Signal Basics

Digital Signal Processing

January 16, 2025

Review: Discrete-Time Signals

A **discrete-time signal** is a function

$$x : \mathbb{Z} \to B,$$

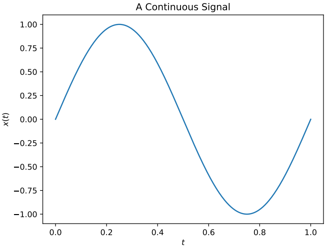
for some output set B (typically $B = \mathbb{R}$ or $B = \mathbb{C}$).

Equivalently, x is a sequence

$$x[n] \in B, \quad -\infty \leq n \leq \infty.$$

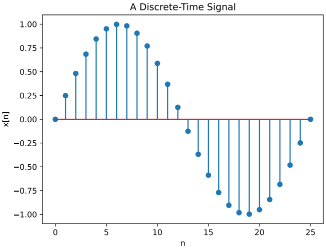
Sampled Continuous Signals

Discrete-time signals often come from continuous signals:



$x_c(t)$

$\rightarrow$

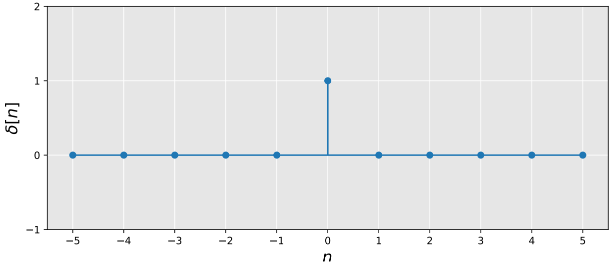


$x[n] = x_c(nT)$

Here, $T \in \mathbb{R}$ is the **sampling period**, $T = (1/25)\mathrm{s} = 0.04\mathrm{s}$

and $\frac{1}{T}$ is the **sampling frequency**. $\frac{1}{T} = 25\mathrm{Hz}$

Unit Sample Function



$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Also known as the **unit impulse function**.

Shifting the Unit Impulse

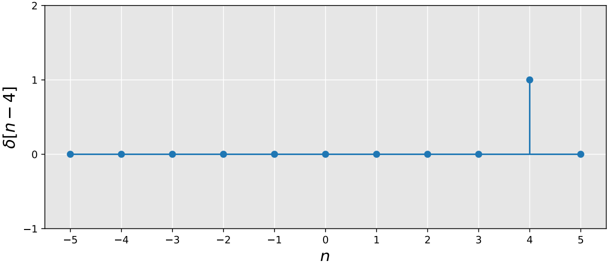
For any integer k,

$$\delta[n-k] = \begin{cases} 1, & n-k=0, \\ 0, & n-k \neq 0. \end{cases}$$

Or, in other words,

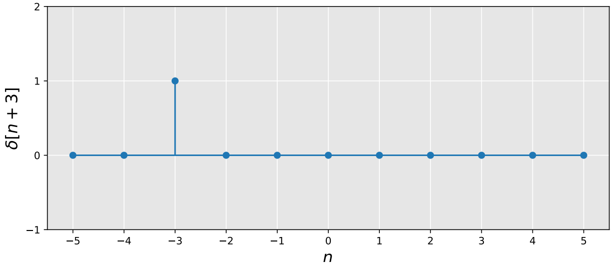
$$\delta[n-k] = \begin{cases} 1, & n=k, \\ 0, & n \neq k. \end{cases}$$

Shifting the Unit Impulse



$$\delta[n-4] = \begin{cases} 1, & n = 4, \\ 0, & n \neq 4. \end{cases}$$

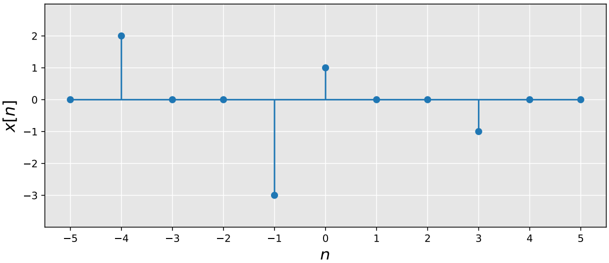
Shifting the Unit Impulse



$$\delta[n+3] = \begin{cases} 1, & n = -3, \\ 0, & n \neq -3, \end{cases}$$

Scaling and Adding Shifted Impulses

We can scale and add shifted impulses to construct signals:



