

Frequency Response of LTI Systems

Digital Signal Processing

March 23, 2023



Review: Transfer Function

Given an LTI system with impulse response $h[n]$:

$$y[n] = x[n] * h[n].$$

The **transfer function** is the z -transform of $h[n]$. It is given by the ratio:

$$H(z) = \frac{Y(z)}{X(z)},$$

where $x[n] \xleftrightarrow{Z} X(z)$ and $y[n] \xleftrightarrow{Z} Y(z)$.

Review: Transfer Function of LCCDE

A linear constant-coefficient difference equation (LCCDE) is an LTI system of the form:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k].$$

It's transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}.$$

Exercise: Moving Average

What is the transfer function for the following moving average system?

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n - k].$$

Answer: Moving Average

Moving Average:

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k].$$

Taking z -transform of both sides:

$$Y(z) = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} X(z).$$

Solving for the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M} \sum_{k=0}^{M-1} z^{-k}.$$

Pole-Zero Plot for Moving Average

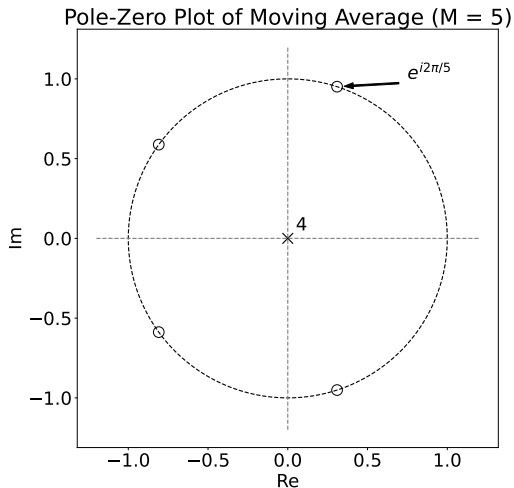
Rearrange the transfer function:

$$\begin{aligned} H(z) &= \frac{1}{M} \sum_{k=0}^{M-1} z^{-k} \\ &= \frac{1}{M} \frac{\sum_{k=0}^{M-1} z^k}{z^{M-1}} \end{aligned} \quad \text{multiply by } \frac{z^{M-1}}{z^{M-1}}$$

Trick: $(1 + z + z^2 + \cdots + z^{M-1})(z - 1) = z^M - 1$.

So, $\sum_{k=0}^{M-1} z^k = \frac{z^M - 1}{(z - 1)}$, which means the zeros are the M th roots of 1, excluding $z = 1$.

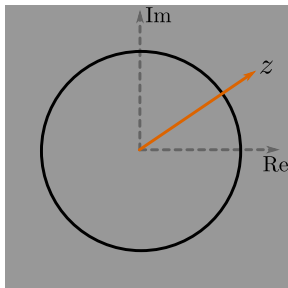
Pole-Zero Plot for Moving Average



Frequency Response of an LTI System

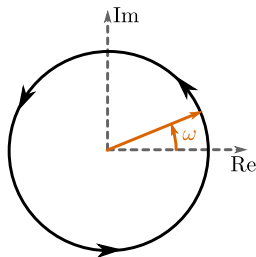
The **frequency response** of an LTI system is the restriction of $H(z)$ to the unit circle, which is the DTFT of the impulse response, $H(e^{i\omega})$.

z -Transform



$$H(z)$$

DTFT



$$H(e^{i\omega})$$

LTI Applied to Complex Sinusoids

Consider a complex sinusoid $e^{i\omega n}$. Applying $T\{\cdot\}$ to it gives:

$$\begin{aligned}T\{e^{i\omega n}\} &= e^{i\omega n} * h[n] \\&= \sum_{k=-\infty}^{\infty} e^{i\omega(n-k)} h[k] \\&= \sum_{k=-\infty}^{\infty} e^{i\omega n} e^{-i\omega k} h[k] \\&= e^{i\omega n} \sum_{k=-\infty}^{\infty} e^{-i\omega k} h[k] && \text{DTFT of } h[k]! \\&= H(e^{i\omega}) e^{i\omega n}\end{aligned}$$

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LTI Applied to Complex Sinusoids

Consider a complex sinusoid $e^{i\omega n}$. Applying $T\{\cdot\}$ to it gives:

$$T\{e^{i\omega n}\} = e^{i\omega n} * h[n]$$

$$= \sum_{k=-\infty}^{\infty} e^{i\omega(n-k)} h[k]$$

$$= \sum_{k=-\infty}^{\infty} e^{i\omega n} e^{-i\omega k} h[k]$$

$$= e^{i\omega n} \sum_{k=-\infty}^{\infty} e^{-i\omega k} h[k]$$

DTFT of $h[k]$!

$$= H(e^{i\omega}) e^{i\omega n}$$

Frequency Response

Let $T\{\cdot\}$ be an LTI system. Recall it's **impulse response** is

$$h[n] = T\{\delta[n]\}.$$

It's **frequency response** is how it responds to a complex sinusoid with a certain frequency $\omega \in [0, 2\pi)$:

$$T\{e^{i\omega n}\} = H(e^{i\omega})e^{i\omega n}.$$

Magnitude and Phase of Frequency Response

Looking at frequency response of an LTI:

$$Y(e^{i\omega}) = H(e^{i\omega})X(e^{i\omega})$$

Remember complex multiplication in Euler form:

