Linear Time-Invariant (LTI) Systems

Digital Signal Processing

January 31, 2023

Linear Systems

Definition

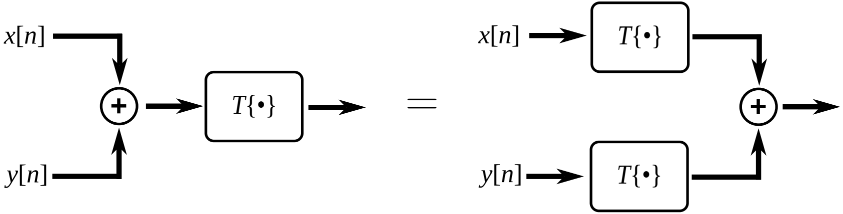
A **linear system** is a system T that satisfies:

1. Additivity: $T\{x[n] + y[n]\} = T\{x[n]\} + T\{y[n]\}$,
2. Scaling: $T\{ax[n]\} = aT\{x[n]\}$,

for all signals $x[n], y[n]$, and all scalar constants, a.

Linearity Property in Diagrams

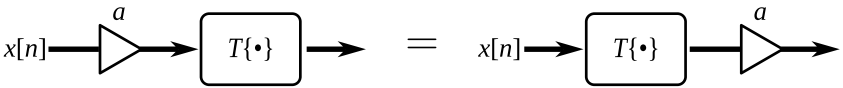
Additivity:



$$T\{x[n] + y[n]\} = T\{x[n]\} + T\{y[n]\}$$

Linearity Property in Diagrams

Scaling:



$$T\{ax[n]\} = aT\{x[n]\}$$

Linear Systems (again)

An *equivalent* definition of linearity combines additivity and scaling into one rule:

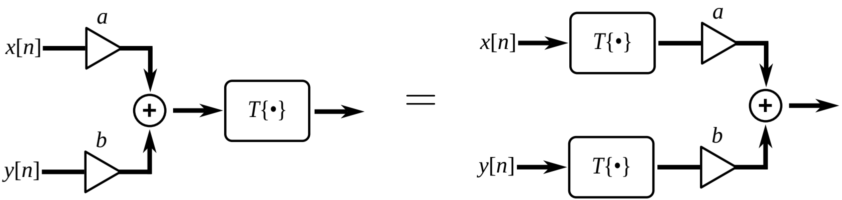
Definition

A **linear system** is a system T that satisfies:

$$T\{ax[n] + by[n]\} = aT\{x[n]\} + bT\{y[n]\},$$

for all signals $x[n], y[n]$, and all scalar constants, a, b.

Linearity Property in Diagrams (again)



$$T\{ax[n] + by[n]\} = aT\{x[n]\} + bT\{y[n]\}$$

Examples

Are the following linear systems or non-linear systems?

- $T\{x[n]\} = 2x[n]$ Linear
- $T\{x[n]\} = x[n-1]$ Linear
- $T\{x[n]\} = x[n]^2$ Non-linear
- $T\{x[n]\} = nx[n]$ Linear
- $T\{x[n]\} = x[2n]$ Linear
- $T\{x[n]\} = x[n] + 1$ Non-linear

Time-Invariant Systems

Definition

A system, T, is called **time-invariant**, or **shift-invariant**, if it satisfies

$$y[n] = T\{x[n]\} \Rightarrow y[n-N] = T\{x[n-N]\},$$

for all signals $x[n]$ and all shifts $N \in \mathbb{Z}$.

Examples

Are the following time-invariant or time-variant systems?

• $T\{x[n]\} = 2x[n]$	Time-invariant
• $T\{x[n]\} = x[n-1]$	Time-invariant
• $T\{x[n]\} = x[n]^2$	Time-invariant
• $T\{x[n]\} = nx[n]$	Time-variant
• $T\{x[n]\} = x[2n]$	Time-variant
• $T\{x[n]\} = x[n] + 1$	Time-invariant

Linear Time-Invariant (LTI) Systems

Definition

A **linear time-invariant (LTI) system** is one that is both linear and time-invariant.

Examples

Are the following LTI or not LTI systems?

- $T\{x[n]\} = 2x[n]$ LTI
- $T\{x[n]\} = x[n-1]$ LTI
- $T\{x[n]\} = x[n]^2$ not LTI
- $T\{x[n]\} = nx[n]$ not LTI
- $T\{x[n]\} = x[2n]$ not LTI
- $T\{x[n]\} = x[n] + 1$ not LTI

LTI Fun Fact

The **only** way to get an LTI system is by composing time shifts and scalings by constants.

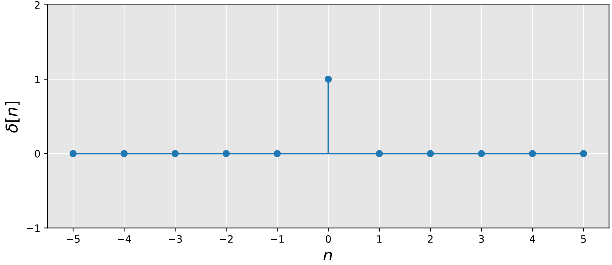
In other words, any LTI system, T, can be written as

$$T\{x[n]\} = \sum_{m=-\infty}^{\infty} a_m x[n-m],$$

for some scalar constants, a_m.

Impulse Response

Recall our **unit sample function** or **impulse**:



$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Impulse Response

Definition

The **impulse response** of a system, T, is the output it produces when given the unit impulse function as input. This is denoted:

$$h[n] = T\{\delta[n]\}.$$

Impulse Response

Recall any sequence, $x[n]$, can be written as a sum of scaled, shifted impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

This is the principle of **superposition**.

Impulse Response for an LTI System

Given an LTI, T:

$$
\begin{aligned}
T\{x[n]\} &= T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} && \text{superposition for } x[n] \\
&= \sum_{k=-\infty}^{\infty} T\left\{x[k]\delta[n-k]\right\} && \text{additivity property} \\
&= \sum_{k=-\infty}^{\infty} x[k]T\left\{\delta[n-k]\right\} && \text{scaling property} \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] && \text{definition of impulse response}
\end{aligned}
$$

Convolution

Definition

The convolution of two sequence, $x[n]$, $h[n]$, is given by

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

With this notation, any LTI system, T, with impulse response, h, can be computed as

$$T\{x[n]\} = (x * h)[n].$$