

Dual-Axis Tilting Quadrotor Aircraft

An investigation into the overactuatedness and control thereof



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May 2017

MSc thesis submitted in fulfillment of the requirements for the degree of Masters of Science in the
Department of Electrical Engineering at the University of Cape Town

Keywords: Non-linear, control, allocation, quadrotor, UAV

”We’re gonna have a superconductor turned up full blast and pointed at you for the duration of this next test. I’ll be honest, we’re throwing science at the wall here to see what sticks. No idea what it’ll do.

Probably nothing. Best-case scenario, you might get some superpowers...”

Cave Johnson -Founder & CEO of Aperture Science

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Abstract

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The aim of this project is to design, simulate and control a novel quadrotor platform which can articulate all 6 Degrees of Freedom by vectoring each propeller's directional thrust. To achieve this the structure of the air-frame must redirect those thrust vectors to any desired orientation. This means it has to transform its configuration during flight, redirecting lift actuators whilst still maintaining stable attitude and position control, despite of such relative motion. In view of this required articulation the proposal is to add 2 axes (degrees) of extra actuation to each propeller. As a result each lift propeller can then be pitched or rolled relative to the body frame. This adaptation, to what is an otherwise well covered and highly researched platform, produces an over-actuated control problem. Actuator control allocation in the context of aerospace platforms is the primary contribution of this dissertation, with unique elements of non-linear (*state-space*) attitude control and plant uncertainty compensation.

The structure of the dissertation first presents the design which the subsequent dynamics and control are derived with respect to. Following that, the kinematics associated with rigid bodies are derived. Any unique effects that could apply to the design like gyroscopic, inertial and aerodynamic responses are investigated and then incorporated into the dynamics. Position and attitude control algorithms are first derived, then simulated and compared based on the plant's dynamics (*which later include filtering state estimation effects on those dynamics*). The relative performance of the controllers are evaluated but regular performance metrics for attitude and position control are ill-suited for such a system. Some time is spent discussing the consequence of this and how the controllers are actually evaluated. Finally the design is built and tested using readily available hobbyist components, thereafter conclusions are drawn on the success or failure of the design.

The purpose of the investigation is evaluation of practicality and feasibility for such a design, most importantly whether the complexity of the mechanical design is an acceptable compromise for the additional degrees of control actuation introduced. The outcome of the build is the assertion of economic feasibility (controller cost and effort) of such a prototype to expand the range of a quadrotor's motion. The design and control derivation presented here are by no means optimal nor the most exhaustive solutions, focus is placed on the system as a whole and not just one aspect of it.

This dissertation report is presented in a logical progression of concepts and information. In some cases the research and results were completed in a different order from how they are listed here...

Acknowledgements

Nomenclature

In order of appearance:

DOF - Degree of Freedom(s)

μ C - micro-controller

UAV - Unmanned aerial vehicle

SISO - Single input single output, control loop

MEMS - Micro-electromechanical system

DIY - Do it yourself

VTOL - Vertical takeoff/landing

IMU - Inertial measurement unit

BLDC - Brushless direct current, motor type

KV - Kilo-volt, BLDC motor rating

μ C - Micro-controller shorthand

PWM - Pulse width modulation

CH - Channel, radio control & PWM signals typically

RC - Radio control

OAT - Opposed active tilting

dOAT - Dual axis opposed active tilting

PD - Proportional derivative, control law

PID - Proportional integral derivative, control law

IBC - Ideal backstepping control

ABC - Adaptive backstepping control

PSO - Particle swarm optimization, gradient free genetic algorithm

BEM - Blade element theory

ESC - Electronic speed controller

MPC - Model predictive control

LQR - Linear quadratic regulator

LCF - Lyupanov candidate function

ITAE - Integral time additive error

TSK - Takagi-Sugeno-kang

I/O - Input/Output

RPM - Revolution Per Minute

RPS - Revolution Per Second

W.R.T - With respect to

LCF - Lyupanov Candidate Function

iff - If and only if

P.D - Positive definite, NOT proportional derivative

S.T - such that

FTC - Fault Tolerant Control

Symbols

Propeller Rotational Speed: Ω_i [rpm] for motors: $i \in [1, 2, 3, 4]$

Rotational speeds in [RPS] are used for Blade Element Theory Calculations in Chapter:3

Net body torque: $\mu \vec{\tau} = [\tau_\phi \ \tau_\theta \ \tau_\psi]^T \in \mathcal{F}^b$

Net body thrust: $\mu \vec{T} = [T_x \ T_y \ T_z]^T \in \mathcal{F}^b$

Body Position: $\vec{E} = [x \ y \ z]^T \in \mathcal{F}^I$

Euler Angles: $\vec{\mathcal{E}} = [\phi \ \theta \ \psi]^T \in \mathcal{F}^{I,v1,b}$

Servo 1 Position: λ_i [rad]

Servo 2 Position: α_i [rad]

Motor module actuator positions: $[\Omega_i \ \lambda_i \ \alpha_i]^T \in \mathcal{F}^{M_i}$

Actuator matrix: $u = [M_1 \ \dots \ M_4]^T \in \mathbb{U}^{12}$

Motor module displacement arm: $\vec{L}_{arm} = 195.16$ [mm]

Euler Rates: $\frac{d}{dt} \vec{\eta} = \dot{\vec{\eta}} = \Phi(\eta) \dot{\omega}_b = [\dot{\phi} \ \dot{\theta} \ \dot{\psi}]^T \in \mathcal{F}^{v1,v2,I}$

Angular Velocity: $\omega = [p \ q \ r]^T \in \mathcal{F}^b$

Linear Velocity: $\nu = [u \ v \ w]^T \in \mathcal{F}^b$

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Chapter 1

Introduction

1.1 Foreword

1.1.1 A Brief Background to the Study

A popular topic for current control and automation research is that of quadrotor UAVs. Attitude control of a quadrotor poses a unique 6-DOF control problem, to be solved with an under-actuated 4-DOF system. As a result the pitch, ϕ , and roll, θ , plants are not directly controllable. The attitude plant is often simplified around a stable operating point. The trimmed operating region is always at the inertial frame's origin; resulting in a zero-set point tracking problem. The highly coupled non-linear dynamics of a rigid body's translational and angular motions arise from gyroscopic torques and Coriolis accelerations (Sec: 3.3.1). Such effects are mostly negligible around the origin, hence the origin trim point decouples the system's nonlinearities. The control system can therefore reduce each state variable, $\vec{x}_b = [x \ y \ z \ \phi \ \theta \ \psi]^T$, to independent single-input single-output (*SISO*) plants. Those simplifications are derived in the Appendix:A.1.

As almost every quadrotor research paper mentions, the recent interest in the platform is due to increased availability of micro-electromechanical systems (*MEMS*) and low-cost microprocessor systems. These technical advancements accomodate onboard state estimation and control algorithm processes in real time. Developmental progress in quadrotors and, to a lesser extent unmanned aerial vehicles (*UAVs*) in general, has led to rapidly growing enthusiast communities. For example; HobbyKing [54] is now a name synonymous with providing custom DIY hobbyist quadrotor assembly kits and frames, no longer retailing only prebuilt commercial products like DJI Phantom [34] or ParrotAR [1] drones.

The avenue for potential application of both fixed wing and vertical take-off and landing (*VTOL*) UAVs is expansive; supporting civil [90], agricultural [95] and security [70] industries and not just recreational hobbyists. The quadrotor design provides a mechanically simple platform on which to test advanced aerospace control algorithms. Commercial drone usage in industry is already emerging as a prolific sector; especially in Southern Africa. Subsequently following the 8th amendment of civil aviation laws [99], commercial use of UAVs is now both legalized and regulated. Research into any non-trivial aspect of the field will therefore be extremely valuable to the field as a whole. Large scale quadrotor, hexrotor and even octocopter UAVs are popular intermediate choices for aerial cinematography and other high payload capacity applications. The cost of a commercial drones such as the SteadiDrone Maverik [78] are significantly less than a chartered helicopter, used to achieve the same panoramic aerial scenes or on-site inspections. One foreseeable issue which may hinder commercial drone progress in the agricultural and civil sectors is the consequential inertial effects from scaling up the aerial structures. When increasing the size of any vehicle, its performance is adversely affected if actuation rates are not proportionately increased.

1.1.2 Research Questions & Hypotheses

The difficulty with quadrotor control is that fundamentally, from their uncertainty and underactuation, they are ill-posed for 6 degree of freedom (*DOF*) setpoint tracking. A quadrotor inherently has only four controllable inputs; each propeller's rotational speed, $\Omega_{1,2,3,4}$, which are then abstracted to a net virtual control input net torque, $\vec{\tau}_\mu = [\tau_\phi \ \tau_\theta \ \tau_\psi]^T$, and a perpendicular heave thrust $\vec{T}_\mu = \sum_{i=1}^4 T(\Omega_i)$ in the \hat{z}_b direction. Those four inputs are then used to effect both the translational X-Y-Z positions, $\vec{x} = [x \ y \ z]^T$, and angular pitch, roll and yaw attitude rotations, $\vec{\eta} = [\phi \ \theta \ \psi]^T$. Pitch and roll torques, τ_ϕ and τ_θ respectively, are produced from differential thrusts of each opposing propellers. Yawing torque, τ_ψ , is induced only by the sum of aerodynamic drag torques about each propeller's rotational axis. Aerodynamic drag and thrust responses are highly non-linear (detailed later in Sec:3.2.1) and difficult to approximate as sources of control torque. As a result the body's yaw channel control is depreciated. Stemming from the system's under-actuation, the attitude control problem becomes a zero set point problem because any other attempt to track attitude is ill-posed.



Figure 1.1: Bell/Boeing V22 Osprey actuation, notations pertinent to patent [96]

The aim of this dissertation is to implement attitude and position dynamic set point tracking on a quadrotor UAV by solving the problem of its inherent under-actuation. Inspired by Boeing/Bell Helicopter's V22 Osprey (Fig:1.1) and the tilting articulation of its propellers, the prototype design proposed here (detail in Sec:2.1) introduces two additional actuators for each of the quadrotor's four lift propellers. Specifically, adding rotations about the \hat{X} and \hat{Y} axes for each motor/propeller pair, the resultant are four separately articulated 3-dimensional thrust vectors instead of a bound perpendicular net heave force. The control problem is then posed as the design and allocation of net forces, $\vec{F}_{net} = [F_x \ F_y \ F_z]^T$, and torques, $\vec{\tau}_{net} = [\tau_\phi \ \tau_\theta \ \tau_\psi]^T$, for a general 6-DOF body such that for any given trajectory, $\vec{x}_d(t) = [x \ y \ z \ \psi \ \theta \ \phi]^T$, the error state $\vec{x}_e(t) = \vec{x}_d(t) - \vec{x}_b(t)$ is asymptotically stable. Trajectory stability is explicitly defined later in the context of Lyapunov stability analysis in Sec:4.4.

$$\lim_{t \rightarrow \infty} \vec{x}_e(t) = \vec{0} \quad \forall \vec{x} \in \mathbb{R}^n \quad (1.1)$$

Where n is number of the degrees of freedom the system has, typically a 6-DOF plant for rigid bodies. The over-actuation brings about the need for a control allocation scheme, one which distributes the 6 commanded system inputs (net torques and forces) among the actuator set (12 actuators) in order to optimize some objective function secondary to that of Eq:1.1. The potential improvement(s) for exploiting those over-actuated elements is the most novel outcome which the project could yield. A cost function aimed at optimizing some aspect unique to aerospace bodies is going to be a completely unique contribution.

Part of the control research question is the multivariable dynamic modeling of the system; making as few assumptions as possible to the non-linear dynamics involved in the quadrotor's motion and its operational conditions. Common linearizations often applied to the quadrotor's control plant will not hold true for the more aggressive maneuvers; they are dependent on small angle approximations and neglect 2nd or higher order effects. To produce a stabilizing control law solution there first needs to be a dynamic model that incorporates both multibody and actuator dynamics, against which the controller efficacy can be tested. The final key outcomes for the project are; the prototype design, its dynamic model and simulation analysis, the resultant control law produced and finally conclusions drawn on all of the above.

For a rigidly connected multibody system with rotational joints between sub-bodies, the induced relative motion between those sub-bodies will produce a lot of unwanted dynamics like inertial and gyroscopic responses, amongst others... A rotating propeller will respond to pitching or rolling much like a Control Moment Gyroscope [130] or a flywheel, producing a precipitating torque cross product. A less trivial aspect which is occasionally considered are the aerodynamic effects produced from the propeller's aerofoil profile. Such induced responses manifest normal to the propeller's rotational axis. Those aspects are not typically compensated for due to a quadrotor's fundamental co-planar propeller counter-rotating pairs which mostly negate such effects. A plant dependent control law is needed for dynamic compensation which reduces uncertainty associated with the subsequent stability proof.

1.1.3 Significance of Study

Owing to the huge popularity of quadrotor platforms as research tools (i.e [10,18,47], etc...), any work that builds on UAV and quadrotor fundamentals will prove to be valuable. With that being said, there is already a plethora of research on the subject of linear and non-linear control techniques for quadrotor platforms (surveyed in Table:1.1). Attitude control loops are the most common topic for research, requiring a unique under-actuated solution and mostly linearized around the origin (Appendix:A.1). Far less common is the application of optimal flight path and trajectory planning to a quadrotor's (*augmented*) autopilot system. The difficulty and ill-posed aspect of a quadrotor's attitude control does not hold true for its position plant, so standard techniques can be applied for waypoint and trajectory planning once the attitude control problem has been addressed.

The most significant aspect of this project is the attitude control, discussed later in Sec:4.5. The over-actuation of the proposed design and, more critically, the manner in which the controller's commanded (virtual) output is distributed among those control effectors would, at the time of writing, appear to be the first of its kind. Otherwise known as control allocation, the requirements of the distribution algorithm(s) are outlined in Sec:4.7. Dynamic setpoint attitude control for aerospace bodies is not a subject heavily researched outside the field of satellite attitude control. Even papers that propose similarly complicated mechanical over-actuation (expanded upon in next in the literature review, Sec:1.2) hardly broach the topic of tracking attitude set points away from the origin.

The control plant presented in this dissertation, developed in Chapter:4, does indeed close both the position and attitude control loops. There is, however, no consideration of trajectory generation nor flight path planning as such topics are well discussed elsewhere. Once closed loop position and attitude control have been achieved, the control algorithms can be adjusted to incorporate higher order state derivative (acceleration, jerk and jounce) tracking needed for nodal waypoint planning. The heuristics involved with flight path planning are well documented and their application is an easily implemented task [48, 79, 111].

Where possible, the system identification and control (*design* and *allocation*) for this project is kept modular and generally applicable. The intention is that its pertinence falls not only within the UAV field but also to any aerospace attitude control plant, rigid or otherwise. Hopefully the investigation can be expanded upon with more focused research on one of the subsystems without compromising

the stability of the whole plant. Provisionally, an obvious outcome which the project could yield is improved yaw control of a quadcopter's attitude. However, if the express purpose was just to improve yaw control, it could be done with a dramatically less complicated design...

Moreover, this dissertation could provide greater insight into higher bandwidth actuation and hence faster control responses for larger aerospace bodies. Any standard quadrotor uses differential thrusts to develop a torque about its body. Such actuation suffers a second order inertial response when the propellers accelerate or decelerate; $\vec{\tau}_p = \mathbb{I}_p \dot{\Omega}_i$ for $i \in [1 : 4]$. Prioritizing pitching the propeller's principle axis of rotation in lieu of changing the rotational speed could potentially improve the actuator plant rate response. This is entirely dependent on how the allocator block is prioritized (presented in Sec:4.7). The exact effects of different actuator prioritization and distribution in the context of aerospace control are, at the time of writing, unique to this research.

1.1.4 Scope and Limitations

Scope

Critical to this project is the conceptualized design and prototyping of a novel actuation suite to be used on a quadrotor platform. The control research question is to apply dynamic attitude setpoint control to the quadrotor platform. Stemming from this is an investigation into the kinematics that are potentially influenced by the design changes and the structure's relative motion. In order to apply correct control theory to achieve the attitude tracking on the physical prototype, plant dynamics must first be identified for the controller to be optimized correctly. Aspects of the mechanical design are detailed in the next chapter; Ch:2.1. There is no scope beyond the cursory investigation for materials analysis or stress testing of the design. This dissertation's scope focuses mainly on the control derivation and embedded systems design/implementation and not the structural integrity of a proposed frame given the forces it may undergo. Physical measurements are only made for critical kinematics such as inertial measurements for the second order gyroscopic and inertial dynamic responses.

As mentioned in the antecedent Sec:1.1.3; trajectory and flight path planning are not ubiquitous with this dissertation. Derivations for the differential equations of motion which dictate a 6-DOF body's movement are applicable to any aerospace body, rigid or otherwise. Some particular standards are used, like Z-Y-X Euler Aerospace rotational matrix sequences, all of which are covered in Sec:2.2. The control plant is stabilized with non-linear state-space control techniques in the time domain, aided and justified by Lyapunov stability theorem [15, 84, 104]. Alternative solutions using model predictive control (*MPC*) or quantitative feedback theory (*QFT*) could provide more refined or effective controllers, however they are not discussed here and remain open to further investigation. Quadrotor attitude control is commonly stabilized with feedback linearizations, decoupling the plant around a trim point so that SISO techniques can be applied. A derivation of such a linearization is included in App:A.1 but beyond that there are no further discussions. Any comparisons between non-zero and zero set-point attitude controller efficacy for quadrotors are difficult as the fundamental objectives are in stark contrast with one another.

Arguably the most important and potentially novel aspect of this project is the control allocation. The system has 12 plant inputs and 6 output variables to be controlled. There is then an entire set of compatible actuator solutions, $u \in \mathbb{U} \in \mathbb{R}^{12}$, which exist for each commanded input. Such a plant is classified as over-actuated. Ergo, there must be some logical process as to how those 12 inputs are combined to achieve the desired 6 control plant inputs, specifically net force \vec{F}_{net} and net torque $\vec{\tau}_{net}$ acting on the system. Appropriate techniques are first derived in Sec:4.7 then simulated and compared before a final solution is implemented in Section:5.4. It is not a comprehensive survey of every possible allocation scheme but rather an analysis of the sub-set of problems and design of what is regarded as a logical and pertinent approach.

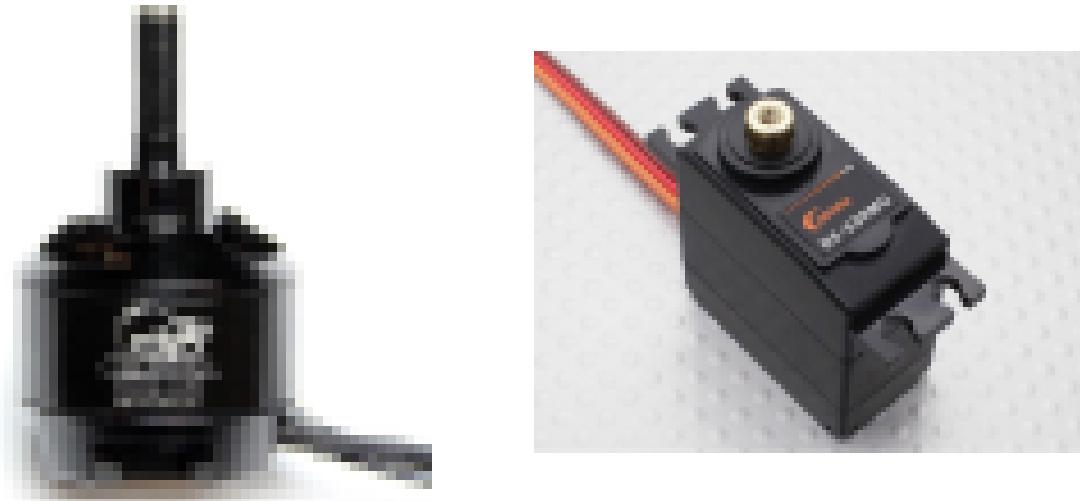
With regards to the prototype design, in Sec2.1, it is assumed that certain aspects are readily available and require no design/development. Particularly the state estimation, updated through a 4-camera positioning system fused with a 6-axis IMU through Kalman Filtering (Sec:5.5), is assumed to accurate and readily disposable at a consistent 50 Hz. Hence state estimation is included but is bereft of intricate detail, this is another topic which remains open to further investigation.

Limitations

The biggest constraint faced by the design is the net weight of the assembled frame. Lift thrusts which are required to keep an aircraft aloft and oppose the net gravitational force are obviously dependent on the body's net weight. The steady state actuator rates ought to be far less than saturation conditions to ensure sufficient actuator headroom to implement control actuators. Conversely the structure's net weight is mostly dependent on the lift motors, often being the heaviest part of the vehicle (batteries included). A trade-off between net weight and actuator effectiveness makes designing the prototype a balancing act of compromise; added actuation is needed to produce the desired thrust vectoring. That added actuation is going to increase the weight which then requires more thrust force to ensure the vehicle remains airborne. Larger motors then need stronger actuators to effect the relative motion and overcome the bodies inertial response. It is a compromise between the weight of the body and the strength/quality of the actuation.

To forego the deliberation detailed above, reducing the possibility of unbounded scope creep, a design limitation is self-imposed on the prototype design. Restricting the propeller diameter, and hence maximum thrust/frame size, will provide a constraint upon which all other design considerations must adhere to. Smaller propellers require far greater rotational speeds to produce similar levels of thrust that their larger diameter counterparts could provide. Electing to use 3 bladed 6×4.5 inch small diameter propellers constrained the maximal overall dimensions of the prototype, but as a consequence required very high RPM motors. Specifically a set of four Cobra-2208/2000KV [30] brushless direct current (*BLDC*) motors are used for lift actuation (Fig:1.2a).

A direct consequence of that decision is, provisionally based on thrust tests, the net thrust actuation disposable to the control loop is limited to around $950 \text{ [g]} \approx 9.5 \text{ [N]}$, per motor at 14.1 [V] . That thrust test data is provided from the official Cobra motor's website, [30], included in App:C.2 and verified independently through testing in Sec:3.2.1. It is critical to ensure the control block does not induce saturation of those BLDC motors. The frame weight needs to be under 50% of the maximum available thrust; or roughly below 2 [kg] . Saturation conditions are detailed later in Sec:4.7.



(a) Cobra CM2208/2000KV BLDC motor [40]

(b) Corona DS-339MG digital servo [54]

Figure 1.2: Mechanical actuators

Another aspect of limitations produced by design decisions made, mostly to reduce prototype costs and weight, is to use of 180° rotation servo motors. Here Corona DS-339MG metal gear digital servos (Fig:1.2b) are used. The servos are for each individual motor's \hat{X}_{M_i} and \hat{X}_{M_i} axial pitch and roll actuators respectively, terms λ_i and α_i to differentiate from body pitch ϕ and roll θ . Servos act in place of either BLDC gimbal or stepper motors with closed loop position control. The latter pair would both accommodate for continuous ($> 2\pi$) rotation but would need their own control design. Continuous rotation (velocity controlled) servos could otherwise be used but would similarly require angular rotation feedback. Any rotations beyond 2π would similarly require slip rings to transmit power throughout rotational movement to avoid mechanical interference from connection lines. Implementing such a design whilst maintaining an acceptable weight would prove too costly nor would it provide additional insight attained from experimental testing. The effect of servo rotation limits can be evaluated in simulation and if it proves to be significant, continuous rotation can be implemented...

Discrete elements for the whole system could potentially limit performance but are mitigated where possible. For example analogue servos have an associated 1 [ms] dead time from their 50 [Hz] refresh rate. That can be addressed by using faster, albeit more expensive, digital servos which samples at 330 [Hz]. The prototype's flight controller needs to provide 12 pulse-width modulated (*PWM*) output compare channels for the 8 servos and 4 BLDC speed controllers. State updates from a ground control station and a fail safe 6-channel (*CH*) radio control receiver module also needs to be processed by the micro-controller (μC) system. Particular attention is paid to the embedded system design and layout in Sec:2.4.

1.2 Literature Review

1.2.1 Existing & Related Work

The field of transformable aerospace frames is not new, with many commercial examples seeing successes over their operational life span. The most notable tilting-rotor vehicle is the Boeing/Bell V22 Osprey [39] aircraft. First introduced into the field in 2007, the Osprey has the ability to pitch its two lift propellers forward to aid translational flight after vertically taking off or landing. In addition to this there have been many papers published on similar tilting bi-rotor UAVs for research purposes.

Birotors



Figure 1.3: General structure for opposed tilting platform, taken from [44]

Research into birotor vehicles (Fig:1.3) with ancillary lift propeller actuation is oft termed *Opposed Active Tilting* or *OAT*. Such a rotorcraft's mechanical design applies either a single *oblique* 45° tilting axis relative to the body; [13, 45, 65], or a *lateral* tilting axis, adjacent to the body; [26, 67, 94, 113]. Leading research is currently focussed on applying doubly actuated tilting axes to birotor UAVs. *Dual axis Opposed Active Tilting* or *dOAT* introduces vectored thrust with independent propeller pitch and roll motions to further expand the actuation suite, [3, 44]. A birotor is sometimes considered preferable to higher degree of freedom multirotor platforms due to their reduced controller effort. However the controller plant derivation, typically requiring feedback linearization and virtual plant abstraction, often detracts from the quality and effectiveness of its stability solution as a result of the birotor's underactuation.

Birotor attitude control mostly introduces plant independent PD [13] and PID [94] stabilizing controller schemes. Sometimes more computationally intensive and plant dependent *ideal* or *adaptive* backstepping controllers are implemented, presented in [65, 113] and [67] respectively. The gyroscopic response of a birotor vehicle's attitude system is more pronounced than that of a quadrotor, derived in Sec:3.3, and so feedback linearisation is almost always used. In an interesting progression from the norm, [73] proposed a unique PID co-efficient selection algorithm for a bi-rotor control block. Using a particle swarm optimization (*PSO*) technique, similar to [134], the coefficients were globally optimized around a given performance metric. However their performance criterion is a standard integral time-weighted absolute error (*ITAE*) term and nothing more appropriate involving effects unique to flight systems was used. *PSO* algorithms iteratively search for a globally optimized solution and offer independent, gradient free based optimization. In subsequent chapters, controller coefficients are optimized using *PSO* algorithms, shown later in Sec:5.1.

Quadruped

Expanding on bi-rotor vehicles, the quadrotor UAV is a popular and well researched multirotor platform due to its mechanical simplicity. The current popularity of quadrotors as research platforms started in 2002, with a control algorithm implemented on what is now known as the X4-Flyer quadrotor [47, 101]. Alternative iterations then followed; like the Microraptor [106] and STARMAC [55] quadcopters which have subsequently been built and tested. A multitude of literature exists around quadrotor kinematics and their control [5, 18, 29, 79, 105], however dedicated rigid body 6-DOF dynamic papers [81, 97] offer better explanations of the kinematics. Often the plant's dynamics are simplified around an origin trim point and assumed to reduce into 6 SISO plants for each degree of freedom (App:A.1). Lately research projects have begun to incorporate non-linear aerodynamic effects like drag and propeller blade-element momentum (*BEM*) theory into the plant model [22, 55, 108]. The higher fidelity models for thrust and propeller responses offer more precision by making less linearisations and assumptions; [7, 55].



(a) Inspire1 articulated upwards

(b) Inspire1 articulated downwards

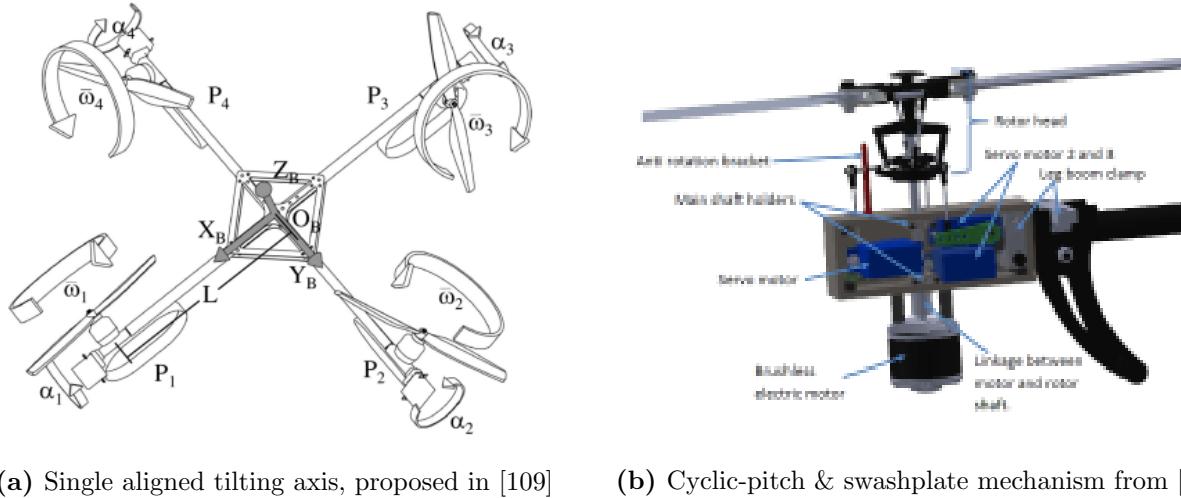
Figure 1.4: DJI Inspire1, the notations are with regards to the DJI patent [131]

At the time of writing, the only commercial UAV multirotor capable of structural transformation is the DJI Inspire1 quadrotor [33], manufactured by Shenzhen DJI Technologies. DJI are better known for their hugely successful DJI Phantom commercial quadrotor [34]. The Inspire1 can articulate its supporting arms up and down as shown in Fig:1.4, both images were sourced from the drone's patent; held by SZ DJI Tech Co [131]. The purpose of such transformations is to both alter the center of gravity and to further expose a belly mounted camera gimbal for panoramic viewing angles. This changes the bodies inertial tensor about its center of gravity, affecting the second order inertial response opposed to changes in angular velocity; $\vec{\tau} = J\Delta\vec{\omega}_b$. That variable inertial matrix is a detrimental effect which makes researchers apprehensive of transformable aerospace frames. The range of transformations which the Inspire1 frame can undergo is limited to just articulating its arms up and down.

In a similar fashion to the progression seen in birotor state-of-the-art, quadrotor research is engaging the topics of single and dual axis propeller module tilting articulations. The extra actuation scheme(s) were first conceptualized and implemented on a prototype related to an ongoing project covered in two reports; [109, 110]. Those authors modified and tested a QuadroXL four rotor helicopter, produced by MikroKopter [41], to actuate a single axis of tilting aligned with the frame's arms (Fig:1.5a). Their proposed control solution, detailed next in Sec:1.2.2, assumes no nominal linearised conditions around hover flight, unlike a similar single axis tilting quadrotor prototype designed by Nemati in [87]. The latter is *simulated* but remains as yet untested.

One approach to improving quadrotor flight response is to alter the manner in which the thrust is mechanically actuated, potentially improving actuator bandwidth (demonstrated in [2, 38]). Drawing from helicopter design, [86] purported a novel quadrotor UAV prototype that used swashplates for varying the propeller pitch and generating torque moments. The aim was a design which was not dependent on rotational speed controlling power electronics (*ESCs*) to actuate variable thrust forces.

Petrol motors were intended for use in place of BLDC motors. Furthermore, the design proposed a single axis of tilt actuation to each of the four motor modules. Whilst mechanically complex, [86] made use of existing off-the-shelf hobbyist helicopter components to design a rotor actuation bracket (Fig:1.5b). The cyclic-pitch swashplates used could apply pitching and rolling torques [88], τ_ϕ and τ_θ , about each propeller's hub, its *principle axis of rotation*. The torques were induced by altering the blades angle of attack throughout the propeller's rotational cycle. The actuation rate of such a configuration is far greater than that of a differential torque produced rolling/pitching motion.



(a) Single aligned tilting axis, proposed in [109]

(b) Cyclic-pitch & swashplate mechanism from [86]

Figure 1.5: Tilt-rotor mechanisms

Irrespective of the strong initial design in the early stages of his project, it would appear that the research in [86] suffered due to time constraints. The introductory derivation on aerodynamic effects and deliberation over the design provide clear insight into the projects goals. However the control solution and system architecture, electronic and software, are severely lacking. A brief introductory proposal of an MPC attitude control system detracted from the comprehensive dynamics discussed. The project ended before testing, simulation or results could be obtained. Unfortunately, despite the novel over-actuated design, there was no discussion given on how that actuator allocation, being the most unique aspect of the project, would be achieved.

**Figure 1.6:** Dual-axis tilt-rotor mechanism used in [38]

Finally, the most crucial research to mention is a project completed by Pau Segui Gasco in [38], which was a dual presented MSc project with Yazan Al-Rihani whose respective research was presented in [2]. At the time of writing, this would appear to be the only project published pertaining to *over-actuation* in aerospace bodies implemented and tested on a quadrotor platform. The research was split between the two authors who completed the electronic/control design and the mechanical design for their respective MSc dissertations. Shown in Fig:1.6, the dual-axis articulation is achieved using an RC helicopter tail bracket and servo push-rod mechanism; reducing the mass of the articulated components but limiting the range of its possible actuation. Considering the propellers as energy

storing flywheels, the induced gyroscopic response was then treated as an additional controllable actuator plant. Their commanded virtual control is distributed by weighted inversion amongst the actuator set, Sec:1.2.2. The whole project justifies the extra actuation as fault tolerance redundancy (*FTC*) but does not necessarily prove how such a redundancy could be beneficial.

1.2.2 Notable Quadrotor Control Implementations

Quadcopter Attitude Control

Note that here $\vec{\eta}$ is not necessarily an Euler angle set but any attitude representative state variable.

Attitude control of a 6-DOF aerospace body, quadrotor or otherwise, is best described by [125] and referred to as *the attitude control problem*. For a rigid body that has an instantaneous Euler attitude state $\vec{\eta}_b$ and a desired state $\vec{\eta}_d$, the problem is to then find a stabilizing torque control $\vec{\tau}_\mu$. The control law is dependent on some feedback error state $\vec{\eta}_e$. Quaternion attitude states later replace Euler angles for attitude representation, $\vec{\eta}_b \Rightarrow Q_b$. The general control law is defined as:

$$\vec{\tau}_\mu = h(\vec{\eta}_d, \dot{\vec{\eta}}_d, \vec{\eta}_b, \dot{\vec{\eta}}_b, t) \quad \in \mathcal{F}^b \quad (1.2a)$$

$$\rightarrow \vec{\tau}_\mu = h(\vec{\eta}_e, \dot{\vec{\eta}}_e, t) \text{ given some error state } \vec{\eta}_e \quad (1.2b)$$

Where the control law designs a net torque such that both the angular position and rates are stabilized with the bounded limits; $\lim \vec{\eta}_b \rightarrow \vec{\eta}_d$ and $\lim \dot{\vec{\eta}}_b \rightarrow \dot{\vec{\eta}}_d$ respectively as $t \rightarrow \infty$. A distinction must be made between euler angular rate vector, $\dot{\vec{\eta}}_b = [\dot{\phi} \ \dot{\theta} \ \dot{\psi}]^T$ and the angular velocity vector $\vec{\omega}_b = [p \ q \ r]^T$. Depending on how the attitude is posed; with rotation matrices [69, 81, 97], quaternions [37, 43, 46, 69] or otherwise (direct cosine matrix etc ...) the error state $\vec{\eta}_e = \vec{\eta}_d - \vec{\eta}_b$ could then differ to a (Hamilton) multiplicative relationship. [125] describes these conventionally different error states.

Simulation and modelling papers often rely on Euler angle based rotation matrices for attitude representation, [16, 18, 80, 87, 107], without addressing the inherent singularity associated with such an attitude representation (known as gimbal lock, [116], Sec:3.2.4). The alternative quaternion attitude representation, first implemented in 2006 on a quadrotor UAV platform in [122], is often used in lieu of rotation matrices. Quaternions do have their own caveat of *unwinding* as a result of the dual-coverage in \mathbb{R}^3 space, discussed in [83] and derived mathematically later in Sec:3.2.6. Quaternions are $\in \mathbb{R}^4$ variables for attitude representations in \mathbb{R}^3 and so a mapping $\mathbb{R}^4 \rightarrow \mathbb{R}^3$ produces an infinite coverage set for each unique attitude state in \mathbb{R}^3 .

Quadrotor plant dynamics, as mentioned previously, are often simplified; especially when represented with a 3-variable Euler angle set, $\vec{\eta} = [\phi \ \theta \ \psi]^T$. The cross-product gyroscopic and Coriolis terms are both neglected when the angular velocity is small, $\vec{\omega}_b \approx \vec{0}$, and the inertial matrix J_b is approximately diagonal, $\text{rank}(J_b) = x$ for $x \in \mathbb{R}^x$. The consequence of such simplifications is the deterioration of both the gyroscopic torque term, $\vec{\tau}_{gyro} = -\vec{\omega}_b \times J_b \vec{\omega}_b \approx \vec{0}$ and the Coriolis force term, $\vec{F}_{cor} = -\vec{\omega}_b \times m\vec{v}_b \approx \vec{0}$ in the body's dynamics (Ch:3 for context). Once the coupled cross-product terms are no longer of consequence, the 6 DOF trajectory, $\vec{x} = [x \ y \ z \ \phi \ \theta \ \psi]^T$, can be treated as a series of independent SISO plants each controlled with an appropriate technique. Quaternion represented attitude plants cannot easily be decomposed into individual single-input-single-output systems (quaternion dynamics in Sec:3.2.5). So a quaternion combined four variable attitude state-space vector is then used, $Q_b = [q_0 \ \vec{q}]^T$, for the major loop trajectory plant of $\vec{x}(t)$.

Opensource and hobbyist flight controller's software (Arducopter [4], Openpilot [74] whose firmware stack is now maintained by LibrePilot, CleanFlight [27], BetaFlight [11], etc ...) for custom fabricated UAV platforms all apply their own flavour of structured attitude controllers and state estimation algorithms, based on onboard hardware sensor fusion. The article *Build Your Own Quadrotor* [75] summarizes the control structures implemented on a range of popular flight controllers.



Figure 1.7: ArduCopter PI control structure for pitch angle θ ; from [75]

The most popular of which, ArduCopter, implements a feed-forward PI compensation controller, whose single channel control loop for an attitude roll channel θ is shown in Fig:1.7. PI, PD and PID controllers are all popular and effective plant independent control solutions for general attitude plants. Table:1.1 lists the common attitude control blocks (not exclusively quadrotors UAVs but MAVs too) and which projects they've been implemented in, after which a critique on the more unique adaptations is given. One ideal backstepping controller listed in Table:1.1, presented in [113], applies an algorithm derived through Hurwitz polynomials unlike the Lyapunov based backstepping control law(s) derived later in Ch:4.

Controller Type	Independent	Dependent	Total Examples
PI	[125]	[125]	2
PD	[2, 79]	[37, 87]	4
PID	[16, 19, 105, 109, 125]	[55, 107, 125]	8
Lead	[29, 101]	N/A	2
LQR	[19]	N/A	1
Backstepping controllers			
Ideal	[80, 113]	[80]	3
Adaptive	[10, 32, 67, 85]		4

Table 1.1: A breakdown of common attitude controllers

In a collection of papers, written by the most prolific early quadrotor author(s) S. Boubdallah and R. Siegward [17–19]; a range of different attitude control implementations are surveyed and tested on the OS4 platform. The final paper, [18], derived and practically tested an integral backstepping attitude controller on the OS4 quadrotor platform. It builds on their research presented earlier in [19] which provides an analysis of PID vs linear quadratic regulator (*LQR*) attitude controllers, specifically in the context of underactuated quadrotor attitude control. LQR controllers aim to optimize the controller effort with actuator inputs $u \in \mathbb{U}$; controller effort is then $\|u\|_2$ or the L_2 norm (magnitude) of the plant input. Although, in theory, solving the associated Riccati cost function may produce a cost optimal, stable and efficient control law it needs exact plant matching. In reality, exact plant matching is difficult to achieve for a quadcopter or any aerospace body for that matter. The resultant controller in [19] achieved asymptotic stability but had poor steady state performance due to low accuracy of the identified actuator dynamics and poor confidence inertial measurements.

Adaptive Backstepping Control (in [129] or any other example in Table:1.1) expands on nominal ideal backstepping fundamentals by introducing disturbance and plant uncertainty terms into the Lyapunov energy function to be used for the backstepping suppression. For Lyapunov iteration the adaptive backstepping process requires a disturbance estimate derivative or *update law* which is often difficult to quantify. Approximation of plant disturbances without *a priori* information is a complex subject. At some point in the design an approximation heuristic must be adopted and that typically involves some compromise of performance or accuracy. One example of disturbance approximation in [32] proposes using a statistical projection operator (or *proj(.)*, [24]). When used in adaptive control, presented similarly in [25], the projection operator *proj(.)* ensures a derivative based estimator can be bound for adaptive regression approximation [100].

Although the control implementation is not explicitly backstepping, in [135] a sliding mode controller was used to compensate for the disturbances in an Unmanned Submersible Vehicle attitude plant. The underwater current disturbances were approximated using a fuzzy logic system, specifically a *zero-order Takagi-Sugeno-kang* fuzzy approximator. The TSK system has been shown in [82] to mimic an artificial neural network approximator; where the fuzzy TSK system is more comprehensible than the latter. Statistical analysis and investigation of approximators without *a priori* knowledge of a system are well beyond the scope of this research but are worth mentioning.

Single/Dual Axis Control & Allocation

The steady state attitude control actuation introduced with either single or dual axis articulation provides room for more control goals to be achieved. Of the few papers published on tilting-axis quadrotors, PD controllers (used in [87] and again in [2,38]) and PID controllers (collectively [109,110]) are the standard fare for attitude control blocks. For either of these systems, there needs to be an allocation rule to distribute a commanded input amongst the actuator set. In the control allocation survey [60] the author describes the control allocation problem for a dynamic plant:

$$\dot{\vec{x}} = f(\vec{x}, t) + g(\vec{x}, \vec{\nu}, t) \quad \vec{x} \in \mathbb{R}^n, \vec{\nu} \in \mathbb{R}^m \quad (1.3a)$$

$$\vec{y} = c(\vec{x}, t) \quad (1.3b)$$

State variables of [60] were changed to match this dissertation's conventions. In the state space equation Eq:1.3a, it is assumed the plant input, $\vec{\nu}$, has a linear multiplicative relationship with the input response, $g(\vec{x}, t, \vec{\nu}) \Rightarrow g(\vec{x}, t)\vec{\nu}$. That linear relationship is a prerequisite for most allocation inversion rules but is not a necessity...

In Eq:1.3a the state $\vec{x} \in \mathbb{R}^n$ has associated plant dynamics $f(\vec{x}, t)$ and a input response $g(\vec{x}, \vec{\nu}, t)$. Setpoint tracking control equates the output variable with the state, in practice only state estimate is available:

$$\vec{y} = c(\vec{x}, t) = h(\vec{x}) = \hat{\vec{x}} \quad (1.4)$$

Therefore the output \vec{y} has the same dimension has the state \vec{x} ; or rather $\vec{x}, \vec{y} \in \mathbb{R}^n$. In an ideal, well posed system the number of actuator inputs equals the number of outputs; that being $\dim(\vec{x}) = \dim(\vec{\nu}) \in \mathbb{R}^n$. In the case where the control input $\vec{\nu} \in \mathbb{R}^m$, if $m > n$ the problem is then over-actuated and an abstraction is needed. The system mechanically commands a physical control input $\vec{\nu}_c$, dependent on explicit actuator positions $u \in \mathbb{U} \in \mathbb{R}^m$ as per some *effectiveness* function derived from the actuator plant's dynamics:

$$\vec{\nu}_c = B(\vec{x}, u, t) \quad \in \mathbb{R}^n \quad (1.5)$$

Assuming that some higher level control law designs well a satisfactory stabilizing virtual control input from the error state(s) $\vec{\nu}_d = h(\vec{x}_d, \dot{\vec{x}}_d, \vec{x}_b, \dot{\vec{x}}_b, t) \in \mathbb{R}^n$. The allocation rule then aims to solve for an explicit actuator position $u \in \mathbb{U} \in \mathbb{R}^m$ derived from $\vec{\nu}_d$ which actuates the physically commanded control input $\vec{\nu}_c$, minimizing the deviation or slack \vec{s} between virtual desired and physical commanded inputs $\vec{\nu}_d$ and $\vec{\nu}_c$ respectively. Allocation is effectively a paradigm which transforms dimensions $\mathbb{R}^m \rightarrow \mathbb{R}^n$ using a commanded actuator matrix position $u \in \mathbb{R}^m$. An over-actuated plant can be summarized in non-linear state space as:

$$\dot{\vec{x}} = f(\vec{x}, t) + g(\vec{x}, \vec{\nu}_c, t) \quad \vec{x} \in \mathbb{R}^n \quad (1.6a)$$

$$\vec{\nu}_c = B(\vec{x}, u, t) \quad \vec{\nu}_c \in \mathbb{R}^n \quad (1.6b)$$

$$\text{with } u \in \mathbb{U}^m \text{ subject to some } \min(\vec{s}) \text{ such that } \vec{s} = \vec{\nu}_d - \vec{\nu}_c \quad (1.6c)$$

$$\vec{\nu}_d = h(\vec{x}_d, \dot{\vec{x}}_d, \vec{x}_b, \dot{\vec{x}}_b, t) \quad \vec{\nu}_d \in \mathbb{R}^n \quad (1.6d)$$

$$\vec{y} = c(\vec{x}, t) = \hat{\vec{x}} \quad (1.6e)$$

The effectiveness function $B(\vec{x}, u, t)$ quantifies how actuator inputs $u \in \mathbb{U}$ correlate to the physically commanded plant input $\vec{\nu}_c$. Inversion based allocation rules which solve for explicit actuator solutions (Sec:4.7) require that $B(\vec{x}, u, t)$ can be abstracted to a linear multiplicative relationship $B(\vec{x}, t)u$ with $B(\vec{x}, t) \in \mathbb{R}^{n \times m}$, such that a generalized inverse of $B(\vec{x}, t)$ can be found. For generic set point tracking the control law will design a desired virtual control input $\vec{\nu}_d$, the allocation rule then has to solve u for $\vec{\nu}_c$ such that for some slack variable $s = \vec{\nu}_c - \vec{\nu}_d$ is minimized:

$$\min_{u \in \mathbb{R}^m, s \in \mathbb{R}^n} \|Q_s\| \text{ subject to } B(\vec{x}, u, t) - h(\vec{x}_e, t) = \vec{\nu}_c - \vec{\nu}_d = s \quad u \in \mathbb{U} \quad (1.7)$$

Which ensures the commanded input $\vec{\nu}_c$ tracks the desired control input $\vec{\nu}_d$; $\vec{\nu}_c \rightarrow \vec{\nu}_d$ as per some cost function of the slack variable Q_s . Mostly the L₂ norm, $Q_s = \|s\|_2$, is used. In an over-actuated system it then follows that there is a whole set of possible inputs for each $\vec{\nu}_c$. A unique actuator solution (rather than a family of solutions) to Eq:1.7 needs a secondary objective function, $j(\vec{x}, u, t)$. Eq:1.7 then becomes;

$$\min_{u \in \mathbb{R}^m, s \in \mathbb{R}^n} (\|Q_s\| + j(\vec{x}, u, t)) \text{ subject to } \vec{\nu}_c - \vec{\nu}_d = s \quad u \in \mathbb{U} \quad (1.8)$$

Those same authors, Johansen and Tjønnås in [60–62], proposed multiple control allocation solutions to a variety of systems. Following [60]; in a subsequent paper [61], the authors introduced a secondary cost function, driving the solution away from the typical linear quadratic programming pseudo and weighted inversion solutions. Aiming for actuator efficiency and not just input saturation, a subsequent paper [62] proposed adaptively allocating actuator positions online. Using a Lyapunov energy equation as the online cost function, the minimization adaptive law was ensured to always settle on a feasible solution.

Over-actuation is not often applied to quadrotors and rather than providing a comprehensive literature review of associated papers here (which are all mostly theoretical derivation), the contextual application and solutions are expanded upon later in Sec:4.7. The only overactuated quadrotor literature which covers allocation of the extra actuators is [2, 38], where the authors apply a weighted pseudo inverse (otherwise known as the Moore-Penrose Inverse [72]) allocation rule. Birotor dual-axis tilting, detailed earlier, results in a critically actuated system and so requires no allocation. As mentioned before, a prerequisite for (*pseudo*) inversion is a multiplicative *linear* control effectiveness relationship for Eq:1.6b.

The only overactuated quadcopter paper which addressed its required control allocation was that of Gasco and Rihani in [2, 38]. Their solution applied weighted inversion, relying on some very specific assumptions to achieve that required input linearity for the system in Eq:1.6b. For the gyroscopic torque response to extra actuator η or γ pitching and rolling movement:

$$\tau = J\Omega\dot{\eta} \in \mathcal{F}^b \quad (1.9)$$

With Ω being that propellers rotational speed and $\dot{\eta}$ being the inducing servos rate. The authors assumed the extra actuators pitch and roll angular rates; $\dot{\eta}$ and $\dot{\gamma}$ respectively, were both proportionally related to their positions η and γ as follows:

$$\dot{\eta} \approx \frac{1}{t_{settle}}\eta \quad \text{and} \quad \dot{\gamma} \approx \frac{1}{t_{settle}}\gamma \quad (1.10)$$

Where t_{settle} is a constant derived the actuator transfer function's settling time from a unit input step. Such an assumption holds true so long as $\Delta\eta$ or $\Delta\gamma$ is smaller than the initial step used to evaluate t_{settle} , a restrictive and unrealistic assumption but implemented nonetheless. It then follows that the gyroscopic first order torque $\tau = -\vec{\omega}_b \times J_b \vec{\omega}_b$ and second order inertial torque $\tau = J_b \dot{\vec{\omega}}_b$ responses are both functions of their associated servo positions η and γ , not respective their derivatives. The extent of that consequence is contrasted with the allocation solution in Sec:4.7.

Satellite Attitude Control

Unconstrained attitude set-point tracking for 6-DOF bodies, quaternion based or otherwise, is a topic well covered in the field of satellite attitude control; [59, 68, 127]. The *status quo* for recent research is on non-linear adaptive attitude backstepping control systems, wherein the adaptive update rule is the novel contribution. Plant uncertainty always adversely affects the confidence in inertial tensor measurements critical to the attitude control of a satellite. In [59] the authors proposed applying adaptive backstepping to compensate for steady state plant uncertainty errors of the (asymmetric) inertial estimations.

Alternatively, instead of deliberating on costly non-orbital prelaunch inertial measurements, [14] developed an algorithm for estimating the inertial tensor using controlled single axis perturbations. Such an approach does assume any initial estimates are sufficiently close to true body values such that they will settle and stability can be ensured, irrespective of how unacceptable the transient performance may be.

Satellite actuator suites mostly include additional redundant effectors, to ensure fault tolerance, and thus require control allocation. Often the extra allocators are control moment gyroscopic actuators, flywheels driven by DC motors, to produce rotational torques. Thrusters have a limited amount of fuel and can actuate the system a finite number of times. The thrusters can then be scheduled with a lower priority, biasing electronic CMG actuators. In [68] the authors address the over-actuation with direct pseudo inversion before applying quaternion based backstepping for attitude control. Such an inversion solves for Eq:1.8:

$$u = B^\dagger \vec{\nu}_d \quad (1.11a)$$

$$B^\dagger = B^T (BB^T)^{-1} \quad (1.11b)$$

$$u \in \mathbb{R}^m, \vec{\nu}_d \in \mathbb{R}^n, B \in \mathbb{R}^{m \times n}, B^\dagger \in \mathbb{R}^{n \times m} \quad (1.11c)$$

Where B is the effectiveness matrix, a static effector form of the effectiveness function $B(\vec{x}, u, t)$. The generalized inverse B^\dagger is such that $BB^\dagger = \mathbb{I}$. Specifically B^\dagger is the general *pseudo* inverse of B (more on inversions in Sec:4.7). Moreover there is an assumed linear multiplicative relationship between the input, $u \in \mathbb{U}$, and the input effectiveness matrix in Eq:1.6b. The controller designed actuator torque $\vec{\nu}_d$ then dictates the input u as per Eq:1.11a. Much like the over-actuation previously discussed with respect to quadcopters; the pseudo inversion method of actuator distribution applies linear quadratic optimization to the allocation slack cost function, Eq:1.7.

The resultant quaternion attitude backstepping controller developed in [68] demonstrated global uniform asymptotic stability. The strength of that backstepping stability lies in the choice of trajectory aiming to be stabilized; $z \rightarrow \vec{0}$. The first Lyapunov trajectory was defined as:

$$z_1 = \begin{bmatrix} 1 - |q_0| \\ \vec{q}_e \end{bmatrix} \quad (1.12a)$$

Such that the Lyapunov energy function candidate is always positive definite and its derivative is positive definite decrescent. The particulars of that stability proof are omitted but it is worth detailing their chosen candidate function;

$$V_1(z) = z_1^T z_1 > 0 \quad \forall [q_0, \vec{q}_e] \quad (1.12b)$$

The absolute quaternion error scalar in Eq:1.12a ensures a global trajectory's asymptotic stability (Sec:4.5.3), not just local stability that would otherwise be gained. The stable equilibrium points at $Q_e = [\pm 1 \quad \vec{0}]^T$ apply settling of the trajectory's *error*, allowing the satellite to track its setpoint. However considering that the controller is an ideally compensating controller, the disturbance rejection and uncertainty compensation of the attitude controller could potentially disrupt that achieved stability. Something which was not discussed in the original paper.

Chapter 2

Prototype Design

2.1 Design



Figure 2.1: Isometric view of the prototype design

The final prototype (Fig:2.1) went through a series of different design iterations, aimed at optimizing engineering time spent on construction and reducing the associated component costs. Significant consideration for the design process was the net weight whose upper limit is inherently limited by the thrust produced from lift motors. Some of the more important design factors, like inertial matrices and associated masses (Sec:2.3), are discussed here in order to give context for the dynamics derived later in Ch:3. The reference frame orientations (which those dynamics are developed with respect to) are detailed here. A brief overview of the electrical systems layout is then given with the components associated and their electrical characteristics included. Finally the actuator suite's functionality and transfer characteristics are quantified. A review of the physical prototype realized and control loop(s) implemented is detailed in Ch:6 along with actual flight test results.

2.1.1 Actuation Functionality

The most important component of the design is the manner of articulation for each concentric gimbal ring which forms the four motor module structures. The control objective is to produce a thrust vectoring actuation set for a quadrotor's control plant. The outcome was a module which independently redirects the thrust generated by the lift propellers (Fig:2.2a). Within each module are servos affixed onto sequential support rings to pitch and roll the substructure's axes. The gyroscope-like frame that surrounds each motor/propeller pair accommodates that relative movement. Aligned with each servo is a coaxial support bearing. The bearing and actuator servos have a mass disparity which results in an eccentric center of mass, producing a net gravitational torque arm. Unfortunately, due to weight constraints, counter balance measures cannot be introduced. Consequences from the center of mass variations must be either compensated for (*plant dependent solution*) or exploited in the dynamics (*additional non-linear actuator plants*). The precise effects are quantified numerically later in Sec:2.3.

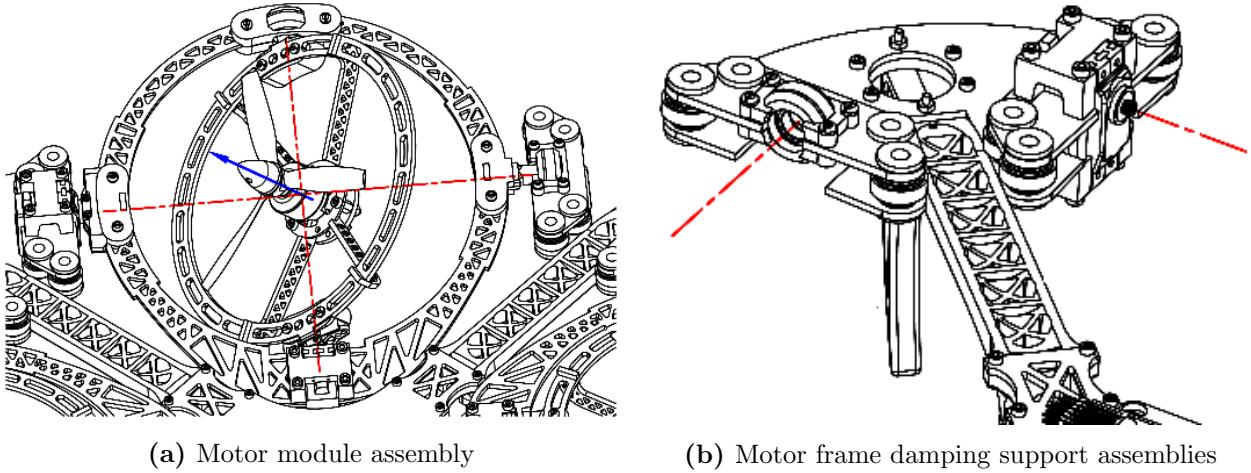


Figure 2.2: Tilting rotor design

Each motor module is positioned such that its produced thrust vector coincides with the intersection of its two rotational axes (Fig:2.2a). As a result there is only a perpendicular displacement, $L_{arm} = 195.16$ mm, co-planar to the body frame's X-Y-Z origin \vec{O}_b (see subsequent Fig:2.8). That length directly affects the differential torque plant; $\vec{\tau}_{diff} = \sum \vec{L}_i \times \vec{T}_i$. An eccentric thrust vector line would make the torque arm displacement a non-orthogonal vector. The center of gravity for each module is time varying and depends on its two servo rotational positions. It is more prudent to ensure intersection of the thrust vector with the rotational center than to balance the masses undergoing rotation. A thrust varying torque is harder to approximate and hence compensate for than a gravitational torque, given the complexity of modeling a propeller's aerodynamic thrust (Sec:3.2.1).

The primary body structure is similar to a traditional quadcopter '+' configuration with adjacent propellers spinning in opposite directions. Each motor module's rotational assembly is suspended by silicone damping balls (Fig:2.2b). A smaller damping assembly in the center of the frame houses all the electronics and power distribution circuitry. All the mounting brackets affixing the motor module rings are 3D printed from CAD models using an Ultimaker V2+ [128]. A complete bill of materials for all parts used, including working drawings for each 3D printed bracket and the laser cut frame(s), is presented in Appendix:B.

The propeller's rotational plane is not aligned exactly with the plane made by the \hat{X}_{M_i} and \hat{Y}_{M_i} rotational servo axes (Fig:2.3). The offset is approximately 23.0 [mm] and must be considered when evaluating pitch/roll inertial and gyroscopic torque responses later in Sec:3.3.1. The propellers are 6 inch (6×4.5) 3-Blade plastic Gemfam propellers, powered by Cobra CM2208-2000KV Brushless DC motors (Fig:2.4a). The thrust produced as a function of angular velocity (in RPS) for the propellers is derived in Sec:3.2.1.



Figure 2.3: Difference between propeller and motor planes

The BLDC motors are controlled with LDPower 20A ESC modules with an in-line OrangeRx RPM Sensor. The ESCs were reflashed with BLHeli [12] firmware. The default firmware on the speed controllers had an unsatisfactory exponential approaching (not linear) input speed curve, in contrast with the linear (unloaded) speed curve in Fig:2.22. The net transfer functions for both ESC modules and the servos are detailed later in Sec:2.4.1. Power for the quadrotor is supplied from a power tether (not from a battery bank). Tethered power will ensure consistent flight time and reduce the concern of payload restriction on the available lift actuation. Power lines to both the BLDC motors and servos are supplied through conventional wiring, however an ideal and more flexible design would see slip-rings for each module's power supply.



