

# Dual-Axis Tilting Quadrotor Aircraft

An investigation into the overactuatedness and control thereof



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*"We're gonna have a superconductor turned up full blast and pointed at you for the duration of this next test. I'll be honest, we're throwing science at the wall here to see what sticks. No idea what it'll do. Probably nothing. Best-case scenario, you might get some superpowers..."*

Cave Johnson -Founder & CEO of Aperture Science

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# Abstract

## Dual-Axis Tilting Quadrotor Aircraft

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The aim of this project is to design, simulate and control a novel quadrotor platform which can articulate all 6 Degrees of Freedom by vectoring each propeller's directional thrust. To achieve this the structure of the air-frame must redirect those thrust vectors to any desired orientation. This means it has to transform its configuration during flight, redirecting lift actuators whilst still maintaining stable attitude & position control, despite of such relative motion. In view of this required articulation the proposal is to add 2 axes (degrees) of extra actuation to each propeller. As a result each lift propeller can then be pitched or rolled relative to the body frame. This adaptation, to what is an otherwise well covered and highly researched platform, produces an over-actuated control problem. Actuator control allocation in the context of aerospace platforms is the primary contribution of this paper, with novel elements of non-linear (*state-space*) attitude control and plant uncertainty compensation.

The structure of the dissertation first presents the design which the subsequent dynamics and control are derived with respect to. Following that, the kinematics associated with rigid bodies are derived. Any unique effects that could apply to the design like gyroscopic, inertial and aerodynamic responses are investigated and then incorporated into the dynamics. Position and attitude control algorithms are first derived, then simulated and compared based on the plant's dynamics (*which include discretionary effects on the system*). The relative performance of the controllers are evaluated but regular performance metrics for attitude and position control are ill-suited for such a system. Some time is spent discussing the consequence of this and how the controllers are actually evaluated. Finally the design is built and tested using readily available RC components and conclusions drawn on the success or failure of the design.

The purpose of the investigation is the practicality and feasibility of such a design, most importantly whether the complexity of the mechanical design is a decent compromise for the added degrees of control actuation. The outcome of the build is to ascertain if it's economically feasible (cost and controller effort) to use such a prototype to expand the range of a quadrotor's motion. The design and control treatment presented here are by no means optimal nor the most exhaustive solutions, focus is placed on the system as a whole and not just one aspect of it.

*This dissertation report is presented in a logical progression of concepts and information. In some cases the research and results were completed in a different order from how they're listed.*

# Acknowledgements

# Nomenclature

DOF - Degree of Freedom(s)  
 $\mu$ C - micro-controller  
UAV - Unmanned Aerial Vehicle  
MAV - Manned Aerial Vehicle  
SISO - Single Input Single Output  
MEMS - Microelectromechanical Systems  
DIY - Do It Yourself  
BLDC - Brushless Direct Current  
VTOL - Vertical Takeoff/Landing  
OAT - Opposed Active Tilting  
dOAT - Dual axis Opposed Active Tilting  
PI - Proportional Integral, control structure  
PD - Proportional Derivative, control structure  
PID - Proportional Integral Derivative, control structure  
IBC - Ideal Backstepping Control  
ABC - Adaptive Backstepping Control  
LQR - Linear Quadratic Regulator  
LCD - Lyapunov Candidate Function  
PSO - Particle Swarm Optimization  
ITAE - Integral Time Additive Error  
BEM - Blade-Element Momentum  
ESC - Electronic Speed Controller  
RC - Radio Control  
CH - Channel  
MPC - Model Predictive Control  
TSK - Takagi-Sugeno-kang  
I/O - Input/Output  
RPM - Revolution Per Minute  
RPS - Revolution Per Second  
W.R.T - With respect to  
PWM - Pulse Width Modulation

# Symbols

Propeller Rotational Speed:  $\Omega_i$  [rpm]

*Rotational speed in RPS is used for Blade Element Theory Calculations in Chapter:3*

Servo 1 Position:  $\lambda_i$  [rad]

Servo 2 Position:  $\alpha_i$  [rad]

Inertial Position:  $\vec{P} = [X_I \ Y_I \ Z_I]^T \in \mathcal{F}^I$

Body Position:  $\vec{E} = [x \ y \ z]^T \in \mathcal{F}^b$

Euler Angles:  $\vec{\mathcal{E}} = [\phi \ \theta \ \psi]^T$

Euler Rates:  $\frac{d}{dt}\vec{\eta} = \dot{\vec{\eta}} = \Phi(\eta)\dot{\omega}_b$

Angular Velocity:  $\omega = [p \ q \ r]^T \in \mathcal{F}^b$

Linear Velocity:  $\nu = [u \ v \ w]^T \in \mathcal{F}^b$



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# Chapter 1

## Introduction

### 1.1 Foreword

#### 1.1.1 A Brief Background to the Study

A popular topic for current control and automation research is that of quadrotor UAVs. Attitude control of a quadrotor poses a unique 6-DOF control problem, to be solved with an under-actuated 4-DOF system. As a result the  $\phi$  pitch and  $\theta$  roll plants aren't directly controllable. The attitude plant is often simplified around a stable operating point. The trimmed operating region is always at the inertial frame's origin; resulting in a zero-set point tracking problem. The highly coupled non-linear dynamics of a rigid body's translational and angular motions arise from gyroscopic torques and Coriolis accelerations [Section: 3.2.1]. Such effects are mostly negligible around the origin, hence the origin trim point decouples the system's nonlinearities. The control system can then reduce<sup>1</sup> each state variable,  $\vec{X}_b = [\phi \ \theta \ \psi \ x \ y \ z]^T$ , to individual SISO plants.

As almost every recent quadrotor research paper mentions, the late interest in the platform is due to recent emergence in the availability of MEMS and low-cost microprocessor systems. These technical advancements accommodate onboard state estimation and control algorithm processing in real time. Developmental progress in quadrotors and, to a lesser extent, UAVs in general has led to rapidly growing enthusiast communities. HobbyKing [41] is now a name synonymous with providing custom DIY hobbyist quadrotor assembly kits, no longer just a retailer for prebuilt commercial products like DJI Phantom or ParrotAR drones.

The avenue for potential application of both fixed wing and VTOL UAVs is expansive, supporting civil [71], agricultural [75] and security [54] industries. The quadrotor design provides a mechanically simple platform on which to test advanced aerospace control algorithms. Commercial drone usage in industry is already emerging as a prolific sector; especially in Southern Africa. Subsequently following the 8<sup>th</sup> amendment of civil aviation laws [78], commercial use of UAVs has been both legalized and regulated. Research into any non-trivial aspect of the field will therefore be to extremely valuable to the field as a whole. Large scale quadrotor, hexrotor and even octotoror UAVs are popular intermediate choices for aerial cinematography and other high payload capacity applications. The cost of a commercial drone like the SteadiDrone Maverik [60] is far less than a chartered helicopter used for the same panoramic aerial scenes or on-site inspections. One foreseeable issue which may hinder commercial drone progress in the agricultural and civil sectors is the consequential inertial effects from scaling up the aerial structures. When increasing the size of any vehicle, its performance is adversely affected if actuation rates aren't proportionately increased.

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<sup>1</sup>Expanded upon in Appendix:A

### 1.1.2 Research Questions & Hypotheses

The difficulty with quadrotor control is that fundamentally they're unstable and under-actuated, *empirically proven later with Layupanov Theorem in Chapter:4*. A quadrotor has only four controllable inputs, namely each propeller rotational speeds,  $\Omega_{1,2,3,4}$ , which are then abstracted<sup>2</sup> to virtual control inputs net torque,  $\mu\vec{\tau} = [\tau_\phi \ \tau_\theta \ \tau_\psi]^T$ , and a perpendicular heave thrust  $\mu\vec{T} = \sum_{i=1}^4 T(\Omega_i)$ . Those four inputs have to effect both the linear X-Y-Z positions,  $\vec{\mathcal{E}} = [x \ y \ z]^T$ , and angular pitch, roll and yaw rotations,  $\vec{\eta} = [\phi \ \theta \ \psi]^T$ . Pitch and roll torques,  $\tau_\phi$  &  $\tau_\theta$ , are induced from differential thrusts of each opposing propeller. Yaw torque,  $\tau_\psi$ , is dependent on net aerodynamic torque about the rotational axes of each propeller (See Section:3.3.1). Aerodynamic responses are non-linear and fluctuating sources of control torques and as such the body's yaw control is depreciated. A result of the under-actuation is that the attitude control problem then becomes a zero set point problem, any other attempt to track attitude is ill-posed.

The aim of this dissertation is to implement attitude and position dynamic set point tracking on a quadrotor UAV by solving the problem of its inherent under-actuation. Inspired by Boeing/Bell Helicopter's V22 Osprey and the tilting articulation of its propellers, the prototype design proposed here (Section:2.1) introduces two additional actuators for each of the four quadrotor's lift propellers. Specifically, adding rotations about the  $\hat{X}$  and  $\hat{Y}$  axes for each motor/propeller pair. The resultant is an articulated 3-dimensional thrust vector instead of a bound perpendicular heave force. The control problem is then posed as the design and allocation of net forces,  $\vec{F}_{net} = [F_x \ F_y \ F_z]^T$ , and torques,  $\vec{\tau}_{net} = [\tau_\phi \ \tau_\theta \ \tau_\psi]^T$ , for a general 6-DOF body such that for any given trajectory,  $\mathbf{X}_d(t) = [x \ y \ z \ \psi \ \theta \ \phi]^T$ , the error state  $\mathbf{X}_e(t) = \mathbf{X}_d(t) - \mathbf{X}_b(t)$  asymptotically tends to  $\vec{0}$ .

$$\lim_{t \rightarrow \infty} \mathbf{X}_e(t) = \vec{0} \quad \forall \mathbf{X} \in \mathbb{R}^n \quad (1.1)$$

Where  $n$  is the degree of freedom the system has. The over-actuation brings about the need for a control allocation scheme which distributes the 6 commanded system inputs (net torques and forces) among the actuator set (12 actuators) in order to optimize some objective function secondary to that of Eq:1.1. The potential improvement(s) for exploiting those over-actuated elements is the most novel outcome that this project will yield. A cost function aimed at optimizing some aspect unique to aerospace bodies is going to be a completely unique contribution.

Part of the control research question is the multivariable treatment of the system; making no assumptions or simplifications to the non-linear dynamics involved in the quadrotors motion and its operational conditions. Standard linearizations applied to the quadrotor's control plant won't hold true for the more aggressive manoeuvres; they're dependent on small angle approximations and negligible  $2^{nd}$  order effects. A stabilizing control law solution will need to expand and simulate the existing kinematic model of an aerial body and apply it to a quadrotor's motion. Following this there must be design, development and control of the new actuator suite which is to be implemented on a quadrotor platform. Final key outcomes for the project are the simulation analysis and prototype construction for the proposed design and the conclusion drawn thereon.

Introducing relative motion within an unconstrained body will produce a lot of unwanted dynamics like inertial and gyroscopic responses, amongst others. A rotating propeller will respond to pitching much like a Control Moment Gyroscope [102] or a flywheel, producing a precipitation torque cross product. A less trivial aspect which is occasionally considered are the aerodynamic effects produced from the propeller's aerofoil profile. Such induced responses occur in planes perpendicular to whatever the propeller's rotation exists in. These aspects aren't normally compensated for due to a quadrotor's fundamental co-planar propeller rotation which negates such effects. It's anticipated that a plant dependent control solution will have to compensate for these dynamics, which if left unaccounted for would cause instability.

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<sup>2</sup>The abstraction of which is explored in Appendix:A



### 1.1.3 Significance of Study

Due to the huge popularity of quadrotor platforms as research tools, any work that expands the UAV & quadrotor general body of knowledge will prove to be valuable. With that being said, there is already a vast amount of existing research on linear and non-linear control techniques for regular quadrotor platforms. The attitude loop is the most common topic for control research, requiring an ingenious under-actuated solution and mostly linearized around the origin (See Appendix:A). Far less common is the application of optimal flight path and trajectory planning to quadrotor control. The uniqueness and difficulty of the quadrotor attitude control does not hold true for its position control, so standard techniques can be used for way point planning and the like once the attitude control problem has been answered.

The most significant aspect of this project is the attitude control, discussed later in Section:4.1. The over-actuation of the proposed design and, more critically, the manner in which the controller's commanded (virtual) output is distributed among those control effectors would appear to be the first of its kind. Otherwise known as control allocation, the requirements of the distribution algorithm(s) are outlined in Section:4.3. Dynamic set point attitude control for aerospace bodies is not a subject heavily researched outside the field of satellite attitude control. Even papers which propose similarly complicated mechanical over-actuation (expanded upon in next in the literature review, Section:1.2) hardly broach the topic of tracking attitude set points away from the origin.

Whilst the control plant (developed in Chapter:4) does indeed close both the position and attitude control loops, there is no consideration of trajectory generation nor flight path planning. Such topics are well discussed elsewhere in a far more concise and deliberate way than this project could ever hope to achieve. Once closed loop position and attitude control have been achieved, the control algorithms can be adjusted to account for higher order state derivative (acceleration, jerk and jounce) tracking needed for nodal way point planning. The heuristics involved with flight path planning are well documented and their implementation is a relatively academic task.

Where possible the system identification and control (design and allocation) for this project is kept both modular and generically applicable. The intention here is that its pertinence falls not only within the UAV field but to any aerospace or free body attitude control. Hopefully this investigation can be expanded upon with more in-depth research on one of the subsystems without compromising the stability of the whole plant. Provisionally, an obvious outcome which the investigation could yield is improved yaw control of a quadcopter's attitude. However, if the express purpose was just to improve yaw control, it could be done with a dramatically less complicated design. . .

Furthermore, the project could provide greater insight into high bandwidth actuation and thus a faster control response for larger aerospace bodies. Any standard quadrotor uses differential thrust to develop a torque about its body. Such actuation suffers a second order inertial response when the propellers accelerate or decelerate,  $\tau_{simplified} = \mathbb{I}_f \dot{\omega}_i$ . Prioritizing pitching the propeller's principle axis of rotation rather than changing the propeller's speed could potentially improve the virtual control response. This is entirely dependent on how the allocator block is prioritized (presented in Section:4.3). The exact effects of different actuator prioritization and distribution in the context of aerospace control are wholly unique to this dissertation.

### 1.1.4 Scope and Limitations

#### Scope

Critical to this project is the conceptualized design and prototyping of a novel actuation suite to be used on a quadrotor platform. The precise requirement of which is to apply dynamic set point attitude control to the body. Stemming from this is an investigation into the kinematics that are potentially

influenced by the design and the structure's relative motion. In order to apply correct control theory to achieve the attitude tracking on a physical prototype, the plant dynamics must first be identified for input responses to be approximated with confidence. Aspects of the mechanical design are covered next in Section:2.1 but, beyond the cursory investigation, there is no scope for materials analysis or stress testing of the design. To the detriment of the project, the design will either produce an over-engineered or catastrophically under-engineered solution. The scope focuses mainly on the control application and embedded systems design, not the structural integrity of a proposed frame given the forces it may undergo. Physical measurements are only made for critical kinematics, such as inertial measurements for the second order gyroscopic and inertial dynamic responses.

As mentioned in the antecedent Section:1.1.3, trajectory & flight path planning are not ubiquitous with this dissertation. Derivations for the differential equations which dictate a 6-DOF body's movement are wholly applicable to any dynamic (rigid or otherwise) aerospace body, although some particular standards are used (sic Z-Y-X Euler Aerospace Sequence, Section:2.2). Similarly the control plant is stabilized with non-linear state space control techniques, aided and justified by Lyapunov theorem. Alternative solutions using Model Predictive Control or Quantitative Feedback Theory could provide more refined or effective controllers, however they aren't presented and remain open to further investigation. Quadrotor attitude control is commonly stabilized with feedback linearizations, decoupling the plant around a trim point so that SISO techniques can be applied. A derivation of such a linearization is included in Appendix:A but beyond that there are no further discussions. Any comparison between non-zero and zero-set point attitude control of quadrotor is difficult as the fundamental objectives are in stark contrast with one another.

Arguably the most important and indeed novel aspect of this project is the control allocation. The system has 12 plant inputs and 6 output variables to be controlled. There is then a family of actuator set solutions,  $u \in \mathbb{U}$ , which exist for each commanded input. Such a plant is classified as over-actuated. Ergo, there must be some logical process as to how those 12 inputs are articulated to achieve the desired 6 control plant inputs. Appropriate techniques are first investigated in Section:4.3 and compared before a final solution is implemented in Section:5.4. It is by no means a comprehensive investigation of every possible allocation scheme but rather an analysis of the sub-set of problems and design of what is regarded as a logical and pertinent approach.

With regards to the actual prototype design, in Section 2.1, it's assumed that certain aspects are a given certainty. Particularly the state estimation, updated through a 4-camera positioning system fused with a 6-axis IMU through Kalman Filtering, is assumed to be precise and readily disposable at a consistent 50 Hz. Hence state estimation is included but is bereft of intricate detail, this is another topic which remains open to further investigation.

## Limitations

The biggest constraint faced by the design is the net weight of the assembled frame. Lift forces required to keep the body aloft are obviously dependent on the all up weight. Conventional wisdom has it that steady state actuator rates ought to be far less than saturation conditions. For stability to be guaranteed at all feasible operating conditions, the actuators must have sufficient headroom to still effect the desired control inputs. Conversely the structure's net weight is mostly dependent on the lift motors, often being the heaviest part of the vehicle (*batteries included*). A trade-off between net weight and actuator efficacy makes designing the prototype a balancing act of compromise; added actuation is needed to produce the desired thrust vectoring. That added actuation is going to increase the weight which then requires more thrust force to ensure the vehicle remains airborne. Larger motors then need stronger actuators to effect the relative motion and overcome the bodies inertial response. It's a compromise between the weight of the body and the strength/quality of the actuation.

To forego the deliberation detailed above, reducing the possibility of unbounded scope creep, a limitation is self-imposed on the prototype design. Restricting the propeller diameter, and hence maximum thrust/frame size, will provide a constraint upon which all other design considerations must adhere to. Smaller propellers require far greater rotational speeds to produce similar levels of thrust that their larger diameter counterparts could provide. Electing to use 3 blade 6X4.5 inch small diameter propellers is going to reduce the overall dimensions of the prototype, but as a consequence will require very high RPM motor. Specifically a set of four Cobra-2208/2000KV [23] Brushless DC motors are to be used for lift actuation (Fig:1.1a)<sup>3</sup>. A direct consequence of this decision is that, provisionally based upon test data<sup>4</sup>, the net thrust disposable for actuation is limited to around 950g,  $\approx 9.5$  N, per motor (see Section:3.3.1). It's critical to ensure the control block doesn't induce over-saturation of the motor actuation, so the frame weight needs to be around 50% of the maximum available thrust, or roughly 2 Kg. Saturation conditions are detailed later in Section: 4.3.

Another aspect of limitations produced by design decisions made, mostly to reduce prototype costs and weight, is to use of 180° rotation servo motors. Here Corona DS-339MG metal gear digital servos (Fig:1.1b)<sup>5</sup> are used. The servos are for each individual motor's  $\hat{X}_{M_i}$  and  $\hat{Y}_{M_i}$  axial pitch and roll actuations respectively. The servos act in lieu of either continuous BLDC (gimbal) or stepper motors. Any non-servo rotations beyond  $2\pi$  will require closed loop position control and, unlike servos, would need slip rings to transmit power throughout rotational movement. However the logistics of implementing such a design whilst maintaining an acceptable weight is almost impossible. Such an implementation is going to dramatically increase the size of the prototype to accommodate for weight increases. Commercial camera stabilizing gimbals already make use of similar configurations but the I/O requirements from the flight controller  $\mu$ C already constricts the amount of expansion available.



(a) Cobra CM2208/2000KV BLDC motor



(b) Corona DS-339MG digital servo

**Figure 1.1:** Mechanical actuators

Discrete elements for the whole system can potentially limit performance but are going to be mitigated if possible. For example analogue servos have an associated 1ms deadband from their 50Hz refresh rate. That can be addressed by using faster, albeit more expensive, digital servos which samples at 330Hz. The prototype's flight controller needs to provide 12 PWM output compare channels for the 8 servos and 4 BLDC speed controllers. State updates from a ground control station and a fail safe 6CH RC receiver module also needs to be processed by the  $\mu$ C system. Particular attention is paid to the embedded system design and layout in Section:2.4.

<sup>3</sup>Image cited from: <http://www.getfpv.com/cobra-cm-2208-2000kv-motor.html>

<sup>4</sup>Official test data from [23] included in Appendix:C.2 and verified independently in Section:3.3.1

<sup>5</sup>From the DS-339MG product page, HobbyKing [41]

## 1.2 Literature Review

### 1.2.1 Existing & Related Work

The field of transformable aerospace frames is not necessarily a new one, with many commercial examples having seen a lot of success over their operational life span. The most notable tilting-rotor vehicle is that of the Boeing/Bell V22 Osprey [30] aircraft. First introduced into the field in 2007, the Osprey has the ability to pitch its two lift propellers forward to aid translational flight after vertically taking off or landing. In addition to this there have been many papers published on similar tilting bi-rotor UAVs for research purposes.

