Resonant characteristics of a steel blade

Nick Simmons

Department of Physics, Simon Fraser University, Burnaby, B.C., V5A 1S6, Canada

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Abstract

In order to ascertain the resonant properties of a steel hacksaw blade undergoing transverse oscillatory motion, we measured the blade's velocity via a mounted induction coil that passed through a magnetic field. By identifying the frequency of free oscillations, both with and without additional damping mass, we were able to extract the damping factors as well as quality factors. Furthermore, driving the system to resonance at the natural frequency of oscillation allowed the phase shift between the driving frequency and the blade velocity to be explored. Excessive transients were noted for the non-driven damped mass as well as for frequencies close to resonance. In both instances the oscillator violated the standard decaying sinusoidal solution for an under damped system.

I. INTRODUCTION

Our goal is to determine if a vibrating blade has similar characteristics to a simple harmonic oscillator. We are curious to see under what, if any, conditions the system can not be modeled using the standard solution for an under damped oscillator. We will attempt to measure γ, and therefore Q, under the free oscillation conditions, as well as the anticipated increase in γ and subsequent decrease in Q after adding a fixed mass to the blade. Furthermore we will explore the systems amplification with the additional mass at, and around, resonance by driving the oscillator through a range of frequencies. Our secondary concern is to verify the findings of Jones, in his mechanical resonance experiment[9]. We found that our range of frequencies in which resonance occurred was comparable, as well as our calculations for the systems Q factor. In addition, the phase difference between velocity and driving frequency as the resonant frequency is approached and passed is examined.

II. THEORY

The equation of a simple harmonic oscillator is given as

$$m\ddot{x} + b\dot{x} + kx = Fcos(\omega t), \text{ (1)}$$

where b is the constant of proportionality between the damping force and the velocity, k is the restoring force per unit distance, and $Fcos(\omega t)$ is the driving force (if present)[1,2]. Since $\omega_o = \sqrt{\frac{k}{m}}$, and $\gamma = \frac{b}{m}$, increasing the mass of the oscillator will decrease the natural frequency of oscillations as well as increase the damping rate. For the case of an under damped oscillator, where $\frac{\gamma}{2} < \omega_o$, the solution to equation 1 is given as

$$x(t) = Ae^{(-\frac{\gamma}{2}t)}cos(\omega t + \phi). \text{ (2)}$$

As the blade / coil assembly itself has mass and is subject to air resistance, γ will have a non-zero value for the free motion of the blade. When a driving force, $Fcos(\omega t)$, is applied to the oscillator (where F is the amplitude of the force and ω is the angular frequency) the velocity of the blade is given by

$$v(t) = \frac{F\omega/m}{\sqrt{\omega^2 - \omega_o^2 + 4\gamma^2\omega^2}}. \quad (3)$$

If the driving frequency matches the natural oscillation frequency of the apparatus a resonant state will occur[2]. At this frequency, the ratio of power applied to power lost per cycle is maximized with a measure of this state being the quality, or Q factor. The Q factor can be extracted via γ with

$$Q = \frac{\omega_o}{2\gamma}\ , \quad (4)$$

giving a quantitative expression for the systems efficiency. The phase difference between the driving force, F, and the measured velocity is given by

$$tan(\phi) = \frac{\omega_o^2 - \omega^2}{2\gamma\omega}, \quad (5)$$

where at resonance ϕ should be zero and changes from $-\frac{\pi}{2}$ at low frequencies as $\omega \to 0$ and $\frac{\pi}{2}$ at high frequencies as $\omega \to \infty$.[8]

III. PROCEDURE

A steel hacksaw blade was attached along it's flat edge to a brick providing a secure and motionless mount. As a method of measuring the motion of the blade's free oscillation was required, an induction coil was attached to the end of the blade. Faraday's law of induction states, $\varepsilon = -N\frac{\mathrm{d}\Phi}{\mathrm{d}t}$[7], as such passing a coil through a fixed magnetic field, a permanent magnet in this case, will induce a current in the coil proportional to the change in magnetic flux. Assuming the magnetic field can be thought of as uniform, an analogy is drawn where $\frac{\mathrm{d}\Phi}{\mathrm{d}t} \simeq \frac{\mathrm{dx}}{\mathrm{d}t}$ with the induced EMF being proportional to the blade's velocity. This signal was then measured by a data acquisition device[3] with the configuration shown in Fig 1. While the additional mass of the coil will increase the damping factor, as well as lower the natural frequency of the apparatus, this frictionless method of analysis served to eliminate most extraneous forces and allowed for concentration of the blades intrinsic properties.

