Phase and Group Delay of Frequency Response

Digital Signal Processing

March 25, 2025

Review: Transfer Function

Given an LTI system with impulse response $h[n]$:

$$y[n] = x[n] * h[n].$$

The **transfer function** is the z-transform of $h[n]$. It is given by the ratio:

$$H(z) = \frac{Y(z)}{X(z)},$$

where $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ and $y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$.

Review: Transfer Function of LCCDE

A linear constant-coefficient difference equation (LCCDE) is an LTI system of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

It's transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

Review: Frequency Response

Looking at frequency response of an LTI:

$$Y(e^{i\omega}) = H(e^{i\omega})X(e^{i\omega})$$

Remember complex multiplication in Euler form:

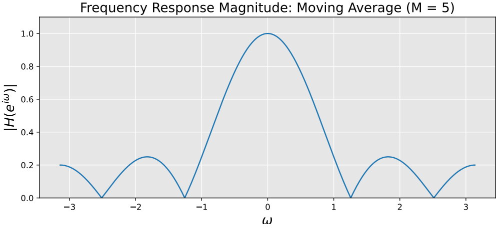
$$re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}$$

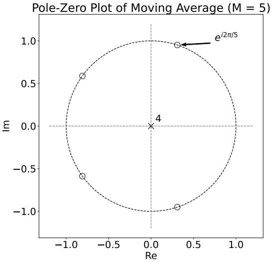
So, we have:

Magnitude: $$|Y(e^{i\omega})| = |H(e^{i\omega})| \cdot |X(e^{i\omega})|$$

Phase: $$\mathrm{Arg}(Y(e^{i\omega})) = \mathrm{Arg}(H(e^{i\omega})) + \mathrm{Arg}(X(e^{i\omega}))$$

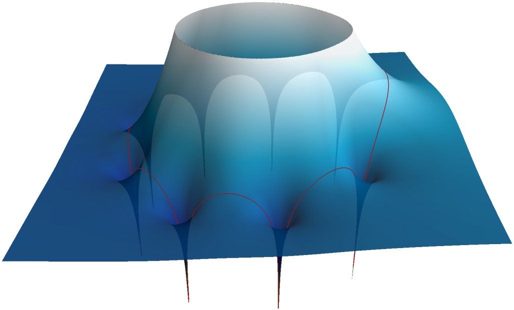
Review: Moving Average (MA)





$$H(z) = \frac{1}{5}\sum_{k=0}^{4} z^{-k} = \frac{\prod_{k=1}^{4}(z - b_k)}{z^4}, \qquad \text{where } b_k = e^{\frac{i2\pi k}{5}}.$$

Moving Average Transfer Function



Magnitude of full z-Transform, $|H(z)|$, as height function.

Frequency response is the red curve.

From https://tttapa.github.io/Pages/Mathematics/Systems-and-Control-Theory/Digital-filters/

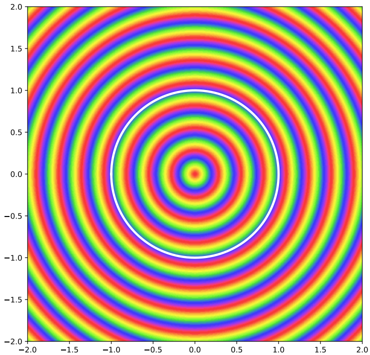
Visualization of Complex Functions

Can't directly plot $H : \mathbb{C} \to \mathbb{C}$ (there are 4 dimensions!)

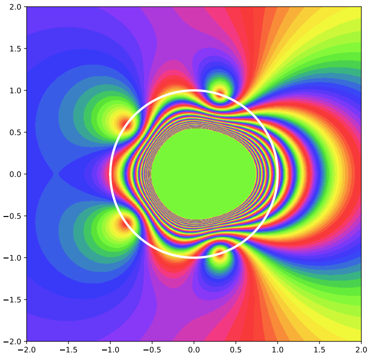
Let $f(z) = (R(z), G(z), B(z))$ be a map that assigns a color (red/green/blue) to every point $z \in \mathbb{C}$.

Then we can display $f(H(z))$ to visualize the complex function $H(z)$.

