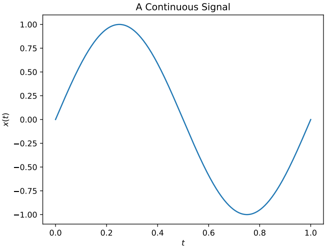
Sampling Continous-Time Signals

Digital Signal Processing

April 8, 2025

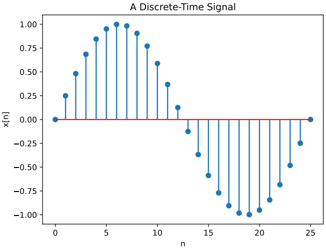
Sampled Continuous Signals

Discrete-time signals often come from continuous signals:



$$x_c(t)$$

$$\rightarrow$$

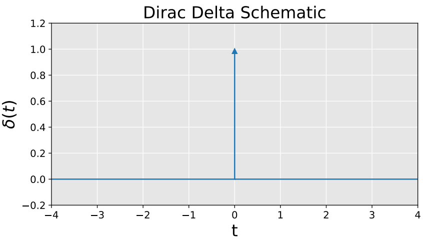


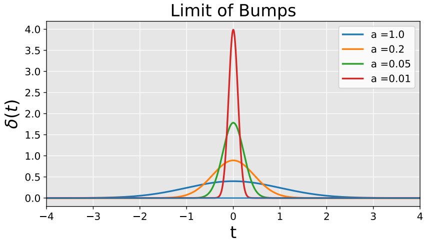
$$x[n] = x_c(nT)$$

Here, $T \in \mathbb{R}$ is the **sampling period**, $\quad T = (1/25)\mathrm{s} = 0.04\mathrm{s}$

and $\frac{1}{T}$ is the **sampling frequency**. $\quad \frac{1}{T} = 25\mathrm{Hz}$

Dirac Delta





- Denoted $\delta(t)$
- Continuous analog to the discrete unit sample function, $\delta[n]$
- Unlike the discrete case, it is **not a function**
- Can be thought of as a limit of “bump” functions:

$$\delta(t) = \lim_{a \to 0} \frac{1}{\sqrt{2\pi a}} \exp\left(-\frac{t^2}{2a}\right)$$

Dirac Delta and Integration

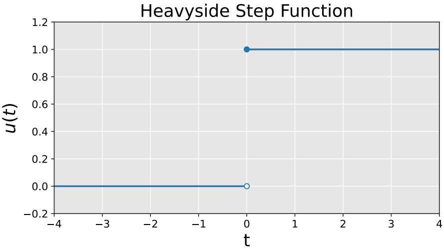
Dirac delta is a *generalized function* (a thing you can integrate):

$$\int_a^b \delta(t)dt = \begin{cases} 1 & \text{if } 0 \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

When we multiply it by a function $f : \mathbb{R} \to \mathbb{R}$, we get:

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0).$$

Integral of the Dirac Delta



Integral of $\delta(t)$ is the continuous unit step function, a.k.a. the Heavyside step function:

$$u(t) = \int_{-\infty}^{t} \delta(s)ds = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

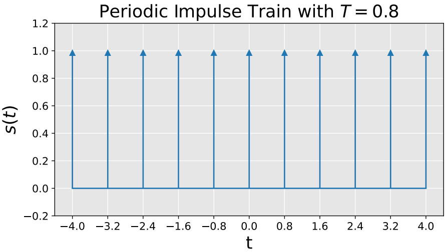
Shifting the Dirac Delta

Shifting a dirac delta evaluates functions at a different time point:

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0).$$

Mathematical idealization of taking a measurement of some continuous process (f) at a particular time (t_0).

Impulse Train



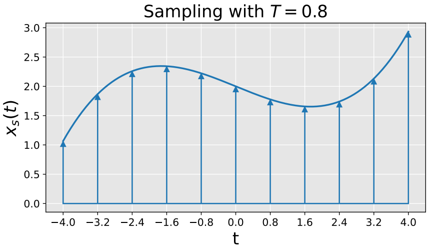
A **periodic impulse train**, a.k.a. a **Dirac comb**, is a sum of dirac deltas, shifted by a sampling period T:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

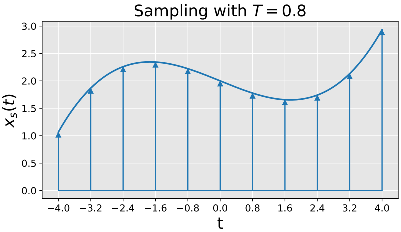
Sampling: First Step

Given a continous signal, $x_c(t)$,
define sampled signal, $x_s(t)$, as:

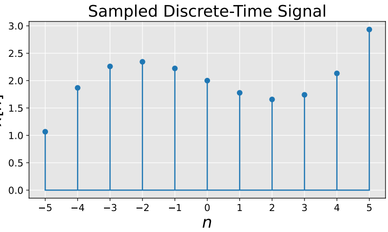
$$
\begin{aligned}
x_s(t) &= x_c(t)s(t) \\
&= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)
\end{aligned}
$$



Sampling: Final Step



$\longrightarrow$



Discrete signal $x[n]$ keeps the sampled values $x_s(nT)$.