#### Biorotors



**Figure 1.2:** General structure for opposed tilting platform

Research into birotor vehicles (Fig:1.2)<sup>6</sup> with ancilliary lift propeller actuation is oft termed Opposed Active Tilting or *OAT*. Such a rotorcraft's mechanical design applies either a single *oblique*  $45^\circ$  tilting axis relative to the body; [10, 34, 50], or a *lateral* tilting axis, adjacent to the body; [19, 51, 74, 88]. Leading research is currently focussed on applying doubly actuated tilting axes to birotor UAVs. Dual axis Opposed Active Tilting or *dOAT* introduces vectored thrust with propeller pitch and roll motions to further expand the actuation suite, [2, 33]. A birotor is sometimes considered preferable to higher order multirotor platforms due to their reduced controller effort. However the controller plant abstraction often detracts from the quality and effectiveness of its stability solution as a result of the birotor's underactuation.

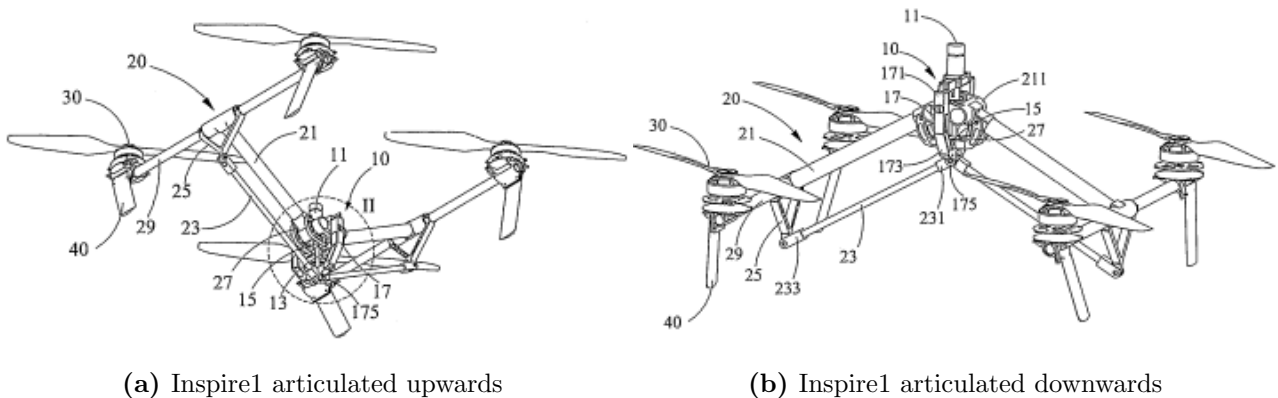
Birotor attitude control typically incorporates plant independent PD [10] and PID [74] controller schemes. Occasionally more computationally intensive and plant dependent Ideal and Adaptive backstepping controllers (*IBC* or *ABC*) are implemented, presented in [50, 88] and [51] respectively. The cross-coupling of a birotor vehicle's attitude system is more pronounced than that of a quadrotor, derived in Section:3.2, and so feedback linearisation is almost always used. In an interesting progression from the norm, Lee et al, [56], proposed a PID co-efficient selection algorithm for a bi-rotor control block. Using a Particle Swarm Optimization technique, similar to [105], the coefficients were globally optimized around a given performance metric. However their performance criterion is a basic

<sup>6</sup>Image from G. Gress: [33]

ITAE<sup>†</sup> term and nothing more appropriate involving effects unique to flight systems. *PSO* algorithms iteratively search for a globally optimized solution and offer independent, derivative free optimization. Later on non-linear controller coefficient are also optimized in this paper using a *PSO* algorithm, shown in Section:5.1.

## Quadrotors

Expanding on multirotor vehicles, the quadrotor UAV is a popular and well researched multirotor platform due to its mechanical simplicity. What would appear to be one of the first quadrotor research implementations, in 2002, is the X4-Flyer quadrotor, [36, 80]. Alternative iterations like the Microraptor [83] and STARMAC [42] quadcopters have subsequently been built and tested. A plethora of literature exists around quadrotor kinematics & control [4, 13, 22, 61, 82], however dedicated rigid body 6-DOF dynamic papers [63, 76] provide better explanations of the kinematics. Often the plant's dynamics are simplified around an origin trim point and assumed to reduce into 6 SISO plants for each degree of freedom (Appendix:A). Lately research projects have begun to incorporate aerodynamic effects like drag and propeller BEM theory into the plant model [16, 42, 85]. Although mostly negligible under standard operating conditions, the higher fidelity models offer more precision without making any linearisations or assumptions, [5, 42].



**Figure 1.3:** DJI Inspire1

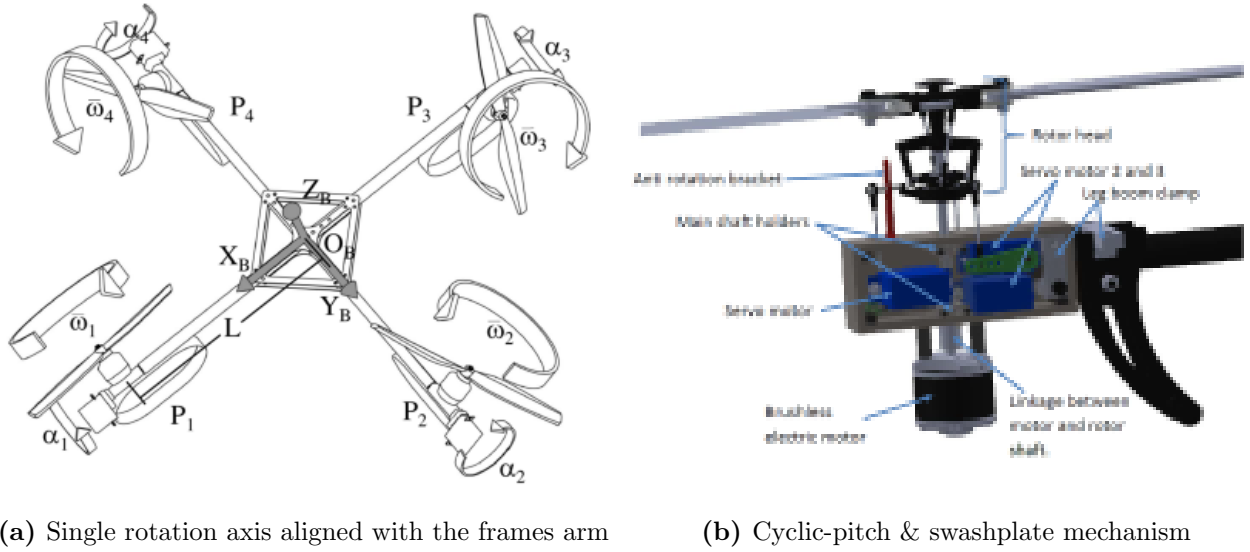
At the time of writing, the only commercial UAV multirotor capable of structural transformation is the DJI Inspire1 quadrotor [25], produced by Shenzhen DJI Technologies (better known for the hugely successful DJI Phantom drone [26]). The Inspire1 can articulate its supporting arms up and down as shown in Fig:1.3 <sup>7</sup>. The aim of such transformations is to both alter the center of gravity and further expose a belly mounted camera gimbal for panoramic viewing angles. This change alters the moment of inertia about the body's center of gravity, in turn affecting the inertial torque response induced by angular velocities, an otherwise detrimental effect which makes researchers apprehensive of transformable aerospace frames. The range of transformations which the frame can undergo is limited to just articulating the arms up and down.

In a similar fashion to the progression seen in birotor state-of-the-art, quadrotor research is engaging the topics of single and dual axis tilting articulations. First conceptualized and implemented on a prototype related to an ongoing project covered in two reports, [86, 87]. The authors M. Ryll et al.(2012, 2013) modified and tested a QuadroXL four rotor helicopter, produced by MikroKopter [31], to actuate only a single axis of tilting aligned with the frame's arms (Fig:1.4a)<sup>8</sup>. Their proposed control solution, detailed next in Section:1.2.2, assumes no nominal linearised conditions around hover flight, unlike a similar single axis tilting quadrotor prototype designed by Nemati, et al. (2012) [68]. The latter remains simulated but as yet untested.

<sup>7</sup>Both images were sourced from the drone's patent, held by SZ DJI Tech Co [103]

<sup>8</sup>Image sourced from Modelling and Control of a Quadrotor UAV with tilting propellers, [86]

One approach to improving quadrotor flight response is to alter the manner in which the thrust is mechanically actuated, potentially improving the actuator bandwidth. Drawing from helicopter design, a project by Napsholm, (2013) [67], purported a quadrotor UAV prototype that used swashplates for varying the propeller pitch and generating torque moments. The aim was a design which wasn't dependent on rotational speed controlling power electronics (*ESCs*) to actuate variable thrust forces. Petrol motors were intended for use in place of BLDC motors. Furthermore, the design proposed a single axis of tilt actuation to each of the four motor modules. Whilst mechanically complex, Napsholm made use of existing RC helicopter components to design a rotor actuation bracket (Fig:1.4b). The cyclic-pitch swashplate used [69] could apply torques,  $\tau_\phi$  and  $\tau_\theta$ , about each propeller's hub, its *principle axis of rotation*, by altering the blades angle of attack throughout the rotational cycle. The actuation rate of such a configuration is far greater than that of a differential torque produced rolling/pitching motion.



**Figure 1.4:** Tilt-rotor mechanisms

Irrespective of the strong initial design in the early stages of his project, it would appear that Napsholm's research suffered due to time constraints. The introductory derivation on aerodynamic effects and deliberation over the design provide clear insight into the projects goals. However the control solution and system architecture, electronic and software, are severely lacking. A brief introductory proposal of an MPC attitude control system detracted from the comprehensive dynamics discussed. The project ended before testing, simulation or results could be obtained. Unfortunately, despite the novel over-actuated design, there was no discussion given on how that actuator allocation, being the most unique aspect, would be achieved.

Finally, the most crucial research to mention is a project completed by Pau Segui Gasco [29], which was a dual presented MSc project with Yazan Al-Rihani [1]. At the time of writing, this would appear to be the only project published pertaining to *over-actuation* in aerospace bodies implemented on a quadrotor platform. The research was split between the two authors who completed the electronic/control design and the mechanical design for their respective MSc dissertations. Shown in Fig:1.5<sup>9</sup>, the dual-axis articulation is achieved using an RC helicopter tail bracket and servo push-rod mechanism; reducing the mass of the articulated component but limiting the range of its actuation. Considering the propellers as a spinning flywheel, the induced gyroscopic response was then treated as a additional controllable actuator plant. Their commanded virtual control is distributed by weighted inversion among the actuator set, Section: 1.2.2. The whole project justifies the extra actuation as fault tolerance redundancy but doesn't necessarily prove how such a redundancy could be beneficial.

<sup>9</sup>Image from Development of a Dual Axis Tilt Rotorcraft UAV: Modelling, Simulation and Control [29]



Figure 1.5: Dual-axis tilt-rotor mechanism

### 1.2.2 Notable Quadrotor Control Implementations

#### Quadcopter Attitude Control

Attitude control of a 6-DOF body, quadrotor or otherwise, is best described by *The Attitude Control Problem* [97]. A rigid body that currently has an attitude state<sup>10</sup>  $\vec{\eta}_b$  and a desired state  $\vec{\eta}_d$ , the problem is to then find a torque control law:

$$\mu\tau = h(\vec{\eta}_b, \vec{\eta}_d, \dot{\vec{\eta}}_b, \dot{\vec{\eta}}_d) \quad (1.2)$$

Such that both the angular position  $\lim \vec{\eta}_b \rightarrow \vec{\eta}_d$  and that angular rates  $\lim \dot{\vec{\eta}}_b \rightarrow \dot{\vec{\eta}}_d$  asymptotically stabilize as  $t \rightarrow \infty$ . A distinction must be made between angular rate vector,  $\dot{\vec{\eta}}_b = [\dot{\phi} \ \dot{\theta} \ \dot{\psi}]^T$  and the angular velocity vector  $\vec{\omega}_b = [p \ q \ r]^T$ . Depending on how the attitude is posed; with rotation matrices [53, 63, 76], quaternions [28, 32, 35, 53] or otherwise (Direct Cosine Matrix etc ...) the error state<sup>11</sup>  $\Delta\vec{E} = \vec{\eta}_d - \vec{\eta}_b$  could then differ to a (Hamilton) multiplicative relationship.

*Note that here  $\vec{\eta}_b$  is not necessarily an Euler set but any attitude representative state variable.*

Simulation and modelling papers often rely on Euler angle based rotation matrices for attitude representation, [12, 13, 62, 68, 84] without addressing the inherent singularity associated with such an attitude representation (sic Gimbal Lock, [89], Section:3.1.2). The alternative quaternion attitude representation, first implemented on a quadrotor UAV in 2006 [94], is often used in lieu of rotation matrices but has its own caveat of *unwinding*, (Section:3.1.4), as a result of quaternions dual-coverage [65] in  $\mathbb{R}^3$  space. Quaternions are  $\in \mathbb{R}^4$  variables for attitude representations and so a mapping  $\mathbb{R}^4 \rightarrow \mathbb{R}^3$  produces a dual coverage for each unique attitude state.

Quadrotor plant dynamics, as mentioned previously, are often simplified; especially when represented with a 3-variable Euler angle set,  $\vec{\eta} = [\phi \ \theta \ \psi]^T$ . The cross-coupled gyroscopic and Coriolis responses are both neglected when the angular velocity<sup>12</sup> is small,  $\vec{\omega}_b \approx 0$ , and the inertial matrix is diagonal,  $rk(\mathbb{I}_b) = x$  for  $x \in \mathbb{R}^x$ . The consequence of which is the ineffectual deterioration of both the gyroscopic term,  $\vec{\tau}_{gyro} = -\vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b \approx 0$  and the Coriolis force term,  $\vec{F}_{cor} = -\vec{\omega}_b \times \vec{a}_b \approx 0$  in the bodies dynamics (Chapter:3 for context). Once the coupled cross-product terms are no longer of consequence, the 6 DOF trajectory,  $\mathbf{X} = [x \ y \ z \ \phi \ \theta \ \psi]^T$ , can be treated as a series of independent SISO plants each controlled with an appropriate technique. Quaternion represented attitude plants cannot easily be decomposed into individual single-input-single-output systems (quaternion dynamics in Section:3.1.3). So a quaternion combined four variable attitude state-space vector is then used,  $Q_b = [q_0 \ \vec{q}]^T$ , for the abstracted major loop plant.

<sup>10</sup>Quaternion attitude states will later replace Euler angles

<sup>11</sup>*The Attitude Control* [97] describes these conventionally different error states

<sup>12</sup>Angular velocity and angular rates are fundamentally different,  $\vec{\omega}_b \neq \dot{\vec{\eta}}$



**Figure 1.6:** ArduCopter PI Euler angle attitude control loop

Commercial flight controller’s software (Arducopter [3], Openpilot [57]<sup>13</sup>, CleanFlight [20], BetaFlight [8], etc ...) for custom fabricated UAV platforms all apply their own flavour of structured attitude controllers and state estimation algorithms, based on onboard hardware sensor fusion. The article *Build Your Own Quadrotor* [58] summarizes the control structures implemented on a range of popular flight controllers. The most popular of which, ArduCopter, implements a feed-forward PI compensation controller (Fig:1.6)<sup>14</sup>. PI, PD and PID controllers are all easy and effective plant independent control solutions for general attitude plants. Table:1.1 collectively lists the common attitude control blocks (not exclusively quadrotors UAVs but MAVs too) and which projects they’ve been implemented in, after which a critique on the more unique adaptations is given.

Controller Type	Independent	Dependent	Total
PI	[97]	[97]	2
PD	[1, 61]	[28, 68]	4
PID	[12, 14, 82, 86, 97]	[42, 84, 97]	8
Lead	[22, 80]	N/A	2
IBC	[62, 88] <sup>15</sup>	[62]	3
ABC	[7, 24, 51, 66]		4
LQR	[14]	N/A	1

**Table 1.1:** A breakdown of common attitude controllers

In a collection of papers, written by Bouabdallah et al ... (2003,2004,2007) arguably the most prolific early quadrotor authors, a range of different control implementations are derived and reviewed. Their last paper (2007) [13] derived and practically tested an Integral Backstepping attitude controller on an OS4 quadrotor platform. It builds on their research from an earlier paper (2003) [14] wherein an analysis of PID vs LQR attitude controllers in the context of quadrotors is posed. LQR controllers aim to optimize the controller effort (with  $u \in \mathbb{U}$ , controller effort is then  $\|u\|$  or the  $L_2$  norm of the plant input). Although, in theory, solving the associated Ricatti<sup>†</sup> cost function may produce an optimal, stable and efficient control law it needs exact plant matching. In reality, exact plant matching is difficult to achieve for a quadcopter or any aerospace body for that matter. The resultant controller in [14] achieved asymptotic stability but had poor steady state performance due to low accuracy of the identified actuator dynamics and poor confidence inertial measurements.

Adaptive Backstepping Control [101](any of the examples in Table:1.1) builds on nominal IBC fundamentals by introducing an additional disturbance state term in the LCF used for the backstepping iteration. The drawback with this form of Backstepping approach is that, from the Lyapunov control theorem, a time derivative for the estimated disturbance (or an *update law*) is needed. Disturbance approximation has been investigated thoroughly but, for a signal without *apriori* information, some heuristic needs to be adopted with the approximation which usually involves some compromise.

<sup>13</sup>NOTE: OpenPilot’s firmware stack is now maintained by LibrePilot

<sup>14</sup>Image sourced from *Build your own Quadrotor* [58]



In one example, [24], the authors implemented a statistical  $proj(\cdot)$  operator based technique. Which, when used in adaptive control, the projection operator [18],  $proj(\cdot)$ , ensures a derivative based estimator is bounded for adaptive regression approximation [79].

Although the control implementation isn't backstepping based, in [106], a sliding mode controller was used to compensate for the disturbances in an Unmanned Submersible Vehicle attitude plant. The underwater current disturbances were approximated using a fuzzy logic system, specifically a *zero-order TSK* fuzzy controller. The TSK system has been proven to act in the same way as an Artificial Neural Network approximator [64]; where the fuzzy TSK system is more comprehensible than the latter. Statistical analysis and investigation of approximators without *a priori* knowledge of a system are well beyond the scope of this research but are worth mentioning.

### Single/Dual Axis Control & Allocation

The extra actuation introduced with single and dual axis articulation provides room for more control goals to be achieved as the order of actuation increases. Of the few papers published on tilting-axis quadrotors, PD controllers (Nemati et al.[2014] [68] and again in Gasco & Rihani [1, 29]) and PID controllers (Ryll et al.[2012,2013] [86,87]) are the norm for control blocks. For either of these systems there needs to be an allocation rule to distribute a commanded input amongst the actuator set. In [46], Johansen et al.[2012] describes the control allocation problem for a dynamic plant:

$$\dot{x} = f(x, t) + g(x, t)\tau \quad (1.3a)$$

$$y = l(x, t) \quad (1.3b)$$

*Note in state space Equation:1.3a, it's assumed the plant input<sup>16</sup>,  $\tau$ , has a linear multiplicative relationship with the input response,  $g(x, t, \tau) \iff g(x, t)\tau$ .*

With a state  $x \in \mathbb{R}^n$  and  $f(x, t)$  &  $g(x, t)$  being the plant's dynamics and input response respectively. In set point tracking, the output is then *tracking* the state  $y = x$ , and hence  $y \in \mathbb{R}^n$ . In an ideal well posed system the number of actuator inputs equals the number of controllable variable outputs; that being  $\dim(x) = \dim(\tau) \in \mathbb{R}^n$ . In the case where the control input  $\tau \in \mathbb{R}^m$ , if  $m > n$  the problem is then overactuated and a level of abstraction is needed; an asymptotically stabilizing virtual control input  $\nu_d$  is designed by a control law  $\nu_d = h(x_e, t)$  to affect dynamics. The goal is to then find a function that maps  $\mathbb{R}^m \rightarrow \mathbb{R}^n$  for an actuator matrix  $u \in \mathbb{U}^m$ . An overactuated plant can be described in state-space as:

$$\dot{x} = f(x, t) + g(x, t)\nu_d, \quad \nu_d \in \mathbb{R}^n \quad (1.4a)$$

$$\nu_c = B(x, t, u) \approx B(x, t)u, \quad u \in \mathbb{U}^m, \quad \nu_c \in \mathbb{R}^n \quad (1.4b)$$

$$y = x \quad (1.4c)$$

$B(x, t, u)$  is the effectiveness function which quantifies how the actuator inputs  $u$  relate to the virtual commanded input  $\nu_c$ .  $B(x, t, u)$  can be abstracted to a multiplicative relationship  $B(x, t)u$  if the plant's dynamics permit it, such that;  $B(x, t) \in \mathbb{R}^{n \times m}$ . For generic setpoint tracking the control law will design a desired virtual control input  $\nu_d$ , the allocation rule then has to solve  $u$  for  $\nu_c$  such that for some slack variable  $s = \nu_c - \nu_d$  is minimized:

$$\min_{u \in \mathbb{R}^m, s \in \mathbb{R}^n} \|Q_s\| \text{ subject to } B(x, t, u) - h(x_e, t) = \nu_c - \nu_d = s, \quad u \in \mathbb{U} \quad (1.5)$$

Which ensures the commanded input  $\nu_c$  tracks the desired control input  $\nu_d$ ;  $\nu_c \rightarrow \nu_d$  as per some cost function of the slack variable  $Q_s$ . Mostly the L2 norm,  $\|Q_s\|$ , is used. In an overactuated system it then follows that there is a set of possible inputs for each  $\nu_c$ . A unique actuator solution (rather than a family solution set) to Eq:1.5 needs a secondary objective function,  $J(x, t, u)$ . Eq:1.5 then becomes;

$$\min_{u \in \mathbb{R}^m, s \in \mathbb{R}^n} (\|Q_s\| + J(x, t, u)) \text{ subject to } \nu_c - h(x_e, t) = s, \quad u \in \mathbb{U} \quad (1.6)$$

---

<sup>16</sup>Disambiguation:  $\tau$  is not necessarily the torque input.

