Non-linear Geometry of Banach Spaces

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The study of non-linear structure of Banach spaces dates back to a result by Kadets [24] in 1963, which says that every two separable infinite dimensional Banach spaces are homeomorphic. In non-linear geometry of Banach spaces, the functions studied are usually uniformly continuous, Lipschitz or coarse Lipschitz. The fundamental question we always ask ourselves is that, given a non-linear map f from a Banach space X to another Banach space Y, suppose that f has a certain property (usually bijective, injective or surjective), is it true that there exists a linear map T from X to Y with the same property?

For example, suppose that X, Y are uniformly homeomorphic Banach spaces, is it true that X, Y are linearly isomorphic? Unfortunately, the answer to this question is negative (Ribe 1984, [35], Theorem 5.1). However, it is of interest to know under what further conditions should be imposed on f (or on X, or on Y) so that the answer becomes positive. For example it is known that if X is uniformly homeomorphic to a Hilbert space, then X is linearly isomorphic to a Hilbert space (Remark 5.4).

We will give a survey article about the Lipschitz and uniform structures of Banach spaces. The main references include the book by Kalton and Albiac [2], the article by Kalton [27] and the paper by Godefroy, Lancien and Zizler [14].

Contents

1 Introduction

The thesis aims to survey major results in the development of non-linear geometry of Banach spaces. The area focused on investigating non-linear maps between Banach spaces. Since a Banach space can be viewed as a topological vector space, or as a metric space, we have a wide class of non-linear maps to discuss. The main references include the book by Kalton and Albiac [2], the article by Kalton [27] and the paper by Godefroy, Lancien and Zizler [14]. In this thesis, we aim to investigate results concerning Lipschitz, uniform and coarse Lipschitz structures of Banach spaces.

In Section 2, we will introduce the basic notations used throughout the paper, and the definitions of the non-linear maps studied, these include Lipschitz maps, uniformly continuous maps and coarse Lipschitz maps. Some basic properties about these categories of maps are also proved.

In Section 3, we will investigate the Lipschitz structure of Banach spaces. We study the differentiability of Lipschitz maps and prove an infinite dimensional version of Rademacher Theorem (Theorem 3.16). This powerful result allows us to prove several important results concerning Lipschitz structures of Banach spaces. It is shown that all the classical Banach spaces ℓ_p, L_p for $1 < p < \infty$ admits unique Lipschitz structure. Then, we move to study the asymptotic structure of Banach spaces, first introduced by Milman [31].

In Section 4 we study extensively the asymptotic structure of c_0. We will introduce a property called the *Lipschitz weak* Kadec Klee* property and shows that this property is equivalent with the asymptotic uniform flatness (Definition 4.1). The main result is that every separable Banach space with this property is contained in c_0. The result allows us to show that c_0 has unique Lipschitz structure.

In Section 5, we will restrict ourselves on classical Banach spaces and study their uniform structures. We will introduce the approximate metric midpoint method and the Kalton-Randrianarivony graph [28] to study the uniform structure of ℓ_p. It is shown that for $1 < p < \infty$, the sequence space ℓ_p has unique uniform structure.

2 Preliminary

2.1 Basic notations and properties

Throughout this survey article, unless otherwise specified, X denotes a Banach space over $\mathbb{R}$. Put B_X, S_X to be the closed unit ball and unit sphere of X respectively. Let $(x_n)_{n=1}^m$ be a finite sequence in X, we put $[x_1, \ldots, x_m]$ to be the closed linear span generated by $(x_n)_{n=1}^m$. If $(x_n)_{n=1}^\infty$ is an infinite sequence, put $[x_n]_{n=1}^\infty$ to be the closed linear span generated by $(x_n)_{n=1}^\infty$. $j_X : X \to X^{**}$ is the natural embedding defined by $x \mapsto j_X(x)$ where $j_X(x)(x^*) = x^*(x)$ for $x^* \in X^*$.

Definition 2.1. Let X, Y be Banach spaces. $T : X \to Y$ be a bounded linear map.

(a) T is said to be a linear isomorphism if T is invertible and T^{-1} is continuous.

(b) T is said to be a linear embedding if T is injective.

(c) X, Y are said to be linearly isomorphic if there exists a linear isomorphism from one space to another. Moreover if there exists a linear isomorphism which is also an isometry, then we say that X, Y are isometric.

(d) We say that X linearly embeds into Y if there exists a linear embedding from X into Y. In this case, we also say that Y contains a copy of X.

(e) If X, Y are linearly isomorphic, the Banach-Mazur distance between X and Y is a number, denoted by $d_{BM}(X, Y)$, defined by

$$d_{BM}(X, Y) = \inf\{||T||\,||T^{-1}|| : T : X \to Y \text{ is a linear isomorphism}\}$$

Sometimes, when referring to the definitions above, we will omit the word linear, for example, when we say that X, Y are isomorphic, it means that X, Y are linearly isomorphic.

Let $1 \leq p < \infty$, put ℓ_p to be the vector space consisting of p-summable sequence of real numbers, i.e.

$$\ell_p = \{(x_n)_{n=1}^\infty : \sum_{n=1}^\infty |x_n|^p < \infty\}$$

The space ℓ_p is a Banach space under the norm

$$||(x_n)_{n=1}^\infty|| = \Big(\sum_{n=1}^\infty |x_n|^p\Big)^{1/p}$$

Put ℓ_∞ to be the space of all bounded sequence of real numbers. ℓ_∞ is a Banach space under the supremum norm $||(x_n)_{n=1}^\infty|| = \sup_n |x_n|$. c_0 is the closed subspace of ℓ_∞ consisting of *null sequences*, that is, it contains $(x_n)_{n=1}^\infty$ such that $\lim_n x_n = 0$. In general, if $(X_n)_{n=1}^\infty$ is a sequence of Banach spaces, $\ell_p(X_n)$ (resp. $c_0(X_n)$) consists of $(x_n)_{n=1}^\infty$ with $x_n \in X_n$ with the norm $||(x_n)_{n=1}^\infty||_p = (\sum_{n=1}^\infty ||x_n||^p)^{1/p}$ (resp. $||(x_n)_{n=1}^\infty||_\infty = \sup_n ||x_n||$). Sometimes we also write $\ell_p(X_n) = (\sum \oplus X_n)_{\ell_p}$ and $c_0(X_n) = (\sum \oplus X_n)_{c_0}$.

For $n = 1, 2, \ldots$ and $1 \le p \le \infty$, ℓ_p^n is the finite dimensional space $\mathbb{R}^n$ endowed with the ℓ_p-norm.

Let $1 \le p < \infty$ and (Ω, Σ, μ) be a measure space. $L_p(\mu)$ is the Banach space of all p-integrable functions on Ω, modulo by almost everywhere equivalence. $L_\infty(\mu)$ is the Banach space of all essentially bounded functions on Ω, modulo by almost everywhere equivalence. When the underlying measure space is $[0, 1]$ with the standard Lebesgue measure, we simply write $L_p(\mu) = L_p$. The spaces c_0, ℓ_p, L_p $(1 \le p \le \infty)$ are one of those earliest known Banach spaces and have been central in the study of Banach space theory.

A Banach space X is *prime* if whenever $E \subset X$ is an infinite dimensional complemented subspace, then E and X are isomorphic. It is well-known that ℓ_p $(1 \le p \le \infty)$ and c_0 are *prime Banach spaces* (see for example, Chapter 2 and 5 of [2]).

It is well-known that ℓ_p is complemented in L_p. To see this, take $A_1 = [0, \frac{1}{2}]$, $A_2 = [\frac{1}{2}, \frac{1}{4}]$, $A_3 = [\frac{1}{4}, \frac{1}{8}]$ and so on, then the map $L_p \to \ell_p$, $f \mapsto (\int_{A_n} f)_{n=1}^\infty$ is a projection.

Suppose f is a real-valued random variable from some probability space $(\Omega, \mathbb{P})$, we denote by $\mathbb{E}(f)$ its *expected value*, i.e.

$$\mathbb{E}(f) = \int_\Omega f(\omega)\, d\mathbb{P}(\omega)$$

The following definition concerns the expected value of $||\sum_{n=1}^m \epsilon_n x_n||^p$, where $(\epsilon_n)_{n=1}^m$ is a finite sequence of mutually independent random variable such that $\mathbb{P}(\epsilon_n = 1) = \mathbb{P}(\epsilon_n = -1) = \frac{1}{2}$. When $m = 2$, $\mathbb{E}||\sum_{n=1}^m \epsilon_n x_n||^p$ is simply the average of $||x_1 + x_2||^p, ||x_1 - x_2||^p$.

Definition 2.2. We say that a Banach space X has type p if there exists a constant $C > 0$ such that for every finite sequence $(x_n)_{n=1}^m$ in X, we have

$$\Big(\mathbb{E}\Big|\Big|\sum_{n=1}^m \epsilon_n x_n\Big|\Big|^p\Big)^{1/p} \le C\Big(\sum_{n=1}^m ||x_n||^p\Big)^{1/p}$$

where $(\epsilon_n)_{n=1}^m$ is a finite sequence of mutually independent random variables on some probability space $(\Omega, \mathbb{P})$ such that $\mathbb{P}(\epsilon_n = 1) = \mathbb{P}(\epsilon_n = -1) = \frac{1}{2}$. We say that X has cotype q if there exists a constant $C > 0$ such that for every finite sequence $(x_n)_{n=1}^\infty$ in X, we have

$$\Big(\sum_{n=1}^{m}||x_n||^q\Big)^{1/q} \leq C\Big(\mathbb{E}\Big|\Big|\sum_{n=1}^{m}\epsilon_n x_n\Big|\Big|^q\Big)^{1/q}$$

By the Kahane-Khintchine inequality [25], if X has type p, then p must be in the range $[1,2]$ and if X has cotype q, then q must be in the range $[2,\infty]$ (the definition of cotype can be naturally extended for the case $q=\infty$). Type and cotype are quantitative descriptions of how far does a space differ from a Hilbert space. Recall that the parallelogram law says that for any finite sequence $(x_n)_{n=1}^m$ in a Hilbert space,

$$\mathbb{E}\Big|\Big|\sum_{n=1}^{m}\epsilon_n x_n\Big|\Big|^2 = \sum_{n=1}^{m}||x_n||^2$$

Thus a Hilbert space has type 2 and cotype 2. Kwapien [29] showed that a Banach space is isomorphic to a Hilbert space if and only if it has type 2 and cotype 2. In general, type and cotype are invariant under isomorphisms. The type and cotype of ℓ_p and L_p are well-known (see Chapter 6 of [2]):

Proposition 2.3. *(i) Let $1 \leq p < 2$. ℓ_p and L_p have type p and cotype 2.*

(ii) Let $2 < p < \infty$. ℓ_p and L_p has type 2 and cotype p.

Definition 2.4. We say that a Banach space X has the Radon-Nikodym property (RNP) if every Lipschitz map $f:[0,1]\to X$ is differentiable almost everywhere.

Finite dimensional spaces are the simplest examples of spaces which have (RNP) because of the classical Radon-Nikodym Theorem. Every separable dual space has (RNP) (see [8]), thus L_p, ℓ_p for $1 < p < \infty$ have (RNP). L_1 and c_0 do not have the (RNP).

Definition 2.5. Let X, Y be Banach spaces. We say that

(a) X is finitely representable in Y if for any finite dimensional subspace $E \subset X$ and any $\epsilon > 0$, there exists a finite dimensional subspace $F \subset Y$ such that $d_{BM}(E,F) < 1+\epsilon$.

(b) X is crudely finitely representable in Y if there exists a constant $\lambda > 1$ such that for any finite dimensional subspace $E \subset X$ and $\epsilon > 0$, there exists a finite dimensional subspace $F \subset Y$ such that $d_{BM}(E,F) < \lambda+\epsilon$.

It is not difficult to show that every Banach space is finitely representable in c_0. To see this, suppose $\epsilon > 0$ and $E \subset X$ is a finite dimensional subspace. Let $\{e_1^*,\ldots,e_N^*\}$ be a

ν-net (ν is later defined) in B_{E^*}. Define $T : E \to c_0$ to be

$$T(x) = (e_1^*(x), e_2^*(x), \dots, e_N^*(x), 0, 0, \dots)$$

Then T is clearly bounded above by 1. Moreover, for $x \in E$, choose $x^* \in B_{E^*}$ with $x^*(x) = ||x||$. Now find some e_k^* with $||e_k^* - x^*|| \leq \nu$.

$$\begin{aligned} ||T(x)|| &\geq |e_k^*(x)| \\ &\geq |x^*(x)| - |e_k^*(x) - e_k^*(x)| \end{aligned}$$

This gives $||T(x)|| \geq ||x||(1-\nu)$. Hence $||T||\,||T^{-1}|| \leq (1-\nu)^{-1}$. The proof is finished by choosing ν to be small.

Using the Principle of Local Reflexivity (see Chapter 12 of [2]), it is deduced that for every Banach space X, X^{**} is always finitely representable in X. Next we introduce *superreflexivity.* This notion was first introduced by James [18] in 1972.

Definition 2.6. A Banach space X is superreflexive if every Banach space Y that is finitely representable in X is reflexive.

Proposition 2.7 (see, for example, Chapter 12 of [2])**.** *Let X, Y be Banach spaces. Suppose X is crudely finitely representable in Y and Y is superreflexive, then X is superreflexive.*

Remark 2.8. *The above shows that superreflexive is preserved under crude finite representability. The result is not true if we replace superreflexivity by reflexivity. To see this, take $X = \ell_1$ or c_0 (a non reflexive sequence space), then X is finitely representable in $(\sum \oplus \ell_1^n)_{\ell_2}$ or $(\sum \oplus \ell_\infty^n)_{\ell_2}$.*

Now, let us introduce the concept of an ultraproduct of a Banach space. Let $\mathcal{I}$ be a set. A collection $\mathcal{U}$ of subsets of $\mathcal{I}$ is called an ultrafilter on $\mathcal{I}$ if

(i) $\varnothing \notin \mathcal{U}$.

(ii) If $A \in \mathcal{U}$ and $A \subset B$ then $B \in \mathcal{U}$.

(iii) If $A, B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$.

(iv) For any $A \subset \mathcal{I}$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Let $f:\mathcal{I}\to X$ be a function. We say that f converges to $x\in X$ through $\mathcal{U}$ if $f^{-1}(\mathcal{U})\in\mathcal{U}$ for every open neighbourhood U of x. In this case we put $\lim_{\mathcal{U}} f = x$.

In practice, we usually consider $\mathcal{I}=\mathbb{N}$, so that every function $f:\mathbb{N}\to X$ is viewed as a sequence. The standard concept of sequential limit corresponds to the convergence through the ultrafilter $\mathcal{F}_\infty$ consisting of all subsets A such that A contains $[n,\infty)\cap\mathbb{N}$ for some n. A free ultrafilter on $\mathbb{N}$ is an ultrafilter which contains $\mathcal{F}_\infty$. Some standard properties of convergence through a free ultrafilter are as follows:

Proposition 2.9. *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Let $(a_n)_{n=1}^\infty$ be a sequence of scalars and $(x_n)_{n=1}^\infty,(y_n)_{n=1}^\infty$ be sequences in X. We have*

(i) If $(a_n)_{n=1}^\infty$ is a bounded sequence, then it converges through $\mathcal{U}$.

(ii) If $(a_n)_{n=1}^\infty$ is bounded and $\lim_{\mathcal{U}} x_n = 0$ then $\lim_{\mathcal{U}} a_nx_n = 0$.

(iii) If $(x_n)_{n=1}^\infty,(y_n)_{n=1}^\infty$ converge through $\mathcal{U}$, then $\lim_{\mathcal{U}} x_n+y_n = \lim_{\mathcal{U}} x_n + \lim_{\mathcal{U}} y_n$.

(iv) If $(x_n)_{n=1}^\infty$ converges to $x\in X$ in the usual order, then $\lim_{\mathcal{U}} x_n = x$.

Put $\ell_\infty(X)$ to be the space of all bounded sequence. According to Proposition 2.9, we can define a semi-norm $||\cdot||_{\mathcal{U}}$ on $\ell_\infty(X)$ by

$$||(x_n)_{n=1}^\infty||_{\mathcal{U}} := \lim_{\mathcal{U}} ||x_n||$$

It follows that when quotiented by $c_{0,\mathcal{U}}(X)$ the space of all sequences $(x_n)_{n=1}^\infty$ such that $\lim_{\mathcal{U}}||x_n||=0$, the space $\ell_\infty(X)/c_{0,\mathcal{U}}(X)$ becomes a Banach space. In this case, the space $\ell_\infty(X)/c_{0,\mathcal{U}}(X)$ is called the ultraproduct of X with respect to the ultrafilter $\mathcal{U}$ and is denoted by $X_{\mathcal{U}}$.

2.2 Various category of non-linear maps

The following classes of functions are the main objects studied throughout this paper. In this section, unless otherwise stated, M,N are metric spaces and their metrics are denoted by d,ρ respectively.

Definition 2.10. Let $(M,d),(N,\rho)$ be metric spaces. $f:M\to N$ be a map. We say that

(a) f is K-Lipschitz if for every $x,y\in M$,

$$\rho(f(x),f(y))\le Kd(x,y)$$

(b) f is Lipschitz if f is K-Lipschitz for some $K > 0$. In this case, put

$$\mathrm{Lip}(f) = \sup\left\{\frac{\rho(f(x), f(y))}{d(x,y)} : x, y \in M,\ x \neq y\right\}$$

(c) f is a Lipschitz embedding if f is Lipschitz, injective and $f^{-1} : f(N) \to M$ is Lipschitz, i.e. there exist $C_1, C_2 > 0$ such that for every $x, y \in M$,

$$C_1 d(x,y) \leq \rho(f(x), f(y)) \leq C_2 d(x,y)$$

In this case, put $\mathrm{dist}(f) = \mathrm{Lip}(f)\mathrm{Lip}(f^{-1})$. We call $\mathrm{dist}(f)$ the distortion constant of f.

(d) f is a Lipschitz isomorphism if f is a surjective Lipschitz embedding.

(e) f is uniformly continuous if for every $\epsilon > 0$, there exist $\delta > 0$ such that for every $x, y \in M$ with $d(x,y) < \delta$, we have $\rho(f(x), f(y)) < \epsilon$.

(f) f is a uniform homeomorphism if f is uniformly continuous, bijective and f^{-1} is uniformly continuous.

(g) f is coarsely Lipschitz if the number $\mathrm{Lip}_\theta(f)$ is finite for some θ, where

$$\mathrm{Lip}_\theta(f) := \sup\left\{\frac{\rho(f(x), f(y))}{d(x,y)} : x, y \in M, d(x,y) \geq \theta\right\}$$

(h) f is a corase Lipschitz embedding if there exists $\theta > 0$ and $C_1, C_2 > 0$ such that for every $x, y \in M$ with $d(x,y) \geq \theta$,

$$C_1 d(x,y) \leq \rho(f(x), f(y)) \leq C_2 d(x,y)$$

Definition 2.11. Let $f : M \to N$ be a map. The modulus of continuity of f is a function $\omega_f : (0, \infty) \to [0, \infty]$ defined by

$$\omega_f(\theta) := \sup\left\{\rho(f(x), f(y)) : x, y \in M, d(x,y) \leq \theta\right\}$$

Proposition 2.12. *Suppose $\omega : [0, \infty) \to [0, \infty]$ is a function such that*

$$\rho(f(x), f(y)) \leq \omega(d(x,y))$$

for every $x, y \in M$ and $w(\theta) \to 0$ as $\theta \to 0+$, then f is uniformly continuous.

Proof. Let $\epsilon > 0$. Choose $\theta > 0$ such that $\omega(\theta) < \epsilon$. Suppose $x, y \in M$ satisfy $d(x, y) < \theta$, then

$$\rho(f(x), f(y)) \leq \omega(d(x, y)) < \omega(\theta) < \epsilon$$

□

Proposition 2.13. *f is K-Lipschitz if and only if $\omega_f(\theta) \leq K\theta$ for all $\theta > 0$.*

Proof. Suppose f is K-Lipschitz. Let $\theta > 0$, suppose $x, y \in M$ satisfy $d(x, y) \leq \theta$, then

$$\rho(f(x), f(y)) \leq Kd(x, y) \leq K\theta$$

Taking supremum over all possible choice of (x, y) gives $\omega_f(\theta) \leq K\theta$. Suppose $\omega_f(\theta) \leq K\theta$ for all θ. Let $x, y \in M$. Then

$$\rho(f(x), f(y)) \leq \omega_f(d(x, y)) \leq Kd(x, y)$$

□

Proposition 2.14. *Let $f : M \to N$ be uniformly continuous on a metrically convex space M, then $\omega_f(\theta) < \infty$ for all $\theta > 0$.*

Proof. Let $\epsilon = 1$. By the uniform continuity of f, there exists $\delta > 0$ such that for every $x, y \in M$ with $d(x, y) < \delta$, we have $\rho(f(x), f(y)) < 1$.