Frequency Analysis of Sampling

Continuous-Time Fourier Transform

Fourier transform of a continuous function, $f(t)$:

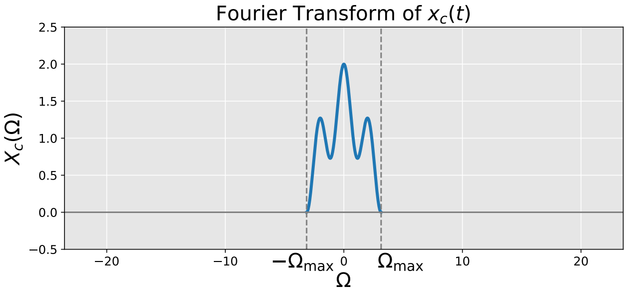
$$F(\Omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\Omega t} f(t) dt$$

Inverse Fourier transform:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\Omega t} F(\Omega) d\Omega$$

Note: Ω is angular frequency, in radians per second.

Bandlimited Continuous Signal



Definition

A continuous signal $x_c(t)$ is **bandlimited** if it has a maximum frequency content Ω_{max}, i.e.,

$$X_c(\Omega) = 0 \qquad \text{for } |\Omega| > \Omega_{\mathrm{max}}.$$

Fourier Transform of a Sampled Signal

Remember, a sampled signal is

$$x_s(t) = x_c(t)s(t)$$

Taking the Fourier transform of both sides gives:

$$X_s(\Omega) = X_c(\Omega) * S(\Omega)$$

So, the Fourier transform of our sampled signal is the convolution of the continuous signal with the Fourier transform of the Dirac comb.

Fourier Transform of a Dirac Comb

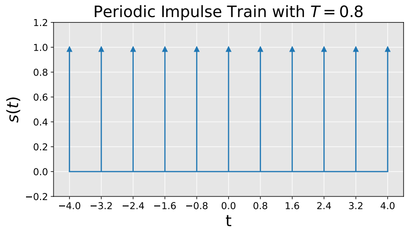
The Fourier transform of a Dirac comb is another Dirac comb:

Time-Domain Comb: $$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT),$$

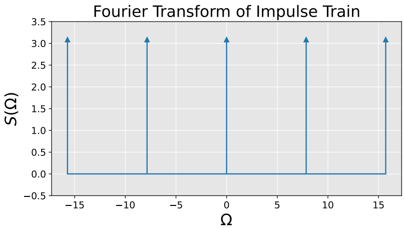
Fourier transform: $$S(\Omega) = \frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s),$$

where $\Omega_s = \frac{2\pi}{T}$ is the angular sampling frequency.

Fourier Transform of a Dirac Comb



$\overset{\mathcal{F}}{\longleftrightarrow}$

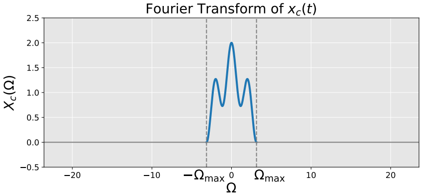


$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$S(\Omega) = \frac{\sqrt{2\pi}}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - k\frac{2\pi}{T}\right)$$

Fourier Transform of a Sampled Signal

$$X_s(\Omega) = X_c(\Omega) * S(\Omega)$$



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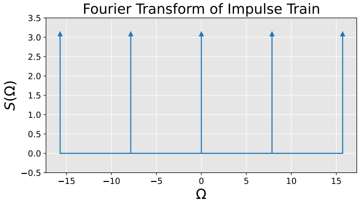

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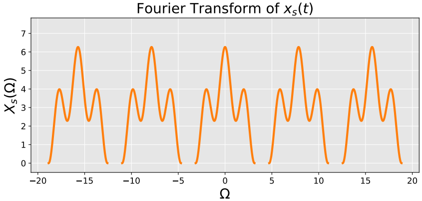


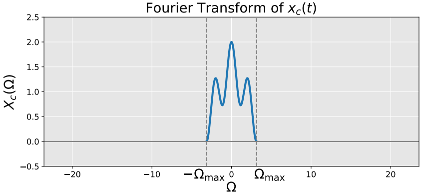
What if We Decrease the Sampling Rate?

Increasing the sampling period from $T = 0.8$ to $T = 1.25$

Decreases the angular sampling rate from $\Omega_s = \frac{2\pi}{0.8} \approx 7.85$ to $\Omega_s = \frac{2\pi}{1.25} \approx 5.03$

What if We Decrease the Sampling Rate?

Increasing the sampling period from $T = 0.8$ to $T = 1.25$:



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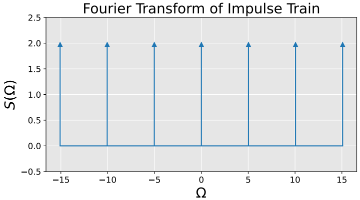

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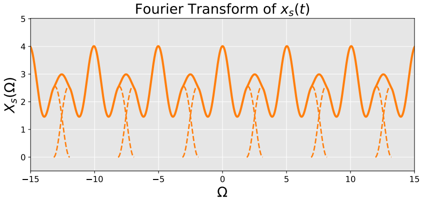
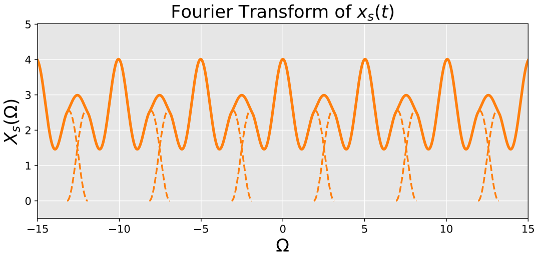


Aliasing and the Nyquist Rate



This is **aliasing**: overlapping frequencies are indistinguishable.

To avoid aliasing (no overlap in frequencies), we need to sample at or above the **Nyquist rate**, which is twice the bandwidth of our signal = $2\Omega_{\max}$.

In this example, the bandlimit is $\Omega_{\max} = \pi$, and the sampling frequency is $\Omega_s = \frac{8}{5}\pi$. Notice, $\Omega_s < 2\Omega_{\max}$.