Those same authors Johansen and Tjnnns [2004,2005,2008] proposed multiple control allocation solutions to a variety of systems. Following [46], in a subsequent paper [47], Johansen and Tjnnns [2005] introduce a secondary cost function, driving the solution away from the typical quadratic programming direct or weighted inversion solution. Aiming for optimal efficiency and not just actuator saturation. In a followup paper [48], the same authors proposed an online adaptive algorithm approach, using a Lyapunov energy function, to ensure the minimization adaptive law settles to a feasible solution.

Over-actuation is not something often applied to quadrotors and as a result rather than providing a comprehensive literature review of associated papers here (which are all mostly theoretical derivation), the contextual application and solutions to the above posed problems are expanded later in Section:4.3.1. The only overactuated quadrotor (birotor dual-axis tilting makes the system critically actuated and so requires no allocation) literature which covers allocation of the extra actuators is [1,29], where the authors apply a weighted pseudo inverse (sic Moore Penrose Inverse [55]) allocation rule. A prerequisite for pseudo inversion is a multiplicative *linear* control effectiveness relationship for Eq:1.4b.

Segui et al. [2012] applied weighted inversion, relying on some very specific assumptions to achieve that linearity relationship in Eq:1.4b. For the net torque response the authors assumed the extra actuators pitch and roll angular rates,  $\dot{\phi}$  and  $\dot{\theta}$  respectively, were proportionally related as follows:

$$\dot{\phi} \approx \frac{\phi}{t_{rise}} \quad (1.7)$$

In which  $t_{rise}$  is the actuators rise time to a set-point. As a result the gyroscopic first order torque  $\tau_{gyro} = -\omega \times \mathbb{I}_f \omega$  and second order inertial torque  $\tau = \mathbb{I} \ddot{\omega}$  are then functions of position  $\phi$  or  $\theta$  and not their derivatives. The extent of that consequence is contrasted with the allocation solution in Section:4.3.

## Satellite Attitude Control

Unconstrained attitude set-point tracking for 6-DOF bodies, quaternion represented or otherwise, is a topic well covered in the field of satellite attitude control; [45,52,99]. The *status quo* for recent research is on non-linear adaptive attitude back-stepping control systems, wherein the adaptive update rule is the novel contribution. Often plant uncertainty affects the inertial tensor of a satellite. In [45], the authors Wang Jia, et al. [2010], proposed applying adaptive back-stepping to compensate for steady state errors of (asymmetric) inertial estimations. Alternatively, instead of deliberating on costly non-orbital prelaunch inertial measurements Bodrany, et al.[2000] [11] developed an algorithm for estimating the inertia tensor based on controlled single axis perturbations. Such an approach does assume any initial estimates are sufficiently close to true body values such that they will settle and stability can be ensured, irrespective of how unacceptable the transient performance may be.

Satellite actuator suites mostly include additional redundant effectors, to ensure fault tolerance, and thus require control allocation. Often the extra allocators are CMG actuators, driven by DC motors, to produce rotational torques. Fuel burning can only actuate for a certain period of time and so thrusters are scheduled to have a lower priority. Seen in the paper [52]; the authors, Kristiansen et al. [2005], address the over-actuation with direct and well-matched inversion before applying quaternion based back-stepping for attitude control. A direct inversion solves to Eq:1.6 such that:

$$u = B^\dagger(\tau_a^b - D\omega_{ib}^b) \quad (1.8a)$$

$$B^\dagger = B^T(BB^T)^{-1} \quad (1.8b)$$

Where  $B$  is the effectiveness matrix and  $B^\dagger$  is such that  $BB^\dagger = \mathbb{I}$ . Specifically  $B^\dagger$  is the general *pseudo* inverse of  $B$  (more on inversions in Sec:4.3). It's assumed there's a multiplicative relationship between the input,  $u \in \mathbb{U}$ , and the input effectiveness matrix in Eq:1.4b. The controller designed actuator torque  $\tau_a^b$  then dictates the input  $u$  as in Eq:1.8a. Much like the over-actuation previously discussed W.R.T quadcopters; the pseudo inversion method of actuator distribution applies quadratic optimization to the allocation slack cost function, Eq:1.5.

## Chapter 2

# Prototype Design

### 2.1 Design



**Figure 2.1:** Isometric view of the prototype design

The final prototype (Fig:2.1) went through a series of different design iterations, all aimed at optimizing engineering time spent on construction and reducing the associated component costs thereof. A significant aspect of consideration for the design process was the net weight whose upper limit, as mentioned before, is inherently limited by the thrust produced from lift motors. Some of the more important design factors, like inertias & masses (Section:2.3), are discussed here in order to give context for the dynamics derived later in Chapter:3. The reference frame orientations which those dynamics are developed with respect to are detailed here. A brief overview of the electrical systems layout is then given with the components associated and their electrical characteristics included. Finally the actuator suite's functionality and transfer characteristics are also quantified. A review of the physical prototype realized and control loop(s) implemented is detailed in Chapter:6 along with actual flight test results.

### 2.1.1 Actuation Functionality

The novel component of the design is the manner of articulation for each concentric gimbal ring which forms the motor module structures. The control objective is to produce a thrust vectoring actuation set for a quadrotor's control plant. The outcome was a module which independently redirects the thrust generated by the lift propellers (Fig:2.2a). Within each module are servos affixed onto sequential support rings to pitch and roll the substructure's axes. The gyroscope-like frame that surrounds each motor/propeller pair accommodates that relative movement. Aligned with each servo is a coaxial support bearing. The bearing and actuator servos have a mass disparity which results in an eccentric center of mass, producing a gravitational torque arm. Unfortunately, due to weight constraints, counter balance measures cannot be introduced. Consequences from the center of mass variations must be either compensated for (*plant dependent solution*) or exploited in the dynamics (*additional non-linear actuator plants*). The precise effects are quantified numerically later in Section:2.3.



**Figure 2.2:** Tilting rotor design

Each motor module is positioned such that its produced thrust vector coincides with the intersection of its two rotational axes. As a result, there's only a perpendicular displacement  $\vec{L}_{arm}$  co-planar to the body frames X-Y-Z origin  $\vec{O}_b$  (see subsequent Fig:2.7). That length directly effects the differential torque plant  $\tau_{diff} = \sum \vec{L}_{arm} \times \vec{T}$ . An off-center thrust vector line would make that arm displacement a non-orthogonal vector. The center of gravity for each module is time varying and depends on its two servo rotational positions. It's more prudent to ensure intersection of the thrust vector with the rotational center than to balance the masses undergoing rotation. A thrust varying torque is harder to approximate and hence compensate for than a gravitational torque, given the complexity with modelling a propeller's aerodynamic thrust (Section:3.3.1).

The primary body structure, similar to a traditional quadcopter '+' configuration, suspends each motor module's rotational assembly with silicon damping balls (Fig:2.2b). For damping to be effective there has to be roughly equatable relative masses between the two damped bodies. A smaller damping assembly in the center of the frame houses all the electronics and power distribution circuitry. All the mounting brackets which affix the motor module rings are 3D printed from CAD models using an Ultimaker V2+ [100]. There is a complete bill of materials for all parts used, including working drawings for each 3D printed bracket and the laser cut frame(s), in Appendix:B.

The propellers rotational plane is not aligned exactly with the plane made by the  $\hat{X}_{M_i}$  and  $\hat{Y}_{M_i}$  rotational servo axes (Fig:2.3). The offset is approximately 23.39 mm and must be considered when evaluating pitch/roll gyroscopic torque responses later in Section:3.2.1. The propellers are 6 inch ( $6 \times 4.5$ ) 3-Blade plastic Gemfam propellers, powered by Cobra CM2208-2000KV Brushless DC motors. The thrust produced as a function of angular velocity (in RPS) for the propellers is derived in Section:3.3.1.



**Figure 2.3:** Difference between propeller and motor planes

The BLDC motors are controlled with LDPower 20A ESC<sup>1</sup> modules with an in-line OrangeRx RPM Sensor. The transfer function for the combined unit is presented next in Section:2.4.1. Power for the quadrotor is supplied not from a battery bank but from a power tether. Tethered power will ensure consistent flight time and reduce the concern of payload restriction on the available lift actuation. Power lines to both the BLDC motors and servos are supplied through conventional wiring, however an ideal construction would see slip-rings for each module's power supply.

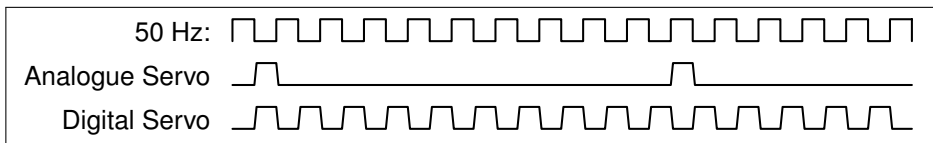


(a) Cobra CM2208-2000KV BLDC Motor Module



(b) Corona DS-339MG Servo Bracket

Metal gear Corona DS-339MG digital servos are used for the two axes of rotation (Fig:2.4b). Each servo has a rotational range of  $\approx \pi$ , positioned such that a zero<sup>th</sup> offset aligns the motor modules, adjacent to the body frame, and has a  $\pm\pi/2$  range. A digital servo updates at 330 Hz, faster than a 50 Hz analogue servo equivalent (Table:2.1). This means the otherwise 20ms zero-order "analogue" sampling becomes a less significant 3.30ms zero-order holding time. Both the  $\hat{X}_{M_i}$  and  $\hat{Y}_{M_i}$  axis servos will be rotating a large loading mass and so their *open loop* plant dynamics are determined empirically in Section:2.4.1.



**Table 2.1:** Analogue & Digital Timing Signals

<sup>1</sup>Flashed with BLHeli [9] firmware. The default firmware on the speed controllers has an unsatisfactory exponential approach (not linear) input speed curve.

## 2.2 Conventions Used

The attitude conventions used for deriving the system's dynamics, in the next Chapter:3, are first briefly discussed here. Often these aspects are assumed to be obvious enough that they're omitted. It's important to clearly and unambiguously define a standard set of framing conventions to avoid uncertainty later. Rotation matrices are included but the focus is on the *contrast* between rotation and transformation operations. Both [35] and [76] provide an in depth and thorough explanation of rotation matrices and DCM attitude representation if such concepts are unfamiliar to the reader. Quaternions are introduced to replace rotation matrices in Section:3.1.3.

### 2.2.1 Reference Frames Convention



**Figure 2.5:** Inertial and Body Reference Frames

Euler (aerospace) frames are used for principle cartesian inertial and body coordinate representation (Fig:2.5). The inertial frame,  $\mathcal{F}^I$ , is aligned such that the  $\hat{X}_I$  axis is in the  $\hat{N}$ orth direction,  $\hat{Y}_I$  is in the  $\hat{E}$ ast direction and  $\hat{Z}_I$  is in the  $\hat{D}$ ownward direction<sup>2</sup>. The body frame,  $\mathcal{F}^b$ , then has both  $\hat{X}_b$  and  $\hat{Y}_b$  aligned obliquely between two perpendicular arms of the quadrotor's body and the  $\hat{Z}_b$  axis in the body's normal direction (Fig:2.8). The body frame's axes and their relation to the prototype design are highlighted next in Section:2.2.2. Frame superscripts  $I$  and  $b$  represent inertial and body frames respectively whilst vector subscripts imply the reference frame in which the vector's coordinates exists or taken relative to. In certain cases a preceding super-script

Relative angular displacement between two frames is commonly measured by the three angle Euler set. The Euler angles  $\vec{\eta} = [\phi \ \theta \ \psi]^T$  represent rotations about the  $\hat{X}$ ,  $\hat{Y}$  and  $\hat{Z}$  axes respectively. Depending on how the rotation sequence is formulated, those angles can be used to construct rotation matrices which give relation to vectors or can transform coordinates. The generic equation to *rotate* a vector  $\vec{v}$  about some (normalized) axis  $\hat{n}$  by some angle  $\mu$  is given by<sup>3</sup>:

$$\vec{v}' = (1 - \cos(\mu))(\vec{v} \cdot \hat{n})\hat{n} + \cos(\mu)\vec{v} + \sin(\mu)(\hat{n} \times \vec{v}) \quad (2.1)$$

Which, when  $\hat{n}$  is either  $\hat{X}$ ,  $\hat{Y}$  or  $\hat{Z}$  axes, can be simplified to produce the fundamental rotation matrices  $\mathbb{R}_x(\phi)$ ,  $\mathbb{R}_y(\theta)$  and  $\mathbb{R}_z(\psi)$ .

<sup>2</sup>In orbital sequences this would be toward the Earth's center. Sometimes referred to as the NED convention

<sup>3</sup>Derived and proven in *Quadrotor Dynamics and Control* [82]



Multiplication by a rotation matrix  $\mathbb{R}_u(\cdot)$  applies a left-handed *rotation* operator about some axis  $\hat{u}$ , the resultant vector still exists in the same reference frame. An  $\hat{X}$  axis rotation by  $\phi$  is;

$$\vec{v}' = \mathbb{R}_x(\phi)\vec{v} \quad (2.2a)$$

$$\vec{v}', \vec{v} \in \mathcal{F}^1 \quad (2.2b)$$

*No subscripts are used in Eq: 2.2 to indicate reference frame ownership because all vectors are in the same frame*

A vector *transformation* changes the resultant vector's reference frame. The transformation is then a rotation by an angle of the *difference* between the resulting and principle reference frames. A transformation from frame  $\mathcal{F}^1$  to  $\mathcal{F}^2$ , differing by an angle of  $\phi$  about the  $\hat{X}$  axis is then:

$$\vec{v}_2 = \mathbb{R}_x(-\phi)\vec{v}_1 \quad (2.3a)$$

$$\vec{v}_2 \in \mathcal{F}^2 \text{ and } \vec{v}_1 \in \mathcal{F}^1 \quad (2.3b)$$

The distinction between Eq:2.2 and Eq:2.3 is the directional sense of the angular operand  $\phi$ , and hence the effect it has on the argument vector. The transformation or rotation of a vector from  $\mathcal{F}^I$  to  $\mathcal{F}^b$  is the product of three sequential operations about each axis. Each subsequent rotation is applied relative to a new intermediate frame; hence each Euler angle is taken relative to a specific intermediate frame. The sequence of axial rotation operations does indeed effect the Euler set. Any consequences of that chosen order is something discussed indepth in *Quaternions and Rotation Sequence*, [53]. In this dissertation the Z-Y-X sequence is used. A transformation of the vector  $\vec{v}$  from the inertial to the body frame,  $\mathcal{F}^I \rightarrow \mathcal{F}^b$ , is then applied by:

$$\mathbb{R}_I^b \triangleq \mathbb{R}_z(\psi)\mathbb{R}_y(\theta)\mathbb{R}_x(\phi) \quad (2.4a)$$

$$\vec{v}_b = \mathbb{R}_I^b(-\psi, -\theta, -\phi)\vec{v}_I \quad (2.4b)$$

$$\Rightarrow \vec{v}_b = \mathbb{R}_z(-\psi)\mathbb{R}_y(-\theta)\mathbb{R}_x(-\phi)\vec{v}_I \quad (2.4c)$$

$$\mathbb{R}_z(-\psi)\mathbb{R}_y(-\theta)\mathbb{R}_x(-\phi) \iff \mathbb{R}_x(\phi)\mathbb{R}_y(\theta)\mathbb{R}_z(\psi) = \mathbb{R}_b^I \quad (2.4d)$$

$$\mathbb{R}_I^b = (\mathbb{R}_b^I)^{-1} = (\mathbb{R}_b^I)^T \quad (2.4e)$$

The relationship in Eq:2.4e is an inversion property (*transpose*) of the rotation matrix. A rotation matrix's inverse can be used interchangeably with its negative counterpart to maintain a positive sense of the argument angle. To ensure clarity throughout this dissertation's mathematics, a negative angular sense implies a *transformation* to a different reference frame. Where applicable, the order of rotation will indicate the sequence direction and an angular sign differentiates the rotation and transformation operators.

The body frame's angular velocity is taken relative to the inertial frame, represented by  $\vec{\omega}_{b/I} \Rightarrow \vec{\omega}_b$ . Seeing that each Euler angle is measured with respect to an intermediary frame, a distinction must then be made between  $d\vec{\eta}/dt$  and  $\vec{\omega}_b$ . All three Euler angles need to be transformed to one common frame. Exploiting vehicle frames 1 & 2, or rather  $\mathcal{F}^{v1}$  &  $\mathcal{F}^{v2}$ , as intermediate frames to respectively describe post  $\mathbb{R}_x(\phi)$  and  $\mathbb{R}_y(\theta)$  operations.

$$\vec{\omega}_b = \frac{d}{dt_b}\vec{\eta} = \frac{d\phi}{dt}\mathbb{R}_{v2}^b(\phi) \begin{bmatrix} \phi \\ 0 \\ 0 \end{bmatrix} + \frac{d\theta}{dt}\mathbb{R}_{v2}^b(\phi)\mathbb{R}_{v1}^{v2}(\theta) \begin{bmatrix} \theta \\ 0 \\ 0 \end{bmatrix} + \frac{d\psi}{dt}\mathbb{R}_{v2}^b(\phi)\mathbb{R}_{v1}^{v2}(\theta)\mathbb{R}_I^{v1}(\psi) \begin{bmatrix} 0 \\ 0 \\ \psi \end{bmatrix} \quad (2.5a)$$

*The vehicle frames in Eq:2.5a and the subsequent rotations between each frame don't necessarily have to be in that order. The equation could change depending on what rotation sequence was used.*

Which then simplifies to the formal relationship between two rotating frames, with  $\vec{\omega}_b = [p \ q \ r]^T$  in  $rad.s^{-1}$ :

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\theta) & \cos(\phi)\sin(\theta) \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (2.5b)$$

$$\Rightarrow \vec{\omega}_b = \Psi(\eta)\dot{\vec{\eta}} \quad (2.5c)$$

$$\Psi(\eta) = \begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\theta) & \cos(\phi)\sin(\theta) \end{bmatrix} \quad (2.5d)$$

$$\Rightarrow \dot{\vec{\eta}} = \Psi^{-1}(\eta)\vec{\omega}_b = \Phi(\eta)\vec{\omega}_b \quad (2.5e)$$

$$\Phi(\mathcal{E}) = \begin{bmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{bmatrix} \quad (2.5f)$$

The termed *Euler* matrix,  $\Phi(\eta)$ , contains a well known and problematic singularity at  $\theta = \pm\pi/2$ ; because  $\tan(\theta), \sec(\theta) \rightarrow \infty$  as  $\theta \rightarrow \pi/2$ . The effect of the rotation matrix singularity is further explored later in Section:3.1.2. Its manifestation in the  $\theta$  angle here is a direct consequence of the Z-Y-X rotation sequence used. Each Euler angle can potentially suffer a singularity depending on how the rotations are sequenced. Indeed quaternions are used for kinematics later in lieu of Euler angles. Euler angular attitude representation is, however, easily understood and well suited to the conventional distinctions made in this Chapter.

Quaternion operations are similarly sequenced in the Z-Y-X order:

$$\mathbb{R}_I^b \iff Q_b \otimes (.) \otimes Q_b^* \quad (2.6a)$$

$$Q_b \triangleq Q_z Q_y Q_x \text{ and } Q_b \triangleq Q_x^* Q_y^* Q_z^* \quad (2.6b)$$

With  $\otimes$  being the Hamilton product<sup>4</sup> (or quaternion multiplication). Each quaternion,  $Q_i$ , is a unit quaternion about that  $\hat{i}^{th}$  axis. It is important to note that a quaternion rotation operates on an argument vector with a zero quaternion scalar component. So then for some vector  $\vec{v}$ , the quaternion rotation operation in Eq:2.6a is equivalent to;

$$Q_{\vec{v}}' = Q \otimes (Q_{\vec{v}}) \otimes Q^* \quad (2.7a)$$

$$\text{Where } Q_{\vec{v}} = \begin{bmatrix} 0 \\ \vec{v} \end{bmatrix}, \quad Q_{\vec{v}}' = \begin{bmatrix} 0 \\ \vec{v}' \end{bmatrix} \quad (2.7b)$$

The quaternion representation in Eq:2.7b ensures that the operation is entirely in  $\mathbb{R}^4$  space. However it is usually omitted, despite  $\mathbb{R}^4$  being implied and as such, Eq:2.7a is then simply:

$$\vec{v}' = Q \otimes (\vec{v}) \otimes Q^* \quad (2.8)$$

Quaternion dynamics, and the quaternion operator, are later expanded upon to replace the use of Euler angles and Rotation matrices as a convention for attitude representation later in Chapter:3

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<sup>4</sup>Later used again to transform inertial tensors



### 2.2.2 Motor Axis Layout

Fundamentally the whole structure consists of multiple rigid bodies with only relative rotations to one another permitted, illustrated previously in the design description in Section:2.1. Those rigid bodies are divided into four inter-connected motor modules,  $\mathbb{M}_{1 \rightarrow 4}$  and a single body  $\mathbb{B}$  (*frame*) structure. Each module consists of two sequential gimbal rings, each with one degree of relative rotation between itself and the next subsequent ring. There needs to be distinct nomenclature used for describing these motor modules such that their dynamic derivations are contextually clear and logical.