Let $\theta > 0$. Suppose that $x, y \in M$ satisfy $d(x, y) \leq \theta$. Fix $L > 0$ be a large number such that $\theta/L < \delta$.

Because M is metrically convex, there are points $x = x_1, x_2, \ldots, x_m = y$ such that $d(x_i, x_{i+1}) \leq d(x, y)/N < \delta$. Hence

$$\rho(f(x), f(y)) = \sum_{i=1}^{m-1} \rho(f(x_i), f(x_{i+1})) \leq N$$

Because N is independent of x, y, taking supremum over all choices of (x, y) finishes the proof. □

Remark 2.15. *It should be noted that a coarse Lipschitz map needs not be continuous, and a coarse Lipschitz embedding needs not be injective. f is coarsely Lipschitz if $Lip_\infty(f) < \infty$, where $Lip_\infty(f) := \lim_{\theta \to \infty} Lip_\theta(f)$.*

Definition 2.16. Let $\alpha, \beta > 0$. Let M be a metric space and $\mathcal{N} \subset M$ be a subset. We say that $\mathcal{N}$ is an α-separated β-net if

(i) $\inf\{d(s_1, s_2) : s_1, s_2 \in \mathcal{N}, s_1 \neq s_2\} \geq \alpha$, and

(ii) $\sup\{d(x, \mathcal{N}) : x \in M\} \leq \beta$.

A subset $\mathcal{N} \subset M$ is called a separated net if it is an α-separated β-net for some $\alpha, \beta > 0$.

Definition 2.17. Let M, N be metric spaces. We say that M, N are net equivalent if there are separated nets $\mathcal{N}_M \subset M, \mathcal{N}_N \subset N$ and a Lipschitz isomorphism $\varphi : \mathcal{N}_M \to \mathcal{N}_N$.

Lemma 2.18. *Every metric space contains an α-separated α-net for all $\alpha > 0$.*

Proof. Let M be a metric space. Let $\mathcal{P}$ be the collection of all $\mathcal{N} \subset M$ α-separated subsets. Then $\mathcal{P}$ is a partially ordered set by inclusion and for every chain $\mathcal{C}$ in $\mathcal{P}$, we can find a maximal element $\cup_{\mathcal{N} \in \mathcal{C}} \mathcal{N}$ in $\mathcal{P}$. A standard application of Zorn's lemma yields a maximal element $\mathcal{N} \in \mathcal{P}$. Suppose $\mathcal{N}$ is not an α-net, then we may add an element to $\mathcal{N}$ and contradicts its maximality. $\square$

We will frequently assume that the underlying metric spaces are metrically convex. This property to some extent resembles an important property in a normed space, namely, if X is a normed space, $x, y \in X$, then X contains every point of the line segment joining x and y.

Definition 2.19. Let M be a metric space. We say that M is metrically convex if for every $x, y \in M$ and $0 < \lambda < 1$, there exists $z_\lambda \in M$ with

$$d(x, z_\lambda) = \lambda d(x, y) \text{ and } d(y, z_\lambda) = (1 - \lambda) d(x, y)$$

The following proposition shows that every uniformly continuous map on a metrically convex metric space is automatically satisfies $\mathrm{Lip}_\theta(f) < \infty$ for all θ.

Proposition 2.20. *Let $f : M \to N$ be uniformly continuous. Assume that M is metrically convex. For every $\theta > 0$, there exists $K_\theta > 0$ such that $\rho(f(x), f(y)) \leq K_\theta d(x, y)$ whenever $d(x, y) \geq \theta$.*

Proof. Let $\theta > 0$. Suppose $x, y \in M$ are two points with $d(x, y) \geq \theta$. Let $m \in \mathbb{N}$ be the smallest positive integer such that $d(x, y)/m < \theta$. Because M is metrically convex, we may find $x = x_1, x_2, \ldots, x_m = y$ with $d(x_i, x_{i+1}) \leq \theta$ for all i. Then

$$\rho(f(x), f(y)) = \sum_{i=1}^{m-1} \rho(f(x_i), f(x_{i+1})) \leq m\omega_f(\theta)$$

Since m is the smallest integer such that $d(x, y)/m < \theta$, we must have $d(x, y)/2m \geq \theta$, so $d(x, y)/2\theta \geq m$. Hence

$$\rho(f(x), f(y)) \leq m\omega_f(\theta) \leq \frac{\omega_f(\theta)}{2} d(x, y)$$

□

Next, we proceed to study the relationship between coarse Lipschitz maps and net structures. These properties both capture the structure of a metric space at *large distance*. It turns out that they are closely related to each other. Before making this statement formal and proving it, let us state two lemmas needed.

Lemma 2.21. *Let M, N be unbounded metric spaces. Let $f : M \to N$ be a map. Suppose that there are $A, B > 0$ such that for all $x, y \in M$,*

$$\frac{1}{A} d(x, y) - B \leq \rho(f(x), f(y)) \leq Ad(x, y) + B$$

Then f is a coarse Lipschitz embedding.

Proof. Let $\theta > 0$ and suppose $x, y \in M$ satisfy $d(x, y) > \theta$. Then

$$\rho(f(x), f(y)) \leq Ad(x, y) + B \leq (A + B\theta^{-1})d(x, y)$$

By restricting $\theta > AB$, we have also

$$\rho(f(x), f(y)) \geq \frac{1}{A} d(x, y) - B \geq (A^{-1} - B\theta^{-1})d(x, y)$$

Hence f is a coarse Lipschitz embedding. □

Lemma 2.22. *Using the notations in Lemma 2.21. Assume further now M is metrically convex. If f is a coarse Lipschitz embedding, then there exist $A, B > 0$ such that*

$$\frac{1}{A} d(x, y) - B \leq \rho(f(x), f(y)) \leq Ad(x, y) + B$$

for all $x, y \in M$.

Proof. Suppose f is a coarse Lipschitz embedding, then there exists $\theta > 0$, $c_1, c_2 > 0$ such that for $x, y \in M$ with $d(x, y) \geq \theta$ we have

$$c_1 d(x, y) \leq \rho(f(x), f(y)) \leq c_2 d(x, y)$$

The inequality in Lemma 2.22 holds for all $x, y \in M$ with $d(x, y) \geq \theta$. So suppose $d(x, y) < \theta$. Because M is unbounded and is metrically convex, we may choose $z \in M$ such that $d(y, z) = 2\theta$. Now $d(x, z) > \theta$ and

$$\begin{aligned} \rho(f(x), f(y)) &\leq \rho(f(x), f(z)) + \rho(f(z), f(y)) \\ &\leq c_2 d(x, z) + c_2 d(z, y) \\ &\leq c_2 d(x, y) + 2c_2 d(z, y) \\ &\leq c_2 d(x, y) + 4c_2\theta \end{aligned}$$

Similarly, we calculate

$$\rho(f(x), f(y)) \geq c_1 d(x, y) - 2(c_1 + c_2)\theta$$

By proof is finished by choosing appropriate constants $A, B > 0$. □

Proposition 2.23. *Let M, N be unbounded metrically convex spaces. The followings are equivalent:*

(i) M, N are net equivalent.

(ii) There is a coarse Lipschitz embedding $f : M \to N$ with $\sup_{y\in N} d(y, f(M)) < \infty$.

Proof. $(i) \Rightarrow (ii)$. Let $\mathcal{N}_M$ be an α-separated β-net of M, $\mathcal{N}_N$ be a δ-separated γ-net of N and $\varphi : \mathcal{N}_M \to \mathcal{N}_N$ be a Lipschitz isomorphism.

First, to each $x \in M$, we may fix $s_x \in \mathcal{N}_M$ with $d(x, s_x) \leq \beta$. Define $f : M \to \mathcal{N}_N$ to be $f(x) = \varphi(s_x)$. We have

$$\rho(f(x), f(y)) = \rho(\varphi(s_x), \varphi(s_y)) \leq \mathrm{Lip}(\varphi) d(s_x, s_y) \leq \mathrm{Lip}(\varphi)(d(x, y) + 2\beta)$$

On the other hand,

$$\rho(f(x), f(y)) = \rho(\varphi(s_x), \varphi(s_y)) \geq \mathrm{Lip}(\varphi^{-1})^{-1}(d(x, y) - 2\beta)$$

By choosing $A = \max(\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi^{-1}))$ and $B = 2\beta$, we see that f is a coarse Lipschitz mapping due to Lemma 2.21. The condition $\sup_{y \in N} d(y, f(M)) < \infty$ clearly holds because $f(M) = \mathcal{N}_N$.

(ii) $\Rightarrow$ *(i)*. Let f be a coarse Lipschitz embedding from M to N. First, there exists $\theta > 0$ such that for all $x, y \in M$ with $d(x, y) \geq \theta$ we have

$$C_1 d(x, y) \leq \rho(f(x), f(y)) \leq C_2 d(x, y)$$

Let $\mathcal{N}_M$ be a θ-separated θ-net of M. By the coarse Lipschitz assumption, $\mathcal{N}_M$ and $f(\mathcal{N}_N)$ are Lipschitz isomorphic. We verify that $\mathcal{N}_N := f(\mathcal{N}_N)$ is a separated net in N.

First, for $s_1, s_2 \in \mathcal{N}_M$, $\rho(f(s_1), f(s_2)) \geq c_1\theta$. This shows $\mathcal{N}_N$ is $(c_1\theta)$-separated. If $y \in N$. there is $x \in M$ such that $\rho(f(x), y) < L$ where $L := \sup_{y \in N} d(y, f(M)) < \infty$. Choose $s \in \mathcal{N}_M$ such that $d(s, x) \leq \theta$. Now,

$$\rho(y, f(s)) \leq \rho(y, f(x)) + \rho(f(x), f(s)) \leq L + A\theta + B$$

where A, B are constants chosen by Lemma 2.22. Hence $\mathcal{N}_N$ is an $(L + A\theta + B)$-net. $\square$

We will now focus on non linear maps between normed spaces. The next result says that a uniform homeomorphism is a coarse Lipschitz embedding. This result is interesting because in general a uniform homeomorphism is not Lipschitz, but when we restrict ourselves at a large distance scale, the uniform homeomorphism actually becomes coarse Lipschitz.

Proposition 2.24. *Let X, Y be normed space. Suppose that $f : X \to Y$ is a uniform homeomorphism. Then f is a coarse Lipschitz embedding.*

Proof. Fix $\theta > 0$. By applying Lemma 2.22 to f, there exists K_θ such that

$$||f(x) - f(y)|| \leq K_\theta ||x - y||$$

for all $x, y \in X$ with $||x - y|| \geq \theta$. Because f^{-1} is uniformly continuous, there exists $\delta > 0$ such that $||x - y|| \geq \theta$ implies $||f(x) - f(y)|| \geq \delta$. By applying Lemma 2.22 to f^{-1}, there exists K_δ such that for all $x, y \in X$ with $||f(x) - f(y)|| \geq \delta$ we have

$$||x - y|| \leq K_\delta ||f(x) - f(y)||$$

Now, for $x, y \in X$ with $||x-y|| \geq \theta$, we have

$$K_\delta^{-1}||x-y|| \leq ||f(x)-f(y)|| \leq K_\theta||x-y||.$$

$\square$

Corollary 2.25. *Every two uniformly homeomorphic normed space are net equivalent.*

Theorem 2.26 (Johnson, Linderstrauss, Schechtman [23])**.** *If X, Y are net equivalent Banach spaces, then they have Lipschitz isomorphic ultraproducts.*

Proof. Let $\mathcal{N}_X$ be a b-separated b-net, $\mathcal{N}_Y$ be a c-separated c-net and $f : \mathcal{N}_X \to \mathcal{N}_Y$ be a Lipschitz isomorphism.

Observe that for any sequence $(x_n)_{n=1}^\infty$ in X, we may find a sequence $(s_n)_{n=1}^\infty$ in $\mathcal{N}_X$ such that

$$||nx_n - s_n|| \leq b$$

If $(x_n), (\overline{x_n})$ are two sequences in X with corresponding $(s_n), (\overline{s_n})$ in $\mathcal{N}_X$, then

$$\begin{aligned}
||f(s_n) - f(\overline{s_n})|| &\leq \mathrm{Lip}(f)||s_n - \overline{s_n}|| \\
&\leq \mathrm{Lip}(f)(||nx_n - n\overline{x_n}|| + ||nx_n - s_n|| + ||n\overline{x_n} - \overline{s_n}||) \\
&\leq n\mathrm{Lip}(f)||x_n - \overline{x_n}|| + 2b\mathrm{Lip}(f)
\end{aligned}$$

That is,

$$\Big|\Big|\frac{f(s_n)}{n} - \frac{f(\overline{s_n})}{n}\Big|\Big| \leq \mathrm{Lip}(f)||x_n - \overline{x_n}|| + \frac{2b}{n}\mathrm{Lip}(f)$$

Put $\overline{x_n} = 0$, we may choose $\overline{s_n} = \overline{s}$ be a constant sequence, so

$$\Big|\Big|\frac{f(s_n)}{n} - \frac{f(\overline{s})}{n}\Big|\Big| \leq \mathrm{Lip}(f)||x_n|| + \frac{2b}{n}\mathrm{Lip}(f)$$

Thus $(f(s_n)/n)_{n=1}^\infty$ is a bounded sequence in Y. Define $F : \ell_\infty(X) \to \ell_\infty(Y)$ to be

$$F((x_n)_{n=1}^\infty) = \Big(\frac{f(s_n)}{n}\Big)_{n=1}^\infty$$

Note that if $(x_n) \in c_{0,\mathcal{U}}(X)$, then $F((x_n)) = 0$, so F may be viewed as a map from $X_\mathcal{U}$ to $Y_\mathcal{U}$. Moreover, we have

$$||F(x_1) - F(x_2)|| \leq \mathrm{Lip}(f)||x_1 - x_2||$$

Using the same method, we can obtain a Lipschitz map $G: Y_{\mathcal{U}} \to X_{\mathcal{U}}$. It remains to check $G \circ F(x) = x$ for $x \in X_{\mathcal{U}}$.

Let $(x_n)_{\mathcal{U}} \in X_{\mathcal{U}}$, $s_n \in \mathcal{N}_X$ such that $||nx_n - s_n|| \le b$. Put $y_n = f(s_n)/n$ and $t_n = f(s_n)$. Then

$$G((y_n)_{\mathcal{U}}) = \Big(\frac{f^{-1}(t_n)}{n}\Big)_{\mathcal{U}} = \Big(\frac{s_n}{n}\Big)_{\mathcal{U}} = (x_n)_{\mathcal{U}}$$

□

Corollary 2.27. *Let X, Y be Banach spaces such that X coarsely Lipschitz embeds into Y. Then for every free ultrafilter $\mathcal{U}$, $X_{\mathcal{U}}$ Lipschitz embeds into $Y_{\mathcal{U}}$.*

Corollary 2.28. *Let X, Y be Banach spaces such that X coarsely Lipschitz embeds into Y. Then for every free ultrafilter $\mathcal{U}$, X coarsely Lipschitz embeds into $Y_{\mathcal{U}}$.*

Proof. This is because $X_{\mathcal{U}}$ always contains an isometric copy of X. Consider $\iota : X \to \ell_\infty(X)$ by $\iota(x) := (x, x, \dots)$. Then $\iota(X)/c_{0,\mathcal{U}}(X)$ is isometric isomorphic to X. □

Theorem 2.29 (Ribe [19])**.** *Let X, Y be Banach space. Suppose X coarsely Lipschitz embeds into Y, then X is crudely finitely representable in Y.*

Proof. By Corollary 2.28, there is a Lipschitz embedding $f : X \to Y_{\mathcal{U}}$ for some free ultrafilter $\mathcal{U}$ of $\mathbb{N}$. By Local Reflexivity, $(Y_{\mathcal{U}})^{**}$ is finitely representable in $Y_{\mathcal{U}}$, so it suffices to show that X is crudely finitely representable in $(Y_{\mathcal{U}})^{**}$.

Let $E \subset X$ be finite dimensional. Then by using Corollary 14.2.24 in [2], there exists a linear isomorphism $T : E \to F \subset (Y_{\mathcal{U}})^{**}$ such that $||T||\,||T^{-1}|| \le \text{dist}(f)$. Therefore X is $(\text{dist}(f) + \epsilon)$-crudely finitely representable in $(Y_{\mathcal{U}})^{**}$. □

Corollary 2.30. *Let X, Y be Banach spaces, $f : X \to Y$ be coarse Lipchitz embedding. If Y has type p (respectively cotype q) then X has type p (respectively cotype q).*

Proof. Recall that Y has type p iff for any finite sequence $(y_i)_{i=1}^N$, we have

$$(\mathbb{E}\Big|\Big|\sum_{i=1}^N \epsilon_i y_i\Big|\Big|^p)^{1/p} \le T_p(Y)(\sum_{i=1}^N ||y_i||^p)^{1/p}$$

Suppose that X coarsely Lipchitz embeds into Y, then X is λ-crudely finitely representable in Y for some λ and Y has type p. Let $(x_i)_{i=1}^N$ be any finite sequence in X. Put $E = [x_1, \ldots, x_N]$, then there is a linear isomorphism $T : E \to F \subset Y$ with $||T|| \, ||T^{-1}|| < \lambda$. Put $y_i = T(x_i)$. Now

$$\begin{aligned}
\Big(\mathbb{E}\Big|\Big|\sum_{i=1}^{N} \epsilon_i x_i\Big|\Big|^p\Big)^{1/p} &\leq ||T^{-1}||\Big(\mathbb{E}\Big|\Big|\sum_{i=1}^{N} \epsilon_i y_i\Big|\Big|^p\Big)^{1/p} \\
&\leq ||T^{-1}||T_p(Y)\Big(\sum_{i=1}^{N} ||y_i||^p\Big)^{1/p} \\
&\leq \lambda T_p(Y)\Big(\sum_{i=1}^{N} ||x_i||^p\Big)^{1/p}
\end{aligned}$$

Hence X has type p. Similarly if Y has cotype q then X has cotype q. □

Corollary 2.31. *If X coarsely Lipschitz embeds into a Hilbert space, then X is isomorphic to a Hilbert space.*

Proof. This is immediate by the Kwapien's Theorem. □

By applying Proposition 2.7, we can also show:

Corollary 2.32. *If X coarsely Lipschitz embeds into a superreflexive Y, then X is superreflexive.*

3 Lipschitz Structure of Banach spaces

In this section, we will study Lipschitz maps between Banach spaces. A natural question in non-linear geometry asks whether a pair of Lipschitz isomorphic Banach spaces is already linearly isomorphic. In general, the answer is negative. Aharoni and Linderstrauss [1] proved there exist a pair of non-separable Banach spaces which are not linear isomorphic. A follow up question to the previous result is that under what conditions would existence of Lipschitz isomorphisms implies existence of linear isomorphisms?

Let $f : X \to Y$ be a Lipschitz map. More generally, we are interested to know what properties does f have (e.g. being an isometry, quotient map, isomorphism, projection etc.) will ensure the existence of the corresponding linear map with the same property.

Here is a (not rigorous) demonstration of the idea that the existence of Lipschitz embedding implies the existence of linear embedding, provided the Lipschitz map is *differentiable* (the meaning of differentiability will be made formally).

Let $f : X \to Y$ be a Lipchitz embedding, i.e. there are $c_1, c_2 > 0$ such that

$$c_1||x_1 - x_2|| \leq ||f(x_1) - f(x_2)|| \leq c_2||x_1 - x_2||$$

Suppose that f is "differentiable" with derivative T at some point $x_0 \in X$, then

$$c_1||tu|| \leq ||f(x_0 + tu) - f(x_0)|| \leq c_2||tu||$$

This gives

$$c_1||u|| \leq ||Tu|| \leq c_2||u||$$

so that T is a linear embedding of X into Y.

In Banach space setting, two notions of differentiability, *Gateaux* differentiability and *Frechet* differentiability, are studied. The former is a generalization of *directional derivatives*. We will see that *Frechet* differentiability is always stronger than the *Gateaux* differentiability.

Definition 3.1. Let $f : X \to Y$ be a map.

(a) f is said to be Gateaux differentiable at a point $x \in X$ if there is a bounded linear operator $T : X \to Y$ such that for every $u \in X$,

$$\lim_{t \to 0} \frac{f(x + tu) - f(x)}{t} = T(u).$$

The uniquely determined operator T is called the Gateaux derivative of f at x and is denoted by $D_f(x)$. The set of points in X where f is Gateaux differentiable will be denoted by Ω_f.