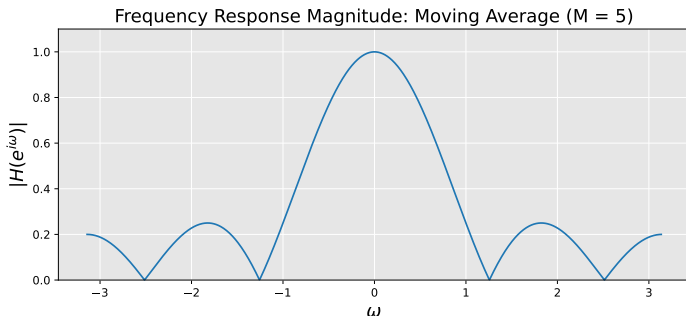
$$re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}$$

So, we have:

Magnitude: $|Y(e^{i\omega})| = |H(e^{i\omega})| \cdot |X(e^{i\omega})|$

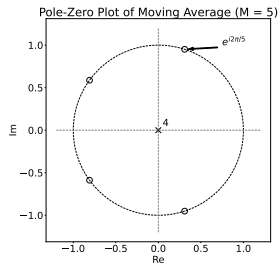
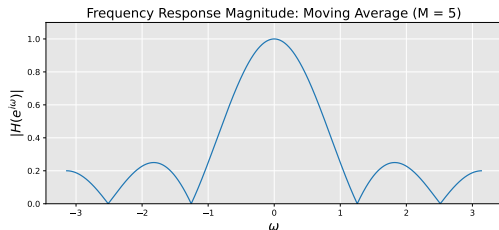
Phase: $\text{Arg}(Y(e^{i\omega})) = \text{Arg}(H(e^{i\omega})) + \text{Arg}(X(e^{i\omega}))$

Frequency Response Magnitude of the Moving Average



This is a **low-pass** filter.

Frequency Response Magnitude of the Moving Average



Note the zeros.

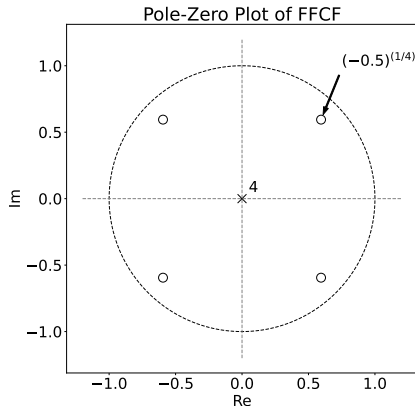
Feedforward Comb Filter (FFCF)

FFCF:

$$y[n] = x[n] + gx[n - k]$$

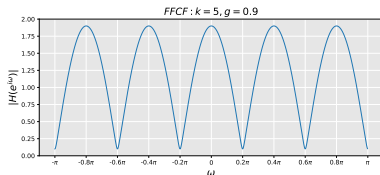
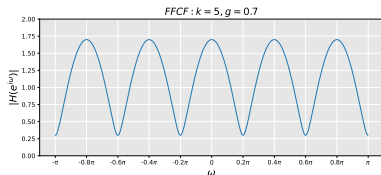
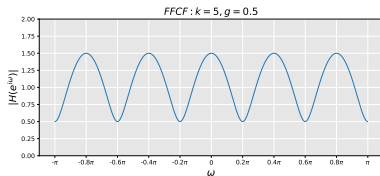
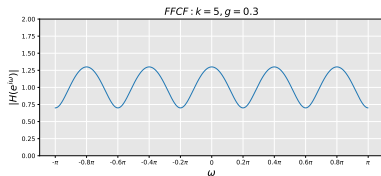
Transfer function:

$$H(z) = \frac{z^k + g}{z^k}$$



$$k = 4, \quad g = 0.5$$

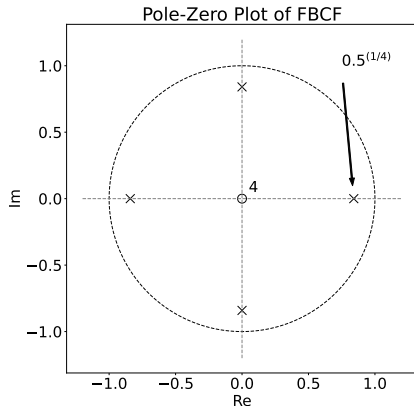
FFCF Frequency Response



Feedback Comb Filter (FBCF)

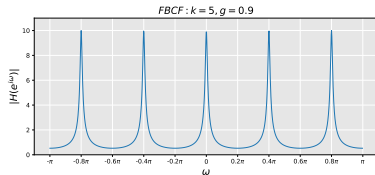
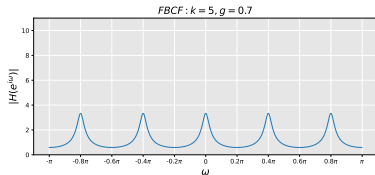
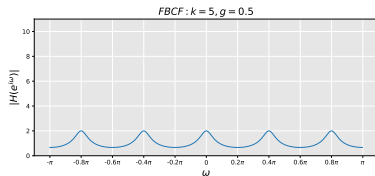
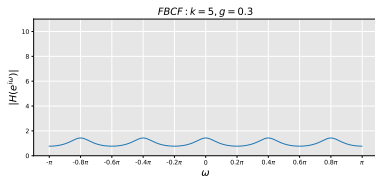
$$y[n] = x[n] + gy[n - k]$$

$$H(z) = \frac{z^k}{z^k - g}$$



$$k = 4, \quad g = 0.5$$

FBCF Frequency Response



Exponential Moving Average

Exponential moving average filter is given by:

$$y[n] = (1 - g)x[n] + gy[n - 1],$$

where $0 < g < 1$.

Note, this is the feedback comb filter with delay 1 and slightly different constants, so it has transfer function:

$$H(z) = \frac{(1 - g)z}{z - g}.$$

Frequency Response of Exponential Moving Average

