The z-Transform, Part II

Digital Signal Processing

February 29, 2024

Properties of the z-Transform

1. Linearity
2. Time-Shift
3. Convolution

1. Linearity

Linearity Property

The z-transform is a linear operator:

If $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$, and $y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$, then

$$ax[n] + by[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX(z) + bY(z),$$

for all complex constants $a, b \in \mathbb{C}$.

Proof of Linearity

$$
\begin{aligned}
\mathcal{Z}\{ax[n] + by[n]\} &= \sum_{n=-\infty}^{\infty} (ax[n] + by[n]) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} ax[n] z^{-n} + \sum_{n=-\infty}^{\infty} by[n] z^{-n} \\
&= a \sum_{n=-\infty}^{\infty} x[n] z^{-n} + b \sum_{n=-\infty}^{\infty} y[n] z^{-n} \\
&= aX(z) + bY(z)
\end{aligned}
$$

2. Time-Shift

Time-Shift Property

Shifting a signal by a time delay of $m \in \mathbb{Z}$ results in a multiplication of the z-transform by z^{-m}:

$$x[n-m] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-m}X(z).$$

Proof of Time-Shift

$$\begin{aligned}
\mathcal{Z}\{x[n-m]\} &= \sum_{n=-\infty}^{\infty} x[n-m] z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x[k] z^{-m-k} && \text{substitute } k = n - m \\
&= \sum_{k=-\infty}^{\infty} x[k] z^{-m} z^{-k} \\
&= z^{-m} \sum_{k=-\infty}^{\infty} x[k] z^{-k} \\
&= z^{-m} X(z)
\end{aligned}$$

3. Convolution

Convolution Property

Convolution of two signals results in the multiplication of their z-transforms:

$$x[n] * y[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)Y(z).$$

Proof of Convolution Property

$$
\begin{aligned}
\mathcal{Z}\{x[n] * y[n]\} &= \sum_{n=-\infty}^{\infty} (x[n] * y[n])\, z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]y[n-k] \right) z^{-n} \\
&= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[k]y[n-k]z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k]z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x[k]z^{-k}Y(z) \\
&= X(z)Y(z)
\end{aligned}
$$

z-Transforms of Linear Time-Invariant Systems

Recall Linear Time-Invariant Systems

Let $T\{\cdot\}$ be an LTI system. Remember this means

$$T\{x[n]\} = x[n] * h[n],$$

where $h[n] = T\{\delta[n]\}$ is the impulse response function.

z-Transform of an LTI System

Definition (System Function)

Consider an LTI system:

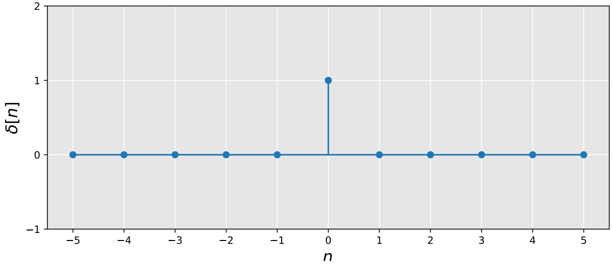
$$y[n] = x[n] * h[n].$$

Using the convolution property of the z-transform, this means

$$Y(z) = X(z)H(z).$$

$H(z)$ is called the **system function** or **transfer function** for T.

Example: Impulse Function



$$\mathcal{Z}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = z^0 = 1$$

Example: Feedforward Comb Filter

Remember the FFCF:

$$y[n] = x[n] + gx[n-k]$$

Its impulse response function is:

$$h[n] = \delta[n] + g\delta[n-k]$$

Using linearity and the time-shift property, we get the system function:

$$H(z) = 1 + gz^{-k}$$

ROC: $|z| > 0$

Example: Feedback Comb Filter

Remember the FBCF:

$$y[n] = x[n] + gy[n-k]$$

To get system function, plug in $x[n] = \delta[n]$:

$$\begin{aligned}
& h[n] = \delta[n] + gh[n-k] & \\
\iff \quad & H(z) = 1 + gz^{-k}H(z) & \text{take } z\text{-transform} \\
\iff \quad & H(z) - gz^{-k}H(z) = 1 & \text{rearrange } H(z) \text{ to left side} \\
\iff \quad & H(z) = \frac{1}{1 - gz^{-k}} & \text{solve for } H(z)
\end{aligned}$$

Alternate Method

Start from the impulse response:

$$h[n] = \sum_{m=0}^{\infty} g^m \delta[n - mk]$$

Using linearity and time-shift property:

$$H(z) = \sum_{m=0}^{\infty} g^m z^{-mk} = \sum_{m=0}^{\infty} \left(gz^{-k}\right)^m = \frac{1}{1 - gz^{-k}},$$

using geometric series formula with $r = gz^{-k}$.

ROC for FBCF

For geometric series to converge, we need $|r| < 1$. Assuming $g > 0$, we have:

$$1 > |r| = |gz^{-k}| = g|z|^{-k},$$

Or

$$|z|^k > g \Rightarrow |z| > g^{\frac{1}{k}}.$$