Figure 2.6: Aligned Motor Frame Axes

Every propeller/motor pair is actuated by two servos. The  $i^{th}$  propeller, directly driven by the motor's rotor, has a rotational speed  $\Omega_i$  [RPS] about the  $\hat{Z}$  stator axis. Two servos are aligned *at rest* with the motor's  $\hat{Y}$  and  $\hat{X}$  axes to pitch and roll the propeller away from its principle frame. Each motor has its own reference frame,  $\mathcal{F}^{M_i}$ , aligned in Fig:2.6 and highlighted with the rotational rings in Fig:2.7.

Motor frames, numbered 1 – 4, transform to the body frame first by an angle of  $\lambda_i$  about the  $\hat{X}_{M_i}$  axis. Then by  $\alpha_i$  about the  $\hat{Y}_{M'_i}$  axis in an intermediate  $M'_i$  frame. The first servo actuates  $\lambda_i$ , rotating  $\mathcal{F}^{M_i}$  to an intermediate  $\mathcal{F}^{M'_i}$  frame. Secondly, the next servo actuates  $\alpha_i$  to produce a second intermediate frame  $M''_i$ . That second servo is affixed in the  $M''_i$  frame. Lastly there's a relative orthogonal rotation about  $\hat{Z}_{M''_i}$  between  $\mathcal{F}^b$  and  $\mathcal{F}^{M''_i}$ . Each module's actuation state is fully described by  $[\Omega_i, \lambda_i, \alpha_i]^T$  for  $i \in [1 : 4]$ . The four motor modules are aligned relative to the body's XYZ axes as shown in Fig:2.8. Modules 1 and 3 have their X-axes in the positive and negative  $\hat{X}_b$  directions of the body frame respectively. Similarly Modules 2 and 4 have their X-axes in the positive and negative  $\hat{Y}_b$  directions of the body frame.



Figure 2.7: Intermediate Motor Frames



Figure 2.8: Body Frame Axes Layout

Not shown in Fig:2.8 is the relative  $\hat{Z}$  axis position with respect to the structure. The  $\hat{Z}$  height of the body's motion centroid is such that its origin is co-planar with the four motor modules rotational centers. The center of motion is not the center of mass, an aspect which is discussed later in Section:??.

Vector transformations from each of the four motor frames to the body frame are characterized as:

$$\vec{v}_b = \mathbb{R}_z(-\sigma_i)\mathbb{R}_y(-\alpha_i)\mathbb{R}_x(-\lambda_i)\vec{v}_{M_i}, \quad \sigma_i \in \frac{1}{2}[0, \pi, 2\pi, 3\pi] \quad (2.9a)$$

With orthogonal rotation matrices  $\mathbb{R}_z$ :

$$\mathbb{R}_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ for } i \in [1, 2, 3, 4] \text{ respectively} \quad (2.9b)$$

The entire actuator space, including propeller speed  $\Omega_i$  [RPS], is then  $\in \mathbb{R}^{12}$ , or rather  $\mathbb{U} \in \mathbb{R}^{12}$ , in contrast with  $\mathbb{U} \in \mathbb{R}^4$  for a normal quadrotor. The actuator input set  $u \in \mathbb{U}$  is then structured as:

$$u_{\in \mathbb{U}} = [\Omega_1 \ \lambda_1 \ \alpha_1 \ \dots \ \Omega_4 \ \lambda_4 \ \alpha_4]^T \quad (2.10)$$

## 2.3 Inertial Matrices & Masses

*Although inertias are presented here rounded to either 2 or 0 decimal places, full floating point numbers are used in simulation and prototype software. In some cases when transforming inertias it's more appropriate to use rotation matrices to apply the transformation and not quaternions. Spatial rotation of inertial tensors are ill suited to quaternion parametrization.*

### Inertias

An undesirable side effect of the relative rotations within a non-rigid body are the inertial responses associated with such movements. Given Newton's Second Law of Rotational Motion<sup>†</sup>, each applied rotation is going to produce an equal but opposite reaction onto the principle inducing frame. Similarly a gyroscopic cross product from rotational velocities is also present. Such first and second order effects are often neglected given that the angular rates which they're dependent on are mostly small enough to approximate as zero,  $\vec{\omega}_b \approx \vec{0}$ . A dynamic set-point (non-zero) attitude tracking plant is, however, going to produce sizable time varying body angular velocities and accelerations. Unlike a traditionally actuated quadrotor, such effects have to be accounted for.



**Figure 2.9:** Inertial Measurement References

The manifestation of those responses are explored next in Section:3.2.1. Both of those effects are dependent on the rotational body's inertial tensor<sup>5</sup> about each respective rotational axis. The magnitude of those inertias are obviously a by-product of the structure's design. Starting with the innermost assembly, in each Motor Frame  $\mathcal{F}^{M_i}$ , the inner ring structure is a 92g body (all components incorporated). The rotational center *roughly* coincides with the center of its mass ( $C.M = [-1.44 \ 0 \ 5.81]^T$  [mm] relative to its rotational center). The inner ring being rotated by  $\lambda_i$  about the  $\hat{X}_{M_i}$  axis then has an inertial matrix (centered and aligned with axes as in Fig:2.9a):

$$\mathbb{I}_{M_i} = \begin{bmatrix} 561.96 & -32.29 & -0.26 \\ -32.29 & 1888.74 & 0.00 \\ -0.26 & 0.00 & 2090.97 \end{bmatrix} \quad [g.cm^2] \quad (2.11a)$$

$$\approx \text{diag}(562, 1889, 2091) \times 10^{-7} \quad [kg.m^2] \quad (2.11b)$$

The effect of rapidly spinning propellers on the inertia in Eq:2.11a is approximated well by a solid disc, hence the inner ring's inertial components are regarded as constant. The moment of inertia about that  $\hat{X}_{M_i}$  rotational axis, pertinent to a  $\lambda_i$  rotation, is then  $\mathbb{I}_\lambda \approx 531 \times 10^{-7} \quad [kg.m^2]$ .

<sup>5</sup>All inertias are assumed symmetrical and calculated in SolidWorks with overridden masses to match physical prototype measurements, all those values are included in Appendix:B

The first  $\lambda_i$  actuating servo and bearing supports are affixed to the intermediate middle ring assembly (Fig:2.9b). The middle ring frame,  $\mathcal{F}^{M'_i}$ , is a 98g body, excluding the inner most ring. Collectively the mass for both the inner and middle rings assemblies is  $m_{module} = 190g$ . The middle ring is rotated by  $\alpha_i$  about its  $\hat{Y}_{M'_i}$  axis. The compound body's inertia about that axis of rotation,  $\hat{Y}_{M_i}$ , is a combination of both the middle ring's inertia and the inner ring's. The latter being a function of the *rotation* (not transformation) angle  $\lambda_i$  which, from the conservation of angular momentum theory [93]<sup>6</sup>, is:

$$\text{If } \mathbb{I}_{middle} = \begin{bmatrix} 2905.70 & 0.02 & 390.89 \\ 0.02 & 8446.41 & 0.01 \\ 390.89 & 0.01 & 11125.74 \end{bmatrix} [g.cm^2] \quad (2.12a)$$

$$\mathbb{I}_{M'_i} = \mathbb{I}_{middle} + \mathbb{R}_x(\lambda_i)(\mathbb{I}_{inner})\mathbb{R}_x^{-1}(\lambda_i) \quad (2.12b)$$

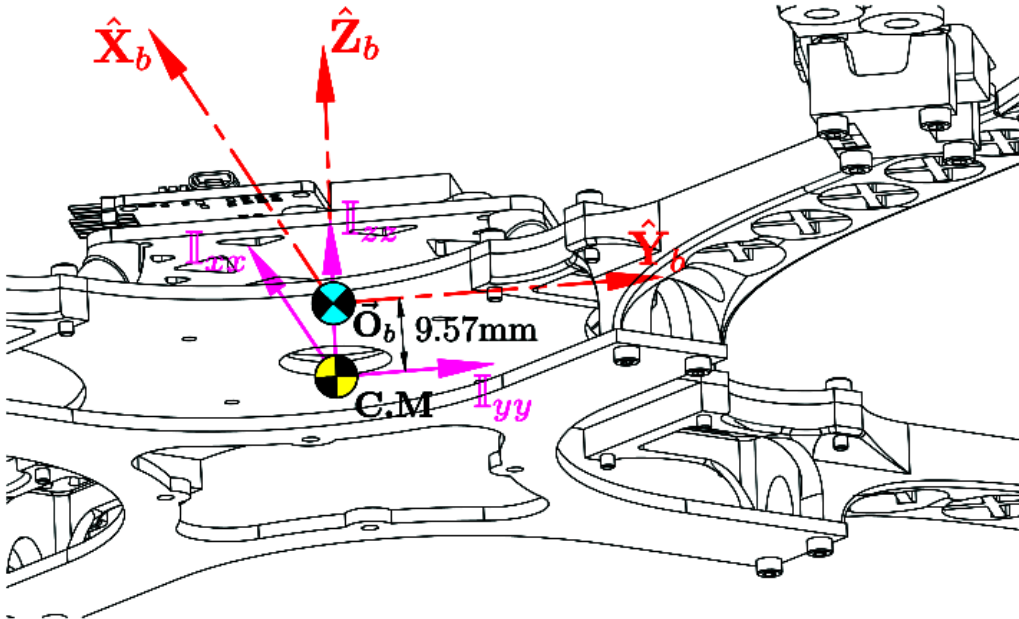
$$\mathbb{I}_{M'_i}(\lambda_i) = \mathbb{I}_{const} + \mathbb{I}_{M_i}(\lambda_i) \quad (2.12c)$$

$$\approx \begin{bmatrix} 3468 & 0 & 391 \\ 0 & 10436 & 0 \\ 391 & 0 & 13155 \end{bmatrix} + \begin{bmatrix} 0 & -32c_\lambda & -32s_\lambda \\ -32c_\lambda & -101c_{2\lambda} & 101s_{2\lambda} \\ -32s_\lambda & 101s_{2\lambda} & 101c_{2\lambda} \end{bmatrix} \times 10^{-7} [kg.m^2] \quad (2.12d)$$

With  $\mathbb{I}_{inner} = \mathbb{I}_{M_i}$  being the inertia from Eq:2.11a, re-orientated through a rotation  $\mathbb{R}_x(\lambda_i)$ . The net inertia is then a combination of an inertia as a function of the rotation angle  $\lambda_i$  and a constant inertia (Eq:2.12c), which together are then simplified<sup>7</sup> to Eq:2.12d. It's important to note the two non-zero products of inertia,  $\mathbb{I}_{yx}$  and  $\mathbb{I}_{yz}$ , which are going to result in a vector torque  $\vec{\tau}_\eta$  response. The inertia then encountered by an  $\alpha_i$  rotation is:

$$\mathbb{I}_\alpha(\lambda) \approx [-32c_\lambda, 10436 - 101c_{2\lambda}, 101s_{2\lambda}]^T \times 10^{-7} [kg.m^2] \quad (2.13)$$

Variable inertias dependent on state input variables are one of many non-trivial aspects unique to the multi-body interconnected design. Control solutions are thus decidedly plant dependent in their formulation. The center of mass for each motor module's compound assembly coincidental with neither rotational axes' intersection. As a result the effective center of mass for each module, and the entire structure, is going to be a time varying function each motor module's angular rotational position.



**Figure 2.10:** Body Frame Center of Mass

<sup>6</sup> $\mathbb{R}_x$  is a full rank and square, so an inverse  $\mathbb{R}_x^{-1}$  always exists.

<sup>7</sup>Eq:2.12d is rounded to no decimal places, seeing that its units are already  $\times 10^{-7}$  and thus some products of inertia are omitted.

The second  $\alpha_i$  rotating servo adjoins the complete motor module (both the inner and middle ring assemblies) to the body structure. The inertial volume of the servo and bearing supports contribute then to the body structure's inertia, whose value excludes any of the four motor modules. Consisting of servo and bearing damping brackets, each "damping" assembly collectively weighs 84g and suspends the motor modules from the body frame with a set of silicon damping balls. The body structure's center of mass (without motor modules, Fig:2.10) coincides with the XY directional axes and lies  $\Delta Z = -9.57 \text{ mm}$  below the Body Frame's origin of motion,  $\vec{\mathbf{O}}_b \in \mathcal{F}^b$ .

*Note: the origin which all motion is calculated with respect to is co-planar to the motor module's rotational centers, not the net center of mass.*

The body's weight, including all four damping assemblies and electronics, totals to 814.7 g. The body's net inertia (*sans* motor modules)  $\mathbb{I}_{body}$ , about its center of mass (Fig:2.10) is:

$$\mathbb{I}_{body} = \begin{bmatrix} 181689.67 & -0.44 & -8.86 \\ -0.44 & 181567.22 & -19.44 \\ -8.86 & -19.44 & 360077.58 \end{bmatrix} \times 10^{-7} \quad [kg.m^2] \quad (2.14a)$$

Using the Parallel Axis theorem<sup>†</sup>, that same net body inertia about the body frame's origin,  $\vec{\mathbf{O}}_b$ , is:

$$\mathbb{I}_{body}' = \mathbb{I}_{body} + m(\vec{d} \cdot \vec{d} + \vec{d} \otimes \vec{d}) \approx \mathbb{I}_{body} + md^2 \quad (2.14b)$$

*Here  $\otimes$  represents the Hamilton product of two 3X3 matrices, it's used elsewhere to indicate the quaternion multiplication operator. The vector  $\vec{d}$  is the difference between the center of mass C.M and the body frame origin  $\vec{\mathbf{O}}_b$ .*

$$\mathbb{I}_{body}' = \begin{bmatrix} 182435.66 & -0.42 & -6.46 \\ -0.42 & 182313.18 & -14.52 \\ -6.46 & -10.41 & 360077.62 \end{bmatrix} \times 10^{-7} \quad [kg.m^2] \quad (2.14c)$$

Net inertia for the compound assembly,  $\mathbb{I}_b$ <sup>8</sup>, about the origin  $\vec{\mathbf{O}}_b$  is a combination of all the relative attached bodies. That being; the four motor modules, transformed and then translated to the body origin, and the body structure itself. That transformation from the motor frame  $\mathcal{F}^{M_i}$  to the body frame  $\mathcal{F}^b$  is analogous to that of Eq: 2.9. Reiterating that the the origin is co-planar to the module's center of rotation, each motor module's inertia,  $\mathbb{I}_{M_i}'$ <sup>9</sup>, is further rotated by  $\alpha_i$  about  $\hat{Y}_{M_i}'$  and finally an orthogonal  $\hat{Z}_{M_i}''$  rotation (aligned with  $\hat{Z}_b$ ) onto  $\mathcal{F}^b$ . Still measured with respect to their individual rotational centers,  $\vec{\mathbf{M}}_i$ , but re-orientated to align with  $\|\vec{\mathbf{O}}_b$ . Contribution of each motor module's inertia, with  $\mathbb{R}_z$  being the same as Eq:2.9b, is then:

$$\mathbb{I}_{i^{th}motor} = \mathbb{R}_z(\sigma_i) \mathbb{R}_z(\alpha_i) (\mathbb{I}_{M_i}'(\lambda_i)) \mathbb{R}_y^{-1}(\alpha_i) \mathbb{R}_z^{-1}(\sigma_i) \quad (2.15a)$$

Or expanding that to separate inner and middle ring components:

$$= \mathbb{R}_z \mathbb{R}_y(\alpha_i) (\mathbb{I}_{middle}) \mathbb{R}_y^{-1}(\alpha_i) \mathbb{R}_z^{-1} + \mathbb{R}_z \mathbb{R}_y(\alpha_i) \mathbb{R}_x(\lambda_i) (\mathbb{I}_{inner}) \mathbb{R}_x^{-1}(\lambda_i) \mathbb{R}_y^{-1}(\alpha_i) \mathbb{R}_z^{-1} \quad (2.15b)$$

$$\text{With axes } \hat{X} \in \mathcal{F}^{M_i}, \hat{Y} \in \mathcal{F}^{M_i}', \hat{Z} \in \mathcal{F}^{M_i}'' \quad (2.15c)$$

*It's at this stage that, despite simplifications, the symbolic inertial equations all become overly cumbersome to include with numeric values... For the sake of brevity, exact calculated inertial values for the input dependent plant are omitted.*

Each module's rotational center ( $[\pm 195.16 \ 0 \ 0]$  &  $[0 \ \pm 195.16 \ 0]$  recalling Fig:2.8) are spaced equally relative to  $\vec{\mathbf{O}}_b$  with a parallel axis arm  $\vec{L}_{arm} = [195.16 \ 0 \ 0]^T \text{ [mm]}$  (Fig:2.11). The net inertial equation about  $\vec{\mathbf{O}}_b$ , dependent on the actuator suite  $\mathbb{U}$  positions, can be calculated as:

$$\mathbb{I}_b(u) = \mathbb{I}_{body} + \sum_{i=1}^4 \mathbb{M}_i \quad [kg.m^2] \quad (2.16a)$$

<sup>8</sup>Disambiguation:  $\mathbb{I}_b$  is *net* body frame's inertia, different from  $\mathbb{I}_{body}$  which is the inertia for *just* the body structure

<sup>9</sup>As defined in Eq:2.12d



Figure 2.11: Inertial Center &amp; Mass Center

$$\mathbb{M}_i = \mathbb{I}_{i^{th}motor} + m_{module}(\vec{L} \cdot \vec{L} - \vec{L} \otimes \vec{L}) \quad (2.16b)$$

Although Eq:2.16 does indeed produce the net body's inertia, the transformations to calculate  $\mathbb{M}_i$  are compounded. Motor module inertias are first translated to their centers of rotation from their respective center of masses and then finally to the body frame's origin. Subsequent transformations are successively going to deteriorate the floating point precision of the resultant inertial tensor. Transforming inertial tensors about each sub-body's center of mass directly to the body frame origin will improve the reliability of the produced inertial equations. It is perhaps more intuitive to consider each sub-body's contribution individually, despite having been derived as combined inertial systems previously.

$$\mathbb{I}_b(u) = \mathbb{I}_{body} + \sum_{i=1}^4 \mathbb{M}_{inner} + \sum_{i=1}^4 \mathbb{M}_{middle} \quad (2.17)$$

The relative movement pertinent to Eq:2.11 and Eq:2.12 are separate from those affecting Eq:2.16. For each inner ring, W.R.T its center of mass measured relative to its center of rotation, different from Eq:2.11a, the inner ring's inertia is calculated as;

$$m_{inner} = 92 \text{ [g]} \quad (2.18a)$$

$$\mathbb{I}_{inner}^{C.M} = \begin{bmatrix} 530.88 & -32.29 & 7.46 \\ -32.29 & 1855.74 & 0 \\ 7.46 & 0 & 2088.87 \end{bmatrix} \text{ [g.cm}^2\text{]} \quad (2.18b)$$

$$C.M_{inner} = [-1.44 \ 0 \ 5.81]^T \text{ [mm]} \quad (2.18c)$$

$$C.M'_{inner} = \mathbb{R}_z \mathbb{R}_y(\alpha_i) \mathbb{R}_x(\lambda_i) (C.M_{inner}) \quad (2.18d)$$

$$\mathbb{I}_{inner} \parallel \vec{O} = \mathbb{R}_z \mathbb{R}_y(\alpha_i) \mathbb{R}_x(\lambda_i) (\mathbb{I}_{inner}) \mathbb{R}_x^{-1}(\lambda_i) \mathbb{R}_y^{-1}(\alpha_i) \mathbb{R}_z^{-1} \quad (2.18e)$$

$$\Delta L = \vec{L}_{arm} - C.M'_{inner} \quad (2.18f)$$

$$\mathbb{M}_{inner} \parallel \vec{O} = \mathbb{I}_{inner} \parallel \vec{O} = \mathbb{I}_{inner} + m_{inner}((\Delta L \cdot \Delta L) \mathbb{I}_{3 \times 3} - \Delta L \otimes \Delta L) \quad (2.18g)$$



Similarly for the middle rings:

$$m_{middle} = 98 \quad [g] \quad (2.19a)$$

$$\mathbb{I}_{middle}^{C.M} = \begin{bmatrix} 2879.06 & 172.29 & 223.58 \\ 172.29 & 6268.97 & 13.33 \\ 223.58 & 13.33 & 8947.52 \end{bmatrix} \quad [g.cm^2] \quad (2.19b)$$

$$C.M_{middle} = [-47.00 \quad 3.74 \quad -3.63]^T \quad [mm] \quad (2.19c)$$

$$C.M'_{middle} = \mathbb{R}_z \mathbb{R}_y(\alpha_i)(C.M_{middle}) \quad (2.19d)$$

$$\mathbb{I}_{middle} = \mathbb{R}_z \mathbb{R}_y(\alpha_i)(\mathbb{I}_{middle})\mathbb{R}_y^{-1}(\alpha_i)\mathbb{R}_z^{-1} \quad (2.19e)$$

$$\Delta L = \vec{L}_{arm} - C.M_{middle}' \quad (2.19f)$$

$$\mathbb{M}_{middle} = \mathbb{I}_{middle} = \mathbb{I}_{middle} + m_{middle}((\Delta L \cdot \Delta L)\mathbb{I}_{3 \times 3} - \Delta L \otimes \Delta L) \quad (2.19g)$$