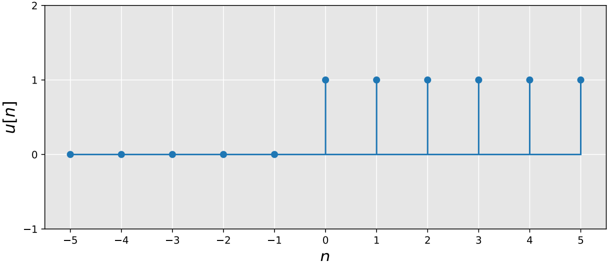
$$x[n] = 2\delta[n+4] - 3\delta[n+1] + \delta[n] - \delta[n-3]$$

Scaling and Adding Impulses

In fact, any sequence, $x[n]$, can be written as a sum of scaled, shifted impulses:

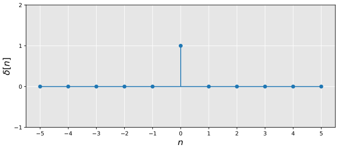
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

Unit Step Function

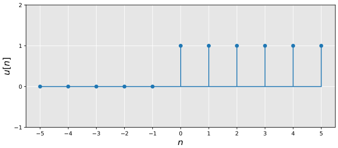


$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Relationship Between Step and Impulse



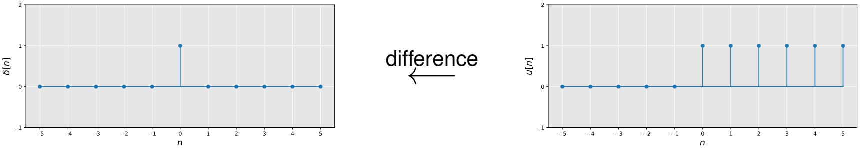
$\xrightarrow{\text{accumulate}}$



$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Discrete analogy to integration

Relationship Between Step and Impulse



$$\delta[n] = u[n] - u[n-1]$$

Discrete analogy to differentiation

Real Exponential Function

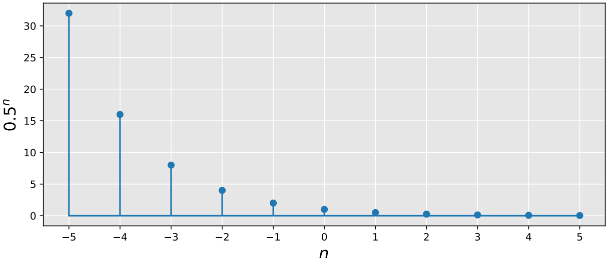
A **real exponential sequence** is of the form

$$x[n] = A\alpha^n,$$

for constants $A \in \mathbb{R}$ and $\alpha \in \mathbb{R}$.

Real Exponential Function

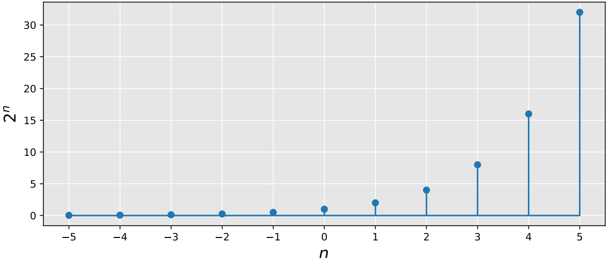
When $0 < \alpha < 1$, we get exponential **decay**:



$$x[n] = 0.5^n$$

Real Exponential Function

When $\alpha > 1$, we get exponential **growth**:



$$x[n] = 2^n$$

Real Exponential Function

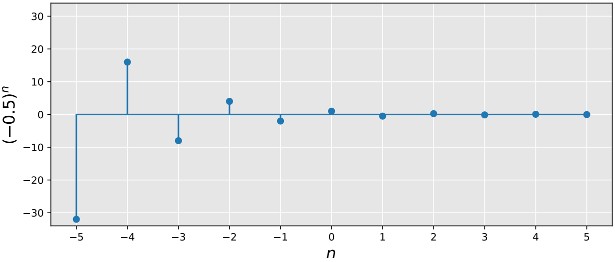
Note: taking reciprocal of α is equivalent to time-reversal:

Let $x[n] = \alpha^n$, then

$$x[-n] = \alpha^{-n} = \left(\alpha^{-1}\right)^n = \left(\frac{1}{\alpha}\right)^n$$

