Frequency Response of LTI Systems

Digital Signal Processing

March 23, 2023

Review: Transfer Function

Given an LTI system with impulse response $h[n]$:

$$y[n] = x[n] * h[n].$$

The **transfer function** is the z-transform of $h[n]$. It is given by the ratio:

$$H(z) = \frac{Y(z)}{X(z)},$$

where $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ and $y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$.

Review: Transfer Function of LCCDE

A linear constant-coefficient difference equation (LCCDE) is an LTI system of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

It's transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

Exercise: Moving Average

What is the transfer function for the following moving average system?

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k].$$

Answer: Moving Average

Moving Average:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k].$$

Taking z-transform of both sides:

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} X(z).$$

Solving for the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k}.$$

Pole-Zero Plot for Moving Average

Rearrange the transfer function:

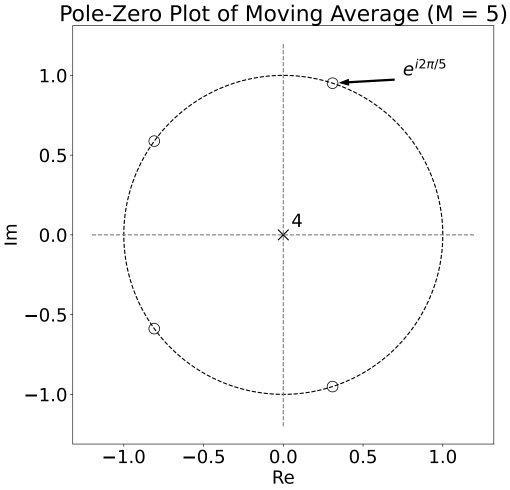
$$H(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k}$$

$$= \frac{1}{M} \frac{\sum_{k=0}^{M-1} z^k}{z^{M-1}} \qquad \text{multiply by } \frac{z^{M-1}}{z^{M-1}}$$

Trick: $(1 + z + z^2 + \cdots + z^{M-1})(z-1) = z^M - 1$.

So, $\sum_{k=0}^{M-1} z^k = \frac{z^M - 1}{(z-1)}$, which means the zeros are the Mth roots of 1, excluding $z = 1$.

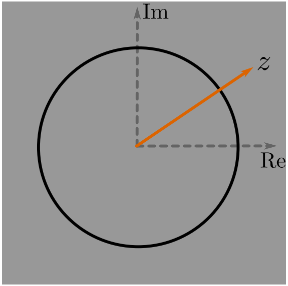
Pole-Zero Plot for Moving Average



Frequency Response of an LTI System

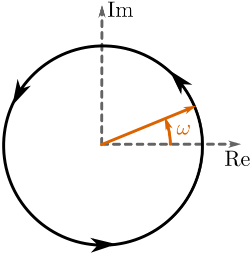
The **frequency response** of an LTI system is the restriction of $H(z)$ to the unit circle, which is the DTFT of the impulse response, $H(e^{i\omega})$.

z-Transform



$H(z)$

DTFT



$H(e^{i\omega})$

LTI Applied to Complex Sinusoids

Consider a complex sinusoid $e^{i\omega n}$. Applying $T\{\cdot\}$ to it gives:

$$
\begin{aligned}
T\{e^{i\omega n}\} &= e^{i\omega n} * h[n] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega(n-k)} h[k] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega n} e^{-i\omega k} h[k] \\
&= e^{i\omega n} \sum_{k=-\infty}^{\infty} e^{-i\omega k} h[k] \qquad \text{DTFT of } h[k]! \\
&= H(e^{i\omega}) e^{i\omega n}
\end{aligned}
$$

LTI Applied to Complex Sinusoids

Consider a complex sinusoid $e^{i\omega n}$. Applying $T\{\cdot\}$ to it gives:

$$
\begin{aligned}
T\{e^{i\omega n}\} &= e^{i\omega n} * h[n] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega(n-k)} h[k] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega n} e^{-i\omega k} h[k] \\
&= e^{i\omega n} \sum_{k=-\infty}^{\infty} e^{-i\omega k} h[k] \qquad \text{DTFT of } h[k]! \\
&= H(e^{i\omega}) e^{i\omega n}
\end{aligned}
$$