The blade was then excited manually while the length of the oscillating section was adjusted to provide free vibrations of approximately 12 Hz, establishing a baseline natural

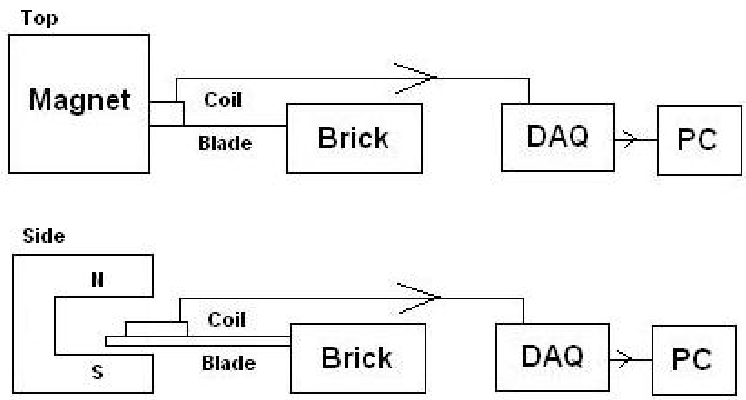

FIG. 1. The blade and induction assembly configured for free oscillations. The blade extended approximately 10 centimeters from the brick and its length adjusted to maintain a frequency of around 12 Hz. The induction coil was placed at rest $\frac{1}{2}$ way into the magnet and angled slightly to provide a perpendicular path between the coils motion and the magnets face.

frequency. Afterwards, a small aluminum plate was mounted to the end of the blade atop the pickup coil providing additional mass and increased damping. Qualitatively, adding the aluminum plate greatly increased the damping rate, which is shown in Fig 3b, as amplitude fell off dramatically and ceased altogether in a short period of time.

The blade was then subjected to forced motion provided by a mechanical driver[5]. Fig. 2 illustrates the positioning of the driver along the blade as well as the function generator[4] used to supply the driver.

The function generator was used as a voltage controlled oscillator, whose frequency was set by an analog voltage sent out by the DAQ. This allowed for fine electronic control of the driving frequency. Initially the blade was driven at the measured natural oscillation frequency in an attempt to establish resonance. Once maximum blade velocity was obtained a systematic sweep from low frequencies to high frequencies was performed in an attempt to map the resonance curve as well as the relative phase shift between the driving frequency and measured blade velocity.

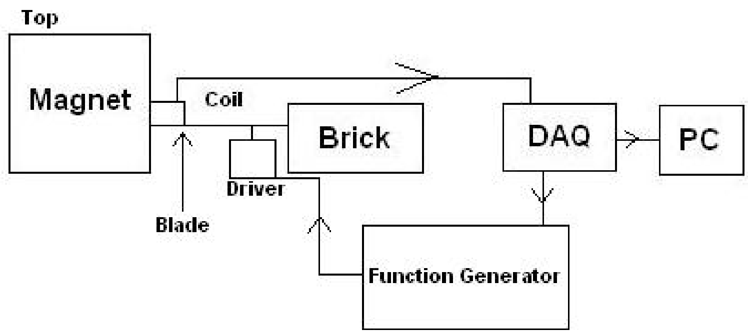

FIG. 2. The mechanical driver was placed close to the brick allowing sufficient torque to be applied with a small driving amplitude. The function generator was setup as a voltage controlled oscillator.

IV. RESULTS

Fig. 3 shows the free oscillations of the non-driven system. Using IGOR[6] the data set was fit using the solution for a damped oscillator, Eq.2. The natural frequency of the blade assembly measured 11.5850 ± 0.0007 Hz. There was a slow and steady decrease in amplitude indicating a non-zero damping factor γ, which measured $0.143 \pm 0.008s^{-1}$, attributed to the mass of the coil, internal torsion and air resistance.
The addition of the aluminum plate increased γ to $0.837 \pm 0.007s^{-1}$, greatly increasing damping over the initial two second interval. After two seconds, the steady state curve is modulated by the transient leading to a series of beats with decaying amplitude, see Fig. 4. A fit with the standard damped equation was not possible.

