

Interpolation and the Sampling Theorem

Digital Signal Processing

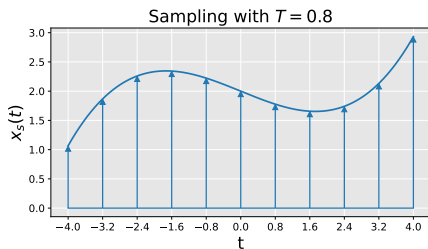
April 2, 2024



Review: Sampling

Given a continuous signal, $x_c(t)$,
define sampled signal, $x_s(t)$, as:

$$\begin{aligned}x_s(t) &= x_c(t)s(t) \\&= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\&= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT) \\&= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)\end{aligned}$$



Review: Fourier Transform of Sampling

Remember, a sampled signal is

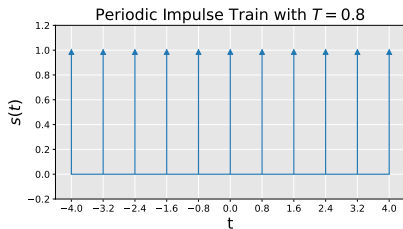
$$x_s(t) = x_c(t)s(t)$$

Taking the Fourier transform of both sides gives:

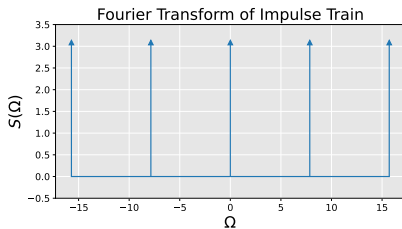
$$X_s(\Omega) = X_c(\Omega) * S(\Omega)$$

So, the Fourier transform of our sampled signal is the convolution of the continuous signal with the Fourier transform of the Dirac comb.

Review: Fourier Transform of Dirac Comb



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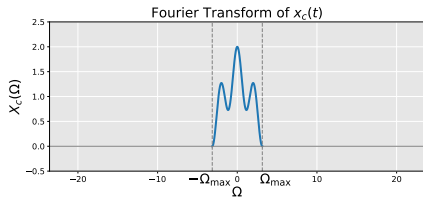


$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

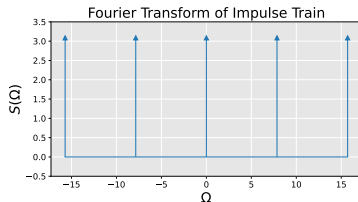
$$S(\Omega) = \frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

Review: Fourier Transform of Sampling

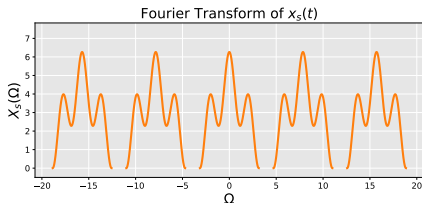
$$X_s(\Omega) = X_c(\Omega) * S(\Omega)$$



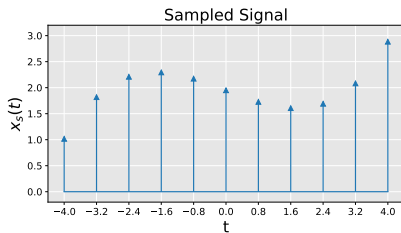
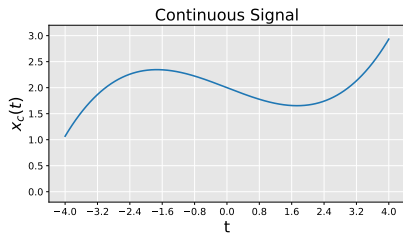
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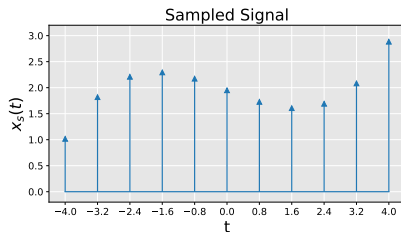
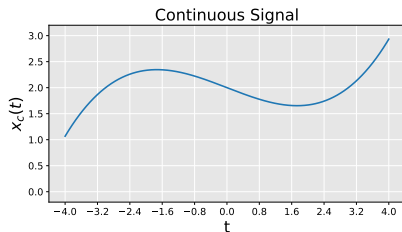
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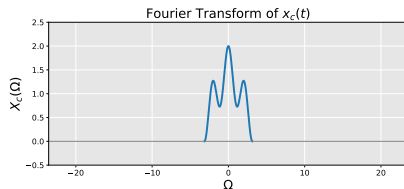
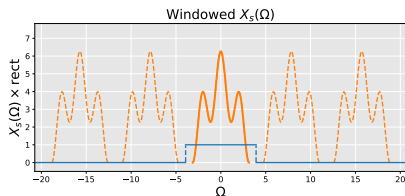
How Do We Reverse the Sampling Process?