(b) If for some fixed $u \in X$ the limit

$$\partial_u f(x) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t}$$

exists, then we say that f has a directional derivative at x in the direction u.

Thus f is Gateaux differentiable at x iff all the directional derivatives $\partial_u f(x)$ exist and the map $u \mapsto \partial_u f(x)$ defines a bounded linear operator from X to Y. In this case $\partial_u f(x) = D_f(x)(u)$.

(c) If the limit in (a) holds uniformly for u in the unit ball, i.e. for every $\epsilon > 0$ there is $\delta > 0$ such that for t with $0 < |t| \leq \delta$, we have: for all $u \in B_X$,

$$\Big|\Big|\frac{f(x+tu) - f(x)}{t} - T(u)\Big|\Big| < \epsilon$$

then we say that f is Frechet differentiable at $x \in X$, and the operator T is then called the Frechet derivative of f at x. So f is Frechet differentiable at x if there is a bounded linear operator $T : X \to Y$ such that

$$f(x+u) = f(x) + T(u) + o(||u||) \text{ as } ||u|| \to 0$$

By definition, it is obvious that *Frechet* differentiability is stronger than *Gateaux* differentiability. The following proposition shows that in finite dimensional settings, the notions of *Frechet* differentiability and *Gateaux* differentiability are the same. Therefore, it is often of interest to restrict ourselves in an infinite dimensional setting.

Proposition 3.2. *Let X be a finite dimensional Banach space, Y be Banach space, $U \subset X$ be open, $f : U \to Y$ be Lipschitz. Suppose that f is Gateaux differentiable at x, then f is Frechet differentiable at x.*

Proof. Let $\epsilon > 0$. Choose an ϵ-net $\{u_i\}_{i=1}^N$ for B_X. Then there is a $\delta > 0$ such that for every t with $0 < |t| \leq \delta$, we have

$$||f(x+tu_i) - f(x) - tD_f(x)(u_i)|| \leq \epsilon|t| \ \ \forall i$$

Let $u \in B_X$. Choose u_i such that $||u - u_i|| < \epsilon$. Now

$$\begin{aligned}||f(x+tu) - f(x) - tD_f(x)(u)|| \leq & ||f(x+tu) - f(x+tu_i)|| \\ & + ||f(x+tu_i) - f(x) - tD_f(x)(u_i)|| \\ & + ||tD_f(x)(u_i) - tD_f(x)(u)||\end{aligned}$$

Note RHS $\leq \epsilon|t|(2\text{Lip}(f) + 1)$. Hence f is Frechet differentiable at x. $\square$

3.1 Existence of Derivatives of Lipschitz Maps

The main goal of this section is to study the differentiability of Lipschitz maps. The motivation goes back to the Rademacher Theorem in real analysis, which states that every Lipschitz $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere. To resolve this problem in Banach space settings, there is a subtle issue to deal with. The most important difficulty is that for an arbitrary Banach space X, there are no standard *Haar* measure. In finite dimensional spaces, we may still use the standard Lebesgue measure on $\mathbb{R}^n$. But in infinite dimensional spaces, *Haar* measures do not even exist as the underlying topology is not locally compact. One way to by-pass this problem is to use an concept called *Haar null sets* introduced by Christensen [7].

In the followings, we try reproduce a differentiability theory as in the real case. The first differentiability result we want to prove is every Lipschitz $f : E \to Y$, where E is finite dimensional, Y is a Banach space with the Radon-Nikodym Property, is Gateaux differentiable outside a measure zero set.

Definition 3.3. Let E be a finite dimensional Banach space. A Lebesgue measure λ on E is the image of the standard Lebesgue measure via some isomorphism from $\mathbb{R}^n$ to E.

Remark 3.4. *Note all Lebesgue measure differs by only a constant, the class of measure zero sets in E is well-defined.*

The next three lemmas below use some convolution technique which essentially aim to smooth a Lipschitz map.

Lemma 3.5. *Let $f : E \to Y$ be a bounded map from a k-dimensional normed space E into a Banach space Y. Take a bump function $\varphi \in C^\infty(E, \mathbb{R})$ that is everywhere positive, compactly supported and has integral 1. Define*

$$g_n(x) = 2^{nk} \int_E f(x - \xi)\varphi(2^n\xi)\, d\lambda(\xi), \quad x \in E, n \in \mathbb{N}$$

Then $\lim_n g_n(x) = f(x)$ *at each Lebesgue point* x *of* f*. In particular,* $\lim_n g_n(x) = f(x)$ *a.e.* $x \in E$ *and* $\lim_n g_n(x) = f(x)$ *at every point of continuity* x *of* f*.*

Lemma 3.6. *Let* $f : X \to Y$ *be a Lipschitz map between Banach spaces. Suppose* E *is a finite dimensional subspace of* X*. Define* $g : X \to Y$ *by*

$$g(x) = \int_E f(x - \xi)\varphi(\xi)\, d\lambda(\xi)$$

where φ *is a bump function. Then:*

(i) g *is Lipschitz and* $Lip(g) \leq textLip(f)$*.*

(ii) For every $x \in X$ *and for every* $u \in E$*,* $\partial_u g(x)$ *exists and* $\partial_u g : X \to Y$ *is continuous.*

Proof.

$$\begin{aligned} ||g(x) - g(y)|| &\leq \int_E ||f(x - \xi) - f(y - \xi)||\varphi(\xi)\, d\lambda(\xi) \\ &\leq textLip(f)||x - y|| \int_E \varphi(\xi)\, d\lambda(\xi) \end{aligned}$$

This shows (i). For (ii), fix x and u. Observe that by the translation invariance of λ,

$$g(x + tu) = \int_E f(x - \xi)\varphi(tu + \xi)\, d\lambda(\xi)$$

Hence

$$\frac{g(x + tu) - g(x)}{t} = \int_E f(x - \xi)\frac{\varphi(tu + \xi) - \varphi(\xi)}{t}\, d\lambda(\xi)$$

Because φ is compactly supported, so $M = \sup_{\xi \in E} |D_\varphi(\xi)(u)| < \infty$. Let $h(\xi, t)$ be the integrand in above. Then for $t \in [-1, 1] \setminus \{0\}$

$$||h(\xi, t)|| \leq M||f(x - \xi)||\chi_K(\xi)$$

where $K = supp(\varphi) + [-1, 1]u$. By Dominated Convergence Theorem, the partial derivative $\partial_u g(x)$ exists and

$$\partial_u g(x) = \int_E \lim_{t \to 0} h(\xi, t)\, d\lambda(\xi) = \int_E f(x - \xi)D_\varphi(\xi)(u)\, d\lambda(\xi)$$

By the proof of (i), $\partial_u g : X \to Y$ is Lipschitz. □

Lemma 3.7. *Let $f : X \to Y$ be Lipschitz where X, Y are Banach. Let E be a k dimensional subspace of X. Define maps (g_n) from X to Y by*

$$g_n(x) = 2^{nk} \int_E f(x - \xi)\varphi(2^n \xi)\, d\lambda(\xi), \;\; x \in X, n \in \mathbb{N}$$

where $\varphi \in C^\infty(E, \mathbb{R})$ is some bump function and λ is a Lebesgue measure on E. Then for all $x \in X$,

$$||g_n(x) - f(x)|| \leq \kappa 2^{-n} Lip(f)$$

where $\kappa = \int_E ||\xi||\varphi(\xi)\, d\lambda(\xi)$. In particular, $\lim_n g_n = f$ *uniformly on X.*

Proof. Write $f(x) = 2^{nk} \int_E f(x)\varphi(2^n\xi)\, d\lambda(\xi)$. Then

$$\begin{aligned} ||g_n(x) - f(x)|| &\leq 2^{nk} \int_E ||f(x-\xi) - f(x)||\varphi(2^n\xi)\, d\lambda(\xi) \\ &\leq \mathrm{Lip}(f) 2^{nk} \int_E ||\xi||\varphi(2^n\xi)\, d\lambda(\xi) \\ &= \kappa 2^{-n} \mathrm{Lip}(f) \end{aligned}$$

□

Theorem 3.8. *Let E be finite dimensional Banach space, Y a Banach space with the Radon-Nikodym property. Then every Lipschitz map $f : E \to Y$ is differentiable almost everywhere.*

Remark 3.9. *According to Proposition 3.2, f is Gateaux differentiable at x_0 if and only if f is Frechet differentiable at x_0. So in Theorem* 3.8*, we do not need to specify which differentiability is discussed.*

Proof. **Claim.** For each $u \in E$, $\partial_u f(x)$ exists for almost all $x \in E$.

WLOG we may assume $E = \mathbb{R}^k$, $u = e_1$ where e_i's are the canonical basis of $\mathbb{R}^k$. Since Y has (RNP), so

$$|\{\xi_1 \in \mathbb{R} : \partial_{e_1} f(\xi_1, \xi_2, \ldots, \xi_k) \text{ does not exist}\}| = 0$$

for every $(\xi_2, \xi_3 ..., \xi_k) \in \mathbb{R}^{k-1}$. Using Fubini theorem we have

$$|\{\xi \in \mathbb{R}^k : \partial_{e_1} f(\xi) \text{ does not exist}\}| = 0$$

Put Ω_u to be the set of points $x \in E$ such that $\partial_u f(x)$ exists.

Fix $u \in E$. For each $t \neq 0$, the map $E \to Y$, $x \mapsto \frac{f(x+tu)-f(x)}{t}$ is strongly measurable and bounded by $\text{Lip}(f)||u||$. It follows that there exists a strongly measurable map $S(\cdot)(u) : E \to Y$ such that $||S(x)(u)|| \leq \text{Lip}(f)||u||$ and for all $x \in \Omega_u$ we have

$$S(x)(u) = \lim_{t \to 0} \frac{f(x+tu) - f(x)}{t}$$

Fix a Lebesgue measure λ on E and let $\varphi \in C^\infty(E, \mathbb{R})$ be a bump function. Let (g_n), $g_n : E \to Y$ to be

$$g_n(x) = 2^{nk} \int_E f(x - \xi)\varphi(2^n \xi)\, d\lambda(\xi)$$

By Lemma 3.6, g_n is continuously differentiable at every $x \in E$, so

$$D_{g_n}(x)(su + tv) = sD_{g_n}(x)(u) + tD_{g_n}(x)(v) \;\; \forall s, t \in \mathbb{R}, \forall u, v \in E$$

Note $g_n(x+tu) = 2^{nk} \int_E f(x-tu-\xi)\varphi(2^n\xi)\, d\lambda(\xi)$. Using Dominated convergence theorem we have

$$D_{g_n}(x)(u) = 2^{nk} \int_E S(x - \xi)(u)\varphi(2^n \xi)\, d\lambda(\xi)$$

By Lemma 3.5, $\lim_{n\to\infty} D_{g_n}(x)(u) = S(x)(u)$ for every Lebesgue point x of $S(\cdot)(u)$

Put $L_u =$ the set of Lebesgue points of $S(\cdot)(u)$.

Choose a basis for E and let G be the $\mathbb{Q}$-span of the basis. Put $\Omega = \cap_{u\in G}\Omega_u$ and $L = \cap_{u\in G} L_u$. Then Ω, L are complement of some measure zero sets. Moreover, for every $x \in L$, for every $s, t \in \mathbb{Q}$, $u, v \in G$, we have

$$S(x)(su + tv) = sS(x)(u) + tS(x)(v)$$

The bounded linear map $S(x) : G \to Y$ therefore extends to a bounded linear map $T(x) : E \to Y$ with $||T(x)|| \leq \text{Lip}(f)$.

Claim. Let $x \in L \cap \Omega$. Then $T(x)(u) = \partial_u f(x)$ for every $u \in E$.

Fix $u \in E$. Let $\epsilon > 0$. Choose $v \in G$ such that $||u - v|| < \epsilon$. Note $\partial_v f(x) = T(x)(v) = S(x)(v)$, so there exists $\delta > 0$ such that for t with $|t| \leq \delta$ we have:

$$||f(x + tv) - f(x) - tT(x)(v)|| < \epsilon|t|$$

Now

$$\begin{aligned}||f(x+tu)-f(x)-tT(x)(u)|| \leq & ||f(x+tu)-f(x+tv)|| \\ & + ||f(x+tv)-f(x)-tT(x)(v)|| \\ & + ||tT(x)(v)-tT(x)(u)||\end{aligned}$$

So $||f(x+tu)-f(x)-tT(x)(u)|| < \epsilon|t|(2\text{Lip}(f)+1)$ for $|t| \leq \delta$. Because $L \cap \Omega$ is a complement of some measure zero sets, the theorem is proved. □

Next, we would like to generalize Theorem 3.8 for separable spaces. As discussed, we must resolve the issue of measures. Following the approach of Christensen [7], we introduce a concept called *Haar null sets* which is one way to overcome the issue that Haar measures may not exist on arbitrary Banach spaces.

Definition 3.10. A Borel subset A of a separable Banach space X is called Haar-null if ther exists a probability measure μ on the σ-algebra $B(X)$ of Borel subsets of X such that $\mu(A+x) = 0$ for all $x \in X$.

Definition 3.11. A Borel-measurable function $h : X \to [0, +\infty]$ on a separable Banach space X is said to be Haar-null if there exists a probability measure μ on $(X, B(X))$ such that

$$\int_X h(x+\xi)\, d\mu(\xi) = 0, \ \ \forall x \in X.$$

A set A is Haar-null iff χ_A is a Haar-null map. Equivalently, h is Haar-null iff there exist a probability measure space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $\eta : \Omega \to X$ such that

$$\mathbb{E}h(x+\eta) = \int_\Omega h(x+\eta(\omega))\, d\mathbb{P}(\omega), \ \ \forall x \in X.$$

We will prove several properties of *Haar null sets* and *Haar null maps.*

Lemma 3.12. *Let $h : X \to [0, +\infty]$ be measurable. Suppose there exists a finite dimensional subspace E of X such that $h(x+\xi) = 0$ a.e. $\xi \in E$ for all $x \in X$, i.e. for all $X \in X$, $\lambda(\{\xi \in E : h(x+\xi) \neq 0\}) = 0$. Then h is Haar-null.*

Proof. Choose a probability measure μ on $(X, B(X))$ such that $\mu(X \backslash E) = 0$ and $\mu(A) = 0$ iff $A \cap E$ has Lebesgue measure zero. Then

$$\int_X h(x+\xi)\, d\mu(\xi) = \int_E h(x+\xi)\, d\mu(\xi) = 0.$$

□

Proposition 3.13. *A measurable function $h : \mathbb{R}^n \to [0, \infty]$ is Haar-null iff $h = 0$ a.e..*

Proof. ($\Rightarrow$) We know that there exists a probability measure μ on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} h(x + \xi)\, d\mu(\xi) = 0$$

It follows that

$$\int_{\mathbb{R}^n} h(x)\, d\lambda(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(x + \xi)\, d\lambda(x)\, d\mu(\xi) = 0$$

The converse is due to Lemma 3.12. □

Lemma 3.14. *Let h be a Haar-null map on a separable infinite dimensional Banach space X. For every $\epsilon > 0$, there exist a probability space $(\Omega, \Sigma, \mathbb{P})$ and a random variable $\eta : \Omega \to X$ with $||\eta|| < \epsilon$ a.e. such that*

$$\mathbb{E}h(x + \eta) = 0, \;\; \forall x \in X.$$

Proof. We know that there exists a probability space $(\Omega', \Sigma', \mathbb{P}')$ and a random variable η' on Ω' such that $\mathbb{E}h(x + \eta') = 0$ for all $x \in X$.

Let $\epsilon > 0$. Because the range of η' is separable, Ω' can be written in a disjoint union of countably many subsets Ω_n such that there exists x_n

$$||x_n - \eta'(\omega)|| < \epsilon, \;\; \forall \omega \in \Omega_n$$

Choose some n such that $\mathbb{P}(\Omega_n) > 0$. The desired probability space and random variable are obtained by putting $\Omega = \Omega_n$, $\mathbb{P}$ on Ω to be defined by

$$\mathbb{P}(A) = \mathbb{P}'(A)/\mathbb{P}(\Omega_n)$$

and $\eta = \eta' - x_n$. □

Lemma 3.15. *Let X be a separable infinite-dimensional Banach space.*

(i) If $(h_n)_{n=1}^{\infty}$ is a sequence of Haar-null maps on X, then $h = \sum_{n=1}^{\infty} h_n$ is Haar-null.

(ii) If $(A_n)_{n=1}^{\infty}$ is a sequence of Haar-null sets, then $A = \cup_{n=1}^{\infty} A_n$ is also Haar-null.

Proof. For (i), for each n we may find a probability space $(\Omega_n, \mathbb{P}_n)$ and $\eta'_n : \Omega_n \to X$ such that $||\eta'_n|| < 2^{-n}$ and $\mathbb{E}h_n(x + \eta'_n) = 0$ for all $x \in X$.

Claim. There exists a probability space $(\Omega, \mathbb{P})$ and a sequence of independent variables $(\eta_n)_{n=1}^{\infty}$ on Ω satisfying $\mathbb{E}h_n(x + \eta_n) = 0$.

Construction. Let $\Omega = \Omega_1 \times \Omega_2 \times \dots$ and $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \dots$. Put $\eta_n : \Omega \to X$ to be

$$\eta_n(\omega_1, \omega_2, \dots) = \eta'_n(\omega_n)$$

Then (η_n) are mutually independent and $\mathbb{E}h_n(x + \eta_n) = 0$ for all $x \in X$.

Let $\eta = \sum_{n=1}^{\infty} \eta_n$. The series is norm convergent in X and hence η is well-defined. By the construction, we also find out that

$$\mathbb{E}h_n(x + \sum_{k=1}^{\infty} \eta_k) = \mathbb{E}h_n(x + \sum_{k \neq n} \eta_k + \eta_n) = 0$$

To see it, take for example $n = 1$. Then by the Fubini's Theorem, we have

$$\begin{aligned}\mathbb{E}h_1(x + \sum_{k=2}^{\infty} \eta_k + \eta_1) &= \int_{\Omega} h_1(x + \sum_{k=2}^{\infty} \eta_k(\omega) + \eta_1(\omega))\, d\mathbb{P}(\omega) \\ &= 0\end{aligned}$$

Therefore $\mathbb{E}h_n(x + \eta) = 0$ for all x and so $\mathbb{E}h(x + \eta) = 0$ for all $x \in X$. Hence h is a Haar-null map. (ii) follows from (i) directly. $\square$

Combining all the previous lemmas, we can prove an infinite dimensional Rademacher Theorem:

Theorem 3.16 (Infinite dimensional Rademacher Theorem)**.** *Let X be a separable Banach space and Y be a Banach space with the Radon-Nikodym Property. Let $F : X \to Y$ be a Lipschitz map. Then the set of points at which f is not Gateaux differentiable is Haar-null.*

Proof. Let $(E_n)_{n=1}^{\infty}$ be a sequence of increasing finite dimensional subspace of X such that $\cup E_n$ is dense in X. Let D_n be the set of points $x \in X$ such that there is a linear operator $T_n : E_n \to Y$ with

$$\lim_{t \to 0} \frac{f(x + tu) - f(x)}{t} = T_n(u), \;\; \forall u \in E_n$$

Put $A_n = X \setminus D_n$. For each z, put $f_z : X \to Y$ to be $f_z(x) = f(x - z)$. Then

$$(z + D_n) \cap E_n = \{x \in E_n : f_z|_{E_n} \text{ is Gateaux differentiable at } x\}$$

By Theorem 3.8, $(z+A_n) \cap E_n$ has measure zero. By Lemma 3.12 A_n is a Haar-null subset of X. By Lemma 3.15, $\cup A_n$ is Haar-null.

Claim. f is Gateaux differentiable at x iff $x \in \cap D_n$.

If f is Gateaux differentiable at x then x is certainly in $\cap D_n$. Conversely, let $x \in \cap D_n$. Then for each n there is a linear map $T_n : E_n \to Y$ such that

$$\lim_{t \to 0} \frac{f(x + tu) - t(x)}{t} = T_n(u)$$

By definition, T_{n+1} extends T_n and so there is a linear map $T : X \to Y$ with $||T|| \leq \mathrm{Lip}(f)$ and $T|_{E_n} = T_n$. Fix $u \in X$, we claim that for all $u \in X$,

$$T(u) = \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t}$$

Let $\epsilon > 0$. Let $v \in \cup E_n$ such that $||u - v|| < \epsilon$. Then there is $\delta > 0$ such that for $|t| \leq \delta$, we have

$$||f(x + tv) - f(x) - tT(v)|| \leq \epsilon |t|$$

It follows that for all $|t| \leq \delta$:

$$||f(x + tu) - f(x) - tT(u)|| \leq \epsilon |t| (2\mathrm{Lip}(f) + 1)$$

□

Remark 3.17. *Theorem 3.16 fails if Gateaux differentiability is replaced by Frechet differentiability. Consider the map $f : \ell_2 \to \ell_2$ defined by*

$$f(x) = (|x_1|, |x_2|, \dots)$$

for $(x_1, x_2, \dots) \in \ell_2$. Clearly f is an isometry and hence a Lipschitz map. We claim that f is nowhere Frechet differentiable.

Proof. Let $x = (x_1, x_2, \dots) \in \ell_2$. Note that f is Gateaux differentiable at x if and only if $x_n \neq 0$ for all n. We may assume that $x_n \neq 0$ for all n. Then for all $u \in \ell_2$ we have

$$D_f(x)(u) = (u_1 \mathrm{sgn}(x_1), u_2 \mathrm{sgn}(x_2), \dots)$$

Suppose that f is Frechet differentiable at x. For $\epsilon = 1$, there exists $\delta > 0$ such that for all $|t| \leq \delta$,

$$\Big|\Big|\frac{f(x+tu)-f(x)}{t} - D_f(x)(u)\Big|\Big|_{\ell_2} \leq \epsilon$$

for all $u \in B_{\ell_2}$. Pick $u = e_k$, $k = 1, 2, \ldots$, then

$$\Big|\frac{|x_k + t| - |x_k|}{t} - \operatorname{sgn}(x_k)\Big| < \epsilon$$

WLOG assume $\operatorname{sgn}(x_k) = 1$ for all k. Take $t = -\delta$. For large k, we have $|x_k - \delta| = \delta - x_k$, so

$$\Big|\frac{\delta - x_k - x_k}{-\delta} - 1\Big| < \epsilon$$

i.e. $|-2 + \frac{x_k}{2\delta}| < \epsilon$ for all k. Take $k \to \infty$, then $x_k \to 0$ and so $2 < \epsilon$. Contradiction. $\square$

Finally, we will prove two *Lipschitz invariant properties* of Banach spaces, namely separability and reflexivity.

Proposition 3.18. *Let X, Y be Lipschitz isomorphic Banach spaces. Suppose X is separable, then Y is separable.*

Proof. This is immediate. $\square$

Proposition 3.19. *Let X, Y be Lipschitz isomorphic Banach spaces. Suppose X is reflexive, then Y is reflexive.*

Proof. Let X, Y be Banach spaces, $f : X \to Y$ be a Lipschitz isomorphism and X be reflexive. By Theorem 3.16, we know that f admits a point x_0 of Gateaux differentiability. Let $D_f(x_0) : X \to Y$ be the derivative. Note there exist constants $c_1, c_2 > 0$ such that for any $x_1, x_2 \in X$,

$$c_2||x_1 - x_2|| \leq ||f(x_1) - f(x_2) \leq c_1||x_1 - x_2||$$

We deduce that

$$c_1||x_1 - x_2|| \leq ||D_f(x_0)(x_1) - D_f(x_0)(x_2)|| \leq c_2||x_1 - x_2||$$

Therefore X is linearly embedded into Y and vice versa. Hence as a closed subspace of a reflexive space, Y is reflexive. $\square$

3.2 Lipschitz Retractions and Linear Projections

A class of functions from a Banach space X to itself, called Lipschitz retractions, will be discussed. The formal definition of Lipschitz retractions is made below. Intuitively speaking, Lipschitz retractions are *non-linear projections*. Certainly we want linear projections are always Lipschitz retractions. We see that under certain conditions, existence of Lipschitz retractions will imply existence of linear projections.

Definition 3.20. A Lipschitz map $r : Y \to Z$ from a metric space Y onto its subset $Z \subset Y$ is called a Lipschitz retraction if $r|_Z = id_Z$. A subset $Z \subset Y$ is called a Lipschitz retract of Y if there exists a Lipschitz retraction from Y onto Z.

Theorem 3.21 (Godefroy, Kalton [12]). *Let X, Y be Banach spaces and X be separable. Let $Q : Y \to X$ be continuous linear map. Suppose that Q has a Lipschitz lifting g, i.e. $g : X \to Y$ is Lipschitz and $Q \circ g = id_X$. Then there is a linear lifting T of Q and $||T|| \leq Lip(g)$.*

Now, let X be a Banach space, $Z \subset X$ a closed subspace, $q : X \to X/Z$ be the quotient map. Suppose that q admits some Lipschitz lifting $f : X/Z \to X$. Then $X, Z \oplus X/Z$ are Lipschitz isomorphic, and a Lipschitz isomorphism is given by $X \to Z \oplus X/Z$, $x \mapsto (x - f \circ q(x), q(x))$, whose inverse is given by $Z \oplus X/Z \to X$, $(a, b) \mapsto f(b) - a$.

Suppose further that X is now separable. By Theorem 3.21, $X, Z \oplus X/Z$ are in fact linearly isomorphic, and a linear isomorphism is given by $X \to Z \oplus X/Z$, $x \mapsto (x - T \circ q(x), q(x))$ where T is a linear lifting of q. Hence in this case Z must be complemented in X.

Recall. Let $f : X \to Y$ be a Lipschitz embedding and is Gateaux differentiable at some point $x_0 \in X$. Then $D_f(x_0) : X \to Y$ is a linear embedding.

Remark 3.22. *Let X, Y Banach. Then X is Lipschitz ismorphic to a Lipschitz retract of Y iff there exist Lipschitz maps $f : X \to Y$, $g : Y \to X$ such that $g \circ f = id_X$. In this case f is a Lipschitz isomorphism between X and $f(X)$ and $f \circ g$ is a Lipschitz retraction.*

Proposition 3.23. *Let $f : X \to Y$ be a Lipschitz embedding. Assume that*

(i) f is Gateaux differentiable at some $x_0 \in X$,

(ii) X is complemented in X^{**},

(iii) $f(X)$ *is a Lipschitz retract of* Y*.*

Then $D_f(x_0)(X)$ *is a Lipschitz retract of* Y*.*

Proof. By considering $f(x+x_0) - f(x_0)$, we may assume that $x_0 = 0$ and $f(0) = 0$. Let $g : Y \to f(X)$ be Lipschitz retraction of $f(X)$, we have $g \circ f = id_X$. For all $n \in \mathbb{N}$ and for all $y_1, y_2 \in Y$ we have

$$||ng(n^{-1}y_1) - ng(n^{-1}y_2)|| \leq \mathrm{Lip}(g)||y_1 - y_2||$$

Since $g(0) = g(f(0)) = 0$, for all $y \in Y$, the sequence $(ng(n^{-1}y))_{n=1}^{\infty}$ is norm bounded in X, thus it is relatively weak* compact in X^{**}.

Fix a nonprincipal ultrafilter $\mathcal{U}$ of $\mathbb{N}$. We may define an operator $Q : Y \to X^{**}$ by

$$Q(y) = w^* \lim_{\mathcal{U}} ng(n^{-1}y)$$

Q is Lipschitz. We want to prove that $Q \circ D_f(0) = j_X$.

Let $x \in X$, put $y = D_f(0)(x)$ then

$$||ng(n^{-1}y) - x|| = n||g(n^{-1}y) - g(f(n^{-1}x))|| \leq \mathrm{Lip}(g)||y - nf(n^{-1}x)||$$

By definition $nf(n^{-1}x) \to y$ as $n \to \infty$, so $||Q(y) - j_X(x)|| = 0$. Hence $Q \circ D_f(0) = id_X$. Finally, let $P : X^{**} \to X$ be a projection, then $D_f \circ (P \circ Q)$ is a Lipschitz retraction from Y onto $D_f(X)$. □

Lemma 3.24. *Let* Z *be a Banach space and* E *be finite dimensional. Let* $f : E \to Z$ *be Lipschitz,* $E_0 \subset E$ *be a subspace such that* $f|_{E_0}$ *is linear. Then there is a linear* $T : E \to Z^{**}$ *such that* $T|_{E_0} = j_Z \circ f|_{E_0}$ *and* $||T|| \leq Lip(f)$*. (Recall that* $j_Z : Z \to Z^{**}$ *is the natural embedding)*

Proof. Fix a Lebesgue measure λ on E_0. Choose $\varphi \in C^\infty(E, \mathbb{R})$ with $\varphi \geq 0$, $\int_{E_0} \varphi \, d\lambda = 1$ and $\varphi(x) = \varphi(-x)$. Put

$$g(x) = \int_{E_0} f(x - \xi)\varphi(\xi) \, d\lambda(\xi), \;\; x \in E$$

By Lemma 3.6, g is Lipschitz, $\mathrm{Lip}(g) \leq \mathrm{Lip}(f)$ and $\partial_u g$ exists for all $u \in E_0$. Let $x \in E_0$, then

$$g(x) = \int_{E_0} (f(x) - f(\xi))\varphi(\xi)\, d\lambda(\xi) = f(x)$$

Choose a decomposition $E = E_0 \oplus E_1$ and a Lebesgue measure μ for E_1. Choose $\psi \in C^\infty(E_1, \mathbb{R})$ with $\int_{E_1} \psi\, d\mu = 1$ and $\psi \geq 0$. Define $g_n : E \to Z$ by

$$g_n(x) = 2^{nk} \int_{E_1} g(x - \xi)\psi(2^n \xi)\, d\mu(\xi)$$

where $k = \dim(E_1)$. By Lemma 3.7, g_n is Lipschitz, $\mathrm{Lip}(g_n) \leq \mathrm{Lip}(g)$ and $\partial_u g_n$ exists and is continuous for all $u \in E_1$. On the other hand, if $u \in E_0$, by Bounded Convergence Theorem,

$$\partial_u g_n(x) = 2^{nk} \int_{E_1} \partial_u g(x - \xi)\psi(2^n \xi)\, d\mu(\xi)$$

Therefore $\partial_u g_n$ exists for all $u \in E$ and is continuous, i.e. $g_n \in C^1(E, \mathbb{R})$. Put $T_n : E \to Z$ to be $T_n(u) = \partial_u g_n(0)$. Note $||T_n(u)|| \leq \mathrm{Lip}(g_n)||u|| \leq \mathrm{Lip}(f)||u||$. The sequence $(T_n(u))_{n=1}^\infty$ is norm bounded in Z, so we may define an operator $T : E \to Z^{**}$ by

$$T(u) = w^* \lim_{\mathcal{U}} T_n(u)$$

where $\mathcal{U}$ is some fixed nonprincipal ultrafilter on $\mathbb{N}$.

We need to check $T|_{E_0} = j_X \circ f|_{E_0}$. If $u \in E_0$, then $T_n(u) = 2^{nk} \int_{E_1} \partial_u g(0 - \xi)\psi(2^n \xi)\, d\mu(\xi)$. By Lemma 3.7,

$$\lim_{n\to\infty} T_n(u) = \partial_u g(0) = \lim_{t\to 0} \frac{g(0 + tu) - g(0)}{t} = f(u)$$

Hence

$$T(u) = w^* \lim_{\mathcal{U}} T_n(u) = w^* \lim_{n\to\infty} T_n(u) = j_X(f(u))$$

□

Theorem 3.25. *Let Y be separable, $Z \subset Y$ closed subspace. If Z is a Lipschitz retract of Y and Z is complemented in Z^{**}, then Z is complemented in Y.*

Proof. Let $r : Y \to Z$ be a Lipschitz retraction. Choose an increasing sequence of finite dimensional subspaces (E_n) of Y such that $\cup E_n$ is dense in Y and $\cup(E_n \cap Z)$ is dense in Z.

For each n consider the map $r|_{E_n} : E_n \to Z$. Note $r|_{E_n \cap Z} = id_{E_n \cap Z}$ is linear. Apply Lemma 3.24 to $r|_{E_n}$, we obtain a linear $T_n : E_n \to Z^{**}$ such that $T_n|_{E_n \cap Z} = j_Z \circ r|_{E_n \cap Z}$ and $||T_n|| \leq \text{Lip}(r|_{E_n}) \leq \text{Lip}(r)$

We want to define an operator $T : Y \to Z^{**}$. Let $y \in Y$, choose a sequence (y_n), $y_n \in E_n$, $y_n \to y$. Then $(T_n(y_n))_{n=1}^{\infty}$ is norm bounded in Z^{**}. Define

$$T(y) = w^* \lim_{\mathcal{U}} T_n(y_n)$$

We must check that T is well-defined, i.e. RHS is independent of the choice of (y_n). Suppose $(y_n), (\hat{y_n})$, $y_n, \hat{y_n} \in E_n$ and both converge to y. Then $\lim_{n\to\infty} T_n(y_n) - T_n(\hat{y_n}) = 0$. So $w^* \lim_{\mathcal{U}} T_n(y_n) = w^* \lim_{\mathcal{U}} T_n(\hat{y_n})$. We have T is linear and $||T|| \leq \text{Lip}(r)$.

We claim that $T|_Z = j_Z$. Pick $z \in Z$ and choose a sequence (z_n), $z_n \in E_n \cap Z$ such that $z_n \to z$. By assumption, $T_n(z_n) = j_Z(z_n)$. Because $j_Z(z_n)$ converges to $j_Z(z)$ in norm, so certainly it converges in weak* topology. Hence

$$T(z) = w^* \lim_{\mathcal{U}} T_n(z_n) = w^* \lim_{\mathcal{U}} j_Z(z_n) = j_Z(z)$$

Finally, let $P : Y^{**} \to Y$ be a linear projection. Then $P \circ T$ is a linear projection from T onto Z. □

3.3 Unique Lipschitz Structures of classical Banach Spaces

We apply results from the previous section to show that under certain conditions on X and Y, Lipschitz isomorphisms are enough to recover the linear structure.

First we study the class of separable reflexive Banach spaces. This class is of interest of study because every separable reflexive space Y always has the Radon-Nikodym Property (RNP). Using differentiation results in Theorem 3.16, every Lipschitz isomorphism $f : X \to Y$ admits a point x_0 of Gateaux differentiability.

Theorem 3.26. *Let X, Y be separable, Y has (RNP) and X is complemented in X^{**}. If X is Lipschitz isomorphic to a Lipschitz retract of Y, then X is linearly isomorphic to a complemented subspace of Y.*

Proof. Let $f : X \to Y$, $g : Y \to X$ be Lipschitz such that $g \circ f = id_X$. By Theorem 3.16, f must be Gateaux differentiable at a point $x_0 \in X$. Note $f(X)$ is a Lipschitz retract of Y, by Proposition 3.23, $D_f(x_0)(X)$ is a Lipschitz retract of Y. By Theorem 3.25 $D_f(x_0)(X)$ is complemented is Y $\square$

Corollary 3.27. *Let X, Y be separable reflexive Banach spaces. Suppose that X, Y are Lipschitz isomorphic, then X is linearly isomorphic to a complemented subspace of Y and Y is linearly isomorphic to a complemented subspace of X.*

Recall the Pelczynski's Theorem:

Theorem 3.28 (Pelczynski [32])**.** *Let X, Y be Banach spaces. Suppose that X is linearly isomorphic to a complemented subspace of Y and Y is linearly isomorphic to a complemented subspace of X. Suppose either*

(i) $X \approx X \oplus X$ *and* $Y \approx Y \oplus Y$*, or*

(ii) $X \approx c_0(X)$ *or* $\ell_p(X)$ *for some* $1 \leq p \leq \infty$*.*

Then X and Y are linearly isomorphic.

Hence with Theorem 3.28, we have:

Theorem 3.29. *Let X, Y be separable reflexive Banach spaces satisfying the Pelczynski's decomposition criteria (as in Theorem 3.28). If X, Y are Lipschitz isomorphic then X, Y are linearly isomorphic.*

As an immediate result, we see that $\ell_p (1 < p < \infty)$ have unique Lipschitz structure.

Corollary 3.30. *Let $1 < p < \infty$. Suppose X is a Banach space Lipschitz isomorphic to ℓ_p (resp. L_p). Then X is linearly isomorphic to ℓ_p (resp. L_p).*

Theorem 3.31. *If X is Lipschitz isomorphic to ℓ_1 and is a dual space. Then X is linearly isomorphic to ℓ_1.*

Proof. X is a dual space implies that X must be complemented in its bidual. Note ℓ_1 has (RNP). Since ℓ_1 is clearly a Lipschitz retract of ℓ_1, by Theorem 3.25, X is linearly isomorphic to a complemented subspace of ℓ_1. But ℓ_1 is a prime Banach space, hence X must be linearly isomorphic to ℓ_1. $\square$

3.4 Asymptotic Uniform Smoothness

In the followings, we will focus on a Banach space property called *Asymptotic Uniform Smoothness.* This notion was originated from Milman's paper [31]. In his paper, Milman introduced many quantities in order to generalize the notion of *uniform smoothness* of Banach spaces. Of which the following quantity was extensively studied:

Definition 3.32. Let X be a Banach space. If $||x|| = 1,\ t > 0,$ and Y is a closed subspace of X, put

$$\bar{\rho}(t, x, Y) = \sup_{\substack{y \in Y \\ ||y|| \leq t}} ||x + y|| - 1$$

and

$$\bar{\rho}(t, x) = \inf_{\substack{Y \subset X \\ \dim(X/Y) < \infty}} \bar{\rho}(t, x, Y)$$

where the infimum is taken over all finite co-dimensional subspace Y of X. Finally, put

$$\bar{\rho}(t) = \sup_{x \in S_X} \bar{\rho}(t, X)$$

The function $\bar{\rho} = \bar{\rho}_X$ is called the modulus of asymptotic uniform smoothness of X. A Banach space X is called asymptotic uniformly smooth (in short: AUS) if $\lim_{t \to 0} \bar{\rho}(t)/t = 0$.

For finite dimensional spaces F, we have always $\bar{\rho}_F(t) = 0$ for all $t \in (0, 1]$. This is because the zero subspace $\{0\}$ is finite co-dimensional in this case. In relation to Lipschitz isomorphisms, we will prove that under some assumption, if X, Y are Lipschitz isomorphic and X is AUS, then Y admits an equivalent AUS norm.

Remark 3.33. *Recall that the modulus of uniform smoothness of X, denoted by ρ_X, is defined by*

$$\rho_X(t) = \sup_{\substack{||x||=1 \\ ||y|| \leq t}} \frac{1}{2}(||x + y|| + ||x - y||) - 1$$

A Banach space X is called uniformly smooth if $\lim_{t \to 0} \rho_X(t)/t = 0$. The modulus of uniform smoothness ρ_X is always a non-negative 1-Lipschitz function. We will prove three properties concerning $\bar{\rho}$ (see [20]):

(i) $\bar{\rho}$ is non-negative increasing 1-Lipschitz function.

(ii) If $X_0 \subset X$ then $\bar{\rho}_{X_0}(t) \leq \bar{\rho}_X(t)$.

(iii) $\bar{\rho}_X(t) \leq 2\rho_X(t)$.

Proof. To prove (i), the part for 1-Lipschitz follows easily from triangle inequality. The fact that $\bar{\rho}$ is increasing is by definition. To prove it is non-negative, let $||x|| = 1$, $Y \subset X$ is finite-codimensional, $||y|| \leq t$. Then

$$1 = ||x|| \leq \frac{1}{2}(||x+y|| + ||x-y||)$$

We have

$$-(||x-y|| - 1) \leq ||x+y|| - 1 \leq \bar{\rho}(t, x, Y)$$

Similarly

$$-\bar{\rho}(t, x, Y) \leq ||x-y|| - 1 \leq \bar{\rho}(t, x, Y)$$

It follows that $\bar{\rho}(t, x, Y) \geq 0$. Hence $\bar{\rho}(t) \geq 0$.

(ii) follows easily from definition, let us prove (iii). If $Y \subset X$ is any closed subspace, we put

$$\hat{\rho}_X(t, x, Y) = \sup_{\substack{||y|| \leq t \\ y \in Y}} \frac{1}{2}(||x+y|| + ||x-y||) - 1$$

Fix $x \in X$. If x^* is a norming functional of X, then $||x-y|| \geq 1$ for all $y \in \ker(x^*)$, we have

$$\begin{aligned}
\hat{\rho}_X(t, x, \ker(x^*)) &\geq \sup_{\substack{||y|| \leq t \\ y \in \ker x^*}} \frac{1}{2}(||x+y|| - 1) \\
&\geq \frac{1}{2}\bar{\rho}_X(t, x, \ker x^*) \\
&\geq \frac{1}{2}\bar{\rho}_X(t, x)
\end{aligned}$$

On the other hand, it is easy to see that

$$\hat{\rho}_X(t, x, \ker x^*) \leq \rho_X(t)$$

Hence $\bar{\rho}_X(t) \leq 2\rho_X(t)$. □

Remark 3.34. *According to Remark 3.33, a uniformly smooth space is always asymptotically uniformly smooth.*

In the following we explicitly calculate $\bar{\rho}_X$ where $X = \ell_p$ $(1 \leq p < \infty)$. The formula gives another proof that the norm on ℓ_p is AUS. (It follows from Remark 3.34 that the norm on ℓ_p is always AUS)

Example 3.35. *Let $1 \leq p < \infty$, $X = \ell_p$. Then*

$$\bar{\rho}_X(t) = (1 + t^p)^{1/p} - 1$$

Proof. Consider $x = e_1$. We first claim that $\bar{\rho}(t, e_1) \geq (1 + t^p)^{1/p} - 1$. (Hence $\bar{\rho}(t) \geq (1 + t^p)^{1/p} - 1$)

Let $Y \subset \ell_p$ be a finite-codimensional subspace. Note that there exists some $y \in Y$ with $||y|| = t$ such that $e_1^*(y) = 0$, or otherwise $e_1^* : Y \to \mathbb{R}$ would have $\ker(e_1^*) = \{0\}$, implying $e_1^*|_Y$ is one-to-one and $\dim(Y) \leq \dim(\mathbb{R})$.

Let $y \in Y$, $||y|| = t$ and $e_1^*(y) = 0$. Then we directly calculate

$$||e_1 + y|| - 1 = (1 + t^p)^{1/p} - 1$$

Therefore

$$\bar{\rho}(t, e_1, Y) \geq (1 + t^p)^{1/p} - 1$$

and hence $\bar{\rho}(t, e_1) \geq (1 + t^p)^{1/p} - 1$

Secondly, we claim that $(1 + t^p)^{1/p} - 1 \geq \bar{\rho}(t)$.

Let $\epsilon > 0$. There exists N such that $\sum_{i>N} |x_i|^p < \epsilon$. Consider $Y = [e_i : i > N]$. For any $y \in Y$ with $||y|| \leq t$, we calculate

$$||x + y|| - 1 = (\sum_{i=1}^{N} |x_i|^p + \sum_{i>N} |x_i - y_i|^p)^{1/p} - 1$$

Notice that by the triangle inequality,

$$\sum_{i>N} |x_i - y_i|^p \leq (\epsilon^{1/p} + t)^p$$

Thus

$$||x + y|| - 1 \leq (1 + (\epsilon^{1/p} + t)^p)^{1/p} - 1$$

It follows that

$$\bar{\rho}(t) \leq \bar{\rho}(t, x, Y) \leq (\epsilon^{1/p} + t)^p)^{1/p} - 1$$

Hence $\bar{\rho}(t) \leq (1 + (\epsilon^{1/p} + t)^p)^{1/p} - 1$. Taking $\epsilon \to 0$ gives the claim. $\square$

Remark 3.36. *Using the same calculation, for $X = c_0$ it is checked that $\bar{\rho}_X(t) = 0$ for all $0 < t \leq 1$. Banach spaces X with the property $\bar{\rho}_X(t) = 0$ for some $0 < t \leq 1$ are called asymptotically uniformly flat. We will have a detailed discussion concerning this class of spaces in Section 4.*

Remark 3.37. *It is an important note that AUS norm is not invariant under linear isomorphism, that is, if X, Y are linearly ismorphic Banach spaces and X is AUS, it is not necessary that Y is also AUS.*

We will give an example to prove this fact. Consider $X = c_0$ and $Y = c$ (the subspace of ℓ_∞ of all convergent sequence). First, X and Y are linearly isomorphic, $X \oplus \mathbb{R} \approx Y$. Note X is AUS (Remark 3.36).

Claim. *$\bar{\rho}_c(t) = t$ for $0 < t \leq 1$. (Hence c_0 admits an equivalent norm which is not AUS)*

Proof. Take $x_0 = (1, 1, \dots)$. Whenever $E \subset c$ is a finite co-dimensional subspace and $y \in Y$ with $||y|| = t$. If $\epsilon > 0$ then there exists N such that

$$|y_N - t| < \epsilon$$

WLOG assume $y_N > 0$ and so $y_N > t - \epsilon$. Now

$$||x_0 + y|| - 1 \geq 1 + t - \epsilon - 1 = t - \epsilon$$

This forces

$$\bar{\rho}_c(t, x_0, E) \geq t - \epsilon$$

Hence $\bar{\rho}_c(t, x) \geq t - \epsilon$ and by taking $\epsilon \to 0$ we obtain the claim. □

Using definition of an AUS norm (Definition 3.32), it is often hard to calculate or estimate the quantity $\bar{\rho}_X$. The following provides an estimate for $\bar{\rho}_X$.

Lemma 3.38. *Let X be a Banach space. Put for $t \in (0, 1]$ and $x \in S_X$*

$$\eta(t, x) = \sup \Big\{ \limsup ||x + x_n|| - 1 \Big\}$$

where the supremum runs over all weakly null sequence $(x_n)_{n=1}^{\infty}$ in X with $||x_n|| \leq t$. We have $\eta(t, x) \leq \bar{\rho}_X(t, x)$.

Proof. Suppose that (x_n) is a weakly null sequence and $||x_n|| \leq t$. If $Y \subset X$ is finite co-dimensional, then $d(x_n, Y) \to 0$ as $n \to \infty$.

So, for all $\epsilon > 0$, there exists $y_n \in T$ such that $||x_n - y_n|| < \epsilon$ for all large n. Therefore $||y_n|| \leq t + \epsilon$. We calculate

$$||x + x_n|| - 1 \leq ||x + y_n|| + ||x_n - y_n|| - 1 \leq \bar{\rho}(t + \epsilon, x, Y) + \epsilon$$

By the continuity of $\bar{\rho}(\cdot, x)$ that

$$\eta(t, x) \leq \bar{\rho}(t, x)$$

□

Lemma 3.39. *Using the notations as in Lemma 3.38, suppose further that X has a separable dual. Then $\eta(t, x) = \bar{\rho}_X(t, x)$.*

Proof. Let $(x_j)_{j=1}^{\infty}$ be a dense sequence in X^*. Put

$$Y_n = \bigcap_{j=0}^{n} \ker x_j^*$$

Then each Y_n is finite co-dimensional and for every $\epsilon > 0$, there exists $y_n \in Y_n$, $||y_n|| \leq t$ such that

$$||x + y_n|| - 1 \geq \bar{\rho}(t, x, Y_n) - \epsilon \geq \bar{\rho}(t, x) - \epsilon$$

It is easily checked that (y_n) is a weakly null sequence in X. Hence $\eta(t, x) \geq \bar{\rho}(t, x)$. □

In order to prove the Lipschitz invariance of (existence of) AUS norm, we will need the following topological result known as the *Gorelik Principle.* For this purpose, we need to recall the Bartle Graves Selection Theorem:

Theorem 3.40 (Bartle-Graves Selection [4])**.** *Let Y be a closed subspace of a Banach space X and $Q : X \to X/Y$ be the quotient map. Then there exists a continuous $s : X/Y \to X$ such that $Q \circ s = 1_{X/Y}$. Moreover, if $\epsilon > 0$, s can be chosen such that*

$$s(B_{X/Y}) \subset (1 + \epsilon)B_X$$

Lemma 3.41 (Gorelik Principle [23])**.** *Let E, X be Banch spaces. Suppose $\varphi : E \to X$ is a homeomorphism such that φ^{-1} is Lipschitz. Let $b, c > 0$ be constants such that*

$$c > Lip(\varphi^{-1})b$$

Then, whenever $E_0 \subset E$ is a finite co-dimensional subspace, there exists a compact $K \subset X$ such that

$$bB_X \subset K + \varphi(2cB_{E_0})$$

Proof. Put $a = \mathrm{Lip}(\varphi^{-1})b$. The main ingredient of the proof is the following claim:

Claim. There exists a compact $\tilde{K} \subset cB_E$ such that for all $\psi : \tilde{K} \to E$ continuous map with $||x - \psi(x)|| \leq a$ for all $x \in \tilde{K}$, then $\psi(\tilde{K}) \cap E_0 \neq \varnothing$.

Let $Q : E \to E/E_0$ be the canonical quotient map. $C := aB_{E/E_0}$. By the Bartle-Graves Selection Theorem (Theorem 3.40), there exists a continuous $s : C \to cB_E$ such that $Q \circ s = 1_C$. Define $F : C \to C$ by

$$F(t) = Q[s(t) - \psi(s(t))] = t - Q(\psi(s(t)))$$

By the Schauder Fixed point theorem, there exists $t_0 \in C$ such that $F(t_0) = t_0$, i.e. $Q(\psi(s(t))) = 0$. It follows that $\psi(s(t_0)) \in E_0$ as Q is a quotient map. Hence $\tilde{K} := s(C)$ satisfies the claim.

Pick any $x_0 \in bB_X$, let $\psi(e) = \varphi^{-1}(x_0 + \varphi(e))$ for $e \in \tilde{K}$. Note

$$e - \psi(e) = \varphi^{-1}(\varphi(e)) - \varphi^{-1}(x_0 + \varphi(e))$$

So $||e - \psi(e)|| \leq b\mathrm{Lip}(\varphi^{-1}) = a$. By the above, there exists $k_0 \in \tilde{K}$ such that $\psi(k_0) \in E_0$.

Put $e_0 = \psi(k_0) = \psi(k_0) - k_0 + k_0 \in E_0$. By triangle inequality, it is checked that $||e_0|| < 2c$. Moreover, observe that $e_0 = \varphi^{-1}(x_0 + \varphi(k_0))$. Therefore $x_0 = \varphi(e_0) - \varphi(k_0)$ and hence $K := -\varphi(\tilde{K})$ satisfies the lemma. □

Theorem 3.42. *Let X, Y be Banach spaces with separable duals. Assume that X is AUS and $f : X \to Y$ is a Lipschitz isomorphism.*

Then there exists an equivalent norm on Y whose modulus of asymptotic uniform smoothness $\bar{\rho}_Y$ satisfies

$$\bar{\rho}_Y(t/4dist(f)) \leq 2\bar{\rho}_X(t)$$

for $t \in (0, 1]$.

Proof. WLOG assume $\mathrm{Lip}(f) = 1$, $\mathrm{Lip}(f^{-1}) = D$. Consider a norm $|||\cdot|||_*$ defined on Y^* by

$$|||y^*|||_* = \sup\left\{\frac{y^*(f(x) - f(x'))}{||x - x'||} : x, x' \in X, x \neq x'\right\}$$

The fact that f is a Lipschitz isomorphism implies that $|||\cdot|||_*$ is an equivalent norm on Y^*. Moreover $|||\cdot|||_*$ is w^*-lower semicontinuous, i.e. $|||y^*|||_* \leq \liminf |||y_n^*|||_*$ for all y^* and for all (y_n^*) w^*-convergent to y^* and $\lim |||y_n^* - y^*|||_* = \ell$. The w^*-lower continuity of $|||\cdot|||_*$ is equivalent to saying that $|||\cdot|||_*$ is the dual norm of some equivalent norm $|||\cdot|||$ on Y. We claim that this norm satisfies the requirement of the theorem.

By Lemma 3.38 and Lemma 3.39, it suffices to check that

$$\eta_Y(t/4D) \leq 2\bar{\rho}_X(t)$$

Let $y \in Y$, $|||y||| = 1$, and (y_n) be a weakly null sequence in Y such that $|||y_n||| \leq t/4D$. The proof is finished once we have proved the following claim:

Claim. $\limsup |||y + y_n||| - 1 \leq 2\bar{\rho}_X(t)$.

By considering a subsequence of (y_n), we may assume that $\lim |||y + y_n|||$ exists. For each n, choose $y_n^* \in Y^*$, $|||y_n^*|||_* = 1$ and $y_n^*(y + y_n) = |||y + y_n|||$.

Note Y^* is separable and so B_{Y^*} is w^*-metrizable (and is w^*-compact). Therefore we may assume that (y_n^*) is w^*-convergent to some $y^* \in Y^*$.

Let $\epsilon > 0$. Then there exists $x, x' \in X$ such that

$$y^*(f(x) - f(x')) \geq |||y^*|||_*(1 - \epsilon)||x - x'||$$

By doing a translation and rescaling on f, we may assume that $x = -x'$, $f(x) = -f(x')$, so that

$$y^*(f(x)) \geq |||y^*|||_*(1 - \epsilon)||x||$$

Specifically, we consider $\tilde{f} : X \to Y$ defined by

$$\tilde{f}(z) = \tilde{f}(z + \frac{x - x'}{2}) - \frac{f(x) - f(x')}{2}$$

Then $\tilde{f}$ is again a Lipschitz isomorphism, with the same Lipschitz constant and there exist $x_0, x_0' \in X$ such that $x_0 = -x_0'$, $\tilde{f}(x_0) = -\tilde{f}(x_0')$ and $\tilde{f}(x_0) - \tilde{f}(x_0') = f(x) - f(x')$.

Let $\beta > \bar{\rho}(t) \geq \bar{\rho}(t, x/||x||)$. Then there exists a finite co-dimensional $X_0 \subset X$ such that $\beta > \bar{\rho}(t, x/||x||, X_0)$. Then, for any $z \in X_0$ with $||z|| \leq t||x||$, we have

$$||x + z|| \leq (1 + \beta)||x||$$

Let $b < t||x||/2D$, $c = t||x||/2$. By the Gorelik Principle (Lemma 3.41), there exists a compact $K \subset Y$ such that

$$bB_Y \subset K + f(2cB_{X_0})$$

Now, consider the sequence $(y_n^* - y^*)$, there exists (v_n) in B_Y such that $\langle y_n^* - y_n^*, v_n\rangle \to \lim |||y^* - y_n^*||| = \ell$. Write $bv_n = k_n + f(z_n)$ where $k_n \in K$ and $z_n \in 2cB_{X_0}$. Note that $(y_n^* - y^*)$ converges to 0 uniformly on K, so

$$\lim\langle y_n^* - y^*, f(z_n)\rangle = -b\ell$$

Put $A_n = y_n^*(f(x) - f(z_n))$. Then

$$A_n \leq |||y_n^*|||_*||x - z_n|| \leq (1 + \beta)||x||$$

Moreover A_n can be written as

$$A_n = 2y^*(f(x)) - y^*(f(z_n) - f(-x)) + b\ell + \epsilon(n)$$

where $\epsilon(n) \to 0$.

Note $y^*(f(z_n) - f(-x)) \leq |||y^*|||_*||z_n + x|| \leq |||y^*|||_*(1 + \beta)||x||$, so

$$A_n \geq 2(1 - \epsilon)|||y^*|||_*||x|| - |||y^*|||_*(1 + \beta)||x|| + b\ell + \epsilon(n)$$

Combining the above inequalities:

$$(1 + \beta)||x|| \geq |||y^*|||_*||x||(1 - \beta - 2\epsilon) + b\ell$$

Take $\beta \to \bar{\rho}(t)$ and $b \to t||x||/2D$ gives

$$(1 + \bar{\rho}(t))||x|| \geq |||y^*|||_*||x||(1 - \bar{\rho}(t) - 2\epsilon) + \frac{t||x||b}{2D}$$

So $|||y^*|||_*$ can be estimated from above by

$$|||y^*|||_* \le 1 + \frac{2\bar{\rho}(t)}{1-\bar{\rho}(t)} - \frac{\ell t}{2D(1-\bar{\rho}(t)}$$

Note $|||y + y_n||| = (y_n^* - y^*)(y) + (y_n^* - y^*)(y_n) + y^*(y + y_n)$, we have

$$\begin{aligned}
\lim |||y + y_n||| &\le \lim |||y_n^* - y^*|||\, |||y_n||| + |||y^*|||_* |||y||| \\
&\le \frac{t}{4D} \lim |||y_n^* - y^*|||_* + |||y^*|||_* \\
&\le \frac{\ell t}{4D} + |||y^*|||_*
\end{aligned}$$

Finally we prove the claim: $\lim |||y + y_n||| - 1 \le 2\bar{\rho}(t)$.

(Case 1) If $\frac{\ell t}{4D} \le 2\bar{\rho}(t)$, then by the above $\lim |||y + y_n||| - 1 \le 2\bar{\rho}(t)$

(Case 2) If $\frac{\ell t}{4D} > 2\bar{\rho}(t)$, then

$$|||y^*|||_* \le 1 + \frac{\ell t}{4D(1-\bar{\rho}(t))} \le 1 - \frac{\ell t}{4D}$$

Hence both cases give $\lim |||y + y_n||| - 1 \le 2\bar{\rho}(t)$. $\square$

4 Asymptotically Uniformly Flat spaces

In this section, we continue our discussion on asymptotic uniform smoothness. We follow the approach of Godefroy, Kalton and Lancien [11] to characterize all asymptotically uniformly flat spaces. The motivation is Remark 3.36, which tells us that the modulus of asymptotic uniform smoothness $\bar{\rho}_{c_0}$ is identically zero, which is the *best* situation for an asymtotically uniformly smooth space. A surprising result in [11] is that all *separable* aysmptotically uniformly flat spaces are in fact closed subspace of c_0 (up to linear isomorphism).

Definition 4.1. Using the notations in Definition 3.32, we say that a Banach space X *asymptotically uniformly flat* (in short: AUF) if $\bar{\rho}_X(t) = 0$ for some $t \in (0, 1]$.

Remark 4.2. *Suppose that* $\bar{\rho}_X(t_0) = 0$ *where* $t_0 \in (0, 1]$*, then it follows from Remark 3.33 that* $\bar{\rho}_X(t) = 0$ *for* $t \in (0, t_0]$.

Example 4.3. *Examples of AUF spaces:*

(i) c_0 *(Remark 3.36).*

(ii) Finite dimensional spaces (see Definition 3.32).

(iii) Subspaces of c_0 *(Remark 3.33).*

4.1 The Lipschitz weak* Kadec-Klee Property

We introduce the Lipschitz weak* Kadec-Klee property of separable Banach spaces. This is a property which is equivalent (see Proposition 4.9 and Proposition 4.12) to asymptotic uniform flatness.

Definition 4.4. Let X be separable. We say that X is LKK* (Lipschitz weak* Kadec-Klee) if there exists $c > 0$ such that for all weak*-null sequence $(x_n^*)_{n=1}^{\infty}$ in X^* and for all $x^* \in X^*$, we have

$$\limsup_{n\to\infty} ||x^* + x_n^*|| \geq ||x^*|| + c \limsup_{n\to\infty} ||x_n^*||$$

We say that X is c-LKK* to specify the constant c in the definition.

Remark 4.5. *According to Remark 3.37, AUF is not invariant under linear isomorphisms.*

Remark 4.6.

(i) If X is LKK*, then the weak* topology and the norm topology on S_{X^*} are the same.

Proof. Because X is separable, the weak* topology on S_{X^*} is metrizable. It needs to show that if (x_n^*) is a sequence in S_{X^*}, $x_n^* \overset{w^*}{\to} x^* \in S_{X^*}$, then $x_n^* \overset{||\cdot||}{\to} x^*$. By the LKK* property we have

$$\limsup_{n\to\infty} ||x_n^*|| \geq ||x^*|| + c \limsup_{n\to\infty} ||x_n^* - x^*||$$

Hence $x_n^* \overset{||\cdot||}{\to} x^*$. □

(ii) If X is LKK*, then X^* is separable.

Proof. This is a consequence of (i) and the fact that the weak* topology of S_{X^*} is separable. □

We summarize the results in [11] and state the following theorem:

Theorem 4.7. *Let X be a separable Banach space. The followings are equivalent:*

(i) X has an AUF renorming.

(ii) X has an equivalent norm which is LKK*.

(iii) X embeds linearly into c_0.

The first objective is to show that X is AUF if and only if X is c-LKK* for some c. Using the idea in [26], we need the following lemma:

Lemma 4.8. *Let X be a separable Banach space which does not contain a copy of ℓ_1. Then for all bounded weak*-null sequence $(x_n^*)_{n=1}^\infty$, there exists a subsequence $(x_{n_k}^*)_{k=1}^\infty$ and a weakly null sequence $(x_n)_{n=1}^\infty$ in X, $||x_n|| \leq 1$ such that*

$$\liminf_{n\to\infty} x_{n_k}^*(x_k) \geq \frac{1}{4} \liminf_{n\to\infty} ||x_n^*||$$

Proof. **Claim 1.** Let $F \subset X^*$ be finite dimensional, $\alpha > \liminf d(x_n^*, F)$. Then there exists a subsequence $(x_{n_k}^*)$, $f^* \in F$ such that for all k:

$$||x_{n_k}^* - f^*|| < \alpha$$

Proof of Claim 1. We may find $(x_{n_k}^*)$ a subsequence of (x_n^*) and (f_k^*) in F such that $||x_{n_k}^* - f_k^*|| < \alpha$. By boundedness of (x_n^*), it follows that (f_k^*) is bounded, so $f_k^* \to f \in F$. So for all large k we have $||x_{n_k}^* - f^*|| \leq \alpha$.

Claim 2. Let $F \subset X^*$ be finite dimensional. Then

$$\liminf d(x_n^*, F) \geq \frac{1}{2} \liminf ||x_n^*||$$

Proof of Claim 2. Let $\alpha > \liminf d(x_n^*, F)$. By claim 1, there exist $(x_{n_k}^*)$ subsequence of (x_n^*) and $f^* \in F$ such that $||x_{n_k}^* - f^*|| \leq \alpha$ for all k. Let x^{***} be any w^*-cluster point of $(x_{n_k}^*)$ in X^{***}. Then $x^{***} \in X^\perp$ (here we view $X \subset X^{**}$). So $||x^{***} - f^*|| \leq \alpha$. By applying the canonical projection P from X^{***} onto X^*, $||f^*|| \leq ||P(f^* - x^{***})|| \leq \alpha$. Hence $||x_{n_k}^*|| \leq 2\alpha$.

Proof of the lemma. WLOG assume $\alpha := \liminf ||x_n^*|| > 0$. Using claim 2, we can inductively construct $(x_{n_k}^*)$ such that

$$\liminf_{k\to\infty} d(x_{n_k}^*, [x_{n_1}^*, \ldots, x_{n_{k-1}}^*]) \geq \frac{\alpha}{2}$$

We may find a sequence (u_k) in B_X such that

- $\liminf_{k\to\infty} x_{n_k}^*(u_k) \geq \frac{\alpha}{2}$
- $x_{n_j}^*(u_k) = 0$ for $j = 1, 2, \ldots, k-1$

By passing to a subsequence we may assume (by the Rosenthal ℓ_1 theorem [36]) that (u_k) is weakly Cauchy, and the desired weakly null sequence (x_n) is given by $x_n := \frac{1}{2}(u_n - u_{n+1})$. □

Proposition 4.9. *Let X be separable. If $\bar{\rho}_X(t) = 0$, then X is $\frac{t}{4}$-LKK*.*

Proof. Let $x^* \in S_{X^*}$, (x_n^*) in X^*, $x_n^* \xrightarrow{w^*} 0$. We need to show

$$\limsup ||x^* + x_n^*|| \geq ||x^*|| + \frac{t}{4} \limsup ||x_n^*||$$

If (x_n^*) is unbounded then it is trivial, so assume (x_n^*) is bounded. We may also assume $\lim ||x_n^*||$ exists. By the Bishop Phelps theorem we may also assume that x^* is norm-attaining, let $x \in S_X$, $x^*(x) = 1$.

Let $\epsilon > 0$. Note $\bar{\rho}_X(t, x) = 0$, so there exists $Y \subset X$ finite co-dimensional such that for all $y \in Y$, $||y|| \leq t$:

$$||x + y|| \leq 1 + \epsilon$$

By Lemma 4.8, choose (y_n) in B_Y, weakly null (in Y) such that

$$\lim x_n^*(y_n) \geq \frac{1}{4} \lim ||x_n^*||$$

Write $\langle x^* + x_n^*, x + ty_n \rangle = 1 + tx^*(y_n) + x_n^*(x) + tx_n^*(y_n)$, we see that

$$\begin{aligned} \lim ||x^* + x_n^*|| &\geq \lim \langle x^* + x_n^*, \frac{x + ty_n}{||x + ty_n||} \rangle \\ &\geq \frac{1}{1+\epsilon} \lim \langle x^* + x_n^*, x + ty_n \rangle \\ &\geq \frac{1 + \frac{t}{4} \lim ||x_n^*||}{1 + \epsilon} \end{aligned}$$

The result follows by taking $\epsilon \to 0$. □

Lemma 4.10. *Let X be separable and c-LKK*. Then for all weakly null sequence $(x_n)_{n=1}^\infty$ and for all x:*

$$\max(||x||, \frac{1}{2-c} \limsup ||x_n||) \leq \limsup ||x + x_n|| \leq \max(||x||, \frac{1}{c} \limsup ||x_n||)$$

Proof. Let (x_n) be weakly null, $x \in X$. We may assume $\lim ||x_n||$, $\limsup ||x + x_n||$ exists. We first show that $\lim ||x + x_n|| \leq \max(||x||, \frac{1}{c} \lim ||x_n||)$.

Pick $y_n^* \in X^*$, $||y_n^*|| = 1$, $y_n^*(x + x_n) = ||x + x_n||$. By passing to subsequences, assume $y_n^* \xrightarrow{w^*} y^* \in X^*$ and $\lim ||y_n^* - y^*||$ exists. Then by c-LKK* property:

$$1 = \lim ||y_n^*|| \geq ||y^*|| + c \lim ||y_n^* - y^*||$$

Note $||x_n + x|| = y_n^*(x) + y^*(x_n) + (y_n^* - y^*)(x_n)$. Therefore

$$\lim ||x + x_n|| \leq ||y^*|| \, ||x|| + (1 - ||y^*||) \, \frac{\lim ||x_n||}{c}$$

Since $||y^*|| \leq 1$, by convexity we obtain $\lim ||x + x_n|| \leq \max(||x||, \frac{1}{c} \lim ||x_n||)$. It remains to show $\max(||x||, \frac{1}{2-c} \lim ||x_n||) \leq \lim ||x + x_n||$.

By weakly null property of (x_n), $||x|| \leq \lim ||x + x_n||$ always holds, so it only needs to check $\frac{1}{2-c} \lim ||x_n|| \leq \lim ||x + x_n||$. Pick $x_n^* \in X^*$, $||x_n^*|| = 1$ and $x_n^*(x_n) = ||x_n||$. Similar to the above, assume $x_n^* \xrightarrow{w^*} x^*$ and $\lim ||x + x_n||$ exists. We have

$$1 - ||x^*|| \geq c \lim ||x_n^* - x^*||$$

Note $\langle x_n^* - x^*, x_n \rangle \to \lim ||x_n||$, so $\lim ||x_n^* - x^*|| \geq \lim \langle x_n^* - x^*, \frac{x_n}{||x_n||} \rangle \geq 1$. So $1 - c \geq ||x^*||$. Write $x_n^*(x + x_n) = ||x_n|| + x_n^*(x)$, we have

$$\begin{aligned}
\lim ||x + x_n|| &\geq \lim x_n^*(x_n + x) \\
&\geq \lim ||x_n| + x^*(x) \\
&\geq \lim ||x_n|| - ||x^*|| \, ||x|| \\
&\geq \lim ||x_n|| - (1 - c)||x||
\end{aligned}$$

The lemma is proved by using the fact that $||x|| \leq \lim ||x + x_n||$ again. □

Remark 4.11. *The constant $\frac{1}{2-c}$ in the left inequality can be replaced by $\frac{1}{2}$ trivially.*

Proposition 4.12. *Let $c \in (0, 1]$ and X is separable, c-LKK*. Then $\bar{\rho}_X(c) = 0$.*

Proof. Recall that X^* must be separable. Since the dual of X is separable, we have for all t:

$$\bar{\rho}_X(t, x) = \eta(t, x) := \sup \Big\{ \limsup ||x + x_n|| - 1 \Big\}$$

where the supremum is taken over all weakly null (x_n) with $||x_n|| \leq t$. According to Lemma 4.10, whenever (x_n) is weakly null with $||x_n|| \leq c$, we have $\limsup ||x + x_n|| - 1 \leq 0$. □

By Proposition 4.12 and Proposition 4.9, we have established the equivalence of the c-LKK* property and asymptotic uniform flatness. As a note, since AUF is not linear invariant, it follows that c-LKK* is not linear invariant. We are ready to prove the main theorem.

Theorem 4.13 (Godefroy, Kalton and Lancien [11])**.** *Let $c \in (0,1]$ and X is separable, c-LKK*. Then for any $\epsilon > 0$, there exists $E \subset c_0$ such that $d(X,E) < \frac{1}{c^2} + \epsilon$.*

The idea of the proof is from Kalton's another paper [26]. The same result without the estimate $d(X,E) < \frac{1}{c^2} + \epsilon$ is proved in Theorem 2.9 of [20] where in their paper, Johnson, Linderstrauss, Preiss and Schechtman gave a proof using the shrinking finite dimensional decomposition argument.

Lemma 4.14. *Let $c \in (0,1]$ and X is separable, c-LKK*.*

(i) If $F \subset X$ is finite dimensional, $\eta > 0$, then there exists a finite dimensional $U \subset X^*$ such that for all $x \in F$, for all $y \in U_\perp$:

$$(1-\eta)\max(||x||, \frac{1}{2}||y||) \leq ||x+y|| \leq (1+\eta)\max(||x||, \frac{1}{c}||y||)$$

(ii) If $G \subset X^*$ is finite dimensional, $\eta > 0$, then there exists a finite dimensional $V \subset X$ such that for all $x^* \in G$, for all $y^* \in V^\perp$:

$$(1-\eta)(||x^*|| + c||y^*||) \leq ||x^* + y^*|| \leq ||x^*|| + ||y^*||$$

Proof. Let (u_j^*) be norm-dense in X^*. Put

$$U_n := \bigcap_{j=1}^{n} \ker u_j^*$$

Suppose (i) fails, then there exists $x_n \in F$, $y_n \in U_n$ such that

$$(1-\eta)\max(||x_n||, \frac{1}{2}||y_n||) > ||x_n + y_n||$$

By normalizing (x_n), assume (x_n) is bounded, then $x_n \to x$ be compactness of B_F, so

$$(1-\eta)\max(||x||, \frac{1}{2}||y_n||) \geq ||x + y_n||$$

By construction, $y_n \in U_n$ and this implies (y_n) is weakly null. By Lemma 4.10:

$$\lim ||x + y_n|| \geq \max(||x||, \frac{1}{2}\limsup ||y_n||)$$

which is a contradiction. Similarly, the right inequality in (i) and hence (ii) is proved. □

Proof of Theorem 4.13. Let $0 < \delta < \frac{1}{3}$. Pick a positive integer t such that $t > \frac{6(1+\delta)}{c^3\delta}$. Let $(\eta_n)_{n\geq 1}$ be a sequence of positive numbers satisfying

- $0 < \eta_n < \frac{\delta}{2}$
- $\prod_{n\geq 1}(1-\eta_n) > 1-\delta$
- $\prod_{n\geq 1}(1+\eta_n) < 1+\delta$

Let (u_n) be a dense sequence in X. Following Kalton (see Theorem 4.2 of [26]), we will inductively construct subspaces (F_n), (F'_n) in X^* and $(E(m,n))_{1\leq m\leq n}$ in X such that

(a) $\dim F_n < \infty$

(b) $F'_n \subset [u_1,\dots,u_n]^\perp \cap \bigcap_{j\leq k\leq n} E(j,k)^\perp$ and $X^* = F_1 \oplus F_2 \oplus \cdots \oplus F_n \oplus F'_n$.

(c) $F'_n = F_{n+1} \oplus F'_{n+1}$

(d) If $x^* \in F_1 + \cdots + F_n$ and $y \in F'_{n+1}$, then

$$(1-\eta_n)(||x^*|| + c||y^*||) \leq ||x^* + y^*|| \leq ||x^*|| + ||y^*||$$

(e) If $x \in (F_1 + \cdots + F_n)_\perp$ and $y \in \sum_{j\leq k\leq n} E(j,k)$, then

$$(1-\eta_n)\max(||x||, \frac{1}{2}||y||) \leq ||x+y|| \leq (1+\eta_n)\max(||x||, \frac{1}{c}||y||)$$

(f) $(F_1 + \cdots + F_{m-1} + F'_n)_\perp \subset E(m,n)$ and $E(m,n) \subset (F_1 + \cdots + F_{m-2})_\perp$

(g) If $x^* \in F_m + \cdots + F_n$, then there exists $x \in E(m,n)$ such that $||x|| \leq 1$ and $x^*(x) \geq c(1-\delta)||x^*||$

We will briefly describe the step: If $n = 1$, choose $F'_1 = [u_1]^\perp$ and F_1 be any complement of F'_1. (At this stage no E-subspace is constructed)

(Inductive step) Suppose we have found $(F_j)_{j\leq n}$, $(F'_j)_{j\leq n}$, $E(j,k)$ for $1 \leq j \leq k \leq n-1$. We first construct $E(m,n)$ for $1 \leq m \leq n$. Suppose $m \geq 3$ (the construction for $m = 1,2$ is similar).

Note if $x^* \in F_1 + \cdots + F_{m-2}$, $y^* \in F_m + \cdots + F_n$ $(\subset F'_{m-1})$,

$$||x^* + y^*|| \geq (1 - \eta_n)(||x^*|| + c||y^*||) \geq c(1 - \frac{\delta}{2})||y^*||$$

By Hahn Banach theorem there exists a finite dimensional subspace $G = G(m,n) \subset (F_1 + \cdots + F_{m-2})_\perp$ and

$$\sup_{g \in B_G} y^*(g) \geq c(1 - \delta)||y^*||$$

for all $y^* \in F_m + \cdots + F_n$. Then we may put $E(m,n) = G + (F_1 + \cdots + F_{m-1} + F'_n)_\perp$. Then $(f), (g)$ are satisfied.

Next we must define F_{n+1}, F'_{n+1}. By Lemma 4.14, there are $H \subset X$, $K \subset X^*$ finite dimensional subspaces such that

(i) If $x \in \sum_{1 \leq m \leq n} E(m,n)$ and $y \in K_\perp$ then

$$(1 - \eta_{n+1}) \max(||x||, \frac{1}{2}||y||) \leq ||x + y|| \leq (1 + \eta_n) \max(||x||, \frac{1}{c}||y||)$$

(ii) If $x^* \in F_1 + \cdots + F_n$ and $y^* \in H^\perp$ then

$$(1 - \eta_{n+1})(||x^*|| + c||y^*||) \leq ||x^* + y^*|| \leq ||x^*|| + ||y^*||$$

Let D be any complement of $F_1 + \cdots + F_n + K$ which is contained in F'_n (This is valid because F'_n is a complement of $F_1 + \cdots + F_n$). Put

$$F'_{n+1} := D \cap (H + \sum_{1 \leq m \leq n} E(m,n) + [u_1, \ldots, u_n]^\perp)$$

Now both F'_n, $K \oplus D$ are complement of $F_1 + \cdots + F_n$. Choose $K' \subset F'_n$ such that $F_1 + \cdots + F_n + K' = F_1 + \cdots + F_n + K$. Finally choose $F_{n+1} \supset K'$ such that $F'_n = F_{n+1} \oplus F'_{n+1}$. This completes the inductive step.

Suppose $(a_n), (b_n)$ are two increasing sequences of integers:

$$a_0 \leq b_0 < a_1 \leq b_1 < a_2 \leq b_2 \leq \ldots$$

Suppose further $b_n + 3 \leq a_{n+1}$ and $x_n \in E(a_n, b_n)$, by $(e), (f)$:

$$\frac{1 - \delta}{2} \max(||x_1||, \ldots ||x_n||) \leq \Big|\Big| \sum_{k=1}^{n} x_k \Big|\Big| \leq \frac{1 + \delta}{c} \max(||x_1 \ldots ||x_n||)$$

Define for $s = 0, \dots t-1$:

$$Y_s = c_0(E(4(n-1)t + 4s + 4, 4nt + 4s + 1)_{n\geq 0})$$

$$Z_s = c_0(E(4nt + 4s + 2, 4nt + 4s + 3)_{n\geq 0})$$

According to the aobve, we can formally define

- $T_s : Y_s \to X$ by $T_s((y_n)_{n\geq 0}) = \sum y_n$
- $R_s : Z_s \to X$ by $R(\xi_0, \dots, \xi_{t-1}) = \sum z_n$

Define

- $T : Y := \ell_\infty((Y_s)_{s=0}^{t-1}) \to X$ by $T(\xi_0 \dots, \xi_{t-1}) = \frac{1}{t}\sum_{s=0}^{t-1} T_s\xi_s$
- $R : \ell_\infty((Z_s)_{s=0}^{t-1}) \to X$ by $R(\xi_0 \dots, \xi_{t-1}) = \sum_{s=0}^{t-1} R_s\xi_s$

We have

$$\forall \xi \in Y_s,\ \frac{1-\delta}{2}||\xi|| \leq ||T_s(\xi)|| \leq \frac{1+\delta}{c}||\xi||$$
$$\forall \xi \in Z_s,\ \frac{1-\delta}{2}||\xi|| \leq ||R_s(\xi)|| \leq \frac{1+\delta}{c}||\xi||$$

It follows that $||T|| \leq \frac{1+\delta}{c}$. Also, we may view R as an operator on the space $c_0(E(4n+2, 4n+3)_{n\geq 0})$ and deduce that $||R|| \leq \frac{1+\delta}{c}$.

Note if $x^* \in X^*$, as $R_s : Z_s \to R_s(Z_s)$ is an isomorphism, we have

$$||x^*|_{R_s(Z_s)}|| \leq \frac{2}{1-\delta}||R_s^* x^*||$$

Using Hahn-Banach Theorem, choose $y^* \in X^*$ such that $R_s^* x^* = R_s^* y^*$ and $||y^*|| \leq 2(1-\delta)^{-1}||R_s^* x^*||$. Then

$$\begin{aligned}
||T_s^* x^*|| &\geq ||T_s^*(y^* - x^*)|| - ||T_s^* y^*|| \\
&\geq c(1-\delta)||y^* - x^*|| - \frac{1+\delta}{c}||y^*|| \\
&\geq c(1-\delta)||x^*|| - c(1-\delta)||y^*|| - \frac{1+\delta}{c}||y^*|| \\
&\geq c(1-\delta)||x^*|| - 2(1-\delta)^{-1}||R_s^* x^*||(c(1-\delta) + \frac{1+\delta}{c})
\end{aligned}$$

By the choice of δ:

$$||T_s^* x^*|| \geq c(1-\delta)||x^*|| - \frac{6}{c}||R_s^* x^*||$$

Therefore

$$\begin{aligned}
||T^* x^*|| &= \frac{1}{t}\sum_{s=0}^{t-1}||T_s^* x^*|| \\
&\geq c(1-\delta)||x^*|| - \frac{6}{ct}||R^* x^*|| \\
&\geq ||x^*||(c(1-\delta) - \frac{6(1+\delta)}{c^2 t})
\end{aligned}$$

Hence $||T^* x^*|| \geq c(1-2\delta)||x^*||$ for all x^*. On the other hand, note $||T|| \leq \frac{1}{c}(1+\delta)$. Hence

$$d(X, \frac{Y}{\ker T}) \leq \frac{1+\delta}{c^2(1-2\delta)} = \frac{1}{c^2} + o(\delta)$$

As a c_0-sum of finite dimensional spaces, Y embeds almost isometrically into c_0. By Alspach Theorem [3], so is its quotient. □

4.2 Lipschitz Structure of c_0

According to Corollary 3.30, ℓ_p $(1 < p < \infty)$ is shown to have unique Lipschitz structure. The proof is based on a well-developed differentiation theory. Unfortunately, the differentiation argument fails for c_0. The differences between c_0 and ℓ_p $(1 < p < \infty)$ are critical. c_0 is not reflexive and does not have (RNP), where ℓ_p $(1 < p < \infty)$ is reflexive and do have (RNP). This makes the differentiation technique not applicable to c_0.

Still, it is shown [11] that c_0 indeed has unique Lipschitz structure. The proof of the result utilizes Theorem 4.7, which tells us that c_0 is contains every separable asymptotically uniformly flat space.

Theorem 4.15. *The class of Banach space X which is linearly isomorphic to a subspace of c_0 is closed under Lipschitz isomorphism.*

Proof. Let $E \subset c_0$, X Banach, $U : E \to X$ be a Lipschitz isomorphism. Put a norm $|||\cdot|||$ on X^* by

$$|||x^*||| = \sup\left\{\frac{|x^*(Ue - Ue')|}{||e-e'||} : e, e' \in E,\, e \neq e'\right\}$$

Because U is a Lipschitz isomorphism, it follows that $|||\cdot|||$ is an equivalent norm on X^*. Moreover, as in the proof of Theorem 3.42 $|||\cdot|||$ is the dual norm of some equivalent norm

$|||\cdot|||$ on X. It can also be checked that $|||\cdot||| \geq ||\cdot||$ (that is, This norm is greater than the original norm of X).

