# The $z$-Transform

Digital Signal Processing

February 22, 2024

# Motivation for the $z$-Transform

- Until now, we've used the Fourier transform to analyze frequency content of **signals**.
- With the $z$-transform, we'll analyze **systems** (specificallly, LTI systems).
- Helps determine properties of a system, such as stability, causality, frequency response, etc.
- Used to design LTI systems (filters).

# Review: Discrete Fourier Transform

## Definition

Let $x[n]$ be a complex-valued, periodic signal with period $L$. The **discrete Fourier transform (DFT)** of $x[n]$ is given by

DFT **analysis**:

$$X[k] = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{-i\omega_0 kn} x[n]$$

DFT **synthesis**:

$$x[n] = \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} e^{i\omega_0 kn} X[k]$$

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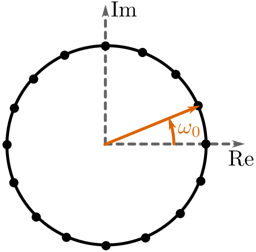
# Transform Domains

**DFT**




$$e^{i\omega_0 k}$$

# Discrete-Time Fourier Transform

## Definition (DTFT Analysis)

Let $x[n]$ be a complex signal for $-\infty < n < \infty$. The **discrete-time Fourier transform (DTFT)** of $x[n]$ is given by

$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} e^{-i\omega n} x[n], \qquad \text{for } \omega \in [-\pi, \pi).$$

- Note range of $n$ is all of $\mathbb{Z}$.
- Note $X(e^{i\omega})$ is defined on the **continuous** unit circle in $\mathbb{C}$.
- Can also think of $X$ as a function of angular frequency, $\omega$.

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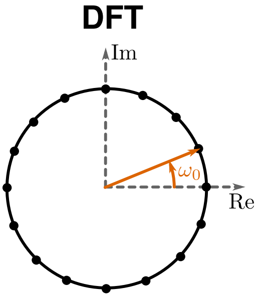
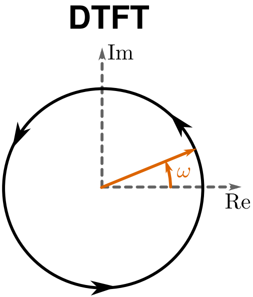



$e^{i\omega_0 k}$




$e^{i\omega}$

# Inverse DTFT

**Definition (DTFT Synthesis)**

Let $X(e^{i\omega})$ be the DTFT of a signal $x[n]$. The **inverse discrete-time Fourier transform** is given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})e^{i\omega n}d\omega.$$

- Note this is an integral around the unit circle in $\mathbb{C}$.

# The $z$-Transform

**Definition ($z$-Transform Analysis)**

Given a complex discrete signal $x[n]$, its $z$**-transform** is given by

$$X(z) = \sum_{n=-\infty}^{\infty} z^{-n} x[n].$$

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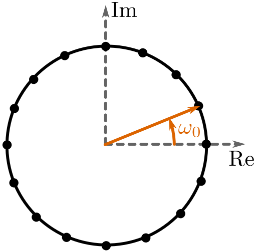
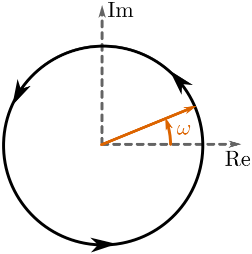
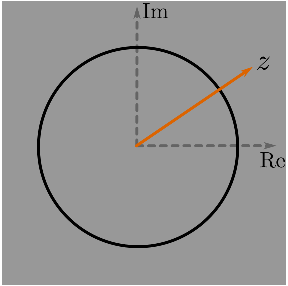
# Transform Domains

| DFT | DTFT | $z$-Transform |
| --- | --- | --- |




$e^{i\omega_0 k}$ $\quad$ $e^{i\omega}$ $\quad$ $z$

# Overloaded Notation!

Notice we reused the notation "$X$" for the DFT, DTFT, and $z$-transform.

The context is clear from the input to $X$:

| | |
|---|---|
| DFT | $X[k]$ |
| DTFT | $X(e^{i\omega})$ |
| $z$-transform | $X(z)$ |

# Quick Side Note: Geometric Series

A geometric series looks like:

$$\begin{aligned} s &= 1 + r + r^2 + r^3 + \cdots \\ &= \sum_{n=0}^{\infty} r^n \end{aligned}$$

The infinite sum evaluates to

$$s = \frac{1}{1-r}, \qquad \text{for } |r| < 1.$$

This holds for $r$ real or complex!