(a) Cobra CM2208-2000KV BLDC motor module

(b) Corona DS-339MG servo bracket

Figure 2.4: Motor module assembly

Metal gear Corona DS-339MG digital servos are used for the two axes of rotation (Fig:2.4b). Each servo has a rotational range of $\approx \pi$, positioned such that a zeroth offset aligns the motor modules, adjacent to the body frame, and has a $\pm\pi/2$ rotational range. A digital servo updates at 330 Hz, faster than a 50 Hz analogue servo equivalent (Fig:2.5). This means the otherwise 20 ms zero-order "analogue" sampling effect is a less significant 3.30 ms zero-order holding time. Both the \hat{X}_{M_i} and \hat{Y}_{M_i} axis servos will be rotating a larger inertial body, as such the open loop plant dynamics are determined empirically in Sec:2.4.1.



Figure 2.5: Digital and analogue servo timing

2.2 Reference Frames Used

Attitude conventions used for deriving the system's dynamics, in Ch:3, are first discussed here. Often these aspects are assumed to be obvious enough that they are omitted. It is important to clearly and unambiguously define a standard set of framing conventions to avoid uncertainty later. Rotation matrices are included but the focus is on the *contrast* between rotation and transformation operations. Both [46] and [97] provide an in-depth and thorough explanation of rotation matrices and direct cosine matrix attitude representation, if such concepts are unfamiliar to the reader. Quaternions are introduced to replace rotation matrix notation for the dynamics in Sec:3.2.5.

2.2.1 Reference Frames Convention

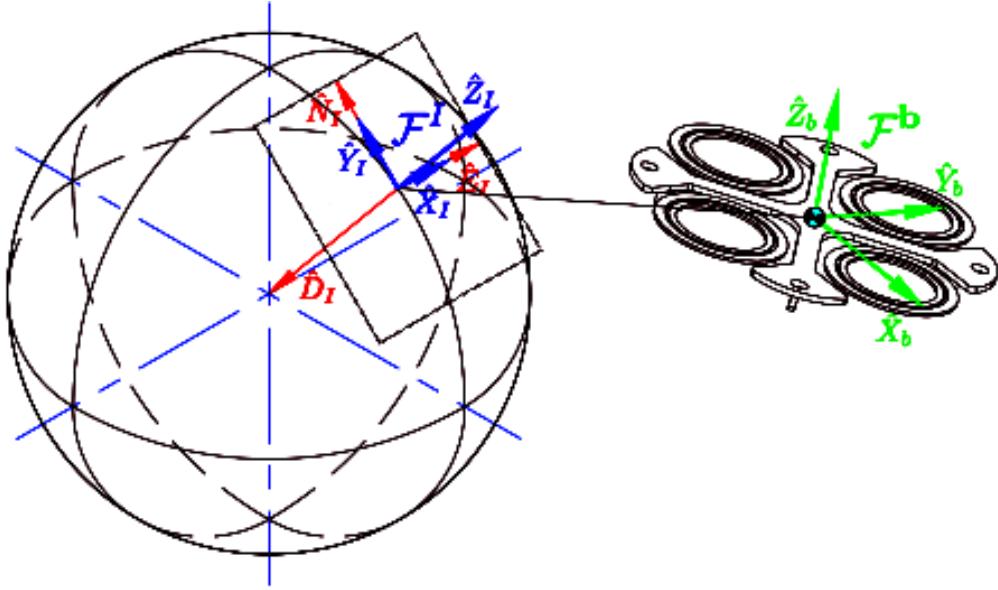


Figure 2.6: Inertial and body reference frames

NASA aerospace frames are used for principle Cartesian inertial and body coordinate representation (Fig:2.6). The inertial frame, \mathcal{F}^I , is aligned such that the \hat{Y}_I axis is in the \hat{N} orth direction, \hat{X}_I is in the \hat{E} ast direction and $-\hat{Z}_I$ is in the \hat{D} ownward direction. In Euler orbital sequences the \hat{Z} direction would be toward the Earth's center, sometimes referred to as the NED convention which differs from the NASA frames used here. The body frame, \mathcal{F}^b , then has both \hat{X}_b and \hat{Y}_b aligned obliquely between two perpendicular arms of the quadrotor's body and the \hat{Z}_b axis in the body's normal upward direction (illustrated in Fig:2.9).

The body frame's axes and center of motion relative to the prototype design's center of mass are both detailed next in Sec:2.2.2. Frame superscripts I and b represent inertial and body frames respectively whilst vector subscripts imply the reference frame in which the vector's coordinates exists or taken relative to. The function $R_I^b(\eta)$ represents a rotation operator of the Euler set $\vec{\eta}$ (expanded on in Eq:2.11) rotating from subscript frame \mathcal{F}^I to superscript frame \mathcal{F}^b .

A vector $\vec{\nu}$ has the relationship between the body and inertial frames:

$$\vec{\nu}_I = R_I^b(\eta) \vec{\nu}_b \quad \vec{\nu}_b \in \mathcal{F}^b, \vec{\nu}_I \in \mathcal{F}^I \quad (2.1)$$

Displacement between the inertial and body frames is given by $\vec{\mathcal{E}}$ defined in the inertial frame:

$$\vec{\mathcal{E}} = [x \ y \ z]^T \quad [\text{m}], \quad \in \mathcal{F}^I \quad (2.2)$$

An axial hat and upper case differentiates axis unit vectors $\hat{X}, \hat{Y}, \hat{Z}$ from position quantities x, y, z in Eq:2.2. The body's position time derivative $\dot{\mathcal{E}}$ refers to the *inertial frame* rate:

$$\frac{d}{dt}\vec{\mathcal{E}} = [\dot{x} \quad \dot{y} \quad \dot{z}]^T \quad [\text{m.s}^{-1}], \quad \in \mathcal{F}^I \quad (2.3)$$

Whereas the body's *velocity* \vec{v} is with respect to the body frame \mathcal{F}^b . Velocity and the inertial position time derivative are related as follows:

$$\vec{v} = R_I^b(\eta) \dot{\vec{\mathcal{E}}} \quad [\text{m.s}^{-1}], \quad \in \mathcal{F}^b \quad (2.4a)$$

$$= R_I^b(\eta) [\dot{x} \quad \dot{y} \quad \dot{z}]^T \quad (2.4b)$$

Relative angular displacement between two frames is commonly measured by the three angle Euler set. The Euler angle set $\vec{\eta} = [\phi \ \theta \ \psi]^T$ represents pitch ϕ , roll θ and yaw ψ rotations about the \hat{X}, \hat{Y} and \hat{Z} axes respectively. Depending on how the rotation sequence is formulated, those angles can be used to construct rotation matrices which give relation to vectors or can transform coordinates.

The general rotation equation to *rotate* some vector \vec{v} about a normalized unit axis \hat{u} through a rotation angle θ [rad] is given by the formula, proven in [31]:

$$\vec{v}' = (1 - \cos(\theta))(\vec{v} \cdot \hat{u})\hat{u} + \cos(\theta)\vec{v} + \sin(\theta)(\hat{u} \times \vec{v}) \quad (2.5)$$

In Eq:2.5, when the unit vector \hat{u} is in the direction of either $\hat{X}; \hat{Y}$ or \hat{Z} axes the equation is simplified to produce the three fundamental rotation matrices $R_x(\phi); R_y(\theta)$ and $R_z(\psi)$. That set of three principle rotation matrices about the Cartesian X-Y-Z axes are defined as:

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} \quad (2.6a)$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \quad (2.6b)$$

$$R_z(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.6c)$$

The notation for a rotation matrix operation is multiplication of the matrix $R_n(\theta)$, applying a left-handed *rotation* operator about some axis \hat{n} by θ . The resultant vector of a rotation operation still exists in the same reference frame. For example an \hat{X} axis rotation by ϕ of some vector \vec{v} is given by:

$$\vec{v}' = R_x(\phi)\vec{v} \quad \vec{v}', \vec{v} \in \mathcal{F}^1 \quad (2.7a)$$

No subscripts are used in Eq: 2.7 to indicate reference frame ownership because all vectors are in the same frame. The time derivative of a rotation matrix about some axis \hat{n} by a rotation θ , $\dot{R}_n(\theta)$ is shown in [105] to be:

$$\frac{d}{dt}(R_n(\theta)) = (\dot{\theta} \cdot \hat{n}) \times R_n \Rightarrow [\dot{\theta} \cdot \hat{n}]_x R_n \quad (2.8a)$$

Where, for some vector \vec{a} , the operator $[\vec{a}]_x$ denotes the cross-product matrix or *skew* matrix. The symmetric skew matrix is a matrix multiplication to replace the cross-product operator, for some other vector \vec{b} ;

$$\vec{a} \times \vec{b} = [\vec{a}]_x \vec{b} \quad (2.8b)$$

$$[\vec{a}]_x \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad (2.8c)$$

A vector *transformation* changes the resultant vector's reference frame. The transformation is then a rotation by an angle of the *difference* (or negative angle) between the resulting and principle reference frames. A transformation from frame \mathcal{F}^1 to \mathcal{F}^2 , differing by an angle of ϕ about the \hat{X} axis is then a negative rotation operation:

$$\vec{\nu}_2 = R_x(-\phi)\vec{\nu}_1 \quad (2.9a)$$

$$\vec{\nu}_2 \in \mathcal{F}^2 \text{ and } \vec{\nu}_1 \in \mathcal{F}^1 \quad (2.9b)$$

The distinction between Eq:2.7 and Eq:2.9 is the directional sense of the angular operand ϕ , and hence the effect it has on the argument vector. The transformation or rotation of a vector from the inertial frame \mathcal{F}^I to the body frame \mathcal{F}^b is the product of three sequential operations about each principle axis. Each subsequent rotation is applied relative to a new intermediate frame; hence each Euler angle is taken relative to a specific intermediate frame and not a global one. The order of those axial rotation operations does affect the Euler set, any consequences of which are detailed in [69]. In this dissertation the Z-Y-X or yaw, pitch, roll rotation sequence is used. A rotation of the vector $\vec{\nu}$ from the inertial to the body frame, $\mathcal{F}^I \rightarrow \mathcal{F}^b$, is then applied by sequential yaw, ψ , pitch, θ , and roll ϕ operations about the \hat{Z} , \hat{Y} and \hat{X} axes respectively:

$$R_I^b(\eta) = R_I^b(\phi, \theta, \psi) \triangleq R_z(\psi)R_y(\theta)R_x(\phi) \quad (2.10a)$$

$$\vec{\nu}' = R_I^b(\phi, \theta, \psi)\vec{\nu} \quad (2.10b)$$

$$\rightarrow \vec{\nu}' = R_z(\psi)R_y(\theta)R_x(\phi)\vec{\nu} \quad (2.10c)$$

It is important to note that in Eq:2.10 both operand and output vectors are *both* in the inertial frame, namely $\vec{\nu}', \vec{\nu} \in \mathcal{F}^I$. A *transformation* of a vector from the inertial to the body frame is the negative counterpart of Eq:2.10, a distinction which is not always explicitly stated.

$$\vec{\nu}_b = R_I^b(-\eta)\vec{\nu}_I = R_I^b(-\phi, -\theta, -\psi)\vec{\nu}_I \quad (2.11a)$$

$$\text{for } \vec{\nu}_b \in \mathcal{F}^b \text{ and } \vec{\nu}_I \in \mathcal{F}^I \quad (2.11b)$$

$$\rightarrow \vec{\nu}_b = R_z(-\psi)R_y(-\theta)R_x(-\phi)\vec{\nu}_I \quad (2.11c)$$

$$= R_x(\phi)R_y(\theta)R_z(\psi)\vec{\nu}_I = R_b^I\vec{\nu}_I \quad (2.11d)$$

$$R_I^b = (R_b^I)^{-1} = (R_b^I)^T \quad (2.11e)$$

The relationship in Eq:2.11e is an inversion property (*transpose*) of the rotation matrix. A rotation matrix's inverse can be used interchangeably with its negative counterpart to maintain a positive sense of the argument angle. To ensure clarity throughout this dissertation's mathematics, a negative angular sense implies a *transformation* to a different reference frame. Where applicable, the order of rotation will indicate the sequence direction whilst the angular sign differentiates the rotation or transformation operations.

The body frame's angular velocity is taken relative to the inertial frame, represented by $\vec{\omega}_{b/I}$ mostly just simplified to $\vec{\omega}_b$ [rad.s⁻¹]. Seeing that each Euler angle is measured with respect to an intermediary frame, a distinction must then be made between $d\vec{\eta}/dt$ and $\vec{\omega}_b$. All three Euler angles need to be transformed to a common frame to define the relationship between Euler and angular rates. Exploiting vehicle frames 1 & 2, or rather \mathcal{F}^{v1} & \mathcal{F}^{v2} , as intermediary frames to describe respectively frames after $R_x(\phi)$ and $R_y(\theta)$ operations and using the rotation matrix derivative from Eq:2.8. The angular velocity $\vec{\omega}_b$ is the time derivative of Euler angles in the body frame:

$$\vec{\omega}_b = [p \quad q \quad r]^T \triangleq \frac{d}{dt}\vec{\eta} = \frac{d}{dt}\vec{\eta}_b \in \mathcal{F}^b \quad (2.12a)$$

$$\vec{\eta}_b \triangleq R_{v2}^b(\phi)\vec{\phi} + R_{v2}^b(\phi)R_{v1}^{v2}(\theta)\vec{\theta} + R_{v2}^b(\phi)R_{v1}^{v2}(\theta)R_I^{v1}(\psi)\vec{\psi} \in \mathcal{F}^b \quad (2.12b)$$

$$\therefore \vec{\omega}_b = \left[\begin{array}{c} \dot{\vec{\phi}} \\ \times \\ R_{v2}^b(\phi) + R_{v2}^b(\phi) \left[\begin{array}{c} \dot{\vec{\theta}} \\ \times \\ R_{v1}^{v2}(\theta) + R_{v2}^b(\phi)R_{v1}^{v2}(\theta) \left[\begin{array}{c} \dot{\vec{\psi}} \\ \times \\ R_I^{v1}(\psi) \end{array} \right] \end{array} \right] \end{array} \right] \in \mathcal{F}^b \quad (2.12c)$$

With Euler vectors $\vec{\phi}$, $\vec{\theta}$ and $\vec{\psi}$ being axis projections onto X-Y-Z axes respectively; $\phi \cdot \hat{i}$, $\theta \cdot \hat{j}$ and $\psi \cdot \hat{k}$. The vehicle frames used for Eq:2.12a and the subsequent rotations between each frame don't necessarily have to be in that order. The equation could change depending on what rotation sequence was used, here Z-Y-Z rotation sequences were used. The Euler rates equation then simplifies to the formal relationship between two rotating frames, with $\vec{\omega}_b = [p \ q \ r]^T$ in [rad.s⁻¹]:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\theta) & \cos(\phi)\sin(\theta) \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (2.12d)$$

$$\Rightarrow \vec{\omega}_b = \Psi(\eta) \dot{\vec{\eta}} \in \mathcal{F}^b \quad (2.12e)$$

$$\Psi(\eta) = \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\theta) & \cos(\phi)\sin(\theta) \end{bmatrix} \quad (2.12f)$$

$$\Rightarrow \dot{\vec{\eta}} = \Psi^{-1}(\eta) \vec{\omega}_b = \Phi(\eta) \vec{\omega}_b \in \mathcal{F}^{v1,v2,I} \quad (2.12g)$$

$$\Phi(\eta) = \begin{bmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{bmatrix} \quad (2.12h)$$

The *Euler* matrix, $\Psi(\eta)$, contains a well known and problematic singularity; at $\theta = \pm\pi/2$ where the determinant of the transformation matrix is zero. The mathematical manifestation of that singularity and its physical consequences are expanded on in Sec:3.2.4. The singularity is present in the middle roll angle θ , which is a direct consequence of the chosen Z-Y-X rotation sequence adopted. Each Euler angle is potentially singular depending on the rotation order used. In later dynamics quaternions are used in lieu of Euler angles (Sec:3.2.5). Attitude in \mathbb{R}^3 , or $SO(3)$, is intuitive and well suited to the conventions defined here.

Quaternions (Sec:3.2.5), despite being in \mathbb{R}^4 , are similarly constructed in the Z-Y-X order following a three rotation sequence. Combined quaternion operations are additive but non-commutative, as such the order is important. The constructed attitude quaternion order will produce the same resultant frame orientation however the quaternion, and its rotation path, will differ. A quaternion Q_b , representing the body's attitude, and some vector $\vec{\nu}_I$ in the inertial frame is related to the body frame \mathcal{F}^b as follows:

$$\vec{\nu}_b = R_I^b(-\eta) \vec{\nu}_I \iff Q_b \otimes \begin{bmatrix} 0 & \vec{\nu}_I \end{bmatrix}^T \otimes Q_b^* \quad (2.13a)$$

$$Q_b \triangleq Q_z \otimes Q_y \otimes Q_x \text{ and it's inverse } Q_b^* \triangleq Q_x^* \otimes Q_y^* \otimes Q_z^* \quad (2.13b)$$

The symbol \otimes represents the Hamilton product, or quaternion multiplication operator. Later the Hamilton product is used again for inertial tensor transformations (Sec:2.3). Each quaternion, Q_i , is always the *unit* quaternion about the i^{th} axis. For the body quaternion, Q_b , it is the unit quaternion rotation about the body's Euler axis, [69]. A quaternion rotation operates on an argument vector with a zero quaternion scalar component. So then for some vector $\vec{\nu}$, the quaternion rotation operation in Eq:2.13a is equivalent to;

$$Q_{\vec{\nu}'} = Q \otimes (Q_{\vec{\nu}}) \otimes Q^* \quad (2.14a)$$

$$\text{Where } Q_{\vec{\nu}} = \begin{bmatrix} 0 & \vec{\nu} \end{bmatrix}^T, \quad Q_{\vec{\nu}'} = \begin{bmatrix} 0 & \vec{\nu}' \end{bmatrix} \quad (2.14b)$$

Quaternion representation in Eq:2.14b ensures that the operation is entirely in \mathbb{R}^4 space. However it is typically omitted, despite \mathbb{R}^4 being implied and as such, Eq:2.14a is then simply:

$$\vec{\nu}' = Q \otimes (\vec{\nu}) \otimes Q^* \quad (2.15)$$

Quaternion dynamics, and the quaternion operator, are later expanded upon to replace the use of Euler angles and rotation matrices as a convention for attitude representation later in Chapter:3.

2.2.2 Motor Axis Layout

The whole structure (previously in Fig:2.1) consists of multiple rigidly connected bodies with only relative rotations between each body permitted by its joints, illustrated previously in the design description in Sec:2.1. Those rigid bodies are categorized into four inter-connected motor modules, $M_{1 \rightarrow 4}$, and a single body structure, \mathbf{B} (*frame* structure, not reference frame). Each module contains two sequential gimbal rings, where each ring has one degree of relative rotation, actuated by a servo, between itself and the subsequent ring. There needs to be distinct nomenclature used for describing these motor modules such that the dynamic derivations later are clear and logical despite the complicated multibody system...



Figure 2.7: Aligned motor frame axes

Every propeller/motor is actuated by a pair of two servos about two subsequent rotational axes (Fig:2.7) in a similar fashion to an Euler rotation sequence. The i^{th} propeller, attached to the BLDC motor's rotor, rotates in frame \mathcal{F}^{M_i} with a rotational speed Ω_i in revolutions per second, or [RPS], about the \hat{Z}_{M_i} stator axis. Fig:2.8 shows the sequential relative module frames.



Figure 2.8: Intermediate motor frames

The motor's stator is affixed to the inner ring assembly which rotates about its \hat{X}_{M_i} axis by λ_i from the module's first servo. The first servo is attached to the middle ring assembly with the frame $\mathcal{F}^{M'_i}$. The middle ring assembly, frame $\mathcal{F}^{M'_i}$, rotates about its $\hat{Y}_{M'_i}$ axis actuated by the second α_i servo. That second servo is affixed to an intermediate $\mathcal{F}^{M''_i}$ frame, finally there's an orthogonal rotation about $\hat{Z}_{M''_i}$ between \mathcal{F}^b and $\mathcal{F}^{M''_i}$. Each module's actuation state is fully described by the rotational speed, both servo positions and all their respective rates; $[\Omega_i, \lambda_i, \alpha_i, \dot{\Omega}_i, \dot{\lambda}_i, \dot{\alpha}_i]^T$ for $i \in [1 : 4]$.

Fig:2.9 shows how the axes of each motor module align with the body frame's axes at rest. The body frame \mathcal{F}^b has the origin \vec{O}_b at the X-Y center of the structure, co-planar to each motor modules' center. *Neither* the body frame's origin *nor* each modules center of rotation is coincidental with either body's center of mass. The exact disparity between the origin(s) of motion and the respective body's center of mass are quantified subsequently in Sec:2.3.



Figure 2.9: Body frame axes layout

The motor module pair 1 and 3 have their \hat{X} -axes in the positive and negative \hat{X}_b directions of the body frame respectively. Similarly Modules 2 and 4 have their \hat{X} -axes in the positive and negative \hat{Y}_b directions of the body frame. Motor modules 1 and 3 have clockwise rotating propellers; denoted by a positive super script or $\Omega_{[1,3]}^+$. Conversely modules 2 and 4 have counter-clockwise rotations; denoted by a negative super script or $\Omega_{[2,4]}^-$.

Not shown in Fig:2.9 is the relative \hat{Z}_b origin position of \vec{O}_b with respect to the entire assembly. The ΔZ height of the body's motion centroid is such that its origin is co-planar with the four motor modules rotational centers. The center of motion is not coincidental with the center of mass.

Each motor is displaced from the body frame origin by $L_{arm} = 195.16$ [mm] (shown in Fig:2.9). Transforming some vector \vec{v}_{M_i} in the motor frame \mathcal{F}^{M_i} to the body frame is given as:

$$\vec{v}_b = R_{M_i}^b \vec{v}_{M_i} = R_z(-\sigma_i)R_y(-\alpha_i)R_x(-\lambda_i)\vec{v}_{M_i} \quad \text{for } \sigma_i \in [0 \quad \frac{\pi}{2} \quad \pi \quad \frac{2\pi}{3}] \quad (2.16a)$$

The orthogonal σ_i rotations about $\hat{Z}_{M'_i}$ are independent of actuator positions, σ_i is a constant determined by the motor module's location. The rotation matrices $R_z(\sigma_i)$ for $\sigma_i = (i-1)\pi/2$ are:

$$R_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for } i \in [1, 2, 3, 4] \text{ respectively} \quad (2.16b)$$

The actuator space, including propeller speed Ω_i [RPS], is then $\in \mathbb{R}^{12}$, or rather $\mathbb{U} \in \mathbb{R}^{12}$, in contrast with $\mathbb{U} \in \mathbb{R}^4$ for a standard quadrotor. The actuator input set $u \in \mathbb{U}$ is then structured as:

$$u_{\mathbb{U}} = [\Omega_1^+ \quad \lambda_1 \quad \alpha_1 \quad \dots \quad \Omega_4^- \quad \lambda_4 \quad \alpha_4]^T \quad \in \mathbb{R}^{12} \quad (2.17)$$

2.3 Inertial Matrices & Masses

When transforming inertias it is more appropriate to use rotation matrices to apply the transformation and not quaternions. Spatial rotation of inertial tensors are ill suited to quaternion parametrization.

An undesirable side effect of the relative rotations within a non-rigid body are the inertial responses associated with such movements. Given Newton's Second Law of Rotational Motion; each applied rotation is going to produce an equal but opposite reaction onto the principle inducing body. Similarly a gyroscopic cross product from rotational velocities is also present when rotating bodies that have their own relative rotation. Typically for most rigid body dynamics, such first and second order effects are negligible given that the angular rates which they depend on are small enough to approximate as zero, $\vec{\omega}_b \approx \vec{0}$. A dynamic set-point (non-zero) attitude tracking plant is, however, going to produce non-zero time varying body angular velocities and accelerations.

The dynamic effects of those torque responses are derived later in Sec:3.3.1. Both inertial and gyroscopic effects are dependent on the considered body's inertia about each respective rotational axis. The magnitude of those inertias are obviously a by-product of the structure's design. The following inertias presented are all calculated from a SolidWorks model with masses to match physical prototype measurements. Each rigidly connected body affected by the same angular velocity is grouped together. Every motor module then contains 3 connected bodies; the propeller/rotor body, the inner ring and finally middle ring assemblies, each of which are now described and expanded.

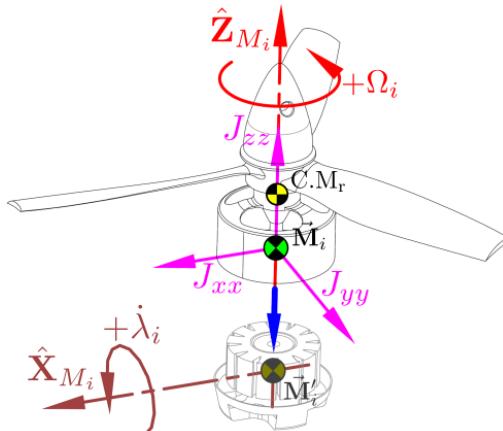


Figure 2.10: Rotor assembly rotational structure

The first rotational body to consider is that of the propeller and rotor assembly (Fig:2.10, excluding the motor's stator). The "rotor" assembly, with subscript r, has a net mass $m_r = 27 \text{ g}$ with a center of mass $C.M_r = [0.0 \ 0.0 \ 15.5]^T \text{ [mm]}$ relative to the entire motor modules center of rotation. The propeller's plane of rotation is $[0.0 \ 0.0 \ 23.0]^T \text{ [mm]}$ relative to the entire body's center of rotation (previously illustrated in Fig:2.3). At high speeds the propeller's inertial contribution to the rotor assembly can be approximated as a solid disc. The entire rotor assembly then has an inertia J_r , with principle inertial axes aligned as in Fig:2.10:

$$J_r = \begin{bmatrix} 105.5 & 0.0 & 0.0 \\ 0.0 & 105.5 & 0.0 \\ 0.0 & 0.0 & 41.8 \end{bmatrix} \quad [\text{g.cm}^2] \quad (2.18)$$

The net angular velocity of the rotor assembly $\vec{\omega}_r \text{ [rad.s}^{-1}\text{]}$ relative to the body frame is produced by the BLDC motor's own rotational velocity Ω_i and both servo rates; $\dot{\lambda}_i$ and $\dot{\alpha}_i$. Here Ω_i and both servo rates are in $[\text{rad.s}^{-1}]$, later Ω_i is needed in $[\text{rev.s}^{-1}]$ for Blade-element momentum theory thrust calculations (Sec:3.2.1). Each servo's angular velocity is *transformed* onto the motor frame \mathcal{F}^{M_i} .

$$\vec{\omega}_r = \begin{bmatrix} 0 \\ 0 \\ \Omega_i \end{bmatrix} + \frac{d\lambda_i}{dt} R_x(-\lambda_i) \begin{bmatrix} \lambda_i \\ 0 \\ 0 \end{bmatrix} + \frac{d\alpha_i}{dt} R_y(-\alpha_i) R_x(-\lambda_i) \begin{bmatrix} 0 \\ \alpha_i \\ 0 \end{bmatrix} \in \mathcal{F}^{M_i} \quad (2.19)$$

Eq:2.19 is later replaced with a quaternion operator. That equation and the remaining angular velocity equations for each body derived here are therefore not expanded further in their current rotation matrix form(s)...



Figure 2.11: Inner ring rotational structure

The next assembly, on which the motor frame \mathcal{F}^{M_i} is attached, is the "inner ring" assembly denoted with subscript n. The inner ring structure has a mass $m_n = 92 \text{ [g]}$, including the rotor assembly in that calculation. The center of mass is positioned $C.M_n = [-1.44 \ 00.0 \ 5.14]^T \text{ [mm]}$ relative to the center of rotation \vec{M}_i . The inner ring, being rotated by the λ_i servo about the \hat{X}_{M_i} axis, then has an inertial matrix (centered and aligned with axes as in Fig:2.11):

$$J_n = J_{M_i} = \begin{bmatrix} 520.9 & -31.7 & -0.3 \\ -31.7 & 1826.3 & 0.0 \\ -0.3 & 0.0 & 2050.8 \end{bmatrix} \quad [\text{g.cm}^2] \quad (2.20)$$

The effect of rapidly spinning propellers, relative to the servo rotational velocity, on the inertia in Eq:2.20 is approximated well by a solid disc. It follows that the inner ring's inertial components can then be regarded as constant with respect to Ω_i ; moreover its center of mass is independent of that propeller's rotation.

The rotational velocity of the collective inner ring assembly $\vec{\omega}_n$ in [rad.s^{-s}] (or $\vec{\omega}_{M_i}$ for the angular velocity of frame \mathcal{F}^{M_i}) is similar to that of Eq:2.19. They both occur in the same frame however the inner ring's angular velocity has no rotor rotation contribution from Ω_i :

$$\vec{\omega}_n = \vec{\omega}_{M_i/b} = \frac{d\lambda_i}{dt} R_x(-\lambda_i) \begin{bmatrix} \lambda_i \\ 0 \\ 0 \end{bmatrix} + \frac{d\alpha_i}{dt} R_y(-\alpha_i) R_x(-\lambda_i) \begin{bmatrix} 0 \\ \alpha_i \\ 0 \end{bmatrix} \in \mathcal{F}^{M_i} \quad (2.21)$$



Figure 2.12: Middle ring rotational structure

That first actuating servo for λ_i and its coaxial support bearing are both affixed to the intermediate "middle ring" assembly, with subscript m (middle ring only Fig:2.12). The intermediate frame $\mathcal{F}^{M'_i}$ is attached to the middle ring body with a mass $m_m = 98$ [g], excluding the inner most ring's contribution. That middle ring body alone has a center of mass $C.M_m = [-4.70 \ 0.37 \ -0.36]^T$ [cm] relative to \vec{M}_i . Together the inner and middle rings make the whole motor module assembly (Fig:2.13), with a subscript p. The net module has a mass $m_p = 190$ [g].



Figure 2.13: Module assembly rotational structure

The center of mass for the motor module, $C.M_p$, varies with the inner ring's rotational position λ_i . That module's center of mass is calculated, with attention paid to units of measurement used:

$$C.M'_n = R_x(\lambda)(C.M_n) \quad (2.22a)$$

$$C.M_p = \frac{m_m(C.M_m) + m_n(C.M'_n)}{m_m + m_n} \quad (2.22b)$$

Substituting physical values in Eq:2.22b for the inner and middle rings' center of masses respectively:

$$\text{C.M}_p(\lambda) = \frac{98 [-4.70 \quad 0.37 \quad -0.36]^T \times 10^{-7} + 92R_x(\lambda) [-1.44 \quad 0.00 \quad 3.06]^T \times 10^{-8}}{190 \times 10^{-3}} \quad (2.22c)$$

Which then has a value at rest, for reference, with the servo $\lambda_i = 0$ at the origin:

$$\text{C.M}_p(0) = [-2.49 \quad 0.19 \quad 0.04]^T \Big|_{\lambda_i=0} \quad [\text{cm}] \quad (2.22d)$$

The complete motor module is rotated by the α_i servo about its $\hat{Y}_{M'_i}$ axis. The module's compound inertia J_p is a combination of the middle ring's inertia J_m and the inner ring's inertia J_n rotated by λ_i about $\hat{X}_{M'_i}$ (aligned as per Fig:2.13). The latter's contribution is dependent on the *rotation* (not transformation) angle λ_i which from the conservation of angular momentum theory, detailed in [98], produces the net inertia of that frame; $J_{M'_i}$ or J_p :

$$\text{With } J_m = \begin{bmatrix} 2905.7 & 0.0 & 390.9 \\ 0.0 & 8446.4 & 0.0 \\ 390.9 & 0.0 & 11125.7 \end{bmatrix} \quad [\text{g.cm}^2] \quad (2.23a)$$

$$J_p(\lambda_i) = J_{M'_i} = J_m + R_x(\lambda_i)(J_n)R_x^{-1}(\lambda_i) \quad (2.23b)$$

And an inertial matrix, for reference, at rest:

$$J_p(0) = \begin{bmatrix} 3365.4 & -0.1 & 390.6 \\ -0.1 & 10210.1 & 0.0 \\ 390.6 & 0.0 & 13118.0 \end{bmatrix} \Big|_{\lambda_i=0} \quad [\text{g.cm}^2] \quad (2.23c)$$

Because R_x is a full rank square matrix, its inverse R_x^{-1} used in Eq:2.23b always exists. The modules inertia could be further divided into constant and variable components which, under certain conditions, could be simplified and neglected...

$$J_p(\lambda_i) = J_{const} + J_{M'_i}(\lambda_i) \quad (2.23d)$$

Noting that $J_{M'_i} = J_n$ is the inner ring's inertia from Eq:2.20, but re-orientated through a rotation $R_x(\lambda_i)$. Finally, the angular velocity experienced by the net motor assembly, $\vec{\omega}_p$ in frame $\mathcal{F}^{M'_i}$, is solely as a result of the α_i servo rotation:

$$\vec{\omega}_p = \vec{\omega}_{M'_i/b} = \frac{d\alpha_i}{dt} R_y(-\alpha_i) \begin{bmatrix} 0 \\ \alpha_i \\ 0 \end{bmatrix} \in \mathcal{F}^{M'_i} \quad (2.23e)$$



Figure 2.14: Damping assembly attached to structure

The second servo which actuates α_i connects the motor module (both the inner and middle ring assemblies) to the body structure. The inertial volume of that second servo and its coaxial bearing support contributes then to the body structure's inertia; whose value excludes any of the four motor modules. Fig:2.14 shows how the complete motor module and its rotational axes (in Fig:2.13) are centered relative to the damping assembly attached to the body structure.

Consisting of servo and bearing damping brackets, each "damping" assembly collectively weighs $m_d = 84$ [g] and suspends the motor modules from the body frame with a set of silicone damping balls. The "body structure", with a subscript y, has center of mass C.M_y (without motor modules attached, Fig:2.15). The center of mass coincides with the \hat{X}_b and \hat{Y}_b directional axes but lies $\Delta Z = -9.52$ [mm] below the body frame's origin of motion $\vec{\mathbf{O}}_b \in \mathcal{F}^b$.



Figure 2.15: Body structure's center of mass

Note: that body frame origin $\vec{\mathbf{O}}_b$ which all motion is calculated with respect to is co-planar to the motor module's rotational centers, not the net center of mass.

The body structure's weight, including all four damping assemblies and electronics, totals to $m_y = 814.70$ [g]. Similarly the body structure's net inertia (*sans* motor modules) J_y , about its center of mass (Fig:2.15), is:

$$J_y = \begin{bmatrix} 181569.7 & 0.4 & -19.4 \\ 0.4 & 181692.2 & 8.9 \\ -19.4 & 8.8 & 360067.2 \end{bmatrix} \times 10^{-7} \quad [\text{kg.m}^2] \quad (2.24a)$$

Using the Parallel Axis theorem to translate that inertia to the origin of motion by $\Delta Z = +9.52$ [mm], the inertia about the origin, $\vec{\mathbf{O}}_b$, is:

$$J' = J + m(\vec{d} \cdot \vec{d} - \vec{d} \otimes \vec{d}) \approx J + md^2 \quad (2.24b)$$

For the general parallel axis transformation in Eq:2.24b, \otimes represents the Hamilton product of two $[3 \times 1]$ matrices. It is used later to indicate quaternion multiplication. The vector \vec{d} is the difference between the center of mass C.M_y and the body frame origin $\vec{\mathbf{O}}_b$.

$$\therefore J'_y = J_y + m_y(\Delta \vec{Z} \cdot \Delta \vec{Z} - \Delta \vec{Z} \otimes \Delta \vec{Z}) \quad (2.24c)$$

That body's inertia J_y at the origin $\vec{\mathbf{O}}_b$ is then:

$$\rightarrow J'_y = \begin{bmatrix} 182307.7 & 0.4 & -14.5 \\ 0.4 & 182430.1 & 6.5 \\ -14.5 & 6.5 & 360067.2 \end{bmatrix} \times 10^{-7} \quad [\text{kg.m}^2] \quad (2.24d)$$

Net inertia for the complete multibody vehicle, J_b , about the origin \vec{O}_b is a combination of all the relative attached bodies. The entire assembly's inertia J_b is the *net* body frame's inertia, different from J_y which is the inertia for *only* the body structure. That collective assembly being the four motor modules, each rotated first by λ_i , then α_i and finally translated to the body frame origin; and the body structure's contribution itself.

Those motor modules' inertial axial transformations from their respective centers of rotation, in frames \mathcal{F}^{M_i} for $i \in [1 : 4]$, to the body frame \mathcal{F}^b are analogous to that of Eq:2.16. Reiterating that \vec{O}_b is *co-planar* to each module's center of rotation; each motor module's inertia, $J_p(\lambda_i)$ or $J_{M'_i}$, defined in Eq:2.23b, is further rotated by α_i about the $\hat{Y}_{M'_i}$ axis and finally an orthogonal $\hat{Z}_{M''_i}$ axis rotation (aligned with \hat{Z}_b) onto \mathcal{F}^b .

For the entire body's net inertia each contributing inertial matrix must be defined with respect to the body's origin; first aligned parallel to the common set of body frame axes \hat{X}_b , \hat{Y}_b and \hat{Z}_b and then translated to the origin \vec{O}_b . For each motor module's inertia, still centered relative to each individual rotational centers \vec{M}_i in Fig:2.16, but re-orientated to align parallel with \vec{O}_b with rotations about axes $\hat{X} \in \mathcal{F}^{M_i}$, $\hat{Y} \in \mathcal{F}^{M'_i}$, $\hat{Z} \in \mathcal{F}^{M''_i}$, is calculated:

$$J_{\vec{M}_i} = R_z(\sigma_i)R_y(\alpha_i)(J_\zeta(\lambda_i))R_y^{-1}(\alpha_i)R_z^{-1}(\sigma_i) \quad (2.25a)$$

$$\text{for bodies } \mathbf{M}_i, i \in [1 : 4] \quad (2.25b)$$

Where the rotation $R_z(\sigma_i)$ was defined as relative orthogonal \hat{Z}_b rotation earlier in Eq:2.16b. Expanding each module's inertia to individual inner and middle ring inertial contributions then yields:

$$\therefore J_{\vec{M}_i} = R_zR_y(\alpha_i)(J_m)R_y^{-1}(\alpha_i)R_z^{-1} + R_zR_y(\alpha_i)R_x(\lambda_i)(J_n)R_x^{-1}(\lambda_i)R_y^{-1}(\alpha_i)R_z^{-1} \quad (2.25c)$$

It's at this stage that, despite simplifications, the symbolic inertial equations all become overly cumbersome to include with numeric values... For the sake of brevity, exact calculated inertial values for the input dependent plant are omitted.

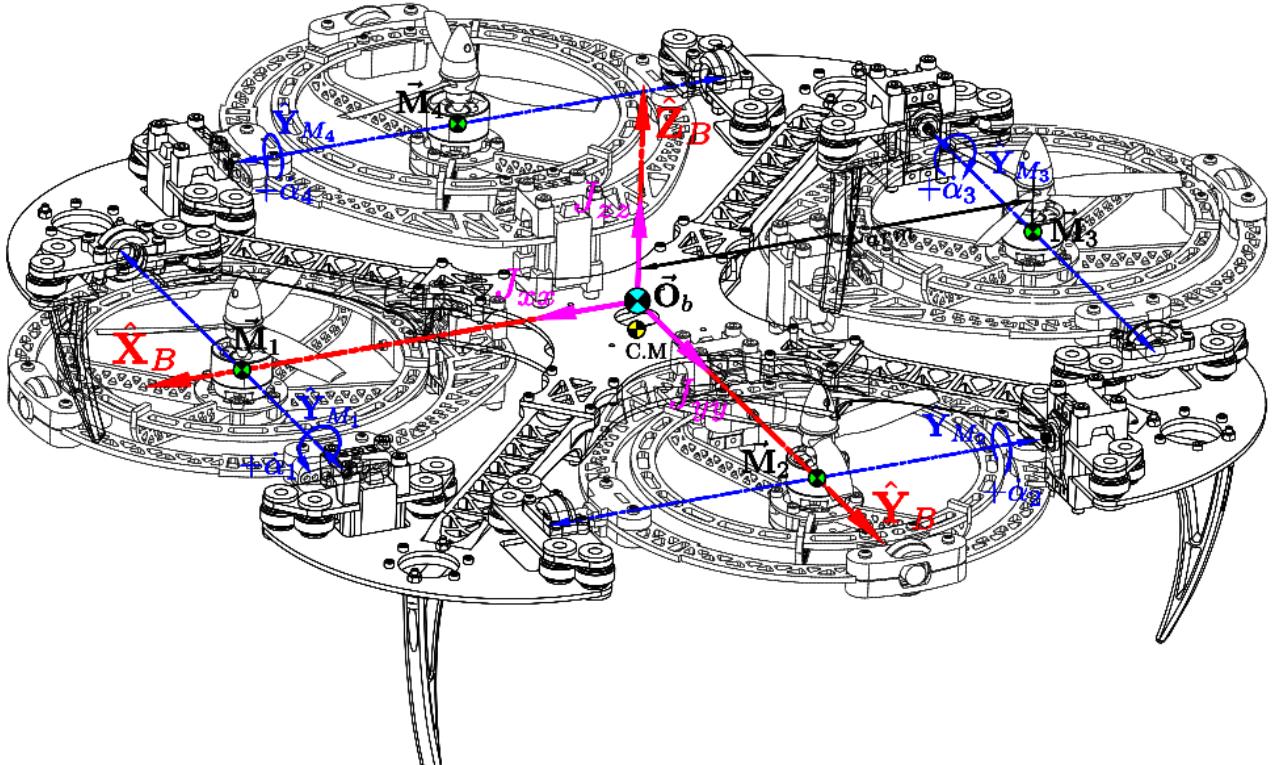


Figure 2.16: Inertial, mass and motor modules respective centers

Each module's rotational center, the vectors $\vec{M}_{1 \rightarrow 4}$, are all spaced equally relative to the origin of motion, \vec{O}_b , with a parallel axis arm $L_{arm} = 195.16$ [mm] (Fig:2.16). To avoid notational confusion the term $\vec{L}_i = [\pm 195.16 \ 0 \ 0]^T$ or $[0 \ \pm 195.16 \ 0]^T$ is used to represent the vector displacement between the origin \vec{O}_b and each motor modules center of rotation $\vec{M}_{1 \rightarrow 4}$. The net inertial equation $J_b(u)$, about the origin \vec{O}_b and depending on the actuator position matrix value $u \in \mathbb{U}$, can be calculated as:

$$\underset{\vec{O}_b}{J_b}(u) = J_y + \sum_{i=1}^4 J'_{\vec{M}_i} \quad [\text{kg.m}^2], \quad u \in \mathbb{U} \quad (2.26a)$$

Where $J'_{\vec{M}_i}$ is the motor module inertia from Eq:2.25 but translated to the origin \vec{O}_b using a parallel axis theorem with $m_p = 190$ [g]:

$$J'_{\vec{M}_i} = J_{\vec{M}_i} + m_p (\vec{L}_i \cdot \vec{L}_i - \vec{L}_i \otimes \vec{L}_i) \quad (2.26b)$$

Although Eq:2.26 produces the entire body's inertia, each equation to calculate $J'_{\vec{M}_i}$ involves cascaded transformations which may deteriorate the resultants certainty. Each motor module's inertia is first translated to their respective centers of rotation then rotated as per the two servos and then finally translated again back to the body frame's origin.

Alternatively the inertia contribution of each sub-body's can be considered separately and translated directly to the body frame's origin from each respective center of mass. This will improve the certainty of the produced inertial equations, each translation/rotation has with it an associated uncertainty and floating point concatenation. It is also perhaps more intuitive for the reader to consider each sub-body's contribution individually, despite having been derived as combined inertial bodies in the above. The vehicles net inertia can the be described as 9 discrete contributing bodies; four inner rings J_n , four middle rings J_m and one body structure J_y :

$$\underset{\vec{O}_b}{J_b}(u) = J'_y + \sum_{i=1}^4 J_n + \sum_{i=1}^4 J_m \quad u \in \mathbb{U} \quad (2.27)$$

Isolating each body and independently considering each inertia; starting with the inner ring's, each having an inertia J_n with respect to its *center of mass* (and not center of rotation) measured *relative* to its center of rotation. The following is then fundamentally different from the process in Eq:2.20, calculating the inner ring's inertial contribution about the origin \vec{O}_b .

For the inner ring only, with a mass m_n and center of mass $C.M_n$ relative to its center of rotation \vec{M}_i . The inner ring(s) contribution then follows:

$$m_n = 92 \quad [\text{g}] \quad (2.28a)$$

$$C.M_n = [-1.44 \ 0.0 \ 5.14]^T \quad [\text{mm}] \quad (2.28b)$$

The inner ring's inertial matrix at that body's center of mass (Fig:2.11) was found to be:

$$\underset{C.M}{J_n} = \begin{bmatrix} 496.6 & -31.7 & 6.6 \\ -31.7 & 1800.1 & 0.0 \\ 6.6 & 0 & 2048.9 \end{bmatrix} \quad [\text{g.cm}^2] \quad (2.28c)$$

Relative to the body frame origin \vec{O}_b the inner ring's center of mass, rotated by λ_i and α_i servos about their respective axes with a relative orthogonal R_z rotation too, is then:

$$C.M'_n = R_z R_y(\alpha_i) R_x(\lambda_i) (C.M_n) \quad (2.28d)$$

So transforming the inertia from Eq:2.28c, still about the center of mass $C.M_n$, but with axes aligned parallel with the body frame, or using the shorthand $\parallel \vec{O}_b$. The inner ring's inertia as a function of both servo angles λ_i and α_i is:

$$\underset{\parallel \vec{O}_b}{J'_n}(\lambda_i, \alpha_i) = R_z R_y(\alpha_i) R_x(\lambda_i) (J_n) R_x^{-1}(\lambda_i) R_y^{-1}(\alpha_i) \mathbb{R}_z^{-1} \quad (2.28e)$$

The vector difference between the new, rotated center of mass $C.M'_n$ with the body origin $\vec{\mathbf{O}}_b$ is given by:

$$\Delta L = \vec{L}_i - C.M'_n \quad (2.28f)$$

Then using the above in a parallel axis translation, adapted from Eq:2.24b, to move the rotated inertia J'_n to the center of the body frame $\vec{\mathbf{O}}_b$:

$$\frac{J_n}{\parallel \vec{\mathbf{O}}_b} = J'_n + m_n((\Delta L \cdot \Delta L) \mathbb{I}_{3 \times 3} - \Delta L \otimes \Delta L) \quad (2.28g)$$

And for reference, when both servos are at rest $\lambda_i = 0$ and $\alpha_i = 0$, the inner ring's inertial contribution about the origin is:

$$\frac{J_n}{\vec{\mathbf{O}}_b} = \begin{bmatrix} 520.9 & -31.0 & 922.6 \\ -31.0 & 36348.5 & 0.0 \\ 922.6 & 0.0 & 36573.0 \end{bmatrix} \times 10^{-7} \Big|_{\lambda_i, \alpha_i=0} \quad [\text{kg.m}^2], \in \mathcal{F}^b \quad (2.28h)$$

Similarly, the same process is applied for the middle ring's rotated and translated inertia. The middle ring *only* (Fig:2.12) has a mass and center of mass relative to the module's center of rotation respectively:

$$m_m = 98 \quad [\text{g}] \quad (2.29a)$$

$$C.M_m = [-47.00 \quad 3.74 \quad -3.63]^T \quad [\text{mm}] \quad (2.29b)$$

The inertial matrix of the middle ring body, excluding the inner ring, about its center of gravity is:

$$\frac{J_m}{C.M} = \begin{bmatrix} 2879.1 & 172.3 & 223.6 \\ 172.3 & 6269.0 & 13.3 \\ 223.6 & 13.3 & 8947.5 \end{bmatrix} \quad [\text{g.cm}^2] \quad (2.29c)$$

Rotated the center of mass only by the α_i servo about the $\hat{Y}_{M'_i}$ axis yields the center of mass $C.M'_m$ relative to $\vec{\mathbf{O}}_b$:

$$C.M'_m = R_z R_y(\alpha_i)(C.M_m) \quad (2.29d)$$

Then the rotated inertial matrix, aligned with axes parallel to the body frame origin $\vec{\mathbf{O}}_b$, follows:

$$\frac{J_m}{\parallel \vec{\mathbf{O}}_b} = R_z R_y(\alpha_i)(J_m) R_y^{-1}(\alpha_i) R_z^{-1} \quad (2.29e)$$

The vector difference from the rotated center of mass to the body frame origin is calculated:

$$\Delta L = \vec{L}_i - C.M'_m \quad (2.29f)$$

Which then leads to the parallel axis translation of the middle ring's inertia to the body origin:

$$\frac{J_m}{\vec{\mathbf{O}}_b} = J'_m + m_m((\Delta L \cdot \Delta L) \mathbb{I}_{3 \times 3} - \Delta L \otimes \Delta L) \quad (2.29g)$$

Again, for reference, at rest with the middle ring servo $\alpha_i = 0$ the middle ring's inertial contribution at $\vec{\mathbf{O}}_b$ is:

$$\frac{J_m}{\vec{\mathbf{O}}_b} = \begin{bmatrix} 2905.7 & 715.4 & -303.9 \\ 715.4 & 27795.7 & 0.0 \\ -303.9 & 0.0 & 30475.0 \end{bmatrix} \times 10^{-7} \Big|_{\alpha_i=0} \quad [\text{kg.m}^2], \in \mathcal{F}^b \quad (2.29h)$$

Then, reiterating Eq:2.27, the instantaneous inertia of the entire body in motion is calculated as the contribution of the connected sub-bodies depending on the actuator matrix $u \in \mathbb{U}$.

$$\frac{J_b(u)}{\vec{\mathbf{O}}_b} = J'_y + \sum_{i=1}^4 \frac{J_n}{\vec{\mathbf{O}}_b} + \sum_{i=1}^4 \frac{J_m}{\vec{\mathbf{O}}_b} \quad u \in \mathbb{U} \quad (2.30a)$$

The mass for the whole vehicle is $m_b = 1574.7$ [g]. For reference, using Eq:2.30a, the inertial matrix for the assembly at the actuator rest conditions, $u = \vec{0}$, about the origin $\vec{\mathbf{O}}_b$ is:

$$J_b(\vec{0}) = \begin{bmatrix} 317448.2 & 0.4 & -14.5 \\ 0.4 & 317570.7 & 6.5 \\ -14.5 & 6.5 & 628257.5 \end{bmatrix} \times 10^{-7} \Big|_{u=\vec{0}} \quad [\text{kg}\cdot\text{m}^2], \in \mathcal{F}^b \quad (2.30b)$$

Unless otherwise specified; any inertia $J_b(u)$ indicates an instantaneous calculated solution to Eq:2.30a given a particular $u(t) \in \mathbb{U}$. The purpose of the derivations for rotated centers of mass in Eq:2.28 and Eq:2.29 is twofold; highlighting both the inertial contributions *and* the variable center of masses for each sub-body.

Seeing that the origin of motion $\vec{\mathbf{O}}_b$ in the body frame \mathcal{F}^b and the net body's center of mass $C.M_b$ are not coincidental, it is important to specify the net center of mass position, which varies with actuator positions $u \in \mathbb{U}$. In the general case for a collection of n bodies, with each body's center of mass at some position \vec{X}_i and each having a mass m_i , resultant center of mass is:

$$C.M = \frac{\sum_{i=1}^n m_i \cdot \vec{X}_i}{\sum_{i=1}^n m_i} \quad (2.31a)$$

Using $C.M'_n$ and $C.M'_m$ as rotated centers of mass defined in Eq:2.28d and Eq:2.29d respectively and $C.M_y$ for the body structure, the vehicle has a variable center of mass $C.M_b(u)$:

$$C.M_b(u) = \frac{m_y C.M_y + \sum_{i=1}^4 m_n C.M'_n + \sum_{i=1}^4 m_m C.M'_m}{m_y + \sum_{i=1}^4 m_n + \sum_{i=1}^4 m_m} \quad (2.31b)$$

So the center of gravity when all actuators are at their zero positions is: $C.M_b(\vec{0}) = [0 \ 0 \ -4.94]^T$ [mm]. Using a gravity vector \vec{G} in the body frame as a result of gravitational acceleration $g = -9.81$ [m.s⁻²]:

$$\vec{G}_b = R_I(\vec{\eta})^b \vec{G}_I \quad [\text{N}], \in \mathcal{F}^b \quad (2.32a)$$

$$= R_I^b(\vec{\eta}) [0 \ 0 \ -9.81(m_b)]^T \quad (2.32b)$$

The resultant gravitational torque about the origin $\vec{\mathbf{O}}_b$ in the body frame \mathcal{F}^b is:

$$\Delta C.G = \vec{\mathbf{O}}_b - C.M_b(u) \quad (2.32c)$$

$$\vec{\tau}_g = \Delta C.G \times m_b \vec{G}_b \quad [\text{N.m}], \tau_g \in \mathcal{F}^b \quad (2.32d)$$

Inertia uncertainty, proven to be destabilizing and detrimental to control efforts in [71,132], can indeed be incorporated into state dependent plant uncertainty like in [10]. Controllers with strong disturbance and uncertainty rejection, like a well designed H_∞ controller, would be ideally suited to controlling an attitude plant without having to explicitly specify all of the above. However it is worth the effort given that Lagrange dynamics are applied to the servo actuator responses later in Sec:3.3. Such equations of motion will later need explicit terms for instantaneous transformed inertias.

2.4 Electronics

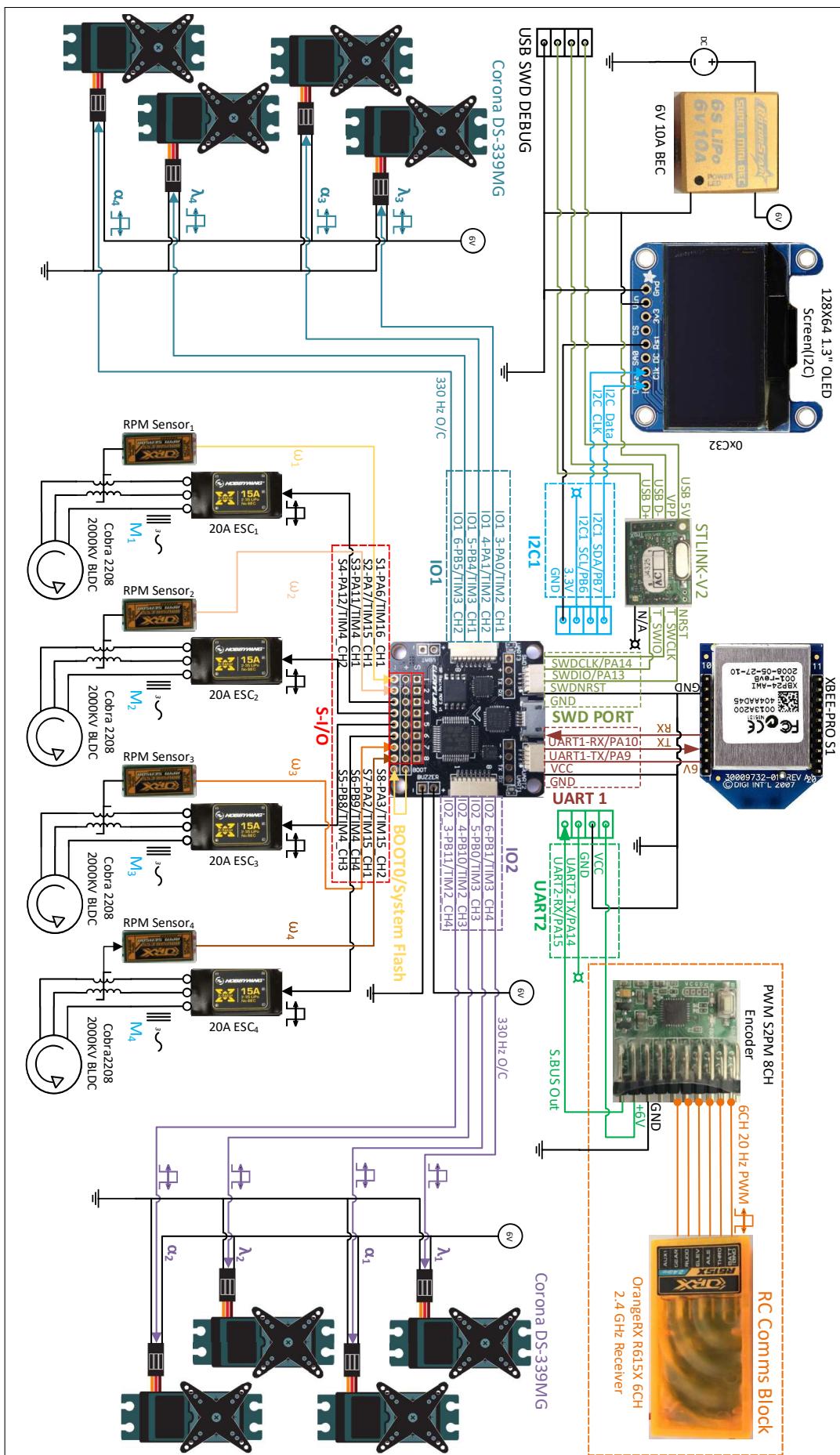
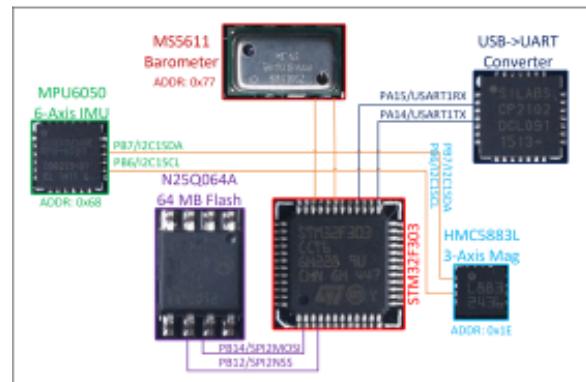


Figure 2.17: Hardware schematic diagram

An abstracted hardware diagram for the (electronic) system layout is shown in Fig:2.17. It is an illustration for the connection of different electronic peripherals used to aid the on-board control system. The structure of the implemented autopilot system and control loops are addressed later. This section aims to provide a brief overview of the specific modules used, their purpose and a description of how they are interfaced. No control loops or code structures are considered yet, those are detailed in Ch:4 and Ch:6 respectively.



(a) SPRacing F3 deluxe flight controller



(b) F3 Deluxe on-board connections

Figure 2.18: SPRacing F3 deluxe layout

The embedded system is constructed around an ARM STM32F303 [120] based microcontroller. The micro-processor board is a commercial flight control board, specifically an SPRacing F3 Deluxe [28] which has had its bootloader removed and custom firmware, unique to this project, developed for it. CleanFlight or BetaFlight opensource software (from [27] and [11] respectively) are typically used for the SPRacing F3; despite open source software its hardware specifications are however not openly available. The reverse engineered electrical schematic for the board is included in App:B.2 but a simplified overview of its internal connections is shown in Fig:2.18b.