Real Exponential Function

When $\alpha < 0$, we get exponential **oscillation**:



$$x[n] = (-0.5)^n$$

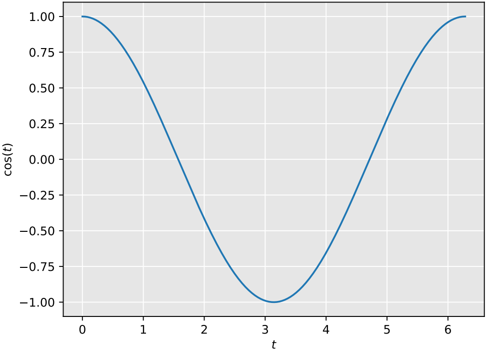
Real Exponential Function

Note: time shift is equivalent to multiplication:

Let $x[n] = \alpha^n$, then

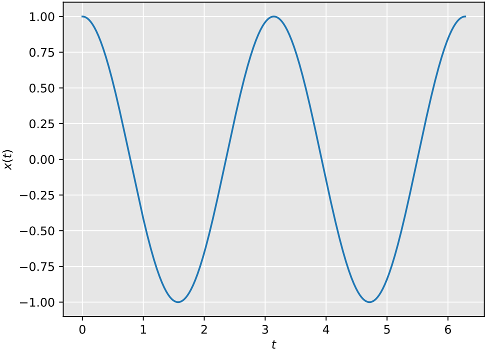
$$x[n-k] = \alpha^{n-k} = \alpha^n \alpha^{-k} = \alpha^{-k} x[n]$$

Remember the Cosine Function



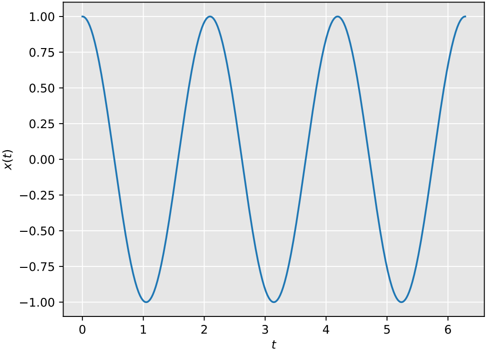
$$x(t) = \cos(t)$$

Remember the Cosine Function



$$x(t) = \cos(2t)$$

Remember the Cosine Function



$$x(t) = \cos(3t)$$

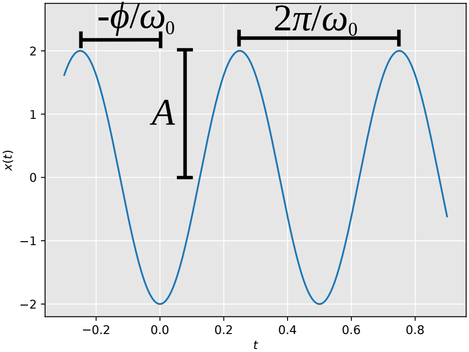
Frequency

$$x(t) = \cos(\omega_0 t)$$

The **frequency,** ω_0, is the number of times the cosine wave repeats in the interval $[0, 2\pi]$.

The **period**, $T_0 = \frac{2\pi}{\omega_0}$, is the length of time the cosine wave takes to repeat, i.e., the time interval between two consecutive peaks.

General Continuous Sinusoid



$$A = 2, \quad \omega_0 = 4\pi, \quad \phi = \pi$$

$$x(t) = A\cos(\omega_0 t + \phi)$$

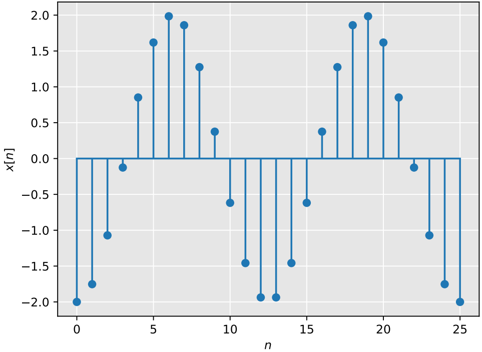
A : amplitude

ω_0 : frequency

ϕ : phase

Discrete-Time Sinusoidal Function

Discrete-time sinusoid is just a sampled continuous sinusoid



$$A = 2, \quad \omega_0 = \frac{4\pi}{25}, \quad \phi = 25\pi$$

$$x[n] = A\cos(\omega_0 n + \phi)$$

A : amplitude

ω_0 : frequency

ϕ : phase

Periodic Discrete-Time Signals

Definition

A signal, $x[n]$, is said to be **periodic** if for some integer $N > 0$,

$$x[n] = x[n + N],$$

for all $n \in \mathbb{Z}$.