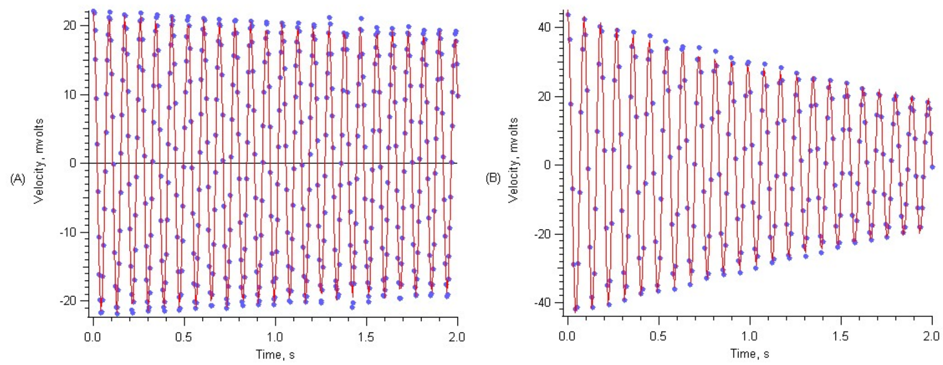

FIG. 3. (A) Undamped oscillator set. Curve fit is a damped sinusoidal function. The coefficients were 0.022 V amplitude, $0.143s^{-1}$ for γ, 11.5850 Hz frequency, 0.00 rad phase offset relative to cosine. (B) Damped oscillator set. Steady state fit coefficients were 0.045 V amplitude, $0.837s^{-1}$ for γ, 11.1202 Hz frequency, 0.07 rad phase offset relative to cosine.

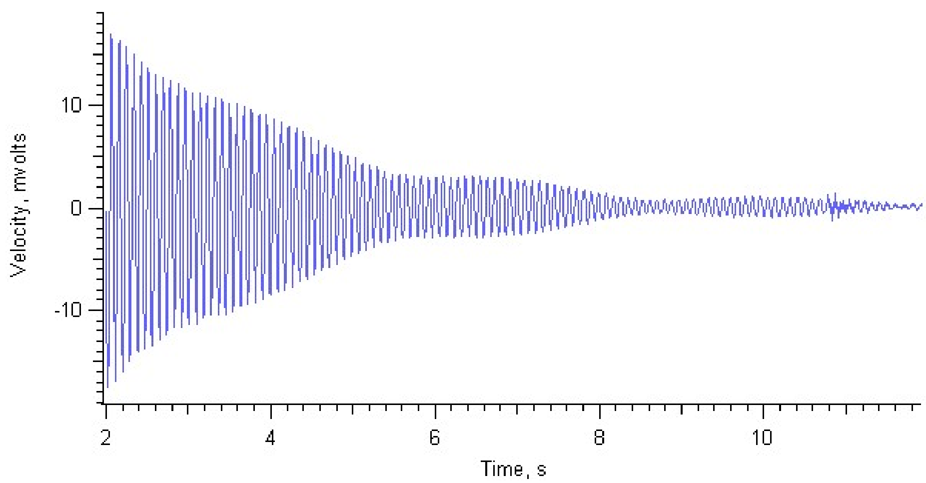

FIG. 4. Damped oscillator with decaying transient amplitude.

The damped oscillation frequency measured 11.1202 ± 0.0005 Hz showing a 4.01% decrease in frequency with the addition of the mass. The addition of the aluminum plate produced nearly a six-fold increase in the damping factor, indicating that the additional mass had significantly more effect on damping than internal or external frictional forces acting on the blade.

Next the blade, with attached aluminum plate, was driven at the measured natural frequency of 11.585 Hz in an attempt to establish resonance. A maximum blade amplitude of 92.47 ± 0.04mV at 11.322 ± 0.004 Hz was recorded when driven with an amplitude of 236.8 ± 0.3 mV at 11.325 ± 0.001 Hz. Fig. 5 illustrates the measured $0.041 \pm 0.002°$ phase shift between the driving frequency and blade velocity. γ in this instance was calculated to be $0.046 \pm 0.005 s^{-1}$, showing a substantial decrease over the free motion damping.

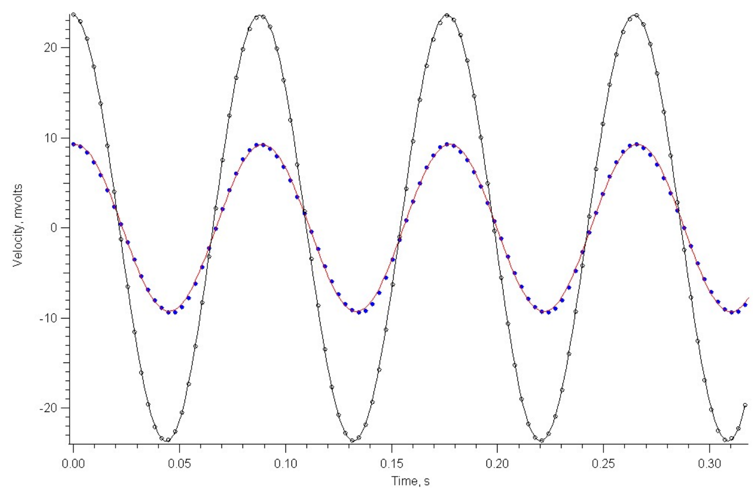

FIG. 5. Blade resonance data set fit with damped equation. The coefficients were 0.09247 V amplitude, $0.046 s^{-1}$ for γ, 11.322 Hz frequency, -0.07 rad phase offset relative to cosine. The driving frequency is plotted for comparison.

