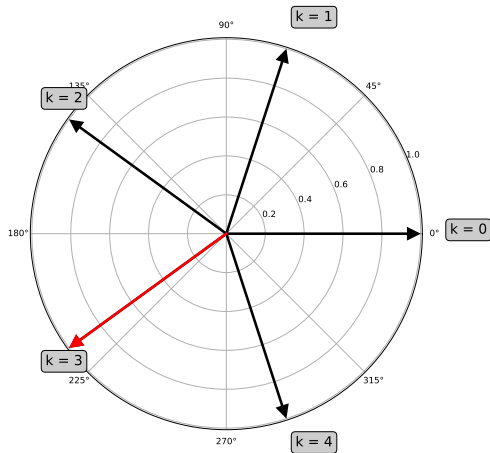$$z_2[2] = (e^{i\omega_0})^4$$

Example: $L = 5$

For $k = 2$,

$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$
$$z_2[1] = (e^{i\omega_0})^2$$
$$z_2[2] = (e^{i\omega_0})^4$$
$$z_2[3] = (e^{i\omega_0})^1$$

For $k = 2$,

$$z_2[0] = (e^{i\omega_0})^0 = 1.0$$
$$z_2[1] = (e^{i\omega_0})^2$$
$$z_2[2] = (e^{i\omega_0})^4$$
$$z_2[3] = (e^{i\omega_0})^1$$
$$z_2[4] = (e^{i\omega_0})^3$$

Discrete Fourier Transform (DFT)

Definition

Now let $x[n]$ be a complex-valued, periodic signal with period L.
The **discrete Fourier transform (DFT)** of $x[n]$ is given by

DFT **synthesis**:

$$x[n] = \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} e^{i\omega_0 kn} X[k]$$

DFT **analysis**:

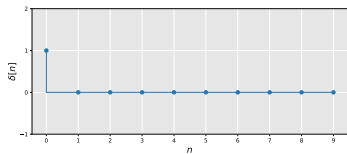$$X[k] = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n]$$

Notation

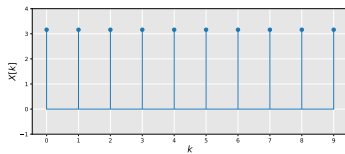If $X[k]$ is the DFT of $x[n]$, we'll denote this as

$$x[n] \xleftrightarrow{\mathcal{DFT}} X[k].$$

These are called **DFT pairs**.

DFT of the Unit Sample Function

$$x[n] = \delta[n] = \begin{cases} 1 & n = 0, \\ 0 & 1 \le n < L. \end{cases}$$

$$X[k] = \frac{1}{\sqrt{L}}, \quad \forall k.$$

DFT of the Unit Sample Function

$$X[k] = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 k n} x[n]$$

$$= \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 k n} \delta[n]$$

$$= \frac{1}{\sqrt{L}} e^0$$

$$= \frac{1}{\sqrt{L}}$$

Properties of DFT

1. Duality
2. Conjugation
3. Linearity
4. Time Shift
5. Modulation

1. Duality

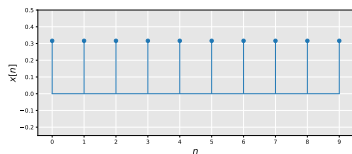The inverse DFT is the same as the forward DFT composed with time-reversal:

If

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k],$$
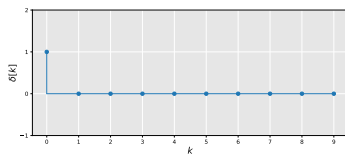
then

$$X[n] \overset{\mathcal{DFT}}{\longleftrightarrow} x[L - k].$$

DFT of a Constant Signal

$$x[n] = \frac{1}{\sqrt{L}}, \quad \forall n.$$

$$X[k] = \delta[k] = \begin{cases} 1 & k = 0, \\ 0 & 1 \leq k < L. \end{cases}$$

Using duality

2. Conjugation

Conjugating a signal results in its DFT being conjugated and time-reversed:

$$\bar{x}[n] \xleftrightarrow{\mathcal{DFT}} \bar{X}[L - k]$$

3. Linearity

The DFT is a linear operator:

If $x[n] \xleftrightarrow{\mathcal{DFT}} X[k]$, and $y[n] \xleftrightarrow{\mathcal{DFT}} Y[k]$, then

$$ax[n] + by[n] \xleftrightarrow{\mathcal{DFT}} aX[k] + bY[k],$$

for all complex constants $a, b \in \mathbb{C}$.

4. Time Shift

Shifting a signal in time results in a modulation (phase shift) of its DFT:

$$x[n - m] \xleftrightarrow{\mathcal{DFT}} e^{-i\omega_0 k m} X[k].$$

5. Modulation

Modulation (phase shift) of a signal results in a time shift of its DFT:

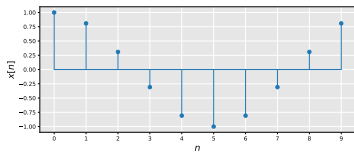$$e^{i\omega_0 nm}x[n] \xleftrightarrow{\mathcal{DFT}} X[k-m].$$

DFT of Real-Valued Signals

Let $x[n]$ be a real-valued signal. In other words, $\mathrm{Im}(x[n]) = 0$.
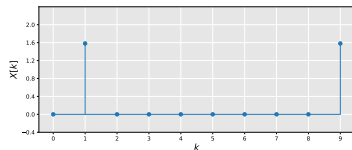
Then the following symmetry in the DFT holds:
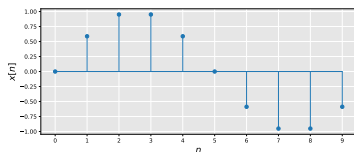
$$X[k] = \bar{X}[L - k].$$

DFT of a Cosine
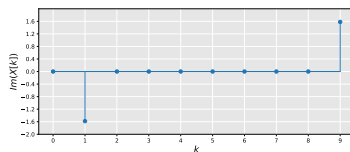

$$x[n] = \cos(i\omega_0 mn)$$

$$X[k] = \frac{\sqrt{L}}{2}(\delta[k - m] + \delta[k - L + m]).$$

DFT of a Sine

$$x[n] = \sin(i\omega_0 mn)$$

$$X[k] = \frac{i\sqrt{L}}{2}(\delta[k - m] - i\delta[k - L + m]).$$