The embedded software implemented is described in Ch:6; the I/O for all the peripherals are however detailed here. The flight-controller has the following onboard peripherals; an I2C MPU-6050 6-axis gyroscope and accelerometer [57] with an I2C connected HMC5883 magnetometer compass [35]; an I2C MS5611 barometer [118] and finally 64 Mb of SPI flash memory.

The combination of above sensors fused for state estimation and their associated filtration algorithms are dealt with in Sec:5.5 of Ch:5.

**Figure 2.19:** SBUS converter & 6CH receiver

Two separate wireless communication loops are used. First; the system relays full state information for a complete 6-DOF X-Y-Z position and $\phi - \theta - \psi$ orientation autopilot system. Sent from an independent ground control station (*GCS*) using 2.4 GHz XBEE S1 module(s) [58] which is connected to the flight controller via USART. Full state-estimation, using a multi-camera system (??), and basic trajectory generation is performed on the GCS for the vehicle to track that trajectory.

Secondly; a partial trajectory (basic orientation) augmented pilot control input system, fail safe and secondary to the autopilot loop, is transmitted through a 6 channel 2.4 GHz radio frequency module. The secondary system allows for physical control without the need of a trajectory generation loop. The 6 CH received signals, otherwise permeated as six individual 20 kHz PWM signals via an OrangeRx R615x receiver [92], are encoded into a single proprietary S.BUS data stream (Fig:2.19).

The need for a serial bus (S.BUS) encoder, specifically using [52], comes about as a consequence of the introduction of the 8 additional servos. As a result, there are no longer 6 free additional timer input/output channels which can be dedicated to input capture of those RC channels. Encoding the received data to a serial data line means the 6CH commands can be processed with a single RX channel by the microcontroller. The encoder implements a USART derivative communications standard called S.BUS. Shown in Fig:2.20 the S.BUS data, captured with a logic analyzer [112], was used to ascertain the data stream's following parameters:

- 25 Bytes per packet
- 8-Bit byte length
- 1 Start byte 0x240
- 1 Byte of state flags
- 1 Stop byte 0x0
- Bytes are:
 - MSB First
 - 1 start & 2 stop bits
 - Even parity bit
 - Inverted
 - 100000 baud ($b.s^{-1}$)
- 22 total bytes of CH data
- Each channel's data is 11 bits long
- 16CH encoded
- Channel data is little endian prioritized
- 14 ms idle time between packets
- Packets are arranged:



Figure 2.20: S.BUS data stream

The received information from the transmitted 6 channels is smoothed with a digital filter, using an infinite impulse response moving average filter. The filters difference equation can be as follows:

$$y_n = \left(1 - \frac{1}{N}\right)y_{n-1} + \frac{1}{N}x_n \quad (2.33)$$

Moving over an average of $N = 5$ samples which, each with a propagation delay of 14 ms due to S.BUS transmission, the filter has a 70 ms zero order holding time. The signal's sampling delays are sufficiently faster than the transfer times of the signals are not of consequence.

Similarly all the measured RPM signals measured by the OrangeRx RPM speed sensors are filtered over 5 samples as well. Any received signals referred to are all post filtration. Filtering for state estimation made without using the inertial-measurement unit (using the camera system) is to be performed separately on the Ground Control Station computer.

Each of the eight digital servo actuators are controlled individually from 330 [Hz] center aligned PWM timer output compare channels (TIM2:CH1→CH4 and TIM3:CH1→CH4). Output pulses typically range from 1 – 2 [ms] to linearly control the rotational position. The servos exact range and transfer function(s) is empirically determined next in Sec:2.4.1. The four 20 [A] brushless DC electronic speed controllers (*ESCs*) are each driven from a 20 [Hz] PWM output (TIM4:CH1→CH4), similarly with 1 – 2 [ms] input pulse widths.

There is a total of 12 PWM output compare signals drawn from the flight controller, 8 for the servos and 4 for the *ESCs*. The servos are powered by a regulated 6 [V] DC 10 [A] power supply [51] whilst the *ESCs* switch unregulated 14.1 [V] DC supplied from an external power tether. The DC supply could be drawn from a battery bank but that would adversely affect the weight of an already heavy platform.

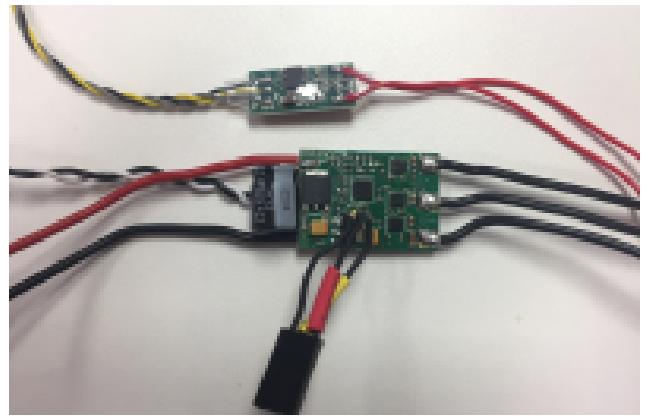
There is no integrated feedback for instantaneous RPM values available from the *ESCs*. Dedicated OrangeRX BLDC RPM sensors, [50], are used to measure each of the four motor's rotational speeds. Despite being termed *brushless DC motors*, the motors are actually 3-phase motors which, when used with an *ESC*, behave like closed loop DC motors. The RPM sensors physically measure switching phases across two of the three motor phases, following that exact RPM can be ascertained. In general, the switching signal of a 3-Phase induction motor is shown by [77] to be proportional to the rotational velocity:

$$F_{rps} = \frac{2 \times F_{poles}}{\text{No. of rotor poles}} \quad [\text{Hz}] \quad (2.34)$$

The output signal generated by the OrangeRx RPM sensors varies the period of a 50% duty cycle square wave, that wave frequency is directly proportional to the motor's pole switching frequency. The sensor output signal has a gain of 7 for the 14 pole BLDC Cobra motors. That gain is verified with the linear relationship(s) physically measured using an optical rotation sensor, plotted in Fig:2.22. Knowing exact RPM rates means the subsequent thrust and aerodynamic torques for the control plant inputs can be calculated with greater certainty.



(a) XRotor 20A ESC connection guide [49]



(b) LDPower 20A ESC with RPM sensor

Figure 2.21: BLDC electronic speed controllers

The *ESCs*, although LDPower 20A devices, are re-flashed with BLHeli firmware [12]. The LDPower *ESCs* (Fig:2.21b) match Hobbywing Xrotor 20A ones (Fig:2.21a) which both use SiLabs F396 microcontrollers; the same firmware can be flashed onto both MCUs. Custom BLHeli software provides greater refinement over configurations like the deflection range of inputs, but default values were used. The plot in Fig:2.22a shows the rotation per second, or otherwise frequency in Hz, speed curve for an unloaded motor; similarly in Fig:2.22b shows the speed curve when loaded for a 6×4.5 prop.



(a) RPM sensor plot - no load

(b) RPM sensor plot - 6X4.5 prop

Figure 2.22: RPM sensor calibration plots

It is worth noting the loaded speed plot in Fig:2.22b is slightly quadratic; the loaded response is from the second order aerodynamic drag term, quadratic with respect to the propellers revolutions per second, expanded on in Sec:3.2.1. Moreover, when the motor is torque loaded by the propeller, the ESC current limits rotational speeds at just over 16×10^3 [RPM].

Timers channels are used to measure the varying frequency output from the RPM sensors. General purpose Timers 15 (TIM15:CH1 \rightarrow CH2), 16 (TIM16:CH1) and 17 (TIM17:CH1) are configured to capture the input PWM signal generated by the speed sensors. Included on the I2C communication line is an I2C O-LED display for debugging and status update purposes.

Any STM32 μ controller is programmed through a dedicated debugging device. The ST-Link V2 [119] is the current proprietary device which, itself, is a specially programmed STM32F10 chip. The chip connects to the dedicated Serial Wire Debugging ports of the target STM (*SWD-CLK*, *SWD-IO* & *SWD-NRST*) and is interfaced via regular USBD+ and USBD- data lines.

2.4.1 Actuator Transfer Functions

Servo Transfer Functions



Figure 2.23: Servo transfer function test rig

The full scale deflection for digital servos are in fact greater than their quoted 180° range. Each servo has a rotational input range of around 230° (Fig:2.24a). In the prototype control loop for the servos are left in open loop; the major loop controller coefficients are expected to account for minor loop actuator dynamics. With that being said, for such an expectation to be validated the simulation would need to represent the servo's response accurately.

Seeing that the 180° limitation was imposed as a design decision, one of the first points of contention is the effect such a constraint would have on the feasible operating trajectories. The control algorithms derived in Ch:4 are first tested with an ideal, continuous rotation servo actuator with similar rate limits and transfer characteristics. Following that servo saturation limitations are introduced and the constraints to feasibly achievable trajectories are investigated.

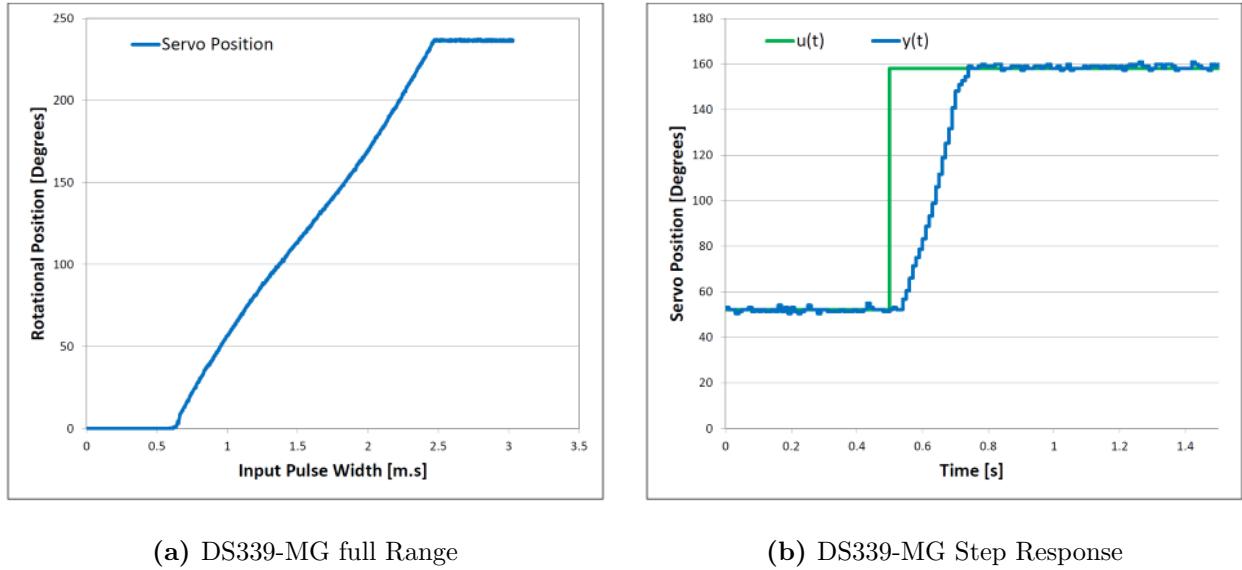


Figure 2.24: Unloaded servo transfer characteristics

For the servos whose rotational range and step response are shown in Fig:2.24, the relationship between the input pulse-width x [m.s] and the rotational output position y [°] is given by:

$$y(x) = \begin{cases} 0^\circ & x < 0.65 \text{ ms} \\ 129.12x - 82.64 & 0.64 \text{ ms} \leq x \leq 2.46 \text{ ms} \\ 230^\circ & x > 2.46 \text{ ms} \end{cases} \quad (2.35)$$

In practice the equation Eq:2.35 is changed such that 0° offset is taken at around a 50% input, making its operational range $\pm 90^\circ$. Each servo is mechanically rate limited to 60°/0.15s or 400 degrees per second with a dead time of $t_d \approx 1.2$ [ms] and a (*negligible*) mechanical deadband of 4 [μs]. Each servo has an approximate (*critically damped*) second order transfer function

$$G_{servo}(s) = e^{-t_{ds}} \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} \quad (2.36a)$$

$$= e^{-0.012s} \frac{(14.869)^2}{s^2 + 2(1)(14.869)s + (14.869)^2} \quad (2.36b)$$

With saturation limits:

$$Y_{servo}(s) = \begin{cases} 0^\circ & |U(s)| < 0.65 \\ G(s) & 0.65 \leq |U(s)| \leq 2.46 \\ 230^\circ & |U(s)| > 2.46 \end{cases} \quad (2.36c)$$

The net transfer block for the servo is shown in Fig:2.25, including saturating non-linearities but neglecting the mechanical deadband...

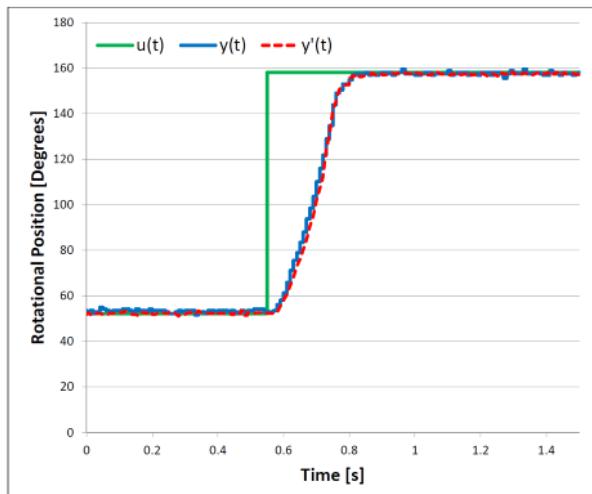


Figure 2.25: Servo block diagram

The plot in Fig:2.24b shows the transfer characteristics, at the shaft output, of an unloaded servo. When rotating the inertial body of the inner ring assembly, arranged as in Fig:2.26a. Plotted in Fig:2.26b is the plant response plot of $y(t)$ which is consistent with the transfer function in Eq:2.36. Despite rotating a load and hence requiring a greater torque. The servo's characteristics remains unchanged, even when the BLDC motor (with a 6×4.5 prop) with a rotational velocity of 6500 RPM is introduced, plotted $y'(t)$, further increasing the torque load of the assembly as a result of the gyroscopic response, Eq:2.18.



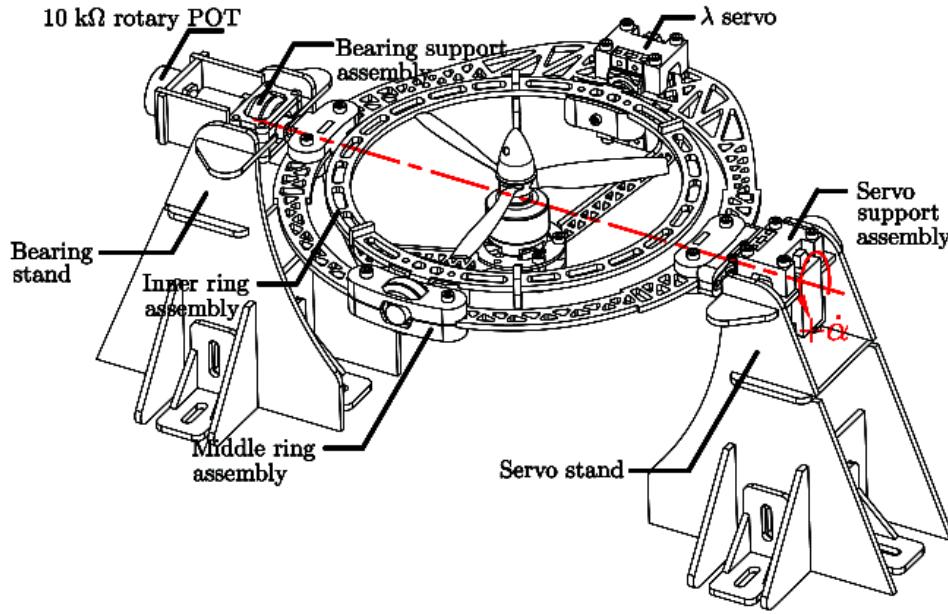
(a) Inner ring servo rig



(b) Servo response plot

Figure 2.26: Inner ring servo characteristics

Similarly Fig:2.27b shows the response for the middle ring servo. Its transients remain the same however oscillations are introduced at the settling point. Those oscillations are as a result of the larger inertia being driven, adding deflection within the frame structure. It is important to specify that the oscillations are not at the servo's output shaft, the rotational position was measured with respect to the bearing supported shaft, coaxial to the servos (Fig:2.27a). Those harmonics are still present under load, plotted in $y'(t)$, despite the frame being tensioned by a lifting thrust. The mechanical oscillations can be accounted for by either introducing a more rigid sub-frame, limiting the maximum angular rate of the servos or simply adding a state dependent input uncertainty term to the control plant.



(a) Middle ring servo test rig



(b) Servo response plot

Figure 2.27: Middle ring servo characteristics

BLDC Transfer Functions

Each Cobra 2208 BLDC motor, when loaded with a 6×4.5 propeller has a quadratic speed curve (plotted in Fig:2.29a). This is as a result of the propeller's opposing aerodynamic drag, approximately proportional to the square of the propellers angular velocity (more on propeller aerodynamics in Sec:3.2.1).



Figure 2.28: BLDC rpm speed calibration and transfer function rig

Using the BLHeli interface, the input range for the motor's speed controllers can be adjusted, but for the purposes of this project were left unchanged. That relationship between input pulse-widths to the ESC and output RPM sensor signal is given by the hybrid state equations for input range limits:

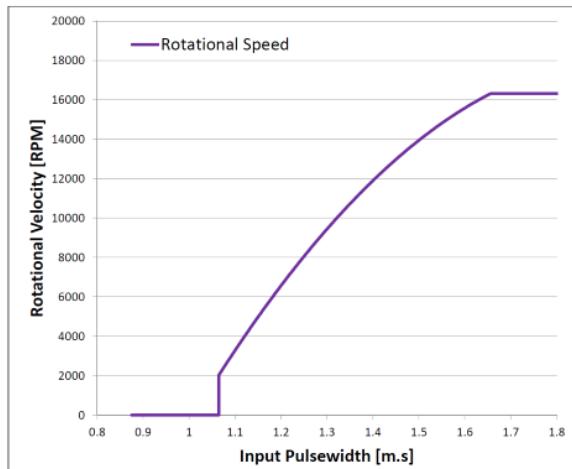
$$y(x) = \begin{cases} 0 & x < 1.065 \text{ ms} \\ -20593x^2 + 80187x - 60004 & 1.065 \text{ ms} \leq x \leq 1.655 \text{ ms} \\ 16300 & x > 1.655 \text{ ms} \end{cases} \quad [\text{RPM}] \quad (2.37)$$

The upper limit in Eq:2.37 and the motor's step response are both governed by the ESC's maximum current limit; in this case 20 [A]. Imposing 10 [A] current limiting, a consequence of using lower power ESCs is plotted $c(t)$ in Fig:2.29b, significantly restricts the motor's transient and steady-state performance. The motor's step response, $y(t)$, has a negligible dead time and 2nd order dynamics, with a transient time constant far faster than the servo's plant. The motor's transfer function for speed in RPM is:

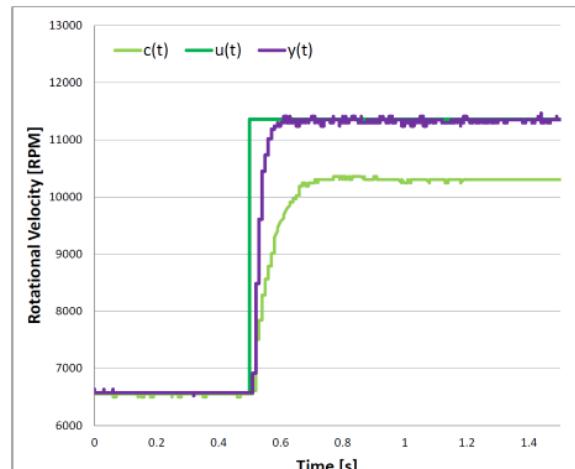
$$G_{BLDC}(s) = \frac{1}{(1 + 1.7583s \times 10^{-3})(1 + 1.7494s \times 10^{-3})} \quad [\text{RPM}] \quad (2.38a)$$

And saturation limits:

$$Y_{BLDC}(s) = \begin{cases} 0 & |U(s)| < 1.065 \\ G(s) & 1.065 \leq |U(s)| \leq 1.655 \\ 16300 & |U(s)| > 1.655 \end{cases} \quad (2.38b)$$



(a) BLDC RPM range



(b) Cobra BLDC step response

Figure 2.29: BLDC motor characteristics

Chapter 3

Kinematics & Dynamics

The following generally applicable, rigid body dynamics are first derived with respect to net forces and torques. Thereafter, those dynamics are adapted to the non-linear, multibody case where only constrained relative rotational motion between bodies is permitted; the same motion which the prototype can undergo. Following that, aerodynamic effects are incorporated into the plant's model. Finally a consolidated, quaternion based plant model is presented which is used for the later control plant development in Ch:4.

3.1 Rigid Body Dynamics

3.1.1 Lagrange Derivation

Fundamentally any body, rigid or otherwise, can undergo two kinds of motion; namely rotational and translational. Often a Lagrangian approach for combined angular and translational movements is used to derive the differential equations of motion for each degree of freedom [102, 124]. The Lagrangian principle ensures that (translational and rotational) kinematic energies and potential energy are conserved throughout the system's trajectory progression. When combined with Euler-Rotational equations, the Euler-Lagrangian formulation from [126] fully defines the aerospace 6-DOF equations.

Lagrangian formalism is regarded as especially useful in non-Cartesian (*spherical etc...*) co-ordinate frames or multi-body systems. With that being said, Cartesian co-ordinates were defined in Sec:2.2.2 for this system. An alternative relative co-ordinate system could be used for implicit-Euler based dynamics, as in [89]. Rigid body dynamics in a Cartesian co-ordinate frame do lend themselves to Newtonian mechanics. The Newton-Euler and Euler-Lagrange formulations both result in the same differential equations of motion, but follow different derivations. The Lagrangian operator, \mathcal{L} , is a scalar term for the difference between kinetic and potential energies, T and U respectively. Considering some generalized path co-ordinates $\vec{\mathbf{r}}(t)$ for a body, with both position $\vec{\xi}$ and attitude \vec{H} included;

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} \vec{\xi} & \vec{H} \end{bmatrix}^T \quad \in \mathcal{F}^\Lambda \quad (3.1)$$

The co-ordinates in Eq:3.1 are generalized and taken with respect to some hypothetical shared frame Λ . Those generalized co-ordinates are later refined to Cartesian body co-ordinates with respect to the inertial frame. The Lagrangian is the difference between kinetic and potential energies, by definition:

$$\mathcal{L}(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}, t) = T(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}) - U(\vec{\mathbf{r}}, \dot{\vec{\mathbf{r}}}) \quad (3.2a)$$

Where the trajectory's kinetic and potential energy function(s) are T and U respectively. Introducing a rigid body's general (translational and rotational) kinetic and potential energies, both defined with respect to the shared reference frame \mathcal{F}^Λ .

There is no angular contribution of potential stored energy, so $U(\vec{r}, \dot{\vec{r}})$ depends only on gravitational potential energy. The gravity vector is defined in the inertial frame $\in \mathcal{F}^I$ as:

$$\vec{G}_I = [0 \ 0 \ -9.81]^T \quad [\text{m.s}^{-2}], \quad \in \mathcal{F}^I \quad (3.2b)$$

Where \vec{G}_I acts in the negative \hat{Z}_I downward direction. The Lagrangian scalar with angular and translational kinetic and potential energies substituted is then:

$$\mathcal{L}(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2} \dot{\vec{\xi}}^T (m_b) \dot{\vec{\xi}} + \frac{1}{2} \dot{\vec{H}}^T (J_b) \dot{\vec{H}} - m \vec{G}_\Lambda z \quad (3.2c)$$

Noting that J_b is the body's generalized inertial matrix aligned and translated with respect to the common frame \mathcal{F}^Λ . The Euler-Lagrange formulation equates partial derivatives of the Lagrangian to any generalized forces, \vec{V} , acting on the system in frame Λ . In the rigid body motion case those *generalized* forces are net forces \vec{F}_μ and net torques $\vec{\tau}_\mu$ in the shared frame $\in \mathcal{F}^\Lambda$.

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} \right) - \frac{\partial \mathcal{L}}{\partial \vec{r}} = \vec{V} = \begin{bmatrix} \vec{F}_\mu \\ \vec{\tau}_\mu \end{bmatrix} \quad \in \mathcal{F}^\Lambda \quad (3.3)$$

Then taking the partial derivatives of Eq:3.2c with respect to the path co-ordinates $\vec{r}(t)$ and path rates $\dot{\vec{r}}(t)$ respectively:

$$\frac{\partial \mathcal{L}}{\partial \vec{r}} = \begin{bmatrix} m_b \vec{G}_\Lambda \\ 0 \end{bmatrix} \quad \in \mathcal{F}^\Lambda \quad (3.4a)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} \right) = \left[\frac{d}{dt} m_b \dot{\vec{\xi}} \quad \frac{d}{dt} J_b \dot{\vec{H}} \right]^T \quad \in \mathcal{F}^\Lambda \quad (3.4b)$$

Where \vec{G}_Λ is the gravitation force transformed to the common frame \mathcal{F}^Λ which $\mathcal{L}(\vec{r}, \dot{\vec{r}})$ is with respect to. The body mass m_b and inertia J_b could potentially have some non-zero time derivative, but here are considered to be constant. Time varying inertias defined in Sec:2.3 are dynamically incorporated subsequently in Sec:3.3.1. For now only the general rigid body case is considered...

Any vector in some rotating coordinate system has a time derivative, relative to another reference frame, as per the Reynolds Transportation Theorem [121]:

$$\frac{d\vec{f}}{dt_a} = \frac{d\vec{f}}{dt_b} + \vec{\omega}_{a/b} \times \vec{f} \quad (3.5)$$

So applying Eq:3.5 to the partial derivatives in Eq:3.4b and further defining the generalized co-ordinates $[\vec{\xi}, \vec{H}]^T$ as 6-DOF Cartesian body co-ordinates with respect to the inertial frame \mathcal{F}^I or the body frame \mathcal{F}^b .

Reiterating that the angular orientations $\vec{\xi}$ are with respect to a common frame \mathcal{F}^Λ , unlike Euler angles $\vec{\eta} \in \mathcal{F}^{v2,v1,I}$. Recalling the definition of attitude in a common frame $\vec{\eta}_b$ from Eq:2.12d, where $\vec{\omega}_b = \dot{\vec{\eta}}_b$ and $\vec{\eta}_b \in \mathcal{F}^b$, the trajectory is refined:

$$\vec{r} = [\vec{\xi} \ \vec{H}]^T \triangleq \begin{bmatrix} \vec{\mathcal{E}} \\ \vec{\eta}_b \end{bmatrix} \quad (3.6a)$$

Which leads to the path rate definition $\dot{\vec{r}}(t)$:

$$\rightarrow \dot{\vec{r}} = [\dot{\vec{\xi}} \ \dot{\vec{H}}]^T \triangleq \frac{d}{dt} \begin{bmatrix} \vec{\mathcal{E}} \\ \vec{\eta}_b \end{bmatrix} = \begin{bmatrix} \vec{v} \\ \vec{\omega} \end{bmatrix} \quad \in \mathcal{F}^b \quad (3.6b)$$

Substituting the changed path co-ordinates from Eq:3.6 into the Lagrangian Eq:3.2c yields a familiar Lagrangian scalar in the body frame \mathcal{F}^b :

$$\mathcal{L} = \frac{1}{2} \vec{v}_b^T (m_b) \vec{v}_b + \frac{1}{2} \vec{\omega}_b^T (J_b) \vec{\omega}_b - m_b \vec{G}_b z \quad \in \mathcal{F}^b \quad (3.7)$$

Where \vec{G}_b is the gravitational force vector from Eq:3.2b transformed to the body frame \mathcal{F}^b and z is the height displacement of the vehicle (in the inertial frame). The time derivative of the substituted path co-ordinates in the partial derivative Eq:3.4b is then:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \vec{r}} \right) = \begin{bmatrix} m_b \frac{d}{dt} \vec{v}_b & J_b \frac{d}{dt} \vec{\omega}_b \end{bmatrix}^T \quad (3.8a)$$

With respective time derivatives using the Reynolds transportation theorem:

$$\rightarrow m_b \frac{d}{dt} \vec{v}_b = m_b \dot{\vec{\nu}} + \vec{\omega}_{b/I} \times m_b \vec{v}_b \quad (3.8b)$$

$$\rightarrow J_b \frac{d}{dt} \vec{\omega}_b = J_b \dot{\vec{\omega}}_b + \vec{\omega}_{b/I} \times J_b \vec{\omega}_b \quad (3.8c)$$

Which, when substituted back into the Euler-Lagrange formulation Eq:3.3, produces the familiar Newton-Euler rigid body equations for translational and rotational motion:

$$\begin{bmatrix} m_b \dot{\vec{v}}_b + \vec{\omega}_{b/I} \times m_b \vec{v}_b \\ J_b \dot{\vec{\omega}}_b + \vec{\omega}_{b/I} \times J_b \vec{\omega}_b \end{bmatrix} - \begin{bmatrix} m_b \vec{G}_b \\ 0 \end{bmatrix} = \vec{\mathbf{V}} = \begin{bmatrix} \vec{F}_\mu \\ \vec{\tau}_\mu \end{bmatrix} \quad (3.9a)$$

$$\rightarrow \vec{F}_\mu = m_b \dot{\vec{\nu}} + \vec{\omega}_b \times m_b \vec{\nu} - m_b R_I^b(-\eta) \vec{G}_I \quad [\text{N}], \quad \in \mathcal{F}^b \quad (3.9b)$$

$$\rightarrow \vec{\tau}_\mu = J_b \dot{\vec{\omega}}_b + \vec{\omega}_b \times J_b \vec{\omega}_b \quad [\text{N.m}], \quad \in \mathcal{F}^b \quad (3.9c)$$

Reiterating that $\vec{\eta}_b \neq \vec{\eta}$ because each Euler Angle is defined in a sequentially rotated reference frame. Four equations are then needed to completely describe a body position's and attitude's state dynamics:

$$\dot{\vec{\mathcal{E}}} = R_b^I(-\eta) \vec{v}_b \quad [\text{m.s}^{-1}], \quad \in \mathcal{F}^I \quad (3.10a)$$

$$\vec{F}_\mu = m_b \dot{\vec{\nu}} + \vec{\omega}_b \times m_b \vec{v}_b - m_b R_I^b(-\eta) \vec{G}_I \quad [\text{N}], \quad \in \mathcal{F}^b \quad (3.10b)$$

$$\dot{\vec{\eta}} = \Phi(\eta) \vec{\omega}_b \quad [\text{rad.s}^{-1}], \quad \in \mathcal{F}^{v2,v1,I} \quad (3.10c)$$

$$\vec{\tau}_\mu = J_b \dot{\vec{\omega}}_b + \vec{\omega}_b \times J_b \vec{\omega}_b \quad [\text{N.m}], \quad \in \mathcal{F}^b \quad (3.10d)$$

Where $\Phi(\eta)$ is the Euler matrix defined previously in Eq:2.12e. State differentials from Eq:3.10 can be simplified to a set of two equations defined in the reference frames of the state variables which they represent. The non-linear form of those equations substitutes $d\vec{\eta}/dt = \Phi(\eta) \vec{\omega}_b$ into the Lagrangian derivative, Eq:3.4b.

$$\frac{d}{dt} \left(\frac{\delta \mathcal{L}}{\delta \vec{r}} \right) = \begin{bmatrix} m_b \frac{d}{dt} \vec{\nu} & J_b \frac{d}{dt} \dot{\vec{\eta}}_b \end{bmatrix}^T \Rightarrow \begin{bmatrix} m_b \frac{d}{dt} \vec{\nu} & J_b \frac{d}{dt} \Phi(\eta) \vec{\omega}_b \end{bmatrix}^T \quad (3.11)$$

Which only affects the angular component as the two kinetic energies are independent of one another. Applying the differential chain rule yields:

$$J_b \frac{d}{dt} \Phi(\eta) \vec{\omega}_b = J_b (\dot{\Phi}(\eta) \vec{\omega}_b + \Phi(\eta) \dot{\vec{\omega}}_b) \quad (3.12)$$

Drawing from [89] and recognizing that J_b must be transformed to the common intermediate Euler axes, $\mathbb{J} = \Psi(\eta)^T J_b \Psi(\eta)$. The state differential equation for angular acceleration in Eq:3.9c, then defined in intermediate (non-inertial) Euler frames for each angle, becomes:

$$M(\eta) \ddot{\vec{\eta}} + C(\eta, \dot{\eta}) \dot{\vec{\eta}} = \Psi(\eta) \vec{\tau}_\mu \quad \in \mathcal{F}^{v2,v1,I} \quad (3.13a)$$

$$M(\eta) = \Psi(\eta)^T J_b \Psi(\eta) \quad (3.13b)$$

$$C(\eta, \dot{\eta}) = -\Psi(\eta) J_b \Psi(\dot{\eta}) + \Psi(\eta)^T [\Psi(\eta) \dot{\vec{\eta}}]_\times J_b \Psi(\eta) \quad (3.13c)$$

The relationship $\dot{\Phi} = \Phi\dot{\Psi}\Phi$ was used to simplify Eq:3.13, the singularity in Φ still remains. The equation in Eq:3.13a fully describes the state derivative $\ddot{\eta}$ in its own reference frame(s), $\mathcal{F}^{v2,v1,I}$. The two differential equations which fully describe the entire body's motion are:

$$\vec{F}_\mu = m_b \dot{\vec{\mathcal{E}}} + R_b^I(-\eta) \vec{\omega}_b \times m_b \dot{\vec{\mathcal{E}}} - m_b \vec{G}_I \quad \in \mathcal{F}^I \quad (3.14a)$$

$$\vec{\tau}_\mu = \Psi(\eta)^{-1} M(\eta) \ddot{\vec{\eta}} + \Psi(\eta)^{-1} C(\eta, \dot{\eta}) \quad \in \mathcal{F}^{v2,v1,I} \quad (3.14b)$$

The generalized net forces and torques acting on the body, \vec{F}_μ and $\vec{\tau}_\mu$ respectively, are the system's controllable inputs which are directly affected by the system's actuators. How the actuator suite affects the net forces and torques depend on the actuator's associated effectiveness function. In the general case, which is refined in Sec:3.2, the control inputs for a quadrotor (Fig:3.1) are typically as follows.

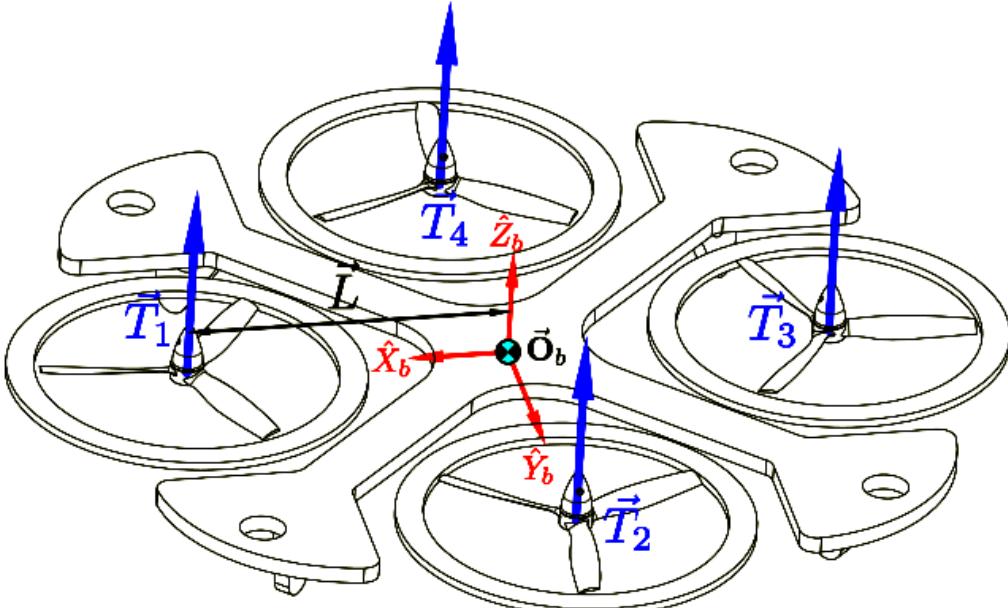


Figure 3.1: Generalized quadrotor net forces and torques

The net force \vec{F}_μ acting on the system is simply the sum of all thrust forces produced by rotating propellers, as some function of those rotational speeds; $\vec{T}(\Omega_i)$.

$$\vec{F}_\mu = \sum_{i=1}^4 \vec{T}(\Omega_i) \quad [\text{N}], \quad \in \mathcal{F}^b \quad (3.15a)$$

Similarly net torque $\vec{\tau}_\mu$ is the sum of all *differential* torque arms produced from opposing propeller thrust vectors. Each torque arm $\vec{L}_{1 \rightarrow 4}$ is relative to the origin of *motion* \vec{O}_b .

$$\vec{\tau}_\mu = \sum_{i=1}^4 \vec{L}_i \times \vec{T}(\Omega_i) \quad [\text{N.m}], \quad \in \mathcal{F}^b \quad (3.15b)$$

In Eq:3.15, the thrust vector $\vec{T}(\Omega_i)$ is a function of the i^{th} motor's rotational velocity Ω_i , fixed in the \hat{z}_b direction. Each thrust vectors could potentially be $\in \mathbb{R}^3$. Similarly \vec{l}_i is that thrust vector's perpendicular displacement from the origin \vec{O}_b .

The above equations are still applicable to any 6 DOF body, common simplifications applied to the system(s) for quadrotor control are explored in App:A.1. Aerodynamic components pertinent for thrust and torque generation relative to Eq:3.15 are now introduced; obviously the contextual focus is on quadrotor and tilting quadrotor platforms...

3.2 Aerodynamics

The relationship between a propeller's rotational speed, Ω_i in [RPS], and its perpendicular thrust vector, $\vec{T}(\Omega_i)$, is more complicated than the quadratic simplification taken at static conditions which most papers suggest (e.g [79, 101] etc...). The thrust produced is mostly dependent on the incident air stream flowing through the propeller's rotational plane; typically being the component of the body velocity normal to that propeller's plane. Fluid flowing *tangentially* across the propeller's plane contributes toward in-plane aerodynamic drag (and hence torque).

The combination of aerodynamic blade-element [93, 108] and fluid-dynamics momentum (*disc actuator*) theories equate an integral term generated across the propeller's length with the produced thrust or torque. A schedule of all aerodynamic effects encountered by a quadrotor's propellers is thoroughly detailed in both [8] and [7]. The following is a review of pertinent aerodynamic theories; vortex ring state and parasitic drag effects are not included as they will be approximately negligible given the aircraft's proposed flight envelope with low translational velocities.

3.2.1 Propeller Torque and Thrust

A possible situation which the prototype could encounter is where an upstream propeller provides the incident fluid flow to another downstream propeller. Such a situation presents a complicated fluid dynamics and vortex wake effect problem. Propeller overlapping effects are discussed in [123] but remain open to further research in the context of the aircraft considered here.

To expedite the system identification process some simplifications are made on the aerodynamics to construct an approximate model; specifically using coefficients in place of complete local chord and pitch based integrals. Such an assumption holds true given that twisted, fixed pitch propellers are used (Fig:3.2a) and not variable pitch swash-plate actuated propellers (Fig:3.2b).



(a) Twisted, fixed pitch



(b) Swash-plate variable pitch; [53]

Figure 3.2: Propeller types

A propeller's profile applies a perpendicular thrust force, T , onto the fluid in which it rotates. To build the following theoretical explanation propellers are first considered in terms of momentum theory; only perpendicular fluid flow through the propeller's plane is regarded. That fluid stream (Fig:3.3) has an incident upstream velocity, v_∞ , and a resultant slip velocity, v_s , downstream relative to the rotational plane. The change of fluid flow as a result of the propeller's rotation can be given as:

$$v_s = \Delta v + v_\infty \quad (3.16)$$

Where Δv is the net change in fluid velocity caused by the propeller blade's rotating aerofoil profile. The propeller induces a velocity directly in front of its rotational plane, v_i , such that the net fluid flow into the plane is $v_b = v_i + v_\infty$. That induced inflowing fluid velocity is different to the net velocity contribution of the propeller; $v_i \neq \Delta v$.

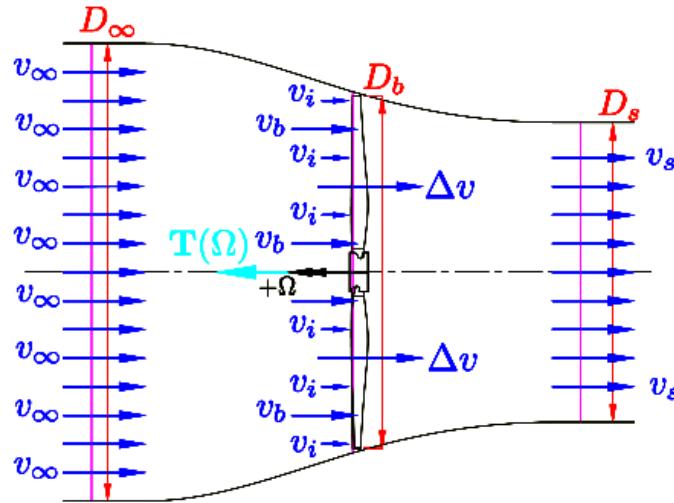


Figure 3.3: Disc Actuator Propeller Planar Flow

It is shown in [8] that, using Bernoulli's pressure theorem, the net fluid flow through the propeller's plane is:

$$v_b = \frac{1}{2}(v_s - v_\infty) = \frac{1}{2}\Delta v = \frac{1}{2}v_s|_{v_\infty=0} \quad (3.17)$$

Stemming from classical disc actuator (fluid *momentum*) theory, [103], the scalar force $T(\Omega)$ acting on the fluid is calculated as a function of mass flow rate with respect to the change in fluid velocity (or *pressure differential*):

$$T = (A_b v_b)\Delta v = \rho\pi R_b^2 v_b \Delta v = \rho\pi R_b^2 (v_i + v_\infty) \Delta v = \frac{1}{2} \rho\pi R_b^2 \Delta v^2 \quad (3.18)$$

Where R_b is the disc (propeller) radius in [m] for the fluid stream under consideration, A_b is the area of that propeller disc. The fluid density of that stream, ρ , is typically 1.225 [kg.m^{-3}] at standard temperature and pressure (*stp*). However, the desired form of thrust generated is as a function of propeller rotational velocity $T(\Omega_i)$, so Eq:3.18 is not satisfactory.

Eq:3.18 can be solved as a function of aerodynamic propulsive power expended, $\Delta P = T\Delta v$. Rotational kinetic energy of a propeller and its transferred propulsive power is difficult to quantify, with compound parasitic losses deteriorating the efficiency of the propeller. Furthermore, the local fluid velocity through the propeller is not purely normal to the propeller plane but is in fact a vector.

In reality fluid flow induced by the propeller's rotation, v_i , directly in front of its plane of rotation is not purely perpendicular but has axial and tangential induced components, termed a and a' respectively. Those induced components for the fluid velocity can be abstracted to induction factors dependent on the incident fluid velocity to the propeller's plane of rotation:

$$v_i = av_\infty \text{ in the axial direction} \quad (3.19a)$$

$$v_\theta = a'\Omega_i R_b \text{ in the tangential direction} \quad (3.19b)$$

From induction factors defined Eq:3.19, the velocity components can be written as factors of upstream velocity v_∞ .

$$v_b = (1 + a)v_\infty \quad (3.20a)$$

$$v_s = (1 + 2a)v_\infty \quad (3.20b)$$

A consequence of the tangential fluid flow is that an angular momentum flow rate exists across the propeller plane. This produces a fluid-momentum torque opposing the rotational motion about the propeller's axis, analogous but perpendicular to Eq:3.18:

$$\vec{Q} = \rho\pi R_b^3 (v_\theta - v_\infty) v_b \quad (3.21)$$

Together, Eq:3.18 and Eq:3.21 comprise propeller momentum theory but cannot be solved on their own. Blade-element theory analyses incremental aerofoil sections of width dr of the propeller profile (the sectional view of which is illustrated in Fig:3.4) at some radius r . Each aerofoil element has a net local fluid velocity \vec{U} across its profile, calculated as:

$$\vec{U} = \sqrt{(v_\infty + v_i)^2 + (v_\Omega + v_\theta)^2} \quad (3.22)$$

Where each profile has a chord length c and an inclination (or *pitch*) θ of the aerofoil *zero-lift line* relative to the horizontal. Local fluid velocities incident to the propeller profile (shown in Fig:3.4) make their own angle of attack ϕ such that a true effective angle of attack α_{eff} is encountered:

$$\phi = \theta - \alpha_{eff} \quad (3.23)$$

That local angle of attack varies with the incident fluid flow magnitude v_∞ and the induced axial velocity v_i . That trigonometric ratio between the two is given as:

$$\phi = \tan^{-1}\left(\frac{v_\infty + v_i}{v_\Omega + v_\theta}\right) = \tan^{-1}\left(\frac{v_\infty(1+a)}{\Omega r(1+a')}\right) \quad (3.24)$$

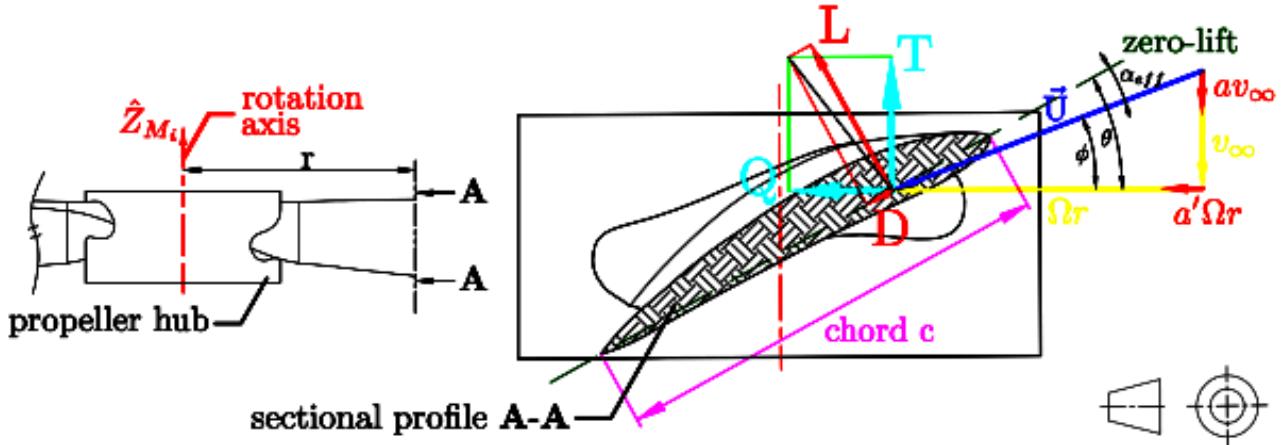


Figure 3.4: Blade element profile at radius r

In-plane fluid flow $\vec{U}(r, \phi)$, for an element at radius r with a local angle of attack ϕ , then contributes towards elemental lift and drag forces as a function of aerofoil's dimensionless lift, C_L , and drag, C_D , coefficients. Those coefficients are determined by the aerofoil's characteristics, but would be constant across the length of a variable pitch, hinged and untwisted hinged prop (Fig:3.2b).

$$\Delta L = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c C_L \quad (3.25a)$$

$$\Delta D = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c C_D \quad (3.25b)$$

With air density ρ typically taken at *stp*. Lift and drag forces, when taken parallel and perpendicular to the plane of rotation, are thrust T and torque F_q forces (Fig:3.4). The in-plane force applies an aerodynamic torque Q at the propellers hub because the force F_q acts at a radius r , [55].

$$dT = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c (C_L \cos(\phi) + C_D \sin(\phi)) . dr \quad (3.26a)$$

$$dF_q = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c (C_L \sin(\phi) + C_D \cos(\phi)) . dr \quad (3.26b)$$

$$\rightarrow dQ = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c (C_L \sin(\phi) + C_D \cos(\phi)) r . dr \quad (3.26c)$$

$$\rightarrow dP = \Omega r dF_x . dr \quad (3.26d)$$

Rotational power expended is a product of angular velocity and the opposing in-plane torque; Eq:3.26d. Power is used in lieu of torque or drag terms in Eq:3.26c or Eq:3.26b respectively. Calculating forces and power terms as per momentum theory for each element, in terms of axial and tangential induction factors:

$$dT = \rho 4\pi r^2 v_\infty (1 + a) a dr \quad (3.27a)$$

$$dP = \rho 4\pi r^2 v_\infty (1 + a) \Omega r (1 + a') dr \quad (3.27b)$$

Equating momentum and element terms produces the blade-element momentum equation(s) for aerodynamic thrust and power from a propeller. Following a few assumptions; most importantly that the lift coefficient C_L is a linear function of the effective angle of attack α_{eff} .

$$C_L = a_L(\theta - \phi) \quad (3.28)$$

Firstly the lift coefficient curve gradient a_L is shown in [56] for an ideally twisted blade, like the fixed pitch propellers under consideration here, to be 2π . An ideal lift coefficient is then a function:

$$C_L = 2\pi(\theta - \phi) \quad (3.29)$$

Secondly assuming tangentially induced velocities, v_θ , are small when compared to the propeller's speed translational speed at radius r , $v(r) = \Omega r$. The tangential induction factor a' is then:

$$a' = \frac{v_\theta}{\Omega r} \ll 1 \quad (3.30)$$

Small angle approximations then apply to Eq:3.26a-3.26c; $\cos(\phi + \alpha_{eff}) \approx 1$ and $\sin(\phi + \alpha_{eff}) \approx \phi + \alpha_{eff}$. Similarly net inflow and axial velocities are $v_\infty + v_i \ll \Omega r$, the following integrals are then found:

$$T = \int_{r=0}^R \frac{1}{2} a_L b c \rho (\Omega r)^2 \left[\theta - \frac{v_\infty + v_i}{\Omega r} \right] dr \quad (3.31a)$$

$$P = \int_{r=0}^R \frac{1}{2} a_L b c \rho (\Omega r)^3 \left[\left(\theta - \frac{v_\infty + v_i}{\Omega r} \right) \left(\frac{v_\infty + v_i}{\Omega r} \right) + C_d \right] dr \quad (3.31b)$$

Where b is the number of blades the propeller has. In practice knowing exact pitch and chord values as a function of r/R is difficult and calculating integrals at each process step is cumbersome. Both Eq:3.31a and Eq:3.31b can be solved by equating element and momentum terms (a full expansion is given in Appendix:A.2). Often dimensionless thrust, torque and power coefficients are defined across the entire blade's length:

$$C_T(J) = \frac{T}{\rho \Omega^2 D^4} \quad (3.32a)$$

$$C_P(J) = \frac{P}{\rho \Omega^3 D^5} \quad (3.32b)$$

Where Ω is the propeller's rotational speed in revolutions per second (*RPS*) and different from other inertial equations like Eq:3.64, with units [rad.s^{-1}]. The propeller diameter D is in [m]. For fixed pitch propellers the thrust and power coefficients are easily determined and remain consistent. Both Eq:3.32a and Eq:3.32b vary as a function of the dimensionless *advance ratio* J .

$$J = \frac{v_\infty}{\Omega R} \quad (3.33)$$

Typically the net upstream velocity v_∞ is simply the perpendicular component (projected onto the plane's normal vector \hat{n} , Eq:3.35) of the vehicle's translational velocity in the body frame, $\vec{v}_b \perp \hat{n}$. For the case of a zero advance ratio, $J = 0$, the conditions are regarded as static. Static thrust and power coefficients are nominal in their values.

Propeller databases like [21] provide comprehensive coefficient values for a range of small and medium diameter propeller types at different advance ratios. Included in the database are blade profiles, pitch and chord lengths; all the results are outcomes of the investigation [22].

The introduction of those coefficients drastically reduces the thrust estimation process. For a typical 6×4.5 inch propeller, the coefficients used were linearly interpolated from similar pitched database results from [21] to match subsequent physical test values. The static thrust and power coefficients are respectively:

$$C_{T0} = 0.191 \quad (3.34a)$$

$$C_{P0} = 0.0877 \quad (3.34b)$$

Fig:3.5 shows plots coefficients for thrust, C_T , and power, C_P , as a function of the advance ratio J . As the incident upstream fluid velocity, v_∞ , increases, the thrust coefficient decreases. So too does the power coefficient and hence the aerodynamic torque. The thrust and power coefficients can be assumed constant for low advance ratios, or in the case considered here, translational velocities.

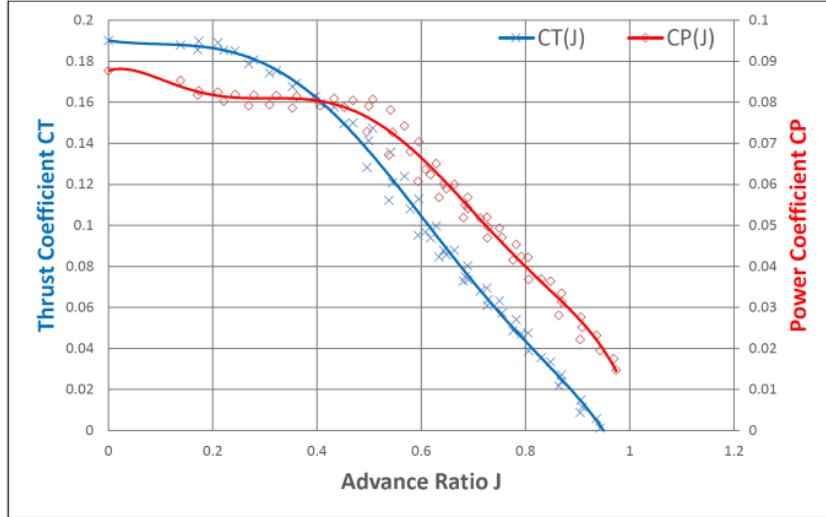
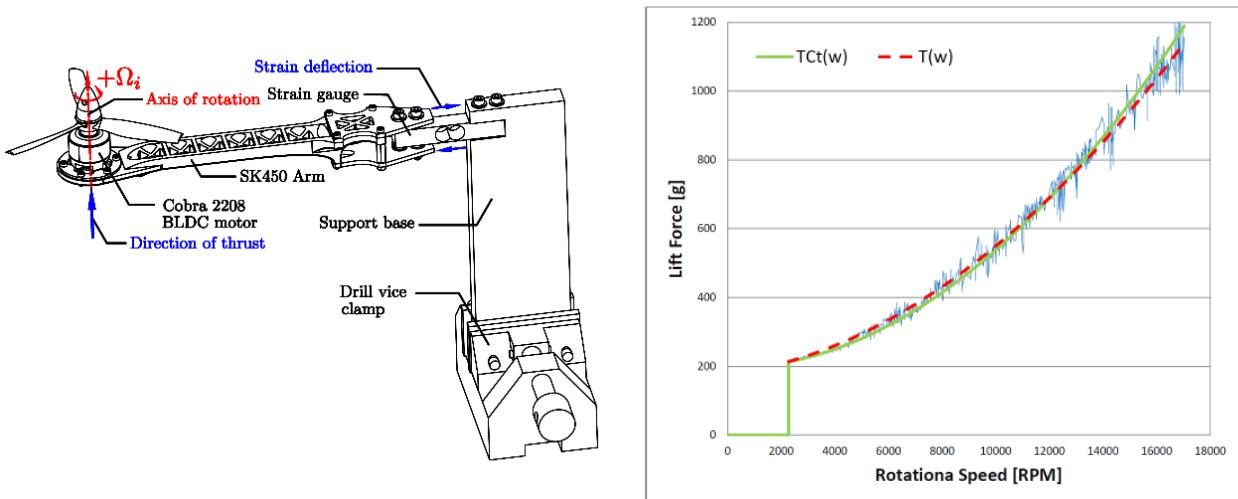


Figure 3.5: Power & thrust coefficients

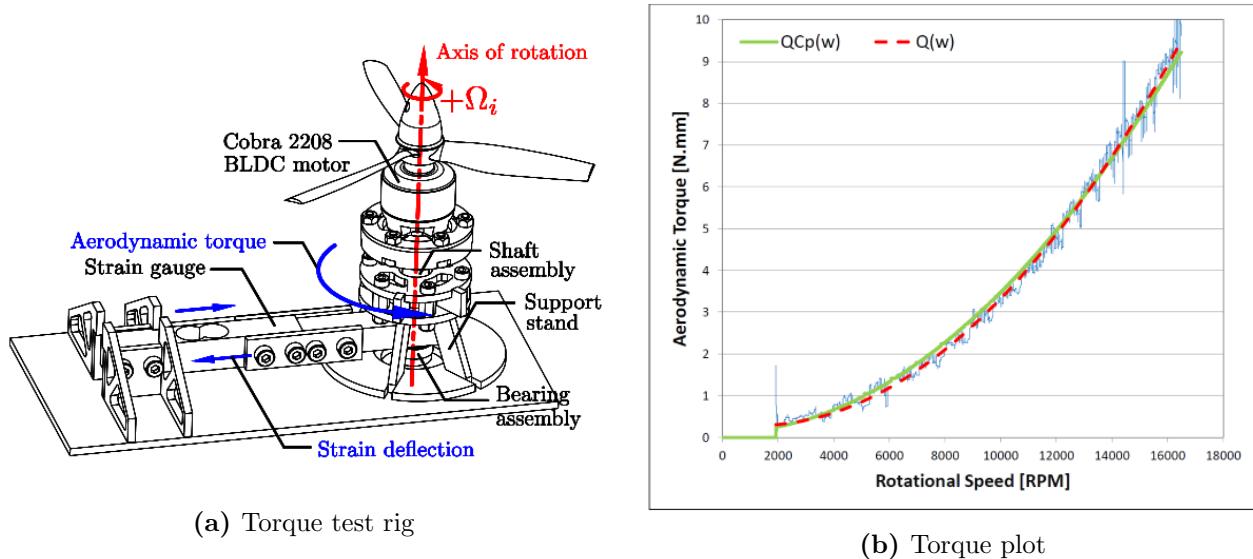
Plotted in Fig:3.6 and Fig:3.7, both thrust and torque test rigs and their static test results are illustrated. Measured values for each test are plotted; $T(\Omega)$ in Fig:3.6b for thrust and $Q(\Omega)$ in Fig:3.7b for torque. The physically tested values are fitted with quadratic trend-lines and plotted against static coefficient estimates using Eq:3.32a for thrust $\hat{T}C_t(\Omega)$ and Eq:3.32b for calculated torque $\hat{Q}C_p(\Omega)$. Results from Fig:3.5 are used as a lookup table and values from Eq:3.32 are calculated, induced propeller thrust and torques can be accurately modeled quadratically, the power term is cubic with respect to its rotational velocity.



(a) Thrust deflection test rig

(b) Static thrust plot

Figure 3.6: Propeller thrust tests

**Figure 3.7:** Static torque tests

Advance ratios, or rather the propeller incident fluid flow(s), are dependent on the vehicle's net translational and angular velocity. Such that the fluid velocity's normal component to the propeller plane is given by:

$$v_\infty = (\vec{v}_b + \vec{L}_{arm} \times \vec{\omega}_b) \cdot \hat{n} \quad \in \mathcal{F}^{M_i} \quad (3.35)$$

Where \vec{v}_b [m.s^{-1}] is the body's transnational velocity and $\vec{\omega}_b$ [rad.s^{-1}] is the body's angular velocity, both transformed to the propeller's frame, $\in \mathcal{F}^{M_i}$. Furthermore $\hat{n}(\lambda_i, \alpha_i)$ is the unit vector normal to the propeller's rotational plane, relative to the body velocity. Then J is calculated from Eq:3.33.

