The DFT and Convolution

Digital Signal Processing

September 18, 2025

Circular Convolution

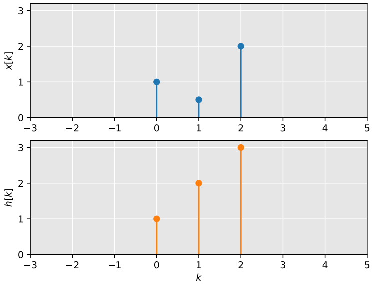
Given two periodic signals, $x[n], h[n]$, both with period L, their **circular convolution** is

$$x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$$

The only thing we've changed is to now "wrap" the index on h.

Circular Convolution Example

Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$

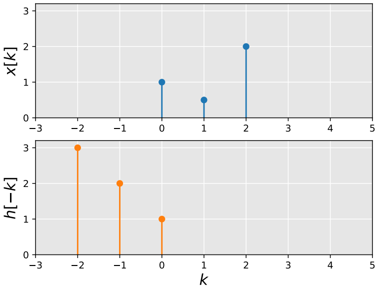


$$x[n] = (1.0, 0.5, 2.0)$$

$$h[n] = (1.0, 2.0, 3.0)$$

Circular Convolution Example

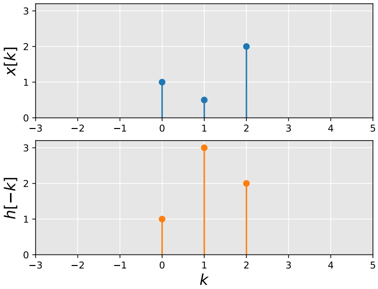
Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$



Flip h about 0

Circular Convolution Example

Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$

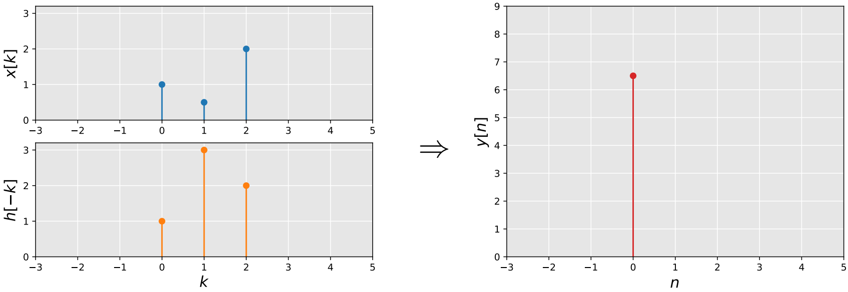


Wrap h

Circular Convolution Example

Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$

For $n = 0$, flip h about 0 to get $h[-k \bmod L]$.

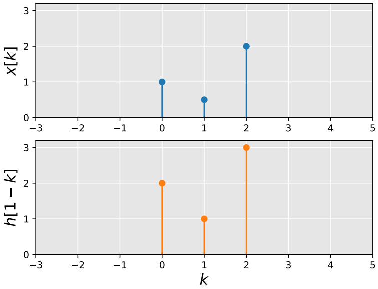


$$\begin{aligned} y[0] &= x[0] \times h[0] + x[1] \times h[2] + x[2] \times h[1] \\ &= 1.0 \times 1.0 + 0.5 \times 3.0 + 2.0 \times 2.0 = 6.5 \end{aligned}$$

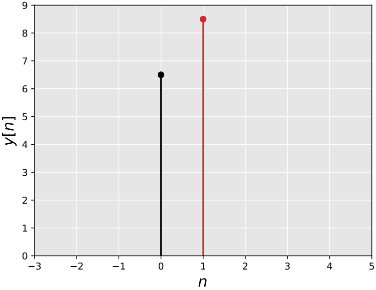
Circular Convolution Example

Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$

For $n = 1$, shift h left by one to get $h[(1-k) \bmod L]$.



$\Rightarrow$

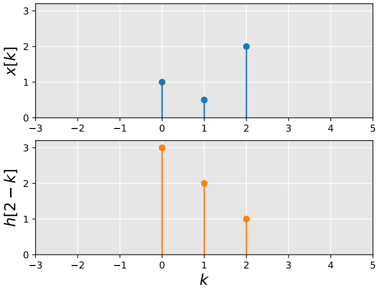


$$
\begin{aligned}
y[1] &= x[0]h[1] + x[1]h[0] + x[2]h[2] \\
&= 1.0 \times 2.0 + 0.5 \times 1.0 + 2.0 \times 3.0 = 8.5
\end{aligned}
$$

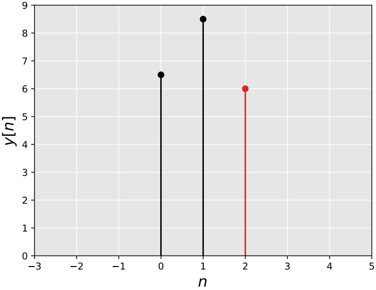
Circular Convolution Example

Computing $y[n] = x[n] * h[n] = \sum_{k=0}^{L-1} x[k]h[(n-k) \bmod L]$

For $n = 2$, shift h left again to get $h[(2-k) \bmod L]$.