Unless otherwise specified; any inertia  $\mathbb{I}_b(u)$ , irrespective of arguments, will refer to an instantaneous calculated solution to Eq:2.17 given a particular  $u(t) \in \mathbb{U}$ . The purpose of the derivations Eq:2.18 & Eq:2.19 is twofold; highlighting both the inertial contributions and the variable center of masses for each sub-body. Seeing that the origin of the motion frame  $\mathcal{F}^b$  and the net body's center of mass aren't coincidental, it's important to quantify the equation for the varying center of mass. If, for a collection of  $n$  bodies, with each body's center at some position  $\vec{X}_i$  and a mass  $m_i$ , the net center of mass is:

$$C.M = \frac{\sum_{i=1}^n m_i \cdot \vec{X}_i}{\sum_{i=1}^n m_i} \quad (2.20a)$$

Such that, with  $\vec{X}_{inner}$  &  $\vec{X}_{middle}$  being rotated centers of mass defined in Eq:2.18d & Eq:2.19d respectively, the entire assembly has a center of mass<sup>†</sup>:

$$C.M(u) = \frac{m_{body} \cdot \vec{X}_{body} + \sum m_{inner} \cdot \vec{X}_{inner} + \sum m_{middle} \cdot \vec{X}_{middle}}{m_{body} + \sum m_{inner} + \sum m_{middle}} \quad (2.20b)$$

Making the resultant gravitational torque<sup>10</sup> about the origin  $\vec{\mathbf{O}}_b$  at any given moment:

$$\Delta C.G = \vec{\mathbf{O}}_b - C.M \quad (2.20c)$$

$$\tau_g = \Delta C.G \times \vec{G}_b \quad [N.m], \tau_g \in \mathcal{F}^b \quad (2.20d)$$

The net mass for the whole assembly is 1574 g. For reference, the center of gravity when all actuators are at their zero positions is:  $C.M = [-0.02 \quad -0.03 \quad -4.5] \quad [mm]$ . Then, according to Eq:2.17, the inertial tensor for the net assembly at the rest conditions,  $u = \vec{0}$ , about the origin  $\vec{\mathbf{O}}_b$  is:

$$\mathbb{I}_b(\vec{0}) = \begin{bmatrix} 317784.78 & -0.42 & -6.46 \\ -0.42 & 317662.31 & -14.52 \\ -6.46 & -14.52 & 628430.75 \end{bmatrix} \quad [g.cm^2] \quad (2.21)$$

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<sup>10</sup>With  $\vec{G}_b = \mathbb{R}_I^b \vec{F}_g \quad [N]$

## 2.4 Electronics

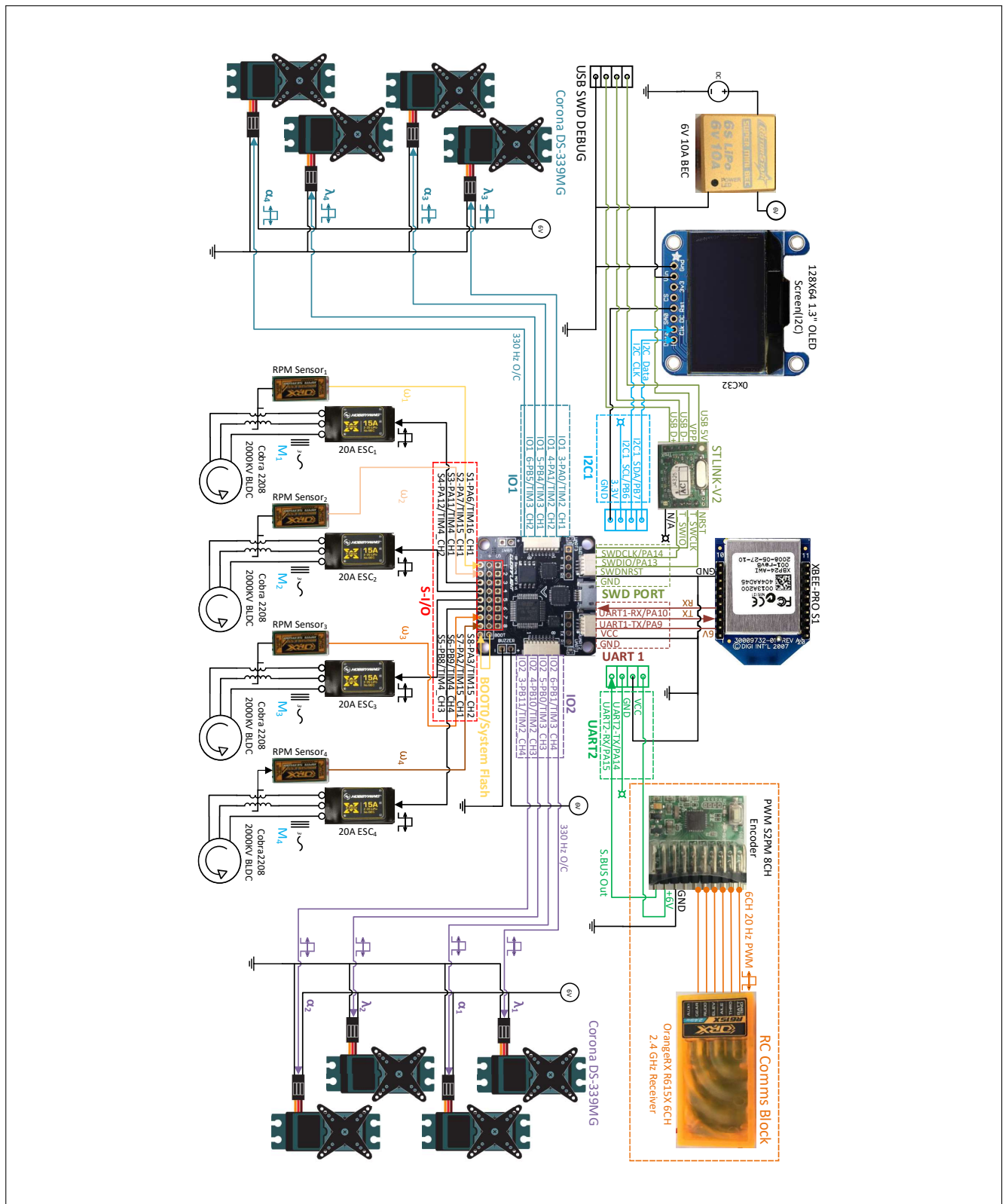


Figure 2.12: Hardware Schematic Diagram



An abstracted hardware diagram for the (electronic) system layout is shown in Fig:2.12. It's an illustration for the connection of different electronic peripherals used to aid the on-board control system. The structure of the autopilot system and control loops are addressed later in Chapter:6. This section aims to provide a brief overview of the specific modules used, their purpose and a description of how they're interfaced. No code structure or control loops are considered yet...



(a) SPRacing F3 Deluxe Flight Controller



(b) SBUS Converter &amp; 6CH Receiver Modules

**Figure 2.13**

The entire system is centered around an ARM STM32F303 [92] based  $\mu$ C. The micro-processor board is a commercial flight control board, specifically an SPRacing F3 Deluxe [21]<sup>11</sup>, which has had its boot-loader removed and custom firmware, unique to this project, developed for it. That software is later described in Chapter:6; the I/O for all the peripherals are detailed here however. The flight-controller has the following onboard peripherals: an I2C MPU-6050 [43] 6-axis gyroscope & accelerometer with an I2C connected HMC5883 [27] magnetometer compass, an I2C MS5611 [90] barometer and finally 64 Mb of SPI flash memory. The electrical schematic of those peripherals and the core STM32F303 is detailed in Appendix:B.2 but their connection(s) are shown in Fig:2.12.

*The combination of above sensors fused for state estimation and their associated algorithms are dealt with in Section:5.3 in Chapter:5.*

**Figure 2.14: S.BUS Data Stream**

Two wireless communication peripherals are used. First the system relays full state information, for a complete 6-DOF autopilot system, from a ground control station using 2.4 GHz XBEE S1 module(s) [44], USART connected. Secondly, an augmented pilot control input system, fail safe and secondary to the autopilot loop, is transmitted through 6 Channel 2.4 GHz R/F comms. The 6 CH received signals, otherwise permeated as six individual 20 KHz PWM signals via an OrangeRx R615x [72] receiver, are encoded into a single proprietary S.BUS data stream.

<sup>11</sup>CleanFlight opensource software is regularly used for the F3 but its hardware specifications are not openly available. The reverse engineered electrical schematic for the board is included in Appendix:B.2

The need for an S.BUS encoder [40] comes about as a consequence of the introduction of the 8 additional servos. As a result, there are no longer 6 free additional timer I/O channels which can be dedicated to input capture. Encoding the received data to a serial communication protocol means the 6CH data can be processed on a single serial RX line. The S.Bus encoder implements a USART derivative communications standard, Fig:2.14 shows the sampled data stream used to ascertain the standard's following parameters:

- 25 Bytes per packet
- 8-Bit byte length
- 1 Start byte 0x240
- 1 Byte of flags
- 1 Stop byte 0x0
- Bytes are:
  - MSB First
  - 1 start & 2 stop bits
  - Even parity bit
  - Inverted
  - 100000 baud (bps)
- 22 bytes of CH data
- Each channel's data is 11 bits long
- 16CH encoded
- Channel data is little endian prioritized
- 14 ms idle time between packets
- Packets are arranged:



The received information from the transmitted 6 channels is filtered through a moving average IIR<sup>†</sup> filter. The filter's difference equation is as follows:

$$y_n = \left(1 - \frac{1}{N}\right)y_{n-1} + \frac{1}{N}x_n \quad (2.22)$$

Moving over an average of  $N = 5$  samples. The signal's sample rate is sufficiently fast enough such that the digital filter's frequency response isn't of consequence. Similarly all the measured RPM signals are filtered as well. Any received signals referred to are all post filtration. Filtering for state estimates is separately performed on the Ground Control Station computer.

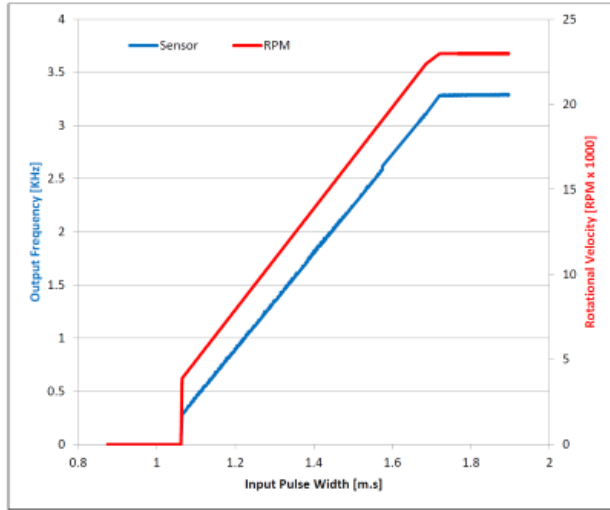
Each of the eight digital servo actuators are driven individually from 330 Hz PWM timer output compare channels (TIM2:CH1→CH4 & TIM3:CH1→CH4). Output pulses typically range from 1ms - 2ms to linearly control the rotary position. The exact range and transfer function is empirically determined next in Subsection:2.4.1. The four 20A brushless DC speed controllers (ESCs) are each driven from a 20 Hz PWM output (TIM4:CH1→CH4), similarly with 1ms - 2ms pulse widths. There are a total of 12 PWM output compare signals drawn from the  $\mu$ C. Servos are powered by a regulated 6V DC 10A power supply [39] whilst the ESCs switch unregulated 15.1 V DC from an externally tethered power supply. The DC supply could potentially be drawn from an on-board battery bank but that would add significant weight to an already heavy platform.

There's no integrated feedback for instantaneous RPM values from the ESCs. Using discrete OrangeRX BLDC RPM sensors [38], that measure switching phases across two of the three motor phases, the exact RPM can be ascertained. The switching signal of a 3-Phase<sup>12</sup> is [59]:

$$F_{rps} = \frac{2 \times F_{poles}}{\text{No. of rotor poles}} \quad [Hz] \quad (2.23)$$

The signal produced by the RPM sensors varies a 50% duty cycle square wave's period, directly proportional to that pole switching frequency. The RPM sensor's output signal is then calibrated to a gain of 7 for the 14 pole BLDC Cobra motors used. That gain is verified with the linear relationship(s) is shown in Figs:2.15. Knowing exact RPM rates means the subsequent thrust and aerodynamic torques for the control plant inputs can be calculated.

<sup>12</sup>Although called BLDC motors, the motors are actually 3-Phase IM motors which, when combined with an ESC, behave in closed loop like BLDC motors.



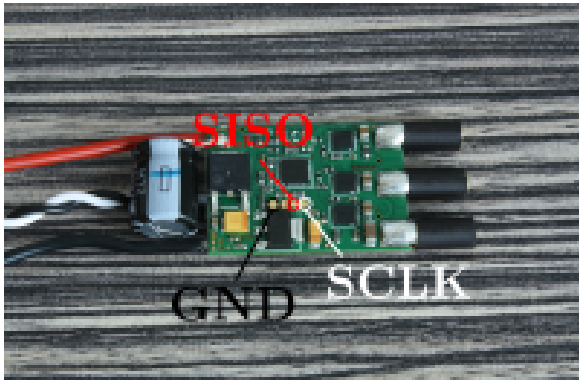
(a) RPM sensor plot - no load



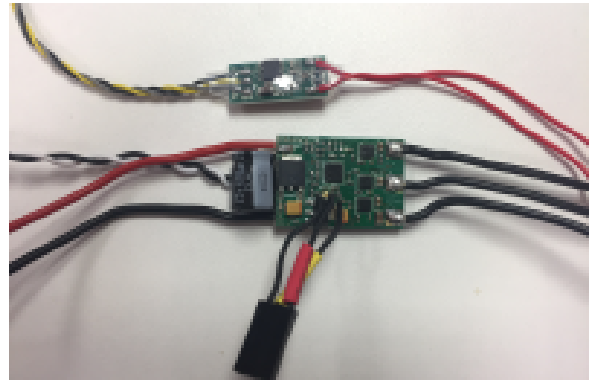
(b) RPM sensor plot - 6" prop

**Figure 2.15:** RPM sensor calibration plots

The speed controllers, although LDPower 20A devices, are all re-flashed with BLHeli<sup>13</sup> [9] firmware. The custom software on the ESC's  $\mu$ C can provide greater refinement over parameter configuration like the deflection range of inputs, however, default values were used. The plot in Fig:2.15a shows the linear RPS (in Hz) speed curve for an unloaded motor; similarly in Fig:2.15b shows the speed curve when loaded for a 6-inch prop. It's interesting to note that the loaded speed curve is slightly parabolic, this is from the aerodynamic drag term which is quadratic with respect to rotational velocity, expanded on in Section:3.3.1. Moreover, when the motor is torque loaded by the propeller, the ESC current limits rotational speeds at just over 16 000 RPM. The sensor feedback is used for minor loop RPM control.



(a) XRotor 20A ESC connection guide [37]



(b) LDPower 20A ESC

**Figure 2.16:** BLDC electronic speed controllers

Timers channels are used to measure the varying frequency of the RPM sensors. General purpose Timers 15 (TIM15:CH1→CH2), 16 (TIM16:CH1) and 17 (TIM17:CH1) are configured to capture the input PWM signal generated by the speed sensors. Included on the I2C communication line is an I2C O-LED display for debugging and status update purposes.

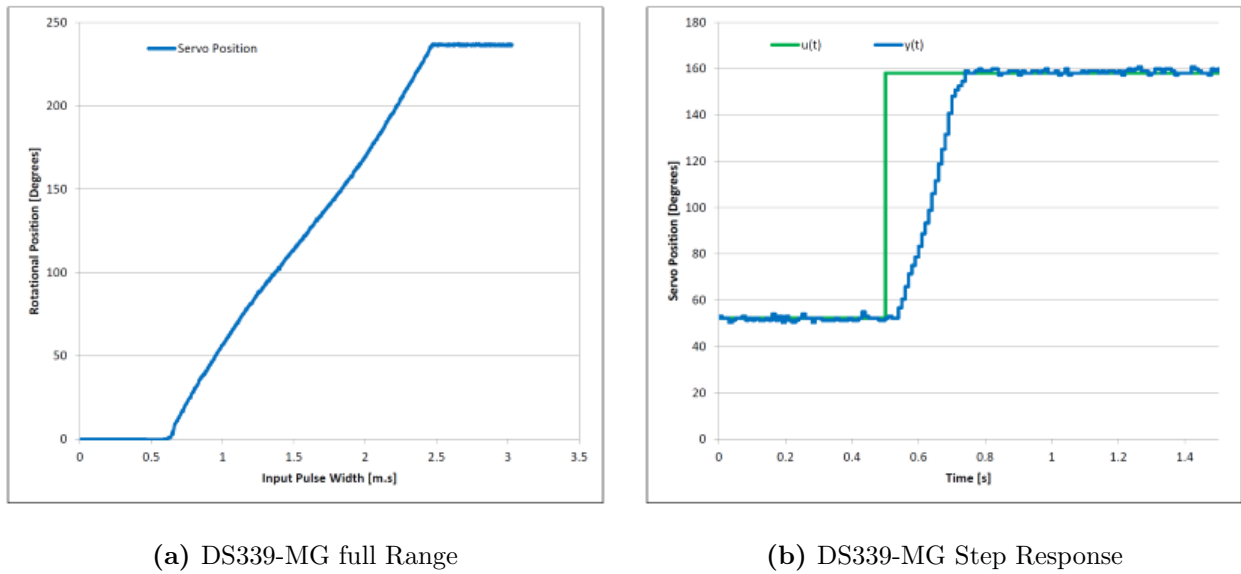
Any STM32  $\mu$ controller is programmed through a dedicated debugging device. The ST-Link V2 [91] is the current proprietary device which, itself, is a specially programmed STM32F10 chip. The chip connects to the dedicated **S**erial **W**ire **D**ebugging ports of the target STM (*SWD-CLK*, *SWD-IO* & *SWD-NRST*) and is interfaced via regular USB+ and USB- data lines.

<sup>13</sup>LDPower 20A ESCs(Fig:2.16b) match Hobbywing Xrotor 20A speed controllers (Fig:2.16a), they both use SiLabs F396 MCUs. Physical rotational values in the plots Fig:2.15 were measured with optical encoders.

### 2.4.1 Actuator Transfer Functions

#### Servo Transfer Functions

The full scale deflection of each digital servo is in fact greater than its quoted  $180^\circ$  range. Each servo has a rotational range of around  $230^\circ$  (Fig:2.17a). The exact characteristics for every servo differ slightly and thus individual transfer functions for each of the 8 servos are used in simulation. In the prototype control loop the servos are left in open loop; the major loop controller coefficients are expected to account for minor loop actuator dynamics. With that being said, for such an expectation the simulation would need to accurately represent the servo's response. Seeing that the  $180^\circ$  limitation was imposed as a design decision; one of the first points of contention is the effect such a constraint would have on the feasible operating trajectories. The control algorithms developed in Chapter:4 are first tested with an ideal, continuous rotation servo actuator with similar rate limits and transfer characteristics, later the servo limitations are introduced.



**Figure 2.17:** Unloaded Servo Characteristics

For the servo <sup>14</sup> whose data range and response are shown in Fig:2.17, the relationship the input pulse-width  $x$  [m.s] and the rotational output position  $y$  degrees is given by:

$$y(x) = \begin{cases} 0^\circ & x < 0.65 \text{ m.s} \\ 129.12x - 82.64 & 0.64 \text{ m.s} \leq x \leq 2.46 \text{ m.s} \\ 230^\circ & x > 2.46 \text{ m.s} \end{cases} \quad (2.24)$$



**Figure 2.18:** Servo block diagram

<sup>14</sup>Servo number 1 tested,  $\lambda_1$

Although, in practice, the equation Eq:2.24 is changed such that  $0^\circ$  is taken at around a 50% input and the operational range is  $\pm 90^\circ$ . Each servo is mechanically rate limited to  $60^\circ/0.15s$  or  $400RPS$  with a dead time of  $1.2\text{ m.s}$  and a mechanical deadband of  $\leq 4\mu s$ . The net transfer block for the servo is shown in Fig:2.18, including non-linearities but neglecting the deadband. Each servo has an approximate (*critically damped*) second order transfer function<sup>†</sup>:

$$G_{servo}(s) = e^{-t_d s} \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} = e^{-0.012s} \frac{(15.717)^2}{s^2 + 2(1)(15.717) + (15.717)^2} \quad (2.25a)$$

With saturation limits:

$$Y_{servo}(s) = \begin{cases} 0^\circ & |U(s)| < 0.65 \\ G(s) & 0.65 \leq |U(s)| \leq 2.46 \\ 230^\circ & |U(s)| > 2.46 \end{cases} \quad (2.25b)$$

The plot in Fig:2.17b is that of an unloaded servo's response. When loaded by an inner ring assembly (Fig:2.19a) the plant response  $y(t)$  is consistent with Eq:2.25. Despite rotating a load mass and hence requiring a greater torque, the servo remains unchanged, even when the BLDC motor (with a  $6 \times 4.5''$  prop) is spun an average rate of 6500 RPM,  $y'(t)$ , further loading the assembly.