Another Visualization of Magnitude

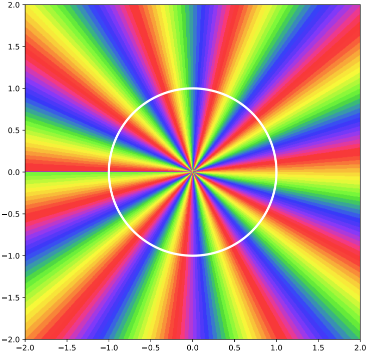


$|z|$

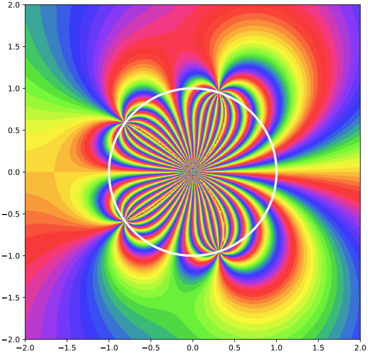


$|H(z)|$ for MA

What About Phase?

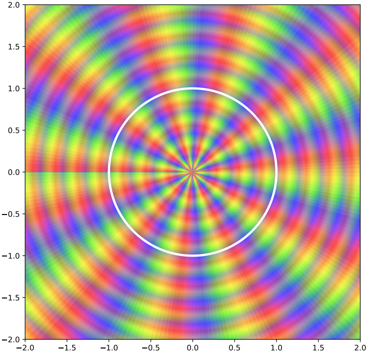


$\mathrm{Arg}(z)$

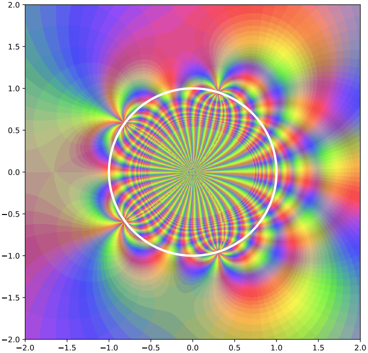


$\mathrm{Arg}(H(z))$ for MA

Visualizing Magnitude and Phase



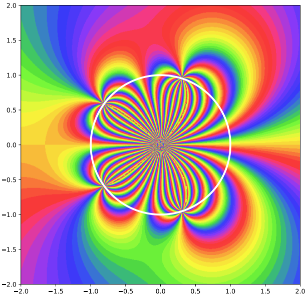
z

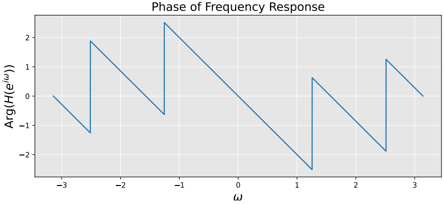


$H(z)$ for MA

Phase Response of MA

Look at phase restricted to unit circle: $\mathrm{Arg}(H(e^{i\omega}))$

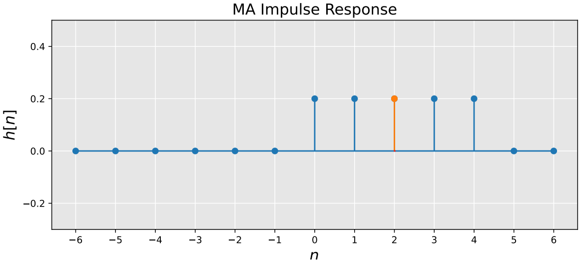




Note discontinuities (jumps by π) when we hit a zero.
Otherwise it is linear with slope = -2.

Phase Response of MA

What is the significance of linear phase with slope = -2?



It tells us the average delay!

Phase Response of Ideal Delay

Ideal Delay:

$$y[n] = x[n-k]$$

Its transfer function is:

$$H(z) = z^{-k}$$

Restricting to unit circle gives the frequency response:

$$H(e^{i\omega}) = e^{-i\omega k}$$

It has phase:

$$\mathrm{Arg}(H(e^{i\omega})) = -\omega k,$$

which is linear in ω, with slope = $-k$.

Phase Delay

This leads us to the following definition:

Definition

The **phase delay** of an LTI system at frequency ω is

$$\tau_{\mathrm{ph}}(\omega) = -\frac{\operatorname{Arg}(H(e^{i\omega}))}{\omega}.$$

Another Way to Get Slope

If $\text{Arg}(H(e^{i\omega})) = -\omega k$, I can also recover the slope, $-k$, as the **derivative** w.r.t. ω. This leads to what is called the group delay.