How Do We Reverse the Sampling Process?



Windowing in the Fourier Domain



Window the Fourier transform of the sampled signal, $X_s(\Omega)$.

Rectangular Function

The **rectangular function**, or **box window** is defined as

$$\text{rect}(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

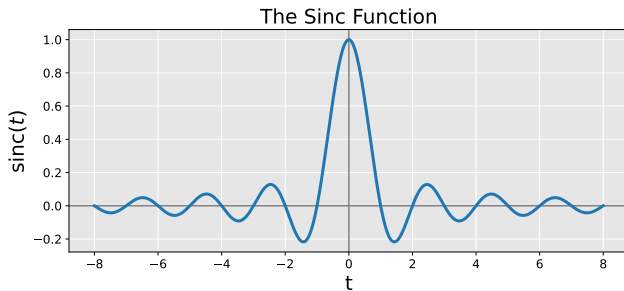
First an Identity

$$\begin{aligned}e^{i\theta} - e^{-i\theta} &= \cos(\theta) + i \sin(\theta) - (\cos(-\theta) + i \sin(-\theta)) \\&= \cos(\theta) + i \sin(\theta) - (\cos(\theta) - i \sin(\theta)) \\&= 2i \sin(\theta)\end{aligned}$$

Inverse Fourier Transform of a Box

$$\begin{aligned}\mathcal{F}^{-1}\{\text{rect}(\xi)\} &= \int_{-\infty}^{\infty} \text{rect}(\xi) e^{i2\pi\xi t} d\xi \\&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i2\pi\xi t} d\xi \\&= \left. \frac{e^{i2\pi\xi t}}{i2\pi t} \right|_{\xi=-\frac{1}{2}}^{\xi=\frac{1}{2}} \\&= \frac{e^{i\pi t}}{i2\pi t} - \frac{e^{-i\pi t}}{i2\pi t} \\&= \frac{2i \sin(\pi t)}{i2\pi t} = \frac{\sin(\pi t)}{\pi t}\end{aligned}$$

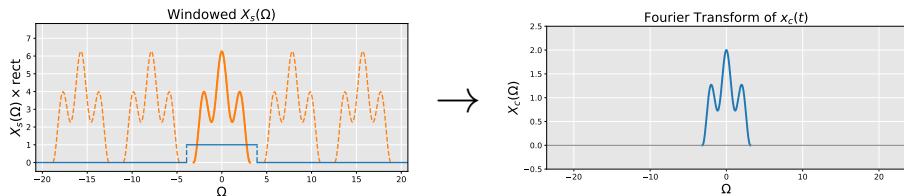
The Sinc Function



$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Note: $\text{sinc}(0) = 1$

Inverse Fourier Transform of a Box



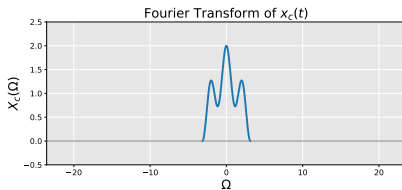
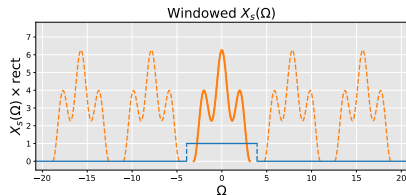
The window we want in the frequency domain is

$$\text{rect}\left(\frac{\Omega}{\Omega_s}\right) = \begin{cases} 1 & \text{if } |\Omega| \leq \frac{\Omega_s}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

The inverse Fourier transform terms of angular frequency:

$$\mathcal{F}^{-1}\left\{\text{rect}\left(\frac{\Omega}{\Omega_s}\right)\right\} = \frac{\sqrt{2\pi}}{T} \text{sinc}\left(\frac{t}{T}\right)$$

Windowing in the Fourier Domain



Windowing the Fourier transform of the sampled signal:

$$X_c(\Omega) = \frac{\sqrt{2\pi}}{\Omega_s} X_s(\Omega) \text{rect} \left(\frac{\Omega}{\Omega_s} \right)$$

The Sampling Theorem

Theorem (Nyquist-Shannon Sampling Theorem)

Let $x_c(t)$ be a continuous signal with Fourier transform $X_c(\Omega)$ that satisfies $X_c(\Omega) = 0$ for $|\Omega| > \Omega_{\max}$. Let $x_s(t)$ be the sampled signal with sampling period T such that

$$\Omega_s > 2\Omega_{\max},$$

where $\Omega_s = \frac{2\pi}{T}$ is the angular sampling frequency. Then $x_c(t)$ is exactly recoverable from $x_s(t)$ as

$$x_c(t) = \sum_{n=-\infty}^{\infty} x_s(nT) \operatorname{sinc} \left(\frac{t - nT}{T} \right).$$