$A_{m,n} => rref(A) = R$

$Rank(A) = r$

Some concepts:

- row echelon form (ref):
  - The first non-zero element of each row is 1 (leading entry)
  - Each leading entry is in a column to the right of the leading entry in the previous row.
  - Rows with all zero elements, if any, are below rows having a non-zero element.
- reduced row echelon form (rref):
  - The matrix is in row echelon form
  - The leading entry in each row is the only non-zero entry in its column.

    $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

    is a ref.

    $$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

    is a rref.
- rref and ref is obtained by apply elimination operation to A.
- pivot columns of reff: the cols that contains leading entry
- free columns of reff: the cols of reff that are not pivot cols.
- pivot rows of reff: the rows that contains leading entry

The four fundamental subspaces of matrix A.

suppose: $rref(A) = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

1. Column space $C(A)$:

$dim(C(A)) = r$

$basis(C(A)) = \text{pivot columns of reff(A)}$

proof:

- If we look at the pivot columns only, we see the r by r identity matrix (in other words, there is no way to combine its rows to give the zero row (except by the combination with all coefficients zero)) => r pivot rows are independent. (1)

- Every other (free) columns is a combination of the pivot columns (it is because of the property of rref) (2)

  (1) (2) => q.e.d

  Note that elimination operations do change the column space of A: $C(A) \neq C(rref(A))$ (because you can see that most of column of rref(A) ends up with 0, this is not true with A)

2. Row space $C(A^T)$:

$dim(C(A^T)) = r$

$basis(C(A^T)) = \text{pivot rows of rref(A)}$

proof:

- If we look at the pivot columns only, we see the r by r identity matrix (in other words, there is no way to combine its rows to give the zero row (except by the combination with all coefficients zero)) => r pivot rows are independent. (1)

  $$rref(A) = \begin{bmatrix} 1^* & 3 & 5 & 0^* & 7 \\ 0^* & 0 & 0 & 1^* & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Because elimination operation does not change the row space of A => pivot rows of rref(A) span the row space of A. (2)

  (1), (2) => q.e.d

Lemma: The number of independent columns equals the number of independent rows.

The above proof give us the conclusion that: number of independent rows of A is r.

We need to prove that number of independent cols of A is also r:

- Ax = 0 <=> Rx = 0
- let p be the basis of C(A), and q be the basis of C(R), we have:

  Ax = 0 <=> $p.x = 0$

  Rx = 0 <=> $q.x = 0$

  => p = q

  => len(p) = len(q)

  => q.e.d

3. Null space $N(A)$:

$dim(N(A)) = n - r$

$basis(N(A)) = \text{special solutions of Ax=0}$

proof:

- the special solution is independent because they contain a identity matrix
- all the solutions is a combination of special solutions: we only need to multiply free variable with a proper scalar, other variable can be determined by equation $rref(A).x = 0$

4. Left null space $N(A^T)$:

$dim(N(A^T)) = m - r$