$\Rightarrow$



$$
\begin{aligned}
y[2] &= x[0]h[2] + x[1]h[1] + x[2]h[0] \\
&= 1.0 \times 3.0 + 0.5 \times 2.0 + 2.0 \times 1.0 = 6.0
\end{aligned}
$$

Convolution and DFT

Theorem (Convolution Theorem)

Given two periodic, complex-valued signals, $x[n], y[n]$*,*

$$\mathcal{DFT}\{x[n] * y[n]\} = \sqrt{L}\left(\mathcal{DFT}\{x[n]\} \times \mathcal{DFT}\{y[n]\}\right).$$

In other words, **convolution** in the time domain becomes **multiplication** in the frequency domain.

Using notation $x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]$ and $y[n] \overset{\mathcal{DFT}}{\longleftrightarrow} Y[k]$, this is:

$$x[n] * y[n] \overset{\mathcal{DFT}}{\longleftrightarrow} \sqrt{L}X[k]Y[k]$$

Proof of Convolution Theorem

$$\mathcal{DFT}\{x[n] * y[n]\}$$

$$= \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} (x[n] * y[n])\, e^{-i\omega_0 nk} \qquad \text{Definition of } \mathcal{DFT}$$

$$= \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} \left(\sum_{m=0}^{L-1} x[m]y[n-m] \right) e^{-i\omega_0 nk} \qquad \text{Definition of } *$$

$$= \frac{1}{\sqrt{L}} \sum_{m=0}^{L-1} \sum_{n=0}^{L-1} x[m]y[n-m] \left(e^{-i\omega_0 mk} e^{-i\omega_0 (n-m)k} \right) \qquad \text{Exponential math}$$

$$= \sum_{m=0}^{L-1} x[m]e^{-i\omega_0 mk} \left(\frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} y[n-m]e^{-i\omega_0 (n-m)k} \right) \qquad \text{Rearrange}$$

$$= \left(\sum_{m=0}^{L-1} x[m]e^{-i\omega_0 km} \right) Y[k] \qquad \text{Definition of } \mathcal{DFT}$$

$$= \sqrt{L} X[k]Y[k] \qquad \text{Definition of } \mathcal{DFT}$$

Convolution and DFT

Theorem (Convolution Theorem II)

Given two periodic, complex-valued signals, $x[n], y[n]$,

$$\mathcal{DFT}\{x[n] \times y[n]\} = \frac{1}{\sqrt{L}} \left(\mathcal{DFT}\{x[n]\} * \mathcal{DFT}\{y[n]\}\right).$$

In other words, the **multiplication** in the time domain becomes **convolution** in the frequency domain.

Again, using notation $x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]$ and $y[n] \overset{\mathcal{DFT}}{\longleftrightarrow} Y[k]$, this is:

$$\sqrt{L}x[n]y[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k] * Y[k]$$

Proof of Convolution Theorem II

$$\mathcal{DFT}^{-1}\{X[k] * Y[k]\}$$

$$= \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} \left(X[k] * Y[k]\right) e^{i\omega_0 nk} \qquad \text{Definition of } \mathcal{DFT}^{-1}$$

$$= \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} \left(\sum_{m=0}^{L-1} X[m]Y[k-m]\right) e^{i\omega_0 nk} \qquad \text{Definition of } *$$

$$= \frac{1}{\sqrt{L}} \sum_{m=0}^{L-1} \sum_{n=0}^{L-1} X[m]Y[k-m] \left(e^{i\omega_0 nm} e^{i\omega_0 n(k-m)}\right) \qquad \text{Exponential math}$$

$$= \sum_{m=0}^{L-1} X[m]e^{i\omega_0 nm} \left(\frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} Y[k-m]e^{i\omega_0 n(k-m)}\right) \qquad \text{Rearrange}$$

$$= \sum_{m=0}^{L-1} x[m]e^{-i\omega_0 km} y[n] \qquad \text{Definition of } \mathcal{DFT}^{-1}$$

$$= \sqrt{L}x[n]y[n] \qquad \text{Definition of } \mathcal{DFT}^{-1}$$