Filter Design Basics

Digital Signal Processing

April 4, 2024

Frequency-Selective Filters

Definition (Frequency-Selective Filter)

A **frequency-selective filter** is a system that passes certain frequencies and supresses certain other frequencies from an input signal to an output signal.

- Note an **ideal** frequency-selective filter would pass desired frequencies unchanged (multiplying by 1), while completely stopping (multiplying by 0) undesired frequencies.
- We'll also think of filters as any system that amplifies desired frequencies and suppress undesired frequencies.

Classes of Frequency-Selective Filters

1. Low-Pass Filters
2. High-Pass Filters
3. Band-Pass Filters
4. Band-Stop Filters

Ideal Low-Pass Filter

If ω_c is our cutoff frequency, we'd like a frequency response that passes every frequency below ω_c and zeros out any frequency above.

So, a rectangular function in the frequency domain:

$$H_{\mathrm{LP}}(e^{i\omega}) = \begin{cases} 1 & \text{for } |\omega| < \omega_c, \\ 0 & \text{otherwise.} \end{cases}$$

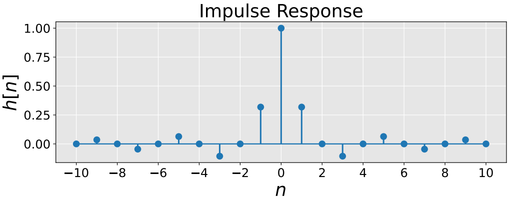
Ideal Low-Pass Filter

The inverse DTFT of a box is

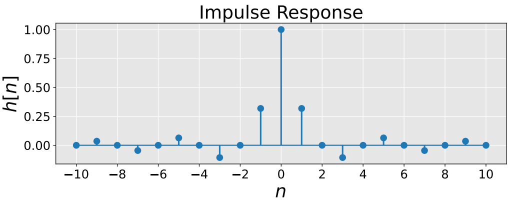
$$h_{\mathrm{LP}}[n] = \mathrm{DTFT}^{-1}\{H_{\mathrm{LP}}(e^{i\omega})\} = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{i\omega n} d\omega$$

$$= \frac{1}{2\pi i n}[e^{i\omega_c n} - e^{-i\omega_c n}]$$

$$= \frac{\sin(\omega_c n)}{\pi n}$$ a sinc.



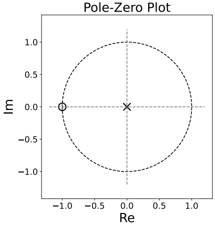
Ideal Low-Pass Filter

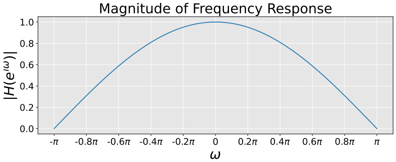


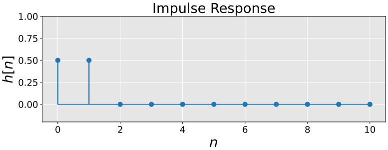
$$h_{\mathrm{LP}}[n] = \frac{\sin(\omega_c n)}{\pi n}$$

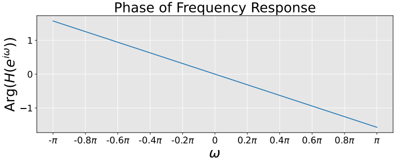
- Can't implement this in practice: infinite extent
- Also, it is not causal

Low-Pass Filter: Single Zero









$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

$$H(z) = \frac{1+z^{-1}}{2}$$

Repeating A Filter

Filter Design Trick

The relative frequency modulations of a filter can often be accentuated by applying it multiple times.

Why? Transfer function multiplies, so composing is: $H(z)H(z)$

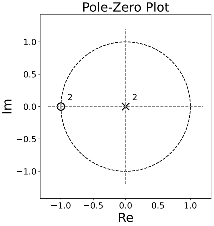
Magnitude of frequency response also multiplies:

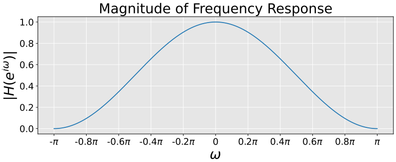
$$|H(e^{i\omega})H(e^{i\omega})| = |H(e^{i\omega})|\,|H(e^{i\omega})|$$

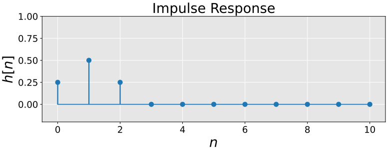
Also, note phase is additive (so, linear phase will stay linear):

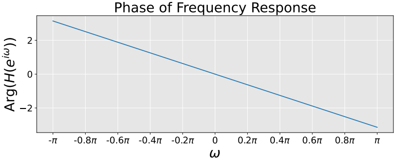
$$\mathrm{Arg}(H(e^{i\omega})H(e^{i\omega})) = 2\mathrm{Arg}(H(e^{i\omega}))$$

Low-Pass Filter: Double Zero





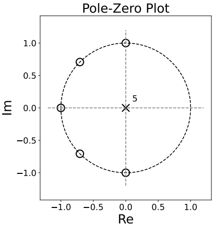


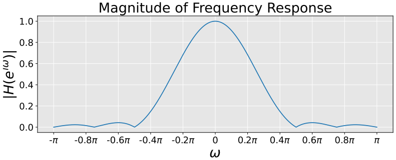


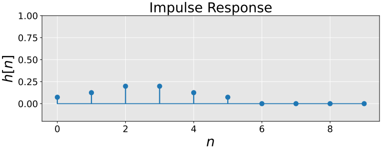
$$y[n] = \frac{1}{4}(x[n] + 2x[n-1] + x[n-2])$$

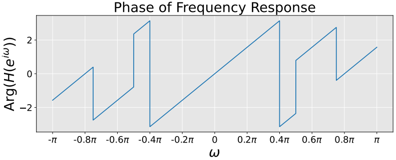
$$H(z) = \frac{1+2z^{-1}+z^{-2}}{4}$$

Low-Pass Filter: Multiple Zeros









$$H(z) = \frac{1}{C}\frac{(z+1)(z-e^{3\pi i/4})(z-e^{-3\pi i/4})(z-i)(z+i)}{z^5}$$

Normalizing A Low-Pass Filter

If we want the constant component to be one, then we need to normalize.

DTFT at $e^{i0} = 1$:

$$H(1) = \sum_{n=-\infty}^{\infty} e^{i0n} h[n] = \sum_{n=-\infty}^{\infty} h[n].$$

So, we need our impulse response function to sum to one.

Normalizing A Low-Pass Filter

$$H(z) = \frac{1}{C}\frac{(z+1)(z-e^{3\pi i/4})(z-e^{-3\pi i/4})(z-i)(z+i)}{z^5}$$

$$= \frac{1+(\sqrt{2}+1)z^{-1}+(\sqrt{2}+2)z^{-2}+(\sqrt{2}+2)z^{-3}+(\sqrt{2}+1)z^{-4}+z^{-5}}{C}$$

The coefficients of the z^{-k} are the weights of the impulse response, so their sum is the constant C that we want:

$$C = 2+2(\sqrt{2}+1)+2(\sqrt{2}+2) = 8+4\sqrt{2} \approx 13.66$$