LTI Applied to Complex Sinusoids

Consider a complex sinusoid $e^{i\omega n}$. Applying $T\{\cdot\}$ to it gives:

$$
\begin{aligned}
T\{e^{i\omega n}\} &= e^{i\omega n} * h[n] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega(n-k)} h[k] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega n} e^{-i\omega k} h[k] \\
&= e^{i\omega n} \sum_{k=-\infty}^{\infty} e^{-i\omega k} h[k] \qquad \text{DTFT of } h[k]! \\
&= H(e^{i\omega}) e^{i\omega n}
\end{aligned}
$$

LTI Applied to Complex Sinusoids

Consider a complex sinusoid $e^{i\omega n}$. Applying $T\{\cdot\}$ to it gives:

$$
\begin{aligned}
T\{e^{i\omega n}\} &= e^{i\omega n} * h[n] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega(n-k)} h[k] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega n} e^{-i\omega k} h[k] \\
&= e^{i\omega n} \sum_{k=-\infty}^{\infty} e^{-i\omega k} h[k] && \text{DTFT of } h[k]! \\
&= H(e^{i\omega}) e^{i\omega n}
\end{aligned}
$$

LTI Applied to Complex Sinusoids

Consider a complex sinusoid $e^{i\omega n}$. Applying $T\{\cdot\}$ to it gives:

$$\begin{aligned} T\{e^{i\omega n}\} &= e^{i\omega n} * h[n] \\ &= \sum_{k=-\infty}^{\infty} e^{i\omega(n-k)} h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{i\omega n} e^{-i\omega k} h[k] \\ &= e^{i\omega n} \sum_{k=-\infty}^{\infty} e^{-i\omega k} h[k] \qquad \text{DTFT of } h[k]! \\ &= H(e^{i\omega}) e^{i\omega n} \end{aligned}$$

LTI Applied to Complex Sinusoids

Consider a complex sinusoid $e^{i\omega n}$. Applying $T\{\cdot\}$ to it gives:

$$
\begin{aligned}
T\{e^{i\omega n}\} &= e^{i\omega n} * h[n] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega(n-k)} h[k] \\
&= \sum_{k=-\infty}^{\infty} e^{i\omega n} e^{-i\omega k} h[k] \\
&= e^{i\omega n} \boxed{\sum_{k=-\infty}^{\infty} e^{-i\omega k} h[k]} \qquad \text{DTFT of } h[k]! \\
&= H(e^{i\omega}) e^{i\omega n}
\end{aligned}
$$

Frequency Response

Let $T\{\cdot\}$ be an LTI system. Recall it's **impulse response** is

$$h[n] = T\{\delta[n]\}.$$

It's **frequency response** is how it responds to a complex sinusoid with a certain frequency $\omega \in [0, 2\pi)$:

$$T\{e^{i\omega n}\} = H(e^{i\omega})e^{i\omega n}.$$

Magnitude and Phase of Frequency Response

Looking at frequency response of an LTI:

$$Y(e^{i\omega}) = H(e^{i\omega})X(e^{i\omega})$$

Remember complex multiplication in Euler form:

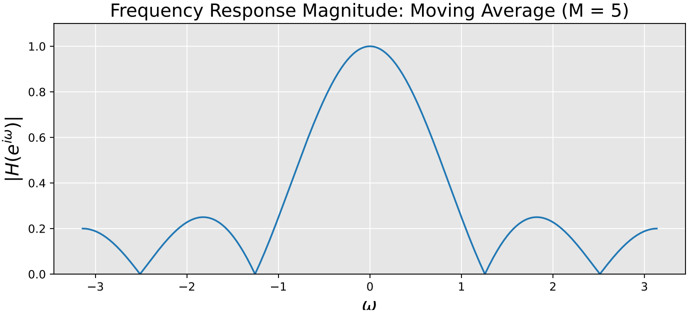
$$re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}$$

So, we have:

Magnitude: $|Y(e^{i\omega})| = |H(e^{i\omega})| \cdot |X(e^{i\omega})|$

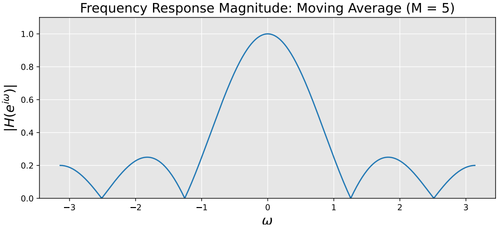
Phase: $\mathrm{Arg}(Y(e^{i\omega})) = \mathrm{Arg}(H(e^{i\omega})) + \mathrm{Arg}(X(e^{i\omega}))$

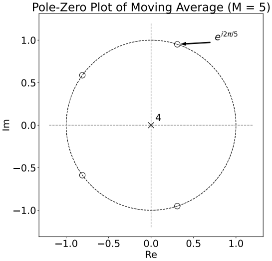
Frequency Response Magnitude of the Moving Average



This is a **low-pass** filter.