It's worth reiterating that the above static coefficients are indeed calculated from physical static tests. However advance ratio coefficient dependencies are linearly interpolated from the closest available matching data (APC Thin-Electric 8X6 propellers) cited from [21].

Clockwise and counterclockwise propellers and rotations were used for both thrust and torque tests. Despite the thrust and test rigs having been designed to isolate each respective response, results from opposing directional tests were averaged in the hopes that unwanted opposing effects were negated. Both positive and negative rotational test results for thrust and torque measurements are included in Appendix:C.1

Discrepancies which emerge between the model or coefficient values derived can be accounted for with lumped uncertainty disturbance term(s). Model uncertainty compensation can easily be incorporated into adaptive backstepping or H_∞ control algorithms. The deviation of the modeled thrust or torques from their true values would be simple to incorporate into a plant dependent Lyapunov candidate function; Sec:4.5.3.

3.2.2 Hinged Propeller Conning & Flapping

Other non-linear effects which adversely effect a propeller's performance have all been well documented in their own right in the context of helicopter aerodynamic and propeller fields [20,114]. Typically such effects are more pronounced when observing hinged variable pitch propellers, fixed pitch propellers have a diminished effect. Moreover, low translational velocities suppress such responses but they're worth mentioning.

Conning and flapping are the two most significant aerodynamic effects encountered by a propeller. Other phenomenon like cyclic vortex ring states are deemed to not be applicable here and fall outside the scope of the investigation.

In translational flight, for a propeller without shrouding or a ducting, each blade encounters varying incident fluid flow throughout its cycle. The advancing blade relative to the body's translational direction encounters a greater fluid flow than the retreating blade, constructive and destructive interference from the body's translational velocity adds to local fluid flows. The effective local angle(s) of attack for the opposing advancing and retreating propeller blades are not symmetrical from such variations. Those unbalanced angles of attack produce a dissymmetry of lift across the propeller blade's surface.



Figure 3.8: Propeller blade flapping; from [55]

Throughout each rotation the blade is forced up and down as it cycles through a varying fluid velocity field, applying a torque moment about the propeller's hub. That torque's magnitude is a function of the body's net translational velocity and the propeller material's stiffness (hence its susceptibility to deflection). The flapping pitches the effective propeller plane (*tip-path plane*), and hence the thrust vector line, away from its principle axis (Fig:3.8).

The propeller's resultant thrust vector is pitched away from its nominal perpendicular state by some deflection angle, α_{1s} in Fig:3.8, toward the direction of translational movement or wind disturbance. Propeller flapping is diminished at low translational velocities with small wind disturbances relative to propeller rotational speed. As such flapping is not applicable to the feasible flight envelope envisaged for the prototype here.

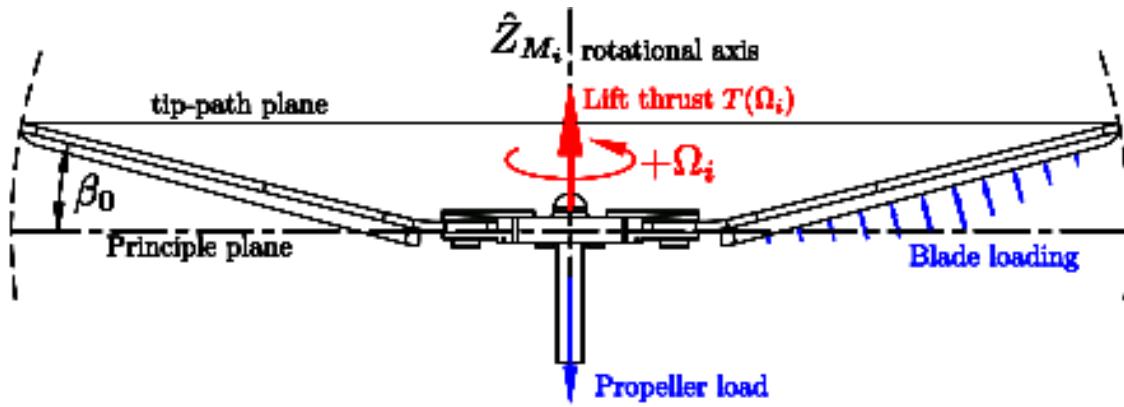


Figure 3.9: Propeller coning

Coning (illustrated in Fig:3.9) is another form of propeller deflection, which again is dependent on the blade material's stiffness. Coning causes both advancing and retreating propeller blades to both deflect upward. Distributed loading on the propeller surface from supporting a body's weight causes the upward deflection. The coning reduces the effective propeller disc's radius, adversely affecting thrust produced, Eq:3.31a. Increased loading accentuates the coning angle experienced by the propellers and as such reduces the tip-path-plane.

Both aerodynamic induced propeller deflection effects can be quantified numerically. Their derivation and resultant equations are cumbersome however. In due course both effects on the produced prototype are not significant enough to produce instability if neglected. The frame could potentially be affected in more adverse ways given certain flight conditions with higher translational velocities or incident wind and fluid flow disturbances...

3.2.3 Drag

For any solid body with some non-zero relative translational velocity motion within a fluid, there is a second order damping response opposing translational velocity. The net drag force, \vec{D}_{net} , although locally dependent on individual component cross-sections can be abstracted to a drag coefficient matrix representing the whole body. For a velocity \vec{v} :

$$\vec{v} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad [\text{m.s}^{-1}] \in \mathcal{F}^b \quad (3.36)$$

The drag equation is given by:

$$\vec{D}_{net}(\vec{v}) = \begin{bmatrix} D_{ii} & D_{ij} & D_{ik} \\ D_{ji} & D_{jj} & D_{jk} \\ D_{ki} & D_{kj} & C_{kk} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}^2 \quad [\text{N}], \quad \in \mathcal{F}^b \quad (3.37)$$

Each drag coefficient's subscript; \hat{i}, \hat{j} and \hat{k} are dependent on the frame's directional cross-section area for each $\hat{X}_b, \hat{Y}_b, \hat{Z}_b$ axis respectively. Given a well designed and symmetrical frame, it can be assumed the off-diagonal elements are of little or no consequence and as such the drag equation can be simplified to a diagonal form:

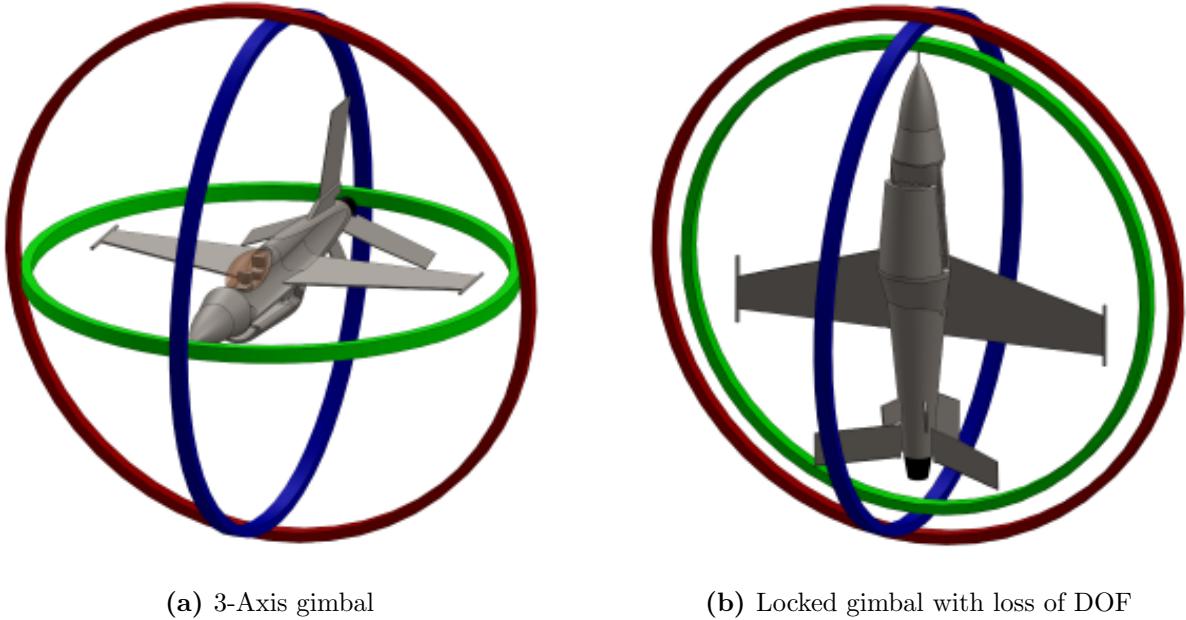
$$\vec{D}_{net}(\vec{v}) \approx \text{diag}(D_{ii}, D_{jj}, D_{kk}) \vec{v}^2 \quad \in \mathcal{F}^b \quad (3.38)$$

Given the second order dependency on translational velocity of the opposing drag forces, such terms can be relegated to a lumped disturbance terms to be adaptively compensated for in the control loop, Sec:4.5.3. The time scale separation between velocity and wind drag effects with the control loop allow for such an assumption. Analogous drag-like effects opposing angular rotation rates exist but, for the intents and purposes of most practical flight envelopes, can be disregarded.

In simulation; if the plant has sufficient disturbance rejection then the drag term in Eq:3.37 would be easily accounted for an adaptive backstepping algorithm. Furthermore it is possible to physically test for the drag coefficients to attain a higher certainty model but, given the flight conditions for this research, such effects will be small if not negligible. As such those tests are outside the scope of investigation here...

3.2.4 Rotation Matrix Singularity

The singularity inherent to Euler Angle parametrization is often mentioned but far less common is the mathematical demonstration of how that singularity manifests itself. A singularity occurs for some matrix A in $\vec{y} = A\vec{x}$ when the matrix has a zero determinate; losing rank and differentiability in terms of \vec{x} . The combined rotation matrix from the \mathcal{F}^I to \mathcal{F}^b is the singular component of an Euler parametrized sequence. Considering the case of a rotational 3-axis gimbal system (Fig:3.10a) which mimics the sequential nature of the Euler set. When the intermediary sequenced rotational angle is at $\pi/2$, the remaining two axes become co-linear (Fig:3.10b). In Z-Y-X rotation sequence adopted in this work, the singularity occurs from the rolling angle θ about \hat{Y} . Both the pitch ϕ or yaw ψ rotations will subsequently have the same rotational effect. Such a situation results in as a loss of a degree of freedom.

**Figure 3.10:** Mechanical gimbal lock

What is clear physically is not necessarily as obvious mathematically. A loss of rank occurs in the Euler Matrix $\Psi(\eta)$, defined previously in Eq:2.12g from Sec:2.2.1. That relation between angular velocity, in the inertial frame or inversely in the body frame, and the angular rates of the Euler Angles has a determinant:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \Phi(\eta)\omega_b \quad (3.39)$$

$$\rightarrow \det(\Phi(\eta)) = \cos(\phi)(\cos(\phi)\sec(\theta)) + \sin(\phi)(\sin(\phi)\sec(\theta)) = \sec(\theta) \quad (3.40)$$

$$\therefore \lim_{\theta \rightarrow \pi/2} |\Phi(\eta)| = \sec(\theta) \rightarrow \infty \quad (3.41)$$

The Euler matrix $\Phi(\eta)$ loses rank as $\theta \rightarrow \pi/2$, loosing differentiability as well. The physical consequence of this is the loss of a degree of freedom. More specifically, if one looks at how the Z-Y-X rotation (or transformation) matrices are formulated, from Eq:2.6.

$$R_I^b(\eta) = R_z(\psi)R_y(\theta)R_x(\phi) = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} \quad (3.42a)$$

$$\therefore R_I^b(\eta) = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\phi c_\theta \end{bmatrix} \quad (3.42b)$$

In the case where $\theta = \pi/2$, and using trigonometric double angles, the following can be reduced;

$$R_I^b(\eta) = \begin{bmatrix} 0 & c_\psi s_\phi - s_\psi c_\phi & c_\psi c_\phi + s_\psi s_\phi \\ 0 & s_\psi s_\phi + c_\psi c_\phi & s_\psi c_\phi - c_\psi s_\phi \\ -1 & 0 & 0 \end{bmatrix} \Big|_{\theta=\pi/2} \quad (3.42c)$$

$$= \begin{bmatrix} 0 & s(\phi - \psi) & c(\phi - \psi) \\ 0 & c(\phi - \psi) & s(\phi - \psi) \\ -1 & 0 & 0 \end{bmatrix} \quad (3.42d)$$

$$= R_{x'}(\phi - \psi) \quad (3.42e)$$

Where the resultant in Eq:3.42e represents an \hat{X}' -axis rotation in a new intermediate frame, post a $\pi/2$ rotation about the \hat{Y} -axis. Through trigonometric double angles a degree of freedom is lost at $\theta = \pi/2$, when ϕ & ψ effect the same angle.

3.2.5 Quaternion Dynamics

An algorithm proposed in [116] suggested a solution to avoid Euler Angle singularities. The heuristic proposed involved switching between sequence conventions (ZYX,ZYZ etc...there are 12 in total) such that the singularity is always avoided. However the implementation of such an algorithm is cumbersome and inefficient. Far more elegant is the use of *quaternion* attitude representations in \mathbb{R}^4 (in [43, 46, 69] amongst others...most notably made popular by [115] for use in animation).

A quaternion is analogous to a rotation matrix in that it represents an attitude difference between two reference frames. An \mathbb{R}^3 attitude is parameterized as one rotation θ about a single unit *Euler* axis \hat{u} (demonstrated using the Rodriguez Formula in [83]). In brief a quaternion consists of a scalar component, q_0 , and complex vector component, $\vec{q} \in \mathbb{C}^3$, such that:

$$Q \triangleq \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} \in \mathbb{R}^4 \quad (3.43)$$

The relationship between an Euler Angles rotation matrix $R_I^b(\eta)$ and a quaternion attitude Q_b is given by the Rodriguez formula:

$$R_I^b(\eta) = R(Q_b) = \mathbb{I}_{3 \times 3} + 2q_0[\vec{q}]_{\times} + 2[\vec{q}]_{\times}^2 \quad (3.44)$$

Where $[\cdot]_{\times}$ is the cross-product matrix defined earlier in Eq:2.8c. All quaternions, unless otherwise specified, are unit quaternions $Q \in \mathbb{Q}_u$. Quaternions with a unity magnitude ensure that rotational operations maintain the vector operand's magnitude. A unit quaternion is defined:

$$\|Q\| = \sqrt{q_0^2 + \vec{q}^2} = 1 \quad (3.45)$$

Quaternion multiplication is distributive and associative, but not commutative. Specifically a quaternion multiplication operation is equivalent to the Hamilton product. For two quaternions, Q & P :

$$Q \otimes P = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} \otimes \begin{bmatrix} p_0 \\ \vec{p} \end{bmatrix} \quad (3.46a)$$

$$\triangleq \begin{bmatrix} q_0 p_0 - \vec{q} \cdot \vec{p} \\ q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p} \end{bmatrix} \quad (3.46b)$$

$$= q_0 p_0 - \vec{q} \cdot \vec{p} + p_0 \vec{q} + q_0 \vec{p} + \vec{q} \times \vec{p} \quad (3.46c)$$

Because the vector component of a quaternion is complex valued, it is natural that there exists a quaternion complex conjugate property, defined as:

$$Q^* = \begin{bmatrix} q_0 \\ -\vec{q} \end{bmatrix} \quad (3.47)$$

It then follows that the fundamental quaternion identity is:

$$Q \otimes Q^* = \mathbb{I}_{4 \times 4} \quad (3.48)$$

A right handed quaternion rotation applied to a vector $\vec{v} \in \mathbb{R}^3$ involves multiplication by two unit quaternions.

$$\begin{bmatrix} 0 \\ \vec{v}' \end{bmatrix} = Q \otimes \begin{bmatrix} 0 \\ \vec{v} \end{bmatrix} \otimes Q^* \quad (3.49)$$

Mostly, the zero scalar components are omitted in a rotation (*or transformation*) operation, it is implied that vector operands are substituted with quaternions.

$$\vec{v}' = Q \otimes (\vec{v}) \otimes Q^* \quad (3.50)$$

In the case of rigid body attitude parameterization with quaternions, Q_b is the quaternion which represents the difference between \mathcal{F}^b and \mathcal{F}^I . A quaternion operator is equivalent to a rotation matrix operation, for some vector $\vec{\nu}_I \in \mathcal{F}^I$;

$$\vec{\nu}_b = R_I^b(\eta) \vec{\nu}_I \iff Q_b \otimes (\vec{\nu}_I) \otimes Q_b^* \quad (3.51)$$

Since quaternions are non-commutative, the construction of a body quaternion Q_b from an Euler angle set $\vec{\eta}$ is sequence dependent. Euler angles, despite being singular, are conceptually simple terms for describing a body's orientation. A Z-Y-X sequenced body quaternion, Q_b , can be constructed from Euler angles as:

$$Q_b = Q_z \otimes Q_y \otimes Q_x = \begin{bmatrix} \cos(\psi/2) \\ 0 \\ 0 \\ \sin(\psi/2) \end{bmatrix} \otimes \begin{bmatrix} \cos(\theta/2) \\ 0 \\ \sin(\theta/2) \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \cos(\phi/2) \\ \sin(\phi/2) \\ 0 \\ 0 \end{bmatrix} \quad (3.52)$$

A quaternion's time derivative, defined in [43], with Q_ω being a quaternion with a vector component equal to angular velocity $\vec{\omega}_{b/I}$ and a zero scalar component, is:

$$\frac{d}{dt} Q_b = \frac{1}{2} Q_b \otimes Q_\omega = \begin{bmatrix} -\frac{1}{2} \vec{q}^T \vec{\omega}_b \\ \frac{1}{2} ([\vec{q}]_\times + q_0 \mathbb{I}) \vec{\omega}_b \end{bmatrix} \quad (3.53)$$

Using quaternions to represent attitudes negates the need for an Euler Matrix, $\Phi(\eta)$, to represent attitudes and their rates. A body quaternion is fully defined in the inertial frame with respect to the body frame or inversely so. The first quaternion time derivative replaces angular velocity rate differentials Eq:3.10a and Eq:3.10c respectively:

$$\dot{\mathcal{E}} = R_b^I(-\eta) \vec{\nu}_b \iff Q_b(-\eta) \otimes \vec{\nu}_b \otimes Q_b^*(-\eta) = Q_b^* \otimes \vec{\nu}_b \otimes Q_b \in \mathcal{F}^I \quad (3.54a)$$

$$\dot{\eta} = \Phi(\eta) \vec{\omega}_b \in \mathcal{F}^{v2,v1,I} \iff \dot{Q}_b = \frac{1}{2} Q_b \otimes Q_\omega \in \mathcal{F}^I \quad (3.54b)$$

Second order time derivatives for quaternion acceleration aren't as useful or concise as their higher order velocity counterparts. The second order derivative is provided here for the sake of completeness. If at all possible, quaternion accelerations are mostly avoided due to their complexity of their calculation. The quaternion analogue for angular acceleration (Eq:3.14b), dependent on net torque acting on a body $\vec{\tau}_\mu$ is given by:

$$\ddot{Q}(\dot{Q}, Q, t) = \dot{Q} \otimes Q^* \otimes \dot{Q} + \frac{1}{2} Q \otimes [J_b^{-1}(\vec{\tau}_\mu - 4(Q^* \otimes \dot{Q}) \times (J_b(Q^* \otimes \dot{Q})))] \quad (3.55)$$

An Euler angle attitude error state used for control plants is defined as the subtracted error between desired and existing attitude orientations $\vec{\eta}_d$ and $\vec{\eta}_b$ respectively. Where $\vec{\eta}_d$ is some attitude produced from a trajectory generation loop.

$$\vec{\eta}_e = \vec{\eta}_d - \vec{\eta}_b \quad (3.56)$$

In contrast with Eq:3.56, a quaternion attitude error is a multiplicative term defined as the difference between two quaternions Q_d and Q_b ;

$$Q_e = Q_d^* \otimes Q_b \quad (3.57)$$

Quaternion attitude control and its stability goals are expanded upon subsequently in Sec:4.5.1.

3.2.6 Quaternion Unwinding

Although quaternions are indeed better than their Euler angle parameterized attitude counterpart(s) and lacking the associated singularity, they do contain one caveat. Because a quaternion $Q = [q_0 \vec{q}]^T$ represents a body's attitude in \mathbb{R}^3 using \mathbb{R}^4 there is an infinite coverage of attitude states, [83].

Each unit quaternion, stemming from Euler-Rodriguez theorem, represents a single Euler-axis rotation of θ about a unit axis \hat{u} such that:

$$Q = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\hat{u} \end{bmatrix} \quad (3.58)$$

That rotation is applied with a quaternion operator, Eq:3.50. For every attitude state in 3-Dimensions there exist two unique quaternions which correspond to the same orientation, differing by their rotational direction about the Euler-axis. The rotation angle θ about the Euler-axis \hat{u} is reciprocal in that $\theta = \theta + 2k\pi$, $k \in \mathbb{N}$; there are then two definitions for Q_b ;

$$Q_b = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\hat{u} \end{bmatrix} \quad (3.59a)$$

$$Q_b = \begin{bmatrix} \cos(\pi - \theta/2) \\ \sin(\pi - \theta/2)\hat{u} \end{bmatrix} = \begin{bmatrix} -\cos(\theta/2) \\ \sin(\theta/2)\hat{u} \end{bmatrix} \quad (3.59b)$$

$$\vec{\eta} \in \mathbb{R}^3 \iff_Q \begin{bmatrix} \pm q_0 \\ \vec{q} \end{bmatrix} \in \mathbb{R}^4 \quad (3.59c)$$

Eq:3.59c asserts that for each attitude in \mathbb{R}^3 there are *two* corresponding quaternions in \mathbb{R}^4 ; $[\pm q_0 \ \vec{q}]^T$. A consequence of this is that two possible error state trajectories exist for every attitude difference. Both a clockwise, $+\theta$, and an anticlockwise, $2\pi - \theta$, rotations which point to the same quaternion attitude error state. This could lead to an erroneous and unnecessary "unwinding" of a complete counter revolution. So for attitude controllers the requirement is that for positive and negative quaternion scalars the control input is consistent:

$$\vec{\nu}_d = h([q_0 \ \vec{q}]^T, t) = h([-q_0 \ \vec{q}]^T, t) \quad (3.60)$$

Or more simply that $Q_e = [|q_0| \ \vec{q}]^T$. The simplest solution adhering to that constraint, which is often used, is to neglect the quaternion scalar component altogether. Using a reduced error state, only the quaternion error vector as an argument for the control law; $h(\vec{q}_e, t)$. Such a solution is an oversimplification and would only ever be locally stable.

An alternative is to use only the absolute quaternion scalar, which ensures the error state represents a right-handed (clockwise) rotation and not necessarily the shortest path. If the resolution of trajectory co-ordinates generated is sufficiently fine the control plant won't encounter a problem.

One proposal presented in [23] suggested using a *signum* operator to design the controller coefficient sign for the desired virtual angular velocity, $\vec{\omega}_d$ control plant input.

$$\vec{\omega}_d = \frac{2}{\Gamma_1} sgn(q_0) \vec{q} \quad (3.61a)$$

Where the signum operator is defined as:

$$sgn(q_0) = \begin{cases} 1 & q_0 \geq 0 \\ -1 & q_0 < 0 \end{cases} \quad (3.61b)$$

Eq:3.61 was shown to be asymptotically stable but only locally in the case where the Euler-axis angle is constrained; $\theta \leq \pm\pi$. That control law would still need the control torques to be calculated from that angular velocity $\vec{\omega}_d$ setpoint.

In [10], the authors used a backstepping controller with a trajectory using the absolute quaternion scalar. The resultant was a global asymptotically stable control law which tracked quaternion setpoints for a satellite's attitude. Controllers presented in Sec:4.5 all incorporate the signed quaternion scalars into the control law; hence relying on the trajectory generation to provide the desired direction of the rotation path.

3.3 Multibody Nonlinearities

The unique component of the prototype's design which facilitates redirection of a propeller's thrust vector (Sec:2.1.1) is also what makes finding the complete equations of motion drastically more complex. The relative (rotary) motion within the multibody system results in torque responses opposing those angular accelerations. Such induced responses, if left unmodelled, would almost definitely destabilize attitude plant. Unmodelled inertia rate responses are shown to be destabilizing in [71]. Typically multibody dynamics are solved (and simulated) as a series of interacting torque and force constraints. There are different schools of thought on the subject, each proposing methodologies for stepping through the systems dynamics; *e.g* Implicit Euler integration [64, 133]...

The prototype investigated here is a multibody system connected with revolute joints, which permit a single degree of relative rotation between each connected rigid body. There are no translational degrees of freedom between each body. Opposed to the angular acceleration applied to a body are *gyroscopic* and *inertial* Newtonian torque responses. The responses from each body are solved independently and those excitation induced torque constraints are imposed onto the structure's sequential rotational joints. Those responses are now quantified and introduced to the dynamic model derived in Sec:3.1.1.

A distinction must be made between torque responses here and those previously in Eq:3.10d. Recalling the classical differential equation of angular motion already derived:

$$\dot{\vec{\omega}}_b = J_b^{-1}(-\vec{\omega}_b \times J_b \vec{\omega}_b + \vec{\tau}_\mu) \quad [\text{m.s}^{-2}], \quad \in \mathcal{F}^b \quad (3.62)$$

Eq:3.62 treats the entire body as rigid; included terms are as a result of the entire multibody's collective motion. What follows is an extension of that attitude state to incorporate relative movements between each connected body. The objective here is to model the multibody dynamic system with clear responses induced from servo rotations of inner and middle ring bodies, $\Delta\lambda_i$ and $\Delta\alpha_i$ respectively. The subsequent derivations are Lagrangian analytical dynamics applied to the multibody system under consideration. For the purposes of this derivation it is assumed that no potential energy can be stored within the structure from material flexure. The only potential energy contribution is as a result of gravitational potential energy.

Alternatively the net dynamics could indeed be derived from a Lagrangian for the *entire* 13 body dynamic system. Where those connected bodies are; four rotor/propeller bodies (Fig:2.10), four inner ring bodies (Fig:2.11), four middle ring bodies (Fig:2.12) and finally the frame structure (Fig:2.16) each with 6 degrees of freedom. Constraints on the assembly's joints would eventually reduce the degrees of freedom and simplify solving for net responses. The purpose here is to model the body's response to changes in the actuation servos' positions, $\Delta\lambda_i$ and $\Delta\alpha_i$, so independent bodies are analyzed first. The final result is, in fact, a Lagrangian for those collective 13 bodies, whose partial derivative with respect to the net angular velocity relative to the inertial frame, $\partial\vec{\omega}_b$, produces the net torque acting on the system.

3.3.1 Relative Rotational Gyroscopic & Inertial Torques

Rotation matrices are used in the following derivations owing to the fact that induced torque responses are dependent on transformed rotational inertias. Quaternions, as mentioned in Sec:2.3, are ill-suited to inertia transformations.

Each of the four motor modules are symmetrical and so the induced torque response characteristics from one module can be extrapolated simply through a \hat{Z}_b reference frame rotation. Each motor module is positioned relative to the body frame's center of motion \vec{O}_b , as in Fig:2.9. Because each relative rotation from the actuator set, $u \in \mathbb{U}$, is actuated separately and upon a different body, their responses are calculated independently too.

Drawing again from Lagrangian theory and considering only the angular energy component for the inner ring assembly attached to frame \mathcal{F}^{M_i} . There is no relative translational motion between each connected body and thus can have no translational kinetic energy contribution. The translational kinetic energy for each module is an extension of body's net kinetic energy in Eq:3.7 and independent of any actuator suite's positions. The motor module's translational motion is incorporated in Eq:3.14a. Considering the i^{th} motor module...

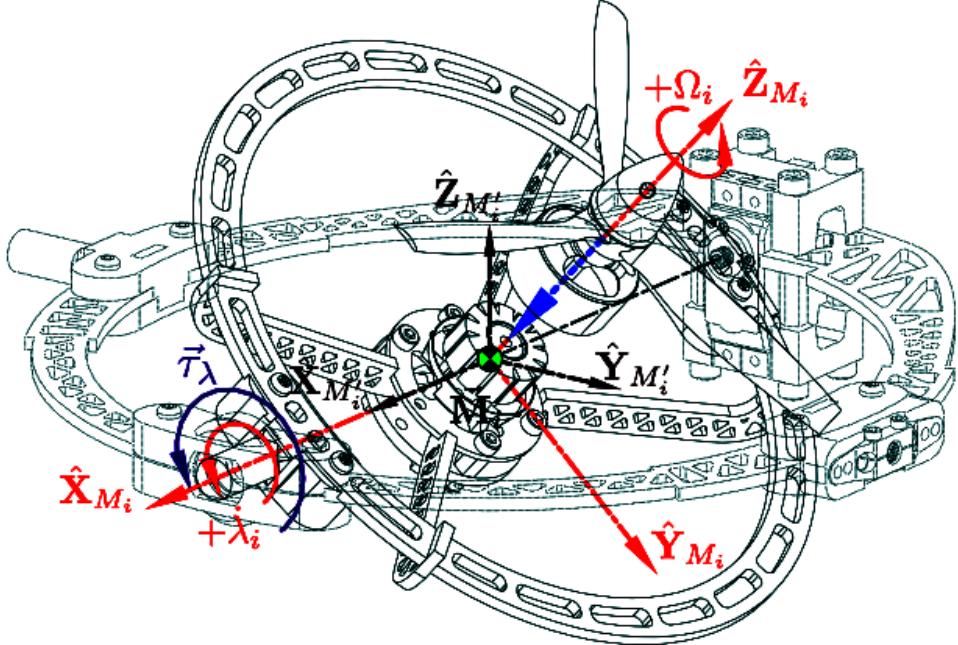


Figure 3.11: Exploded inner ring inertial bodies for $\vec{\tau}_\lambda$

Deriving dynamic responses for changes in the λ_i servo, acting on the inner ring frame \mathcal{F}^{M_i} relative to the middle ring frame $\mathcal{F}^{M'_i}$, requires a relative path co-ordinate to be defined. Seeing that the only path variable between the two frames is that servo's rotational position λ_i about the \hat{X}_{M_i} axis; the path co-ordinates $\vec{u}(t) = [\lambda_i \ 0 \ 0]^T$ are used to produce the Lagrangian for the inner ring's energy relative the middle ring frame, $\mathcal{L}_{M'_i} \in \mathcal{F}^{M'_i}$.

The inner ring assembly consists of two separate bodies (exploded in Fig:3.11). Each with relative rotational motion and independent kinetic energies those bodies are; the rotor assembly with an inertia J_r (defined earlier in Eq:2.18) and the inner ring (*sans* rotor assembly) which has an inertia J_{ir} .

$$J_{ir} \triangleq J_n - J_r = J_{M_i} - J_r \quad \in \mathcal{F}^{M_i} \quad (3.63)$$

Where J_n or J_{M_i} is the net inertia for the inner ring assembly, defined numerically in Eq:2.20. The rotor assembly has an angular velocity $\vec{\omega}_{r/M'_i}$ relative to the middle ring frame $\mathcal{F}^{M'_i}$ from the BLDC motor's rotation Ω_i and the inner ring's servo rate $\dot{\lambda}_i$:

$$\vec{\omega}_{r/M'_i} = R_x(\lambda) \vec{\Omega}_i + \frac{d\lambda}{dt}(\vec{\lambda}_i) \quad \in \mathcal{F}^{M'_i} \quad (3.64a)$$

$$= R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \quad (3.64b)$$

With the propeller's angular velocity vector; $\vec{\Omega}_i = [0 \ 0 \ \Omega_i]^T \in \mathcal{F}^{M_i}$ and measured in $[\text{rad.s}^{-1}]$, not in revolutions per second. The servo position is defined as a vector in the $\hat{X}_{M'_i}$ axis; $\vec{\lambda}_i = [\lambda_i \ 0 \ 0]^T$ measured in $[\text{rad}]$. Next, the inner ring's angular velocity $\vec{\omega}_{M_i/M'_i}$ relative to the frame $\mathcal{F}^{M'_i}$ is only as a result of $\dot{\lambda}_i$:

$$\vec{\omega}_{M_i/M'_i} = \frac{d\lambda}{dt}(\vec{\lambda}_i) = \dot{\vec{\lambda}}_i \quad \in \mathcal{F}^{M'_i} \quad (3.65)$$

The Lagrangian for the inner ring's energy $\mathcal{L}_{M'_i}$, relative to the middle ring frame $\mathcal{F}^{M'_i}$, consists purely of rotational kinetic energy from angular velocities described in Eq:3.64 and Eq:3.65. The relative potential energy as a result of the rotated center of motion for the inner ring is neglected here as it is already included in Eq:2.32d and shown to simplify out later (subsequently in Eq:??) when considering the entire system as a whole. Both inertias for the rotor and inner ring bodies, J_r and J_{ir} respectively, are transformed to align with the middle ring frame $\mathcal{F}^{M'_i}$ using an $R_x(\lambda)$ rotation:

$$\mathcal{L}_{M'_i} = \frac{1}{2} \vec{\omega}_{r/M'_i}^T (J'_r) \vec{\omega}_{r/M'_i} + \frac{1}{2} \vec{\omega}_{M_i/M'_i}^T (J'_{ir}) \vec{\omega}_{M_i/M'_i} \quad (3.66a)$$

The rotor and inner ring inertias, J_r and J_{ir} respectively, are transformed to align with frame $\mathcal{F}^{M'_i}$:

$$J'_r = R_x(\lambda)(J_r)R_x^{-1}(\lambda) \quad \text{and} \quad J'_{ir} = R_x(\lambda)(J_{ir})R_x^{-1}(\lambda) \quad (3.66b)$$

Then expanding the Lagrangian $\mathcal{L}_{M'_i}$ in Eq:3.66a with the above definitions:

$$\begin{aligned} \rightarrow \mathcal{L}_{M'_i} = & \frac{1}{2} \left(R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \right)^T \left(R_x(\lambda)(J_r)R_x^{-1}(\lambda) \right) \left(R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \right) \\ & + \frac{1}{2} \dot{\vec{\lambda}}_i^T \left(R_x(\lambda)(J_{ir})R_x^{-1}(\lambda) \right) \dot{\vec{\lambda}}_i \end{aligned} \quad (3.66c)$$

Noting that the inner ring's inertia J_{ir} is an independent body from the rotor assembly J_r . Recalling the Euler-Lagrange formulation from Eq:3.3 using path co-ordinates $\vec{u}(t)$ for the inner ring frame, $\mathcal{F}^{M'_i}$, relative to the middle ring frame, $\mathcal{F}^{M'_i}$. The generalized torque forces \vec{U} acting on the middle ring are then:

$$\vec{U} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M'_i}}{\partial \dot{\vec{u}}} \right) - \frac{\partial \mathcal{L}_{M'_i}}{\partial \vec{u}} \in \mathcal{F}^{M'_i} \quad (3.67)$$

From [9] the partial derivative of a rotation matrix $R_x(\lambda)$ (and by extension the *transformation matrix* $R_x(-\lambda)$) is linearized using a Taylor series expansion. It follows that for some small perturbation $\partial\theta$ away from the nominal angle $\bar{\theta}$, a generalized rotation matrix about an axis \hat{u} by that angle θ becomes a first order approximation:

$$R_u(\bar{\theta} + \partial\theta) \approx \underbrace{(1 - [\Phi_u(\bar{\theta})\partial\theta]_{\times})}_{\text{infinitesimal rot}} R_u(\bar{\theta}) \quad (3.68)$$

Where $\Phi_u(\bar{\theta})$ is a generalized Euler matrix derivative analogous to Eq:2.12h. The consequence of Eq:3.68 is that, for the transformed rotational inertias in Eq:3.66, both $R_x(\lambda)(J_r)R_x^{-1}(\lambda)$ and $R_x(\lambda)(J_{ir})R_x^{-1}(\lambda)$ can be approximated using their instantaneous transformation with no partial derivatives.

$$R_x(\lambda)(J_r)R_x^{-1}(\lambda) = J'_r \rightarrow \frac{\partial}{\partial \vec{u}} J'_r = \frac{\partial}{\partial \vec{\lambda}_i} J'_r \approx 0 \quad (3.69a)$$

Similarly for the inner ring without the rotor body's contribution:

$$R_x(\lambda)(J_{ir})R_x^{-1}(\lambda) = J'_{ir} \rightarrow \frac{\partial}{\partial \vec{u}} J'_{ir} = \frac{\partial}{\partial \vec{\lambda}_i} J'_{ir} \approx 0 \quad (3.69b)$$

Simplifications in Eq:3.69 are expanded and shown to be reasonable assumptions next in Sec:3.3.2. It follows that partial derivatives of the Lagrangian Eq:3.66 with respect to \vec{u} are negligible; $\partial \mathcal{L}_{M'_i} / \partial \vec{u} \approx 0$. Only the partial derivative with respect to the path rate $\dot{\vec{u}}$ remain:

$$\therefore \vec{U} \approx \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M'_i}}{\partial \dot{\vec{u}}} \right) = \frac{d}{dt} \left((J'_r) \left(R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \right) + (J'_{ir}) \dot{\vec{\lambda}}_i \right) \quad (3.70)$$

Transformed inertial rates \dot{J}'_r and \dot{J}'_{ir} must first be defined before evaluating the simplified Lagrangian derivative in Eq:3.70. Those inertial derivatives cannot be separated by time scale from the remainder of Eq:3.70 given that $\dot{\vec{\lambda}}_i$ determines both inertial rates \dot{J}'_r and \dot{J}'_{ir} but is also a component of the kinetic energy in Eq:3.66c.

Starting with the general case; for some transformed inertia J to be aligned relative to a frame \mathcal{F}^b where the inertia is originally defined with respect to a frame \mathcal{F}^a . If the two frames differ by some rotation angle θ about an Euler axis \hat{u} , the generalized rotation matrix from frame \mathcal{F}^a to \mathcal{F}^b is given by $R_{\hat{u}}(\theta)$ calculated from Eq:2.7. The transformed inertia is then calculated as:

$$J' = R_{\hat{u}}(\theta)(J)R_{\hat{u}}^{-1}(\theta) \quad (3.71a)$$

Which, from the product rule and the rotation matrix time derivative definition in Eq:2.8, that transformed inertia has its own derivative:

$$\dot{J}' = \frac{d}{dt} \left(R_{\hat{u}}(\theta)(J)R_{\hat{u}}^{-1}(\theta) \right) \quad (3.71b)$$

$$= \frac{d}{dt} \left(R_{\hat{u}}(\theta) \right) (J) R_{\hat{u}}^{-1}(\theta) + R_{\hat{u}}(\theta) \left(\frac{d}{dt} (J) \right) R_{\hat{u}}^{-1}(\theta) + R_{\hat{u}}(\theta) (J) \frac{d}{dt} \left(R_{\hat{u}}^{-1}(\theta) \right) \quad (3.71c)$$

$$= [\dot{\vec{\theta}}] \times R_{\hat{u}}(\theta) (J) R_{\hat{u}}^{-1}(\theta) + R_{\hat{u}}(\theta) (\dot{J}) R_{\hat{u}}^{-1}(\theta) - R_{\hat{u}}(\theta) (J) [\dot{\vec{\theta}}] \times R_{\hat{u}}^{-1}(\theta) \quad (3.71d)$$

Where $\dot{\vec{\theta}} = \dot{\theta} \cdot \hat{u}$ is the angular velocity vector between the two frames. In most cases, the inertia won't be changing in its principle frame, or rather that $\dot{J} = 0$. Both the rotor assembly and inner ring inertias are constant in their principle frames, the transformed inertias however have the following derivatives:

$$\rightarrow \dot{J}'_r = \frac{d}{dt} \left(R_x(\lambda)(J_r)R_x^{-1}(\lambda) \right) \quad (3.72a)$$

$$= [\dot{\vec{\lambda}}_i] \times R_x(\lambda)(J_r)R_x^{-1}(\lambda) - R_x(\lambda)(J_r)[\dot{\vec{\lambda}}_i] \times R_x^{-1}(\lambda) \quad (3.72b)$$

Similarly for the inner ring's transformed inertial rate \dot{J}'_{ir} , without the rotor's contribution, is:

$$\rightarrow \dot{J}'_{ir} = \frac{d}{dt} \left(R_x(\lambda)(J_{ir})R_x^{-1}(\lambda) \right) \quad (3.73a)$$

$$= [\dot{\vec{\lambda}}_i] \times R_x(\lambda)(J_{ir})R_x^{-1}(\lambda) - R_x(\lambda)(J_{ir})[\dot{\vec{\lambda}}_i] \times R_x^{-1}(\lambda) \quad (3.73b)$$

Inserting those transformed inertial derivatives into Eq:3.70 and using Reynolds transportation theorem, Eq:3.5 for a vector's derivative in a rotating reference frame. The product rule then yields:

$$\begin{aligned} \rightarrow \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M'_i}}{\partial \dot{\vec{\mathbf{u}}}'} \right) &= \left[(\dot{J}'_r)(R_x(\lambda)\vec{\Omega}_i + \dot{\vec{\lambda}}_i) + (J'_r)R_x(\lambda)\dot{\vec{\Omega}}_i + \vec{\omega}_{M_i/M'_i} \times (J'_r)R_x(\lambda)\vec{\Omega}_i + (J'_r)\ddot{\vec{\lambda}}_i \right. \\ &\quad \left. + \vec{\omega}_{M_i/M'_i} \times (J'_r)\dot{\vec{\lambda}}_i \right] + \left[(\dot{J}'_{ir})\dot{\vec{\lambda}}_i + (J'_{ir})\ddot{\vec{\lambda}}_i + \vec{\omega}_{M_i/M'_i} \times (J'_{ir})\dot{\vec{\lambda}}_i \right] = \vec{\mathbf{U}} \end{aligned} \quad (3.74)$$

Recombining inertial bodies with the same angular velocity ($J'_r + J'_{ir} = J'_n$) and recognizing that, from Eq:3.65, $\vec{\omega}_{M_i/M'_i} = \dot{\vec{\lambda}}$; the generalized net torque encountered by a rotation of λ is:

$$\therefore \vec{\mathbf{U}} = (J'_r)\vec{\Omega}'_i + (J'_r)\dot{\vec{\Omega}}'_i + \dot{\vec{\lambda}}_i \times (J'_r)\vec{\Omega}'_i + (J'_n)\dot{\vec{\lambda}}_i + (J'_n)\ddot{\vec{\lambda}}_i + \dot{\vec{\lambda}}_i \times (J'_n)\dot{\vec{\lambda}}_i \in \mathcal{F}^{M'_i} \quad (3.75a)$$

Where both $\vec{\Omega}'_i$ and $\dot{\vec{\Omega}}'_i$ are the respective transformed angular velocity and acceleration of the propeller in the middle ring frame:

$$\vec{\Omega}'_i = R_x(\lambda)\vec{\Omega}_i \in \mathcal{F}^{M'_i} \quad (3.75b)$$

$$\dot{\vec{\Omega}}'_i = \frac{d\vec{\Omega}}{dt}(R_x(\lambda)\vec{\Omega}_i) = R_x(\lambda)\dot{\vec{\Omega}}_i \in \mathcal{F}^{M'_i} \quad (3.75c)$$

The net torque response, $\hat{\tau}_\lambda$ from an $\Delta\lambda_i$ rotation, induced in the middle ring frame $\mathcal{F}^{M'_i}$, can be grouped into *inertial rates*, second order *inertial* and first order *gyroscopic* components;

$$\hat{\tau}_\lambda = \underbrace{(\dot{J}'_r)\vec{\Omega}'_i + (J'_n)\dot{\vec{\lambda}}_i}_{\text{Inertial rates}} + \underbrace{(J'_r)\dot{\vec{\Omega}}'_i + (J'_n)\ddot{\vec{\lambda}}_i}_{\text{Inertial}} + \underbrace{\dot{\vec{\lambda}}_i \times (J'_r)\vec{\Omega}'_i + \dot{\vec{\lambda}}_i \times (J'_n)\dot{\vec{\lambda}}_i}_{\text{Gyroscopic}} \in \mathcal{F}^{M'_i} \quad (3.76)$$

In Eq:3.76, a hat accent is used to denote the inner ring's servo torque vector; $\hat{\tau}_\lambda$. That equation represents an *estimation* of the true torque $\vec{\tau}_\lambda$, calculated using actuator states $\vec{\Omega}_i$ and $\dot{\vec{\lambda}}_i$ with their derivatives. The true physical torque $\vec{\tau}_\lambda$ would have to be physically measured with a torque transducer.

Similarly for the middle ring frame $\mathcal{F}^{M'_i}$ relative to the intermediary frame $\mathcal{F}^{M''_i}$, the only relative path variable is $\vec{v}(t) = [0 \ \alpha_i \ 0]^T$. The entire motor module structure consists of three separate rotating bodies each with their own relative angular velocities; the *rotor* assembly, *inner* and *middle* ring structures (exploded in Fig:3.12).

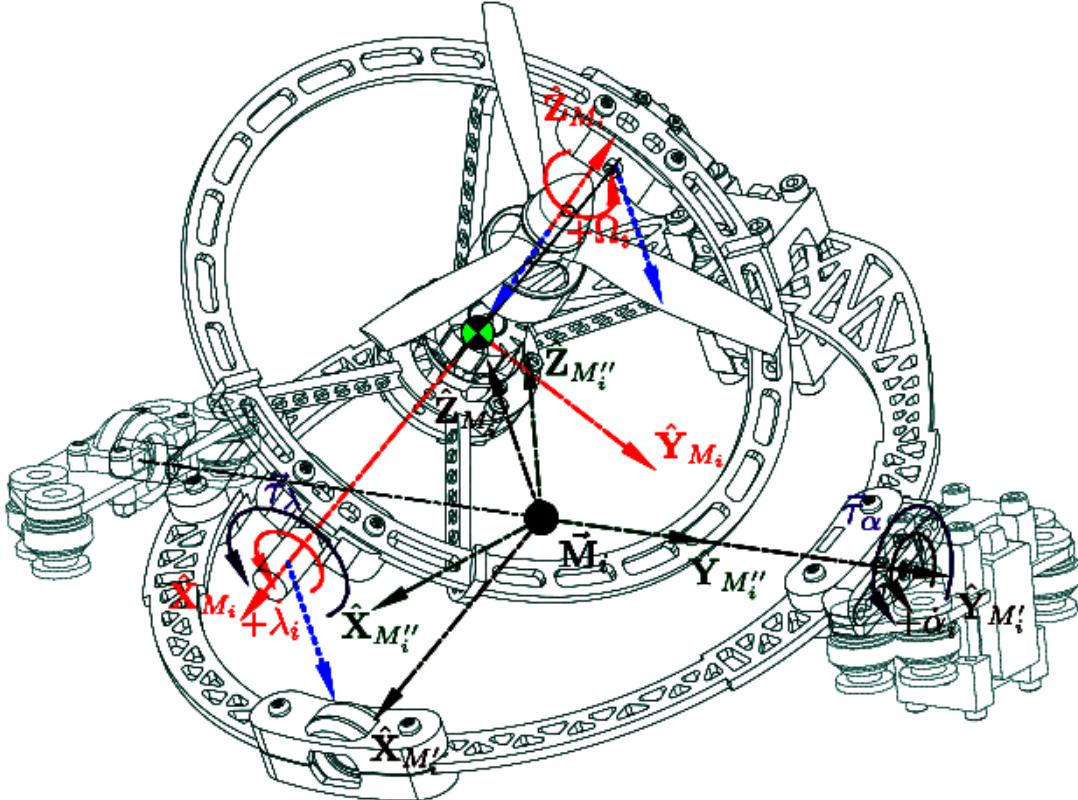


Figure 3.12: Exploded middle ring inertial bodies for $\vec{\tau}_\alpha$

Applying the same process to evaluate the α_i servo's response the middle ring assembly Lagrangian $\mathcal{L}_{M''_i}$ is constructed, but with respect to the intermediary frame $\mathcal{F}^{M''_i}$. First transforming the inertias; the rotor assembly, further rotated by α_i about its $\hat{Y}_{M'_i}$ axis, has an inertia aligned with axes in $\mathcal{F}^{M''_i}$:

$$J''_r = R_y(\alpha)(J'_r)R_y^{-1}(\alpha) = R_y(\alpha)R_x(\lambda)(J_r)R_x^{-1}(\lambda)R_y^{-1}(\alpha) \in \mathcal{F}^{M''_i} \quad (3.77a)$$

Which has a derivative \dot{J}''_r :

$$\dot{J}''_r = R_y(\alpha)(\dot{J}'_r)R_y^{-1}(\alpha) + [\vec{\alpha}_i] \times R_y(\alpha)(J'_r)R_y^{-1}(\alpha) - R_y(\alpha)(J'_r)[\dot{\vec{\alpha}}_i] \times R_y^{-1}(\alpha) \quad (3.77b)$$

The inner ring structure has an inertia, still *without* including the rotor assembly:

$$J''_{ir} = R_y(\alpha)(J'_{ir})R_y^{-1}(\alpha) = R_y(\alpha)R_x(\lambda)(J_{ir})R_x^{-1}(\lambda)R_y^{-1}(\alpha) \in \mathcal{F}^{M''_i} \quad (3.78a)$$

Similarly with a derivative \dot{J}''_{ir} :

$$\dot{J}''_{ir} = R_y(\alpha)(\dot{J}'_{ir})R_y^{-1}(\alpha) + [\vec{\alpha}] \times R_y(\alpha)(J'_{ir})R_y^{-1}(\alpha) - R_y(\alpha)(J'_{ir})[\dot{\vec{\alpha}}] \times R_y^{-1}(\alpha) \quad (3.78b)$$

Finally the middle ring structure's inertia from Eq:2.23a, with neither the rotor's nor the inner ring's contributions:

$$J'_m = R_y(\alpha)(J_m)R_y^{-1}(\alpha) \quad (3.79a)$$

Which, when using the collective motor module inertia J_p from Eq:2.23b, expands to:

$$= R_y(\alpha)(J_p)R_y^{-1}(\alpha) - R_y(\alpha)R_x(\lambda)(J_n)R_x^{-1}(\lambda)R_y^{-1}(\alpha) \quad (3.79b)$$

$$= J'_p - J''_n = J'_p - (J''_{ir} + J''_r) \in \mathcal{F}^{M''_i} \quad (3.79c)$$

Which has a derivative purely as a result of $\dot{\alpha}$:

$$\dot{J}'_m = [\dot{\vec{\alpha}}_i] \times R_y(\alpha)(J_m)R_y^{-1}(\alpha) - R_y(\alpha)(J_m)[\dot{\vec{\alpha}}_i] \times R_y^{-1}(\alpha) \quad (3.79d)$$

However; that derivative \dot{J}'_m , using \dot{J}''_r and \dot{J}''_{ir} from Eq:3.77b and Eq:3.78b respectively, expands to:

$$\dot{J}'_m = [\dot{\vec{\alpha}}_i] \times R_y(\alpha)(J_p)R_y^{-1}(\alpha) - R_y(\alpha)(J_p)[\dot{\vec{\alpha}}_i] \times R_y^{-1}(\alpha) - (\dot{J}''_{ir} + \dot{J}''_r) \quad (3.79e)$$

Note that introducing the relations of Eq:3.79c and Eq:3.79e to the collective body inertia J_p is to simplify the subsequent equations. Each body then has its own relative angular velocity with respect to the intermediate frame $\mathcal{F}^{M''_i}$. For the rotor; $\vec{\omega}_{r/M''_i}$ is the relative angular velocity of that assembly from the motor Ω_i and both inner and middle servo rates $\dot{\lambda}_i$ and $\dot{\alpha}_i$:

$$\vec{\omega}_{r/M''_i} = R_y(\alpha)R_x(\lambda)\vec{\Omega}_i + \frac{d\lambda}{dt}(R_y(\alpha)\vec{\lambda}_i) + \frac{d\alpha}{dt}(\vec{\alpha}_i) \in \mathcal{F}^{M''_i} \quad (3.80a)$$

$$= R_y(\alpha)R_x(\lambda)\vec{\Omega}_i + R_y(\alpha)\dot{\vec{\lambda}}_i + \dot{\vec{\alpha}}_i \quad (3.80b)$$

$$\rightarrow \vec{\omega}_{r/M''_i} = \vec{\Omega}''_i + \dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i \quad (3.80c)$$

Where $\vec{\Omega}''_i$ and $\dot{\vec{\lambda}}'_i$ are respectively propeller and inner servo velocities transformed to the frame $\mathcal{F}^{M''_i}$. Next, the inner ring has an angular velocity $\vec{\omega}_{M_i/M''_i}$ relative to the intermediate frame $\mathcal{F}^{M''_i}$ from the two servo rates $\dot{\lambda}_i$ and $\dot{\alpha}_i$:

$$\vec{\omega}_{M_i/M''_i} = \frac{d\lambda}{dt}(R_y(\alpha)\vec{\lambda}_i) + \frac{d\alpha}{dt}(\vec{\alpha}_i) \in \mathcal{F}^{M''_i} \quad (3.81a)$$

$$= R_y(\alpha)\dot{\vec{\lambda}}_i + \dot{\vec{\alpha}}_i = \dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i \quad (3.81b)$$

Finally the middle ring has an angular velocity $\vec{\omega}_{M'_i/M''_i}$ relative to the intermediary frame only as a result of $\dot{\alpha}_i$:

$$\vec{\omega}_{M'_i/M''_i} = \frac{d\alpha}{dt}(\vec{\alpha}_i) = \dot{\vec{\alpha}}_i \in \mathcal{F}^{M''_i} \quad (3.82)$$

Using the relative path co-ordinate $\vec{v}(t)$, the Lagrangian $\mathcal{L}_{M''_i}$ can be constructed for the combined motor module relative to the frame $\mathcal{F}^{M''_i}$ with kinetic energies of the rotor assembly, inner and middle ring structures respectively:

$$\mathcal{L}_{M''_i} = \frac{1}{2}\vec{\omega}_{r/M''_i}^T(J''_r)\vec{\omega}_{r/M''_i} + \frac{1}{2}\vec{\omega}_{M_i/M''_i}^T(J''_{ir})\vec{\omega}_{M_i/M''_i} + \frac{1}{2}\vec{\omega}_{M'_i/M''_i}^T(J'_m)\vec{\omega}_{M'_i/M''_i} \quad (3.83)$$

Where Eq:3.83 again does not include any potential gravitational energies as such quantities were already accounted for in Eq:2.32d. Therefore the Langragian from Eq:3.83 expands to:

$$\begin{aligned} \therefore \mathcal{L}_{M''_i} = & \frac{1}{2} \left[R_y(\alpha)R_x(\lambda)\vec{\Omega}_i + R_y(\alpha)\dot{\vec{\lambda}}_i + \dot{\vec{\alpha}}_i \right]^T (J''_r) \left[R_y(\alpha)R_x(\lambda)\vec{\Omega}_i + R_y(\alpha)\dot{\vec{\lambda}}_i + \dot{\vec{\alpha}}_i \right] \\ & + \frac{1}{2} \left[R_y(\alpha)\dot{\vec{\lambda}}_i + \dot{\vec{\alpha}}_i \right]^T (J''_{ir}) \left[R_y(\alpha)\dot{\vec{\lambda}}_i + \dot{\vec{\alpha}}_i \right] + \frac{1}{2} \dot{\vec{\alpha}}_i^T (J'_m) \dot{\vec{\alpha}}_i \end{aligned} \quad (3.84)$$

Again, justifying the rotation matrix linearization using Eq:3.68, matrices J''_r , J''_{ir} and J'_m are all instantaneous transformed inertias. The Euler-Lagrange formulation then simplifies, with the partial derivative $\partial\mathcal{L}_{M''_i}/\partial\vec{v} \approx 0$. So the generalized forces \vec{V} are:

$$\vec{V} = \frac{d}{dt} \left(\frac{\partial\mathcal{L}_{M''_i}}{\partial\dot{\vec{v}}} \right) - \frac{\partial\mathcal{L}_{M''_i}}{\partial\vec{v}} \approx \frac{d}{dt} \left(\frac{\partial\mathcal{L}_{M''_i}}{\partial\dot{\vec{v}}} \right) \in \mathcal{F}^{M''_i} \quad (3.85)$$

Finding the partial derivative of the Lagrangian $\mathcal{L}_{M''_i}$ in Eq:3.84 with respect to $\dot{\vec{v}}$ yields:

$$\frac{\partial \mathcal{L}_{M''_i}}{\partial \dot{\vec{v}}} = (J''_r) [\vec{\Omega}''_i + \dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i] + (J''_{ir}) [\dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i] + (J'_m) \dot{\vec{\alpha}}_i \quad (3.86a)$$

Which, with relative angular velocity definitions from Eq:3.80,3.81 and 3.82, expands to:

$$= (J''_r) [R_y(\alpha) R_x(\lambda) \vec{\Omega}_i + R_y(\alpha) \dot{\vec{\lambda}}_i + \dot{\vec{\alpha}}_i] + (J''_{ir}) [R_y(\alpha) \dot{\vec{\lambda}}_i + \dot{\vec{\alpha}}_i] + (J'_m) \dot{\vec{\alpha}}_i \quad (3.86b)$$

Then taking that parital derivative's time derivative and using inertial rates defined in Eq:3.77b,3.78b and 3.79e; split into product ruled derivative components:

$$\begin{aligned} \rightarrow \vec{V} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M''_i}}{\partial \dot{\vec{v}}} \right) &= [(J''_r) (\vec{\Omega}''_i + \dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i)] \\ &+ [(J''_r) \dot{\vec{\Omega}}''_i + \vec{\omega}_{M_i/M''_i} \times (J'_r) \vec{\Omega}''_i + (J''_r) \ddot{\vec{\lambda}}'_i + \vec{\omega}_{M_i/M''_i} \times (J''_r) \dot{\vec{\lambda}}'_i + (J''_r) \ddot{\vec{\alpha}}_i + \vec{\omega}_{M'_i/M''_i} \times (J''_r) \dot{\vec{\alpha}}_i] \\ &+ [(J''_{ir}) (\dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i)] + [(J''_{ir}) \ddot{\vec{\lambda}}'_i + \vec{\omega}_{M_i/M''_i} \times (J''_{ir}) \dot{\vec{\lambda}}'_i + (J''_{ir}) \ddot{\vec{\alpha}}_i + \vec{\omega}_{M'_i/M''_i} \times (J''_{ir}) \dot{\vec{\alpha}}_i] \\ &+ [(J'_m) \dot{\vec{\alpha}}_i] + [(J'_m) \ddot{\vec{\alpha}}_i + \vec{\omega}_{M'_i/M''_i} \times (J'_m) \dot{\vec{\alpha}}_i] \end{aligned} \quad (3.86c)$$

With relative frame angular velocities; $\vec{\omega}_{M_i/M''_i}$ of the inner ring relative to the intermediate frame, and $\vec{\omega}_{M'_i/M''_i}$ of the middle ring relative to the intermediate frame. Both are defined respectively as:

$$\vec{\omega}_{M_i/M''_i} = R_y(\alpha) \dot{\vec{\lambda}}_i + \dot{\vec{\alpha}}_i = \dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i \in \mathcal{F}^{M''_i} \quad (3.86d)$$

$$\vec{\omega}_{M'_i/M''_i} = \dot{\vec{\alpha}}_i \in \mathcal{F}^{M''_i} \quad (3.86e)$$

Eq:3.86c is an ominous and decidedly complicated result to try expand and make sense of. However it can be simplified; recognizing that generalized torques in Eq:3.86c contain kinetic energies already introduced in Eq:3.76, but transformed to the frame $\mathcal{F}^{M''_i}$. After some mathematics, Eq:3.86c can be simplified with responses pertinent to $\Delta\alpha_i$ and then the transformed generalized force response $R_y(\alpha)\hat{\tau}_\lambda$:

$$\begin{aligned} \vec{V} = R_y(\alpha) \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M'_i}}{\partial \dot{\vec{u}}} \right) &+ (R_y(\alpha)(J'_r) R_y^{-1}(\alpha)) \dot{\vec{\alpha}} + (J''_r - R_y(\alpha)(J'_r) R_y^{-1}(\alpha)) (\vec{\Omega}''_i + \dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i) + (J''_r) \ddot{\vec{\alpha}}_i \\ &+ \dot{\vec{\alpha}}_i \times (J''_r) (\vec{\Omega}''_i + \dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i) + (R_y(\alpha)(J'_{ir}) R_y^{-1}(\alpha)) \dot{\vec{\alpha}} + (J''_{ir} - R_y(\alpha)(J'_{ir}) R_y^{-1}(\alpha)) (\dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i) \\ &+ (J''_{ir}) \ddot{\vec{\alpha}}_i + \dot{\vec{\alpha}}_i \times (J''_{ir}) (\dot{\vec{\lambda}}'_i + \dot{\vec{\alpha}}_i) + (J'_m) \dot{\vec{\alpha}}_i + (J'_m) \ddot{\vec{\alpha}}_i + \dot{\vec{\alpha}}_i \times (J'_m) \dot{\vec{\alpha}}_i \end{aligned} \quad (3.86f)$$

Paying special attention to differentiate J''_r and J''_{ir} from Eq:3.77b and Eq:3.78b respectively with $R_y(\alpha)(J'_r) R_y^{-1}(\alpha)$ and $R_y(\alpha)(J'_{ir}) R_y^{-1}(\alpha)$. Where the latter two are inertia rates from Eq:3.72 and Eq:3.73 both transformed to the frame $\mathcal{F}^{M''_i}$.

