ECE 4750/6750 + CS 4501/6501: Digital Signal Processing **Due Wed 4/5**

Homework 3: The z-Transform

Instructions: Submit a single Jupyter notebook (.ipynb) of your work to Collab by 11:59pm on the due date. All code should be written in Python. **Be sure to show all the work involved in deriving your answers! If you just give a final answer without explanation, you may not receive credit for that question.**

You may discuss the concepts with your classmates, but write up the answers entirely on your own. Do not look at another student's answers, do not use answers from the internet or other sources, and do not show your answers to anyone. **Cite any sources you used outside of the class material (webpages, etc.), and list any fellow students with whom you discussed the homework concepts.**

1. In class, we have seen the right-sided exponential and the left-sided exponential LTI systems. Their impulse response functions are given by:

$$\begin{aligned} \text{Right-sided:} \quad & h_R[n] = a^n u[n], \\ \text{Left-sided:} \quad & h_L[n] = -b^n u[-n-1], \end{aligned}$$

   where $a, b \in \mathbb{C}$ are constants, and $u[n]$ is the unit step function.

   Now consider an LTI system that is defined by the impulse response:

$$h[n] = h_R[n] + h_L[n].$$

   (a) What is the transfer function, $H(z)$, for this system?

   (b) What are the poles and zeros of this system?

   (c) What is the region of convergence (ROC) for this system? **Hint:** This will be a region defined in terms of the magnitudes, $|a|$, $|b|$.

   (d) Is this a causal system? Why or why not?

   (e) For constant values $a = 0.5$ and $b = 1.5$, plot the magnitude of the frequency response. What can you say about how this system would effect the frequencies of an input signal?

2. Consider the following transfer function:

$$H(z) = \frac{(z-1)(z+1)}{(z-0.5)(z+0.5)}.$$

   (a) What are the poles and zeros of this system?

   (b) Write down an equation for this system as a linear constant-coefficient difference equation (LCCDE). In other words, write an equation as a sum of time-shifted and scaled copies of the input $x[n]$ and the ouput $y[n]$.

   (c) Again, plot the magnitude of the frequency response for this system. What can you say about how this system would effect the frequencies of an input signal?

(d) Write a Python function to implement your system equation in part (b). Test it on real-valued sinusoid signals, $x[n] = \sin(\omega n)$, for angular frequencies of $\omega = \frac{\pi}{8}$, $\omega = \frac{\pi}{4}$, and $\omega = \frac{\pi}{3}$. Verify that you get sinusoids as output. Using the frequency response function, $H(e^{i\omega})$, what is the expected change in magnitude (amplitude) and phase of these sinusoids? Again, verify that your system produced these amplitude and phase changes.

3. Let's use LTI systems to predict the weather. Take the exponential averaging system that we discussed in class:

$$y[n] = (1 - g)x[n] + gy[n - 1].$$

Given a time series $x[n]$ for $n = 0, \ldots, L - 1$, we can use this as a prediction model for the next timepoint $x[n + 1]$ (that our model has not seen yet) by taking the last output, $y[n]$, as our prediction for $x[n + 1]$. Implement this exponential averaging system and test it on the provided data of average daily temperatures in Charlottesville this year (`cville-temps.csv`). Do the following:

(a) Make predictions for $x[n]$ (using $y[n-1]$) for $n = 1, \ldots, L-1$. Plot these predictions over a plot of the original data. Do this three times, with gain parameters $g = 0.25, 0.5, 0.75$. What difference do you see with the three different parameters?

(b) Compute the mean absolute error (MAE) of the three different models ($g = 0.25, 0.5, 0.75$). The MAE is

$$MAE = \frac{1}{L - 1} \sum_{i=1}^{L-1} |x[n] - y[n - 1]|.$$

Which choice of g gave the best prediction (lowest MAE)?

4. **(Required for graduate students only, extra credit for undergraduates)** Repeat the previous prediction experiment, but using a discrete Taylor series to approximate the time series:

$$y[n] = x[n] + D\{x[n]\} + \frac{1}{2}D\{D\{x[n]\}\},$$

where $D\{x[n]\} = x[n] - x[n - 1]$ is the backward difference. (Note there is no parameter g here, so this is just one model.) Begin by expanding the above formula until it is in the form of an LCCDE. Also, plot the frequency response function, $H(e^{i\omega})$, for this system. Remember the exponential moving average was a low-pass filter, does this system also look like a low-pass filter?