Periodicity of Sinusoids

If $x[n] = x[n+N]$ for a sinusoid, then

$$\begin{aligned} A\cos(\omega_0 n + \phi) &= A\cos(\omega_0(n+N) + \phi) \\ &= A\cos(\omega_0 n + \omega_0 N + \phi) \end{aligned}$$

This will only be true when

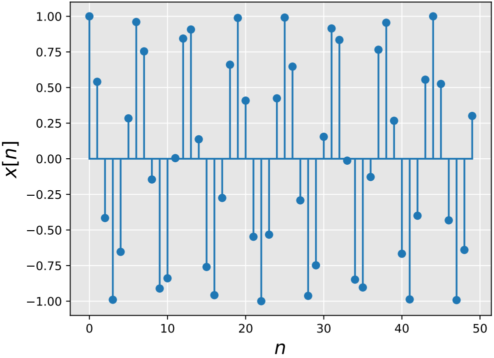
$$\omega_0 N = 2\pi k, \qquad \text{for some integer } k.$$

So, $\omega_0 = 2\pi\frac{k}{N}$ must be 2π times a rational number.

Periodic or Aperiodic?

$$x[n] = \cos(n)$$

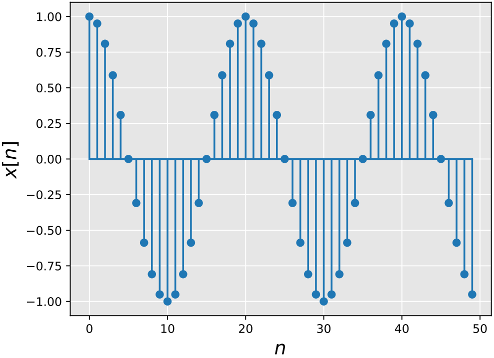
Not periodic!



Periodic or Aperiodic?

$$x[n] = \cos\left(\frac{n\pi}{10}\right)$$

Periodic!



Computing the Period of a Sinusoid

To get the period, N, write the frequency as

$$\omega_0 = 2\pi \frac{k}{N},$$

where $\frac{k}{N}$ is a *simplified* fraction.

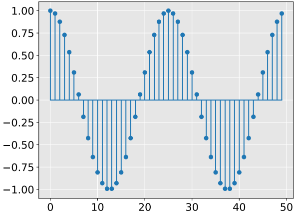
For the previous example:

$$\omega_0 = \frac{\pi}{10} = 2\pi \frac{1}{20},$$

so $N = 20$.

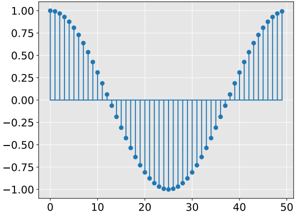
Weirdness with Discrete-Time Periodicity

$$\omega_0 = 2\pi\frac{48}{50}$$



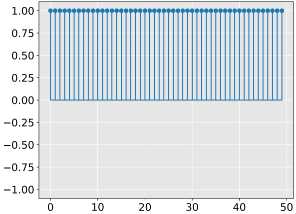
Period = 25

$$\omega_0 = 2\pi\frac{49}{50}$$



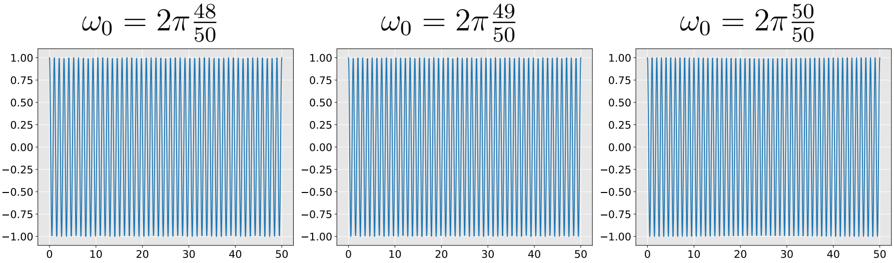
Period = 50

$$\omega_0 = 2\pi\frac{50}{50}$$



Period = 1

Continuous Versions of These Cosine Waves



Energy of a Signal

Definition

The **energy** of a signal, $x[n]$, is defined as

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

Power of a Signal

Definition

The **power** of a signal, $x[n]$, is defined as

$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2.$$

If $x[n]$ is periodic, with period N, then

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2.$$