A Survey of Partitions

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Combinatorics of Partitions

Definition

Fix n a nonnegative interger. A *partition* of n is a string $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ with λ_i nonnegative integers and

$$n = \sum_{i=1}^{n} \lambda_i.$$

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Example

$n=2$	$n=3$	$n=4$
2 $1+1$	3 $2+1$ $1+1+1$	4 $3+1$ $2+2$ $2+1+1$ $1+1+1+1$

Formula? Not so much...

- Generating Function: Let $p(n)$ be the number of partitions of n, then

$$\begin{aligned}\sum_{n\geq 0} p(n)x^n &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots \\ &= 1+x+2x^2+3x^3+5x^4+7x^5+\cdots\end{aligned}$$

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- Asymptotic Formula: (Hardy and Ramanujan 1918)

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}, \ n \to \infty$$

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$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}, \; n \to \infty$$

- "Closed" Formula: (Rademacher 1937)
  Exists but horrible (see Wikipedia's "Partition" page).

Young diagrams

Definition

Let λ be a partition n. The *Young diagram* $[\lambda]$ is the array with λ_i boxes in row i (matrix coordinates), *e.g.*

$$[(2,2,2,1)] = \ydiagram{2,2,2,1} \qquad [(4,3)] = \ydiagram{4,3}$$

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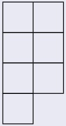
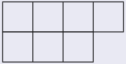
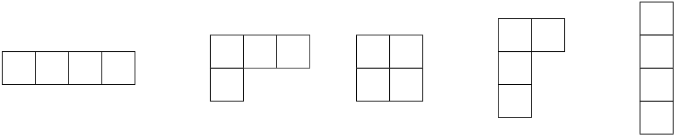
$[(2,2,2,1)] =$  $[(4,3)] =$ 

A *tableau* is a filling of $[\lambda]$ with numbers $\{1, 2, \cdots, n\}$. A tableau is called *standard* if the entries of each row and column are increasing.

$$\text{tableau: } \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 6 & 2 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array} \qquad \text{standard: } \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 & 7 \\ \cline{1-2}\end{array}$$

Standard tableaux $n = 4$.



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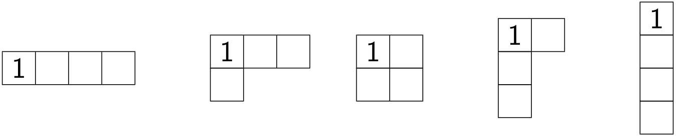

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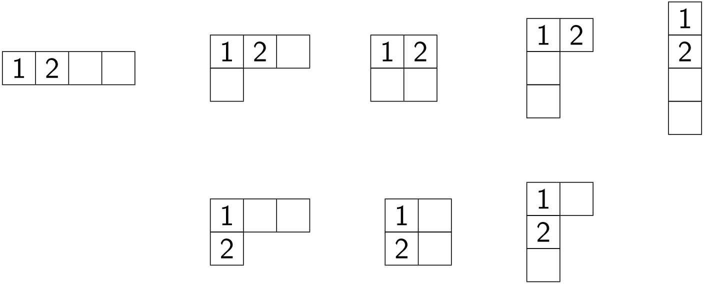

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$$\begin{array}{|c|c|c|c|}\hline 1&2&3&\\ \hline\end{array}
\qquad
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\qquad
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$$\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau}$$

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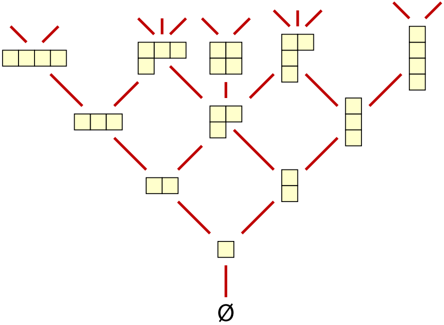
$$\begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} \qquad \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau}$$

Young's Lattice

Young's lattice is the lattice built by connecting two Young diagrams if they differ by only one box.



Exercise: The number of paths from $\emptyset$ to $[\lambda]$ is the number of standard tableau on $[\lambda]$.

Application I: Symmetric functions

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A polynomial in variables $x_1, x_2 \cdots, x_n$ is called *symmetric* if any permutation of the variables leaves it unchanged.

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- $x_1 + 2x_2 + x_3$ is not symmetric

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- $x_1 + 2x_2 + x_3$ is not symmetric
- For each λ a partition of n, let $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$, then the *monomial symmetric functions*

$$m_\lambda = \sum_{\alpha \sim \lambda} x^\alpha$$

and every symmetric function is a sum of the m_λ.

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$$m_\lambda = \sum_{\alpha \sim \lambda} x^\alpha$$

  and every symmetric function is a sum of the m_λ.
- $m_{(2,1,0)} = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$

Example (Some bases: Let λ be a partition of n.)

- Elementary: e_λ

$$e_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$$

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Exercise: Show $e_\lambda = \sum_\mu M_{\lambda\mu} m_\mu$ where $M_{\lambda\mu}$ is the number of $0-1$ matrices with row sums λ_i and column sums μ_j.

Schur functions

IMHO the most important basis, but they are
poorly named (studied earlier by Jacobi and then by Frobenius).

Carl G.J. Jacobi
1804-1851

Ferdinand Frobenius
1849-1917

Issai Schur
1875 -1941

$$
\begin{aligned}
s_{(2,1,0)} &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3 \\
&= m_{(2,1)} + 2m_{(1,1,1)}
\end{aligned}
$$

Schur functions

Definition

Let $[\lambda]$ be a Young diagram. A $[\lambda]$-tableau is called *semistandard* if the row entries are weakly increasing and column entries are strictly increasing.
If T is a semistandard $[\lambda]$-tableau the *content* of T is $c(T) = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ if for each i there are α_i $i's$.