Definition

The **group delay** of an LTI system at frequency ω is

$$\tau_{\text{gr}}(\omega) = -\frac{d}{d\omega}\text{Arg}(H(e^{i\omega}))$$

Why Two Types of Delays?

For an ideal delay, $y[n] = x[n-k]$, we have seen both phase and group delay are the same:

$$\tau_{\mathrm{ph}}(\omega) = \tau_{\mathrm{gr}}(\omega) = k.$$

More generally, these are the same whenever we have a **linear phase response** system:

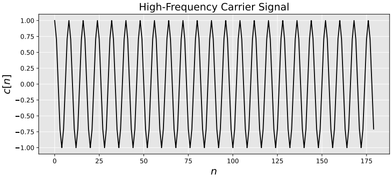
$$\mathrm{Arg}(H(e^{i\omega})) = -\omega k.$$

But if the phase is not linear in ω, these won't be the same:

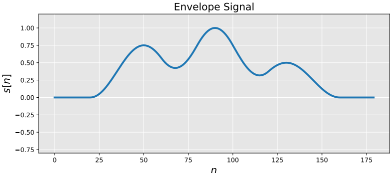
$$\tau_{\mathrm{ph}}(\omega) = -\frac{\mathrm{Arg}(H(e^{i\omega}))}{\omega} \neq -\frac{d}{d\omega}\mathrm{Arg}(H(e^{i\omega})) = \tau_{\mathrm{gr}}(\omega).$$

Sinusoid Signal with Envelope

Consider a high-frequency sinusoid: $c[n] = \cos(\omega_0 n)$, which we'll call the **high-frequency carrier**:



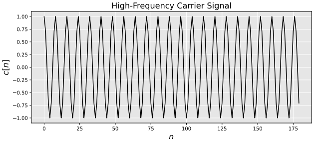
And some lower-frequency **envelope**, $s[n]$:



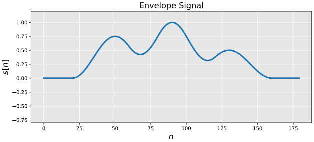
Sinusoid Signal with Envelope

Now multiply them together to get the signal:

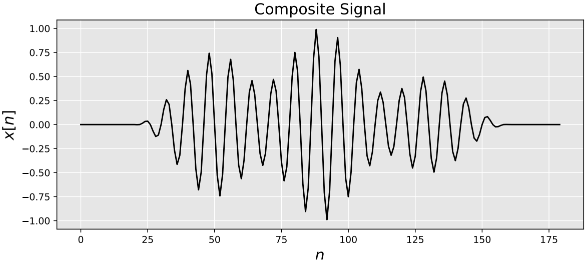
$$x[n] = s[n]\cos(\omega_0 n).$$



$\times$



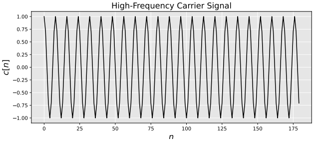
$=$



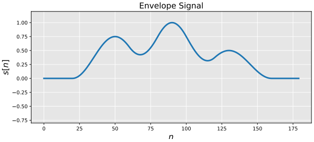
Sinusoid Signal with Envelope

Now multiply them together to get the signal:

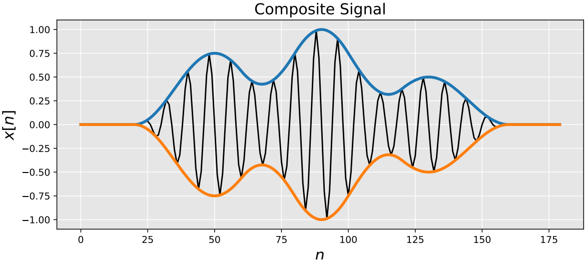
$$x[n] = s[n]\cos(\omega_0 n).$$



$\times$



$=$

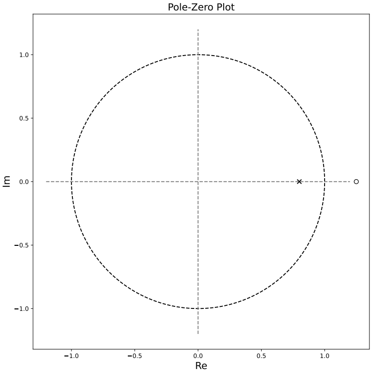


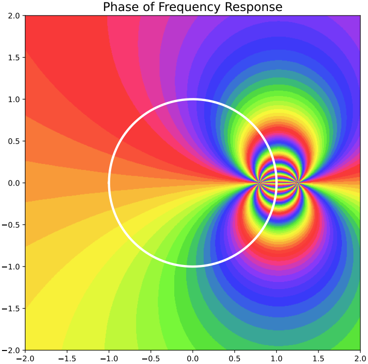
Phase and Group Delay

Now when applying a general LTI system, $H(z)$, the phase delay will act on the high-frequency carrier, while the group delay will act on the envelope:

$$y[n] \approx |H(e^{i\omega_0})| s[n - \tau_{\mathrm{gr}}(\omega_0)] \cos(\omega_0 n + \tau_{\mathrm{ph}}(\omega_0)).$$

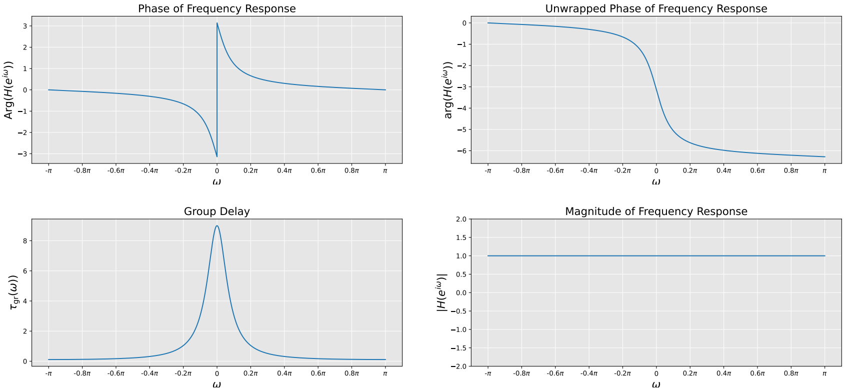
Pole-Zero Inverse Pairs





$$H(z) = \left(\frac{4}{5}\right) \frac{1 - \frac{5}{4} z^{-1}}{1 - \frac{4}{5} z^{-1}}$$

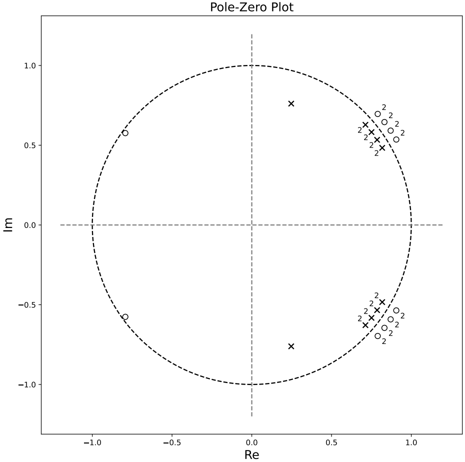
Pole-Zero Inverse Pairs



Pole-zero inverse pairs cause a narrowband group delay, while leaving the magnitude fixed.

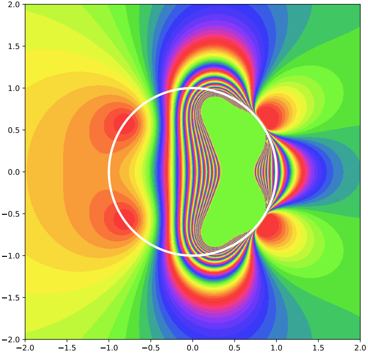
Example[1]

Consider an LTI system with the following pole-zero plot:

