1. Why Gauss elimination does not change the null space of matrix A?

let M be the product of elementary matrixes of A.

- suppose $x \in N(A)$, so Ax = 0. we have:

  $MAx = M(Ax) = M.0 = 0$

  so $x \in N(MA)$ as well.
- suppose $y \in N(MA)$, so (MA).y=0. we have:

  $Ay = M^{-1}MAy = M^{-1}.0 = 0$

  so $y \in N(A)$ as well.

That means when you apply Gauss elimination, you do change the solution space.

2. Why every elementary is invertible?

https://en.wikipedia.org/wiki/Elementary_matrix

https://math.vanderbilt.edu/sapirmv/msapir/prleelementary.html

3. If n vectors are independent, is their span R^n ?

Yes. Here is why:

Proposition. Let W be a subspace of a finite-dimensional vector space V. If $dim(W) = dim(V)$, then W=V.

If we are given linearly independent $\{v_1, \ldots, v_n\} \in R^n$, then $span\{v_1, \ldots, v_n\}$ is an n-dimensional subspace of R^n. Since $dim(R^n) = n$, the proposition implies that $span\{v_1, \ldots, v_n\} = R^n$

4. Proof: A invertible <=> A's columns are independent & A is squared.

$A = [v_1, v_2, v_3 \ldots, v_n] : v_i \text{is column vector}$

- Suppose A is invertible => A is squared

  $Ax = 0 <=> A^{-1}Ax = A^{-1}0 <=> x = 0$

  => q.e.d
- Suppose A's columns are independent & A is squared.

  $\{v_1, v_2, \ldots, v_n\}$ are independent => $\{v_1, v_2, \ldots, v_n\}$ is a basis of R^n (see Q3). So:

  $$I_n = A * \begin{bmatrix} b_{1,1} & \ldots b_{1,n} \\ b_{2,1} & \ldots b_{2,n} \\ \ldots & \\ b_{n,1} & \ldots b_{n,n} \end{bmatrix} = A * B => B = A^{-1}$$

5. Proof: A's columns are independent => A^TA invertible.

- A^TA is squared.
- A's columns are independent <=> N(A)={0}. Let $\vec{v} \in N(A^TA)$ .We have:

$$
\begin{aligned}
& A^T A.\vec{v} = \vec{0} \\
<=> \; & v^T A^T A v = v^T.\vec{0} = 0 \\
<=> \; & (Av)^T.(Av) = 0 \\
<=> \; & Av = 0 \\
<=> \; & \begin{cases} \vec{v} = \vec{0} \\ \vec{v} \in N(A) \end{cases} \\
<=> \; & N(A) = N(A^T A) = \{0\}
\end{aligned}
$$

=> q.e.d

6. Proof: Orthonormal set of vector are independent.

https://www.quora.com/Is-an-orthonormal-set-of-vectors-a-linearly-independent-set/answer/Supreeth-Narasimhaswamy

7. Why different eigenvalues produce independent eigenvectors?

Suppose we have k = 2 eigenvalues and some combination of $x_1 \text{ and } x_2$ produces 0:

$c_1x_1 + c_2x_2 = 0 \quad (1)$. We will prove that $c_1 = c_2 = 0$:

$Ac_1x_1 + Ac_2x_2 = 0 <=> c_1\lambda_1x_1 + c_2\lambda_2x_2 = 0 \quad (2)$

Subtract $(1)\lambda_2 \text{ to } (2)$:

$x_1c_1(\lambda_1 - \lambda_2) = 0 <=> c_1 = 0$ because $x_1 \neq 0$ and eigenvalues are different.

Similarly, we can show that $c_2 = 0$. This same argument extends to any number of eigenvalue.

8. Why A is singular <=> A has an eigenvalue equal 0?

Because: $det(A) = \lambda_1..\lambda_n$

Eigenvectors of a symmetric matrix A corresponding to different eigenvalues are orthogonal.

http://www.maths.manchester.ac.uk/~peter/MATH10212/notes10.pdf