(a) Inner Ring Servo Response



(b) Middle Ring Servo Response

**Figure 2.19:** Inner and middle servo characteristics

However, in Fig:2.19b, the response for a servo driving the middle ring is shown. Its transients remains the same but oscillations are introduced by the larger mass being driven. These are product of the structure's flex within the middle ring assembly<sup>15</sup>, *not from the servo plant*. The oscillations are still present under load,  $y'(t)$ , despite the frame being tensioned by a thrust vector. The harmonics can be accounted for by either introducing a more rigid sub-frame, limiting the maximum angular rate or applying a damping minor loop controller. The latter would be a *virtual* closed loop with an approximated error rate as the prototype structure doesn't incorporate positional feedback for each motor module.

## BLDC Transfer Functions

Each Cobra 2208 BLDC motor, when loaded with a  $6 \times 4.5$  propeller has a quadratic speed curve, Fig:2.20a. This is as a result of the propeller's opposing aerodynamic drag, *approximately* proportional

<sup>15</sup>The rotational position was measured with respect to the bearing supported output shaft, coaxial to the servos, and *not* the servo's output shaft

to the square of the propellers angular velocity (Section:3.3.1). The relationship<sup>16</sup> between input pulse-width to the ESCs and output RPM sensor signal (Fig:2.20a) is:

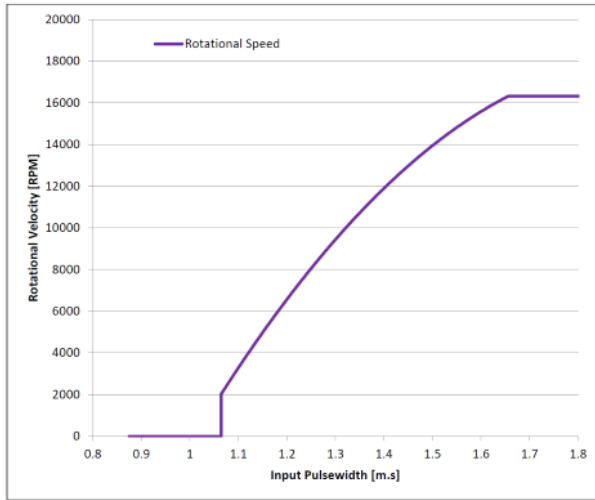
$$y(x) = \begin{cases} 0 & x < 1.065 \text{ m.s} \\ -20593x^2 + 80187x - 60004 & 1.065 \text{ m.s} \leq x \leq 1.655 \text{ m.s} \\ 16300^\circ & x > 1.655 \text{ m.s} \end{cases} \quad (2.26)$$

The upper limit in Eq:2.26 and the motor's step response are both governed by the ESC's maximum current limit; in this case 20A. Imposing 10A current limiting (a consequence of using lower power ESCs), the plot for  $\mathbf{c(t)}$  in Fig:2.20b, significantly restricts the motor's transient and steady-state performance. The motor's step response,  $\mathbf{y(t)}$  has a negligible dead time and 2<sup>nd</sup> order dynamics<sup>17</sup>, far faster than the servo's plant. The motors transfer function for speed in RPM is:

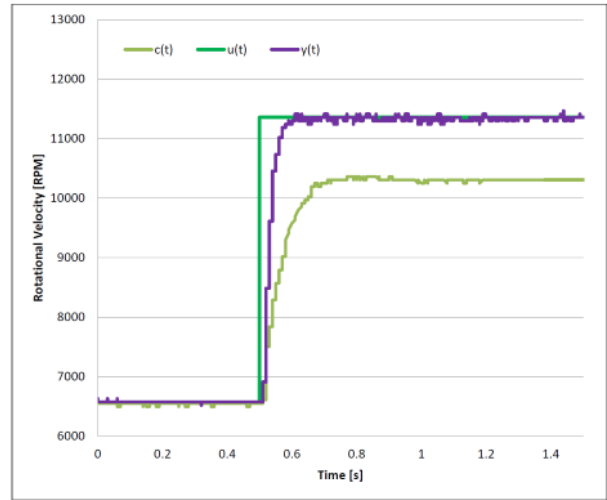
$$G_{BLDC}(s) = \frac{1}{(1 + 1.7583s \times 10^{-3})(1 + 1.7494s \times 10^{-3})} \quad (2.27a)$$

And saturation limits:

$$Y_{BLDC}(s) = \begin{cases} 0^\circ & |U(s)| < 1.065 \\ G(s) & 1.065 \leq |U(s)| \leq 1.655 \\ 16300 & |U(s)| > 1.655 \end{cases} \quad (2.27b)$$



(a) BLDC RPM range



(b) Cobra BLDC step response

**Figure 2.20:** BLDC motor characteristics

<sup>16</sup>The input range can be adjusted in BLHeli ESC software to improve input resolution.

<sup>17</sup>It can't be stressed enough how much the BLHeli ESC firmware improved dynamic response of the motors

## Chapter 3

# Kinematics & Dynamics

Generally applicable rigid body dynamics are first derived. Thereafter, those dynamics are adapted to the non-linear multibody case where constrained relative internal rotational is permitted. Following that, aerodynamic effects are investigated and incorporated into the plant's model. Finally a consolidated, quaternion based plant model is presented which is used for the later control plant development next in Chapter:4.

### 3.1 Rigid Body Dynamics

#### 3.1.1 Lagrange Derivation

Fundamentally any body, rigid or otherwise, can undergo two kinds of movements, namely rotational and translation motions. Often a Lagrangian [81,96] approach for combined angular and translational movements is used to derive the differential equations of motion for each degree of freedom. The Lagrangian principle ensures that (translational and rotational) kinematic energies and potential energy are conserved throughout the system's trajectory progression. When combined with Euler-Rotational equations, the Euler-Lagrangian [98] formulation fully defines the aerospace 6-DOF equation set.

Lagrangian formalism is regarded as especially useful in non-cartesian (*spherical etc...*) co-ordinate frames or multi-body systems. With that being said, a cartesian co-ordinate system was already defined in Section:2.2.2. Rigid body dynamics in a cartesian co-ordinate frame do lend themselves to Newtonian mechanics. The Newton-Euler or Euler-Lagrange formulations both stipulate the same resultant differential equations of motion. The Lagrangian operator,  $\mathcal{L}$ , is a term consisting of the difference between kinetic and potential energies,  $T$  and  $U$  respectively. Considering some generalized path co-ordinates  $\mathbf{r}(t)$ , for both linear  $\vec{\mathcal{E}}$  and angular  $\vec{\eta}$  relative positions;

$$\mathbf{r}(t) = \begin{bmatrix} \vec{\mathcal{E}} \\ \vec{\eta} \end{bmatrix} \quad (3.1)$$

The co-ordinates in Eq:3.1 are generalized here, despite being symbols commonly used to represent linear and angular positions. The generalized co-ordinates are later be refined as Cartesian body co-ordinates with respect to the inertial frame. The Lagrangian is, by definition:

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, t) = T(\mathbf{r}, \dot{\mathbf{r}}) - U(\mathbf{r}, \dot{\mathbf{r}}) \quad (3.2a)$$

With kinetic and potential energy function(s)  $T$  and  $U$  respectively. Then introducing a rigid body's general kinetic (angular & linear) and potential energies, in some shared reference frame. In this case

the only potential energy is gravitational<sup>1</sup> potential energy:

$$\mathcal{L} = \frac{1}{2} \begin{bmatrix} \dot{\vec{\mathcal{E}}}^T (m) \dot{\vec{\mathcal{E}}} \\ \dot{\vec{\eta}}^T (\mathbb{I}_b) \dot{\vec{\eta}} \end{bmatrix} - \begin{bmatrix} m \vec{G}_z \\ 0 \end{bmatrix} \quad (3.2b)$$

Noting that  $\mathbb{I}_b$  is the inertial tensor of the body aligned w.r.t to whichever reference frame is used. The Euler-Lagrange formulation equates partial derivatives of the Lagrangian to any generalized forces,  $\mathbf{V}$ , acting on the system. In this case the generalized forces or, more specifically a net force  $\vec{F}_{net}$  and a net torque  $\vec{\tau}_{net}$ .

$$\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\mathbf{r}}} \right) - \frac{\delta L}{\delta \mathbf{r}} = \mathbf{V} = \begin{bmatrix} \vec{F}_{net} \\ \vec{\tau}_{net} \end{bmatrix} \quad (3.3)$$

And taking the partial derivatives of Eq:3.2b with respect to the path co-ordinates  $\mathbf{r}$ :

$$\frac{\delta L}{\delta \mathbf{r}} = \begin{bmatrix} m \vec{G}_x \\ 0 \end{bmatrix} \quad (3.4a)$$

$$\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\mathbf{r}}} \right) = \begin{bmatrix} m \frac{d}{dt} \dot{\vec{\mathcal{E}}} & \mathbb{I} \frac{d}{dt} \dot{\vec{\eta}} \end{bmatrix}^T \quad (3.4b)$$

Where  $\vec{G}_x$  is the gravitation force in whichever reference frame ( $\mathcal{F}^x$ ) the lagrangian is with respect to. In any generalized coordinate system a rotating vector's time derivative which, according to the Reynolds Transportation Theorem [77], is given by:

$$\frac{d\vec{f}}{dt_a} = \frac{d\vec{f}}{dt_b} + \vec{\omega}_{a/b} \times \vec{f} \quad (3.5)$$

So applying that theorem (Eq:3.5) to the partial derivatives in Eq:3.4b and further defining the generalized co-ordinates as cartesian body coordinates with respect to an inertial origin (the body frame  $\mathcal{F}^b$  and inertial frame  $\mathcal{F}^I$ ). Noting that in Eq:3.4b the place holders used for linear ( $\vec{\mathcal{E}}$ ) and angular positions ( $\vec{\eta}$ ) all exist in a common shared frame<sup>2</sup>, and hence:

$$\frac{d}{dt} \begin{bmatrix} \vec{\mathcal{E}} \\ \vec{\eta} \end{bmatrix} \triangleq \begin{bmatrix} \vec{v} \\ \vec{\omega} \end{bmatrix} \in \mathcal{F}^b \quad (3.6)$$

It then follows that the Lagrangian Eq:3.2b changes to:

$$\mathcal{L} = \frac{1}{2} \begin{bmatrix} \vec{v}^T (m) \vec{v} \\ \vec{\omega}^T (\mathbb{I}) \vec{\omega} \end{bmatrix} - \begin{bmatrix} m \vec{G}_b z \\ 0 \end{bmatrix} \quad (3.7a)$$

$$\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\mathbf{r}}} \right) = \begin{bmatrix} m \frac{d}{dt} \vec{v} & \mathbb{I} \frac{d}{dt} \vec{\omega} \end{bmatrix}^T \quad (3.7b)$$

$$\rightarrow m \frac{d}{dt} \vec{v} = m \dot{\vec{v}} + \vec{\omega}_{I/b} \times \vec{v} \quad (3.7c)$$

$$\rightarrow \mathbb{I}_b \frac{d}{dt} \vec{\omega} = \mathbb{I}_b \dot{\vec{\omega}} + \vec{\omega}_{I/b} \times \mathbb{I}_b \vec{\omega} \quad (3.7d)$$

Which, when substituted back into the Euler-Lagrange formulation Eq:3.3, results in the familiar Newton-Euler equations for linear and angular differentials, both in the body frame;

$$\vec{F}_{net} = m \dot{\vec{v}} + \vec{\omega}_b \times m \vec{v} - m \mathbb{R}_I^b (-\eta) \vec{G}_I \quad (3.8a)$$

$$\vec{\tau}_{net} = \mathbb{I}_b \dot{\vec{\omega}}_b + \vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b \quad (3.8b)$$

It's important to recall that  $\vec{\omega}_b \neq \dot{\vec{\eta}}$  where  $\vec{\eta} = [\phi \ \theta \ \psi]^T$ , seeing that Euler Angles are defined in sequentially rotated reference frames. So then four differential equations are often used to completely describe the entire set of state derivatives, namely:

$$\dot{\vec{\mathcal{E}}} = \mathbb{R}_b^I (-\eta) \vec{v} \in \mathcal{F}^I \quad (3.9a)$$

$$\vec{F}_{net} = m \dot{\vec{v}} + \vec{\omega}_b \times m \vec{v} - m \mathbb{R}_I^b (-\eta) \vec{G}_I \in \mathcal{F}^b \quad (3.9b)$$

<sup>1</sup>Here  $G = [0 \ 0 \ -9.81]^T \text{ m.s}^{-2}$  in the Inertial frame,  $\in \mathcal{F}^I$

<sup>2</sup>In this case  $\vec{\eta} \neq [\phi \ \theta \ \psi]^T$  seeing that the angular position  $\vec{\eta}$  is defined in a common frame.  $\vec{\eta}$  is NOT an Euler angle set.



$$\dot{\vec{\eta}} = \Psi(\eta)\vec{\omega}_b \in \mathcal{F}^{v2}, \mathcal{F}^{v1}, \mathcal{F}^I \quad (3.9c)$$

$$\vec{\tau}_{net} = \mathbb{I}_b \dot{\vec{\omega}}_b + \vec{\omega}_b \times \mathbb{I}_b \vec{\omega}_b \in \mathcal{F}^b \quad (3.9d)$$

The state differentials in Eq:3.9 can be reduced to a set of two equations. Those differentials are defined in reference frames of the state variables which they represent. The non-linear form of those equations substitutes<sup>3</sup>  $d\vec{\eta}/dt = \Phi(\eta)\vec{\omega}_b$  in the Lagrangian derivative, Eq:3.4b.

$$\frac{d}{dt} \left( \frac{\delta \mathcal{L}}{\delta \dot{\mathbf{r}}} \right) = \left[ m \frac{d}{dt} \vec{v} \quad \mathbb{I} \frac{d}{dt} \dot{\vec{\eta}} \right]^T \Rightarrow \left[ m \frac{d}{dt} \vec{v} \quad \mathbb{I} \frac{d}{dt} \Phi(\eta)\vec{\omega}_b \right]^T \quad (3.10)$$

This only affects the angular component as the two kinetic energies are independent of one another. And so applying the differential chain rule:

$$\mathbb{I} \frac{d}{dt} \Phi(\eta)\vec{\omega}_b = \mathbb{I} (\Phi(\dot{\eta})\vec{\omega}_b + \Phi(\eta)\dot{\vec{\omega}}_b) \quad (3.11)$$

Drawing from [70] and recognizing that  $\mathbb{I}$  must be transformed to the common intermediate Euler axes,  $\mathbb{J} = \Psi(\eta)^T \mathbb{I} \Psi(\eta)$ . The controllable differential equation for angular acceleration in Eq:3.8b, then in intermediate Euler frames for each angle becomes<sup>4</sup>:

$$M(\eta)\ddot{\vec{\eta}} + C(\eta, \dot{\eta})\dot{\vec{\eta}} = \Psi(\eta)\vec{\tau}_{net} \quad (3.12a)$$

$$M(\eta) = \Psi(\eta)^T \mathbb{I} \Psi(\eta) \quad (3.12b)$$

$$C(\eta, \dot{\eta}) = -\Psi(\eta)\mathbb{I}\dot{\Psi}(\eta) + \Psi(\eta)^T [\Psi(\eta)\dot{\vec{\eta}}]_{\times} \mathbb{I} \Psi(\eta) \quad (3.12c)$$

Equation 3.12a fully describes the state derivative  $\ddot{\vec{\eta}}$  in its own frame(s). The two differential equations which describe the entire bodies motion are then:

$$\vec{F}_{net} = m\dot{\vec{\mathcal{E}}} + \mathbb{R}_b^I(-\eta)\vec{\omega}_b \times m\dot{\vec{\mathcal{E}}} - m\vec{G}_I \in \mathcal{F}^I \quad (3.13a)$$

$$\vec{\tau}_{net} = \Psi(\eta)^{-1} M(\eta)\ddot{\vec{\eta}} + \Psi(\eta)^{-1} C(\eta, \dot{\eta}) \in \mathcal{F}^{v2, v1, I} \quad (3.13b)$$

The generalized net forces effecting the system,  $\vec{F}(u)$  and  $\vec{\tau}(u)$ , are the system's controllable inputs and are going to be affected directly the systems actuators and their associated effectiveness function. In the general case, which is expanded upon in Section:3.3, the control inputs are typically as follows: The net force acting on the system is just the sum of all thrust vectors produced by rotating propellers,  $T(\Omega_i)$ .

$$\mu \vec{F} = \sum_{i=1}^4 \vec{T}(\Omega_i) \quad (3.14a)$$

Secondly the net torque is the sum of all torque arms produced from those propeller thrust vectors.

$$\mu \vec{\tau} = \sum_{i=1}^4 \vec{l}_i \times \vec{T}(\Omega_i) \quad (3.14b)$$

Where  $\vec{T}(\Omega_i)$  is the  $i^{th}$  motor's thrust vector, not necessarily in  $\mathbb{R}^3$ , and typically bound to the  $\hat{Z}_b$  axis,  $\vec{l}_i$  is that thrust vector's perpendicular displacement from the origin  $\mathbf{O}_b$ . The above equations are still applicable to any 6 DOF body, common simplifications applied to the system(s) for quadrotor control are explored in Appendix:A. Aspects unique to (multibody) aerospace frames are now introduced. Obviously the focus is on quadrotor and tilting quadrotor platforms...

<sup>3</sup>Originally introduced in Eq:2.5e

<sup>4</sup>The relationship  $\dot{\Phi} = \Phi \dot{\Psi} \Phi$  was used to simplify Eq:3.12, the singularity in  $\Phi$  still remains...

### 3.1.2 Rotation Matrix Singularity

The Euler Angle singularity is often mentioned but far less common is the demonstration of exactly how that singularity *mathematically* manifests itself. By definition, a singularity occurs when a loss of differentiability is encountered. In the case of an affixed 3-axis gimbal (Fig:3.1a), when an intermediary rotational angle, for example the rolling angle  $\theta$ , is at  $\pi/2$  then the remaining two axes become co-linear (Fig:3.1b). That being both pitch  $\phi$  or yaw  $\psi$  rotations will subsequently have the same effect. Such a situation results in what is termed as a loss of a degree of freedom.



**Figure 3.1:** Gimbal lock

What is clear in a physical system is not necessarily as clear mathematically. An obvious loss of differentiability is manifested in the Euler Matrix  $\Psi(\eta)$ , Eq:2.5e from Section:2.2.1. The relation between angular velocity, in the inertial frame or inversely in the body frame, and the angular rates of the Euler Angles.

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \Phi(\eta)\omega_b \quad (3.15)$$

$$\text{As } \lim_{\theta \rightarrow \pi/2} \sec(\theta), \tan(\theta) \rightarrow \infty \quad (3.16)$$

Or that  $\Phi(\eta)$  is undefined at  $\theta = \pi/2$ . It's clear to see that in Eq:3.15 there exists an undefined singularity as  $\theta \rightarrow \pi/2$ . The physical consequence of this is the loss of a degree of freedom. More specifically, if one looks at how the Z-Y-X rotation (or transformation) matrices are formulated:

$$\mathbb{R}_I^b = \mathbb{R}_z \mathbb{R}_y \mathbb{R}_x = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} \quad (3.17a)$$

$$\mathbb{R}_I^b = \begin{bmatrix} c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\ s_\psi c_\theta & s_\psi s_\theta s_\phi + c_\psi c_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \quad (3.17b)$$

In the case where  $\theta = \pi/2$ , and using trigonometric double angles;

$$= \begin{bmatrix} 0 & c_\psi s_\phi - s_\psi c_\phi & c_\psi c_\phi + s_\psi s_\phi \\ 0 & s_\psi s_\phi + c_\psi c_\phi & s_\psi c_\phi - c_\psi s_\phi \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & s(\phi - \psi) & c(\phi - \psi) \\ 0 & c(\phi - \psi) & s(\phi - \psi) \\ -1 & 0 & 0 \end{bmatrix} = \mathbb{R}_{x'}(\phi - \psi) \quad (3.17c)$$

Where the resultant in Eq:3.17c represents an X-axis rotation in a new intermediate frame, post a  $\pi/2$  rotation in the y-axis. Through trigonometric double angles a degree of freedom is lost at  $\theta = \pi/2$ , when  $\phi$  &  $\psi$  effect the same angle.

### 3.1.3 Quaternion Dynamics

An algorithm proposed in *How To Avoid a Singularity When Using Euler Angles?* [89] suggested a solution to the problem of Euler Angle singularities. The proposed heuristic was to switch between sequencing conventions (ZYX, ZYZ etc... there are 12 in total) such that the singularity is always avoided. However the implementation of such an algorithm is cumbersome and inefficient. Far more elegant is the use of *quaternion* attitude representations in  $\mathbb{R}^4$  ([32, 35, 53] amongst others...).