# DTFT Example: Right-Sided Exponential

Right-sided exponential signal, for some constant $a \in \mathbb{C}$, is:

$$x[n] = \begin{cases} a^n & \text{for } n \geq 0, \\ 0 & \text{for } n < 0, \end{cases}$$
$$= a^n u[n].$$

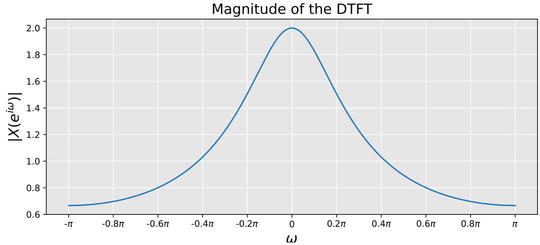
The DTFT is

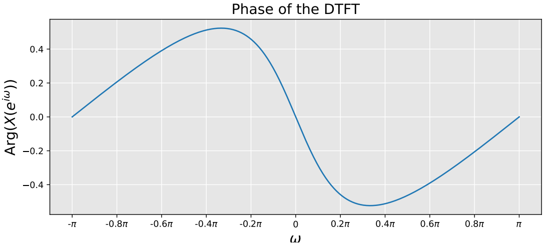
$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-i\omega n} = \sum_{n=0}^{\infty} (ae^{-i\omega})^n$$
$$= \frac{1}{1 - ae^{-i\omega}}, \qquad \text{for } |ae^{-i\omega}| < 1, \text{ or } |a| < 1.$$

Diverges for $|a| \geq 1$.

# Plotting the DTFT

$$x[n] = 0.5^n u[n] \qquad \leftrightarrow \qquad X\left(e^{i\omega}\right) = \frac{1}{1-0.5e^{-i\omega}}$$



# $z$-Transform of Right-Sided Exponential

The $z$-transform of $x[n] = a^n u[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}}, \qquad \text{for } |az^{-1}| < 1, \quad \text{or} \quad |z| > |a|$$

$$= \frac{z}{z - a}.$$

# Region of Convergence (ROC)

## Definition (ROC)

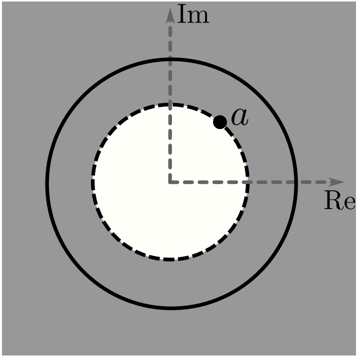
The **region of convergence** for the $z$-transform of a signal $x[n]$ is defined as all points $z \in \mathbb{C}$ where

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty.$$

# ROC for Right-Sided Exponential

For $x[n] = u[n]a^n$, we had

$$X(z) = \frac{z}{z-a}, \qquad \text{for } |z| > |a|.$$



# Left-Sided Exponential

Left-sided exponential for a constant $a \in \mathbb{C}$ is

$$x[n] = \begin{cases} -a^n & \text{for } n < 0 \\ 0 & \text{for } n \geq 0 \end{cases}$$
$$= -a^n u[-n-1]$$

The $z$-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n$$
$$= -\sum_{m=1}^{\infty} a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m$$
$$= 1 - \frac{1}{1 - a^{-1}z}, \qquad \text{for } |a^{-1}z| < 1, \quad \text{or} \quad |z| < |a|$$

# Left-Sided Exponential

Further simplifying:

$$\begin{aligned} X(z) &= 1 - \frac{1}{1 - a^{-1}z} \\ &= \frac{1 - a^{-1}z}{1 - a^{-1}z} - \frac{1}{1 - a^{-1}z} \\ &= \frac{-a^{-1}z}{1 - a^{-1}z} \\ &= \frac{z}{z - a}, \qquad \text{for } |z| < |a|. \end{aligned}$$

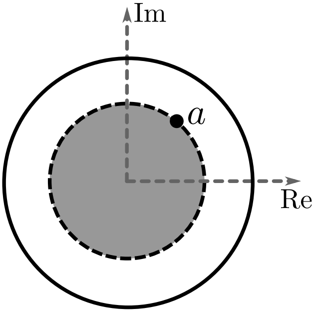
Same as right-sided exponential, but different ROC!

Right-sided ROC: $|z| > |a|$, Left-sided ROC: $|z| < |a|$

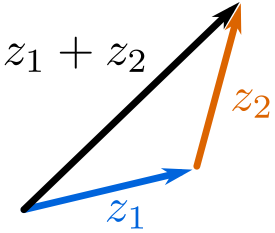
# ROC for Left-Sided Exponential

For $x[n] = -a^n u[-n-1]$, we had

$$X(z) = \frac{z}{z-a}, \qquad \text{for } |z| < |a|.$$



# Quick Side Note: Triangle Inequality




Complex Addition

Triangle inequality tells us:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Do this iteratively to get:

$$\left|\sum_{n=-\infty}^{\infty} z_n\right| \leq \sum_{n=-\infty}^{\infty} |z_n|.$$

# General ROC of $z$-Transform

In general, we have:

$$\begin{aligned} |X(z)| &= \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| \\ &= \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} \end{aligned}$$

Note this only depends on the magnitude $r = |z|$, not the angle of $z$.

# General ROC of $z$-Transform

ROC is an annulus:

$$0 \leq r_R < |z| < r_L \leq \infty$$

$r_R$: right-side radius

$\sum_{n=0}^{\infty} |x[n]| r^{-n}$ diverges when $r < r_R$.

$r_L$: left-side radius

$\sum_{n=-\infty}^{-1} |x[n]| r^{-n}$ diverges when $r > r_L$.