Generalized torques in Eq:3.86f can be further simplified by introducing combined inertial bodies $J_n = J_r + J_{ir}$ for the *entire* inner ring, from Eq:2.20, and $J_p = J_m + R_x(\lambda)(J_n)R_x^{-1}(\lambda)$ for the *entire* motor module's inertia, from Eq:2.23b. Using $J'_p = R_y(\alpha)(J_p)R_y^{-1}(\alpha)$ and $J'_n = R_y(\alpha)(J_n)R_y^{-1}(\alpha)$ for the net modules inertia and the entire inner ring inertia both respectively aligned with the frame $\mathcal{F}^{M''_i}$:

$$\begin{aligned} \rightarrow \vec{V} = R_y(\alpha) \vec{U} &+ (R_y(\alpha)(J'_n) R_y(\alpha)) \dot{\vec{\alpha}}_i + (J'_p - R_y(\alpha)(J_p) R_y^{-1}(\alpha)) \dot{\vec{\alpha}}_i \\ &+ (J''_n - R_y(\alpha)(J'_n) R_y^{-1}(\alpha)) \dot{\vec{\lambda}}'_i + (J''_r - R_y(\alpha)(J'_r) R_y^{-1}(\alpha)) \vec{\Omega}''_i \\ &+ J'_p \ddot{\vec{\alpha}}_i + \dot{\vec{\alpha}}_i \times ((J'_p) \dot{\vec{\alpha}} + (J''_n) \dot{\vec{\lambda}}'_i + (J''_r) \vec{\Omega}''_i) \end{aligned} \quad (3.86g)$$

Noting that $\dot{J}_p = \dot{J}'_r + \dot{J}'_{ir} + \dot{J}_m$ and that $\dot{J}_m = 0$, it follows that $\dot{J}_p = \dot{J}'_n$. Isolating the servo's torque response from $\Delta\alpha$, and again grouping inertial bodies with shared angular velocities together. The *inertial rates*, second order *inertial* and first order *gyroscopic* responses are then:

$$\begin{aligned}\hat{\tau}_\alpha(\lambda) = & \underbrace{\left(\dot{J}'_p \dot{\alpha}_i + \left(J''_n - R_y(\alpha)(\dot{J}'_n)R_y^{-1}(\alpha) \right) \dot{\lambda}'_i + \left(J''_r - R_y(\alpha)(\dot{J}'_r)R_y^{-1}(\alpha) \right) \dot{\Omega}''_i \right)}_{\text{Inertial rates}} \\ & + \underbrace{\left(J'_p \ddot{\alpha}_i + \dot{\alpha}_i \times \left((J'_p) \dot{\alpha}_i + (J''_n) \dot{\lambda}'_i + (J''_r) \dot{\Omega}''_i \right) \right)}_{\text{Inertial}} \quad \underbrace{\left((J'_p) \dot{\alpha}_i + (J''_n) \dot{\lambda}'_i + (J''_r) \dot{\Omega}''_i \right)}_{\text{Gyroscopic}} \quad \in \mathcal{F}^{M''_i} \quad (3.87)\end{aligned}$$

The servo response $\hat{\tau}_\alpha(\lambda)$ as a function with an argument of the inner ring's servo position is *not* the same as the generalized torque \vec{V} described in Eq:3.86g. The latter contains terms for the inner ring's servo response. A hat accent is used again to denote the torque response seeing that is an estimation, not a true measurement. Careful inspection could have yielded the inertial and gyroscopic components of both Eq:3.76 and Eq:3.87, however the effect of inertial rates on the torque system is a far less obvious result. The assumption in Eq:3.68 that rotated inertias can be linearized is shown to hold true next in Sec:3.3.2 where simulations and physical tests corroborate the above models.

Both servo's respective induced torques, $\vec{\tau}_\lambda$ and $\vec{\tau}_\alpha(\lambda_i)$, occur in sequential gimbal-like frames. The opposing negative responses to induced relative rotations effect the angular state dynamics in Eq:3.10d, and must be transformed to the common body frame. Motor numbers for $i \in [1 : 4]$ are implied:

$$\hat{\tau}_Q(u) = - \sum_{i=1}^4 \left(R_z(\sigma) R_y(\alpha) \hat{\tau}_\lambda + R_z(\sigma) \hat{\tau}_\alpha(\lambda_i) \right) \quad [\text{N.m}], \quad \in \mathcal{F}^b \quad (3.88)$$

The final non-trivial torque term associated with the multibody motion which needs to be accounted for is as a result of the entire system's motion. Specifically considering the responses which relative rotations λ_i and α_i have to the net angular velocity of the entire multibody system $\vec{\omega}_b$. Such responses are an extension of the fundamental rigid 6-DOF differential equation for angular motion, reiterated from Eq:3.62:

$$\dot{\vec{\omega}}_b = (J_b^{-1}) \left(-\vec{\omega}_b \times (J_b) \vec{\omega}_b + \vec{\tau}_\mu \right) \quad [\text{rad.s}^{-2}], \quad \in \mathcal{F}^b \quad (3.89)$$

Before continuing with a Lagrangian formulation applied to the entire multibody system; it is worth first establishing a Lemma to add some clarity to the steps which follow. Consider the hypothetical, non-inertial, 2-D system illustrated in Fig:3.13.

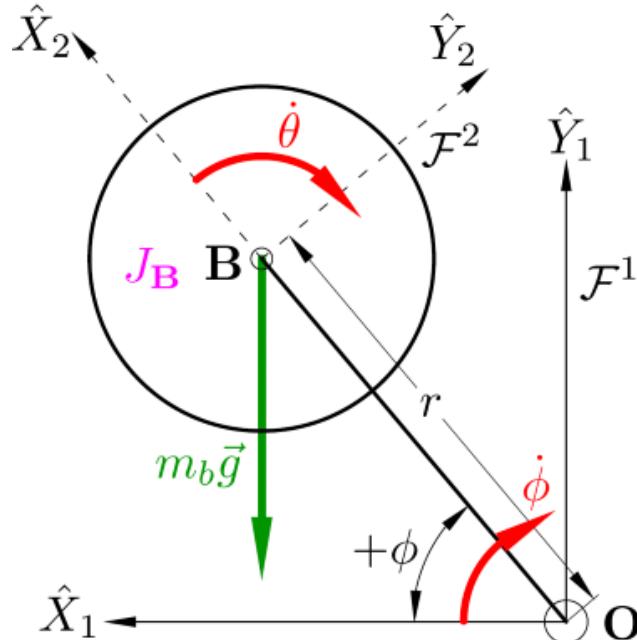


Figure 3.13: Rotating system

A massless rod of length r connects some rotational body, with a mass m_b , at point \mathbf{B} to a center point \mathbf{O} . The principle frame \mathcal{F}^1 has axes \hat{X}_1 and \hat{Y}_1 as illustrated. The arm has a rotational velocity $\dot{\phi}$ relative to \mathcal{F}^1 , applied by some "motor". Attached to the end of the rod is a secondary frame \mathcal{F}^2 with an \hat{X}_2 axis co-linear to the rod and a perpendicular \hat{Y}_2 . The rotational body, centered at point \mathbf{B} , has a rotational inertia $J_{\mathbf{B}}$ about the point (or axis) \mathbf{B} . That rotating body has a rotational velocity $\dot{\theta}$ from another "motor" relative to \mathcal{F}^2 . The question is then how to find the net torque acting on the system in terms of angular velocities $\dot{\phi}$ and $\dot{\theta}$ and their derivatives (or accelerations)?

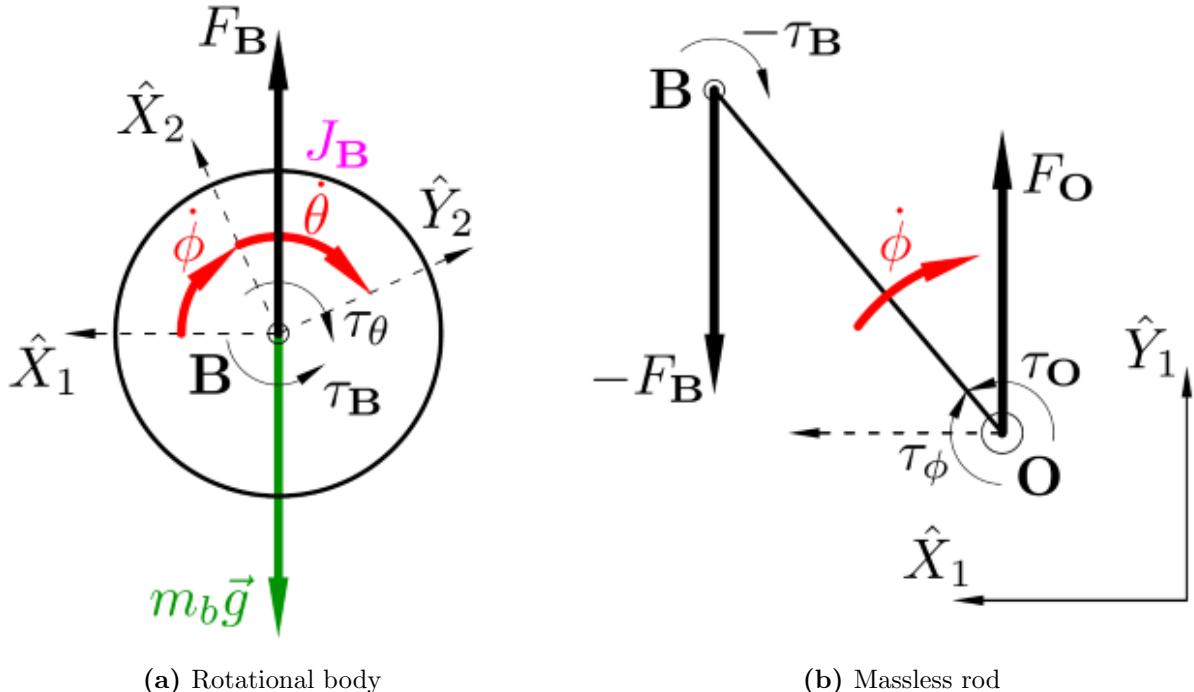


Figure 3.14: Free-body diagram for rotational system

Isolated free body diagrams for each body under consideration are illustrated in Fig:3.14. Starting with the rotational body, Fig:3.14a, the torque acting about point \mathbf{B} is simply an inertial response to combined angular accelerations $\ddot{\theta}$ and $\ddot{\alpha}$:

$$\tau_{\mathbf{B}} = -\tau_{\theta} = -J_{\mathbf{B}}(\ddot{\theta} + \ddot{\phi}) \quad (3.90)$$

The net force is purely the gravitational force acting at point \mathbf{B} as a result of the mass m_b :

$$\mathbf{F}_{\mathbf{B}} = -G = -m_b \vec{g} \quad (3.91)$$

That torque and force pair, $F_{\mathbf{B}}$ and $\tau_{\mathbf{B}}$, are transferred to the massless rod connecting through point \mathbf{B} to \mathbf{O} , Fig:3.14b. The net torque acting around point \mathbf{O} is then comprised of three components; inferred torque from $\tau_{\mathbf{B}}$, a torque arm from force $F_{\mathbf{B}}$ and an inertial torque response to the effective "point-mass" at point \mathbf{B} relative to \mathbf{O} :

$$\tau_{\mathbf{O}} = -\tau_{\phi} = -\tau_{\mathbf{B}} - F_{\mathbf{B}}r \cos \phi + m_b r^2 (\ddot{\phi}) \quad (3.92)$$

The net response force acting at point \mathbf{O} , $F_{\mathbf{O}}$, is of no consequence to the calculation of net torques. The "motor" applies a torque τ_{ϕ} to the rod to induce some angular acceleration $\ddot{\phi}$ on the whole system. Opposed to that angular acceleration is the torque $\tau_{\mathbf{O}}$ which acts against that "motor". The torque τ_{ϕ} acting on the system can then be simplified:

$$\tau_{\phi} = J_{\mathbf{B}}(\ddot{\theta} + \ddot{\phi}) + m_b r^2 (\ddot{\phi}) - m_b \vec{g} r \cos \phi \quad (3.93)$$

That result would not be as obvious when inferred from an energy equation. The equivalent Lagrangian for net kinetic and potential energy of the system, T and U respectively, would be:

$$\mathcal{L} = T(\theta, \phi) - U(\theta, \phi) \quad (3.94a)$$

$$\mathcal{L} = \frac{1}{2}\vec{\omega}_{\mathbf{B}}^T(J_{\mathbf{B}})\vec{\omega}_{\mathbf{B}} + \frac{1}{2}\vec{\omega}_{\mathbf{O}}^T(J_{\mathbf{O}})\vec{\omega}_{\mathbf{O}} - m_b\vec{g}r \sin \phi \quad (3.94b)$$

Where $\vec{\omega}_{\mathbf{B}}$ and $\vec{\omega}_{\mathbf{O}}$ are net angular velocities of the rotational body and massless connection rod respectively. The important thing to consider is that $J_{\mathbf{O}}$, the net rotational inertia about the point \mathbf{O} , is simply the point mass inertia $m_b r^2$ and NOT the expected parallel axis theorem $J_{\mathbf{O}} \neq J'_{\mathbf{B}} = J_{\mathbf{B}} + m_b r^2$. Expanding Eq:3.94b and applying the Euler-Lagrange formulation yields:

$$\rightarrow \mathcal{L} = \frac{1}{2}(\dot{\theta} + \dot{\phi})^T (J_{\mathbf{B}})(\dot{\theta} + \dot{\phi}) + (\dot{\phi})(m_b r^2)(\dot{\phi}) - m_b(-g)r \sin \phi \quad (3.94c)$$

$$\therefore \text{Generalized forces} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = \vec{\tau}_{\phi} \quad (3.94d)$$

$$= \frac{d}{dt}\left((J_{\mathbf{B}})(\dot{\theta} + \dot{\phi}) + (m_b r^2)(\dot{\phi})\right) - m_b g r \cos \phi \quad (3.94e)$$

$$\therefore \tau_{\phi} = J_{\mathbf{B}}(\ddot{\theta} + \ddot{\phi}) + m_b r^2(\ddot{\phi}) - m_b g r \cos \phi \quad (3.94f)$$

$$= J_b \ddot{\theta} + J'_b \ddot{\phi} + \tau_g \quad (3.94g)$$

Where J'_b is the parallel axis inertia and τ_g is the gravitational torque arm contribution. The above then leads to the corollary ascertained from the system in Fig:3.13:

Lemma 3.3.1. *A torque response opposed to angular acceleration of a doubly rotating body can be found as the contribution of the principle rotational inertia about the first axis of rotation with only the first rotational acceleration and a parallel axis inertia about the second rotational axis with the second, independent rotational acceleration.*

Or the same torque can be found as the inertial opposition to net angular acceleration (sum of both rotations) about the first axis and a point mass inertia opposed to the second rotation about its respective axis.

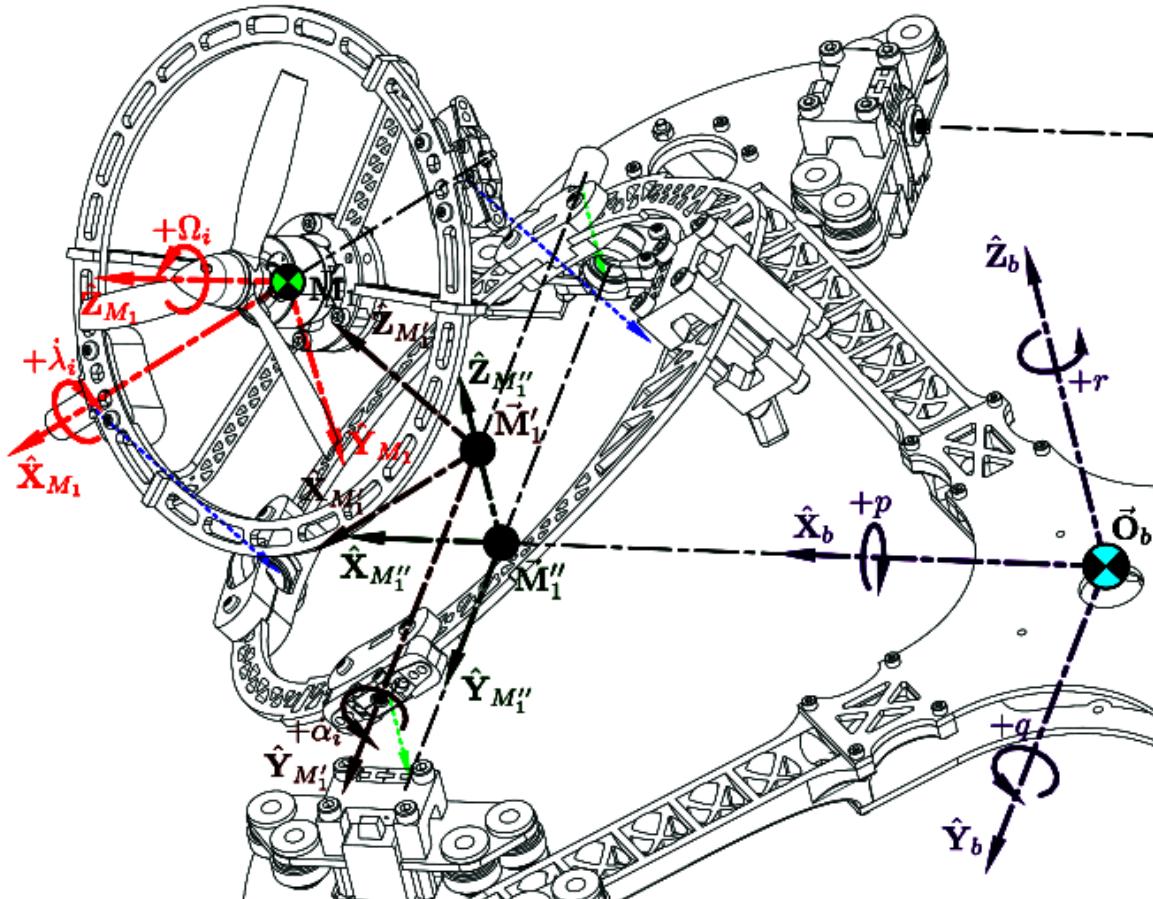


Figure 3.15: Exploded motor module inertial bodies for $\vec{\omega}_b$ response

Returning then to the net multibody system and separating the motor module from the entire body structure first (exploded bodies for *motor module 1* in Fig:3.15). Considering only the additional contribution $\vec{\omega}_b$ has on a singular motor module and later introducing the entire combined system; the Lagrangian derivation for motion relative to the inertial frame then follows...

The relative $\hat{Z}_{M_i''}$ rotation is what differentiates the intermediate frame $\mathcal{F}^{M_i''}$ (used for calculations pertinent to Fig:3.12) and the body frame \mathcal{F}^b . The now familiar rotor assembly, inner and middle ring structure's inertias from Eq:3.77a,3.78a and 3.79a have respective counterparts aligned with \mathcal{F}^b :

$$J_r''' = R_z(\sigma)(J_r'')R_z^{-1}(\sigma) = R_z(\sigma)R_y(\alpha)R_x(\lambda)(J_r)R_x^{-1}(\lambda)R_y^{-1}(\alpha)R_z^{-1}(\sigma) \quad (3.95a)$$

$$J_{ir}''' = R_z(\sigma)(J_{ir}'')R_z^{-1}(\sigma) = R_z(\sigma)R_y(\alpha)R_x(\lambda)(J_{ir})R_x^{-1}(\lambda)R_y^{-1}(\alpha)R_z^{-1}(\sigma) \quad (3.95b)$$

$$J_m'' = R_z(\sigma)(J_m')R_z^{-1}(\sigma) = R_z(\sigma)R_y(\alpha)(J_m)R_y^{-1}(\alpha)R_z^{-1}(\sigma) \quad (3.95c)$$

Where σ_i in Eq:3.95 is the relative orthogonal $\hat{Z}_{M_i''}$ difference between frames $\mathcal{F}^{M_i''}$ and \mathcal{F}^b defined before in Eq:2.16 and illustrated previously in Fig:2.9. Because σ_i is constant for each $i \in [1 : 4]$, the inertial rate for each component of the motor module is simply transformations of $\dot{J}_r'', \dot{J}_{ir}''$ and \dot{J}_m'' previously in Eq:3.77b,3.78b and 3.79e. Or more generally, for some inertia J :

$$\frac{d}{dt}(R_z(\sigma)(J)R_z^{-1}(\sigma)) = 0 \quad (3.96a)$$

So, dropping the σ argument, here $R_z(\sigma) \Rightarrow R_z$ is implied; the rotor, inner and middle inertial rates then follow respectively:

$$\dot{J}_r''' = R_z(\dot{J}_r'')R_z^{-1} \quad (3.96b)$$

$$\dot{J}_{ir}''' = R_z(\dot{J}_{ir}'')R_z^{-1} \quad (3.96c)$$

$$\dot{J}_m'' = R_z(\dot{J}_m')R_z^{-1} \quad (3.96d)$$

Similarly, the angular velocities for each separate body (rotor, inner and middle rings) in \mathcal{F}^b but relative to the inertial frame \mathcal{F}^I are then, first for the rotor:

$$\vec{\omega}_{r/I} = \vec{\Omega}_i''' + \dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}_i' + \vec{\omega}_{b/I} \quad (3.97a)$$

$$= R_z R_y(\alpha) R_x(\lambda) \vec{\Omega} + R_z R_y(\alpha) \dot{\vec{\lambda}}_i + R_z \dot{\vec{\alpha}}_i + \vec{\omega}_b \quad \in \mathcal{F}^b \quad (3.97b)$$

Extending that to the inner ring:

$$\vec{\omega}_{M_i/I} = \dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}_i' + \vec{\omega}_{b/I} \quad (3.98a)$$

$$= R_z R_y(\alpha) \dot{\vec{\lambda}}_i + R_z \dot{\vec{\alpha}}_i + \vec{\omega}_{b/I} \quad \in \mathcal{F}^b \quad (3.98b)$$

And lastly the middle ring structure has a relative angular velocity:

$$\vec{\omega}_{M'_i/I} = \dot{\vec{\alpha}}_i' + \vec{\omega}_{b/I} \quad (3.99a)$$

$$= R_z \dot{\vec{\alpha}}_i + \vec{\omega}_b \quad \in \mathcal{F}^b \quad (3.99b)$$

Noting that Lemma:3.3.1 and the parallel axis term in Eq:3.94g refer to the parallel axis difference between the *center of mass* and the resultant rotational axis, the vector difference between the rotated center of mass for a motor module, $C.M'_p$, and the body frame origin \vec{O}_b is defined:

$$C.M'_p = \frac{m_n C.M''_n + m_m C.M'_m}{m_p} \quad (3.100a)$$

With $C.M'_n$ and $C.M'_m$ being rotated inner and middle ring centers of mass respectively from Eq:2.28d and Eq:2.29d:

$$\therefore C.M'_p = \frac{m_n R_z R_y(\alpha) R_x(\lambda) C.M_n + m_m R_z R_y(\alpha) C.M_m}{m_n + m_m} \quad (3.100b)$$

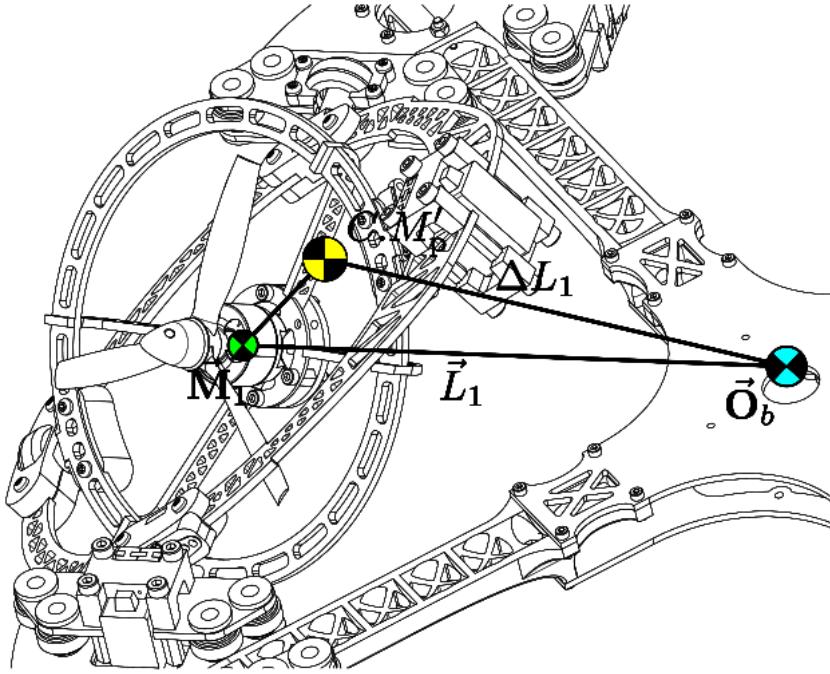


Figure 3.16: Illustration of rotated center of gravity $C.M'_p$

Which leads to the vector difference ΔL_i , illustrated in Fig:3.16.

$$\Delta L_i = \vec{L}_i - C.M'_p \quad (3.100c)$$

The time derivative of that module's moving center of gravity, $C.M'_p$ relative to the origin $\vec{\mathbf{O}}_b$, is:

$$\Delta \dot{L}_i = -\frac{d}{dt}(C.M'_p) \quad (3.100d)$$

$$= -\frac{1}{m_p} \left(m_n (R_z([\vec{\alpha}_i] \times R_y(\alpha) R_x(\lambda) C.M_n + R_y(\alpha) [\vec{\lambda}_i] \times C.M_n)) + m_m R_z[\vec{\alpha}_i] \times C.M_m \right) \quad (3.100e)$$

Then, extended from Lemma:3.3.1, the motor module's *point-mass* inertia J_H about the origin $\vec{\mathbf{O}}_b$ is defined, with motor module mass $m_p = m_n + m_m$, using masses m_n and m_m from Eq:2.28a and Eq:2.29a:

$$J_H \triangleq m_p ((\Delta L_i \cdot \Delta L_i) \mathbb{I}_{3 \times 3} - \Delta L_i \otimes \Delta L_i) \quad (3.101a)$$

Or, using the definition of inner and outer products:

$$= m_p ([\Delta L_i]^T [\Delta L_i] - [\Delta L_i] [\Delta L_i]^T) \quad (3.101b)$$

Which leads to the that point mass's inertial rate $d/dt(J_H)$:

$$\dot{J}_H = m_p ([\Delta \dot{L}_i]^T [\Delta L_i] + [\Delta L_i]^T [\Delta \dot{L}_i] - [\Delta \dot{L}_i] [\Delta L_i]^T - [\Delta L_i] [\Delta \dot{L}_i]^T) \quad (3.101c)$$

Unfortunately that inertial rate, \dot{J}_H in Eq:3.101c, cannot be simplified further to a more concise form. The Lagrangian for the energy of a single motor module, about the origin $\vec{\mathbf{O}}_b$, \mathcal{L}_{p_i} can then be constructed. This time, *including* the gravitational potential energy component:

$$\begin{aligned} \mathcal{L}_{p_i} = & \frac{1}{2} \vec{\omega}_{r/I}^T (J_r'') \vec{\omega}_{r/I} + \frac{1}{2} \vec{\omega}_{M_i/I}^T (J_{ir}'') \vec{\omega}_{M_i/I} + \frac{1}{2} \vec{\omega}_{M'_i/I}^T (J_m'') \vec{\omega}_{M'_i/I} + \vec{\omega}_{b/I}^T (J_H) \vec{\omega}_{b/I} \\ & - m_p \vec{G}_b \cdot (R_I^b(\eta) \vec{\mathcal{E}} + \Delta L_i) \end{aligned} \quad (3.102)$$

Where the term $m_b \vec{G}_b \cdot (R_I^b(\eta) \vec{\mathcal{E}} + \Delta L_i)$ is the vector analogue of gravitational potential energy mgh with $R_I^b(\eta) \vec{\mathcal{E}}$ being the relative X-Y-Z inertial frame position in the body frame \mathcal{F}^b relative to the body origin $\vec{\mathbf{O}}_b$. Expanding \mathcal{L}_{p_i} with terms defined previously:

$$\begin{aligned} \rightarrow \mathcal{L}_{p_i} = & \left[\vec{\Omega}_i''' + \dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}'_i + \vec{\omega}_b \right]^T (J_r''') \left[\vec{\Omega}_i''' + \dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}_i + \vec{\omega}_b \right] + \left[\dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}'_i + \vec{\omega}_b \right]^T (J_{ir}''') \left[\dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}'_i + \vec{\omega}_b \right] \\ & \left[\dot{\vec{\alpha}}'_i + \vec{\omega}_b \right]^T (J_m'') \left[\dot{\vec{\alpha}}'_i + \vec{\omega}_b \right] + \vec{\omega}_b^T (m_p ([\Delta L_i]^T [\Delta L_i] - [\Delta L_i] [\Delta L_i]^T)) \vec{\omega}_b \\ & - m_p \vec{G}_b \cdot (R_I^b(\eta) \vec{\mathcal{E}} + \Delta L_i) \end{aligned} \quad (3.103)$$

Applying partial derivatives of the Lagrangian formulation to \mathcal{L}_{p_i} relative to the angular path coordinates $\vec{\eta}_b$ and $\vec{\omega}_b$ to find generalized forced $\vec{\mathbf{W}}$. Recalling $\vec{\eta}_b$ is the angular orientation from Eq:2.12d, defined entirely in the body frame \mathcal{F}^b , and similarly assuming that $\partial/\partial\vec{\eta}_b(\Delta L_i) \approx 0$:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}_{p_i}}{\partial \dot{\vec{\eta}}_b} \right) - \frac{\partial \mathcal{L}_{p_i}}{\partial \vec{\eta}_b} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{p_i}}{\partial \vec{\omega}_b} \right) - \frac{\partial \mathcal{L}_{p_i}}{\partial \vec{\eta}_b} = \vec{\mathbf{W}} = \vec{\tau}_M \quad (3.104a)$$

$$\begin{aligned} = & \frac{d}{dt} \left((J_r''') \left[\vec{\Omega}_i''' + \dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}'_i + \vec{\omega}_b \right] + (J_{ir}''') \left[\dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}'_i + \vec{\omega}_b \right] + (J_m'') \left[\dot{\vec{\alpha}}'_i + \vec{\omega}_b \right] + (J_H) \left[\vec{\omega}_b \right] \right) \\ & + m_p \vec{G}_b \times \Delta L_i \end{aligned} \quad (3.104b)$$

Then using inertial rate derivatives from Eq:3.96b-3.96d and \dot{J}_H from Eq:3.101c, and inserting relative angular velocities from Eq:3.97-3.99:

$$\begin{aligned} = & \left[(J_r''') (\vec{\Omega}_i''' + \dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}'_i + \vec{\omega}_b) \right] + \left[(J_r''') \dot{\vec{\Omega}}_i''' + \vec{\omega}_{M_i/I} \times (J_r''') \vec{\Omega}_i''' + (J_r''') \ddot{\vec{\lambda}}_i'' + \vec{\omega}_{M_i/I} \times (J_r''') \dot{\vec{\lambda}}_i'' \right. \\ & \left. + (J_r''') \ddot{\vec{\alpha}}'_i + \vec{\omega}_{M'_i/I} \times (J_r''') \dot{\vec{\alpha}}'_i + (J_r''') \dot{\vec{\omega}}_b + \vec{\omega}_{b/I} \times (J_r''') \vec{\omega}_b \right] + \left[(J_{ir}''') (\dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}'_i + \vec{\omega}_b) \right] + \left[(J_{ir}''') \ddot{\vec{\lambda}}_i'' \right. \\ & \left. + \vec{\omega}_{M_i/I} \times (J_{ir}''') \dot{\vec{\lambda}}_i'' + (J_{ir}''') \ddot{\vec{\alpha}}'_i + \vec{\omega}_{M'_i/I} \times (J_{ir}''') \dot{\vec{\alpha}}'_i + (J_{ir}''') \dot{\vec{\omega}}_b + \vec{\omega}_{b/I} \times (J_{ir}''') \vec{\omega}_b \right] + \left[(J_m'') (\dot{\vec{\alpha}}'_i + \vec{\omega}_b) \right] \\ & \left[(J_m'') \ddot{\vec{\alpha}}'_i + \vec{\omega}_{M'_i/I} \times (J_m'') \dot{\vec{\alpha}}'_i + (J_m'') \dot{\vec{\omega}}_b + \vec{\omega}_{b/I} \times (J_m'') \vec{\omega}_b \right] + \left[(\dot{J}_H) \vec{\omega}_b \right] + \left[(J_H) \dot{\vec{\omega}}_b + \vec{\omega}_{b/I} \times (J_H) \vec{\omega}_b \right] \\ & + [m_p \vec{G}_b \times \Delta L_i] \end{aligned} \quad (3.104c)$$

After expanding relative angular velocity terms; $\vec{\omega}_{M_i/I}$, $\vec{\omega}_{M'_i/I}$, $\vec{\omega}_{M''_i/I}$ and $\vec{\omega}_{b/I}$ and applying some mathematics, Eq:3.104a is shown to include a transformed component of Eq:3.86c.

$$\begin{aligned} \rightarrow \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{p_i}}{\partial \vec{\omega}_b} \right) - \frac{\partial \mathcal{L}_{p_i}}{\partial \vec{\eta}_b} = & R_z \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M_i''}}{\partial \dot{\vec{\mathbf{v}}}} \right) + (J_r''') \vec{\omega}_b + \vec{\omega}_b \times (J_r''') \vec{\Omega}_i''' + \vec{\omega}_b \times (J_r''') \dot{\vec{\lambda}}_i'' + \vec{\omega}_b \times (J_r''') \dot{\vec{\alpha}}'_i \\ & + \vec{\omega}_b \times (J_r''') \vec{\omega}_b + J_r''' \dot{\vec{\omega}}_b + (J_{ir}''') \vec{\omega}_b + \vec{\omega}_b \times (J_{ir}''') \dot{\vec{\lambda}}_i'' + \vec{\omega}_b \times (J_{ir}''') \dot{\vec{\alpha}}'_i + \vec{\omega}_b \times (J_{ir}''') \vec{\omega}_b + (J_{ir}''') \dot{\vec{\omega}}_b \\ & (J_m'') \vec{\omega}_b + \vec{\omega}_b \times (J_m'') \dot{\vec{\alpha}}'_i + \vec{\omega}_b \times (J_m'') \vec{\omega}_b + (J_m'') \dot{\vec{\omega}}_b + (\dot{J}_H) \vec{\omega}_b + (J_H) \dot{\vec{\omega}}_b + \vec{\omega}_b \times (J_H) \vec{\omega}_b \\ & + m_p \vec{G}_b \times \Delta L_i \end{aligned} \quad (3.104d)$$

Combining inertial bodies with the same angular velocities and introducing terms $\vec{\tau}_\lambda$ and $\vec{\tau}_\alpha$ from Eq:3.76 and Eq:3.87 respectively:

$$\begin{aligned} \therefore \vec{\mathbf{W}} = \vec{\tau}_M = & R_z \vec{\tau}_\alpha(\lambda) + R_z R_y(\alpha) \vec{\tau}_\lambda + (J_r'''' + J_{ir}'''' + J_m'' + J_H) \vec{\omega}_b + (J_r'''' + J_{ir}'''' + J_m'' + J_H) \dot{\vec{\omega}}_b \\ & + \vec{\omega}_b \times (J_r'''' + J_{ir}'''' + J_m'' + J_H) \vec{\omega}_b + \vec{\omega}_b \times \left((J_r''') (\vec{\omega}_i''' + \dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}'_i) + (J_{ir}''') (\dot{\vec{\lambda}}_i'' + \dot{\vec{\alpha}}'_i) + (J_m'') (\dot{\vec{\alpha}}'_i) \right) \\ & + m_p \vec{G}_b \times \Delta L_i \end{aligned} \quad (3.104e)$$

And recognizing that $(J_r'''' + J_{ir}'''' + J_m'' + J_H)$ can be simplified to a parallel axis translation of the transformed net motor module inertia J'_p from Eq:2.23b; analogous to the net motor module inertia defined in Eq:2.26b:

$$\begin{aligned} (J_r'''' + J_{ir}'''' + J_m'' + J_H) = & R_z R_y(\alpha) R_x(\lambda) (J_r) R_x^{-1}(\lambda) R_y^{-1}(\alpha) R_z^{-1} \\ & + R_z R_y(\alpha) R_x(\lambda) (J_{ir}) R_x^{-1}(\lambda) R_y^{-1}(\alpha) R_z^{-1} + R_z R_y(\alpha) (J_m) R_y^{-1}(\alpha) R_z^{-1} + J_H \end{aligned} \quad (3.105a)$$

$$= R_z(J_p)R_z^{-1} + m_p \left([\Delta L_i]^T [\Delta L_i] - [\Delta L_i][\Delta L_i]^T \right) = J'_{\vec{M}_i} \quad (3.105b)$$

Moreover, the above can be applied to the associated inertia rates $\dot{J}_r''', \dot{J}_{ir}''', \dot{J}_m''$ and \dot{J}_H . Using Eq:3.96b,3.96c,3.96d and 3.101c it can be shown that:

$$(\dot{J}_r''' + \dot{J}_{ir}''' + \dot{J}_m'' + \dot{J}_H) = J'_{\vec{M}_i} \quad (3.105c)$$

The generalized torque acting on a single motor module, $\vec{\tau}_M$ from Eq:3.104e, is then found as combinations of responses to servos λ_i and α_i , the changing inertial rates $J'_{\vec{M}_i}$ as a result of those rotations and finally the net response to the entire frames angular velocity $\vec{\omega}_b$.

$$\begin{aligned} \vec{\tau}_M = & R_z \vec{\tau}_\alpha(\lambda) + R_z R_y(\alpha) \vec{\tau}_\lambda + (J'_{\vec{M}_i}) \vec{\omega}_b + (J'_{\vec{M}_i}) \dot{\vec{\omega}}_b + \vec{\omega}_b \times (J'_{\vec{M}_i}) \vec{\omega}_b \\ & + \vec{\omega}_b \times \left((J''_p) \dot{\vec{\alpha}}'_i + (J'''_n) \dot{\vec{\lambda}}''_i + (J'''_r) \vec{\Omega}'''_i \right) + m_p \vec{G}_b \times \Delta L_i \in \mathcal{F}^b \end{aligned} \quad (3.106)$$

Considering the rigid body response for the body structure's motion, J_y . That structure's inertia J_y is a constant which represents the vehicle's constant, rigid inertias which are not included in the motor module; defined previously in Eq:2.24d.

$$\vec{\tau}_y = (J_y) \dot{\vec{\omega}}_b + \vec{\omega}_b \times (J_y) \vec{\omega}_b \in \mathcal{F}^b \quad (3.107)$$

The net response for the *entire* multibody system is then a sum of Eq:3.106 for modules $i \in [1 : 4]$ and $\vec{\tau}_y$ in Eq:3.107. By inspection, without constructing a complete Lagrangian for the entire system, the net torque acting on the body frame \mathcal{F}^b is:

$$\vec{\tau}_\mu = (J_y) \dot{\vec{\omega}}_b + \vec{\omega}_b \times (J_y) \vec{\omega}_b + \sum_{i=1}^4 \vec{\tau}_M(i) \in \mathcal{F}^b \quad (3.108)$$

Recalling the net vehicles rotational inertia $J_b(u)$, calculated as a function of the actuation matrix u , which was defined previously in 2.30a. It follows that Eq:3.108 reduces to:

$$\begin{aligned} \vec{\tau}_\mu = & (J_b(u)) \dot{\vec{\omega}}_b + \vec{\omega}_b \times (J_b(u)) \vec{\omega}_b \\ & - \sum_{i=1}^4 \left[R_z \vec{\tau}_\alpha(\lambda) + R_z R_y(\alpha) \vec{\tau}_\lambda + (J'_{\vec{M}_i}) \vec{\omega}_b + \vec{\omega}_b \times \left((J''_p) \dot{\vec{\alpha}}'_i + (J'''_n) \dot{\vec{\lambda}}''_i + (J'''_r) \vec{\Omega}'''_i \right) \right] \\ & - m_p \vec{G}_b \times \sum_{i=1}^4 \Delta L_i \end{aligned} \quad (3.109)$$

The final sum of gravitational torque arms can be reduced to $\vec{\tau}_g$ from Eq:2.32d which considers the *net* resultant center of gravity. Then, extending the angular differential equation Eq:3.10d to incorporate the multibody responses derived above:

$$\vec{\tau}_\mu = (J_b) \dot{\vec{\omega}}_b + \vec{\omega}_b \times (J_b) \vec{\omega}_b - \vec{\tau}_b - \vec{\tau}_g \quad (3.110a)$$

Using the net gravitational torque arm $\vec{\tau}_g$ defined earlier in Eq:2.32d:

$$\vec{\tau}_g \triangleq \Delta C.G \times m_b \vec{G}_b \quad (3.110b)$$

And defining a new response torque $\vec{\tau}_b$ which represents collective responses from internal rotations relative each body, i^{th} arguments for motor modules are implied:

$$\vec{\tau}_b \triangleq \dot{J}_b(u) \vec{\omega}_b + \sum_{i=1}^4 \left[R_z \vec{\tau}_\alpha(\lambda) + R_z R_y(\alpha) \vec{\tau}_\lambda + \vec{\omega}_b \times \left((J''_p) \dot{\vec{\alpha}}'_i + (J'''_n) \dot{\vec{\lambda}}''_i + (J'''_r) \vec{\Omega}'''_i \right) \right] \quad (3.110c)$$

Noting that $\dot{J}_b(u)$ is another introduced term which is the sum of all module inertia rates from Eq:3.105c, given that body structures inertia J_y is constant:

$$\dot{J}_b(u) \triangleq \sum_{i=1}^4 (\dot{J}'_{\bar{M}_i}) \quad (3.111)$$

The torque $\vec{\tau}_b$ in Eq:3.110c is the most important result here, however; definitions of $\vec{\tau}_\alpha(\lambda)$ and $\vec{\tau}_\lambda$ in Eq:3.76 and Eq:3.87 respectively were necessary to simplify and isolate different components of Eq:3.110c. Depending on the magnitude of λ_i and $\dot{\alpha}_i$; $\vec{\tau}_b$ could be further simplified to an approximation $\hat{\tau}_b$:

$$\hat{\tau}_b \triangleq \dot{J}_b(u) \vec{\omega}_b \approx \vec{\tau}_b \Big|_{\alpha, \lambda \ll 1} \quad (3.112)$$

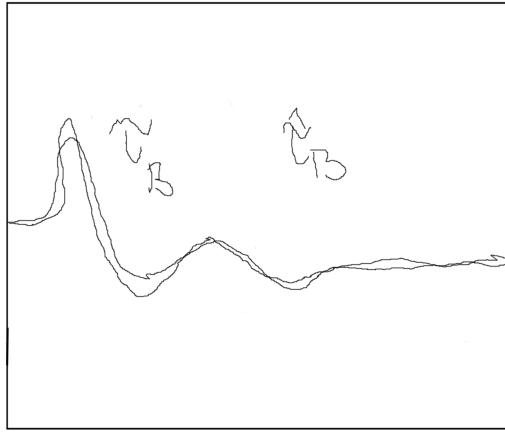


Figure 3.17: Approximated and true body torque responses

Moreover, that inertial rate for \dot{J}_b could be further approximated simply as the difference of instantaneous calculations for $J_b(u)$ at some sample time n ; if a constant sample rate or time difference Δt is known:

$$\dot{J}_b(u) \approx (J_b(u_n) - J_b(u_{n-1})) / \Delta t \quad (3.113)$$

The difference between an approximated torque $\hat{\tau}_b$ and the true body torque $\vec{\tau}_b$ for a typical flight envelope is plotted in Fig:3.17. Using Eq:3.112 and Eq:3.110c respectively for $\hat{\tau}_b$ and $\vec{\tau}_b$

Uncertainties associated with the above inertial models or their particular values can easily be compensated for as disturbances. More specifically uncertainties of the above equations system are modelled as plant dependent state uncertainties; and could be adaptively compensated for accordingly...

3.3.2 Simulation and verification of induced model

Linearization simulation and comparison

Previously, in Sec:3.3.1, the proposed Lagrangian energy functions for both $\Delta\lambda$ and $\Delta\alpha$ servo rotations and responses to body angular velocities $\vec{\omega}_b$ were derived. Reiterating those energy equations; the inner ring net (kinetic) energy Lagrangian from Eq:3.66 is:

$$\mathcal{L}_{M'_i} = \frac{1}{2} \vec{\omega}_{r/M'_i}^T (J'_r) \vec{\omega}_{r/M'_i} + \frac{1}{2} \vec{\omega}_{M_i/M'_i}^T (J'_{ir}) \vec{\omega}_{M_i/M'_i} \quad (3.114a)$$

$$= \frac{1}{2} \left(R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \right)^T \left(R_x(\lambda) (J_r) R_x^{-1}(\lambda) \right) \left(R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \right) + \frac{1}{2} (\dot{\vec{\lambda}}_i)^T \left(R_x(\lambda) (J_{ir}) R_x^{-1}(\lambda) \right) (\dot{\vec{\lambda}}_i) \quad (3.114b)$$

And similarly for the middle ring's net (kinetic) energy from Eq:3.83:

$$\mathcal{L}_{M_i''} = \frac{1}{2} \vec{\omega}_{r/M_i''}^T (J_r'') \vec{\omega}_{r/M_i''} + \frac{1}{2} \vec{\omega}_{M_i'/M_i''}^T (J_{ir}'') \vec{\omega}_{M_i'/M_i''} + \frac{1}{2} \vec{\omega}_{M_i'/M_i''}^T (J_m') \vec{\omega}_{M_i'/M_i''} \quad (3.114c)$$

$$\begin{aligned} &= \frac{1}{2} \left(R_y(\alpha) R_x(\lambda) \vec{\Omega}_i + R_y(\alpha) \dot{\vec{\lambda}}_i + \ddot{\vec{\lambda}}_i \right)^T \left(R_y(\alpha) (J_r') R_y^{-1}(\alpha) \right) \left(R_y(\alpha) R_x(\lambda) \vec{\Omega}_i + R_y(\alpha) \dot{\vec{\lambda}}_i + \ddot{\vec{\lambda}}_i \right) \\ &\quad \frac{1}{2} \left(R_y(\alpha) \dot{\vec{\lambda}}_i + \ddot{\vec{\alpha}}_i \right)^T \left(R_y(\alpha) (J_{ir}') R_y^{-1}(\alpha) \right) \left(R_y(\alpha) \dot{\vec{\lambda}}_i + \ddot{\vec{\alpha}}_i \right) \\ &\quad + \frac{1}{2} \left(\ddot{\vec{\alpha}}_i \right)^T \left(R_y(\alpha) (J_m) R_y^{-1}(\alpha) \right) \left(\ddot{\vec{\alpha}}_i \right) \end{aligned} \quad (3.114d)$$

Solving for the generalized forces acting on each system requires application of Euler-Lagrange formulation, using partial derivatives relative to generalized path co-ordinates. Both the inner and middle ring systems were defined with relative co-ordinate paths for the angular servo position $\vec{u} = [\lambda_i \ 0 \ 0]^T$ and $\vec{v} = [0 \ \alpha_i \ 0]^T$ respectively. The generalized forces for both systems are then, using the Euler-Lagrange formulation:

$$\underbrace{\frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M_i'}}{\partial \dot{\vec{u}}} \right) - \frac{\partial \mathcal{L}_{M_i'}}{\partial \vec{u}}}_{\text{Inner ring}} = \vec{U} \quad \text{and} \quad \underbrace{\frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M_i''}}{\partial \dot{\vec{v}}} \right) - \frac{\partial \mathcal{L}_{M_i''}}{\partial \vec{v}}}_{\text{Middle ring}} = \vec{V} \quad (3.115)$$

The assumption in Eq:3.68, presented in [9], is used to linearize and reduce partial derivatives of the respective inner and middle ring Lagrangians in Eq:3.115. Those simplifications are such that Eq:3.115 becomes:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M_i'}}{\partial \dot{\vec{u}}} \right) = \vec{U} \Big|_{(\partial \mathcal{L}_{M_i'}/\partial \vec{u}) \approx 0} \quad \text{and} \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M_i''}}{\partial \dot{\vec{v}}} \right) = \vec{V} \Big|_{(\partial \mathcal{L}_{M_i''}/\partial \vec{v}) \approx 0} \quad (3.116)$$

Considering first the case of the inner ring Lagrangian $\mathcal{L}_{M_i'}$ in Eq:3.114b; the partial derivative with respect to path variable $\vec{u} = \vec{\lambda}_i$ was simplified:

$$\frac{\partial \mathcal{L}_{M_i'}}{\partial \vec{u}} = \frac{\partial}{\partial \vec{\lambda}_i} (\mathcal{L}_{M_i'}) \approx 0 \quad (3.117)$$

Expanding Eq:3.117 and finding the partial derivative of that Lagrangian, $\mathcal{L}_{M_i'}$ in Eq:3.114b, with respect to $\vec{u} = \vec{\lambda}_i$:

$$\begin{aligned} \frac{\partial \mathcal{L}_{M_i'}}{\partial \vec{\lambda}_i} &= \frac{1}{2} \left(\frac{\partial}{\partial \vec{\lambda}_i} \left(R_x(\lambda) \vec{\Omega}_i \right) \right)^T \left(R_x(\lambda) (J_r) R_x^{-1}(\lambda) \right) \left(R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \right) \\ &\quad + \frac{1}{2} \left(R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \right)^T \left(\frac{\partial}{\partial \vec{\lambda}_i} \left(R_x(\lambda) (J_r) R_x^{-1}(\lambda) \right) \right) \left(R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \right) \\ &\quad + \frac{1}{2} \left(R_x(\lambda) \vec{\Omega}_i + \dot{\vec{\lambda}}_i \right)^T \left(R_x(\lambda) (J_r) R_x^{-1}(\lambda) \right) \left(\frac{\partial}{\partial \vec{\lambda}_i} \left(R_x(\lambda) \vec{\Omega}_i \right) \right) \\ &\quad + \frac{1}{2} \left(\dot{\vec{\lambda}}_i \right)^T \left(\frac{\partial}{\partial \vec{\lambda}_i} \left(R_x(\lambda) (J_{ir}) R_x^{-1}(\lambda) \right) \right) \left(\dot{\vec{\lambda}}_i \right) \end{aligned} \quad (3.118)$$

Testing that assumption, presented in Eq:3.68; the partial derivatives components in Eq:3.118 are explicitly defined. First considering the rotor's transformed inertia $J_r' = R_x(\lambda) (J_r) R_x^{-1}(\lambda)$ with a partial derivative as per the differential product rule:

$$\frac{\partial}{\partial \vec{u}} J_r' = \frac{\partial}{\partial \vec{\lambda}_i} \left(R_x(\lambda) (J_r) R_x^{-1}(\lambda) \right) \quad (3.119a)$$

$$= \frac{\partial}{\partial \vec{\lambda}_i} \left(R_x(\lambda) \right) (J_r) R_x^{-1}(\lambda) + R_x(\lambda) (J_r) \frac{\partial}{\partial \vec{\lambda}_i} \left(R_x^{-1}(\lambda) \right) \quad (3.119b)$$

Similarly for the inner ring's transformed inertial contribution, $J'_{ir} = R_x(\lambda)(J_{ir})R_x^{-1}(\lambda)$ from Eq:3.114b, however *without* including the rotor body's inertia:

$$\frac{\partial}{\partial \vec{u}} J'_{ir} = \frac{\partial}{\partial \vec{\lambda}_i} \left(R_x(\lambda) \right) (J_{ir}) R_x^{-1}(\lambda) + R_x(\lambda) (J_{ir}) \frac{\partial}{\partial \vec{\lambda}_i} \left(R_x^{-1}(\lambda) \right) \quad (3.119c)$$

The final partial derivative to take note of is that for the rotor's angular velocity $\vec{\Omega}_i$ but transformed into the inner ring frame $\mathcal{F}^{M'_i}$ which the Lagrangian in Eq:3.114a is taken with respect to:

$$\frac{\partial}{\partial \vec{\lambda}_i} \left(R_x(\lambda) \vec{\Omega}_i \right) \quad (3.119d)$$

The application of the rotation matrix linearization, from Eq:3.68, to the partial derivative rotations in Eq:3.119b-3.119b is:

$$R_x(\bar{\lambda}_i + \partial \lambda_i) \approx \left(1 - [\Phi_x(\bar{\lambda}_i) \partial \lambda_i]_{\times} \right) R_x(\bar{\lambda}_i) \quad (3.120a)$$

Where $\Phi_x(\bar{\lambda}_i)$ is simply the partial derivative of the Euler matrix, $\Phi(\eta)$ from Eq:2.12h, with respect to an \hat{X} axis rotation. That being:

$$\Phi_x(\bar{\lambda}_i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sin \bar{\lambda}_i & \cos \bar{\lambda}_i \\ 0 & -\cos \bar{\lambda}_i & \sin \bar{\lambda}_i \end{bmatrix} \quad (3.120b)$$

So then, for some nominal $\bar{\lambda}_i$ perturbed by some small deviation $\partial \lambda_i$, Eq:3.120a expands to:

$$\therefore R_x(\bar{\lambda}_i + \partial \lambda_i) \approx R_x(\bar{\lambda}_i) - \begin{bmatrix} 0 & 0 & 0 \\ 0 & (c^2 \bar{\lambda}_i - s^2 \bar{\lambda}_i) & (-c \bar{\lambda}_i s \bar{\lambda}_i - s \bar{\lambda}_i c \bar{\lambda}_i) \\ 0 & (s \bar{\lambda}_i c \bar{\lambda}_i + c \bar{\lambda}_i s \bar{\lambda}_i) & (c^2 \bar{\lambda}_i - s^2 \bar{\lambda}_i) \end{bmatrix} \partial \lambda_i \quad (3.120c)$$

Which, obviously, when the perturbations $\partial \lambda_i$ away from a nominal $\bar{\lambda}_i$ are small, it follows that:

$$\rightarrow R_x(\bar{\lambda}_i + \partial \lambda_i) \approx R_x(\bar{\lambda}_i) \Big|_{\partial \lambda_i \ll 1} \quad (3.120d)$$

That simplification then applies to Eq:3.119b, 3.119c and 3.119d, for a small $\partial \lambda_i$:

$$\frac{\partial}{\partial \vec{\lambda}_i} J'_r \approx 0 \quad (3.121a)$$

$$\frac{\partial}{\partial \vec{\lambda}_i} J'_{ir} \approx 0 \quad (3.121b)$$

$$\frac{\partial}{\partial \vec{\lambda}_i} R_x(\lambda) \vec{\Omega}_i \approx 0 \quad (3.121c)$$

It can therefore be said that the assumption in Eq:3.117 is not without merit, or that:

$$\frac{\partial \mathcal{L}_{M'_i}}{\partial \vec{u}} \approx 0 \rightarrow \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M'_i}}{\partial \dot{\vec{u}}} \right) - \frac{\partial \mathcal{L}_{M'_i}}{\partial \vec{u}} \approx \frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M'_i}}{\partial \dot{\lambda}_i} \right) = \vec{U} = \dot{\tau}_\lambda \quad (3.122)$$

A final verification of the suggested simplifications is done with a **Simulink** model, shown in Fig:3.18.

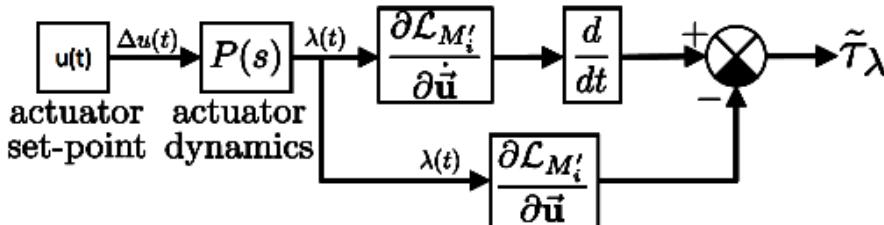


Figure 3.18: Simulink Lagrangian block

The model applies a direct comparison of $\tilde{\tau}_\lambda$, originally from Eq:3.76, both *with* and *without* the rotation matrix linearization. Convention has it that an estimated or simulated value is denoted with an hat accent. Therefore a calculated inner ring torque response, *with linearized* partial derivatives as per Eq:3.121, is termed as $\hat{\tau}_\lambda$. The true representation of the induced torque response, $\tilde{\tau}_\lambda$, is calculated from the block model detailed in Fig:3.18. The simulated $\tilde{\tau}_\lambda$ applies no partial derivative simplifications or linearizations, calculating the full inner ring Euler-Lagrange formulation from Eq:3.115.