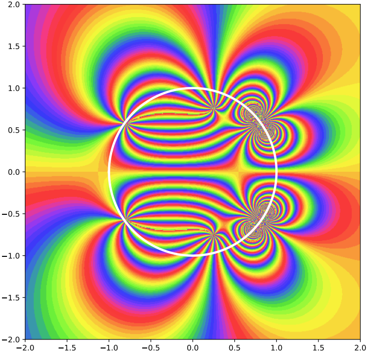


[1] From Oppenheim & Schafer, pg. 278

Example: Transfer Function

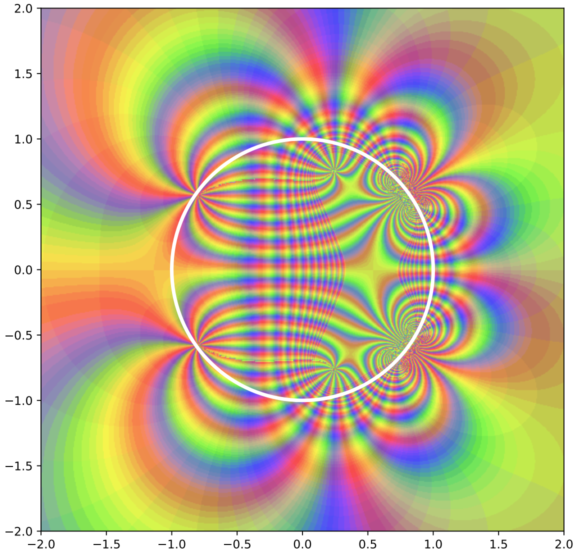


$$|H(z)|$$



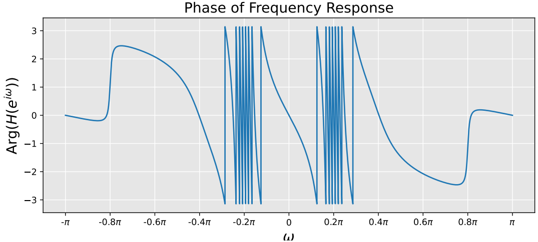
$$\mathrm{Arg}(H(z))$$

Example: Transfer Function

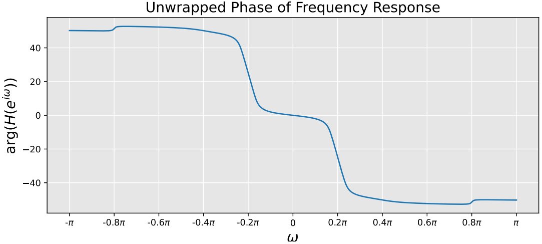


Example: Phase of Frequency Response

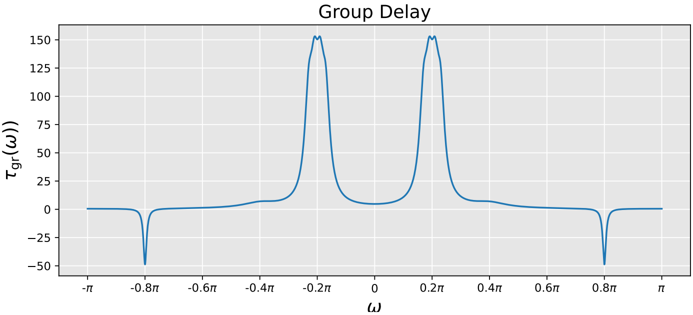
Wrapped:



Unwrapped:

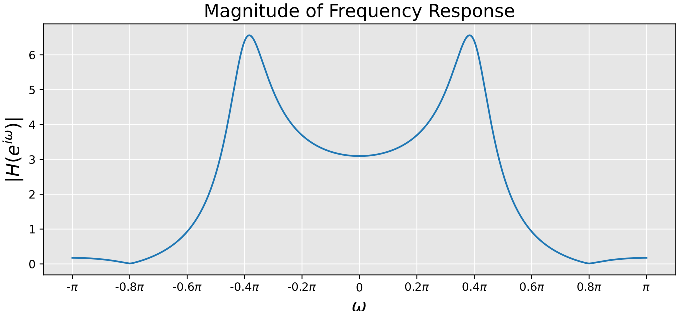


Example: Group Delay



$$\tau_{\mathrm{gr}}(\omega) = -\frac{d}{d\omega}\mathrm{Arg}(H(e^{i\omega}))$$

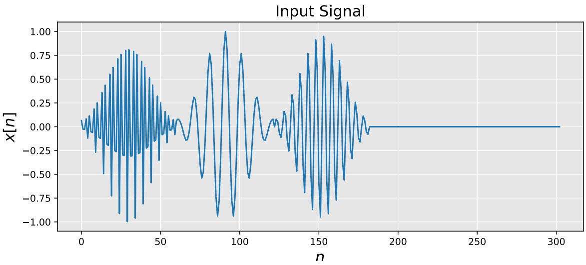
Example: Magnitude of Frequency Response



Example: Input and Output

Input signal has three frequencies: $0.8\pi, 0.2\pi, 0.4\pi$

Input:



Output: