All-Pass and Minimum Phase Systems

Digital Signal Processing

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UNIVERSITY of VIRGINIA

Review: Rational Transfer Functions

A **rational transfer function** looks like

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Applying the FTOA, we can factor the numerator and denominator:

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=0}^{M}(1 - c_k z^{-1})}{\prod_{k=0}^{N}(1 - d_k z^{-1})}$$

The c_k are **zeros** of $H(z)$ (zeros of the numerator).
The d_k are **poles** of $H(z)$ (zeros of the denominator).

All-Pass Systems

Definition

An **all-pass system** is an LTI system whose frequency response has magnitude equal to one. In other words, its frequency response, $H(e^{i\omega})$, satisfies:

$$|H(e^{i\omega})| = 1, \qquad \text{for all } \omega \in [-\pi, \pi).$$

Simplest All-Pass System

Consider a transfer function $H(z)$ of the form:

$$H(z) = \frac{z^{-1} - c}{1 - \bar{c}z^{-1}}.$$

The magnitude of its frequency response is:

$$\begin{aligned}
|H(e^{i\omega})| &= \frac{|e^{-i\omega} - c|}{|1 - \bar{c}e^{-i\omega}|} && \text{plug in } z = e^{i\omega} \\
&= \frac{|e^{-i\omega}||1 - ce^{i\omega}|}{|1 - \bar{c}e^{-i\omega}|} && \text{pull out } e^{-i\omega} \text{ factor} \\
&= \frac{|1 - ce^{i\omega}|}{|1 - \bar{c}e^{-i\omega}|} && |e^{-i\omega}| = 1 \\
&= \frac{|\overline{1 - \bar{c}e^{-i\omega}}|}{|1 - \bar{c}e^{-i\omega}|} = 1 && \text{conjugates}
\end{aligned}$$

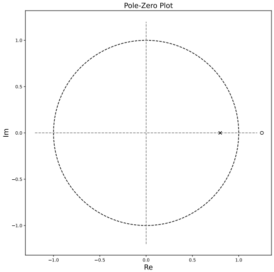
Simplest All-Pass System

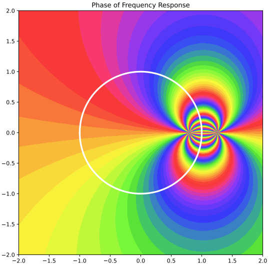
Rearranging the simple all-pass system:

$$H(z) = \frac{z^{-1} - c}{1 - \bar{c}z^{-1}} = -c\,\frac{1 - c^{-1}z^{-1}}{1 - \bar{c}z^{-1}}.$$

We get a zero at c^{-1} and a pole at $\bar{c}$.

Let $\bar{c} = re^{-i\omega}$. Then $c^{-1} = \frac{1}{r}e^{-i\omega}$. This is the pole-zero inverse pair relationship we saw last time:





Real Coefficent All-Pass System

The transfer function $H(z) = \frac{z^{-1} - c}{1 - \bar{c} z^{-1}}$ represents the system:

$$y[n] = -cx[n] + x[n-1] + \bar{c} y[n-1].$$

It has complex coefficients.

If we want a real-valued system, we add a conjugate zero / pole

$$H(z) = \frac{(z^{-1} - c)(z^{-1} - \bar{c})}{(1 - \bar{c} z^{-1})(1 - c z^{-1})} = \frac{z^{-2} - 2\mathrm{Re}(c) z^{-1} + |c|^2}{1 - 2\mathrm{Re}(c) z^{-1} + |c|^2 z^{-2}},$$

which now is a system with only real-valued coefficients.

General Real-Valued All-Pass System

The general form of a real-valued all-pass system is

$$H(z) = \prod_{j=1}^{J} \frac{z^{-1} - d_j}{1 - d_j z^{-1}} \prod_{k=1}^{K} \frac{(z^{-1} - c_k)(z^{-1} - \bar{c_k})}{(1 - \bar{c_k} z^{-1})(1 - c_k z^{-1})},$$

where $c_j \in \mathbb{C}$ and $d_j \in \mathbb{R}$.

Inverse Systems

Can we "undo" an LTI system? That is, given an output

$$y[n] = h[n] * x[n],$$

can we get back the input signal $x[n]$?

This means we want an inverse system, $h^{-1}[n]$, such that:

$$h^{-1}[n] * (h[n] * x[n]) = x[n], \qquad \text{for all signals } x[n].$$

This implies $h^{-1}[n] * h[n] = \delta[n]$.

Taking the z-transform, we have

$$H^{-1}(z)H(z) = 1 \quad \Longrightarrow \quad H^{-1}(z) = \frac{1}{H(z)}$$

Inverse of a Rational Transfer Function

Let

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=0}^{M}(1 - c_k z^{-1})}{\prod_{k=0}^{N}(1 - d_k z^{-1})}$$

Then the inverse just flips the numerator and denominator:

$$H^{-1}(z) = \frac{1}{H(z)} = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=0}^{N}(1 - d_k z^{-1})}{\prod_{k=0}^{M}(1 - c_k z^{-1})}$$

The poles of $H(z)$ become zeros of $H^{-1}(z)$, and zeros become poles.

Exercise: Inverse of FBCF

What is the inverse system for the FBCF?

$$y[n] = x[n] + gy[n-k]$$

Solution

Transfer function for FBCF:

$$H(z) = \frac{1}{1 - gz^{-k}}.$$

Inverse transfer function:

$$H^{-1}(z) = 1 - gz^{-k}.$$

This is the FFCF! (with negative gain)

Minimum-Phase Systems

Definition

A **minimum-phase system** is an LTI system that is stable, causal, and whose inverse is also stable and causal.

Because the poles and zeros flip roles in the inverse, **a minimum-phase system must have all of its poles and zeros inside the unit circle.**

Decomposition of Stable, Causal Systems

Theorem

Let $H(z)$ be the transfer function for a stable, causal LTI system. Then $H(z)$ can be decomposed into a product

$$H(z) = H_{\mathrm{min}}(z)H_{\mathrm{ap}}(z),$$

where $H_{\mathrm{min}}(z)$ is a minimum-phase system, and $H_{\mathrm{ap}}(z)$ is an all-pass system.