The block $u(t)$ in Fig:3.18 represents some commanded change in actuator position, within the actuator space $u \in \mathbb{U}$. That space consists of propeller speeds Ω_i and servo positions λ_i and α_i for $i \in [1 : 4]$; as detailed in Eq:2.17. Each actuator has its own transfer function, driven by the dynamic block $P(s)$, such transfer characteristics were empirically determined in Sec:2.4.1. That actuator argument $u(t)$ then leads to some time varying inner ring servo position $\lambda_i(t)$, with a rate $\dot{\lambda}_i$. Both of which are then be used to calculate the complete Euler-Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M'_i}}{\partial \dot{\mathbf{u}}} \right) - \frac{\partial \mathcal{L}_{M'_i}}{\partial \mathbf{u}} = \tilde{\tau}_\lambda \quad (3.123)$$

Seeing that Eq:3.123 produces a 3-Dimensional vector result and not a scalar; vector magnitudes $\|\hat{\tau}_\lambda\|$ and $\|\tilde{\tau}_\lambda\|$ are discussed. The objective here is to quantify the effect a rotation matrix linearization has on the estimated generalized torque calculations detailed above. Plotted in Fig:3.19a are both estimated $\|\hat{\tau}_\lambda\|$ and simulated $\|\tilde{\tau}_\lambda\|$ torques, calculated with a nominal constant angular propeller speed $\bar{\Omega}_i = 6000$ [RPM]. Increasing positive and negative step sizes for changes in $\Delta\lambda$ are shown, resulting in larger torque responses.

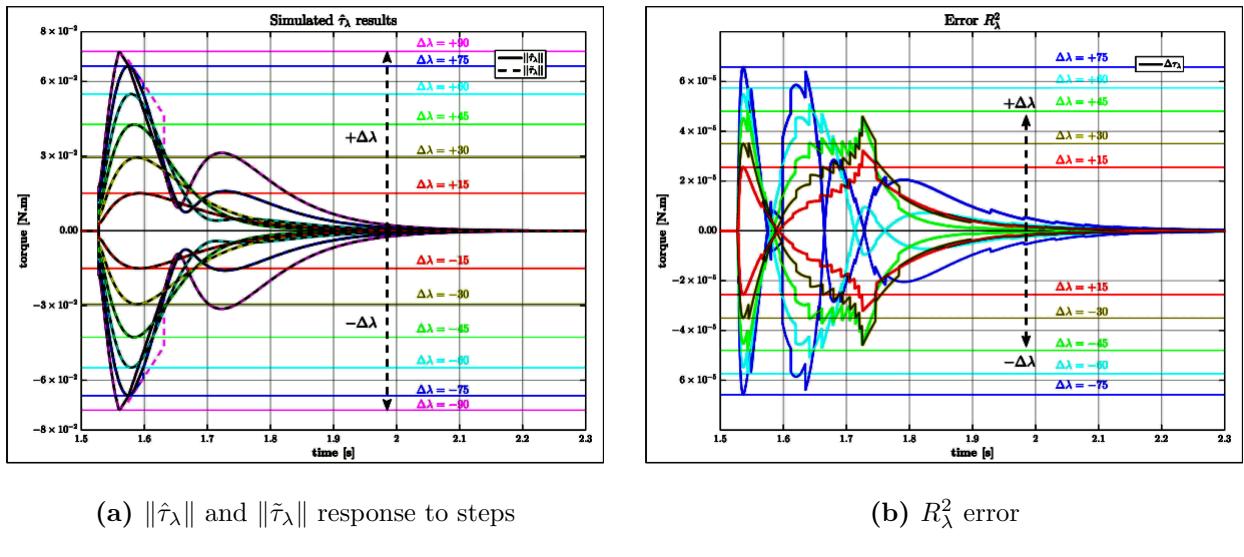


Figure 3.19: Inner ring induced torque responses for $\Delta\lambda$

The regressive R_λ^2 error between $\hat{\tau}_\lambda$ and $\tilde{\tau}_\lambda$ is plotted in Fig:3.19b. That difference between $\hat{\tau}_\lambda$ and $\tilde{\tau}_\lambda$ is precisely the partial derivative contribution $\partial \mathcal{L}_{M'_i} / \partial \mathbf{u}$. Or rather that, mathematically:

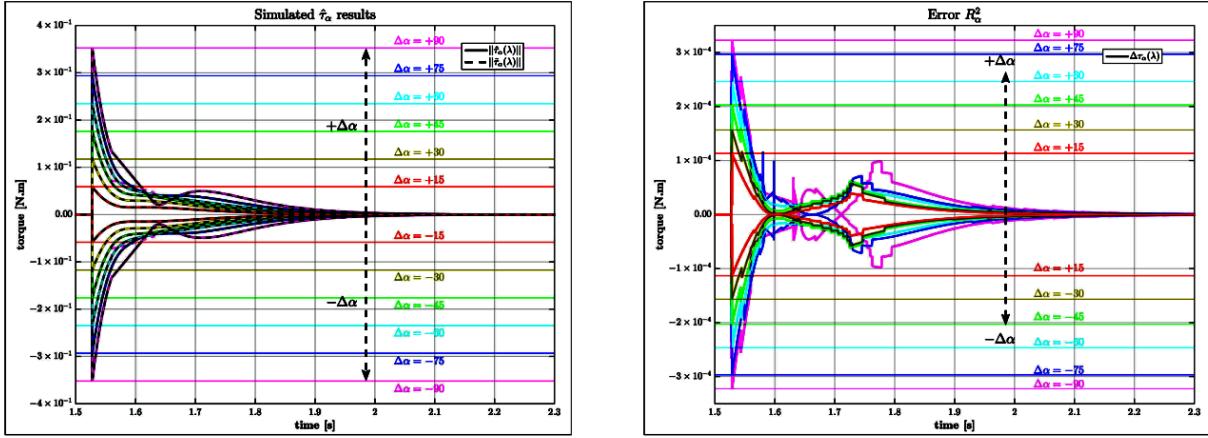
$$R_\lambda^2 = \|\hat{\tau}_\lambda\| - \|\tilde{\tau}_\lambda\| \quad (3.124a)$$

$$= \left(\frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M'_i}}{\partial \dot{\mathbf{u}}} \right) \right) = \left(\frac{d}{dt} \left(\frac{\partial \mathcal{L}_{M'_i}}{\partial \dot{\mathbf{u}}} \right) - \frac{\partial \mathcal{L}_{M'_i}}{\partial \mathbf{u}} \right) \quad (3.124b)$$

$$\therefore R_\lambda^2 = \frac{\partial \mathcal{L}_{M'_i}}{\partial \mathbf{u}} \quad (3.124c)$$

However the simulation for R_λ^2 suffered from tolerance errors in the integral approximator, this was as a result of the small deviations which were being calculated. Despite that; the differences, using the inertial matrices and dimensions defined for the prototype in Sec:2.3, were typically in the order $\times 10^{-5}$ [Nm] for small steps in $\Delta\lambda$.

Only for large angular changes in $\Delta\lambda$ does the approximation begin to deteriorate. Mostly both $\hat{\tau}_\lambda$ and $\tilde{\tau}_\lambda$ were three orders of magnitude greater than their errors; torques were in the range of $\times 10^{-2}$ [Nm]. The error for $\lambda = \pi/2$ was not included in Fig:3.20b because it was the only error which did not fit on the $\times 10^{-5}$ [N.m] scale.

(a) $\|\hat{\tau}_\alpha\|$ and $\|\tilde{\tau}_\alpha\|$ response to steps(b) R_α^2 error**Figure 3.20:** Middle ring induced torque responses for $\Delta\alpha$

The same process was then applied to the middle ring Lagrangian, $\mathcal{L}_{M_i''}$ from Eq:3.114d, to evaluate $\hat{\tau}_\alpha(\lambda)$. Those results are plotted collectively in Fig:3.20. Note that both $\|\hat{\tau}_\alpha\|$ and $\|\tilde{\tau}_\alpha\|$ are plotted, not the generalized torques \vec{V} acting on the system. The generalized torque response \vec{V} from Eq:3.115 and expanded in Eq:3.86g include the inner ring energy component \vec{U} or $\vec{\tau}_\lambda$ whereas $\hat{\tau}_\alpha$ and $\tilde{\tau}_\alpha$ do not. From Eq:3.87, $\hat{\tau}_\alpha$ is defined as a function of \vec{V} and \vec{U} :

$$\hat{\tau}_\alpha(\lambda) \triangleq \vec{V}(\alpha, \lambda) - R_y(\lambda) \vec{U}(\lambda) \quad (3.125)$$

Plots for the net generalized torque response $\vec{V}(\alpha, \lambda)$ acting on the system with combined changes in $\Delta\alpha$ and $\Delta\lambda$ are included in App:???. Moreover only a constant value for λ_i Was used. The same constant propeller speed $\vec{\Omega}_i = 6000$ [RPM] was used together with a constant $\lambda_i = 0$ [rad] origin position for the inner ring's servo.

The initial torque spike for $\tau_\alpha(\lambda)$ shown in Fig:3.20a is as a result of the significantly larger rotational inertia, J_p for the entire motor module, encountered by a second order angular acceleration $\ddot{\alpha}$. The response is depreciated in the case of the inner ring for τ_λ because of how much smaller that rotational inertia, J_n for the inner ring only, physically is. As per power calculations from the servo; the angular acceleration is rate limited to 472 [rad.s⁻²], but neither tests come close to reaching saturation.

The second torque peak which begins to manifest in the inner ring response for τ_λ shown in Fig:3.19a are as a result of the angular velocity rate, $\dot{\lambda}_{max} = 7.4799$ [rad.s⁻¹], limit being encountered. The same limit is encountered for the middle ring response τ_α in Fig:3.20a but is far less significant in relation to the initial second order acceleration torque peak. The deviation between $\hat{\tau}_\alpha$ and $\tilde{\tau}_\alpha$ is only of the order of 10^{-4} [Nm] whilst typical induced torque values are again three orders of magnitude greater, or of the order 10^{-1} [Nm]. Only at large angular changes for both $\Delta\lambda_i$ and $\Delta\alpha_i$ do the simplifications proposed begin to deteriorate.

From plots in both Fig:3.19a and Fig:3.20a; it is clear that a linearized rotation matrix to reduce complexity of generalized torque calculations in Eq:3.115 is a fair simplification. The linearization(s) proposed in Eq:3.116 hold true so long as the step size for $\Delta\lambda_i$ or $\Delta\alpha_i$ are small enough, typically having an error three orders of magnitude less than the induced response considered. The control loop will only ever be dealing with minor step size changes for servo positions and so, the linearization is an appropriate one that will reduce interval computational complexity.

Dynamic model verification

In spite of the rigorous mathematical approach applied to the multibody system above, physical corroboration of the proposed model(s) is still required. The systems described in Eq:3.76 for $\hat{\tau}_\lambda$, Eq:3.87 for $\hat{\tau}_\alpha$ and Eq:3.88 for $\hat{\tau}_b$ require further verification before a structured and accurate simulation can be constructed based upon them. Two rigs were designed and constructed (Fig:3.21 and Fig:3.23) to physically measure the induced torques in question. The first test rig recreates the relative motion of the inner ring actuated by λ_i . Similarly the second test platform mimics the middle ring's motion driven by the outer α_i servo. The net body response, $\hat{\tau}_b$ relating to net angular body velocity $\vec{\omega}_b$, is harder to recreate on an isolated test rig. Such results are only discussed in the context of simulation and a MatLab SimScape model (see App:C.3), the latter is introduced subsequently...

Considering first the inner most ring assembly, Fig:3.21 shows the test rig used to isolate and measure $\vec{\tau}_\lambda$ responses to $\Delta\lambda$ rotations. The inner ring is supported by two bearing assemblies; an extended shaft in the $-\hat{X}_{M_i}$ direction connects the inner ring to the driving servo block. Rotational torque $\vec{\tau}_\lambda$ is transferred through the shaft extension from the servo to the inner ring. The servo block is secured only by a vertically aligned (and calibrated) strain gauge. The deflection of the strain gauge is then proportional to the torque applied by the servo to rotate the inner ring structure.

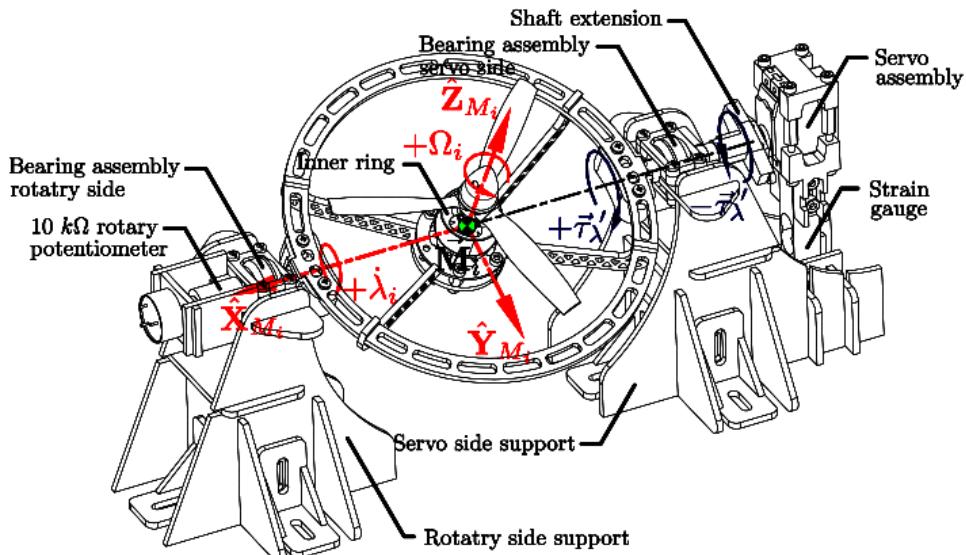


Figure 3.21: Inner ring torque test rig

It is important to mention that whilst the bearing assembly accommodates for the transfer of the servo's rotational torque, the assembly isolates only the \hat{X}_{M_i} component of the induced torque. If $\vec{\tau}'_\lambda$ is the deflection measured, its relationship with the induced torque vector $\vec{\tau}_\lambda$ is given by:

$$\vec{\tau}'_\lambda = \vec{\tau}_\lambda \cdot \hat{X}_{M_i} \in \mathcal{F}^{M_i} \quad (3.126)$$

One final thing to consider is that the equations for $\hat{\tau}_\lambda$ in Eq:3.76 *does not* account for the gravitational torque arm as a result of an eccentric center of gravity relative to the center of rotation. For the derivations presented previously in Sec:3.3.1, net gravitational torque for a resultant effective center of gravity $\vec{\tau}_g$ was later incorporated into Eq:3.110a. The torque response opposed to angular accelerations of $\ddot{\lambda}_i$ induced by the servo is then, from Eq:3.76, with an introduced gravitational component:

$$\begin{aligned} \vec{\tau}_\lambda = & (J'_r)\vec{\Omega}'_i + (J'_n)\dot{\vec{\lambda}}_i + (J'_r)\dot{\vec{\Omega}}'_i + (J'_n)\ddot{\vec{\lambda}}_i + \dot{\vec{\lambda}}_i \times (J'_r)\vec{\Omega}'_i + \dot{\vec{\lambda}}_i \times (J'_n)\dot{\vec{\lambda}}_i \\ & + m_n C.M'_n \times \vec{G}_{M_i} \in \mathcal{F}^{M'_i} \end{aligned} \quad (3.127)$$

The term $m_n C.M'_n \times \vec{G}_{M_i}$ is the gravitational torque arm from a rotated center of mass, $C.M'_n$; first defined in Eq:2.28d. It is the effective torque contribution of a single inner ring to the net gravitational torque $\vec{\tau}_g$ from Eq:3.110b. Note the strain gauge measured response encountered will be $-\hat{\tau}_\lambda$.

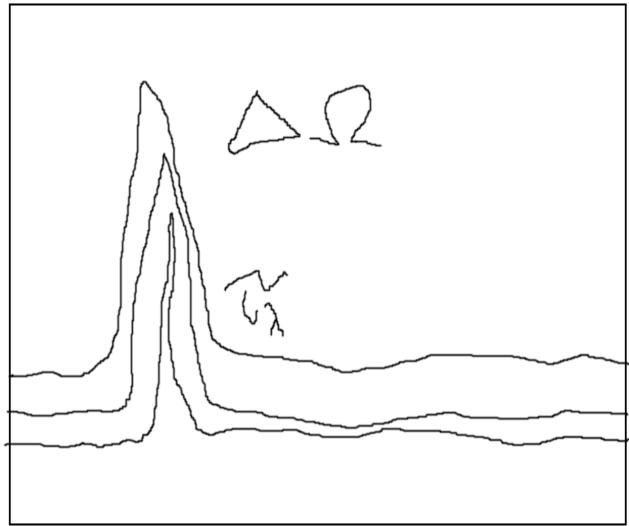


Figure 3.22

The plot in Fig:3.22 shows simulated values of $\hat{\tau}_\lambda$ at increments of propeller rotational speeds Ω_i . The torque peak changes with increased rotation rates of Ω_i , motors 1 & 3 have clockwise rotations ('+'), motors 2 & 4 are counter-clockwise ('-'), illustrating the gyroscopic torque effect from the propeller's rotation. A positive, clockwise rotational sense was used in the above tests. As per convention, in the plot Fig:3.22, $\vec{\tau}_\lambda$ represents the physically measured torque on the test rig illustrated in Fig:3.21. Lastly $\tilde{\tau}_\lambda$ is a torque simulated in **MathWorks SimScape**. App:C.3 details how the SimScape simulation was constructed from a SolidWorks model to match the physical system.

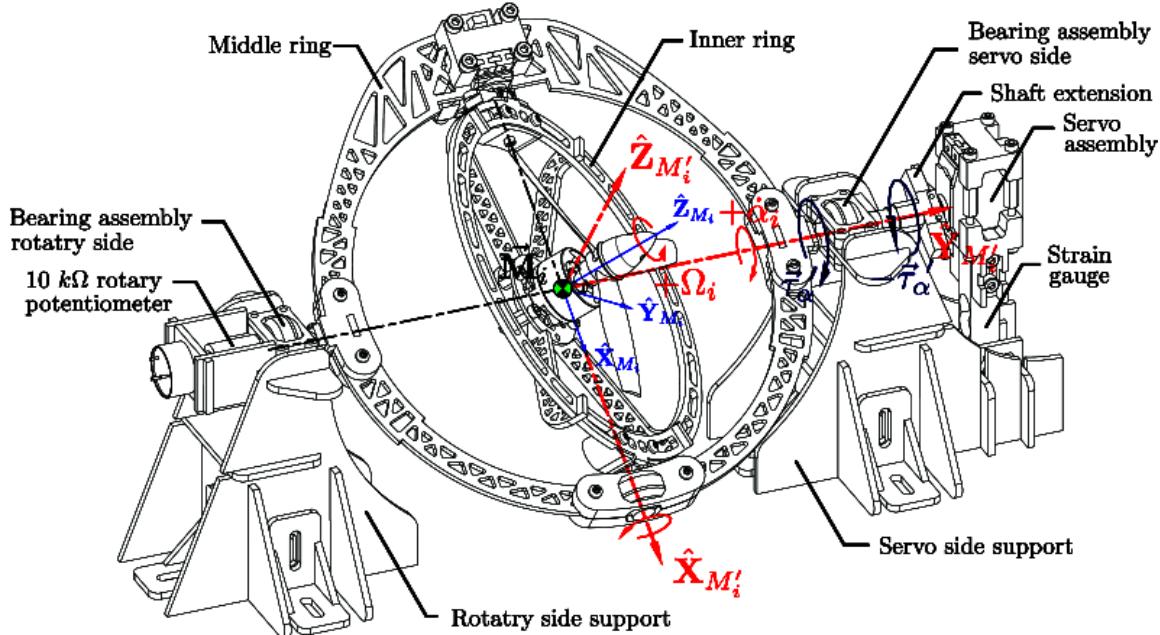


Figure 3.23: Middle ring torque test rig

Similarly the middle ring test structure (Fig:3.23) contains inner and middle ring assemblies representing the entire motor module. The inner ring servo λ_i is held at constant values to represent varying inertias for the net module J_b described in Eq:2.23b. The outer servo α_i applies an accelerating torque $\vec{\tau}_\alpha$ with an opposing $-\vec{\tau}_\lambda$ response. Only the $\hat{Y}_{M'_i}$ component deflects the strain gauge and as such the torque measured by the strain gauge is:

$$\vec{\tau}'_\alpha(\lambda) = \vec{\tau}_\alpha(\lambda) \cdot \hat{Y}_{M'_i} \in \mathcal{F}^{M'_i} \quad (3.128)$$

Reiterating the equation for the expected simulated torque $\hat{\tau}_\alpha(\lambda)$ from Eq:3.87 with an included

gravitational component and induced torque as a result of the inner ring's rotation:

$$\begin{aligned}\hat{\tau}_\alpha(\lambda_i) = & R_x(\lambda)\vec{\tau}_\lambda + (\dot{J}'_p)\dot{\vec{\alpha}}_i + \left(\dot{J}''_n - R_y(\alpha)(\dot{J}'_n)R_y^{-1}(\alpha) \right) \vec{\lambda}'_i + \left(\dot{J}''_r - R_y(\alpha)(\dot{J}'_r)R_y^{-1}(\alpha) \right) \vec{\Omega}''_i \\ & + (J'_p)\ddot{\vec{\alpha}}_i + \dot{\vec{\alpha}}_i \times \left((J'_p)\dot{\vec{\alpha}}_i + (J''_n)\vec{\lambda}'_i + (J''_r)\vec{\Omega}''_i \right) + m_p C.M'_p \times \vec{G}_{M_i} \quad \in \mathcal{F}^{M''_i}\end{aligned}\quad (3.129)$$

Where the term $C.M''_p$ is the net rotated center of gravity for the entire motor module:

$$C.M'_p = \frac{m_n R_y(\alpha) R_x(\lambda) C.M_n + m_m R_y(\alpha) C.M_m}{m_n + m_m} \quad (3.130)$$

Moreover, the included torque term $\hat{\tau}_\lambda$ is going to be zero for the case where the propeller's rotational speed and the inner ring's servo speed are both constant; $\dot{\vec{\Omega}}_i = 0$ and $\dot{\lambda}_i = 0$. Plotted in Fig:3.24 shows results for measured torques $\vec{\tau}'_\alpha(\lambda)$, expected simulated torque values $\hat{\tau}_\alpha(\lambda)$ and resultant SimScape simulation values for $\tilde{\tau}_\alpha(\lambda)$. Tests are conducted with a constant propeller rotational velocity $\vec{\Omega}_i = 6500$ [RPM] but differing rotational positions for the inner ring servo λ_i . That angular position affects the net encountered rotational inertial $J_p(\lambda)$.



Figure 3.24: Tests for $\vec{\tau}_\lambda$

The final simulated module is that of $\hat{\tau}_b$

The above responses are pertinent to simulation and plant dependent feedback compensation. The simulation environment is structured such that the torques are produced as responses from Newtonian movement at every step interval. In due course it would be more efficient (and less stiff) for the simulation to exploit an implicit Euler [64, 133] coordinate system in lieu of the cartesian response equations developed above. However this was not implemented in Chapter:5 and remains open to further testing and simulation...

3.4 Consolidated Model

Reiterating the different aspects detailed above and consolidating the state equations from Eq:3.10a-3.10d. Then lifting the attitude states to \mathbb{R}^4 space with the use of quaternions. Also introducing the non-linear inertial and gyroscopic responses to induced perturbations, $\vec{\tau}_\lambda$ and $\vec{\tau}_\alpha$ from Eq:3.76 and Eq:3.87 respectively, with non-linear inertial matrix terms $J_b(u)$ from Section:2.3. Finally replacing

net *virtual* plant inputs, $\vec{\tau}_\mu$ and \vec{F}_μ , with higher fidelity thrust models. The exact effectiveness actuator relationships are explored in Section:4.3. The following are state differential equations then used for subsequent control system development and simulation...

$$\dot{\mathcal{E}} = Q_b^* \otimes \vec{v}_b \otimes Q_b \quad \in \mathcal{F}^I \quad (3.131a)$$

$$\dot{\vec{v}}_b = m^{-1} \left(-\vec{\omega}_b \times m\vec{v}_b + Q_b \otimes m\vec{G}_I \otimes Q_b^* - \vec{D}_{net}(\vec{v}_b) + \vec{F}_\mu(u) \right) \quad \in \mathcal{F}^b \quad (3.131b)$$

$$\dot{Q}_b = \frac{1}{2} Q_b \otimes \vec{\omega}_b \quad \in \mathcal{F}^I \quad (3.131c)$$

$$\dot{\vec{\omega}}_b = \mathbb{I}_b(u)^{-1} \left(-\vec{\omega}_b \times \mathbb{I}_b(u)\vec{\omega}_b - \vec{\tau}_Q(u) + \vec{\tau}_g(u) + \sum_{i=1}^4 \vec{Q}(\Omega, \lambda, \alpha) + \mu \vec{\tau}(u) \right) \quad \in \mathcal{F}^b \quad (3.131d)$$

$$u = [\Omega_1^+, \lambda_1, \alpha_1, \dots \Omega_4^-, \lambda_4, \alpha_4] \quad \in \mathbb{U} \in \mathbb{R}^{12} \quad (3.131e)$$

With net force and torque control plant inputs, \vec{F}_μ and $\vec{\tau}_\mu$ respectively. Both are later abstracted to virtual control inputs next in Ch:4, motor number subscripts $i \in [1 : 4]$ are implied in the following.

$$\vec{F}_\mu(u) = \sum_{i=1}^4 \vec{T}(\Omega, \lambda, \alpha) = \sum_{i=1}^4 Q_{M_i}^* \otimes T(\Omega) \otimes Q_{M_i} \quad \in \mathcal{F}^b \quad (3.132a)$$

$$\vec{\tau}_\mu(u) = \sum_{i=1}^4 \vec{l} \times \vec{T}(\Omega, \lambda, \alpha) = \sum_{i=1}^4 \vec{l} \times (Q_{M_i}^* \otimes T(\Omega) \otimes Q_{M_i}) \quad \in \mathcal{F}^b \quad (3.132b)$$

Scalar thrust $T(\Omega)$ is a function of the propeller's rotational velocity whereas $\vec{T}(\Omega, \lambda, \alpha)$ is a vector in \mathbb{R}^3 . The thrust vector is redirected similarly to Eq:2.16a and transformed to the frame \mathcal{F}^b . Equivalently $Q(\Omega)$ is the scalar aerodynamic torque in \mathcal{F}^{M_i} about each motor's rotor \hat{Z} -axis, $\vec{Q}(\Omega, \lambda, \alpha)$ is the torque vector counterpart in \mathcal{F}^b . Both thrust and aerodynamic propeller torque terms are calculated from their respective coefficients (plotted in Fig:3.5):

$$T(\Omega) = C_T(J) \rho \Omega^2 D^4 \quad [\text{N}] \quad (3.133a)$$

$$Q(\Omega) = C_P(J) \rho \Omega^3 D^5 (1/R\Omega) \quad [\text{Nm}] \quad (3.133b)$$

Noting that Ω for aerodynamic calculations in Eq:3.133a and Eq:3.133b has units [RPS]. Inertial torque responses from actuator input rates from Eq:3.88 are introduced as feedback compensation terms;

$$\tau_Q(u) = \sum_{i=1}^4 -Q_{M_i} \otimes \tau_{\lambda_i}(u) \otimes Q_{M_i}^* - Q_{M'_i} \otimes \tau_{\alpha_i}(u) \otimes Q_{M'_i}^* \quad \in \mathcal{F}^b \quad (3.134)$$

Then including variable gravitational torque arm from Eq:2.32d, dependent on net actuator positions u :

$$\vec{\tau}_g(u) = \Delta C.G \times \vec{G}_b \quad \in \mathcal{F}^b \quad (3.135)$$

And finally, the entire vehicles instantaneous inertia, taken from Eq:2.27, is given as:

$$J_b(u) = J_y' + \sum_{i=1}^4 J_n + \sum_{i=1}^4 J_m \quad u \in \mathbb{U} \quad (3.136)$$

It is possible to bundle both attitude states (either euler angles $\vec{\eta}$ or quaternions Q_b) together with the linear translational position $\vec{\mathcal{E}}$ into a single state \vec{x} . Which then has its own combined control law. This could potentially exploit the cross-product coupling terms between angular and linear displacements for control benefits.

Chapter 4

Controller Development

4.1 Stability

Stables

4.2 Control Loop

The control problem for this dissertation is, as outlined in Chater:1; to achieve dynamic (*attitude*) set point tracking on a quadrotor by solving the problem of its inherent underactuation. For the purposes of the subsequent controller development, the plant is described in the following non-linear state space form:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) + g(\mathbf{x}, \vec{\nu}, t) \quad (4.1a)$$

$$y = c(\mathbf{x}, t) + d(\mathbf{x}, \vec{\nu}, t) \quad (4.1b)$$

Where the plant dynamics are governed by $f(\mathbf{x}, t)$ and the plant's input response by $g(\mathbf{x}, \vec{\nu}, t)$, for a given control input $\vec{\nu}$. The latter is not necessarily a function based relationship and could take the multiplicative form; $g(\mathbf{x}, t)\vec{\nu}$. The objective for setpoint tracking is for the output to track the state; namely $y = c(\mathbf{x}, t) = \mathbf{x}$. As such, the control problem is to design a stabilizing control law for an error state \mathbf{x}_e ¹:

$$\vec{\nu}_d = h(\mathbf{x}_e, t) \quad (4.2)$$

Such that the control plant is globally asymptotically stable or that $\lim_{t \rightarrow \infty} \mathbf{x}_e = 0$. It is possible to combine attitude and position states into a single common trajectory state such that:

$$\mathbf{x} = \begin{bmatrix} \vec{\mathcal{E}} \\ Q_b \end{bmatrix} \quad (4.3)$$

The body's trajectory is then fully described by $\mathbf{x}(t)$. Separate control laws are developed for attitude and position tracking and hence those states aren't combined. However for the purposes of discussing the control plant, a single major loop is considered. The designed control input, $\vec{\nu}_d$, is then implemented by actuator suite $u \in \mathbb{U}$ through its effectiveness function:

$$\nu_c = B(\mathbf{x}, u, t) \quad (4.4)$$

The exact relationship of the virtual control input and commanded input, $\nu_c \rightarrow \nu_d$, is governed by the allocation algorithm. That allocation function, B^\dagger , can be *roughly* referred to as the effectiveness

¹Ignoring how the state error is formulated for the time being...

inverse². The actuator positions are then solved for, avoiding saturation, subject to some constraint:

$$\underset{\in \mathbb{U}}{u} = B^\dagger(\mathbf{x}, \nu_d, t) \quad (4.5)$$

The control allocation requirements and schemes are expanded upon subsequently in Sec:4.7. Multiple attitude controllers are presented whose stability is proven with Lyapunov[†] stability theorem. Each controller is compared in the context of an over actuated quadrotor plant. Similarly a series of proposed allocation schemes are evaluated too. Those comparisons, their details and how controller efficacy and stability are evaluated is all presented next in Chapter:5.

²Direct (*pseudo*) inversion is the typical allocation scheme.

A generalized over-actuated control loop is split into a series of cascaded control blocks, each with an individual function, as illustrated in Fig:4.1. From the error state of the generated trajectory, \mathbf{x}_e , the control law designs a virtual control input, $\vec{\nu}_d$, which is cast as the argument to the allocation block. From the allocation law, $B^\dagger(\mathbf{x}, \vec{\nu}_d, t)$, physical actuator positions are obtained; $u \in \mathbb{U}$. Those actuator positions effect a virtual plant input, $\vec{\nu}_c = B(\mathbf{x}, u, t)$, which is an input to the state function's dynamics, Eq:3.131. Not shown, but implied in Fig:4.1, is the state derivative feedback of $\dot{\mathbf{x}}$ to the plant transfer function. Finally the output tracking state is estimated with some filtration paradigm, $\hat{\mathbf{x}} = A(\mathbf{x}, t)$, and fed back to the error state.

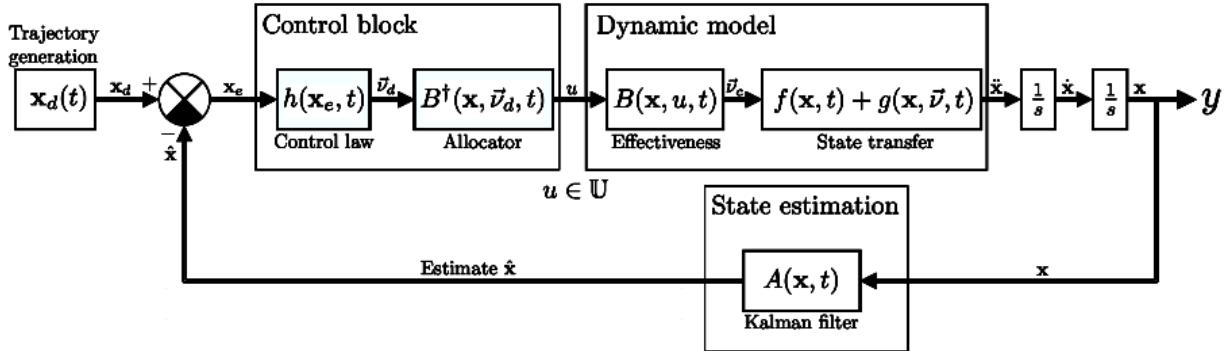


Figure 4.1: Generalized control loop with allocation

4.3 Control Plant Inputs

Thus far control plant inputs for the set of differential state equations, from Eq:3.131, have mostly been described with net forces and torques; $\mu\vec{F}$ & $\mu\vec{\tau}$. The relationship between each propeller's rotational speed & servo positions and its resultant output thrust direction is calculated as a quaternion transformation of produced lift force, as in Eq:3.132.

$$\mu\vec{F}(u) = \sum Q_{M_i}^*(\lambda_i, \alpha_i) \otimes T(\Omega_i) \otimes Q_{M_i}(\lambda_i, \alpha_i) \quad \in \mathcal{F}^b \quad (4.6a)$$

$$\mu\vec{\tau}(u) = \sum \vec{l} \times (Q_{M_i}^*(\lambda_i, \alpha_i) \otimes T(\Omega_i) \otimes Q_{M_i}(\lambda_i, \alpha_i)) \quad \in \mathcal{F}^b \quad (4.6b)$$

To accommodate comparison of each controller and allocation scheme, the error state control law(s) design net plant inputs $\mu\vec{F}$ and $\mu\vec{\tau}$. The allocation rule then takes both net inputs as an argument to find actuator positions to effect those net inputs. As such each control law can be tested against various allocation rules and *vise versa*. However typical allocation algorithms, like pseudo-inversion, require a multiplicative relationship between plant and control inputs...

The actuator effectiveness functions in Eq:4.6 aren't readily reducible to a single multiplicative relationship with the actuator matrix $u \in \mathbb{U}$. Thusly the effectiveness function needs an extra layer of abstraction to incorporate a multiplicative relationship. Rather than calculating actuator positions directly from $\vec{\nu}_d$, a set of four 3-dimensional thrust vectors, $\vec{T}_{1 \rightarrow 4}$ for each motor module, are calculated first.

$$\vec{\nu}_d = \begin{bmatrix} \mu\vec{F} \\ \mu\vec{\tau} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ [\vec{l}_1]_\times & [\vec{l}_2]_\times & [\vec{l}_3]_\times & [\vec{l}_4]_\times \end{bmatrix} \begin{bmatrix} \vec{T}_1 \\ \vec{T}_2 \\ \vec{T}_3 \\ \vec{T}_4 \end{bmatrix} \quad (4.7)$$

Where $[\vec{l}_i]_\times$ is the cross product vector of the i^{th} torque arm. Individual actuator positions for each module, $[\Omega_i, \lambda_i, \alpha_i]^T$, can be calculated from those thrust vectors \vec{T}_i for $i \in [1 : 4]$ with some trigonometry, ensuring that they only adhere to Eq:4.6. That trigonometric inversion³ can be described as the function R^\dagger :

$$[\Omega_i, \lambda_i, \alpha_i]^T = R^\dagger(\mathbf{x}, \vec{F}_i, t) \quad \text{for } i \in [1 : 4] \quad (4.8)$$

³Inverting either rotation matrix operations or quaternions to solve for angular servo positions, Eq:4.109 in Sec:4.7.

To summarize; each allocation rule decomposes net force and torque vectors into four directional thrust vectors for each, or 12 directional components. The force components are an abstracted allocation layer in place of explicit actuator positions, which are subsequently solved for...

$$B^\dagger(\mathbf{x}, \vec{\nu}_d, t) = [T_{1x}, T_{1y}, T_{1z}, \dots, T_{4x}, T_{4y}, T_{4z}]^T \quad (4.9)$$

The control block in the loop (Fig:4.1) is then modified to incorporate the extra allocation abstraction level, shown in Fig:4.2. The output from that control block is still the same actuator matrix $u \in \mathbb{U}$. The block merely accommodates for comparison of various $B^\dagger(\mathbf{x}, \vec{\nu}_d, t)$ allocation rules without having to redesign the remainder of the loop's structure.

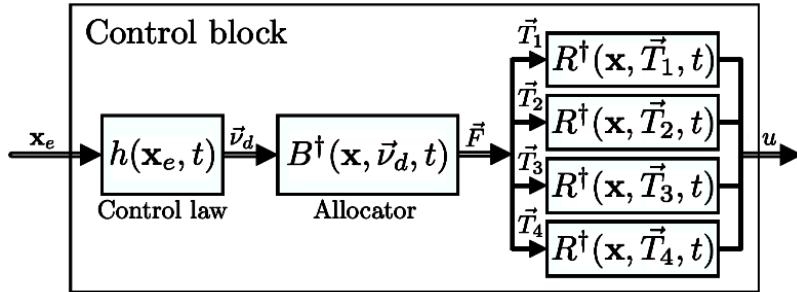


Figure 4.2: Abstracted control block

Al allocation algorithms proposed follow the same input/output structure described in Fig:4.2. Only one allocation algorithm does, however, circumvent the virtual abstraction level of thrust vector's for each module to directly calculate actuator positions, Section:4.7.5.

Each control law is co-dependent on an accompanying allocation algorithm. Traditional control loops (under-actuated or well matched) typically have a unity allocation rule and as such require no consideration so they're mostly disregarded. Separate control laws for attitude ad position control are presented next in Section:4.5 and 4.6 respectively. Thereafter a series of allocation rules are proposed in Section:4.7. Although presented independently, the controller and allocation laws are mutually inclusive. The stability of each control law is proven objectively but actual controller tuning and optimization takes place only in the following Chapter:5, in Sec:5.1.

Model Dependent & Independent Controllers

Two classes of controllers are presented, attitude and position control laws. The former being the primary focus of this research project and containing a more complete schedule of control treatment and controller comparison. Both control categories consider MIMO state vector loops for attitude and position states \mathcal{E} & Q_b . The allocation algorithm combines both virtual control inputs $\vec{\nu}_d = [\mu \vec{F} \mu \vec{\tau}]^T$ generated from the two control categories to calculate actuator positions.

The control dependency on the system plant is as a consequence of the prominent actuator response dynamics, as derived previously in Sec:3.3.1. Whilst not a prerequisite for stability, plant dependent compensation certainly improves controller performances. Independent and dependent cases are only considered for one type of controller; the most basic case PD controller in Section:4.5.2. It's shown that for an independent (PD) controller to achieve global stability some stringent assumptions must first be met.

Inherent plant dependency makes backstepping controllers an attractive control paradigm in this dissertation's context. The proposed plant dependent control laws compensate for undesirable dynamics their design, basic PD & PID control structures (*and the like*) will not. The first and most basic control solution, used as a reference case, is a PD controller for attitude and position with direct-inversion⁴ allocation, both plant dependent and independent PD controllers are compared.

⁴Pseudo-inversion or Moore-Penrose inversion

4.4 Lyapunov Stability Theorem

Lyapunov's stability theory is a critical aspect of non-linear controller design. An abundance of literature has been written on the subject⁵ spanning through the progression of control engineering. Typically linear systems are proven[†] to be stable using the frequency domain with Laplace transforms, the same is not true for non-linear systems. Lyapunov's stability theorem proves (*global*) asymptotic stability for continuous time invariant systems, linear or otherwise.

The theorem applies analysis of a generalized energy function representative of a system's autonomous trajectory. A negative trajectory energy derivative will ensure the system's energy is always dissipating toward a stable settling point. Lyapunov analysis is a popular method for stability verification because system's trajectory itself needn't be explicitly defined for stability to be ascertained. Proof of Lyapunov's theorem is done with a contradiction disproof and, as such, the theoretical underpinning is somewhat cumbersome.

Despite the conceptually difficult proof, it's worth reiterating its fundamentals given that backstepping controllers are proposed later in Sec:4.5.3 for attitude control. A backstepping controller iteratively enforces Lyapunov stability criterion onto the system through the control structure. In general, given a non-linear time invariant system that follows some continually differentiable trajectory $\mathbf{x}(t)$, typically the trajectory is going to progress subject to some rule:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) \quad (4.10)$$

Then, constructing a generalized positive-definite function (generalized energy or *Lyapunov candidate* function) $V(x)$ for a trajectory $x = \mathbf{x}(t)$. A positive definite matrix, M , is defined such that $z^T M z \geq 0 \forall z$. As such an LCF typically has the form:

$$V = \mathbf{x}^T P \mathbf{x} \quad (4.11)$$

Given that, by its definition, the trajectory is continually differentiable; there is a partial gradient matrix for each component of $V(x)$ in the form:

$$\nabla V(x) = \left[\frac{\delta V(x)}{\delta x_1} \frac{\delta V(x)}{\delta x_2} \dots \frac{\delta V(x)}{\delta x_n} \right] \quad x \in \mathbb{R}^n \quad (4.12)$$

The energy function's derivative, otherwise referred to as the *Lie derivative*[†], is calculated as follows:

$$\dot{V}(x) = \nabla V(x)^T f(x) = \frac{\delta V(x)}{\delta x_1} f_1(x) + \frac{\delta V(x)}{\delta x_2} f_2(x_2) + \dots + \frac{\delta V(x)}{\delta x_n} f_n(x) \quad (4.13)$$

Lyapunov's theorem states that *iff* the candidate function $V(x)$ is positive definite with $\dot{V}(0) = 0$ and its derivative is negative definite; $\dot{V}(x) < 0 \forall x \neq 0$, the system is then globally asymptotically stable. Mathematically that means, for any $\mathbf{x}(t) \geq 0$:

$$V(\mathbf{x}(t)) = V(\mathbf{x}(0)) + \int_0^t \dot{V}(\mathbf{x}(t)).dt \leq V(\mathbf{x}(t)) \quad (4.14)$$

Which can be physically interpreted as the system's generalized energy function always dissipating, irrespective of trajectory path taken. With a continually decreasing energy function, the system will inevitably settle to some stable point, hence the trajectory exists in some bounded $\{x|V(x) \leq V(\mathbf{x}(t))\}$, which is defined as global asymptotic stability. Every trajectory of $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$ converges to the zero⁶ setpoint as $t \rightarrow \infty$.

The asymptotic stability proof can be extended to exponential stability boundedness, such that *iff* the same conditions are met and there exists some positive coefficient $\alpha > 0$ such that $\dot{V}(x) \leq -\alpha V(x)$. That implies the system is globally exponentially stable as is bound in such a way that:

$$\|\mathbf{x}(t)\| \leq M e^{-\alpha t/2} \|\mathbf{x}(0)\| \quad (4.15)$$

⁵Included in almost every meritable textbook and papers; [84, 104], amongst others...

⁶Adapted to zero error state tracking in lieu or zero set point settling.

4.5 Attitude Control

4.5.1 The Attitude Control Problem

Set point tracking control of the attitude plant is to then design a stabilizing control torque $\mu\vec{\tau} = h(\mathbf{x}_e, t)$ such that; for any desired attitude quaternion, $\forall Q_d \in \mathbb{Q}$, and an instantaneous attitude body quaternion, similarly $\forall Q_b \in \mathbb{Q}$, the error state asymptotically stabilizes to 0; $Q_e \rightarrow [1 \vec{0}]^T$. Or that:

$$\mu\vec{\tau} = h(Q_e, \dot{Q}_e) \text{ such that } \lim_{t \rightarrow \infty} Q_e = \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} \quad (4.16)$$

Quaternion error states are defined as the Hamilton product (*difference*) between the desired and instantaneous quaternion attitude states. Quaternion error states are in contrast with the subtractive relationship for Euler angle error states. The attitude error state is calculated as:

$$Q_e = Q_d^* \otimes Q_b \quad (4.17)$$

The relative angular velocity error between the body frame, \mathcal{F}^b , and the trajectory's desired frame, \mathcal{F}^d , is given as $\vec{\omega}_e$. The body angular velocity, $\vec{\omega}_b$ is subject to the differential Eq:3.131d. As such there's an angular rate error:

$$\vec{\omega}_e = \vec{\omega}_d - \vec{\omega}_b \quad (4.18a)$$

The desired angular rate is taken with respect to the desired angular attitude frame, and so it must be transformed back onto the existing body frame.

$$\vec{\omega}_e = Q_e^* \otimes \vec{\omega}_d \otimes Q_e - \vec{\omega}_b \quad (4.18b)$$

Typically for the trajectories generated here the desired angular velocity is zero; $\vec{\omega}_d = \vec{0}$. It follows that the angular rate error is then simply the negative body angular velocity. It would be easy to incorporate a non-zero angular velocity setpoint to accommodate for higher order state derivative tracking trajectories.

$$\vec{\omega}_e = -\vec{\omega}_b \Big|_{\vec{\omega}_d=\vec{0}} \quad (4.18c)$$

The time derivative of the quaternion error state is given by Eq:3.53. The derivative \dot{Q}_e is then dependent on the angular velocity error and calculated as follows:

$$\dot{Q}_e = \frac{1}{2}Q_e \otimes \vec{\omega}_e = -\frac{1}{2}Q_e \otimes \vec{\omega}_b \Big|_{\vec{\omega}_d=\vec{0}} \quad (4.19)$$

4.5.2 Linear Controllers

PD Controller

The control law which is used as a basic reference for comparison is a simple Proportional-Derivative structured attitude controller. Specifically, a stability proof derived from the one presented *The Attitude Control Problem* [125] is used for asymptotic stability verification. An attitude PD control law, proportional to the vector quaternion error only⁷ and angular rate error, designs the control torque as:

$$\mu\vec{\tau}_{PD} = K\vec{\omega}_e + \alpha\vec{q}_e \quad (4.20)$$

Where both K and α are positive definite symmetrical 3×3 coefficient matrices still to be determined. This control law neglects the quaternion scalar error and is susceptible to unwinding. Then using a candidate Lyapunov energy function V_{PD} :

$$V_{PD}(\vec{q}_e, \vec{\omega}_e) = \alpha\vec{q}_e^T \vec{q}_e + \alpha(q_0 - 1)^2 + \frac{1}{2}\vec{\omega}_e^T \mathbb{I}_b \vec{\omega}_e \quad (4.21)$$

⁷Such that the error is $\in \mathbb{R}^3$.

And recalling from Eq:3.131d that body's the angular velocity differential $\dot{\vec{\omega}}_b$ is:

$$\dot{\vec{\omega}}_b = \mathbb{I}_b^{-1}(-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} + \mu \vec{\tau}) \in \mathcal{F}^b \quad (4.22)$$

With actuator inputs $u \in \mathbb{U}$ implied and \vec{Q} being a simplified representation of the net aerodynamic torque experienced by the body from the rotating propellers, drawn from Eq:3.133b. Then, exploiting a unit quaternion's inherent property, it follows that:

$$\|Q\| = \vec{q}^T \vec{q} + q_0^2 = \vec{q}^2 + q_0^2 = 1 \quad (4.23)$$

Substituting the angular velocity error state, $\vec{\omega}_e = -\vec{\omega}_b$, the proportional derivative LCF in Eq:4.21 is simplified⁸ to:

$$V_{PD} = \alpha \vec{q}_e^2 + \alpha q_0^2 - 2q_0 + 1 + \frac{1}{2} \vec{\omega}_e^T \mathbb{I}_b \vec{\omega}_e \quad (4.24a)$$

$$= 2\alpha(1 - q_0) + \frac{1}{2} \vec{\omega}_b^T \mathbb{I}_b \vec{\omega}_b \quad (4.24b)$$

Similarly, using the fact that for a quaternion's derivative:

$$\dot{Q} = \begin{bmatrix} -\frac{1}{2} \vec{q}^T \vec{\omega} \\ \frac{1}{2} (\vec{q} \times + q_0 \mathbb{I}) \vec{\omega} \end{bmatrix} \quad (4.25)$$

Then, substituting the above into the derivative of the LCF, \dot{V}_{PD} , yields:

$$\dot{V}_{PD} = 2\alpha \frac{1}{2} \vec{q}_e^T \vec{\omega}_e + \frac{1}{2} \dot{\vec{\omega}}_b^T \mathbb{I}_b \vec{\omega}_b + \frac{1}{2} \vec{\omega}_b \mathbb{I}_b \dot{\vec{\omega}}_b^T \quad (4.26a)$$

$$= -\alpha \vec{q}_e^T \vec{\omega}_b + \vec{\omega}_b^T \mathbb{I}_b \dot{\vec{\omega}}_b \quad (4.26b)$$

Simplifying the angular acceleration $\dot{\vec{\omega}}_b$ and introducing the PD control law Eq:4.20, $\mu \vec{\tau}_{PD}$:

$$\vec{\omega}_b^T \mathbb{I}_b \dot{\vec{\omega}}_b = \vec{\omega}_b^T (-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} - K \vec{\omega}_b + \alpha \vec{q}_e) \quad (4.27a)$$

$$\rightarrow \dot{V}_{PD} = -\alpha \vec{q}_e^T \vec{\omega}_b + \vec{\omega}_b^T (-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} - K \vec{\omega}_b + \alpha \vec{q}_e) \quad (4.27b)$$

$$= -\alpha \vec{q}_e^T \vec{\omega}_b + \alpha \vec{\omega}_b^T \vec{q}_e - \vec{\omega}_b^T K \vec{\omega}_b + \vec{\omega}_b^T (-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q}) \quad (4.27c)$$

It follows that the transpose term $\vec{q}_e^T \vec{\omega}_b \iff \vec{\omega}_b^T \vec{q}_e$ is interchangeable as its resultant product is the same. The LCF derivative then simplifies to:

$$\dot{V}_{PD} = -\vec{\omega}_b^T K \vec{\omega}_b + \vec{\omega}_b^T (-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q}) \quad (4.27d)$$

Then, under specific circumstances the following assumptions can be made to ensure the asymptotic stability proof can be applied. The stability obviously breaks down if any of the assumptions fail, as such the stability is not global...

1. The inertial matrix, \mathbb{I}_b , is approximately diagonal. Which, given the symmetrical design and similarly that the angular rate can be made small with appropriately slow trajectory updates, is a fair assumption then:

$$\vec{\omega}_b^T (\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b) \approx \vec{0}$$

2. The actuator rate torque responses, $\vec{\tau}_Q$, are all second order effects dependent on \dot{u} . Typically the actuator rates are going to be kept small and so any of the inertial responses to those position changes are small enough to be considered negligible. The approximation is made:

$$\vec{\tau}_Q \approx \vec{0}$$

3. Finally, for the sake of the stability proof, the eccentric gravitational torque arm is neglected, $\vec{\tau}_g \approx \vec{0}$. Such a situation only holds true if $u \approx \vec{0}$ or that servo actuator positions⁹ are close to their zero positions.

⁸The quaternion scalar q_0 in Eq:4.24 is implied to be the quaternion error state scalar

⁹Excluding propeller rotational speeds, considering only the servo positions

All of these assumptions are made under extraneous circumstances and can't be assumed for almost all of the prototype's flight envelope. The plant independent case is considered and simulated purely for contrition; mainly to demonstrate the need for plant dependent compensation. All subsequent control laws compensate for the plant dynamic response torques introduced in Section:3.3.

If each of the assumptions made hold true, then the Lyapunov energy function's derivative is approximately negative definite.

$$\dot{V}_{PD} \approx -\alpha \vec{q}_e^T \vec{\omega}_b + \vec{\omega}_b^T (-K \vec{\omega}_b + \alpha \vec{q}_e) \quad (4.28a)$$

$$\Rightarrow \dot{V}_{PD} = -\vec{\omega}_b^T K \vec{\omega}_b = -K \|\vec{\omega}_b\|^2 < 0 \quad \forall (\vec{\omega}_e, Q_b) = \mathbf{z}(t) \quad (4.28b)$$

Where $\mathbf{z}(t)$ is a generalized attitude trajectory which includes $\vec{\omega}_b$ & Q_b and K is a symmetrical¹⁰ positive (*definite*) 3X3 coefficient matrix. Then from Lyapunov stability theorem the limits exist; $\lim_{t \rightarrow \infty} \vec{\omega}_e = \vec{0}$, $\lim_{t \rightarrow \infty} \vec{q}_e = 0$ and $\lim_{t \rightarrow \infty} (1 - q_0) = 0$. Hence $Q_e \rightarrow [1 \ \vec{0}]^T$ as $t \rightarrow \infty$, asymptotically stabilizing the attitude error state.

Introducing model dependent compensation to the PD control law in Eq:4.20 alleviates the stringent requirements on assumptions 1 through 3.

$$\mu \vec{\tau}_{PD} = \vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b - \vec{\tau}_Q - \vec{\tau}_g - \vec{Q} + K \vec{\omega}_b + \alpha \vec{q}_e \quad (4.29)$$

The resultant stability proof for Eq:4.29 is much the same as that for the independent case, Eq:4.20, and uses the identical LCF from Eq:4.21. The resultant dependent control law is no longer reliant on the very broad assumptions needed for independent stability to be achieved. The dynamic compensation in Eq:4.29 improves control response, especially considering the form of unwanted dynamics which have already quantified previously and modelled with *relative* confidence.

Auxiliary Plant Controller

Expanding on what has, in practice¹¹, proven to be a very popular and effective control law for attitude stabilization, McGilvray et al. [2006] [122] suggested introducing an auxiliary plant term to a Proportional-Derivative structure. Most significantly, their altered PD controller adds auxiliary terms proportional to the quaternion time derivative error. The critical component of that change is the part of the auxiliary plant proportional to the quaternion scalar. The scalar term is otherwise neglected in the previous PD control law (Sec:4.5.2) and prevents unwinding if incorporated.

The modified (*auxilliarily*) PD control torque is a function of errors states for quaternions, angular rates and quaternion rates. The compensating plant dependent control law is given as:

$$\mu \vec{\tau}_{XPD} = \underbrace{-\Gamma_2 \tilde{\Omega} - \Gamma_3 \vec{q}_e + \mathbb{I}_b \dot{\tilde{\Omega}}}_{Independent} + \underbrace{\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q}}_{Compensation} \quad (4.30)$$

In which case the coefficients¹² Γ_2 & Γ_3 are both diagonal positive definite coefficient matrices and Γ_1 , introduced subsequently in Eq:4.32, is a p.d symmetrical coefficient matrix. The auxiliary plants $\tilde{\Omega}$ & $\dot{\tilde{\Omega}}$ are defined as follows and draw on Eq:4.25 for definition of some aspects. For the first auxiliary plant $\tilde{\Omega}$ is proportional to the quaternion error and hence $\dot{\tilde{\Omega}}$ is a quaternion derivative term:

$$\tilde{\Omega} = -\Gamma_1 \vec{q}_e \Rightarrow \dot{\tilde{\Omega}} = -\Gamma_1 \dot{\vec{q}}_e \quad (4.31a)$$

$$\dot{\tilde{\Omega}} = -\frac{1}{2} \Gamma_1 (q_0 \mathbb{I}_{3X3} + [\vec{q}_e]_\times) \vec{\omega}_e \quad (4.31b)$$

$$= \frac{1}{2} \Gamma_1 (q_0 \mathbb{I}_{3X3} + [\vec{q}_e]_\times) \vec{\omega}_b \quad (4.31c)$$

¹⁰Symmetry, unlike the subsequent Auxiliary controller, is not a prerequisite for stability...

¹¹Practical examples of various quadrotor attitude PD controllers listed in Table:1.1 from Sec:1.2.1.

¹²Reiterating that exact coefficient values are determined in Chapter:5...