A quaternion is analogous to a rotation matrix in that it represents an attitude difference between two reference frames. An  $\mathbb{R}^3$  position is parameterized as a single rotation  $\theta$  about a unit axis  $\hat{u}$  (Sic Rodriguez Formula [65]). Without deliberating too much on their proof or details, a quaternion consists of a scalar component,  $q_0$ , and complex vector component,  $\vec{q} \in \mathbb{C}^3$ , such that;

$$Q \triangleq \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} \in \mathbb{R}^4 \quad (3.18)$$

The relationship between an Euler Angles rotation matrix  $\mathbb{R}_I^b(\eta)$  and a quaternion attitude  $Q_b$  is given by the Rodriguez formula:

$$\mathbb{R}_I^b(\eta) = \mathbb{R}(Q_b) = \mathbb{I} + 2q_0[\vec{q}]_\times + 2[\vec{q}]_\times^2 \quad (3.19)$$

Any and all quaternions, unless otherwise stated, in this dissertation are all unit quaternions<sup>5</sup>,  $Q \in \mathbb{Q}_u$ . The need for quaternions with unity magnitude is such to ensure rotational operations don't affect the magnitude of the vector operand. A unit quaternion is defined as:

$$||Q|| = \sqrt{q_0^2 + \vec{q}^2} = 1 \quad (3.20)$$

Quaternion multiplication is distributive and associative, but not commutative. Specifically a quaternion multiplication operation is equivalent to the Hamilton product. For two quaternions,  $Q$  &  $P$ :

$$Q \otimes P = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} \otimes \begin{bmatrix} p_0 \\ \vec{p} \end{bmatrix} \quad (3.21a)$$

$$= q_0 p_0 - \vec{q} \cdot \vec{p} + p_0 \vec{q} + q_0 \vec{p} + \vec{q} \times \vec{p} \quad (3.21b)$$

Seeing that the vector component of a quaternion is complex valued, it is natural that there exists a quaternion conjugate property. Namely:

$$Q^* = \begin{bmatrix} q_0 \\ -\vec{q} \end{bmatrix} \quad (3.22)$$

It then follows that<sup>6</sup>:

$$Q \otimes Q^* = \mathbb{I}_{4 \times 4} \quad (3.23)$$

<sup>5</sup>Unit quaternions are a subset of the quaternion space

<sup>6</sup>Disambiguation:  $\mathbb{I}$  in this context is a  $4 \times 4$  identity matrix, not an inertial matrix

To apply quaternion rotations to a vector  $\vec{v} \in \mathbb{R}^3$  involves multiplication by two unit quaternions.

$$\begin{bmatrix} 0 \\ \vec{v}' \end{bmatrix} = Q \otimes \begin{bmatrix} 0 \\ \vec{v} \end{bmatrix} \otimes Q^* \quad (3.24)$$

Mostly, the zero scalar components are omitted in a rotation (*or transformation*) operation, such that it is recognized the vector operands are substituted with quaternions.

$$\vec{v}' = Q \otimes (\vec{v}) \otimes Q^* \quad (3.25)$$

In the case of rigid body attitude representation,  $Q_b$  is the quaternion which represents the difference between  $\mathcal{F}^b$  and  $\mathcal{F}^I$ . A quaternion operator is equivalent to a rotation matrix operation:

$$\mathbb{R}_I^b \xleftrightarrow{Q} Q_b \otimes (\cdot) \otimes Q_b^* \quad (3.26)$$

A quaternion time derivative, with  $Q_\omega$  being a quaternion with a vector component equal to angular velocity  $\omega \in \mathcal{F}^b$  and a zero scalar component, is given by:

$$\frac{d}{dt}Q_b = \frac{1}{2}Q_b \otimes Q_\omega = \begin{bmatrix} -\frac{1}{2}\vec{q}^T \vec{\omega}_b \\ \frac{1}{2}([\vec{q}]_\times + q_0 \mathbb{I})\vec{\omega}_b \end{bmatrix} \quad (3.27)$$

Using quaternions to represent attitudes negates the need for an Euler Matrix,  $\Phi(\eta)$ , to represent attitudes and their rates. A body quaternion is fully defined in the body frame. The first quaternion time derivative replaces Eq:3.9a& Eq:3.9c;

$$\dot{\mathcal{E}} = \mathbb{R}_b^I(-\eta)\vec{v} \xleftrightarrow{Q} Q_b \otimes \vec{v} \otimes Q_b^* \quad (3.28a)$$

$$\dot{\eta} = \Phi(\eta)\vec{\omega}_b \xleftrightarrow{Q} \dot{Q} = \frac{1}{2}Q_b \otimes Q_\omega \quad (3.28b)$$

Second order derivatives for quaternion acceleration aren't as useful as their velocity counterparts. The second order derivative is mentioned here however it's only relevant to quaternion backstepping in the control chapter. If possible, quaternion accelerations are mostly avoided due to their complexity;

$$\ddot{Q}(\dot{Q}, Q, t) = \dot{Q} \otimes Q^* \otimes \dot{Q} + \frac{1}{2}Q \otimes [\mathbb{I}_b^- 1(\tau - 4(Q^* \otimes \dot{Q}) \times (\mathbb{I}_b(Q^* \otimes \dot{Q})))] \quad (3.29)$$

### 3.1.4 Quaternion Unwinding

Although quaternions are better than Euler angles and lack the associated singularity, they do contain one caveat. Seeing that a quaternion  $Q = [q_0 \ \vec{q}]^T$  represents an attitude orientation of a body in  $\mathbb{R}^3$  using  $\mathbb{R}^4$  variables there exists what is called dual coverage [65]. Each unit quaternion, stemming from Euler-Rodriguez theorem, is parametrized such that the quaternion operation represents an eigenaxis rotation of  $\theta$  about an axis  $\hat{u}$  such that:

$$Q = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\hat{u} \end{bmatrix} \quad (3.30)$$

That rotation is executed through the quaternion operator Eq:3.25. As a result its clear to see that for each unique attitude in 3-Dimensions there exist two quaternions which correlate to the same position. Namely:

$$Q = \begin{bmatrix} q_0 \\ \vec{q} \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\hat{u} \end{bmatrix}. \quad (3.31a)$$

And seeing that  $\theta = 2\pi - \theta$ , then;

$$Q = \begin{bmatrix} \cos(\pi - \theta/2) \\ \sin(\pi - \theta/2)\hat{u} \end{bmatrix} = \begin{bmatrix} -\cos(\theta/2) \\ \sin(\theta/2)\hat{u} \end{bmatrix} \quad (3.31b)$$

Every physical attitude in  $\mathbb{R}^3$  has two corresponding quaternions in  $\mathbb{R}^4$ ;  $[\pm q_0 \vec{q}]^T$ . A consequence of this is two possible error state trajectories for every attitude difference. A clockwise  $\theta$  rotation and an anticlockwise  $2\pi - \theta$  negative rotation. This could lead to an erroneous and unnecessary "unwinding" of a complete counter revolution as a result of a dual covered error state.

Often the sign scalar component of the attitude quaternion error (Section:4.1.2) is simply neglected or assumed positive. As such for attitude controllers the requirement is that for positive and negative scalars the control input is consistent:

$$\nu_d = h([q_0 \vec{q}]^T, t) = h([-q_0 \vec{q}]^T, t) \quad (3.32)$$

Or more simply that  $Q_e = [|q_0| \vec{q}]^T$ . The most simple solution which adheres to that constraint is to simply neglect the scalar component and use  $h(\vec{q}_e, t)$ . A positive quaternion scalar will always ensure that an error state represents a right-handed clockwise rotation. If the resolution of trajectory co-ordinates generated is sufficiently high enough, the control plant will never encounter a problem.

One proposal in *Nonlinear Quadcopter Attitude Control* [17] suggested using a *signum* operator to design the signs of the controller coefficients for the virtual control plant input.

$$\vec{\omega}_d = \frac{2}{\tau} \text{sgn}(q_0) \vec{q} \quad (3.33a)$$

$$\text{sgn}(\vec{q}) = \begin{cases} 1 & \vec{q} \geq 0 \\ -1 & \vec{q} < 0 \end{cases} \quad (3.33b)$$

The resultant *hybrid* controller provides global asymptotic stability, but only in the case that the eigenaxis angle  $\theta \leq \pm\pi$ . The control law described in Eq:3.33 would still need the control torques to be designed from the angular velocity virtual control input.

An alternative proposal [65] was to lift the quaternion error-state back into  $\mathbb{R}^3$  using the Rodriguez formula, Eq:3.19. The mapping back to  $\mathbb{R}^3$  effectively ensures that  $\theta$  is minimized, or that the error-state imposes the shortest possible rotation between the reference and desired body frames.

## 3.2 Multibody Nonlinearities

Typically multibody dynamics are solved (and simulated) as a series of interactions and responses. There are different schools of thought which have proposed various methodologies for stepping through the systems dynamics [Sic Implicit Euler [49, 104]]. For the prototype design here, only relative rotational motion is permissible between the interconnected rigid bodies. Each body is considered independently, as free and rigid, whose constraint torques induced from excitation are imposed on sequential rotational joints. Opposed to those torques are Newtonian responses of importance which manifest as what is termed *gyroscopic* and *inertial* torques.

A distinction must be made between torque responses here and those of Eq:3.9d. The latter being a response to be compensated for and the former being something which can be exploited by the control allocation algorithm in Section:4.3. The multibody analysis which follows is a very Newtonian approach in that each body involved is resolved independently and relative responses are transferred onto the inducing body.

### 3.2.1 Relative Rotational Gyroscopic & Inertial Torques

The torque response induced from relative rotations, the only permissible relative motion, between each connected rigid body are transferred from interacting bodies as a result of Newtons second law of rotational motion. For each of the motor modules' pitching or rolling motion, the servo motors

apply some torque for that rotation. Opposed to the rotational motion are both inertial and gyroscopic response(s) of that body being acted upon. The latter being a consequence of a vector's time derivative in a rotating frame, Eq:3.5.

Each of the four motor modules are symmetrical and as such the induced torque response characteristics for one module can be extrapolated through a reference frame rotation. Seeing that each relative rotation from the actuator set  $u \in \mathbb{U}$  is actuated independently and upon a different body, their responses are calculated separately too.

Drawing again from Lagrangian theory<sup>7</sup> and considering only the rotational kinetic energy for the inner ring assembly  $\mathcal{F}^{M_i}$ , there are no permissible transnational motions between each body and as such there is no linear kinetic energy contribution. The Lagrangian for the inner ring is formed, with concern on the effect  $\lambda_i$  has on the system:

$$\mathcal{L}_{M_i} = \frac{1}{2} \vec{\Omega}_i^T (\mathbb{I}_p) \vec{\Omega}_i + \frac{1}{2} \dot{\vec{\lambda}}_i^T (\mathbb{I}_\lambda) \dot{\vec{\lambda}}_i \quad (3.34)$$

Where  $\mathbb{I}_p$  is the propeller's rotational inertia about its  $\hat{Z}$  axis and  $\mathbb{I}_\lambda$  being the inner ring's inertia defined in Eq:2.11a. Noting that  $\vec{\Omega}_i = [0 \ 0 \ \Omega_i]^T \in \mathcal{F}^{M_i}$  and  $\dot{\vec{\lambda}}_i = [\dot{\lambda}_i \ 0 \ 0]^T \in \mathcal{F}^{M'_i}$ , the two contributors are not in a common frame. As such the equation<sup>8</sup> changes to:

$$\mathcal{L}_{M_i} = \frac{1}{2} \vec{\Omega}_i^T (\mathbb{I}_p) \vec{\Omega}_i + \frac{1}{2} \dot{\vec{\lambda}}_i^T (\mathbb{I}_\lambda) \dot{\vec{\lambda}}'_i \quad (3.35a)$$

$$\dot{\vec{\lambda}}'_i = Q_x^*(-\lambda_i) \otimes (\dot{\vec{\lambda}}_i) \otimes Q_x(-\lambda_i) \Rightarrow \dot{\vec{\lambda}}'_i = \dot{\vec{\lambda}}_i \quad (3.35b)$$

Where both  $\mathbb{I}_p$  and  $\mathbb{I}_\lambda$  are taken W.R.T to their rotational centre(s). Recalling the Euler-Lagrange formulation from Eq:3.3 with generalized co-ordinates<sup>9</sup>  $\mathbf{u}(t)$  of  $\mathcal{F}^{M_i}$  relative to  $\mathcal{F}^b$ .

$$\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\mathbf{u}}} \right) - \frac{\delta L}{\delta \mathbf{u}} = \mathbf{V} = \vec{\tau}_{net} \quad (3.36)$$

Then:

$$\frac{d}{dt} \left( \frac{\delta \mathcal{L}}{\delta \dot{\mathbf{u}}} \right) = \frac{d}{dt} \mathbb{I}_p \vec{\Omega}_i + \vec{\omega}_{M_i/b} \times \mathbb{I}_p \vec{\Omega}_i + \frac{d}{dt} \mathbb{I}_\lambda \dot{\vec{\lambda}}_i + \vec{\omega}_{M_i/b} \times \mathbb{I}_\lambda \dot{\vec{\lambda}}_i \quad (3.37a)$$

With  $\vec{\omega}_{M_i/b}$  being the net angular velocity of the inner ring frame relative to the body frame:

$$\vec{\omega}_{M_i/b} = Q_x^*(-\lambda_i) Q_y^*(-\alpha_i) \otimes \dot{\vec{\alpha}}_i \otimes Q_y(-\alpha_i) Q_x(-\lambda_i) + Q_x^*(-\lambda_i) \otimes \dot{\vec{\lambda}}_i \otimes Q_x(-\lambda_i) \quad (3.37b)$$

The net torque from a  $\lambda_i$  rotation, induced in the motor module frame  $\mathcal{F}^{M_i}$ , can be grouped into second order *Inertial* and first order *Gyroscopic* components. Depending on the fidelity of the model or aggressiveness of control actions taken, higher order induced terms could be ignored.

$$\vec{\tau}_\lambda = \underbrace{\mathbb{I}_p \ddot{\vec{\Omega}}_i + \mathbb{I}_\lambda \ddot{\vec{\lambda}}_i}_{Inertial} + \underbrace{\vec{\omega}_{M_i/b} \times \mathbb{I}_p \vec{\Omega}_i + \vec{\omega}_{M_i/b} \times \mathbb{I}_\lambda \dot{\vec{\lambda}}_i}_{Gyroscopic} \in \mathcal{F}^{M_i} \quad (3.37c)$$

Similarly for the middle ring, with inertia a function of the  $\lambda$  angle  $\mathbb{I}_\alpha(\lambda_i)$  from Eq:2.12c, with generalized co-ordinates  $\mathbf{v}(t)$  of  $\mathcal{F}^{M'_i}$  relative to  $\mathcal{F}^b$ :

$$\mathcal{L}_{M'_i} = \frac{1}{2} \dot{\vec{\alpha}}_i^T (\mathbb{I}_\alpha(\lambda_i)) \dot{\vec{\alpha}}_i \quad (3.38a)$$

<sup>7</sup>The generalized linear kinetic energy for each module is an extension of that in Eq:3.13a and is independent of any of the actuator positions.

<sup>8</sup> $\dot{\vec{\lambda}}'_i$  is superfluous but included for completeness

<sup>9</sup>Relative to the body frame and not the inertial frame because Eq:3.9d accounts for the inertial response of the entire body frame. Here the induced relative responses are of concern

$$\frac{d}{dt_b} \left( \frac{\delta L}{\delta \dot{\mathbf{v}}} \right) = \frac{d}{dt_{M'_i}} \mathbb{I}_\alpha(\lambda_i) \dot{\tilde{\alpha}}_i + \vec{\omega}_{M'_i/b} \times \mathbb{I}_\alpha(\lambda_i) \dot{\tilde{\alpha}}_i \quad (3.38b)$$

$$\vec{\omega}_{M'_i/b} = Q_y^*(-\alpha_i) \otimes \dot{\tilde{\alpha}}_i \otimes Q_y(\alpha_i) \quad (3.38c)$$

Which are also grouped into first and second order components.

$$\vec{\tau}_\alpha(\lambda_i) = \underbrace{\mathbb{I}_\alpha(\lambda_i) \ddot{\tilde{\alpha}}_i}_{Inertial} + \underbrace{\vec{\omega}_{M'_i/I} \times \mathbb{I}_\alpha(\lambda_i) \dot{\tilde{\alpha}}_i}_{Gyroscopic} \in \mathcal{F}^{M'_i} \quad (3.38d)$$

Each of the induced torques,  $\vec{\tau}_\lambda$  and  $\vec{\tau}_\alpha(\lambda_i)$ , occur in intermediary frames associated with the inner and middle ring assemblies. As such, their negative responses effect<sup>10</sup> Eq:3.9d, and need to be transformed to the body frame.

$$\tau(u) = \sum_{i=1}^4 -Q_{M_i} \otimes \tau_{\lambda_i}(u) \otimes Q_{M_i}^* - Q_{M'_i} \otimes \tau_{\alpha_i}(u) \otimes Q_{M'_i}^* \in \mathcal{F}^b \quad (3.39)$$

The above responses are pertinent to simulation and plant dependent compensation. The simulation environment is structured such that the torques are produced as responses from Newtonian movement at every step interval. In due course it would be more efficient (and less stiff) to exploit an implicit Euler [49,104] coordinate system in substitution for the cartesian response equations developed above. However this was not implemented in Chapter:5.

### 3.3 Aerodynamics

The relationship between a propeller's rotational speed,  $\Omega_i$ , and its produced thrust,  $\vec{T}(\Omega_i)$ , is more complicated than the quadratic simplification taken at static conditions which most papers puport. The thrust is mostly dependent on the incident air velocity entering the propellers rotational plane, typically being the component of the body velocity normal to the propeller's plane. Fluid flow parallel to the propeller contributes to aerodynamic drag. The combination of aerodynamic Blade-element [73,85] and fluid-dynamics Momentum (*disc actuator*) theories stipulates an integral term taken across the propellers length which accurately models the aerodynamic thrust and torque functions. A verbose presentation of all aerodynamic effects experienced by a quadrotor's propeller(s) is thoroughly detailed in [6] or [5]. The following is a review of pertinent aerodynamic theories. Some phenomena aren't included, like Vortex Ring States or parasitic drag like effects, which were deemed to not be pertinent.

#### 3.3.1 Propeller Torque and Thrust

*A feasible situation which the prototype could encounter is where an upstream propeller provides to the incident fluid flow to another downstream propeller. Such a situation presents a complicated fluid dynamics & wake effects problem<sup>11</sup> but remains open to further research. To expedite the system I.D process some simplifications are applied to the aerodynamics to construct an approximate model. Other responses are regarded as inconsequential and are compensated for in a lumped disturbance term.*

The rotation of a propeller applies a thrust force,  $\vec{T}$ , on the fluid stream<sup>12</sup> in which it acts. That fluid stream (Fig:3.2) has an incident head velocity,  $v_\infty$ , and slip stream velocity relative to the rotational plane,  $v_s$ . There exists some relationship about the change of fluid flow applied by the propeller's rotation. That relationship can then be given by:

$$v_s = \Delta v + v_\infty \quad (3.40)$$

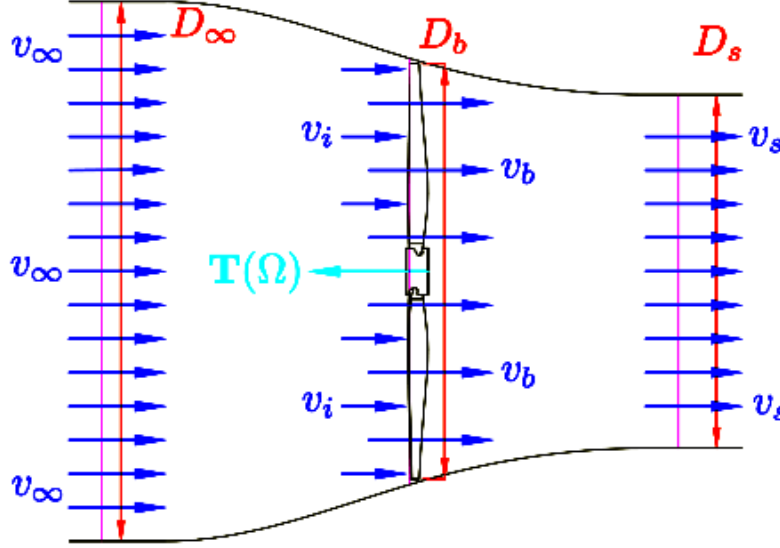
<sup>10</sup>Depending on dynamic equations used it could effect Eq:3.13b. However the equations Eq:3.13 are inconsequential when using quaternion dynamics.

<sup>11</sup>Propeller overlapping effects are investigated in [95]

<sup>12</sup>Only perpendicular mass flow across the propeller's plane is considered here...