Let $x^* \in X^*$, $(x_n^*)_{n=1}^\infty$ be a weak* null sequence in X^*, by Theorem 4.7, we want to show that

$$\limsup |||x^* + x_n^*||| \geq |||x^*||| + c \limsup |||x_n^*|||$$

where c is a constant to be determined later (c must be independent of x^* and x_n^*). If $\limsup |||x_n^*||| = 0$, then we are done. Suppose already $\limsup |||x_n^*||| > 0$ and there exist $\epsilon > 0$ such that $|||x_n^*||| \geq \epsilon > 0$ for all n. Let $\delta > 0$, there exists $e, e' \in E$ such that

$$\frac{|x^*(Ue - Ue')|}{||e - e'||} \geq (1 - \delta)|||x^*|||$$

By considering a translation of U, assume $e = -e'$ and $Ue = -Ue'$.

Specifically, we consider $\tilde{U} : E \to X$ defined by

$$\tilde{U}(x) = U(x + \frac{e - e'}{2}) - \frac{Ue - Ue'}{2}$$

Then $\tilde{U}$ is again a Lipschitz isomorphism, with the same Lipschitz constant, and there exists $e_0, e_0' \in E$ such that $e_0 = -e_0'$, $\tilde{U}e_0 = -\tilde{U}e_0'$ and $\tilde{U}e_0 - \tilde{U}e_0' = Ue - Ue'$.

Because E is a subspace of c_0, it is AUF, there exists a finite co-dimensional subspace E_0 of E such that for all $f \in E_0$, $||f|| \leq ||e||$,

$$||e \pm f|| \leq (1 + \delta)||e||$$

Put $C := \frac{||e||}{2}$ and $b < \frac{||e||}{2C}$. We apply the Gorelik Principle (Lemma 3.41) to the map $U : E \to (X, |||\cdot|||)$, there exists a compact $K \subset X$ such that

$$bB_X \subset K + U(||e||B_{E_0})$$

Choose x_n in $B_{(X,|||\cdot|||)}$ such that $\liminf x_n^*(x_n) \geq \epsilon$. So there exists (f_k) in $||e||B_{E_0}$ such that

$$\liminf x_k^*(-Uf_k) \geq \epsilon b$$

Note $x^*(Uf_k + Ue) \leq |||x^*|||(1 + \delta)||e||$ and $|x^*(Ue)| > (1 - \delta)|||x^*|||\,||e||$. So $x^*(Uf_k) \leq$

$2\delta||e||\,|||x^*|||$. Now

$$\liminf (x^* + x_k^*)(e - f_k) = \liminf x^*Ue - x^*Uf_k + x_k^*Ue - x^*Uf_k$$
$$\geq (1-3\delta)||e||\,|||x^*||| + \epsilon b$$

Hence

$$\liminf |||x^* + x_k^*||| \geq \frac{1-3\delta}{1+\delta}|||x^*||| + \frac{\epsilon b}{(1+\delta)||e||}$$

Take $b \to \frac{||e||}{2C}$, $\delta \to 0$ we get

$$\liminf |||x^* + x_k^*||| \geq |||x^*||| + \frac{1}{2C}\epsilon$$

Finally take $\epsilon \to \limsup |||x_n^*|||$, which proves $|||\cdot|||$ is $(2C)^{-1}$-LKK*. $\square$

Before showing the main result, we state a definition and two results needed.

Definition 4.16. Let $1 \leq p \leq \infty$. Let X be a Banach space and $\lambda > 0$. X is called an $\mathcal{L}_p^\lambda$ space if for all finite dimensional subspace E of X, there exists a finite dimensional subspace $F \subset X$ containing E with $d_{BM}(F, \ell_p^{\dim(F)}) \leq \lambda$. X is called an $\mathcal{L}_p$ space if X is an $\mathcal{L}_p^\lambda$ space for some $\lambda > 0$.

Theorem 4.17 (Heinrich, Mankiewicz [17])**.** *The class of $\mathcal{L}_\infty$ Banach spaces is closed under uniform homeomorphisms.*

Theorem 4.18 (Johnson, Zippin [22])**.** *Let X be a closed subspace of c_0. Suppose X is an $\mathcal{L}^\infty$ space, then X is isomorphic to c_0.*

Below is the main theorem in this section.

Theorem 4.19. *c_0 has unique Lipschitz structure.*

Proof. Now, suppose that X is Lipschitz isomorphic to c_0, clearly c_0 is $\mathcal{L}_\infty$. It follows from Theorem 4.17 that X is linearly isomorphic to an $\mathcal{L}^\infty$ subspace E of c_0. By Theorem 4.18, E is linearly isomorphic to c_0. $\square$

4.3 Lipschitz quotient maps

Following [9], to further extend the study of Lipschitz structure of c_0, we now turn to quotients of c_0. The main question we want to resolve is the following: suppose that a Banach sapce X is Lipschitz isomorphic to a linear quotient of c_0, is X necessarily linearly isomorphic to some linear quotient of c_0? As in the proof of Theorem 4.13, the Alspach Theorem [3] says that linear quotients of c_0 are in fact linear subspaces of c_0. Thus if X is Lipschitz isomorphic to a linear quotient of c_0, it follows that X must be a linear subspace of c_0. The main difficulty here is that it is not obvious whether this linear subspace is already a linear quotient of c_0. We will provide a sufficient condition so that such subspace X is indeed a linear quotient.

Definition 4.20. Let X, Y be Banach sapce, $C > 0$. $f : X \to Y$ be a map.

(a) Y is called a C-linear quotient of X if there exists a closed subspace X_0 of X and a linear isomorphism $T : X/X_0 \to Y$ such that $||T||\,||T^{-1}|| \leq C$.

(b) f is called C-co-Lipschitz if for all $x \in X$, for all $r > 0$:

$$B_Y(f(x), \frac{r}{C}) \subset f(B_X(x, r))$$

(c) f is called a C-Lipschitz quotient map if f is γ-Lipschitz, γ'-co-Lipschitz and $\gamma\gamma' \leq C$. In this case Y is called a C-Lipschitz quotient of X.

Remark 4.21. *(i) Y is a C-linear quotient of X iff there exists $G : X \to Y$ is a surjective linear operator such that the induced isomorphism $\tilde{G} : X/\ker G \to Y$ has $||\tilde{G}||\,||\tilde{G}^{-1}|| \leq C$. In this case we say G is a C-linear quotient map.*

(ii) If f is co-Lipschitz, then f is automatically onto.

Lipschitz isomorphisms and linear projections are Lipschitz quotient maps. In fact for the results in this section, almost all Lipschitz quotient maps concerned are functions of these types, or compositions of functions of these types.

To proceed, let us state a differentiability-type result (see Theorem 3.1 of [20]).

Theorem 4.22. *Let $\epsilon > 0$, X be superreflexive, $f : c_0 \to X$ be Lipschitz. Then there exists $x_0 \in c_0, \ell \in L(c_0, X)$ and $\delta > 0$ such that*

$$||f(x_0 + h) - f(x_0) - \ell(h)|| \leq \epsilon ||h||$$

for all $h \in c_0$, $||h|| \le \delta$.

Remark 4.23. *Using the terminology in [20], the above condition is saying that f has a point of ϵ-Frechet differentiability.*

Proposition 4.24. *Let X be finite dimensional. Suppose that X is a C-Lipschitz quotient of c_0, then X is a K-linear quotient of c_0 for all $K > C$.*

Proof. Let $K > C$. $f : c_0 \to X$, γ-Lipschitz, γ'-co-Lipschitz and $\gamma\gamma' \le C$.

Let $\epsilon > 0$ such that $\gamma'\epsilon < 1$ and $(\gamma + \epsilon)(\gamma'^{-1} - \epsilon)^{-1} \le K$. Let $x_0 \in c_0, \delta > 0, \ell \in L(c_0, X)$ by the lemma. We claim that ℓ is onto.

Suppose not. Put $Z = \ell(c_0)$. By Riesz Lemma there exists $y \in X$, $||y|| = 1$ and $d(y, Z) > \gamma'\epsilon$. Because f is γ'-co-Lipschitz, we have

$$B_X(f(x_0), \frac{\delta}{\gamma'}) \subset f(B_{c_0}(x_0, \delta))$$

so there exists $z \in B_{c_0}(0, \delta)$ such that $f(x_0) + \frac{\delta}{\gamma'}y = f(x_0 + z)$. Now

$$||f(x_0 + z) - f(x_0) - \ell(z)|| \le \epsilon||z||$$

which implies $||y - \ell(\frac{\gamma' z}{\delta})|| \le \gamma'\epsilon$. Hence $d(y, Z) \le \gamma'\epsilon$ which is a contradiction. Now ℓ is onto, it induces a canonical linear isomorphism $L : c_0/\ker \ell \to X$ which satisfies $||L||\,||L^{-1}|| \le (\gamma + \epsilon)(\gamma'^{-1} - \epsilon)^{-1} \le K$ (see the following note). $\square$

Note. In the above, using $||f(x_0 + z) - f(x_0) - \ell(z)|| \le \epsilon||z||$ for all $||z|| \le \delta$, we can show that $||L^{-1}|| \le (\gamma'^{-1} - \epsilon)^{-1}$.

Assume $x_0 = 0$, $f(x_0) = 0$, $\delta = 1$.

Pick $y \in B(0, 1) \subset f(B(0, \gamma'))$, there exists x_1, $||x_1|| \le \gamma'$ such that $y = f(x_1)$. We have

$$||y - \ell(x_1)|| \le \epsilon\gamma'$$

Note $y - \ell(x_1) \in B(0, \epsilon\gamma') \subset f(B(0, \gamma^2\epsilon))$, there exists x_2, $||x_2|| \le \gamma^2\epsilon$ such that $y - \ell(x_1) = \ell(x_2)$

$$||y - \ell(x_1 - x_2)|| \le \epsilon^2\gamma'^2$$

Following this argument (same as in the open mapping theorem), it is shown that $y \in \ell(B(0, \frac{\gamma'}{1-\gamma'\epsilon}))$.

Now we turn to the main result. The proof rely on two results concerning the approximation property. Recall that a Banach space X has *bounded approximation property* (BAP) if there exists a sequence of finite-rank operator $(T_n)_{n=1}^{\infty}$ from X to X such that $x = \lim_n T_n x$. X is said to have *metric approximation property* (MAP) if T_n's can be chosen to have $||T_n|| = 1$.

Theorem 4.25 (Grothendieck [16])**.** *Suppose a separable dual space has the approximation property, then it has the metric approximation property.*

Lemma 4.26 (Godefroy, Kalton, lemma 3.1 of [13])**.** *Let $Y \subset c_0$, Y has MAP, $C > 1$. Then there exists $E_1, E_2, \dots$ finite dimensional subspaces of Y such that Y is a C-linear quotient of $(\sum \oplus E_n)_{c_0}$ and E_i's are C-linear quotient of Y.*

Theorem 4.27. *Let X be Banach. Assume that X^* has approximation property. Suppose X is Lipschitz isomorphic to a quotient of c_0, then X is linearly isomorphic to a quotient of c_0.*

According to [27], it is unknown whether the assumption that X^* has approximation property can be dropped. The key point of the assumption on X is that it allows us to apply several known results concerning the approximation property. We state the two results needed.

Proof. Let Z be a quotient of c_0. $f : Z \to X$ be Lipschitz isomorphism. Let $\beta \geq \mathrm{Lip}(f)\mathrm{Lip}(f^{-1})$.

We know that quotients of c_0 are in fact linearly isomorphic to subspaces of c_0. Hence X is in fact linearly isomorphic to a subspace of c_0. Let $T : X \to Y$ be a linear isomorphism. Put $\theta = ||T||\,||T^{-1}||$.

Since X^* has AP, so Y^* has AP. By Grothendieck's result Theorem 4.25, Y has MAP, let (E_n) be subspaces of Y chosen as in Kalton's result Lemma 4.26. Put

- $Q : c_0 \to Z$ be a quotient map.
- $T_n : Y \to E_n$ be C-linear quotient map.

Then $T_n \circ T \circ f \circ Q$ is $(C\theta\beta)$-Lipschitz quotient map from c_0 onto E_n. Let $K > C$. By the previous proposition E_n is a $(K\theta\beta)$-linear quotient of c_0. Let Q_n be a $(K\theta\beta)$-linear quotient map from c_0 onto E_n. Then

$$(\sum \oplus c_0)_{c_0} \to (\sum \oplus E_n)_{c_0},\ (x_n)_{n=1}^{\infty} \mapsto (Q_n(x_n))_{n=1}^{\infty}$$

is a $(K\theta\beta)$-linear quotient map. Hence Y is a $(CK\theta\beta)$-linear quotient of c_0 and X is $(CK\theta^2\beta)$-linear quotient of c_0. $\square$

5 Uniform and Coarse Lipschitz Structure of Banach Spaces

5.1 Introduction

In this section, we will study maps which are uniformly continuous or coarsely Lipschitz. As before, we want to ask whether two uniformly homeomorphic Banach spaces are already linearly isomorphic. A result by Ribe [35] shows the case fails even for separable cases.

Theorem 5.1 (Ribe [35])**.** *Let $(p_n)_{n=1}^{\infty}$ be a strictly decreasing sequence in $(1, \infty)$ such that $p_n \to 1$. Let $X = (\sum \oplus L_{p_n})_{\ell_2}$ be the ℓ_2-sum of the L_{p_n} spaces. Then X and $X \oplus L_1$ are uniformly homeomorphic.*

Remark 5.2. *By Ribe's result Theorem 5.1, reflexivity is not preserved by uniform homeomorphisms (compare with Proposition 2.7).*

By Proposition 2.24, every uniform homeomorphism is coarse Lipschitz isomorphism (**Warning**: despite of its name, a coarse Lipschitz *isomorphism* needs not be injective). Therefore, suppose X, Y are uniformly homeomorphic Banach spaces, they must be coarsely Lipschitz embedded into each of another. According to Theorem 2.29, X must be crudely finitely representable in Y and vice versa. Hence, using Corollary 2.30, we know that X, Y must be of the same type and cotype.

Inspired by Corollary 2.30 that type and cotype are preserved by uniform homeomorphisms, we have several results concering uniform structures. The first one is that separable Hilbert space has unique uniform structure.

Proposition 5.3. *Let X be a Banach space such that it is uniformly homeomorphic to ℓ_2, then X is linearly isomorphic to ℓ_2.*

Proof. By Kwapien's Theorem [29], every space with type 2 and cotype 2 is linearly isomorphic to a Hilbert space. To finish to proof, we must show X is separable. But this is clear because (uniform) homeomorphisms preserve separability. □

Remark 5.4. *By Kwapien's Theorem, if X is uniformly homeomorphic to a Hilbert space, then X is linearly isomorphic to a Hilbert space.*

The next result, also deduced by using type and cotype argument, is a combination of two individual results by Enflo and Linderstrauss.

Theorem 5.5 (Enflo [10], Linderstrauss [30]). *Let $1 \leq p, q < \infty$. If $L_p(\mu_1), L_q(\mu_2)$ are uniformly homeomorphic, then either they have the same finite dimension or $p = q$.*

Proof. If $L_p(\mu_1)$ is finite dimensional. Because $L_q(\mu_2)$ is crudely finitely representable in $L_p(\mu_1)$, it follows that $\dim(L_q(\mu_2)) \leq \dim(L_p(\mu_1))$. The reverse inequality is the same.

Assume now they are both infinite dimensional. Because $L_p(\mu_1), L_q(\mu_2)$ have the same type and cotype. This can happen if and only if $p = q$ (Proposition 2.3). □

It is natural to ask whether ℓ_p and L_p are uniformly homeomorhic for $1 \leq p < \infty$ (the answer for $p = \infty$ is trivial, since ℓ_∞ and L_∞ are isomorphic, see Chapter 5 of [2]). This question is resolved by Bourgain [6] for the case $1 < p < 2$, Enflo (see Chapter 10 of [5]) for the case $p = 1$ and Gorelik [15] for the case $2 < p$.

Theorem 5.6. *If $1 \leq p \neq 2 < \infty$, then ℓ_p, L_p are not uniformly homeomorphic.*

In the followings, we will restrict ourselves on classical Banach spaces and study the uniform and coarse Lipschitz structrues of those spaces. We will start by looking at coarse Lipchitz maps from ℓ_p to ℓ_q.

In linear theory, it is well-known that ℓ_p and ℓ_q $(p \neq q)$ have distinct linear structure. However, the approach to the cases $(i) : q < p$ and $(ii) : p < q$ are different. The former one is a consequence of the Pitt's theorem:

Theorem 5.7 (Pitt [33]). *Suppose $q < p$, then every bounded linear operator $T : \ell_p \to \ell_q$ is compact.*

where for the case $p < q$, we can at most deduce that every bounded linear operator $T : \ell_p \to \ell_q$ is *strictly singular*, i.e. T is not an isomorphism when restricted on any infinite dimensional subspace (see for example, Theorem 2.1.9 of [2]).

In the following sections, we aim to prove that ℓ_p and ℓ_q $(p \neq q)$ have distinct coarse Lipschitz structure. It turns out that as in the linear case, the methods in approaching the two cases ($q < p$ and $p < q$) are completely different, where the former case is handled using the *approximate metric mid-point method* and the latter case is handled using the *Kalton-Randrianarivony graph* [28].

5.2 The approximate metric midpoint method

Definition 5.8. If M is a metric space, $x, y \in M$, a *metric midpoint of* x, y is a point $z \in M$ such that

$$d(z, x) = d(z, y) = \frac{1}{2}d(x, y)$$

In a general metric space M and $x, y \in M$ be distinct, it is not necessary that a metric midpoint exists. Moreover, suppose that there is some metric midpoint between x and y, it is not necessary that such midpoint is unique. To see this fact, we can consider the metric space consisting of the four vertices of the unit square under the ℓ_1-metric. Specifically, $M = \{(0, 1), (1, 0), (1, 1), (0, 0)\}$. Then $z_1 = (0, 0)$ and $z_2 = (1, 1)$ are metric midpoints of $x = (1, 0)$ and $y = (0, 1)$.

Definition 5.9. Let M be a metric space. Let $x, y \in M$, $\delta > 0$. *The set of approximate metric midpoint between* x, y is the set

$$\mathrm{Mid}(x, y, \delta) = \{z \in M : max(d(x, z), d(y, z)) \leq (1 + \delta)\frac{d(x, y)}{2}\}$$

Suppose M, N are metric spaces and $f : M \to N$ such that it is *strongly norm-attaining*, i.e. there exist $x, y \in M$ such that

$$d_N(f(x), f(y)) = \mathrm{Lip}(f)\, d_M(x, y)$$

then in this case, f maps metric midpoints of x, y to metric midpoints of $f(x), f(y)$.

From now on, the metric spaces concerned are Banach spaces. The first result will be a perturbation result of approximate metric midpoints under coarse Lipschitz maps which holds for general Banach spaces.

Proposition 5.10. *Let X, Y be Banach spaces. $f : X \to Y$ be coarse Lipschitz map. If $Lip_\infty(f) > 0$, then for every $\theta > 0, \epsilon > 0, 0 < \delta < 1$, there exist $x, y \in X$, $||x - y|| \geq \theta$ and*

$$f(Mid(x, y, \delta)) \subset Mid(f(x), f(y), (1 + \epsilon)\delta)$$

Proof. Let θ, ϵ, δ be given.

Let $\eta > 0$ be later specified. We may choose $\theta' > \theta$ such that

$$\mathrm{Lip}_{\theta'}(f) < (1 + \eta)\mathrm{Lip}_\infty(f)$$

Choose $x, y \in X$ such that

(i) $||x-y|| > 2\theta'(1-\delta)^{-1}$

(ii) $\frac{||f(x)-f(y)||}{||x-y||} \geq \frac{1}{(1+\eta)^2}\mathrm{Lip}_{\theta'}(f)$.

Let $u \in \mathrm{Mid}(x,y,\delta)$. Then by triangle inequality

$$||y-u|| \geq \frac{1-\delta}{2}||x-y|| \geq \theta'$$

Now

$$\begin{aligned} ||f(y)-f(u)|| &\leq \mathrm{Lip}_{\theta'}(f)||y-u|| \\ &\leq \mathrm{Lip}_{\theta'}(f)\frac{1+\delta}{2}||x-y|| \\ &\leq (1+\eta)^2\frac{1+\delta}{2}||f(x)-f(y)|| \end{aligned}$$

The same estimate holds for $||f(x)-f(u)||$. Now if η is chosen small, then RHS$<$ $\frac{(1+\epsilon)\delta}{2}||f(x)-f(y)||$. □

In the cases for classical sequence spaces, we have a good lower bound and upper bound of the set of approximate metric midpoints.