The second auxiliary plant, $\tilde{\Omega}$, is a term proportional to a combined quaternion vector and angular velocity error state.

$$\tilde{\Omega} = \vec{\omega}_e - \bar{\Omega} = \vec{\omega}_e + \Gamma_1 \vec{q}_e \quad (4.32a)$$

$$= -\vec{\omega}_b + \Gamma_1 \vec{q}_e \quad (4.32b)$$

Using an LCF similar to the basic one V_{PD} from Eq:4.21, but introducing an auxiliary term $\tilde{\Omega}$ into the candidate function V_{XPD} :

$$V_{XPD}(\vec{q}_e, \tilde{\Omega}) = \vec{q}_e^T \vec{q}_e + (q_0 - 1)^2 + \frac{1}{2} \tilde{\Omega}^T (\Gamma_3^{-1} \mathbb{I}_b) \tilde{\Omega} \quad (4.33)$$

Again using the simplification from a quaternion's inherent properties in Eq:4.23, the LCF from Eq:4.33 then simplifies with the following derivative:

$$V_{XPD} = 2(1 - q_0) + \frac{1}{2} \tilde{\Omega}^T (\Gamma_3^{-1} \mathbb{I}_b) \tilde{\Omega} \quad (4.34a)$$

$$\dot{V}_{XPD} = 2 \frac{1}{2} \vec{q}_e^T \vec{\omega}_e + \frac{1}{2} \dot{\tilde{\Omega}}^T (\Gamma_3^{-1} \mathbb{I}_b) \tilde{\Omega} + \frac{1}{2} \tilde{\Omega}^T (\Gamma_3^{-1} \mathbb{I}_b) \dot{\tilde{\Omega}} \quad (4.34b)$$

$$\dot{V}_{XPD} = -\vec{q}_e^T \vec{\omega}_b + \frac{1}{2} \dot{\tilde{\Omega}}^T (\Gamma_3^{-1} \mathbb{I}_b) \tilde{\Omega} + \frac{1}{2} \tilde{\Omega}^T (\Gamma_3^{-1} \mathbb{I}_b) \dot{\tilde{\Omega}} \quad (4.34c)$$

It follows that from Eq:4.32 then the auxiliary plant derivative $\dot{\tilde{\Omega}}$ is:

$$\dot{\tilde{\Omega}} = -\dot{\vec{\omega}}_b + \Gamma_1 \dot{\tilde{\Omega}} \Rightarrow \dot{\vec{\omega}}_b = \mathbb{I}_b^{-1} (-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} + \mu \vec{\tau}) \quad (4.35a)$$

$$\therefore \dot{\tilde{\Omega}} = -\mathbb{I}_b^{-1} (-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} + \mu \vec{\tau}) - \Gamma_1 \dot{\tilde{\Omega}} \quad (4.35b)$$

Substituting the auxiliary PD control law, $\mu \vec{\tau}_{XPD}$ from Eq:4.30, into the auxiliary plant derivative then yields:

$$\rightarrow \dot{\tilde{\Omega}} = \mathbb{I}_b^{-1} (\mathbb{I}_b \dot{\tilde{\Omega}} - \Gamma_2 \tilde{\Omega} - \Gamma_3 \vec{q}_e) - \dot{\tilde{\Omega}} \quad (4.35c)$$

$$= \mathbb{I}_b^{-1} (-\Gamma_2 \tilde{\Omega} - \Gamma_3 \vec{q}_e) \quad (4.35d)$$

From the positive symmetric (or *diagonal*) properties of the coefficient matrices Γ_1, Γ_2 & Γ_3 , the auxiliary plant's transpose is then:

$$\dot{\tilde{\Omega}}^T = \mathbb{I}_b^{-1} (-\Gamma_2 \tilde{\Omega}^T - \Gamma_3 \vec{q}_e^T) \quad (4.36)$$

It then follows that the P.D auxiliary plant component in the LCF, Eq:4.33, simplifies:

$$\frac{1}{2} \dot{\tilde{\Omega}}^T (\Gamma_3^{-1} \mathbb{I}_b) \tilde{\Omega} = \frac{1}{2} (-\Gamma_2 \tilde{\Omega}^T - \Gamma_3 \vec{q}_e^T) \Gamma_3^{-1} \tilde{\Omega} \quad (4.37a)$$

$$= \frac{1}{2} (-\tilde{\Omega}^T \Gamma_2 \Gamma_3^{-1} \tilde{\Omega} - \vec{q}_e^T \tilde{\Omega}) \quad (4.37b)$$

And substituting Eq:4.32, $\vec{q}_e^T \tilde{\Omega} = -\vec{q}_e^T \vec{\omega}_b + \Gamma_1 \vec{q}_e^T$:

$$\frac{1}{2} (-\tilde{\Omega}^T \Gamma_2 \Gamma_3^{-1} \tilde{\Omega} + \vec{q}_e^T \vec{\omega}_b - \vec{q}_e^T \Gamma_1 \vec{q}_e) \quad (4.37c)$$

Similarly, for the transposed energy function counterpart:

$$\frac{1}{2} \tilde{\Omega}^T (\Gamma_3^{-1} \mathbb{I}_b) \dot{\tilde{\Omega}} = \frac{1}{2} (-\tilde{\Omega} \Gamma_2 \Gamma_3^{-1} \tilde{\Omega}^T + \vec{q}_e^T \vec{\omega}_b - \vec{q}_e^T \Gamma_1 \vec{q}_e) \quad (4.37d)$$

Which, when substituted back into Eq:4.34c, then simplifies the LCF derivative to negative definite:

$$\Rightarrow \dot{V}_{XPD} = -\vec{q}_e^T \Gamma_1 \vec{q}_e - \tilde{\Omega} \Gamma_2 \Gamma_3^{-1} \tilde{\Omega}^T < \vec{0} \quad \forall (\vec{q}_e, \tilde{\Omega}) \quad (4.38)$$

As such, the control law $\mu \vec{\tau}_{XPD}$ asymptotically stabilizes the attitude plant. Both $\tilde{\Omega}$ and \vec{q}_e tend to $\vec{0}$, or more specifically the following limits exist:

$$\lim_{t \rightarrow \infty} \vec{q}_e = 0 \text{ and } \lim_{t \rightarrow \infty} \tilde{\Omega} = 0 \quad (4.39a)$$

Then, from the auxiliary plant definition in Eq:4.32, the extended limits present themselves;

$$\lim_{t \rightarrow \infty} \vec{\omega}_b = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \tilde{\Omega} = 0 \quad (4.39b)$$

The stability proof for V_{XPD} can then be extended to a stable exponentially bounded trajectory. From a quaternion's inherent definition it follows that $0 \leq q_0 \leq 1$. It can then be stated that:

$$1 - q_0 \leq 1 - q_0^2 = \|\vec{q}_e\|^2 \quad (4.40)$$

Seeing that exponential stability is a maximal boundedness proof, the relationship Eq:4.40 can then replace the quaternion scalar term $2(1 - q_e)$ in V_{XPD} . For the stability proof the LCF is rewritten in terms of its component's norm(s) to produce the inequality:

$$V_{XPD} = \vec{q}_e^T \vec{q}_e + (q_0 - 1)^2 + \frac{1}{2} \tilde{\Omega}^T (\Gamma_3^{-1} \mathbb{I}_b) \tilde{\Omega} \quad (4.41a)$$

$$\rightarrow V_{XPD} \leq 2 \|\vec{q}_e\|^2 + \frac{1}{2} \Gamma_3^{-1} \mathbb{I}_b \|\tilde{\Omega}\|^2 \quad (4.41b)$$

Similarly the LCF derivative can be written in terms of its norms as:

$$\dot{V}_{XPD} = -\Gamma_2 \Gamma_3^{-1} \|\tilde{\Omega}\|^2 - \Gamma_1 \|\vec{q}_e\|^2 \quad (4.41c)$$

V_{XPD} then gains a maximum such that:

$$V_{XPD} \leq \max \left\{ 2, \frac{\lambda_{\max}(\Gamma_3^{-1} \mathbb{I}_b)}{2} \right\} (\|\vec{q}_e\|^2 + \|\tilde{\Omega}\|^2) \quad (4.42)$$

Where the function λ_{\max} represents the maximum eigenvalue of its argument, in this case $\Gamma_3^{-1} \mathbb{I}_b$. Similarly the *negative definite* LCF derivative is bound by the minimum:

$$\dot{V}_{XPD} \leq -\min \{ \lambda_{\min}(\Gamma_1), \lambda_{\min}(\Gamma_2 \Gamma_3^{-1}) \} (\|\vec{q}_e\|^2 + \|\tilde{\Omega}\|^2) \quad (4.43)$$

Therefore there exists some ratio $\alpha > 0$ that satisfies the relationship requirement between the LCF and its derivative; $\dot{V}_{XPD} \leq -\alpha V_{XPD}$, where α is defined as the ratio of those maxima¹³:

$$\alpha = \frac{\min \{ \lambda_{\min}(\Gamma_1), \lambda_{\min}(\Gamma_2 \Gamma_3^{-1}) \}}{\max \{ 2, \frac{\lambda_{\max}(\Gamma_3^{-1} \mathbb{I}_b)}{2} \}} \quad (4.44)$$

The attitude trajectory $(\vec{q}_e(t), \tilde{\Omega}(t))$ is then exponentially bounded by:

$$(\|\vec{q}_e(t)\|, \|\tilde{\Omega}(t)\|) \leq M e^{-\alpha t/2} (\|\vec{q}_e(0)\|, \|\tilde{\Omega}(0)\|) \quad (4.45)$$

The above stability proof for the auxiliary attitude controller was expanded upon and derived from McGilvray et al. [2006] [122] and adapted to fit attitude setpoint tracking. The fact that the auxiliary plant controller introduces the quaternion error, which is dependent on the quaternion scalar, dramatically improves controller performance. The exponential stability notably improves settling times and overshoot errors, seen next in Chapter:5.

Interestingly an earlier paper by Joshi, et al. [1995] [63] was the precursor for PD based attitude plants with asymptotic exponential stability. Joshi's control law first proposed didn't make use of any defined auxiliary plants, unlike [122], but equivalent terms were effectively incorporated. That control law developed for spacecraft attitude tracking proposed a very similar exponentially stabilizing control scheme to that of $\mu \vec{\tau}_{XPD}$. That controller, when changed to the notational convention used above, designs body torque as:

$$\mu \vec{\tau}'_{XPD} = -\frac{1}{2} \left[([\vec{q}_e]_\times + q_0 \mathbb{I}_{3 \times 3}) \Gamma_1 + \alpha (1 - q_0 \mathbb{I}) \right] \vec{q}_e - \Gamma_2 \vec{\omega}_b \quad (4.46)$$

¹³A maximum and minimum

4.5.3 Non-linear Controllers

Backstepping controllers(presented in [10, 66, 68],etc...) are a popular choice for non-linear attitude control plants. The process, through iterative design, enforces Lyapunov stability criteria to ensure asymptotic stability. In a report [129] Van Kampen, et al. [2008] describes fundamental backstepping algorithms. Ideal backstepping control is a precise control solution which requires exact plant matching, something that is difficult to achieve in practice. Another caveat of IBC control is poor disturbance rejection, being especially susceptible to plant uncertainty. The ideal backstepping algorithm can then be extended to incorporate non-idealities. The disturbance and uncertainty (*estimate error*) terms are incorporated into the LCF energy function. By Lyapunov's theorem their respective estimation error terms are stabilized.

Ideal Backstepping Controller

Starting with the ideal case for the first proposed backstepping controller; it's assumed the attitude plant described in Eq:3.131d, from the consolidated model in Sec:3.4, absolutely matches the dynamics of the physical prototype. The ideal backstepping controller aims to perfectly compensate for the plant's dynamic response to trajectory inputs. Ignoring any uncertainty associated with the dynamic equation, the aim here is to apply a stabilizing torque design. Recalling the quaternion tracking error from Eq:4.17; $Q_e = Q_d^* \otimes Q_b$. Then considering the first LCF proposal:

$$V_1(\vec{q}_e) = \vec{q}_e^T \vec{q}_e + (q_0 - 1)^2 \quad (4.47)$$

Which, after substituting in the quaternion derivatives and *without* using the quaternion simplification in Eq:4.24, has a Lie derivative:

$$\dot{V}_1 = 2\vec{q}_e^T \dot{\vec{q}}_e + 2(q_0 - 1)\dot{q}_0 \quad (4.48a)$$

$$= 2\vec{q}_e^T \frac{1}{2} ([\vec{q}_e]_{\times} + q_0 \mathbb{I}_{3 \times 3}) \vec{\omega}_e - 2(q_0 - 1) \frac{1}{2} \vec{q}_e^T \vec{\omega}_e \quad (4.48b)$$

$$= \vec{q}_e^T ([\vec{q}_e]_{\times} + q_0 \mathbb{I}_{3 \times 3}) \vec{\omega}_e - q_0 \vec{q}_e^T \vec{\omega}_e + \vec{q}_e^T \vec{\omega}_e \quad (4.48c)$$

$$= \vec{q}_e^T [\vec{q}_e]_{\times} \vec{\omega}_e + \vec{q}_e^T \vec{\omega}_e \quad (4.48d)$$

$$= -\vec{q}_e^T [\vec{q}_e]_{\times} \vec{\omega}_b - \vec{q}_e^T \vec{\omega}_b \quad (4.48e)$$

Then choosing the first stabilizing function, z_1 , with a virtual backstepping control input Ω_d . It's important to note that Ω_d is used here to differentiate the backstepping *desired* value from the trajectory instructed $\vec{\omega}_d$ from Eq:4.18a, or any auxiliary plants defined previously for the Auxiliary PD controller in Sec:4.5.2.

$$\vec{\omega}_b \Rightarrow \Omega_d = \Gamma_1 \vec{q}_e \quad (4.49)$$

Where Γ_1 is the first symmetric positive definite coefficient matrix, a fact that is important to stress due to positive definite matrix's invertability. That stabilizing law then simplifies the LCF derivative \dot{V}_1 to the negative definite term:

$$\dot{V}_1 = -\vec{q}_e^T [\vec{q}_e]_{\times} \Omega_d - \vec{q}_e^T \Omega_d \quad (4.50a)$$

$$= -\vec{q}_e^T [\vec{q}_e]_{\times} \Gamma_1 \vec{q}_e - \vec{q}_e^T \Gamma_1 \vec{q}_e \quad (4.50b)$$

$$= -\vec{q}_e^T \Gamma_1 \vec{q}_e \quad (4.50c)$$

A vector cross product with itself has a zero resultant. However, that stabilizing virtual plant input Ω_d has its own associated error, z_1 , which then needs to be stabilized as well:

$$z_1 = \vec{\omega}_b - \Omega_d = \vec{\omega}_b - \Gamma_1 \vec{q}_e \quad (4.51a)$$

$$\rightarrow \vec{\omega}_b = z_1 - \Gamma_1 \vec{q}_e \quad (4.51b)$$

Introducing that error z_1 into a second LCF, which expands the first proposed V_1 . Here is where it's important that Γ_1 is p.d & symmetrical:

$$V_2(\vec{q}_e, z_1) = V_1(\vec{q}_e) + \frac{1}{2} z_1^T \Gamma_1^{-1} z_1 \quad (4.52a)$$

$$= \vec{q}_e^T \vec{q}_e + (q_0 - 1)^2 + \frac{1}{2} z_1^T \Gamma_1^{-1} z_1 \quad (4.52b)$$

That first error z_1 has its own derivative, and recalling $\dot{\vec{\omega}}_b$ from earlier with an as yet undefined controllable input $\mu\vec{\tau}$, which still has plant dependency compensation.

$$\dot{z}_1 = \dot{\vec{\omega}}_b - \Gamma_1 \dot{\vec{q}}_e \quad (4.53a)$$

$$= \dot{\vec{\omega}}_b - \frac{\Gamma_1}{2} ([\vec{q}_e]_{\times} + q_0 \mathbb{I}_{3 \times 3}) \vec{\omega}_e \quad (4.53b)$$

$$= \mathbb{I}_b^{-1} (-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} + \mu \vec{\tau}) + \frac{\Gamma_1}{2} ([\vec{q}_e]_{\times} + q_0 \mathbb{I}_{3 \times 3}) \vec{\omega}_b \quad (4.53c)$$

So then, following from Eq:4.53c, finding the derivative \dot{V}_2 :

$$\begin{aligned} \dot{V}_2(\vec{q}_e, z_1) &= \vec{q}_e^T (z_1 - \Gamma_1 \vec{q}_e) + z_1^T \Gamma_1^{-1} \left(\mathbb{I}_b^{-1} (-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} + \mu \vec{\tau}) \right. \\ &\quad \left. + \frac{\Gamma_1}{2} ([\vec{q}_e]_{\times} + q_0 \mathbb{I}_{3 \times 3}) \vec{\omega}_b \right) \end{aligned} \quad (4.54a)$$

$$\begin{aligned} &= -\vec{q}_e^T \Gamma_1 \vec{q}_e + z_1^T \Gamma_1^{-1} \left(\Gamma_1 \vec{q}_e + \mathbb{I}_b^{-1} (\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} + \mu \vec{\tau}) \right. \\ &\quad \left. + \frac{\Gamma_1}{2} ([\vec{q}_e]_{\times} + q_0 \mathbb{I}_{3 \times 3}) \vec{\omega}_b \right) \end{aligned} \quad (4.54b)$$

Then proposing the compensated stabilizing backstepping control law:

$$\mu \vec{\tau}_{IBC} = \vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b - \vec{\tau}_Q - \vec{\tau}_g - \vec{Q} - \mathbb{I}_b \Gamma_1 \vec{q}_e - \frac{\mathbb{I}_b \Gamma_1}{2} ([\vec{q}_e]_{\times} + q_0 \mathbb{I}_{3 \times 3}) \vec{\omega}_b - \mathbb{I}_b \Gamma_2 z_1 \quad (4.55)$$

With Γ_2 being another positive definite symmetric coefficient matrix. Then with the control law $\mu \vec{\tau}_{IBC}$ introduced into the LCF derivative \dot{V}_2 simplifies to negative definite:

$$\dot{V}_2 = -\vec{q}_e^T \Gamma_1 \vec{q}_e - z_1^T \Gamma_2 z_1 \leq 0 \quad \forall (\vec{q}_e, z_1) \quad (4.56)$$

As such $\vec{q}_e \rightarrow 0$ & $z_1 \rightarrow 0$ as $t \rightarrow \infty$. Similarly $z_1 \rightarrow 0$, which leads to the limit:

$$\lim_{t \rightarrow \infty} (\vec{\omega}_b - \Gamma_1 \vec{q}_e) = 0 \quad (4.57)$$

Because the quaternion error vector already tends to 0; $\vec{q}_e \rightarrow 0$, it follows that $\vec{\omega}_b \rightarrow 0$. It can also be said that, from the definition of $\vec{\omega}_e$, that the angular velocity error stabilizes too. There is a distinct similarity in the structure of $\mu \vec{\tau}_{IBC}$ from Eq:4.55 and that of the auxiliary PD controller presented in Eq:4.30. Furthermore, using the same reasoning from Eq:4.41, the exponential stability proof then follows:

$$V_{IBC} \leq V_2 = 2 \|\vec{q}_e\|^2 + \frac{\Gamma_1^{-1}}{2} \|z_1\|^2 \quad (4.58a)$$

$$\dot{V}_{IBC} \leq \dot{V}_2 = -\Gamma_1 \|\vec{q}_e\|^2 - \Gamma_2 \|z_1\|^2 \quad (4.58b)$$

Then both the energy function and its derivative are bound respectively by:

$$V_{IBC} \leq \max \left\{ 2, \frac{\lambda_{\max}(\Gamma_1^{-1})}{2} \right\} (\|\vec{q}_e\|^2 + \|z_1\|^2) \quad (4.59a)$$

$$\dot{V}_{IBC} \leq \min \{ \lambda_{\min}(\Gamma_1), \lambda_{\min}(\Gamma_2) \} (\|\vec{q}_e\|^2 + \|z_1\|^2) \quad (4.59b)$$

Which then leads to a similar exponential stability trajectory boundedness such that:

$$\dot{V}_{IBC} \leq \alpha V_{IBC} \quad (4.60a)$$

$$\therefore (\|\vec{q}_e(t)\|, \|z_1(t)\|) \leq M e^{-\alpha t/2} (\|\vec{q}_e(0)\|, \|z_1(0)\|) \quad (4.60b)$$

Adaptive Backstepping Controller

As effective as the control law defined above in Section:4.5.3 may be, it lacks suitable disturbance rejection properties. Any plant uncertainties or disturbances encountered would adversely affect the controller in a dramatic manner (Sec:5.4). Introducing a term for lumped uncertainty/disturbance torques, \vec{L} , into the dynamic equations leads to:

$$\dot{\vec{\omega}}_b = \mathbb{I}_b^{-1}(-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} + \vec{L} + \mu \vec{\tau}) \quad (4.61)$$

It would obviously be easy to simply introduce a compensation term for $-\vec{L}$ into the control law. In practice, however, it is very difficult to approximate a disturbance term without *apriori* knowledge about any of its properties. Noise compensation in sensors can be done easily due to the known frequency bandwidth which that noise occurs in, the same cannot be said for wind disturbances and the like.

An approximate estimation term \hat{L} has to be used for that disturbance compensation in the designed control torque $\mu \vec{\tau}$. That estimate term is then going to have its own error from the physical disturbance affecting the system:

$$\tilde{L} = \vec{L} - \hat{L} \quad (4.62)$$

The purpose of adaptive backstepping is to introduce that estimate error term into an LCF and develop a derivative term for $\dot{\hat{L}}$, or a disturbance update law, such that even the estimate error asymptotically stabilizes. Typically, that disturbance update rule is the contribution of satellite and general attitude control papers. Similar terms can be introduced for plant uncertainty which can similarly be adapted for but are not included here...

The estimate error is then introduced into the LCF from an ideal backstepping control, in order for it to be dissipated as per Lyapunov theorem.

$$V_{ABC}(\vec{q}_e, z_1, \tilde{L}) = V_{IBC}(\vec{q}_e, z_1) + \frac{1}{2} \tilde{L}^T \Gamma_L^{-1} \tilde{L} \quad (4.63a)$$

$$= \vec{q}_e^T \vec{q}_e + (q_0 - 1)^2 + \frac{1}{2} z_1^T \Gamma_1^{-1} z_1 + \frac{1}{2} \tilde{L}^T \Gamma_L^{-1} \tilde{L} \quad (4.63b)$$

Where the positive symmetric matrix $\Gamma_L \geq 0 \in \mathbb{R}^{3 \times 3}$ is termed as the adaptation gain coefficient matrix. Those particular coefficients determine how responsive the system is to disturbances and the rate at which it adapts to compensate for them. Then, to prove stability one starts with the Lie derivative \dot{V}_{ABC} :

$$\dot{V}_{ABC}(\vec{q}_e, z_1, \tilde{L}) = \dot{V}_{IBC}(\vec{q}_e, z_1) + \frac{1}{2} \dot{\tilde{L}}^T \Gamma_L^{-1} \tilde{L} + \frac{1}{2} \tilde{L}^T \Gamma_L^{-1} \dot{\tilde{L}} \quad (4.64)$$

Recalling the definition of \tilde{L} from Eq:4.62. For its derivative $\dot{\tilde{L}}$ it's reasonable to assume the dynamics of the physical disturbance \vec{L} are far slower than the time constant of the control system, or that $\dot{\vec{L}} \ll \dot{\hat{L}}$. Then it follows:

$$\dot{\tilde{L}} = \dot{\vec{L}} - \dot{\hat{L}} \approx \vec{0} - \dot{\hat{L}} = -\dot{\hat{L}} \quad (4.65)$$

Substituting that estimation error rate back into the derivative \dot{V}_{ABC} , which expands upon Eq:4.54, yields:

$$\begin{aligned} \dot{V}_{ABC}(\vec{q}_e, z_1, \tilde{L}) &= \vec{q}_e^T (z_1 - \Gamma_1 \vec{q}_e) + z_1^T \Gamma_1^{-1} \left(\mathbb{I}_b^{-1} (-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_Q + \vec{\tau}_g + \vec{Q} + \vec{L} + \mu \vec{\tau}) \right. \\ &\quad \left. + \frac{\Gamma_1}{2} ([\vec{q}_e]_\times + q_0 \mathbb{I}_{3 \times 3}) \vec{\omega}_b \right) - \tilde{L}^T \Gamma_L^{-1} \dot{\hat{L}} \end{aligned} \quad (4.66a)$$

And using a similar control law to $\mu \vec{\tau}_{IBC}$, which has a disturbance estimate compensation term:

$$\mu \vec{\tau}_{ABC} = \vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b - \vec{\tau}_Q - \vec{\tau}_g - \vec{Q} - \hat{L} - \mathbb{I}_b \Gamma_1 \vec{q}_e - \frac{\Gamma_1 \mathbb{I}_b}{2} ([\vec{q}_e]_\times + q_0 \mathbb{I}_{3 \times 3}) \vec{\omega}_b - \mathbb{I}_b \Gamma_2 z_1 \quad (4.67a)$$

Which reduces the energy function's derivative to:

$$\dot{V}_{ABC} = -\vec{q}_e^T \Gamma_1 \vec{q}_e - z_1^T \Gamma_2 z_1 + z_1^T \Gamma_1^{-1} \left(\mathbb{I}_b^{-1} (\vec{L} - \hat{L}) \right) - \tilde{L}^T \Gamma_L^{-1} \dot{\hat{L}} \quad (4.67b)$$

$$= -\vec{q}_e^T \Gamma_1 \vec{q}_e - z_1^T \Gamma_2 z_1 + z_1^T (\Gamma_1^{-1} \mathbb{I}_b^{-1}) \tilde{L} - \tilde{L}^T \Gamma_L^{-1} \dot{\hat{L}} \quad (4.67c)$$

$$= -\vec{q}_e^T \Gamma_1 \vec{q}_e - z_1^T \Gamma_2 z_1 + \tilde{L}^T \Gamma_L^{-1} (\Gamma_1^{-1} \Gamma_L \mathbb{I}_b^{-1} z_1 - \dot{\hat{L}}) \quad (4.67d)$$

The decision then needs to be made as to how the disturbance estimate is going to be updated, or what $\dot{\hat{L}}$ is defined as. The clear choice would be to compensate for the final term in the LCF, making it purely negative definite:

$$\dot{\hat{L}} = \Gamma_1^{-1} \Gamma_L \mathbb{I}_b^{-1} z_1 = \Gamma_1^{-1} \Gamma_L \mathbb{I}_b^{-1} \vec{\omega}_b - \Gamma_L \mathbb{I}_b^{-1} \vec{q}_e \quad (4.68)$$

The disturbance is therefore compensated for and the estimate error is ensured to have asymptotic stability seeing that V_{ABC} is positive definite.

$$\dot{V}_{ABC} = -\vec{q}_e^T \Gamma_1 \vec{q}_e - z_1^T \Gamma_2 z_1 < \vec{0} \quad \forall (\vec{q}_e, z_1, \tilde{L}) \quad (4.69)$$

Exponential stability for the plant however cannot be proven with the above control and disturbance laws, there is no non-zero estimate error coefficient in the LCF derivative. A lot of work has been done on the statistical nature of disturbance approximation and how best to adapt a non-linear control system to the influence of unwanted disturbances. An interesting approach would be to use the previous disturbance estimate, $\vec{L} = \hat{L}_{n-1}$, such that:

$$\tilde{L}' = \vec{L} - \hat{L} = (\hat{L}_{n-1} - \hat{L}_n) \quad (4.70a)$$

$$\dot{\tilde{L}'} = \Gamma_1^{-1} \Gamma_L \mathbb{I}_b^{-1} z_1 + \tilde{L}' \quad (4.70b)$$

$$\dot{\hat{L}} = \Gamma_1^{-1} \Gamma_L \mathbb{I}_b^{-1} \vec{\omega}_b - \Gamma_L \mathbb{I}_b^{-1} \vec{q}_e + (\hat{L}_{n-1} - \hat{L}_n) \quad (4.70c)$$

$$\therefore \dot{V}'_{ABC} = -\vec{q}_e^T \Gamma_1 \vec{q}_e - z_1^T \Gamma_2 z_1 - \tilde{L}'^T \Gamma_L^{-1} \tilde{L}' \quad (4.70d)$$

Given that the starting estimate $\hat{L}_0 = \vec{0}$ and that the change of disturbance over a single control cycle is going to be small once the approximator has settled, its fair to assume the following:

$$\lim_{t \rightarrow \infty} \tilde{L}' = (\hat{L}_{n-1} - \hat{L}_n) \rightarrow \tilde{L} \quad (4.71)$$

Then, it leads to the following LCF derivative which can then prove exponential stability. It clear that a coefficient $\dot{V}_{ABC} \leq \alpha V_{ABC}$ exists and can be found:

$$\dot{V}_{ABC}(\vec{q}_e, z_1, \tilde{L}) = -\vec{q}_e^T \Gamma_1 \vec{q}_e - z_1^T \Gamma_2 z_1 - \tilde{L}^T \Gamma_L^{-1} \tilde{L} \quad (4.72)$$

The assumption in Eq:4.71 is going to need to be tested in simulation later in Chapter:5; the adaptive gain matrix Γ_L is something that will similarly need to be designed. For control coefficients a separate optimization loop will be run, later disturbance will be introduced and the adaptive gain will independently be attained and optimized.

4.6 Position Control

Only two control laws for position control are proposed. Due to the nature of Coriolis cross-coupling, an attitude plant can be stabilized independently from the position plant, the converse is however not true. A basic Proportional-Derivative control structure is presented as the reference case, thereafter a more complicated adaptive backstepping control algorithm is derived...

The dynamics for position control, Eq:3.131b, include a coupled angular velocity element.

$$\dot{\vec{v}}_b = m^{-1}(-\vec{\omega}_b \times m\vec{v}_b + Q_b^* \otimes m\vec{G}_I \otimes Q_b + \mu\vec{F}) \in \mathcal{F}^b \quad (4.73)$$

Typically, given the standard operating conditions of a quadrotor, it's assumed that $\vec{\omega}_b \approx \vec{0}$. As such the inherent angular velocity coupled dynamics are negligible; $\vec{\omega}_b \times m\vec{v}_b \approx 0$. If the entire state vector, both attitude and position $\mathbf{x}(t) = [\mathcal{E}, Q_b]^T$, of the plant is known then it's easy to compensate for those dynamics rather than making assumptions about their influence on the system given particular operating conditions. That plant dependency can be introduced in the control force $\mu\vec{F}$.

The translational velocity, \vec{v}_b , defined in the body frame is related to the inertial frame through a quaternion transformation:

$$\dot{\mathcal{E}} = Q_b \otimes \vec{v}_b \otimes Q_b^* \in \mathcal{F}^I \quad (4.74)$$

The difference in reference frames is an important distinction between the position and attitude control loops. Position error is calculated purely as a subtractive term:

$$\mathcal{E}_e = \mathcal{E}_d - \mathcal{E}_b \in \mathcal{F}^I \quad (4.75)$$

With $\mathcal{E}_d(t)$ being some desired position designed by the trajectory generation block. The translational velocity error can be similarly calculated but, in the same way angular velocity $\vec{\omega}_d = \vec{0}$, the desired translational velocity is zero.

$$\dot{\mathcal{E}}_e = \dot{\mathcal{E}}_d - \dot{\mathcal{E}}_b = -\dot{\mathcal{E}}_b \Big|_{\dot{\mathcal{E}}_d=\vec{0}} \quad (4.76)$$

The objective for position setpoint tracking is analogous to that of the attitude setpoint tracking. In particular the aim is to produce a stabilizing control law that ensures the position tracking error asymptotically tends to $\vec{0}$:

$$\mu\vec{F} = g(\mathcal{E}_e, \dot{\mathcal{E}}_e) \text{ such that } \lim_{t \rightarrow \infty} \mathcal{E}_e = \vec{0} \quad (4.77)$$

Where $\mu\vec{F}$ is the control force to effect Eq:4.73 $\in \mathcal{F}^b$.

4.6.1 PD Controller

Starting with a simple PD structure to use as a reference case. A plant dependent controller designs the net force proportional to both the position error and the first derivative velocity error¹⁴.

$$\mu\vec{F}_{PD} = K\dot{\mathcal{E}}_e + \alpha\mathcal{E}_e + \vec{\omega}_b \times m\vec{v}_b - m\vec{G}_b \in \mathcal{F}^b \quad (4.78)$$

For the stability proof the error states must be transformed to the body frame \mathcal{F}^b such that the control input and error states all act in a common frame. So defining an error state in the body frame X_e :

$$X_e = Q_b^* \otimes (\mathcal{E}_d - \mathcal{E}_b) \otimes Q_b = X_d - X_b \quad (4.79a)$$

$$\dot{X}_e = Q_b^* \otimes (\dot{\mathcal{E}}_d - \dot{\mathcal{E}}_b) \otimes Q_b = -Q_b^* \otimes \dot{\mathcal{E}}_b \otimes Q_b = -\vec{v}_b \Big|_{\dot{\mathcal{E}}_d=\vec{0}} \quad (4.79b)$$

As such the control law from Eq:4.78, despite being $\in \mathcal{F}^b$ has arguments $\mathcal{E}, \dot{\mathcal{E}} \in \mathcal{F}^I$, which must similarly transform to:

$$\mu\vec{F}_{PD} = K\dot{X}_e + \alpha X_e + \vec{\omega}_b \times m\vec{v}_b - m\vec{G}_b \quad (4.80a)$$

$$= -K\vec{v}_b + \alpha X_e + \vec{\omega} \times m\vec{v}_b - m\vec{G}_b \quad (4.80b)$$

Then using a p.d Lyapunov candidate function:

$$V_{PD}(X_e, \dot{X}_e) = \frac{\alpha}{2} X_e^T X_e + \frac{m}{2} \dot{X}_e^T \dot{X}_e = \frac{\alpha}{2} X_e^T X_e + \frac{m}{2} \vec{v}_b^T \vec{v}_b \quad (4.81)$$

¹⁴The same P and D coefficient symbols are used for continuity.

Then calculating the LCF derivative with the PD control law substituted:

$$\dot{V}_{PD} = \alpha X_e^T \dot{X}_e + \vec{v}_b^T m \dot{\vec{v}}_b = -\alpha X_e^T \vec{v}_b + \vec{v}_b^T m \dot{\vec{v}}_b \quad (4.82a)$$

$$= -\alpha X_e^T \vec{v}_b + \vec{v}_b^T (-\vec{\omega}_b \times m \vec{v}_b + m \vec{G}_b + \mu \vec{F}_{PD}) \quad (4.82b)$$

$$= -\alpha X_e^T \vec{v}_b + \vec{v}_b^T (-K \vec{v}_b + \alpha X_e) \quad (4.82c)$$

$$\Rightarrow \dot{V}_{PD} = -K \vec{v}_b^T \vec{v}_b < \vec{0} \quad \forall (X_e, \dot{X}_e) \quad (4.82d)$$

It then follows that the following global asymptotically¹⁵ stabilizing limits exist:

$$\lim_{t \rightarrow \infty} X_e = Q_b^* \otimes (\mathcal{E}_d - \mathcal{E}_b) \otimes Q_b = \vec{0} \quad (4.83a)$$

$$\therefore \lim_{t \rightarrow \infty} \mathcal{E}_b = \mathcal{E}_d \quad (4.83b)$$

$$\lim_{t \rightarrow \infty} \dot{X}_e = Q_b^* \otimes (\dot{\mathcal{E}}_d - \dot{\mathcal{E}}_b) \otimes Q_b = -\vec{v}_b \Big|_{\dot{\mathcal{E}}_e=0} = 0 \quad (4.83c)$$

4.6.2 Adaptive Backstepping Controller

An adaptive backstepping algorithm, similar the attitude controller derived previously in Sec:4.5.3, is now applied to position control. The disturbance term, $\vec{D} \in \mathcal{F}^b$, introduced to the differential Eq:4.73 represents any lumped drag and wind forces encountered by the body which weren't quantified numerically in Sec:3.2.3. The backstepping iterations of the position control loop first need to stabilize the position error and then compensate for those disturbances...

$$\dot{\vec{v}}_b = m^{-1} (-\vec{\omega}_b \times m \vec{v}_b + m \vec{G}_b + \vec{D}_b + \mu \vec{F}) \in \mathcal{F}^b \quad (4.84)$$

Obviously the compensation for \vec{D} is going to be an approximation of that physical disturbance term; \hat{D} . Beginning the backstepping process for position with the position state tracking error:

$$z_1 = \mathcal{E}_d - \mathcal{E}_b \quad (4.85)$$

Which then has its own derivative:

$$\dot{z}_1 = \dot{\mathcal{E}}_d - \dot{\mathcal{E}}_b = Q_b \otimes (\vec{0} - \vec{v}_b) \otimes Q_b^* = -Q_b \otimes \vec{v}_b \otimes Q_b^* \quad (4.86)$$

Transforming that error, z_1 , to the body frame \mathcal{F}^b , in the same way as Eq:4.79a, makes the stability proof more concise. That reference frame transformation doesn't affect the Lie derivative as the energy function's gradient depends on its partial derivative w.r.t it's positional trajectory only, namely $\mathcal{E}_e(t)$.

$$\hat{z}_1 = X_e = Q_b^* \otimes z_1 \otimes Q_b = Q_b^* \otimes (\mathcal{E}_d - \mathcal{E}_b) \otimes Q_b \quad (4.87a)$$

$$\therefore \dot{\hat{z}}_1 = Q_b^* \otimes \dot{z}_1 \otimes Q_b = Q_b^* \otimes (\dot{\mathcal{E}}_d - \dot{\mathcal{E}}_b) \otimes Q_b = -\vec{v}_b \quad (4.87b)$$

Then proposing the first positive definite LCF, $V_1(\hat{z}_1)$, in terms of that tracking error:

$$V_1(\hat{z}_1) = \frac{1}{2} \hat{z}_1^T \hat{z}_1 \quad (4.88a)$$

$$\Rightarrow \dot{V}_1 = \hat{z}_1^T \dot{\hat{z}}_1 = -\hat{z}_1^T \vec{v}_b \Big|_{\dot{\mathcal{E}}_d=\vec{0}} \quad (4.88b)$$

The first stabilizing velocity function¹⁶, Ω_d , and its associated error, \hat{z}_2 , can be defined as:

$$\vec{v}_b \Rightarrow \Omega_d = \Gamma_1 \hat{z}_1 \quad (4.89a)$$

¹⁵Not exponentially stabilizing however.

¹⁶Using Ω_d to differentiate from \vec{v}_d which would otherwise be the translational velocity produced from the desired trajectory...

$$\hat{z}_2 = \Omega_d - \vec{v}_b = \Gamma_1 \hat{z}_1 - \vec{v}_b \quad (4.89b)$$

$$\therefore \vec{v}_b = \Gamma_1 \hat{z}_1 - \hat{z}_2 \quad (4.89c)$$

So that second error state \hat{z}_2 has a derivative:

$$\dot{\hat{z}}_2 = \dot{\Omega}_d - \dot{\vec{v}}_b = \Gamma_1 \dot{\hat{z}}_1 - m^{-1}(-\vec{\omega}_b \times m\vec{v}_b + m\vec{G}_b + \vec{D}_b + \mu\vec{F}) \quad (4.90a)$$

$$= -\Gamma_1 \vec{v}_b - m^{-1}(-\vec{\omega}_b \times m\vec{v}_b + m\vec{G}_b + \vec{D}_b + \mu\vec{F}) \quad (4.90b)$$

Introducing that second error \hat{z}_2 into a new LCF V_2 :

$$V_2(\hat{z}_1, \hat{z}_2) = V_1(\hat{z}_1) + \frac{1}{2} \hat{z}_2^T \hat{z}_2 = \frac{1}{2} \hat{z}_1^T \hat{z}_1 + \frac{1}{2} \hat{z}_2^T \hat{z}_2 \quad (4.91)$$

Which has a derivative:

$$\dot{V}_2 = \hat{z}_1^T \dot{\hat{z}}_1 + \hat{z}_2^T \dot{\hat{z}}_2 = -\hat{z}_1^T \vec{v}_b + \hat{z}_2^T \dot{\hat{z}}_2 \quad (4.92a)$$

$$= -\hat{z}_1^T \vec{v}_b + \hat{z}_2^T \left(-\Gamma_1 \vec{v}_b - m^{-1}(-\vec{\omega}_b \times m\vec{v}_b + m\vec{G}_b + \vec{D}_b + \mu\vec{F}) \right) \quad (4.92b)$$

And substituting Eq:4.89c for \vec{v}_b into only the first energy term of the LCF derivative. Specifically; $-\hat{z}_1^T \vec{v}_b = -\hat{z}_1^T (\Gamma_1 \hat{z}_1 - \hat{z}_2)$. The remaining terms for \vec{v}_b are left unchanged:

$$= -\hat{z}_1^T (\Gamma_1 \hat{z}_1 - \hat{z}_2) + \hat{z}_2^T \left(-\Gamma_1 \vec{v}_b - m^{-1}(-\vec{\omega}_b \times m\vec{v}_b + m\vec{G}_b + \vec{D}_b + \mu\vec{F}) \right) \quad (4.92c)$$

$$= -\hat{z}_1^T \Gamma_1 \hat{z}_1 + \hat{z}_2^T \left(-\hat{z}_1 - \Gamma_1 \vec{v}_b - m^{-1}(-\vec{\omega}_b \times m\vec{v}_b + m\vec{G}_b + \vec{D}_b + \mu\vec{F}) \right) \quad (4.92d)$$

An ideal backstepping control law, with the assumption that \vec{D}_b is precisely known, is then:

$$\mu\vec{F}_{IBC} = \vec{\omega}_b \times m\vec{v}_b - m\vec{G}_b - \vec{D}_b - m\hat{z}_1 - m\Gamma_1 \vec{v}_b + m\Gamma_2 \hat{z}_2 \quad (4.93a)$$

$$= \vec{\omega}_b \times m\vec{v}_b - m\vec{G}_b - \vec{D}_b + (\Gamma_1 \Gamma_2 - m)\hat{z}_1 - m(\Gamma_1 + \Gamma_2)\vec{v}_b \quad (4.93b)$$

$$\Rightarrow \dot{V}_{IBC} = \dot{V}_2 = -\hat{z}_1^T \Gamma_1 \hat{z}_1 - \hat{z}_2^T \Gamma_2 \hat{z}_2 < 0 \quad \forall (\hat{z}_1, \hat{z}_2) \& \forall (z_1, z_2) \quad (4.93c)$$

Which clearly leads to asymptotic (*extended to exponential next*) stability under the assumption that the disturbance term \vec{D}_b is known and can be compensated for well. In the controller both Γ_1 & Γ_2 are symmetric positive definite control coefficient matrices to be determined later...

Adjusting the backstepping rule and proposed LCF to incorporate an adaptive disturbance approximation term \hat{D} , similar to the adaptive backstepping attitude controller previously in Sec:4.5.3. That approximation leads to an estimation error \tilde{D} , once again assuming the physical disturbance dynamics \vec{D}_b are far slower than the control dynamics; $\dot{\vec{D}}_b \ll \dot{\hat{D}}$.

$$\tilde{D} = \vec{D}_b - \hat{D} \in \mathcal{F}^b \quad (4.94a)$$

$$\dot{\tilde{D}} = \dot{\vec{D}}_b - \dot{\hat{D}} \approx \vec{0} - \dot{\hat{D}} = -\dot{\hat{D}} \quad (4.94b)$$

$$\rightarrow \mu\vec{F}_{ABC} = \vec{\omega}_b \times m\vec{v}_b - m\vec{G}_b - \hat{D} - m\hat{z}_1 - m\Gamma_1 \vec{v}_d + m\Gamma_2 \hat{z}_2 \quad (4.94c)$$

$$= \vec{\omega}_b \times m\vec{v}_b - m\vec{G}_b - \hat{D} + (\Gamma_1 \Gamma_2 - m)\hat{z}_1 - m(\Gamma_1 + \Gamma_2)\vec{v}_b \quad (4.94d)$$

Then proposing an LCF which includes that disturbance estimate error \tilde{D} and finding its derivative:

$$V_{ABC} = \frac{1}{2} \hat{z}_1^T \hat{z}_1 + \frac{1}{2} \hat{z}_2^T \hat{z}_2 + \frac{1}{2} \tilde{D}^T \Gamma_D^{-1} \tilde{D} \quad (4.95a)$$

$$\Rightarrow \dot{V}_{ABC} = \hat{z}_1^T \dot{\hat{z}}_1 + \hat{z}_2^T \dot{\hat{z}}_2 + \tilde{D}^T \Gamma_D^{-1} \dot{\tilde{D}} \quad (4.95b)$$

$$= -\hat{z}_1^T \Gamma_1 \hat{z}_1 + \hat{z}_2^T \left(-\hat{z}_1 - \Gamma_1 \vec{v}_b - m^{-1}(-\vec{\omega}_b \times m\vec{v}_b + m\vec{G}_b + \vec{D}_b + \mu\vec{F}_{ABC}) \right) - \tilde{D} \Gamma_D^{-1} \dot{\tilde{D}} \quad (4.95c)$$

$$= -\hat{z}_1^T \Gamma_1 \hat{z}_1 + \hat{z}_2^T \left(-\Gamma_2 \hat{z}_2 - m^{-1} (\vec{D}_b - \hat{D}) \right) - \tilde{D}^T \Gamma_D^{-1} \dot{\hat{D}} \quad (4.95d)$$

$$= -\hat{z}_1^T \Gamma_1 \hat{z}_1 - \hat{z}_2^T \Gamma_2 \hat{z}_2 - \hat{z}_2^T m^{-1} \tilde{D} - \tilde{D}^T \Gamma_D^{-1} \dot{\hat{D}} \quad (4.95e)$$

$$= -\hat{z}_1^T \Gamma_1 \hat{z}_1 - \hat{z}_2^T \Gamma_2 \hat{z}_2 - \tilde{D}^T \Gamma_D^{-1} (m^{-1} \Gamma_D \hat{z}_2 + \dot{\hat{D}}) \quad (4.95f)$$

Then, a self-evident choice for the disturbance update law would be; $\dot{\hat{D}} = -m^{-1} \Gamma_D \hat{z}_2$, which would ensure asymptotic stability. Expanding on that, an interesting solution which could potentially enforce exponential stability would be to use \hat{D}_{n-1} as a disturbance estimate:

$$\dot{\hat{D}} = -m^{-1} \Gamma_D \hat{z}_2 + (\hat{D}_{n-1} - \hat{D}) \quad (4.96a)$$

$$\therefore \dot{V}_{ABC} = -\hat{z}_1^T \Gamma_1 \hat{z} - \hat{z}_2^T \Gamma_2 \hat{z}_2 - \tilde{D}^T \Gamma_D^{-1} (\hat{D}_{n-1} - \hat{D}_n) \quad (4.96b)$$

$$\approx -\hat{z}_1^T \Gamma_1 \hat{z}_1 - \hat{z}_2^T \Gamma_2 \hat{z}_2 - \tilde{D}^T \Gamma_D^{-1} \tilde{D} < \vec{0} \quad \forall (\hat{z}_1, \hat{z}_2, \tilde{D}) \quad (4.96c)$$

Similarly for the suggested exponentially stable adaptive backstepping controller for the attitude plant, using \hat{D}_{n-1} for an existing disturbance estimate will need to be simulated in order to ascertain if it's a suitable conjecture. Note here that adaptive control laws refer to adaptability to unknown disturbances and not plant model uncertainty which would take a different approach to the backstepping algorithm.

4.7 Controller Allocation

Following the higher level control laws, a distribution algorithm is needed to *allocate* out the desired virtual control inputs, $\vec{\nu}_d = [\mu \vec{F} \ \mu \vec{r}]^T$, to commanded actuator positions, $u_c \in \mathbb{U}$. The allocation block, $B^\dagger(\mathbf{x}, \vec{\nu}_d, t)$, from the control loop, Fig:4.1, constructs physical actuator positions from the virtual control input. For regular, unconstrained control allocation the solution is posed as an optimization¹⁷ problem; aiming to minimize deviation between the virtual and commanded control inputs, $\vec{\nu}_d$ & $\vec{\nu}_c$ respectively.

$$\min_{u \in \mathbb{U}^{12}, s \in \mathbb{R}^6} (\|Q_s\|) \text{ such that } \vec{\nu}_d - \vec{\nu}_c = h(\mathbf{x}_e, t) - B(\mathbf{x}, t, u) = s \quad (4.97)$$

Where Q_s is some cost function to prioritize the slack variable, s , requirements. Typically that cost function will just be the L_2 norm of the slack. In Eq:4.97 a generalized controller, $h(\mathbf{x}_e, t)$, is used; in the context of a 6-DOF control loop. That controller would combined virtual control inputs $\mu \vec{F}$ and $\mu \vec{r}$ for position and attitude control respectively. In the over-actuated case, there exists an entire family of suitable actuator positions u which are all solutions to Eq:4.97. The over-allocation solution is to introduce a secondary cost function¹⁸, $J(\mathbf{x}, t, u)$, into the optimization problem of Eq:4.97.

$$\min_{u \in \mathbb{U}^{12}, s \in \mathbb{R}^6} (\|Q_s\| + J(\mathbf{x}, u, t)) \text{ such that } h(\mathbf{x}_e, t) - B(\mathbf{x}, u, t) = s \quad (4.98)$$

That secondary control objective $J(\mathbf{x}, t, u)$ and its associated *explicit* solution to Eq:4.98 is the subject of control allocation research. Not much work has been done on over-allocation for aerospace vehicles outside the field of satellite attitude control (Section:1.2.2 for examples). Often satellites are over actuated for the sake of fault tolerance and redundancy [6, 76]. Actuator rate constraints can be further introduced such that u is limited by Δu , constraining sequential actuator position changes.

$$\begin{aligned} \therefore \min_{u \in \mathbb{U}^{12}, s \in \mathbb{R}^6} (\|Q_s\| + J(\mathbf{x}, u, t)) \text{ s.t. } h(\mathbf{x}_e, t) - B(\mathbf{x}, u, t) = s \\ \text{subject to } u = u_{n-1} + \Delta u, \Delta u \in \mathbb{C} \end{aligned} \quad (4.99)$$

¹⁷Control Allocation by Johansen, et al. [2012] [60] and Control allocation for over actuated systems by Oppenheimer, et al. [2006] [91] both detail the nature of generalized nonlinear allocation loops.

¹⁸Or control objective.

Most control allocation paradigms assume a linear, multiplicative relationship with the effectiveness function, hence the abstraction layer which was introduced previously in Eq:4.7. The allocator effectiveness function, when abstracted to a linear matrix multiplication, reduces to:

$$\vec{\nu}_d = h(\mathbf{x}_e, t) \iff B(\mathbf{x}, u, t) = B(\mathbf{x}, t)u = \vec{\nu}_c \quad (4.100)$$

With $\vec{\nu}_d$ & $\vec{\nu}_c \in \mathbb{R}^n$, $u \in \mathbb{U} \in \mathbb{R}^m$, $B \in \mathbb{R}^{m \times n}$. That assumption makes addressing the allocation conceptually simpler, accommodating the use of inversion based allocation laws (Sec:4.7.1,4.7.2,4.7.3). That abstraction to a multiplicative relationship with decomposed thrust components was suggested in Eq:4.7. The subsequent rotation "inversion" function, $R^\dagger(\mathbf{x}, \vec{F}_i, t)$, to solve for physical actuator positions $(\Omega_i, \lambda_i, \alpha_i)$, is however as yet undefined.

Assuming for the moment there is some allocation rule that, from $\vec{\nu}_d$, designs well four decomposed stabilizing 3-dimensional thrust vectors $\vec{T}_{1 \rightarrow 4}$ to be produced by each motor module. It then follows that each of those four thrust vectors then relate¹⁹ to their individual associated actuator positions through a quaternion *rotation*:

$$\vec{T}_i = Q_{M_i} \otimes \vec{T}(\Omega_i) \otimes Q_{M_i}^* \in \mathcal{F}^b \quad (4.101a)$$

$$= Q_z(\sigma_i)Q_y(\alpha_i)Q_x(\lambda_i) \otimes \vec{T}(\Omega_i) \otimes Q_x^*(\lambda_i)Q_y^*(\alpha_i)Q_z^*(\sigma_i) \quad (4.101b)$$

Where each motor thrust vector, $\vec{T}(\Omega_i)$, is:

$$\vec{T}(\Omega_i) = \begin{bmatrix} 0 \\ 0 \\ T(\Omega_i) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ C_T(J)\rho\Omega_i^2 D^4 \end{bmatrix} \in \mathcal{F}^{M_i} \quad (4.101c)$$

In the transformation Eq:4.101a the angle σ_i is an orthogonal \hat{Z} transformation from $\mathcal{F}^b \rightarrow \mathcal{F}^{M_i''}$ from Eq:2.16a. The thrust function $T(\Omega_i)$ is the BEM theory equation using thrust coefficients, Eq:3.133a, in the direction of the rotor shaft's axis of rotation, bound to \hat{Z}_{M_i} . Seeing that quaternion rotation (*transformation*) operators change the reference frame whilst retaining the vector operand's magnitude, it follows that $T(\Omega_i)$, and by extension the propeller speed Ω_i , can be solved for:

$$|\vec{T}_i| = \sqrt{\|[T_x \ T_y \ T_z]\|} = \sqrt{T_x^2 + T_y^2 + T_z^2} = |T(\Omega_i)| = |C_T(J)\rho\Omega_i^2 D^4| \quad (4.102a)$$

$$\rightarrow \Omega_i = \sqrt{\frac{|\vec{T}_i|}{C_T(J)\rho D^4}} = \sqrt{\frac{\sqrt{T_x^2 + T_y^2 + T_z^2}}{C_T(J)\rho D^4}} \quad (4.102b)$$

Then reversing (or *undoing*) the transformation from motor module to body frame in Eq:4.101a:

$$\vec{T}(\Omega_i) = Q_z^*(\sigma_i)Q_y^*(\alpha_i)Q_x^*(\lambda_i) \otimes \vec{T}_i \otimes Q_x(\lambda_i)Q_y(\alpha_i)Q_z(\sigma_i) \in \mathcal{F}^{M_i} \quad (4.103a)$$

$$\rightarrow \vec{T}(\Omega_i) = Q_{M_i}^* \otimes \vec{T}_i \otimes Q_{M_i} \in \mathcal{F}^{M_i} \quad (4.103b)$$

Knowing only $\vec{T}(\Omega_i)$ and \vec{T}_i in the motor frame and body frame respectively requires solving for a quaternion which relates the two. If both vectors are of unit length, \hat{T}_i & $\hat{T}(\Omega_i)$; then the following relationship can be exploited to find a relative quaternion:

$$\hat{T}_i = \frac{\vec{T}_i}{|\vec{T}_i|} = \frac{\vec{T}_i}{\sqrt{T_x^2 + T_y^2 + T_z^2}} \in \mathcal{F}^b \quad (4.104a)$$

$$\hat{T}(\Omega_i) = \frac{\vec{T}(\Omega_i)}{|\vec{T}(\Omega_i)|} = \frac{\vec{T}(\Omega_i)}{|C_T(J)\rho\Omega^2 D^4|} = [0 \ 0 \ 1]^T \in \mathcal{F}^{M_i} \quad (4.104b)$$

$$Q_{M_i} = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} = \begin{bmatrix} 1 + \hat{T}_i \cdot \hat{T}(\Omega_i) \\ -\hat{T}_i \times \hat{T}(\Omega_i) \end{bmatrix} \quad (4.104c)$$

¹⁹Using the quaternion analogue of rotation $\mathcal{F}^{M_i} \rightarrow \mathcal{F}^b$ from Eq:2.16a.

Where Eq:4.104c is an extension of the inherent quaternion definition which rotates a vector around a single Euler axis, Eq:3.46c, when applied to two unit vectors. That quaternion can indeed be used to solve for relative pitch, roll and yaw Euler angles (Appendix:A.3). The problem is that Eq:4.104c solves for the **most direct, shortest path** rotation from one vector to another. In most cases, a sequenced Z-Y-X rotation is by no means the shortest possible path. As a result solutions for $[\phi, \theta, \psi]^T$ from Eq:4.104c won't be meaningful for trying to resolve suitable servo positions λ_i and α_i .

The associated $[\phi, \theta, \psi]^T$ solutions to Eq:A.15 are then of no consequence in trying to solve for sequence of rotation angles²⁰ $[\lambda_i, \alpha_i, \sigma_i]^T$. Furthermore, when considering a sequenced Z-Y-X quaternion, no further insight can be extracted without applying cumbersome trigonometric inversions;

$$Q_b = \begin{bmatrix} \cos \frac{\psi}{2} \\ 0 \\ 0 \\ \sin \frac{\psi}{2} \end{bmatrix} \otimes \begin{bmatrix} \cos \frac{\theta}{2} \\ 0 \\ \sin \frac{\theta}{2} \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \cos \frac{\phi}{2} \\ \sin \frac{\phi}{2} \\ 0 \\ 0 \end{bmatrix} \quad (4.105a)$$

$$= \begin{bmatrix} c \frac{\psi}{2} c \frac{\theta}{2} c \frac{\phi}{2} + s \frac{\psi}{2} s \frac{\theta}{2} s \frac{\phi}{2} \\ c \frac{\psi}{2} c \frac{\theta}{2} s \frac{\phi}{2} - s \frac{\psi}{2} s \frac{\theta}{2} c \frac{\phi}{2} \\ c \frac{\psi}{2} s \frac{\theta}{2} c \frac{\phi}{2} + s \frac{\psi}{2} c \frac{\theta}{2} s \frac{\phi}{2} \\ s \frac{\psi}{2} c \frac{\theta}{2} c \frac{\phi}{2} - c \frac{\psi}{2} s \frac{\theta}{2} s \frac{\phi}{2} \end{bmatrix} = \begin{bmatrix} q_0 \\ q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} \quad (4.105b)$$

$$\rightarrow \vec{T}_i = \begin{bmatrix} c \frac{\psi}{2} c \frac{\theta}{2} c \frac{\phi}{2} + s \frac{\psi}{2} s \frac{\theta}{2} s \frac{\phi}{2} \\ c \frac{\psi}{2} c \frac{\theta}{2} s \frac{\phi}{2} - s \frac{\psi}{2} s \frac{\theta}{2} c \frac{\phi}{2} \\ c \frac{\psi}{2} s \frac{\theta}{2} c \frac{\phi}{2} + s \frac{\psi}{2} c \frac{\theta}{2} s \frac{\phi}{2} \\ s \frac{\psi}{2} c \frac{\theta}{2} c \frac{\phi}{2} - c \frac{\psi}{2} s \frac{\theta}{2} s \frac{\phi}{2} \end{bmatrix} \otimes \vec{T}(\Omega_i) \otimes \begin{bmatrix} s \frac{\psi}{2} s \frac{\theta}{2} s \frac{\phi}{2} + c \frac{\psi}{2} c \frac{\theta}{2} c \frac{\phi}{2} \\ s \frac{\psi}{2} s \frac{\theta}{2} c \frac{\phi}{2} - c \frac{\psi}{2} c \frac{\theta}{2} s \frac{\phi}{2} \\ -c \frac{\psi}{2} s \frac{\theta}{2} c \frac{\phi}{2} - s \frac{\psi}{2} c \frac{\theta}{2} s \frac{\phi}{2} \\ c \frac{\psi}{2} s \frac{\theta}{2} s \frac{\phi}{2} - s \frac{\psi}{2} c \frac{\theta}{2} c \frac{\phi}{2} \end{bmatrix} \quad (4.105c)$$

Instead; returning to rotation matrices for the inverse transformation and reiterating that Euler angle equivalents for the servos are; $[\phi, \theta, \psi]^T \iff [\lambda_i, \alpha_i, \sigma_i]^T$. It then follows (where i^{th} motor subscripts $1 \rightarrow 4$ are implied):

$$\vec{T}_i = \begin{bmatrix} c\sigma & -s\sigma & 0 \\ s\sigma & c\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\alpha & 0 & s\alpha \\ 0 & 1 & 0 \\ -s\alpha & 0 & c\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\lambda & -s\lambda \\ 0 & s\lambda & c\lambda \end{bmatrix} \vec{T}(\Omega_i) \quad (4.106a)$$

$$\Rightarrow \vec{T}_i = \begin{bmatrix} c\sigma c\alpha & c\sigma s\alpha s\lambda - s\sigma c\lambda & c\sigma s\alpha c\lambda + s\sigma s\lambda \\ s\sigma c\alpha & s\sigma s\alpha s\lambda + c\sigma c\lambda & s\sigma s\alpha c\lambda - c\sigma s\lambda \\ -s\alpha & cas\lambda & cac\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ T(\Omega_i) \end{bmatrix} \quad (4.106b)$$

$$\Rightarrow \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} s\sigma s\lambda + c\sigma s\alpha c\lambda \\ s\sigma s\alpha c\lambda - c\sigma s\alpha \\ cac\lambda \end{bmatrix} T(\Omega_i) \quad (4.106c)$$

Where σ is an orthogonal multiple which rotates the vector about the \hat{Z}_b axis. The fact that the principle thrust vector $\vec{T}(\Omega_i)$ has only a \hat{Z}_{M_i} component in the motor frame makes the solution for servo angles dramatically less complex to solve. Then Eq:4.106c simplifies even further to the following four trigonometric relations respectively for each motor module:

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \left[\begin{bmatrix} sac\lambda \\ -s\lambda \\ cac\lambda \end{bmatrix}, \begin{bmatrix} s\lambda \\ sac\lambda \\ cac\lambda \end{bmatrix}, \begin{bmatrix} -sac\lambda \\ s\lambda \\ cac\lambda \end{bmatrix}, \begin{bmatrix} -s\lambda \\ -sac\lambda \\ cac\lambda \end{bmatrix} \right] T(\Omega_i) \quad \text{for } i \in [1, 2, 3, 4] \quad (4.107)$$

It then becomes a simple case of inverse trigonometry to solve for both λ_i and α_i respectively, for the example case of $i = 1$, the following holds true and can be similarly applied to the remaining motor modules.

²⁰ σ_i is already known to be an orthogonal multiplicate...

Using $T(\Omega_i) = \|\vec{T}_i\|$ and the four quadrant secondary arctangent function²¹, $\text{arctan2}(x, y)$, for both inversion solutions to get full quadrature²² results:

$$\lambda = \text{arctan2}\left(-T_y, \sqrt{\|\vec{T}_i\|^2 - T_y^2}\right) \quad (4.108a)$$

$$\alpha_i = \text{arctan2}(T_x, T_z) \quad (4.108b)$$

Therefore, the secondary component of the control allocation block, $R^\dagger(\mathbf{x}, \vec{T}_i, t)$ from Fig:4.2 is then summarized as a single rotation inversion function (for motor module $i = 1$):

$$\begin{bmatrix} \Omega_i \\ \lambda_i \\ \alpha_i \end{bmatrix} = R^\dagger(\mathbf{x}, \vec{T}_i, t) = \begin{bmatrix} \left(\sqrt{T_x^2 + T_y^2 + T_z^2}/C_T(J)\rho D^4\right)^{\frac{1}{2}} \\ \text{atan2}(T_x^2, \|\vec{T}_i\| \sqrt{\|\vec{T}_i\|^2 - T_x^2}) \\ -\text{atan2}(T_x, T_z \|\vec{T}_i\|) \end{bmatrix} \quad (4.109)$$

Further simplifications could be drawn from the definitions of each element included in Eq:4.109, but it would just be superfluous as each servo angle can be solved for. At this stage the only remaining undefined component of the entire control block is the abstracted allocation algorithms, $B^\dagger(\mathbf{x}, \vec{\nu}_d, t)$, which are now addressed...

