

All-Pass and Minimum Phase Systems

Digital Signal Processing

March 30, 2023



Review: Rational Transfer Functions

A **rational transfer function** looks like

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

Applying the FTOA, we can factor the numerator and denominator:

$$H(z) = \left(\frac{b_0}{a_0} \right) \frac{\prod_{k=0}^M (1 - c_k z^{-1})}{\prod_{k=0}^N (1 - d_k z^{-1})}$$

The c_k are **zeros** of $H(z)$ (zeros of the numerator).

The d_k are **poles** of $H(z)$ (zeros of the denominator).

All-Pass Systems

Definition

An **all-pass system** is an LTI system whose frequency response has magnitude equal to one. In other words, its frequency response, $H(e^{i\omega})$, satisfies:

$$|H(e^{i\omega})| = 1, \quad \text{for all } \omega \in [-\pi, \pi).$$

Simplest All-Pass System

Consider a transfer function $H(z)$ of the form:

$$H(z) = \frac{z^{-1} - c}{1 - \bar{c}z^{-1}}.$$

The magnitude of its frequency response is:

$$\begin{aligned} |H(e^{i\omega})| &= \frac{|e^{-i\omega} - c|}{|1 - \bar{c}e^{-i\omega}|} && \text{plug in } z = e^{i\omega} \\ &= \frac{|e^{-i\omega}| |1 - ce^{i\omega}|}{|1 - \bar{c}e^{-i\omega}|} && \text{pull out } e^{i\omega} \text{ factor} \\ &= \frac{|1 - ce^{i\omega}|}{|1 - \bar{c}e^{-i\omega}|} && |e^{i\omega}| = 1 \\ &= \frac{|1 - \bar{c}e^{-i\omega}|}{|1 - \bar{c}e^{-i\omega}|} = 1 && \text{conjugates} \end{aligned}$$

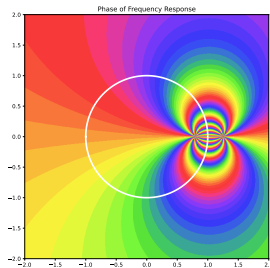
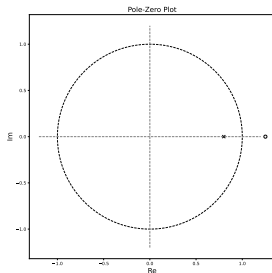
Simplest All-Pass System

Rearranging the simple all-pass system:

$$H(z) = \frac{z^{-1} - c}{1 - \bar{c}z^{-1}} = -c \frac{1 - c^{-1}z^{-1}}{1 - \bar{c}z^{-1}}.$$

We get a zero at c^{-1} and a pole at \bar{c} .

Let $\bar{c} = re^{-i\omega}$. Then $c^{-1} = \frac{1}{r}e^{-i\omega}$. This is the pole-zero inverse pair relationship we saw last time:



Real Coefficient All-Pass System

The transfer function $H(z) = \frac{z^{-1}-c}{1-\bar{c}z^{-1}}$ represents the system:

$$y[n] = -cx[n] + x[n-1] + \bar{c}y[n-1].$$

It has complex coefficients.

If we want a real-valued system, we add a conjugate zero / pole

$$H(z) = \frac{(z^{-1} - c)(z^{-1} - \bar{c})}{(1 - \bar{c}z^{-1})(1 - cz^{-1})} = \frac{z^{-2} - 2\text{Re}(c)z^{-1} + |c|^2}{1 - 2\text{Re}(c)z^{-1} + |c|^2z^{-2}},$$

which now is a system with only real-valued coefficients.

General Real-Valued All-Pass System

The general form of a real-valued all-pass system is

$$H(z) = \prod_{j=1}^J \frac{z^{-1} - d_j}{1 - d_j z^{-1}} \prod_{k=1}^K \frac{(z^{-1} - c_k)(z^{-1} - \bar{c}_k)}{(1 - \bar{c}_k z^{-1})(1 - c_k z^{-1})},$$

where $c_j \in \mathbb{C}$ and $d_j \in \mathbb{R}$.

Inverse Systems

Can we “undo” an LTI system? That is, given an output

$$y[n] = h[n] * x[n],$$

can we get back the input signal $x[n]$?

This means we want an inverse system, $h^{-1}[n]$, such that:

$$h^{-1}[n] * (h[n] * x[n]) = x[n], \quad \text{for all signals } x[n].$$

This implies $h^{-1}[n] * h[n] = \delta[n]$.

Taking the z -transform, we have

$$H^{-1}(z)H(z) = 1 \quad \implies \quad H^{-1}(z) = \frac{1}{H(z)}$$

Inverse of a Rational Transfer Function

Let

$$H(z) = \left(\frac{b_0}{a_0} \right) \frac{\prod_{k=0}^M (1 - c_k z^{-1})}{\prod_{k=0}^N (1 - d_k z^{-1})}$$

Then the inverse just flips the numerator and denominator:

$$H^{-1}(z) = \frac{1}{H(z)} = \left(\frac{a_0}{b_0} \right) \frac{\prod_{k=0}^N (1 - d_k z^{-1})}{\prod_{k=0}^M (1 - c_k z^{-1})}$$

The poles of $H(z)$ become zeros of $H^{-1}(z)$, and zeros become poles.

Exercise: Inverse of FBCF

What is the inverse system for the FBCF?

$$y[n] = x[n] + gy[n - k]$$

Solution

Transfer function for FBCF:

$$H(z) = \frac{1}{1 - gz^{-k}}.$$

Inverse transfer function:

$$H^{-1}(z) = 1 - gz^{-k}.$$

This is the FFCF! (with negative gain)

Minimum-Phase Systems

Definition

A **minimum-phase system** is an LTI system that is stable, causal, and whose inverse is also stable and causal.

Because the poles and zeros flip roles in the inverse, a **minimum-phase system must have all of its poles and zeros inside the unit circle.**

Decomposition of Stable, Causal Systems

Theorem

Let $H(z)$ be the transfer function for a stable, causal LTI system. Then $H(z)$ can be decomposed into a product

$$H(z) = H_{\min}(z)H_{\text{ap}}(z),$$

where $H_{\min}(z)$ is a minimum-phase system, and $H_{\text{ap}}(z)$ is an all-pass system.