Finally, in Fig 6, as the frequency was swept both below resonance and above resonance the system showed decreasing amplitude as well as a negative phase shift for frequencies below resonance and a positive phase shift for frequencies above resonance. In particular amplitude was 48.4 ± 0.2mV at 11.253 ± 0.003 Hz with a $-36.83 \pm 0.04°$ phase shift. Above resonance at 13.486 ± 0.003 Hz a $158.49 \pm 0.04°$ phase shift was measured with amplitude decreasing to 9.7 ± 0.1mV.

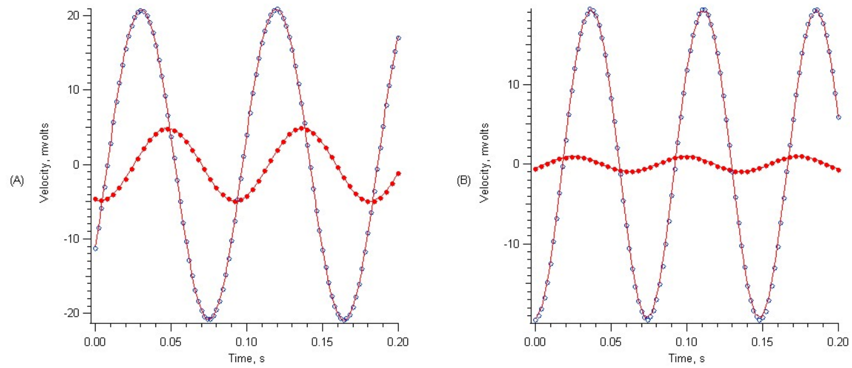

FIG. 6. (A) Below resonance at 11.253 Hz, with a -36.83° phase shift. (B) Above resonance at 13.486 Hz, with a 158.49° phase shift.

V. DISCUSSION

Qualitative analysis of the oscillator shows a rapid energy loss indicating a decreasing Q factor with increased damping. Table 1 lists the results of our calculations for Eq.4

Table 1

	Q	Frequency
Undamped	41 ± 2	11.5850 ± 0.0007 Hz.
Non-driven Damped	6.6 ± 0.6	11.1202 ± 0.0005 Hz.
Resonance	$1.2 \times 10^2 \pm 1 \times 10^1$	11.322 ± 0.004 Hz.

Addition of the aluminum plate decreased the Q factor from 41 to 6.6 a factor of 6.21, which is similar to the increase in damping of 5.85. The application of the driving force to the system at resonance supplied additional energy eliminating nearly all damping and served to increase the Q factor to 120. To investigate the exact nature of where resonance occurs a plot of velocity vs. frequency is shown in Fig 7. Clear asymptotic behavior is noted around resonance of 11.322 Hz with a steep decline as frequency is changed. Occurrence of resonance appears to be limited to a very narrow range of ≈ 10.67 to 13.26 Hz.

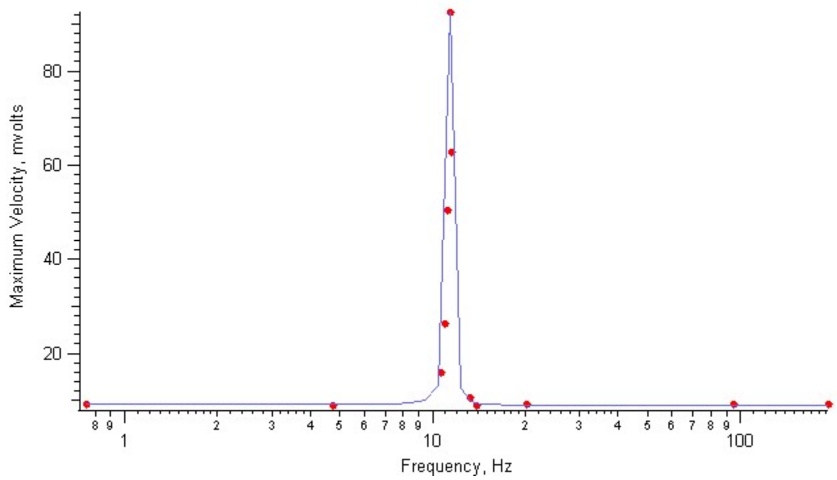

FIG. 7. Resonance curve for the damped oscillator from Fig. 3b. Peak amplitude of 92.47mV at 11.322 Hz fitted with Eq. 3