4.7.1 Pseudo Inverse Allocator

Conceptually the simplest control allocation solutions to Eq:4.98 stems from what are categorized as "inversion" based cost optimizations (the first three proposed allocators in Sec:4.7.1, 4.7.2 & 4.7.3). As alluded to previously, the requirements for inversion based allocation is that the effectiveness function $B(\mathbf{x}, u, t)$ is a linear relationship which can be abstracted to $B(\mathbf{x}, t)u$. The objective of inversion is that for the control problem $\vec{\nu}_c = B(\mathbf{x}, t)u$ to find some matrix $B^\dagger(\mathbf{x}, t)$ such that for some $\vec{\nu}_d$:

$$\vec{\nu}_c = \vec{\nu}_d = B(\mathbf{x}, t)u \Rightarrow B^\dagger(\mathbf{x}, t)\vec{\nu}_d = B^\dagger(\mathbf{x}, t)B(\mathbf{x}, t)u \quad (4.110a)$$

With the inversion identity:

$$B(\mathbf{x}, t)B^\dagger(\mathbf{x}, t) = \mathbb{I}_{m \times m} \quad (4.110b)$$

$$\rightarrow u = B^\dagger(\mathbf{x}, t)\vec{\nu}_d \quad (4.110c)$$

Or more generally, without the dependency of linearity:

$$u = B^\dagger(\mathbf{x}, \vec{\nu}_d, t) \quad (4.110d)$$

Where $B(\mathbf{x}, t) \in \mathbb{R}^{m \times n}$. When the B matrix has full rank; that being $m > n$, the inversion of B^\dagger , short of online iterative techniques to solve for the inversion, is not so trivial. A linear least squares technique is the basis of the direct inversion allocation schemes. The secondary control objective, $J(\mathbf{x}, u, t)$, is chosen to be a quadratic cost function that can be solved as an explicit least squares problem. The net effect of which aims to minimize controller effort (*magnitude*), such that:

$$J(\mathbf{x}, u, t) = \min_{u \in \mathbb{U}} \frac{1}{2} (u - u_p)^T W (u - u_p) \text{ such that } \vec{\nu}_c = B(\mathbf{x}, u, t) \quad (4.111)$$

Its worth mentioning that Eq:4.111 has no slack variable to be optimized, unlike Eq:4.98. The matrix W is a positive definite matrix of weighting²³ coefficients that prioritises different actuators in the actuator matrix u higher than others. Similarly u_p is the preferred actuator position matrix²⁴ which the system tends towards.

²¹More on the arctan2 function in Appendix:A.3.

²²Exploiting the fact that $\text{arctan}(x) = \text{arcsin}(x/\sqrt{1-x^2})$.

²³Discussed in full next in Sec:4.7.2.

²⁴Priority acutator positions are detailed in Sec:4.7.3.

The least squares solution [42] to Eq:4.111 for that inversion matrix $B^\dagger(\mathbf{x}, t)$ is then:

$$u \in \mathbb{U} = (\mathbb{I} - CB(\mathbf{x}, t))u_p + C\vec{\nu}_d \quad (4.112a)$$

$$C = W^{-1}B^T(\mathbf{x}, t)(B(\mathbf{x}, t)W^{-1}B^T(\mathbf{x}, t))^{-1} \quad (4.112b)$$

The solution in Eq:4.112 is referred to as the generalized inverse with weighted and preferred actuators positions. In the special case where there are no weightings, $W = \mathbb{I}_{n \times n}$, and no preferred actuator values are given, $u_p = \vec{0}$, the solution reduces to:

$$u = B^T(\mathbf{x}, t)(B(\mathbf{x}, t).B^T(\mathbf{x}, t))^{-1}\vec{\nu}_d \quad (4.113a)$$

$$= B^\dagger(\mathbf{x}, t)\vec{\nu}_d \quad (4.113b)$$

The simplified unique case of Eq:4.113 is termed a Moore-Penrose or pseudo-inversion of the actuator effectiveness matrix $B(\mathbf{x}, t)$. The pseudo-inversion is the most basic of allocation techniques, with a least squares minimization of controller effort. For an effectiveness $B(\mathbf{x}, t)$ matrix from Eq:4.8 relating to the layout described in Fig:2.9, the pseudo-inversion allocator is:

$$B(\mathbf{x}, t) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & 0 & -L \\ 0 & 0 & -L & 0 & 0 & 0 & 0 & 0 & L & 0 & 0 & 0 \\ 0 & L & 0 & -L & 0 & 0 & 0 & -L & 0 & L & 0 & 0 \end{bmatrix} \in \mathbb{R}^{12 \times 6} \quad (4.114a)$$

$$\Rightarrow B^\dagger(\mathbf{x}, t) = B^T(B.B^T)^{-1} \quad (4.114b)$$

$$= \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4L} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{-1}{2L} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{-1}{4L} \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2L} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{-1}{4L} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2L} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4L} \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{-1}{2L} & 0 & 0 \end{bmatrix} \in \mathbb{R}^{6 \times 12} \quad (4.114c)$$

Such that the pseudo-inversion allocation rule $u = B^\dagger(\mathbf{x}, t)\vec{\nu}_d$ produces a feasible set of control thrust vectors, $\vec{T}_{1 \rightarrow 4}$, for some virtual control input $\vec{\nu}_d = h(\mathbf{x}_e, t)$. Those thrust vectors, numbered $1 \rightarrow 4$, are then solved for as explicit actuator positions $[\Omega_i, \lambda_i, \alpha_i]^T = R^\dagger(\mathbf{x}, \vec{T}_i, t)$ to construct the actuator matrix $u \in \mathbb{U} \in \mathbb{R}^{12}$. Noting that $B(\mathbf{x}, t)$ does not necessarily have to be static with respect to either the state vector \mathbf{x} or time t . The allocation rule in Eq:4.114 is the most simplified case of the least squares quadratically optimized equation for Eq:4.98 and is used as the base reference allocation law to which all other proposed rules are compared against.

The direct (*pseudo*) inversion solution guarantees the commanded virtual control input is met and that actuators aren't necessarily saturated. In certain cases it may be desired to completely saturate certain actuators before exploiting other actuator plant inputs. That would entail an iterative "daisy chaining" [60] allocation to be performed numerically online, enforcing saturation for atleast some actuators and achievement of control objectives. That particular approach is avoided here as completely saturating an actuator isn't desirable; in the context of thrust generation (or vectoring) with propeller's saturation is something to be avoided...

4.7.2 Weighted Pseudo Inverse Allocator

A weighted inversion still treats the preferred actuator positions as negligible, or that $u_p = \vec{0}$ in Eq:4.112, but adds priority to different actuators in the form of a W matrix. The positive definite (*symmetrical*) weighting matrix is square with respect to the actuator dimension, $W \in \mathbb{R}^{12 \times 12}$ (*or more generally* $W \in \mathbb{R}^{m \times m}$). The Moore-Penrose inversion assumes that all actuators are equally weighted and purely diagonal, $W = \mathbb{I}$. A time dependent adaptive weighting matrix could prioritize actuators following control faults or actuator deterioration. The control objective of a weighted inversion is to design the explicit weighting coefficients as per some preferred heuristic or optimization²⁵.

Each weighting coefficient determines how the least squares solution to Eq:4.98 preferentially biases a particular actuator, in this case the weighting matrix's divisions correlate to mixed actuator thrust vector values. The 3×3 diagonal groupings $W_{1 \rightarrow 4}$ relate to individual thrust component biasing (T_{ix}, T_{iy}, T_{iz}) whilst off-centre 3×3 groupings mix separate thrust terms $\vec{T}_{1 \rightarrow 4}$.

Pseudo-inversion, previously, will exactly match the virtual control input $\vec{\nu}_d = B(\mathbf{x}, u, t) = \vec{\nu}_c$ so long as the actuators aren't saturated. Biasing actuators with an explicit weighting matrix could otherwise introduce a slack between the desired control requirements and their commanded counterparts. Such a case could result in instability given that trajectory tracking is stabilized through Lyapunov's theorem in the design of $\vec{\nu}_d$, not solving for allocated actuator positions. Short of iteratively²⁶ processing variable weights until a viable solution is found, a constraint on the nature of the weighting matrix needs to be introduced.

$$\begin{aligned} \vec{T}_1 &\Downarrow & \vec{T}_2 &\Downarrow & \vec{T}_3 &\Downarrow & \vec{T}_4 &\Downarrow \\ \vec{T}_1 &\Rightarrow \begin{bmatrix} W_{1:1}W_{1:2}W_{1:3} \\ W_{1:4}W_{1:5}W_{1:6} \\ W_{1:7}W_{1:8}W_{1:9} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} W_{5:1}W_{5:2}W_{5:3} \\ W_{5:4}W_{5:5}W_{5:6} \\ W_{5:7}W_{5:8}W_{5:9} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \vec{T}_2 &\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} W_{2:1}W_{2:2}W_{2:3} \\ W_{2:4}W_{2:5}W_{2:6} \\ W_{2:7}W_{2:8}W_{2:9} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} W_{6:1}W_{6:2}W_{6:3} \\ W_{6:4}W_{6:5}W_{6:6} \\ W_{6:7}W_{6:8}W_{6:9} \end{bmatrix} \\ \vec{T}_3 &\Rightarrow \begin{bmatrix} W_{5:1}W_{5:2}W_{5:3} \\ W_{5:4}W_{5:5}W_{5:6} \\ W_{5:7}W_{5:8}W_{5:9} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} W_{3:1}W_{3:2}W_{3:3} \\ W_{3:4}W_{3:5}W_{3:6} \\ W_{3:7}W_{3:8}W_{3:9} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \vec{T}_4 &\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} W_{6:1}W_{6:2}W_{6:3} \\ W_{6:4}W_{6:5}W_{6:6} \\ W_{6:7}W_{6:8}W_{6:9} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} W_{4:1}W_{4:2}W_{4:3} \\ W_{4:4}W_{4:5}W_{4:6} \\ W_{4:7}W_{4:8}W_{4:9} \end{bmatrix} \end{aligned}$$

Figure 4.3: Weighting matrix biasing

So long each horizontal and vertical weighting groups contributing to each thrust vector, $W_{T_i} \in \mathbb{R}^{3 \times 12}$, each have a unit norm, the designed control torque and force inputs will be met. Physically the resultant thrusts and torque (thrust differentials) would be balanced amongst similarly directed components. Furthermore, an additional restraint is that only permissible thrust vector mixings are between opposing pairs; $\vec{T}_1 \& \vec{T}_3$ and $\vec{T}_2 \& \vec{T}_4$. Such a constraint simplifies the time spent optimizing weighting coefficients in Sec:5.2.

²⁵Not considered or discussed are adaptive weightings as those are out of the scope of this work and pertain more to FTC [6].

²⁶Online iterative solutions are avoided given their increased computational complexity and the possibility that, given an infinite processing time, a solution may not necessarily be found.

The physical consequences of giving priority biasing to thrust vector components in the \hat{X}_b & \hat{Y}_b ²⁷ directions is that the allocation block prioritizes using pitch or roll servos, λ_i & α_i respectively, before changing the propeller's rotational speed Ω_i . Similarly balancing the off-diagonal thrust vector mixing blends controller effort amongst opposing actuators.

The explicit weighting coefficients are to be optimized iteratively in simulation, Sec:5.2; aiming to minimize some performance metric. That metric, which evaluates relative performance of a proposed set of weighting coefficients, is penalized²⁸ from actuator slew rate times and a slack variable norm;

$$\int (a \|t^{\nu_d - \nu_c} - 1\| + b \|s\|) dt \quad (4.115)$$

Where the integral is run until $t \rightarrow \infty$ over the length of a single simulation cycle. As such, the weighting matrix coefficients try to reduce the transient time taken for the actuator block to settle whilst ensuring stability isn't compromised. Optimization iterations for the weight coefficients are completely independent from the controller coefficient loops to be run in Sec:5.1...

4.7.3 Priority Norm Inverse Allocator

The last allocator based on typical inversion applies a non-zero preferred actuator position from Eq:4.98; or that $u_p \neq \vec{0} \in \mathbb{U}$. The preferred actuator position is the matrix value of u which the allocator naturally tends toward. An obvious choice for that value are the conditions required for stable hovering, those which simply keep the quadcopter airborne. There are however some intricacies which must be discussed with respect to what hovering conditions are.

For a generalized body of weight m , a net gravitational force opposes upward movement in the inertial frame; $\vec{M} = [0, 0, -G.m]^T \in \mathcal{F}^I$. At the hover state there are no net forces or torques²⁹ acting on the system, all dynamics are balanced. As such the hovering conditions are then simply:

$$\begin{bmatrix} \mu \vec{F}_p \\ \mu \vec{\tau}_p \end{bmatrix} = \begin{bmatrix} \vec{M} \\ \vec{0} \end{bmatrix} \in \mathcal{F}^I \quad (4.116)$$

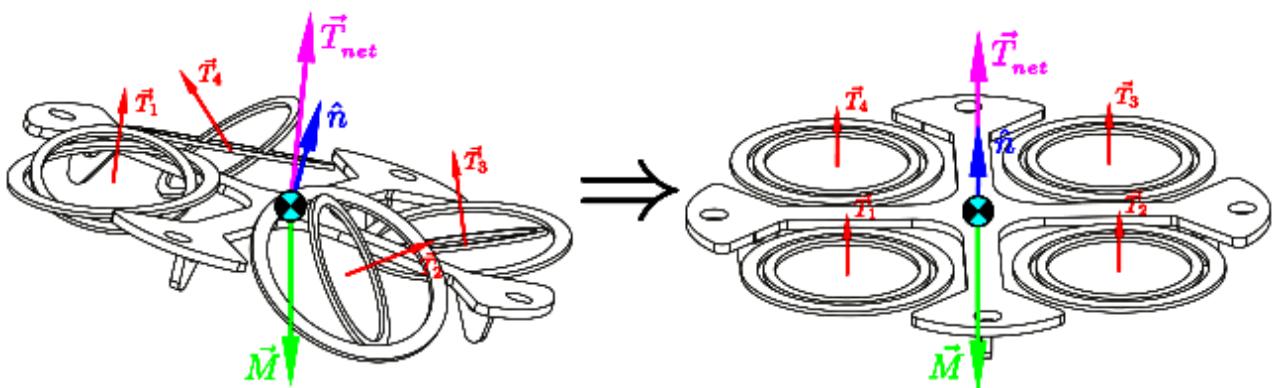


Figure 4.4: Hover conditions W.R.T the inertial frame \mathcal{F}^I

²⁷Recalling that the allocator block designs \vec{T}_{1-4} in the body frame, $\in \mathcal{F}^b$. Then the rotation inversion block $R^\dagger(\mathbf{x}, \vec{T}_i, t)$ from Eq:4.109 finds $(\Omega_i, \lambda_i, \alpha_i)$ to transform $\vec{T}(\Omega_i)$ to the body frame; effectively mapping $\mathcal{F}^{M_i} \rightarrow \mathcal{F}^b$.

²⁸More on simulations and optimizations next in Chapter:5-Simulations & Results.

²⁹Unwanted system dynamics like torques from an eccentric gravitational center or disturbances are compensated for in a plant dependent control law $\mu \vec{\tau} = h(\mathbf{x}_e, t)$.

However, calculating hover conditions purely in the inertial frame give no indication on what attitude the body has. Two options present themselves on how to solve for hover values. First; take hover conditions with respect to the inertial frame, such that the body's attitude tends to the origin always. The free body diagram in Fig:4.4 illustrates this.

$$\vec{\nu}_I = \begin{bmatrix} \mu \vec{F}_p \\ \mu \vec{\tau}_p \end{bmatrix} = \begin{bmatrix} \vec{M} \\ \vec{0} \end{bmatrix} \in \mathcal{F}^b \quad (4.117)$$

Conversely, the second option is to take hover conditions with respect to the body frame (Fig:4.5). The difference is that the body's preferred actuator positions are dependent on each instantaneous orientation. That attitude stays constant whilst the actuators are redirected to produce inertial hovering conditions; irrespective of the attitude. The preferred hovering conditions are then always dependent on the commanded attitude trajectory.

$$\vec{\nu}_b = \begin{bmatrix} \mu \vec{F}_p \\ \mu \vec{\tau}_p \end{bmatrix} = \begin{bmatrix} Q_b^* \otimes \vec{M} \otimes Q_b \\ \vec{0} \end{bmatrix} \in \mathcal{F}^b \quad (4.118)$$

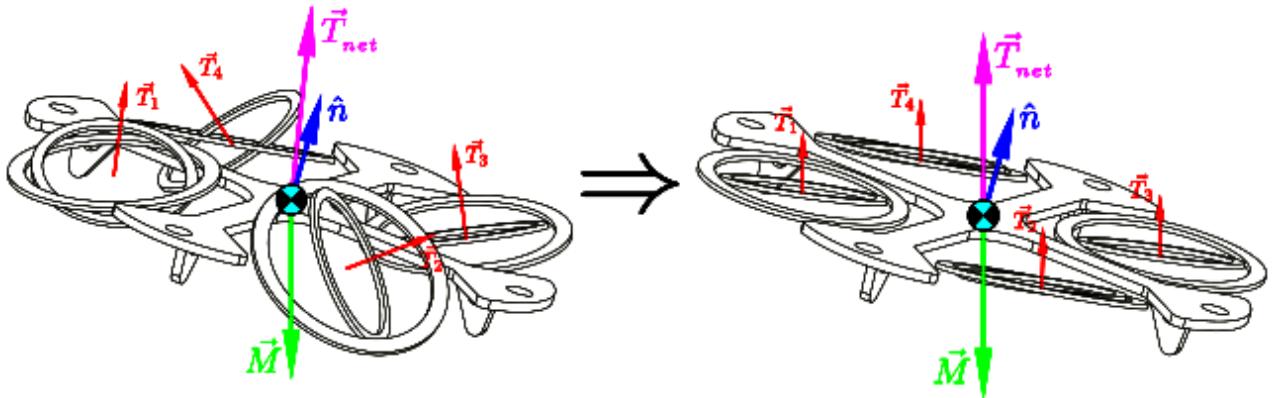


Figure 4.5: Hover conditions W.R.T the body frame \mathcal{F}^b

Explicit actuator positions are then solved for Eq:4.117 & Eq:4.118 using pseudo inversion from Eq:4.113. The two solutions are then as follows:

$$u_p^I = R^\dagger(\mathbf{x}, (B^\dagger(\mathbf{x}, \vec{\nu}_I, t)), t) \quad (4.119a)$$

$$u_p^b = R^\dagger(\mathbf{x}, (B^\dagger(\mathbf{x}, \vec{\nu}_b, t)), t) \quad (4.119b)$$

Where the inverse rotation operator, R^\dagger from Eq:4.119, is applied to all four thrust vectors produced by the allocation operator B^\dagger . Both actuator matrices are then applied to Eq:4.112 and could be combined with a non-diagonal weighting matrix.

$$u_{\mathbb{U}} = (\mathbb{I} - CB(\mathbf{x}, t))u_p + C\vec{\nu}_d \quad (4.120a)$$

$$C = W^{-1}B^T(\mathbf{x}, t)(B(\mathbf{x}, t)W^{-1}B^T(\mathbf{x}, t))^{-1} \quad (4.120b)$$

The physical consequences of either preferred actuator positions are demonstrated in simulation in Sec:5.4.3. Priority actuator positions aren't simulated together with weighting matrices, the two are compared independently...

4.7.4 Non-linear Plant Control Allocation

Despite the added actuation, each complex dynamic response from an actuator excitation is not fully exploited. The dynamics of an actuator's motor module, Sec:3.3.1, until now has been treated as an element to be compensated for in feedback structure. An alternative approach, seen in Gasco, et al. [2012] [2, 38], is to use the actuator reactions as additional non-linear actuator plants. In [2, 38] the actuator plants and their resultant dynamics were introduced as additional dimensions to the actuator matrix $u \in \mathbb{U}$.

Such an approach was achievable because the authors, despite adding two extra degrees of freedom for each propeller, hadn't vectored the propeller thrust. The non-linear proposal here is to first calculate a Pseudo-inversion actuator solution *without* plant compensation³⁰, then introducing those induced actuator responses from such an excitation to alleviate the control plant requirement. A subsequent revised virtual control plant input is used iteratively to find a subsequent pseudo-inversion solution; the process is cycled until the control requirements are met.

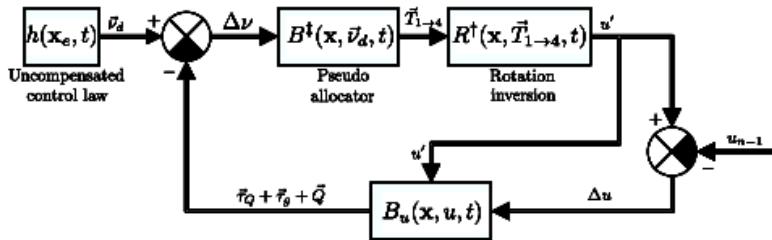


Figure 4.6: Allocation loop iteration

In Fig:4.6 the iteration loop is shown, each iteration is run online and settles at a balance point. In the loop, the block $B_u(\mathbf{x}, u, t)$ is a combination of non-linear actuator response terms from Eq:3.134,2.32d & 3.133b; those being $\vec{\tau}_Q$, $\vec{\tau}_g$ & \vec{Q} respectively. The settling point, where possible, a portion of the commanded control input \vec{v}_d is achieved from the otherwise compensated for actuator response dynamics.

4.7.5 Online Optimized Secondary Goal Allocator

³⁰Disregarding $\vec{\tau}_Q$, $\vec{\tau}_g$ & \vec{Q} .

Chapter 5

Simulations & Results

5.1 Controller Tuning

5.1.1 Partical Swarm Based Optimization

5.1.2 Performance Metric

5.1.3 Global & Local Minima

5.1.4 Fmincon Differences

5.1.5 PD

5.1.6 XPD

5.1.7 IBC

5.1.8 ABC

5.2 Allocator Optimization

5.3 Simulation Block

5.4 Optimized Controller Results

5.4.1 Attitude Control

5.4.2 Position Control

5.4.3 Allocator Performance

5.5 State Estimation

5.5.1 Attitude Control Results

5.5.2 Attitude Outputs

Chapter 6

Prototype Flight Results

Chapter 7

Conclusion

- Lagrange dynamics for multibody system could have produced a more concise model etc . . .
- Particule multibody dynamics with interactions could provide a more verbose simulation environment rather than the very newtonian simulation loop constructed
- Implicit equation dynamics in simulation may improve optimization loops in PSO algorithm - Firmware changes to ESC drastically improved transfer function time constant, made assumption that servos would improve actuator response times redundant. -Difficulty with non-linear multibody in Sec:3.3 causes stiffness in control optimization. Same troubles with time varying inertias caused Osprey inspiration issues too...

Appendix A

Expanded Equations

A.1 Standard Quadrotor Dynamics

Following the 6-DOF fundamental derivations in Section:3.1.1, the common reductions typically applied for a generic "+" configuration quadrotor are now presented. Reiterating those four differential equations, Eq:3.10, which describe a rigid body's motion (using rotation matrices and not quaternions):

$$\dot{\vec{v}} = \mathbb{R}_b^I(-\eta)\vec{v}_b \quad \in \mathcal{F}^I \quad (\text{A.1a})$$

$$\dot{\vec{v}}_b = m^{-1} [-\vec{\omega}_b \times m\vec{v} + m\mathbb{R}_I^b(-\eta)\vec{G}_I + \vec{F}_{net}] \quad \in \mathcal{F}^b \quad (\text{A.1b})$$

$$\dot{\vec{\eta}} = \Psi(\eta)\vec{\omega}_b \quad \in \mathcal{F}^{v2}, \mathcal{F}^{v1}, \mathcal{F}^I \quad (\text{A.1c})$$

$$\dot{\vec{\omega}}_b = \mathbb{I}_b^{-1} [-\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b + \vec{\tau}_{net}] \quad \in \mathcal{F}^b \quad (\text{A.1d})$$

With the Euler matrix, $\Psi(\eta)$, defined in Eq:2.12f. The net lift force produced by motors $i = [1 : 4]$, bound perpendicularly to the \hat{Z}_b axis, is given by:

$$\vec{T} = \sum_{i=1}^4 F_i \hat{k} \quad \in \mathcal{F}^b \quad (\text{A.2a})$$

The simplified relationship between the thrust force \vec{F}_i and the propellers rotational speed Ω_i is approximately quadratic:

$$\vec{F}_i = k_1 \Omega_i^2 \quad (\text{A.2b})$$

Similarly the aerodynamic torque opposing each rotating propeller is:

$$Q = k_2 \Omega_i^2 \quad (\text{A.3})$$

Coefficients k_1 & k_2 are typically determined from physical test results. The control pitch and roll torques, τ_ϕ & τ_θ respectively, are generated by opposing differential lift forces. Lastly the yaw torque is generated as a net response to the rotational aerodynamic propeller torques. The control torque inputs are then defined as:

$$\tau_\phi = \vec{L}_{arm}(\vec{F}_1 - \vec{F}_3) \quad (\text{A.4a})$$

$$\tau_\theta = \vec{L}_{arm}(\vec{F}_2 - \vec{F}_4) \quad (\text{A.4b})$$

$$\tau_\psi = \sum_{i=1}^4 (-1)^i k_2 \Omega_i \quad (\text{A.4c})$$

Then expanding the linear translational position and attitude differentials, Eq:A.1b & Eq:A.1d, to their component forms:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} + \begin{pmatrix} -g\sin(\theta) \\ g\cos(\theta)\sin(\phi) \\ g\cos(\theta)\cos(\phi) \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ T \end{pmatrix} \in \mathcal{F}^b \quad (\text{A.5a})$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{\mathbb{I}_y - \mathbb{I}_z}{\mathbb{I}_x} qr \\ \frac{\mathbb{I}_z - \mathbb{I}_x}{\mathbb{I}_y} pr \\ \frac{\mathbb{I}_x - \mathbb{I}_y}{\mathbb{I}_z} pq \end{pmatrix} + \mathbb{I}_b^{-1} \begin{pmatrix} \tau_\phi \\ \tau_\theta \\ \tau_\psi \end{pmatrix} \in \mathcal{F}^b \quad (\text{A.5b})$$

Considering the size of the average angular velocity $\omega_b \approx \vec{0}$, the gyroscopic and Coriolis effects on the body (namely both cross product terms) are depreciated and can be regarded as negligible. Assuming too that the body has a (*roughly*) diagonal inertial matrix. Then:

$$\begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} \approx \vec{0} \quad \text{and} \quad \begin{pmatrix} \frac{\mathbb{I}_y - \mathbb{I}_z}{\mathbb{I}_x} qr \\ \frac{\mathbb{I}_z - \mathbb{I}_x}{\mathbb{I}_y} pr \\ \frac{\mathbb{I}_x - \mathbb{I}_y}{\mathbb{I}_z} pq \end{pmatrix} \approx \vec{0} \quad (\text{A.6})$$

As a result, equations from Eq:A.5 then simplify to:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -g\sin(\theta) \\ g\cos(\theta)\sin(\phi) \\ g\cos(\theta)\cos(\phi) \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ T \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{\mathbb{I}_x} \tau_\phi \\ \frac{1}{\mathbb{I}_y} \tau_\theta \\ \frac{1}{\mathbb{I}_z} \tau_\psi \end{pmatrix} \quad (\text{A.7})$$

Similarly, around the origin and at hovering conditions, $\Psi(\eta) \approx \vec{1}$ for $\eta \approx \vec{0}$ and so from Eq:A.1c, then $\dot{\eta} \approx \omega_b$. Or in component form:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} \approx \begin{pmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{pmatrix} \quad (\text{A.8})$$

As such, the differential equations Eq:A.5 are then simplified to the following six SISO controllable plants:

$$\ddot{x} = (-\cos(\phi)\sin(\theta)\cos(\psi) - \sin(\phi)\sin(\psi)) \frac{1}{m} T \quad (\text{A.9a})$$

$$\ddot{y} = (-\cos(\phi)\sin(\theta)\sin(\psi) + \sin(\phi)\cos(\psi)) \frac{1}{m} T \quad (\text{A.9b})$$

$$\ddot{z} = g - (\cos(\phi)\cos(\theta)) \frac{1}{m} T \quad (\text{A.9c})$$

$$\ddot{\phi} = \frac{1}{\mathbb{I}_x} \tau_\phi \quad (\text{A.9d})$$

$$\ddot{\theta} = \frac{1}{\mathbb{I}_y} \tau_\theta \quad (\text{A.9e})$$

$$\ddot{\psi} = \frac{1}{\mathbb{I}_z} \tau_\psi \quad (\text{A.9f})$$

Typically, the simplified states in Eq:A.9 are abstracted to what is termed as an "augmented pilot control system". As such the controllable inputs are abstracted to T , $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$. Wherein the pilot can dictate the attitude and net heave thrust of the quadrotor, mostly with various flavours of PID control.

A.2 Blade-Element Momentum Expansion

Expanding on the Blade-Element Momentum equations from Eq:3.27 & Eq:3.31a. Reiterating the equations, they are:

$$dT = \rho 4\pi r v_\infty (1+a) a dr \quad (\text{A.10a})$$

$$dT = \frac{1}{2} a_L b c \rho (\Omega r)^2 \left(\theta - \frac{v_\infty + v_i}{\Omega r} \right) dr \quad (\text{A.10b})$$

Equating the two and defining an inflow ratio term $\lambda = \frac{v_\infty + v_i}{\Omega r} = \frac{v_\infty (1+a)}{\Omega r}$ yields the following quadratic equation:

$$\lambda^2 + \left(\frac{\sigma a_L}{8} + \lambda_c \right) \lambda - \frac{\sigma a_L}{8} \theta \frac{r}{R} = 0 \quad (\text{A.11})$$

Where λ_c is the nominal free-stream inflow ratio when $v_i = 0$. Another term, σ , is defined as the propeller solidity and is given by:

$$\sigma = \frac{bc}{\pi R} \quad (\text{A.12})$$

Then, solving Eq:A.11 for λ :

$$\lambda = \sqrt{\left(\frac{\sigma a_L}{16} - \frac{\lambda_c}{2} \right)^2 + \frac{\sigma a_L}{8} \theta \frac{r}{R}} - \left(\frac{\sigma a_L}{16} - \frac{\lambda_c}{2} \right) \quad (\text{A.13})$$

So then the inflow ratio can be solved as a function of the propeller element's aerofoil profile and its static inflow factor. In static conditions, the inflow factor is:

$$\lambda = \frac{v_i}{\Omega r} = \sqrt{\frac{C_{T0}}{2}} \quad (\text{A.14})$$

Then substituting λ back into Eq:3.31a and solving the integral produces an instantaneous thrust value. The difficulty of solving the blade-element momentum integrals is knowing the exact chord profile and local angle of attack.

A.3 Euler-Angles from Quaternions

The solution for Euler angles from an attitude quaternion is an easy trigonometric inversion. Noting that the transformation from the body frame to each motor frame follows the Z-Y-X sequence, and using an inversion solution adapted from [117], where the transformation to quaternions is based on Shoemake's [115] definition. Each quaternion can be constructed from sequenced Euler angles, as in Eq:3.52. Then, solving for each euler angle using simultaneous solutions and inverse trigonometry:

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \arctan2(2(q_0 q_x + q_y q_z), 1 - 2(q_x^2 + q_y^2)) \\ \arcsin(2(q_0 q_y - q_x q_z)) \\ \arctan2(2(q_0 q_z + q_x q_y), 1 - 2(q_y^2 + q_z^2)) \end{bmatrix} \quad (\text{A.15})$$

Where $\arctan2(x, y)$ is the four-quadrant tangent inverse [36], producing the principle argument of the complex operands;

$$\arctan2(x, y) = PR \arg(x + yi) = Arg(x + yi) \quad (\text{A.16})$$

The use of an full quadrature arctangent function is to find solutions for Euler angles that are not only acute.

Appendix B

Design Bill of Materials

B.1 Parts List

Part Name	No. Used	Unit Weight[g]
Electronics		
SPRacing F3 Deluxe Flight Controller	1	8
OrangeRx 615X 2.4 GHz 6CH Receiver	1	9.8
Signal Converter SBUS-PPM-PWM	1	5.0
STLink-V2 Debugger	1	3
RotorStar Super Mini S-BEC 10A	1	30
128x96" OLED Display	1	7
XBee-Pro S1	2	4
HobbyWing XRotor 20A Opto ESC	4	15
OrangeRX RPM Sensor	4	2
HobbyKing Multi-Rotor Power Distribution Board	1	49
Motors		
Corona DS-339MG	8	32
Cobra 2208 2000KV Brushless DC	4	44.2
Frame Components		
APM Flight Controller Damping Platform	1	7
HobbyKing SK450 Replacement Arm (2 pcs)	2	51
SK450 Extended Landing Skid	1	23.25
Alloy Servo Arm (FUTABA)	8	4
10X18X6 Radial Ball Bearing	8	5
80g Damping Ball	32	≈ 0
Plastic Retainers for Damping Balls	32	≈ 0
3/5mm Aluminum Prop Adapter	4	≈ 1
6x4.5 Gemfam 3-Blade Propeller	4	6
M3 6mm Hex Nylon Spacer	8	≈ 0
M3 16mm Hex Nylon Spacer	32	≈ 0
M3 25mm Nylon Screw	128	≈ 0.08
M2.5x10mm Socket Head Cap Screw	36	≈ 0.2
M2.5x25mm Socket Head Cap Screw	20	≈ 0.6
M2.5 A-Lok Nut	16	≈ 0

Table B.1: Parts List

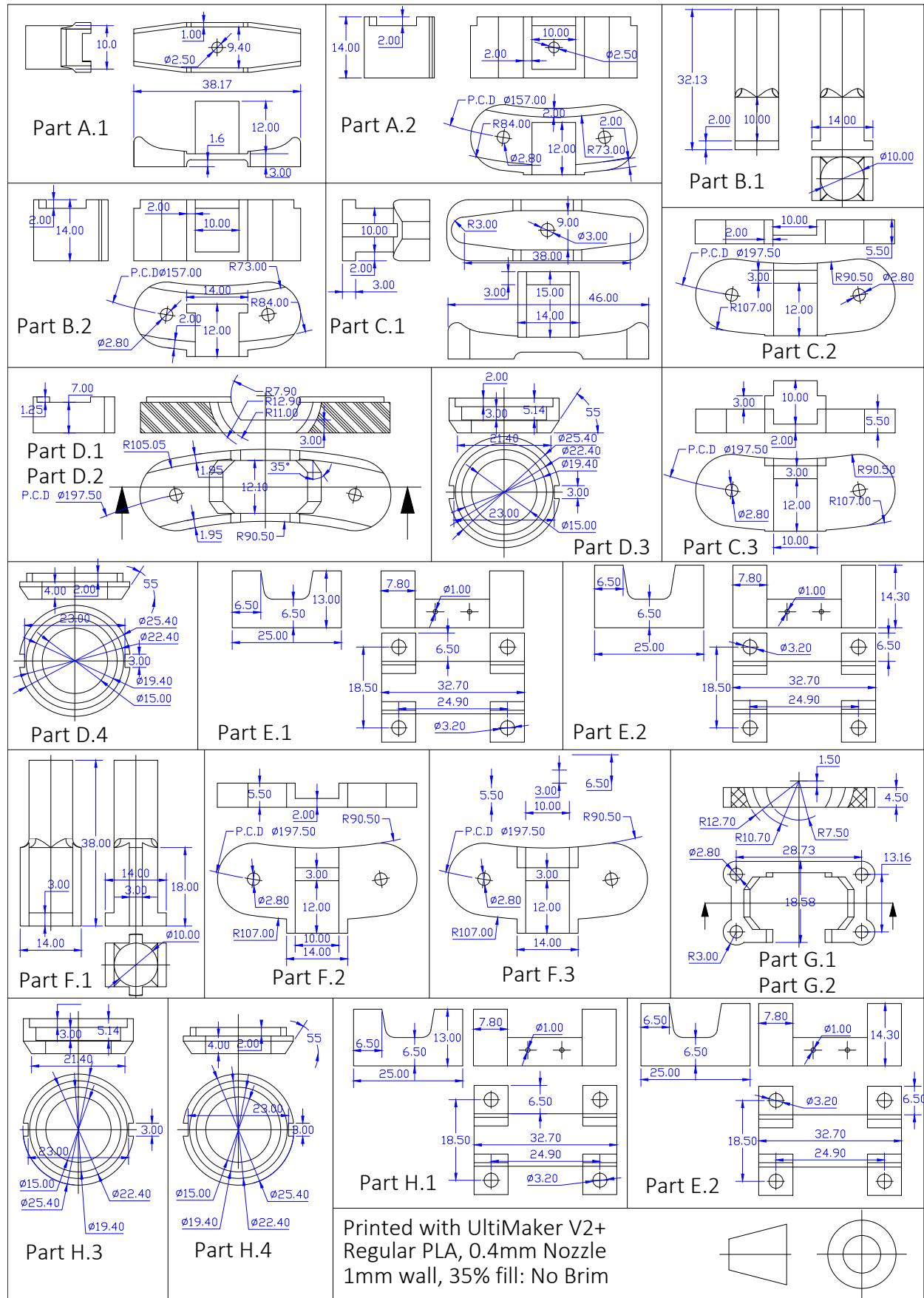


Table B.2: 3D Printed Parts

Bracket Assemblies 2

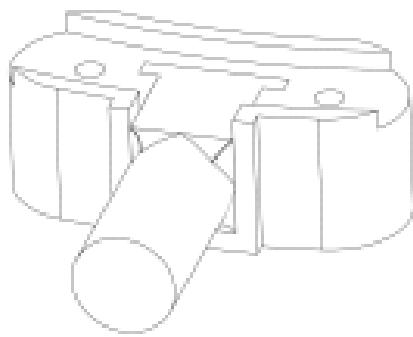


Figure B.1: Bearing Bracket Inner Ring Assembly
Parts: A.1, A.2

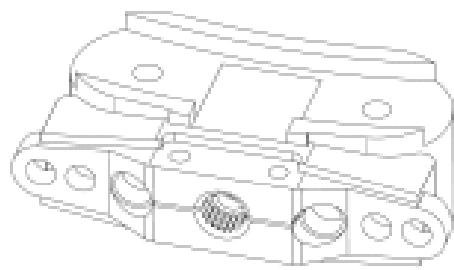


Figure B.2: Servo Bracket Inner Ring Assembly
Parts: B.1, B.2, M3 Servo Horn

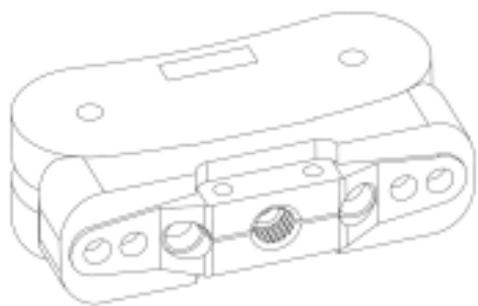


Figure B.3: Servo Bracket Middle Ring Assembly
Parts: C.1, C.2, C.3, M3 Servo Horn



Figure B.4: Bearing Holder Middle Ring Assembly
Parts: D.1, D.2, D.3, D.4, 18-10 Bearing

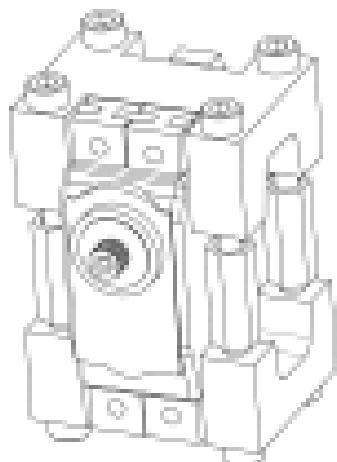


Figure B.5: Servo Mount Middle Ring Assembly
Parts: E.1, E.2, Corona Servo & Fasteners

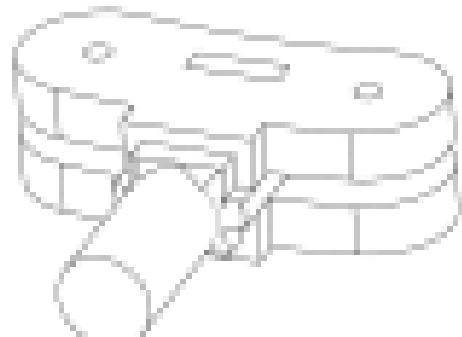


Figure B.6: Bearing Shaft Middle Ring Assembly
Parts: F.1, F.2, F.3

Table B.3: Inner & Middle Ring Assemblies

Bracket Assemblies 2

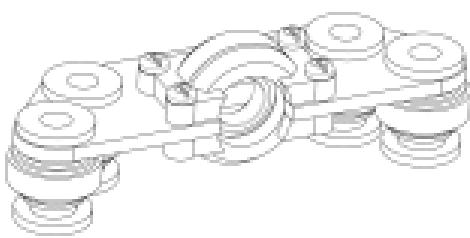


Figure B.7: Bearing Holder Damping Assembly
Parts: G.1, G.2, G.3, G.4, 18-10 Bearing, 80g Damping Balls, Bearing Holder Damping Bracket

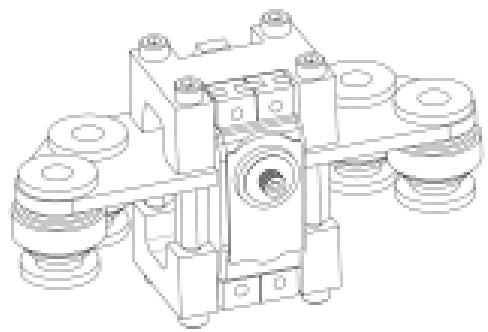


Figure B.8: Servo Mount Damping Assembly
Parts: H.1, H.2, Corona Servo & Fasteners, 80g Damping Balls, Servo Mount Damping Bracket

Table B.4: Damping Assemblies

Laser Cut Brackets

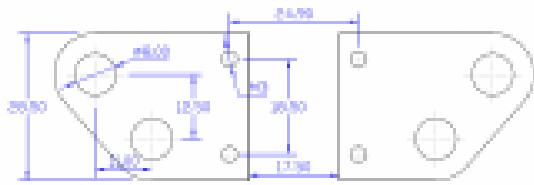


Figure B.9: Servo Mount Damping Bracket

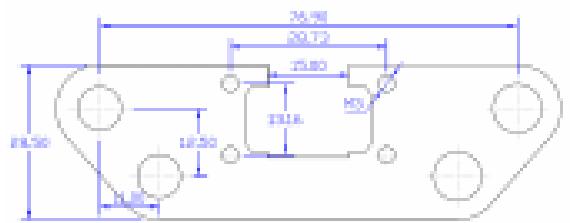


Figure B.10: Bearing Holder Damping Bracket

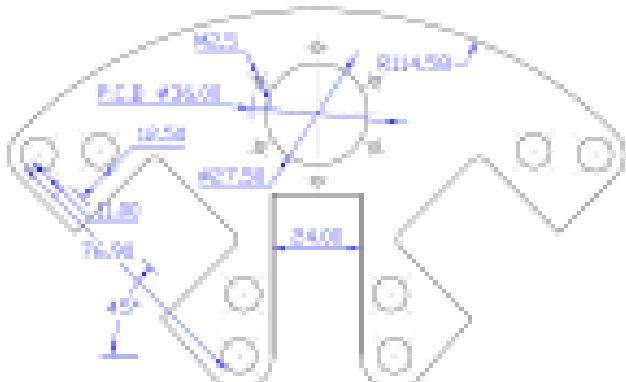


Figure B.11: Arm Mount Damping Bracket

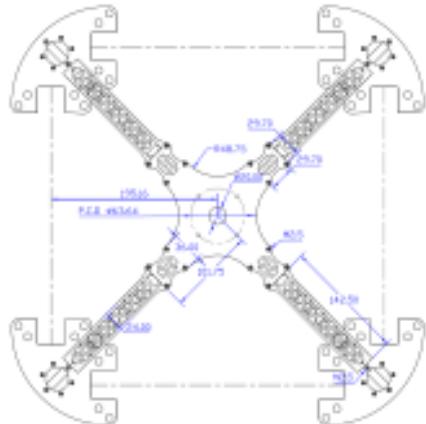


Figure B.12: Frame Brackets

Table B.5: Laser Cut Damping Brackets

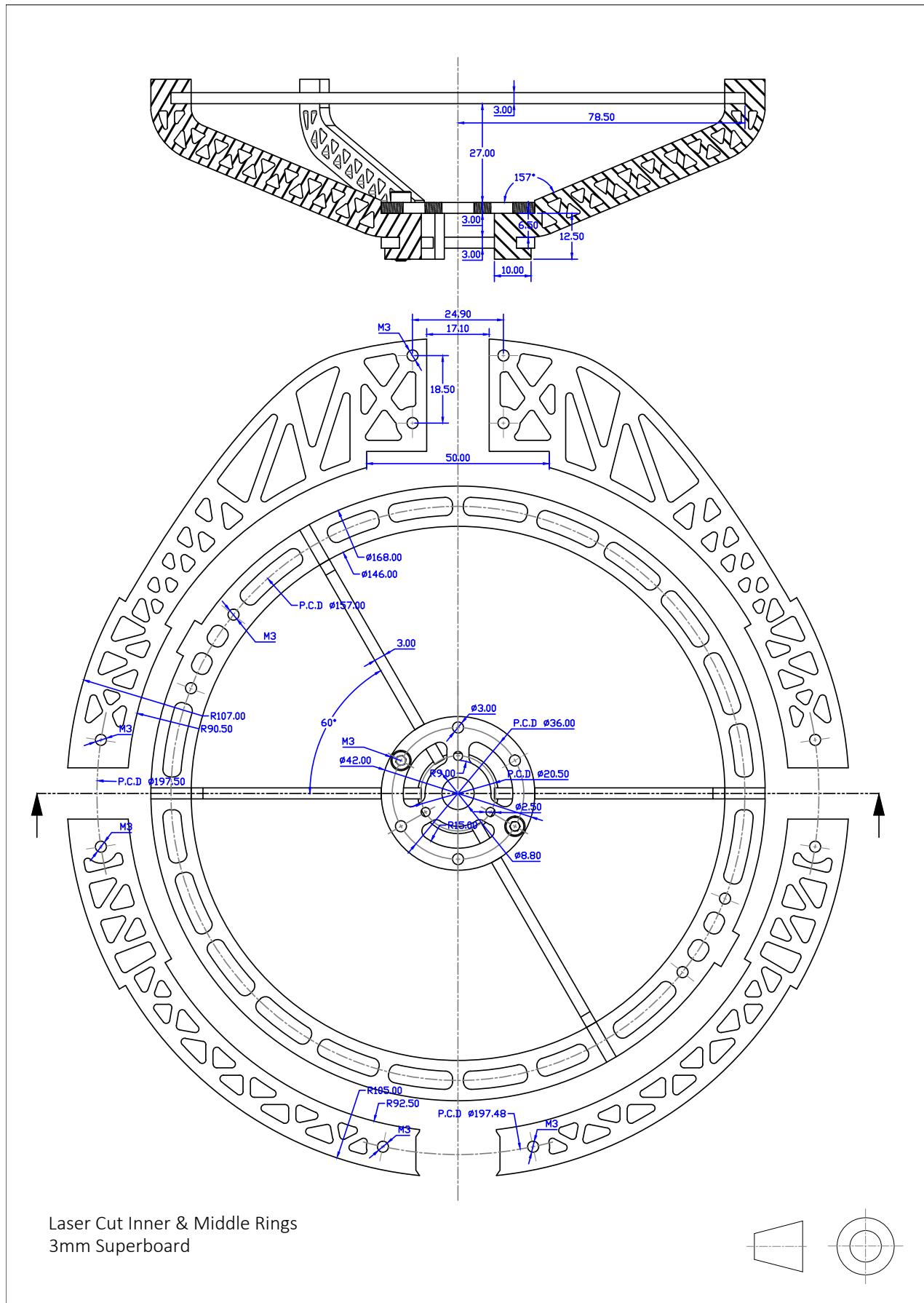


Table B.6: Laser Cut Parts

B.2 F3 Deluxe Schematic Diagram

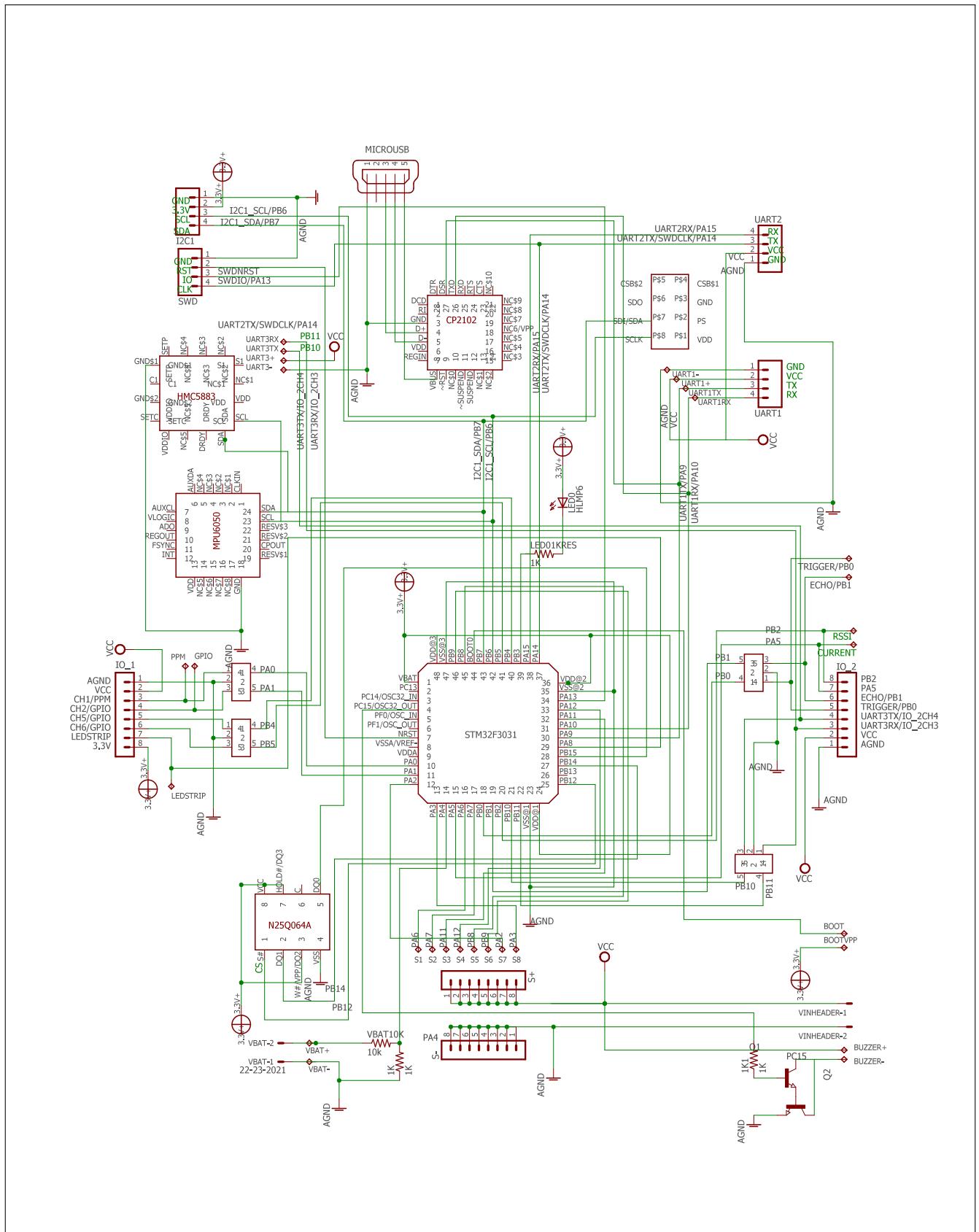
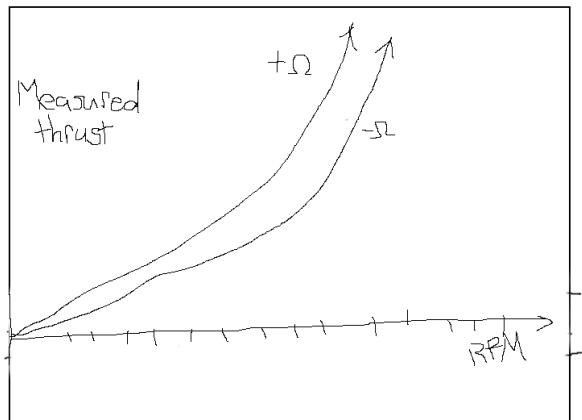


Figure B.13: F3 Deluxe Flight Controller Hardware Schematic

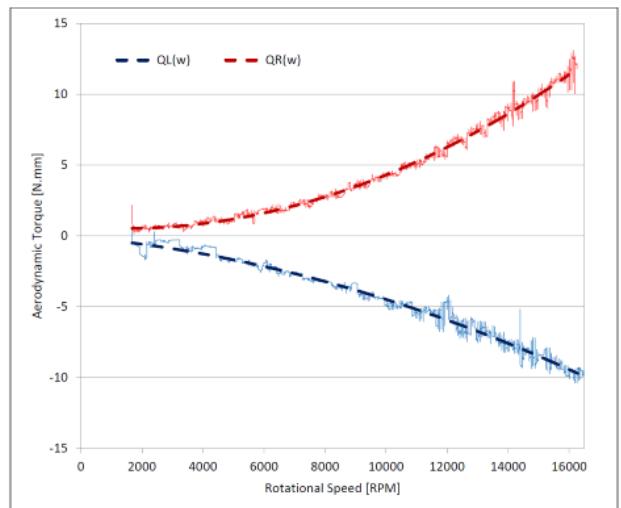
Appendix C

System ID Test Data

C.1 Thrust and Torque Test Data



(a) Thrust tests



(b) Torque tests

Figure C.1: Clockwise and counterclockwise rotation tests

C.2 Cobra CM2208-200KV Thrust Data

Cobra CM-2208/20 Motor Propeller Data										
Magnets 14-Pole	Motor Wind 20-Turn Delta	Motor Kv 2000 RPM/Volt		No-Load Current $I_o = 0.77$ Amps @ 10v	Motor Resistance $R_m = 0.076$ Ohms		I Max 20 Amps	P Max (3S) 220 W		
Stator 12-Slot	Outside Diameter 27.7 mm, 1.091 in.	Body Length 24.0 mm, 0.945 in.		Total Shaft Length 45.2 mm, 1.780 in.	Shaft Diameter 3.17 mm, 0.125 in.		Motor Weight 44.2 gm, 1.56 oz			
Test Data From Sample Motor		Input	6.0 V	8.0 V	10.0V	12.0V	Measured Kv value	Measured Rm Value		
	Io Value	0.59 A	0.67 A	0.77 A	0.87 A		1988 RPM/Volt @ 10v	0.076 Ohms		
Prop Manf.	Prop Size	Li-Po Cells	Input Voltage	Motor Amps	Input Watts	Prop RPM	Pitch Speed in MPH	Thrust Grams	Thrust Ounces	Thrust Eff. Grams/W
APC	5.25x4.75-E	3	11.1	13.34	148.1	17,507	78.7	451	15.91	3.05
APC	5.5x4.5-E	3	11.1	13.67	151.7	17,388	74.1	456	16.08	3.01
APC	6x4-E	3	11.1	14.87	165.1	17,003	64.4	630	22.22	3.82
APC	7x4-SF	3	11.1	21.82	242.2	13,985	53.0	840	29.63	3.47
APC	7x5-E	3	11.1	24.02	266.6	13,272	62.8	797	28.11	2.99
FC	5x4.5	3	11.1	8.66	96.1	19,061	81.2	428	15.10	4.45
FC	5x4.5x3	3	11.1	12.38	137.4	17,825	76.0	534	18.84	3.89
FC	6x4.5	3	11.1	15.47	171.7	16,792	71.6	721	25.43	4.20
GemFan	5x3	3	11.1	6.67	74.0	19,801	56.3	374	13.19	5.05
HQ	5x4	3	11.1	7.13	79.1	18,182	68.9	373	13.16	4.71
HQ	5x4x3	3	11.1	9.25	102.7	17,401	65.9	449	15.84	4.37
HQ	5x4.5-BN	3	11.1	11.17	124.0	16,902	72.0	487	17.18	3.93
HQ	6x3	3	11.1	7.34	81.5	18,128	51.5	419	14.78	5.14
HQ	6x4.5	3	11.1	13.53	150.2	16,206	69.1	645	22.75	4.29
HQ	6x4.5x3	3	11.1	17.60	195.4	15,137	64.5	762	26.88	3.90
HQ	7x4	3	11.1	20.71	229.9	14,250	54.0	850	29.98	3.70
HQ	7x4.5	3	11.1	20.31	225.4	14,351	61.2	865	30.51	3.84
Prop Manf.	Prop Size	Li-Po Cells	Input Voltage	Motor Amps	Input Watts	Prop RPM	Pitch Speed in MPH	Thrust Grams	Thrust Ounces	Thrust Eff. Grams/W
APC	5.25x4.75-E	4	14.8	17.29	255.9	20,560	92.5	603	21.27	2.36
APC	5.5x4.5-E	4	14.8	17.87	264.5	20,436	87.1	635	22.40	2.40
APC	6x4-E	4	14.8	20.15	298.2	19,829	75.1	837	29.52	2.81
FC	5x4.5	4	14.8	10.89	161.2	22,511	95.9	588	20.74	3.65
FC	5x4.5x3	4	14.8	16.43	243.2	20,828	88.8	718	25.33	2.95
FC	6x4.5	4	14.8	20.09	297.3	19,809	84.4	998	35.20	3.36
HQ	4x4.5-BN	4	14.8	10.45	154.7	22,661	96.6	477	16.83	3.08
HQ	5x3	4	14.8	6.88	101.8	23,580	67.0	442	15.59	4.34
HQ	5x4	4	14.8	10.22	151.3	22,739	86.1	589	20.78	3.89
HQ	5x4x3	4	14.8	13.26	196.2	21,763	82.4	710	25.04	3.62
HQ	5x4.5-BN	4	14.8	16.10	238.3	20,899	89.1	744	26.24	3.12
HQ	6x3	4	14.8	11.06	163.7	22,512	64.0	679	23.95	4.15
HQ	6x4.5	4	14.8	19.62	290.4	19,948	85.0	982	34.64	3.38

Figure C.2: Official Test Results for Cobra Motors

C.3 Combined Torque Responses

Figure C.3: Net generalized torque response \vec{V} to steps in $\Delta\lambda$ only

Figure C.4: Net generalized torque response \vec{V} to steps in $\Delta\alpha$ only

C.4 SimScape**Modelling**

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