Lemma 5.11. *Let $1 \leq p < \infty$. $(e_i)_{i=1}^{\infty}$ be the natural basis of ℓ_p. For $N \in \mathbb{N}$, put $E_N = [e_i : i > N]$.*

Let $x, y \in \ell_p, \delta \in (0,1)$. Put $u = \frac{x+y}{2}, v = \frac{x-y}{2}$. Then:

1. *there exists $N \in \mathbb{N}$ such that*

$$u + \delta^{1/p}||v||B_{E_N} \subset Mid(x,y,\delta)$$

2. *there exists a compact $K \subset \ell_p$ such that*

$$Mid(x,y,\delta) \subset K + 2\delta^{1/p}||v||B_{\ell_p}$$

Proof. Let $0 < \eta < 1$ be later defined. Pick $N \in \mathbb{N}$ such that $\sum_{i=1}^{N}|v_i|^p \geq (1-\eta^p)||v||^p$. Pick $z \in E_N$ such that $||z||^p \leq \delta||v||^p$.

Claim. We may choose a small η such that $||x-(u+z)||^p \leq (1+\delta)^p||v||^p$

Note $||x-(u+z)|| = ||v-z||$. The case for $p=1$ is easily obtained. Assume $p>1$. We have $||v-z||^p = \sum_{i=1}^{N} |v_i|^p + \sum_{i>N} |v_i - z_i|^p$. Notice that

$$(\sum_{i>N} |v_i - z_i|^p)^{1/p} \leq (\sum_{i>N} |z_i|^p)^{1/p} + (\sum_{i>N} |v_i|^p)^{1/p} \leq ||z|| + \eta ||v||$$

Therefore

$$||x-(u+z)||^p \leq ||v||^p + (||z|| + \eta||v||)^p \leq ||v||^p (1 + (\delta^{1/p} + \eta)^p)$$

Since $\lim_{\eta \to 0} 1 + (\delta^{1/p} + \eta)^p = 1 + \delta < (1+\delta)^p$, so we obtain the claim. The calculation for $||y-(u+z)||$ is the same, hence $u+z \in \mathrm{Mid}(x,y,\delta)$.

For *(ii)*, suppose that $u+z \in \mathrm{Mid}(x,y,\delta)$. Write $z = z^{(1)} + z^{(2)}$, where $z^{(1)} \in F_N := [e_i : 1 \leq i \leq N]$ and $z^{(2)} \in E_N$.

Note $||v-z||, ||v+z|| \leq (1+\delta)||v||$. By convexity (triangle inequality):

$$||z^{(1)}|| \leq ||z|| \leq (1+\delta)||v||$$

Therefore $u + z^{(1)} \in u + (1+\delta)||v|| B_{F_N} =: K$. On the other hand, by convexity we also have

$$\max(|z_i|^p, |v_i|^p) \leq \frac{1}{2}(|v_i - z_i|^p + |v_i + z_i|^p)$$

Therefore

$$\sum_{i=1}^{N} |v_i|^p + \sum_{i>N} |z_i|^p \leq \frac{1}{2}(||v-z||^p + ||v+z||^p)$$

It follows that $(1-\eta^p)||v||^p + ||z^{(2)}||^p \leq (1+\delta)^p ||v||^p$, i.e. $||z^{(2)}||^p \leq ||v||^p((1+\delta)^p - (1-\eta^p))$. $\lim_{\eta \to 0}(1+\delta)^p - (1-\eta^p) = (1+\delta)^p - 1 < 2^p \delta$.

Hence by choosing η small, $||z^{(2)}||^p \leq ||v||^p 2^p \delta$, we have

$$\mathrm{Mid}(x,y,\delta) \subset K + 2\delta^{1/p} ||v|| B_{\ell_p}$$

□

The main tool needed is a topological result of coarse Lipschitz maps from ℓ_q to ℓ_p where $p < q$.

Proposition 5.12. *Let $1 \leq p < q < \infty$. $f : \ell_q \to \ell_p$ be coarse Lipschitz. Then for every $t, \epsilon > 0$, there exists $u \in \ell_q, \tau > t, N \in \mathbb{N}$ and compact $K \subset \ell_p$ such that*

$$f(u + \tau B_{E_N}) \subset K + \epsilon\tau B_{\ell_p}$$

where $E_N = [e_i : i > N]$.

Proof. Assume $\mathrm{Lip}_\infty(f) > 0$. Choose $\delta > 0$ be later defined. Pick $\theta > 0$ large (later defined) such that

$$\mathrm{Lip}_\theta(f) < 2\mathrm{Lip}_\infty(f)$$

By Proposition 5.10, there exist $x, y \in \ell_q$, $||x - y||_q \geq \theta$ such that

$$f(\mathrm{Mid}(x, y, \delta)) \subset \mathrm{Mid}(f(x), f(y), 2\delta)$$

Let $u = \frac{x+y}{2}, v = \frac{x-y}{2}, \tau = \delta^{1/q}||v||$. By Lemma 5.11, there exists $N \in \mathbb{N}$ such that

$$u + \tau B_{E_N} \subset \mathrm{Mid}(x, y, \delta)$$

and a compact set $K \subset \ell_p$ such that

$$\mathrm{Mid}(f(x), f(y), 2\delta) \subset K + (2\delta)^{1/p}||f(x) - f(y)||B_{\ell_p}$$

Combining the above gives:

$$f(u + \tau B_{E_N}) \subset K + (2\delta)^{1/p}||f(x) - f(y)||B_{\ell_p}$$

If δ is chosen small:

$$\begin{aligned}(2\delta)^{1/p}||f(x) - f(y)|| &\leq (2\delta)^{1/p}2\mathrm{Lip}_\infty(f)||x - y||_q \\ &= 4\mathrm{Lip}_\infty(f)2^{1/p}\delta^{\frac{1}{p}-\frac{1}{q}}\tau \\ &< \epsilon\tau\end{aligned}$$

Also, if θ is chosen large, then $\tau > \frac{1}{2}\tau^{1/q}\theta > t$. □

Remark 5.13. *In the above proof, we wee that the role of $p < q$ is critical. That is, the estimate of $(2\delta)^{1/p}||f(x) - f(y)||$ in the last step.*

Corollary 5.14. *If $1 \leq p < q < \infty$, then ℓ_q does not coarsely Lipschitz embed into ℓ_p.*

Proof. Suppose on the contrary that ℓ_q coarse Lipschitz embeds into ℓ_p, and $f : \ell_q \to \ell_p$ is a coarse Lipschitz embedding. Then there exists $\theta > 0$, $c_1, c_2 > 0$ such that

$$c_1||x-y||_q \leq ||f(x)-f(y)||_p \leq c_2||x-y||_q$$

whenever $x, y \in \ell_q$ with $||x-y||_q \geq \theta$.

Let $\epsilon > 0$. By previous proposition, there exist $\tau > \theta, u \in \ell_q, N \in \mathbb{N}$ and a compact $K \subset \ell_p$ such that

$$f(u+\tau B_{E_N}) \subset K + \epsilon\tau B_{\ell_p}$$

We may choose a sequence (u_n) in $u+\tau B_{E_N}$ satisfying $||u_n - u_m|| \geq \tau > \theta$. Write $f(u_n) = k_n + \epsilon\tau v_n$ where $k_n \in K$, $v_n \in B_{\ell_p}$. Since K is compact, say (k_{n_k}). WLOG assume $||k_{n_k} - k_{n_j}|| < \epsilon\tau$ for all k, j. Now

$$||f(u_{n_k}) - f(u_{n_j})|| \leq 3\epsilon\tau$$

As a result, LHS$>$ $c_1||u_{n_k} - u_{n_k}|| \geq c_1\tau$. Hence $c_1 < 3\epsilon$, which forces $c_1 = 0$ and a contradiction. □

5.3 The Kalton-Randrianarivony Graph

Following the approach [28], we resolve the case where $p < q$ by using the Kalton-Randrianarivony Graph:

Definition 5.15 (Kalton-Randrianarivony Graph)**.** Let $\mathbb{M} \subset \mathbb{N}$ be an infinite subset. Put

$$G_k(\mathbb{M}) = \{\bar{n} = (n_1, \ldots, n_k) : n_i \in \mathbb{M} \text{ and } n_1 < n_2 < \cdots < n_k\}$$

with the metric

$$d(\bar{n}, \bar{m}) = |\{j : n_j \neq m_j\}|$$

for $\bar{n} = (n_1, \ldots, n_k)$ and $\bar{m} = (m_1, \ldots, m_k)$.

Remark 5.16. *$G_k(\mathbb{M})$ is a bounded metric space with diameter k.*

Theorem 5.17 (Kalton, Randrianarivony [28])**.** *Suppose $1 < p < \infty$. Let X be a reflexive Banach space with the property that whenever $x \in X$ and $(x_n)_{n=1}^{\infty}$ is weakly null in X, then*

$$\limsup ||x - x_n||^p \leq ||x||^p + \limsup ||x_n||^p$$

Then, if $\mathbb{M}$ is an infinite subset of $\mathbb{N}$, $\epsilon > 0$ and $f : G_k(\mathbb{M}) \to X$ is any Lipschitz map, there exists an infinite $\mathbb{M}' \subset \mathbb{M}$ such that

$$diam f(G_k(\mathbb{M}')) \leq 2k^{1/p} Lip(f) + \epsilon$$

Proof. **Claim.** Let $f : G_k(\mathbb{M}) \to X$ be any Lipschitz map, $\epsilon > 0$. We claim that there exists $\mathbb{M}' \subset \mathbb{M}$ and $u \in X$ such that

$$||f(n_1 \ldots, n_k) - u|| < \mathrm{Lip}(f) + \frac{\epsilon}{2}$$

for all $(n_1, \ldots, n_k) \in G_k(\mathbb{M}')$

The proof of the claim is by induction. When $k = 1$, by the weak compactness of B_X, there exists $\mathbb{M}_0 \subset \mathbb{M}$ such that

$$\lim_{n \in \mathbb{M}_0} f(n) = u$$

exists weakly in X. So

$$\limsup_{n \in \mathbb{M}_0} ||f(n) - u|| = \limsup_{n \in \mathbb{M}_0} \limsup_{m \in \mathbb{M}_0} ||f(n) - f(m)|| \leq \mathrm{Lip}(f)$$

Therefore by removing finitely many terms in $\mathbb{M}_0$, we have

$$||f(n) - u|| < \mathrm{Lip}(f) + \frac{\epsilon}{2}$$

Suppose the claim is proved for any $\tilde{f} : G_{k-1}(\mathbb{M}) \to X$. If $f : G_k(\mathbb{M}) \to X$ is Lipschitz, by a diagonal argument we may find an infinite $\mathbb{M}_0 \subset \mathbb{M}$ such that

$$\lim_{n_k \in \mathbb{M}_0} f(n_1, \ldots, n_{k-1}, n_k) = \tilde{f}(n_1 \ldots, n_{k-1})$$

exists weakly for all $(n_1, \ldots, n_{k-1}) \in G_{k-1}(\mathbb{M}_0)$. Applyinh the induction hypothesis to $\tilde{f} : G_{k-1}(\mathbb{M}_0) \to X$ we may find $\mathbb{M}_1 \subset \mathbb{M}_0$ and $u \in X$ such that for all $\bar{n} \in G_{k-1}(\mathbb{M}_1$:

$$||\tilde{f}(\bar{n}) - u|| \leq \mathrm{Lip}(\tilde{f})(k-1)^{1/p} + \epsilon/2$$

Now

$$\begin{aligned}
\limsup_{n_k \in \mathbb{M}_1} ||f(\bar{n}, n_k) - u||^p &= \limsup_{n_k \in \mathbb{M}_1} ||f(\bar{n}, n_k) - \tilde{f}(\bar{n}) + \tilde{f}(\bar{n}) - u||^p \\
&\leq ||\tilde{f}(\bar{n}) - u||^p + \limsup_{n_k \in \mathbb{M}_1} ||f(\bar{n}, n_k) - \tilde{f}(\bar{n})||^p \\
&\leq (\mathrm{Lip}(f)(k-1)^{1/p} + \epsilon/2)^p + \limsup_{n_k \in \mathbb{M}_1} ||f(\bar{n}, n_k) - \tilde{f}(\bar{n})||^p
\end{aligned}$$

Note

$$\limsup_{n_k \in \mathbb{M}_1} ||f(\bar{n}, n_k) - \tilde{f}(\bar{n})||^p \leq \limsup_{n_k \in \mathbb{M}_1} \limsup_{n'_k \in \mathbb{M}_1} ||f(\bar{n}, n_k) - f(\bar{n}, n'_k)||^p \leq \mathrm{Lip}(f)^p$$

It follows that

$$\lim_{n_k \in \mathbb{M}_1} ||f(\bar{n}, n_k) - u|| \leq \mathrm{Lip}(f) k^{1/p} + \epsilon/2$$

By induction the claim is proved and the theorem follows immediately. □

Corollary 5.18. *If $1 \leq q < p < \infty$, then ℓ_q does not coarsely Lipschitz embed into ℓ_p.*

Proof. Suppose not, then we may find a $f : \ell_q \to \ell_p$ and a $B > 1$ such that

$$||x - y|| \leq ||f(x) - f(y)|| \leq B||x - y||$$

for all $x, y \in \ell_q$ with $||x - y|| \geq 1$.

Let $\varphi : G_k(\mathbb{N}) \to \ell_q$ be defined by $\varphi(n_1, \ldots, n_k) = e_{n_1} + \cdots + e_{n_k}$. Observe that φ is Lipschitz, $\mathrm{Lip}(\varphi) \leq 2$, and $||\varphi(\bar{n}) - \varphi(\bar{m})|| \geq 1$ whenever $\bar{n} \neq \bar{m}$. Therefore $f \circ \varphi$ is Lipschitz with $\mathrm{Lip}(f \circ \varphi) \leq 2B$.

By Theorem 5.17, we may find $\mathbb{M} \subset \mathbb{N}$ such that

$$\mathrm{diam}(f \circ \varphi)(G_k(\mathbb{M})) \leq 6Bk^{1/p}$$

But $\mathrm{diam}\varphi(G_k(\mathbb{M}) = (2k)^{1/q}$, the above implies $(2k)^{1/q} \leq 6Bk^{1/p}$, which is a contradiction.

□

We conclude this section by applying the previous results to show that classical sequence spaces have unique uniform structure. This result is first explicitly stated in [23]. Before the proof, let us state two results needed.

Theorem 5.19 (Ribe, (Theorem 2, Section 5 of [34]))**.** *The class of $\mathcal{L}_p$-space (recall Definition 4.16) is stable under uniform homeomorphisms. Moreover, the class of complemented subspace of L_p ($1 < p < \infty$) is stable under uniform homeomorphisms.*

Theorem 5.20 (Johnson, Odell [21])**.** *Let $1 < p < \infty$ and X be a separable $\mathcal{L}_p$-space. Suppose X does not contain a copy of ℓ_2, then X is linearly isomorphic to ℓ_p.*

Theorem 5.21 (Johnson, Linderstrauss, Schechtman [23]). *Let $1 < p \neq 2 < \infty$. Suppose that X, ℓ_p are uniformly homeomorphic, then X, ℓ_p are linearly isomorphic.*

Proof. Let $p \neq 2$. Let X be a Banach space uniformly homeomorphic to ℓ_p. Then according to Theorem 5.19, X is an $\mathcal{L}_p$-space and is complemented in L_p. By Theorem 5.20, it suffices to show that X does not contain a copy of ℓ_2.

Suppose on the contrary that X contains a copy of ℓ_2, then it follows that ℓ_2 coarsely Lipschitz embeds into ℓ_p. However, this must contradict to either Corollary 5.14 or Corollary 5.18. $\square$

References

[1] Aharoni I., Linderstrauss J., *Uniform equivalence between Banach spaces.* Proc. Amer. Math. Soc., **84**, 281-283, 1978.

[2] Albiac F., Kalton N., *Topics in Banach Space Theory, Second Edition*, Springer-Verlag New York, **GTM 233**, 2016.

[3] Alspach D., *Quotients of* c_0 *are almost isometric to subspaces of* c_0, Proc. Amer. Math. Soc., **76**, 285-288, 1979.

[4] Bartle R.G., Graves L.M., *Mappings between function spaces*, Proc. Amer. Math. Soc., **72**, 400-413, 1952.

[5] Benyamini Y., Lindenstrauss J., *Geometric Nonlinear Functional Analysis, Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, 2000.

[6] Bourgain J., *Remarks on the extension of Lipschitz maps defined on discrete sets and uniform homeomorphisms*, Geometric Aspects of Functional Analysis, 157-167, 1985.

[7] Christensen J.P.R., *On sets of Haar measure zero in Abelian Polish groups*, Israel Journal of Mathematics, **13**, 255-260, 1972.

[8] Diestel J., Uhl J.J., *Vector measures*, Amer. Math. Soc., 1977.

[9] Dutrieux Y., *Quotients of* c_0 *and Lipschitz Homeomorphisms.* Houston Journal of Mathematics, **27**, 2001.

[10] Enflo P., *On the nonexistence of uniform homeomorphisms between* L_p*-spaces*, Ark. Mat., **8**, 103-105, 1969.

[11] Godefroy G., Kalton N., Lancien G., *Subspaces of* $c_0(\mathbb{N})$ *and Lipschitz isomorphisms*, Geometric & Functional Analysis GAFA, **10**, 798-820, 2000.

[12] Godefroy G., Kalton N., *Lipschitz-free Banach spaces*, Stud. Math., **159**, 121-141, 2003.

[13] Godefroy G., Kalton N., Li D., *On subspaces of* L^1 *which embed into* ℓ^1, J. Reine Angew. Math., **471**, 43-75, 1996.

[14] Godefroy G., Lancien G., Zizler V., *The non-linear geometry after Nigel Kalton*, Rocky Mountain J. Math., **44**, 1529 - 1583, 2014.

[15] Gorelik E., *The uniform nonequivalence of* L_p *and* ℓ_p, Isr. Math., **87**, 1-8, 1994.

[16] Grothendieck A., *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16**, 1955.

[17] Heinrich S., Mankiewicz P., *Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces*, Studia Math., **73**, 225-251, 1982.

[18] James R.C., *Super-reflexive Banach spaces*, Can. J. Math., **24**, 896-904, 1972.

[19] James R.C., *On uniformly homeomorphic normed spaces. II*, Ark. Math., **16**, 1-9, 1978.

[20] Johnson W.B., Linderstrauss J., Preiss D., Schechtman G., *Almost Frechet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces*, Proc. London Math. Soc., **84**, 711-746, 2002.

[21] Johnson W.B., Odell E., *Subspaces of L_p which embeds into ℓ_p*, Compositio Math., **28**, 37-49, 1974.

[22] Johnson W.B., Zippin M., *On subspaces of quotients of $(\sum G_n)_{\ell_p}$ and $(\sum G_n)_{c_0}$*, Isarel J. Math., **13**, 311-316, 1972.

[23] Jonhson W.B., Linderstrauss J., Schechtman G., *Banach spaces determined by their uniform structures*, Geom. Funct. Anal. **6**, 430-470, 1996.

[24] Kadets M.I., *A proof of the topological equivalence of all separable infinite-dimensional Banach spaces*, Funkcional. Anal. i Prilozen, **1**, 61-70, 1967.

[25] Kahane J.P., *Sur les sommes vectorielles $\sum \pm u_n$*, C. R. Acad. Sci. Paris, **259**, 2577-2580, 1964.

[26] Kalton N., *Property (M), M-ideals and almost isometric structure*, Journal für die reine und angewandte Mathematik, 461, 1993.

[27] Kalton N., *Non linear geometry of Banach Spaces*, Rev. Mat. Complut., **21**, 7-60, 2008.

[28] Kalton N., Randrianarivony N.L., *The coarse Lipschitz geometry of $\ell_p \oplus \ell_q$*, Math. Ann., **341**, 223-237, 2008.

[29] Kwapien S., *Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients*, Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity VI, **44**, 583-595, 1972.

[30] Linderstrauss J., *On nonlinear projections in Banach spaces*, Mich. Math. J., **11**, 263-287, 1964.

[31] Milman V., *Geometric theory of Banach spaces. II. Geometry of the unit ball.* Uspekhi Mat. Nauk 26, 1971.

[32] Pelczynski A., *Projections in certain Banach spaces*, Stud. Math., **19**, 209-228, 1960.

[33] Pitt H.R., *A note on bilinear forms*, J. Lond. Math. Soc., **11**, 174-180, 1932.

[34] Ribe M., *On uniformly homeomorphic normed spaces, II*, Ark. Math., **1**, 1-9, 1978.

[35] Ribe M., *Existence of separable uniformly homeomorphic nonisomorphic Banach spaces*, Isarel J. Math., **48**, 139-147, 1984.

[36] Rosenthal H.P., *A characterization of Banach spaces containing* ℓ_1, Proc. Nat. Acad. Sci., **71**, 241-243, 1974.