By plotting velocity phase shift vs. frequency in Fig 8, the connection between phase and resonance is explored. At resonance, a $0.041 \pm 0.002°$phase shift is noted, while at low frequencies ($\omega \to 0$), velocity leads the driving frequency and the phase is shifted towards $-\frac{\pi}{2}$ radians. Conversely at high frequencies ($\omega \to \infty$) the velocity lags behind the driving frequency and a phase shift of $+\frac{\pi}{2}$ radians occurs.

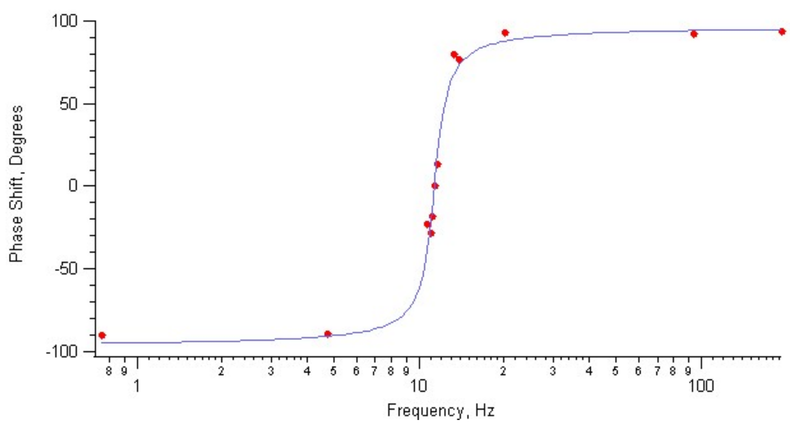

FIG. 8. Velocity phase relative to the driving frequency. The data set is fit with Eq.5

A final observation, which was briefly presented in Fig 4., was the presence of beats when the driving frequency did not closely match the resonant frequency. Representing the superposition of the steady state and transient solutions to Eq. 1[8] , these beats were readily produced at 11.51 ± 0.02 Hz shown in Fig. 9.

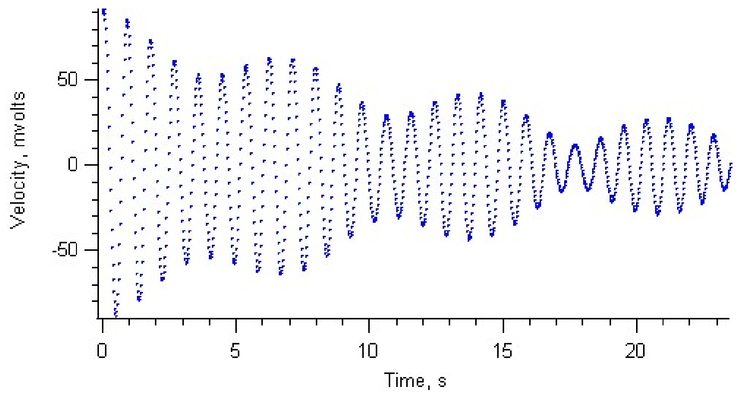

FIG. 9. "Beating" occurred at 11.51 Hz as the steady state vector and decaying transient vector are combined.

VI. CONCLUSION

This report attempts to explore the established laws regarding a mechanical oscillator and the limitations of such laws. By constructing an apparatus to measure the particular characteristics of a vibrating hacksaw blade, we have shown that the addition of mass to the oscillator produced an unacceptable transient response and series of beats which do not conform to Eq. 2. This increased the damping factor slightly, decreasing the natural frequency of oscillation, while the Q factor of the system decreased proportionally to the increase in damping. Driving the system to resonance showed a rapid increase in amplitude over a very narrow range of frequencies as determined by the distinct combination of components in this system. If resonance was not precisely matched, the presence of beats and non conformance was again noted. The energy input by the driver served to increase the Q factor of the system by 18 times effectively amplifying the freely oscillating apparatus and minimizing γ. The velocity phase offset plotted against the driving frequency showed as expected a $-\frac{\pi}{2}$ offset at low frequencies as $\omega \to 0$ and a $+\frac{\pi}{2}$ offset at frequencies above resonance as $\omega \to \infty$.[8] A further area of inquiry would be to measure the systems response with a wider selection of damping plates. The transient beating response may be unique to this particular combination of components.

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