Frequency Response Magnitude of the Moving Average





Note the zeros.

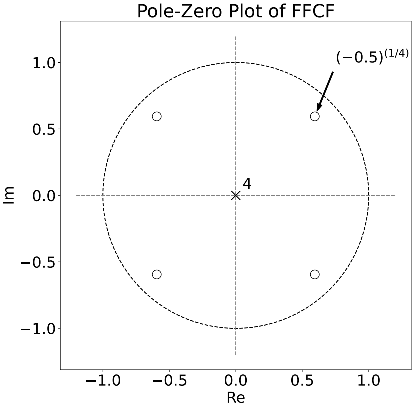
Feedforward Comb Filter (FFCF)

FFCF:

$$y[n] = x[n] + gx[n-k]$$

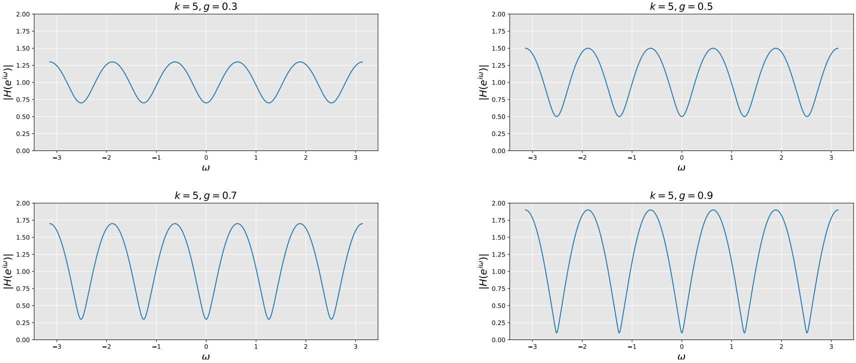
Transfer function:

$$H(z) = \frac{z^k + g}{z^k}$$



$$k = 4, \;\; g = 0.5$$

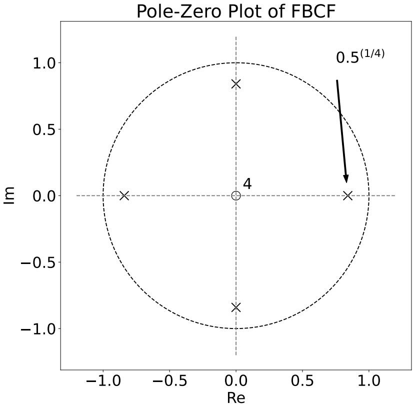
FFCF Frequency Response



Feedback Comb Filter (FBCF)

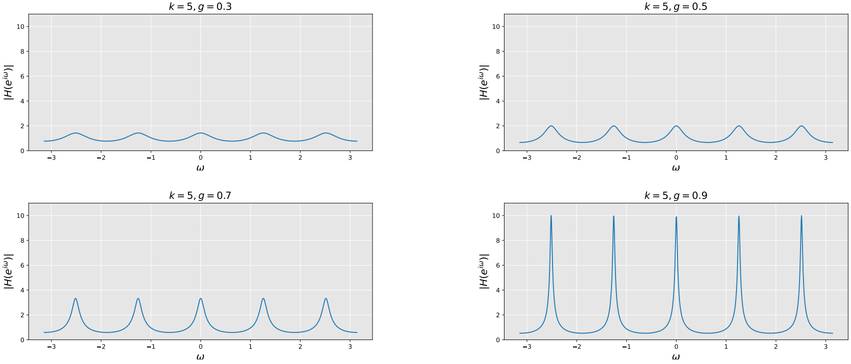
$$y[n] = x[n] + gy[n-k]$$

$$H(z) = \frac{z^k}{z^k - g}$$



$$k = 4,\ g = 0.5$$

FBCF Frequency Response



Exponential Moving Average

Exponential moving average filter is given by:

$$y[n] = (1-g)x[n] + gy[n-1],$$

where $0 < g < 1$.

Note, this is the feedback comb filter with delay 1 and slightly different constants, so it has transfer function:

$$H(z) = \frac{(1-g)z}{z-g}.$$

Frequency Response of Exponential Moving Average