Midterm Exam

CSCI–UA 0480–074: Quantum Computing Spring 2024

March 5, 2024

1. (a) Express the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the Hadamard basis. [2]

   (b) Suppose that we *measure* the state $|\psi\rangle$ in the Hadamard basis. What are the outcome probabilities and post-measurement states, in terms of α and β? [4]

   (c) For each of the following states, determine whether the state is entangled or separable. Then write the mixed state on the first qubit (i.e., after discarding the second qubit), either as a distribution over pure states, or as a density matrix. Justify your answer in each case. [8]

   i. $\frac{|01\rangle+|10\rangle}{\sqrt{2}}$.

   ii. $\frac{|00\rangle+|10\rangle}{\sqrt{2}}$.

   iii. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + i\,|11\rangle)$.

   iv. $\frac{1}{2}(|00\rangle + e^{i\pi/4}|01\rangle + e^{3i\pi/4}|10\rangle - |11\rangle)$.

2. The "Bell states" are the four two-qubit states

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \qquad |\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

   (a) Design a circuit that prepares a chosen Bell state. That is, design a circuit that realises the unitary mapping:

$$|00\rangle \mapsto |\Phi^+\rangle \qquad |10\rangle \mapsto |\Phi^-\rangle \qquad |01\rangle \mapsto |\Psi^+\rangle \qquad |11\rangle \mapsto |\Psi^-\rangle$$

   Explain why your circuit is correct. [6]

   (b) Explain how your answer to the previous part implies that the Bell states are orthogonal (i.e., they form a basis for $\mathbb{C}^4$). [3]

   (c) Write the state $|+\rangle \otimes |-\rangle$ in the Bell basis; i.e., as a linear combination of Bell states. [2]

   (d) Design a circuit that measures a two-qubit state in the Bell basis. That is, design a circuit that implements the projective measurement

$$\mathcal{M} = (|\Phi^+\rangle\langle\Phi^+|, |\Phi^-\rangle\langle\Phi^-|, |\Psi^+\rangle\langle\Psi^+|, |\Psi^-\rangle\langle\Psi^-|).$$

   Your circuit should measure only in the computational basis. Prove that your circuit is correct. [5]

3. Consider the following state on three qubits:

$$|\psi\rangle = \sqrt{\frac{1}{5}}\,|001\rangle + \sqrt{\frac{2}{5}}\,|010\rangle + \sqrt{\frac{2}{5}}\,|100\rangle$$

(a) Consider measuring

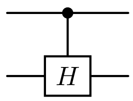
i. the *first* qubit [6]

ii. the *third* qubit [6]

of $|\psi\rangle$ in the computational basis. In each case, compute the probability of each outcome and the post-measurement state. For each post-measurement state, determine whether the two *unmeasured* qubits are entangled.

(b) Design a circuit that creates the state $|\psi\rangle$ from the state $|0\rangle^{\otimes 3}$. [8]

In addition to standard gates $X, Z, H, S, \mathsf{CNOT}$, you may use the following special gates:

i. the controlled-H gate  , which maps $|0\rangle\,|\psi\rangle \mapsto |0\rangle\,|\psi\rangle$ and $|1\rangle\,|\psi\rangle \mapsto |1\rangle\,(H\,|\psi\rangle)$;

ii. the gate R, which maps $|0\rangle \mapsto \frac{1}{\sqrt{5}}\,|0\rangle + \frac{2}{\sqrt{5}}\,|1\rangle$ and $|1\rangle \mapsto -\frac{2}{\sqrt{5}}\,|0\rangle + \frac{1}{\sqrt{5}}\,|1\rangle$.

4. Consider the mixed states

$$\rho = \begin{cases} |0\rangle & \text{with probability } 1/6 \\ |1\rangle & \text{with probability } 1/6 \\ |+\rangle & \text{with probability } 2/3 \end{cases} \qquad \sigma = \begin{cases} |+\rangle & \text{with probability } p \\ |-\rangle & \text{with probability } 1-p \end{cases}$$

Find p such that ρ and σ are indistinguishable.