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Example

The semistandard $[(2,1)]$-tableaux:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

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For each λ be a partition of n the *Schur function* is

$$s_\lambda = \sum_T x^{c(T)} = \sum_T x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

where the sum is over all semistandard $[\lambda]$-tableaux.

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Application II: Representation theory of $\mathfrak{S}_n$

The symmetric group $\mathfrak{S}_n$ is the set of bijections (permutations)

$$w : \{1, 2, \cdots, n\} \longrightarrow \{1, 2, \cdots, n\}.$$

We denote a $w \in \mathfrak{S}_n$ as

$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w(1) & w(2) & \cdots & w(n) \end{pmatrix}, \text{e.g. } w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

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We "multiply" $w, v \in \mathfrak{S}_n$ by function composition

$$w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, v = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \text{then}$$

$$w \circ v = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} = id$$

Application II: Representation theory of $\mathfrak{S}_n$

Alternatively, we could present the $w \in \mathfrak{S}_n$ as invertible matrices.

For example, let $e_i^T = (0, 0, \cdots, 0, 1, 0, \cdots 0) \in \mathbb{R}^n$ where 1 is in spot i. For $w \in \mathfrak{S}_n$ define the $n \times n$ matrix $[w]$ by the equations

$$[w] \cdot e_i = e_{w(i)}.$$

E.g. if $w = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, then

$$[w] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} e_2 & e_3 & e_1 \end{pmatrix}.$$

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Astonishingly this respects function composition!

$$[w \circ v] = [w] \cdot [v]$$

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Question: What other ways can we "represent" permutations as invertible matrices?

Intermission

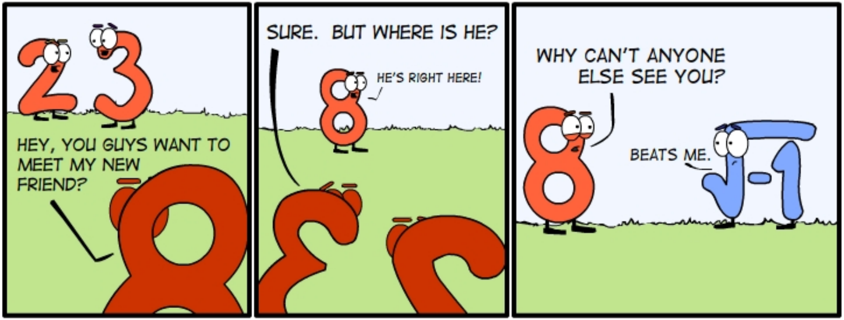

Figure: http : //mrchasemath.wordpress.com/category/complex/

Application II: Representation theory of $\mathfrak{S}_n$

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A *representation* of $\mathfrak{S}_n$ is a set M of $k \times k$ invertible matrices which satisfy for all $w, v \in \mathfrak{S}_n$

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- For $w \in \mathfrak{S}_n$, define $triv(w) = 1$.

Invariant sub-spaces

Definition

Let M be a representation consisting of $k \times k$ matrices. We say a subspace $V \subset \mathbb{R}^k$ is *$\mathfrak{S}_n$-invariant* if for all $w \in \mathfrak{S}_n$ and $\vec{v} \in V$

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Example

Consider the standard representation [ ], and let

$$V = \operatorname{span}(e_1 + e_2 + \cdots + e_n) \subset \mathbb{R}^n.$$

Then V is $\mathfrak{S}_n$-invariant because

$$[w] \cdot \sum_i re_i = \sum_i r[w] \cdot e_i = \sum_i re_{w(i)} = \sum_j re_j.$$

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E.G. $\mathit{triv} = S^{(n)}$, $\mathit{sgn} = S^{(1,1,\cdots,1)}$, *and* $M^{\perp} = S^{(n-1,1)}$.

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Example

- Peri Rule: If M is representation of $\mathfrak{S}_n$, by considering the permutations which fix n we make M a representation of $\mathfrak{S}_{n-1}$, then

$$Res_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} S^\lambda = \bigoplus_{\lambda^-} S^{\lambda^-}$$

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- $\lambda = \ydiagram{2,1}$, then $\lambda_1^- = \ydiagram{2}$ and $\lambda_2^- = \ydiagram{1,1}$

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where λ^- are the partitions of $n-1$ connected to λ in Young's lattice.

- $\lambda = \begin{array}{|c|c}\hline & \\ \hline & \\ \cline{1-1} \end{array}$, then $\lambda_1^- = \begin{array}{|c|c|}\hline & \\ \hline \end{array}$ and $\lambda_2^- = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$
- Exercise: Use induction and the Peri rule to show

$$\dim S^{\lambda} = \#\,\text{standard } [\lambda]\text{-tableaux.}$$

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- We have seen that both Λ_n (symmetric functions) and $Rep(\mathfrak{S}_n)$ are *vector spaces* (we exhibited several bases for them).

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- We have seen that both Λ_n (symmetric functions) and $Rep(\mathfrak{S}_n)$ are *vector spaces* (we exhibited several bases for them).
- They have more structure as inner product spaces (think dot product from calculus) and we have some orthonormal bases

$$\langle s_\lambda, s_\mu \rangle_{\Lambda_n} = \delta_{\lambda\mu} = (S^\lambda, S^\mu)_{Rep(\mathfrak{S}_n)}.$$

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- $\mathrm{ch}(S^\lambda) = s_\lambda$ is an isometry (plus some more!).
- Let $X \in \Lambda_n$ and $X = \sum_\lambda c_\lambda s_\lambda$ with c_λ non-negative integers (Schur positive), then there is some representation M such that $\mathrm{ch}(M) = X$!

Any questions?