Where  $\Delta v$  is the change in velocity added to the fluid by the propeller blade's aerofoil profile. The propeller induces a velocity directly in front of it's rotational plane,  $v_i$ , such that the net fluid flow into the plane is  $v_b = v_i + v_\infty$ . Bernoulli's principle<sup>†</sup> has it that net fluid flow through that plane is

$$v_b = \frac{1}{2}(v_s - v_\infty) = \frac{1}{2}\Delta v = \frac{1}{2}v_s|_{v_\infty=0} \quad (3.41)$$



**Figure 3.2:** Disc Actuator Propeller Planar Flow

And as such, stemming from classical Disc Actuator<sup>†</sup> (*momentum*) theory, the force,  $T(\Omega)$ , acting on the fluid is a function of mass flow rate and change in velocity (pressure differential).

$$\vec{T} = (A_b v_b) \Delta v = \rho \pi R_b^2 v_b \Delta v = \rho \pi R_b^2 (v_i + v_\infty) \Delta v = \frac{1}{2} \rho \pi R_b^2 \Delta v^2 \quad (3.42)$$

Indeed Eq:3.42 can be solved as a function of aerodynamic propulsive power expended,  $P = \vec{T}v$ . However that relationship between rotational kinetic power transferred is tenuous at best, compounded parasitic losses deteriorate the power transferred. Furthermore, the local fluid velocity over the propeller isn't purely normal to its plane. There are axial and tangential induced velocity components,  $a$  and  $a'$  respectively. Fluid velocity induction factors can then be defined as a dependents of incident velocity:

$$v_i = a v_\infty \quad (3.43a)$$

$$v_\theta = a' \Omega R \quad (3.43b)$$

From Eq:3.43 the velocity components can be written as functions of free stream velocity  $v_\infty$ .

$$v_b = (1 + a)v_\infty \quad (3.44a)$$

$$v_s = (1 + 2a)v_\infty \quad (3.44b)$$

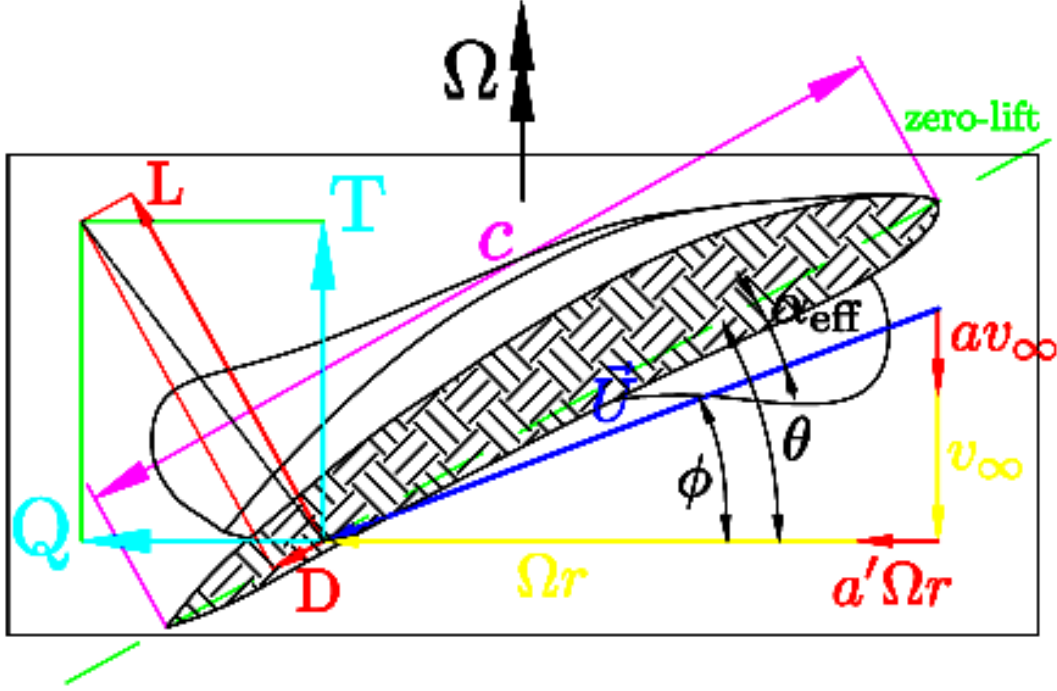
And from the tangential fluid flow there is then an angular moment rate across the propeller plane too. This produces a torque response to the rotational motion about the propeller's axis of rotation, analogous to Eq:3.42.

$$\vec{Q} = \rho \pi R_b^3 (v_\theta - v_\infty) v_b \quad (3.45)$$

Blade-element theory analyses incremental aerofoil sections  $dr$  of a propeller profile (Fig:3.3) at some radius  $r$ . Net local fluid velocity across a single elemental aerofoil profile  $\vec{U}$  is calculated as:

$$\vec{U} = \sqrt{(v_\infty + v_i)^2 + (v_\infty + v_\theta)^2} \quad (3.46)$$





**Figure 3.3:** Blade element profile at radius  $r$

Each elemental profile, of chord length  $c$ , has a local pitch,  $\theta$ , of its aerofoil profile relative to the horizontal. Local fluid velocities (again in Fig:3.3) encountered by the propeller make their own an angle of attack  $\phi$  such that:

$$\phi = \theta - \alpha_{effective} \quad (3.47)$$

That local angle of attack changes with the inflow magnitude  $v_\infty$  and the induced axial velocity  $v_i$ . That trigonometric ratio is given as:

$$\phi = \tan^{-1} \left( \frac{v_\infty + v_i}{v_\Omega + v_\theta} \right) = \tan^{-1} \left( \frac{v_\infty(1 + a)}{\Omega r(1 + a')} \right) \quad (3.48)$$

The in-plane fluid flow  $\vec{U}(r, \phi)$ , for an element at radius  $r$  with a local angle of attack  $\phi$ , then contributes towards elemental lift and drag forces as a function of aerofoil's dimensionless lift,  $C_L$ , and drag,  $C_D$ , coefficients<sup>13</sup>.

$$\Delta L = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c C_L \quad (3.49a)$$

$$\Delta D = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c C_D \quad (3.49b)$$

With air density  $\rho$ <sup>14</sup> and local chord length  $c$ . Those lift and drag forces are taken as components parallel and perpendicular to the plane of rotation. Those components are then thrust  $T$  and torque  $F_x$  forces (Fig:3.3).

$$dT = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c (C_L \cos(\phi) + C_D \sin(\phi)) . dr \quad (3.50a)$$

$$dF_x = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c (C_L \sin(\phi) + C_D \cos(\phi)) . dr \quad (3.50b)$$

$$\rightarrow dQ = \frac{1}{2} \rho \vec{U}(r, \phi)^2 c (C_L \sin(\phi) + C_D \cos(\phi)) r . dr \quad (3.50c)$$

<sup>13</sup>The lift and drag coefficients are determined by the aerofoil's characteristics, but would be constant across the length of a variable pitch, non-twisted hinged propeller...

<sup>14</sup>Typically  $\rho = 1.225 \text{ kg/m}^3$

$$\rightarrow dP = \Omega r dF_x . dr \quad (3.50d)$$

Typically a power term, Eq:3.50d, is given in lieu of torque or drag terms, Eq:3.50c or Eq:3.50b. Then calculating forces and torques as per momentum theory for each element, in terms of tangential and axial induction factors:

$$dT = \rho 4\pi r^2 v_\infty (1 + a) a . dr \quad (3.51a)$$

$$dP = \rho 4\pi r^2 v_\infty (1 + a) \Omega r (1 + a') . dr \quad (3.51b)$$

Then equating momentum and element terms together produces the blade-element momentum equation(s) for thrust and power produced by a propeller. Following a few assumptions, most importantly that the lift coefficient  $C_L$  is a linear function of the effective angle of attack  $\alpha_{eff}$ . The lift curve gradient  $a_L$ , for an ideally twisted blade, like the fixed pitch propellers under consideration here, is typically  $2\pi$  such that  $C_L = 2\pi(\theta - \phi)$ . And assuming that tangential induced velocities  $v_\theta$  are small (or that the tangential induction factor  $a' \ll 1$ ) when compared to the propeller's speed  $\Omega r$ . Similarly the net inflow and axial induced velocities  $v_\infty + v_i \ll \Omega r$ .<sup>15</sup>

$$T = \int_{r=0}^R \frac{1}{2} a_L b c \rho (\Omega r)^2 \left( \theta - \frac{v_\infty + v_i}{\Omega r} \right) . dr \quad (3.52a)$$

$$P = \int_{r=0}^R \frac{1}{2} a_L b c \rho (\Omega r)^3 \left[ \left( \theta - \frac{v_\infty + v_i}{\Omega r} \right) \left( \frac{v_\infty + v_i}{\Omega r} \right) + C_d \right] . dr \quad (3.52b)$$

Generally knowing the exact pitch and chord values as a function  $r/R$  is difficult and calculating integrals at each process step is cumbersome. Both Eq:3.52a & Eq:3.52b can be solved by equating element and momentum terms (Appendix:??). Often dimensionless thrust, torque and power coefficients are defined:

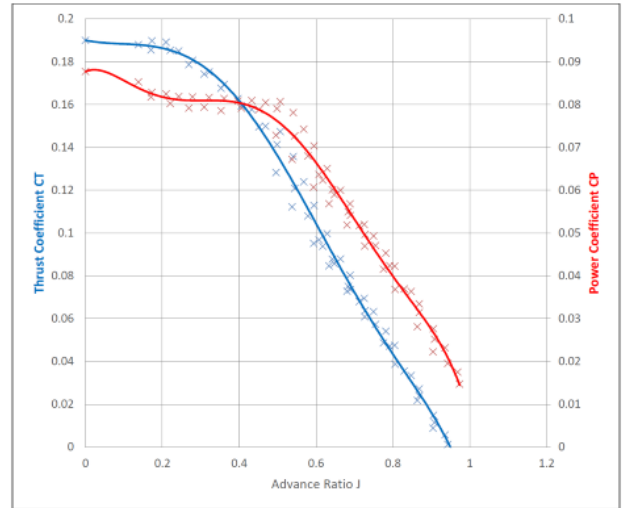
$$C_T(J) = \frac{T}{\rho \Omega^2 D^4} \quad (3.53a)$$

$$C_P(J) = \frac{P}{\rho \Omega^3 D^5} \quad (3.53b)$$

Where  $\Omega$  is the propellers rotational speed in RPS and  $D$  is the propellers diameter. For fixed pitch propellers the thrust and power coefficients are easily determined. Eq:3.53a and Eq:3.53b both vary due to the *advance ratio*  $J$ .

$$J = \frac{v_\infty}{\Omega R} \quad (3.54)$$

In most cases, the net head stream velocity  $v_\infty$  is the perpendicular component (projected onto the plane's normal vector  $\hat{n}$ ) of the vehicles transnational velocity in the body frame,  $\vec{v}_b \cdot \hat{n}$ . For the case of a zero advance ratio,  $J = 0$ , the conditions are regarded as static. Static thrust and power coefficients are nominal in their values.



**Figure 3.4:** Power & thrust coefficients

Propeller databases like [15]<sup>16</sup> provide comprehensive values for a range of propeller types at different advance ratios. The introduction of those coefficients greatly simplifies the thrust estimation process. For a typical 6X4.5 inch propeller<sup>17</sup>, the static thrust and power coefficients respectively are:

$$C_{T0} = 0.191 \quad (3.55a)$$

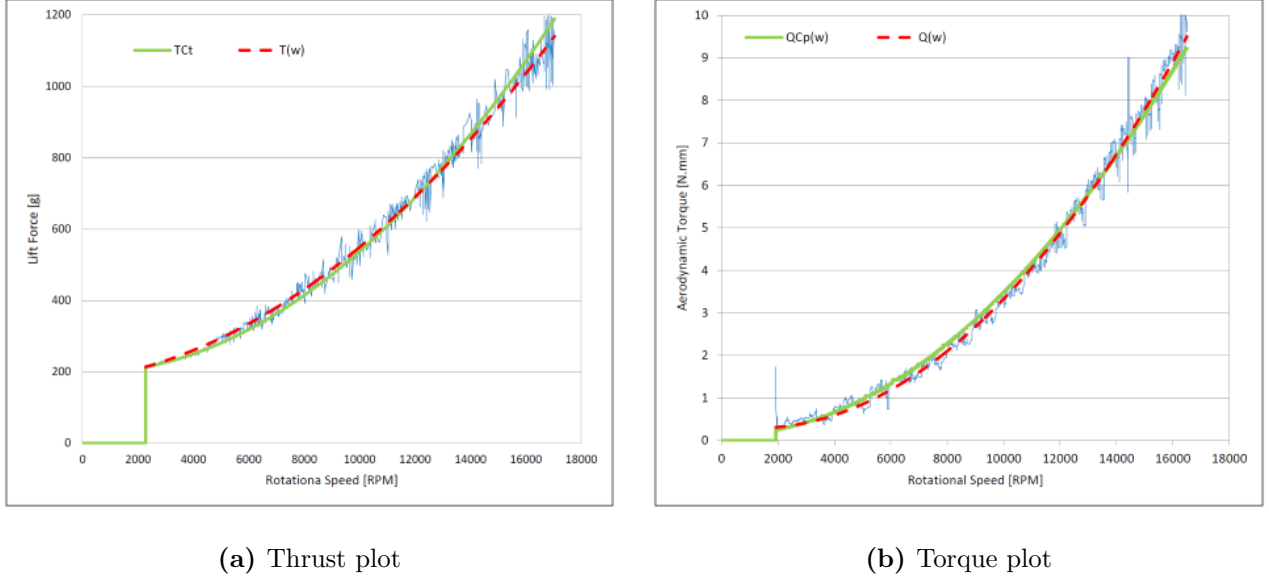
<sup>15</sup>Small angle approximations then apply to  $\cos(\phi + \alpha_{eff}) \approx 1$  and  $\sin(\phi + \alpha_{eff}) \approx \phi + \alpha_{eff}$

<sup>16</sup>The UIUC database also includes blade profiles, pitch and chord lengths. The database is the outcome of [16].

<sup>17</sup>linearly interpolated from similar pitched database results to match physical test values

$$C_{P0} = 0.0877 \quad (3.55b)$$

Fig:3.4 shows the thrust,  $C_T$ , and power,  $C_P$ , coefficients as a function of the advance ratio  $J$ . As the incident head fluid velocity,  $v_\infty$ , increases, the thrust coefficient decreases. So too does the power coefficient and hence the aerodynamic torque.



**Figure 3.5:** Static propeller tests

In Fig:3.5, the results both of static thrust and torque tests are plotted. In each test the measured values are shown (with quadratic trend-lines) and an estimated value dependent on static coefficients. Using the results from the plot in Fig:3.4 as a lookup table and calculating the values from Eq:3.53, induced propeller thrust and torques can be accurately modeled (*quadratically*<sup>18</sup>). Instantaneous advance ratios, or rather the propeller incident fluid flow(s), are dependent on the vehicle's net translational and angular velocity. Such that the fluid velocity's normal component to the propeller plane is given by:

$$v_\infty = (\vec{v}_b + \vec{L}_{arm} \times \vec{\omega}_b) \cdot \hat{n} \quad (3.56)$$

Where  $\vec{v}_b$  is the body's translational velocity and  $\vec{\omega}_b$  is the body's angular velocity, both transformed to the propeller's frame,  $\in \mathcal{F}^{M_i}$ . Additionally  $\hat{n}$  is a unit vector normal to the propeller's rotational plane. Then  $J$  is calculated as in Eq:3.54.

*It's worth noting that the above static coefficients are indeed calculated from physical static tests. However advance ratio coefficient dependencies are linearly interpolated from the closest available matching data (APC Thin-Electric 8X6 propellers). Discrepancies between the model or coefficient values derived can be accounted for with lumped uncertainty and disturbance term(s). Model uncertainty compensation is elegantly incorporated in adaptive backstepping or  $H_\infty$  control algorithms. The deviation of the modeled thrust or torques from their true values would be easy to incorporate into a plant dependent Lyapunov candidate function; Section:??.*

### 3.3.2 Hinged Propeller Conning & Flapping

Non-linear effects which deteriorate a propeller's performance are well documented in the field of helicopter propeller and aerodynamic theory. Typically such effects are more pronounced when observing hinged variable pitch<sup>19</sup> propellers. Conning and flapping are the two most significant aerodynamic

<sup>18</sup>The power term is cubic W.R.T its coefficient

<sup>19</sup>Twisted fixed pitched propellers are used on the prototype here and as such effects detailed in Section:3.3.2 are diminished. Moreover, low translational velocities suppress such responses but are worth mentioning.

responses produced by a propeller.

In translational flight, the blades of each propeller encounter varying incident fluid flow. The advancing blade, relative to the body's translational direction, encounters a greater fluid flow. The resultant is that the effective local angle of attack for advancing and retreating blades aren't symmetrical. There then exists a dissymmetry of lift across the propeller's surface. As such each propeller is forced up and down throughout one rotational cycle, the extent of that deflection being dependent on translational velocities. The flapping then pitches the effective propeller plane, and hence the thrust vector, away from its principle angle of rotation, Fig:3.6<sup>20</sup>.

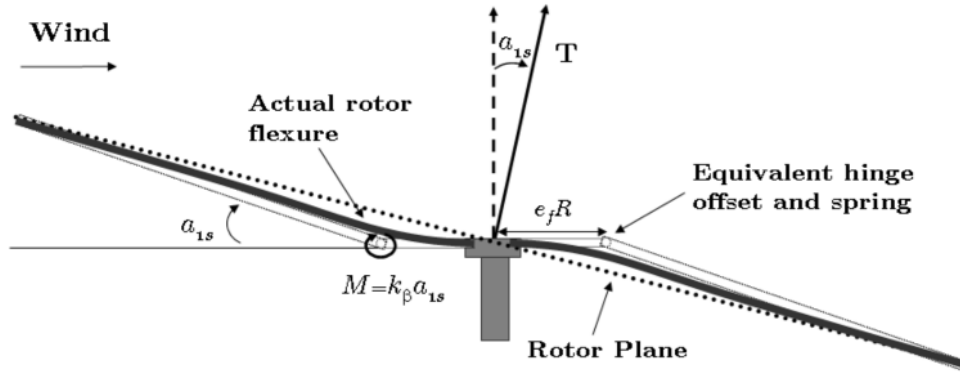


Figure 3.6: Propeller blade flapping

The net effect is that the propeller's thrust vector is pitched marginally away from an exact perpendicular vector by some deflection angle. The phenomenon is diminished at low translational velocities and as such, isn't applicable to the range of flight envelopes the prototype for the project will experience.

Coning (Fig:3.7)<sup>21</sup> is another form of propeller deflection which, dependent on the blades stiffness properties, causes the propeller blades (advancing and retreating) to both deflect upward. The coning reduces the effective propeller's disc area adversely affecting thrust produced. Increased loading accentuates the coning angle experienced by the propellers and as such alters the tip-path-plane.

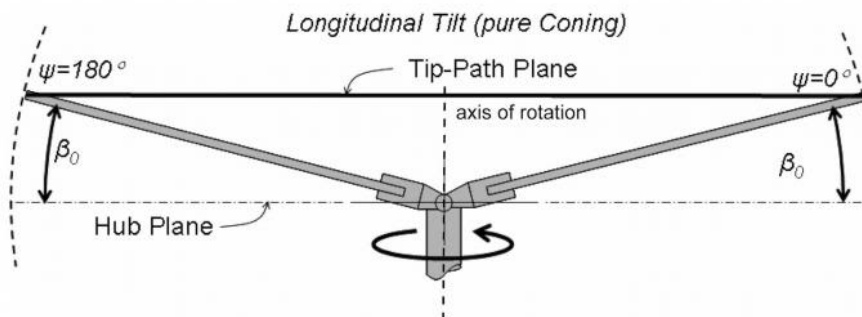


Figure 3.7: Propeller coning

Both effects can be quantified mathematically but their derivation and resultant equations are cumbersome. In due course their effect on the produced prototype which this project investigates aren't of consequence...

<sup>20</sup>Diagram cited from Hoffman et al.(2007) [42]

<sup>21</sup>Misc Image [?]

### 3.3.3 Drag

## 3.4 Consolidated Model

After reviewing Consolidating all components derived previously into a set of differential equations to be used for control plant development...

## Chapter 4

# Control Treatment

### Control Plant & Discussion

Previously, in the Design Chapter:??, it was shown that the uniqueness of the prototype considered here stems from its over-actuation. There are 12 actuator positions which are required to effect 6 degrees of freedom. Owing to the complexities which arise as a result of control plant allocation, the control algorithm is first abstracted to virtual control inputs  $\tau_{net}$  and  $F_{net}$ . Those virtual inputs are then distributed via the allocation algorithm. The control block is completely independent from the allocation block and as such the two can be altered independently.

### Control Plant Inputs

Until now, the control plant inputs for state equations Eq:?? have been described with net forces and torques. Previously in Section:?? the relationship between servo positions and thrust vector directions were quantified. In Section:?? that thrust vector magnitude was found to be *approximately* quadratically dependent on the propellers rotational speed.

Appropriate control allocation techniques are required to manipulate all 12 actuators to effect the 6 net forces and torques. The control plant is split into two stages, namely a higher level set point tracking controller which applies

For the control plant development, the plant inputs are abstracted to a virtual plant force and torque inputs. A secondary control layer applies the

## Model Dependent & Independent Controllers

### 4.1 Attitude Control

#### 4.1.1 The Attitude Control Problem

#### 4.1.2 Quaternion Based Error States

#### PD Controller

#### Auxilliary Plant Controller

#### PID Controller

#### 4.1.3 Non-linear Controllers

#### Ideal Back-stepping Controller

#### Adaptive Back-stepping Controller

#### Disturbance Update Law

#### Lyupanov Derived Ideal Controller

### 4.2 Position Control

#### 4.2.1 Backstepping Position Controller

### 4.3 Controller Allocation

#### 4.3.1 Non-linear Plant Control Allocation

#### 4.3.2 Pseudo Inverse Allocator

#### 4.3.3 Weighted Pseudo Inverse Allocator

#### 4.3.4 Priority Norm Inverse Allocator

#### 4.3.5 Online Optimized Secondary Goal Allocator

## Chapter 5

# Simulations & Results

### 5.1 Controller Tuning

#### 5.1.1 Partical Swarm Based Optimization

#### 5.1.2 Performance Metric

#### 5.1.3 Global & Local Minima

#### 5.1.4 Fmincon Differences

### 5.2 Simulation Block

### 5.3 State Estimation

### 5.4 Optimized Controller Comparisons

#### 5.4.1 Allocator Performance

#### 5.4.2 Attitude Control Results

#### 5.4.3 Autopilot Outcome



## Chapter 6

# Prototype Flight Results

## Chapter 7

## Conclusion

- Lagrange dynamics for multibody system could have produced a more concise model etc ...
- Particule multibody dynamics with internactions could provide a more verbose simulation environment rather than the very newtonian simulation loop constructed
- Implicit equation dynamics in simulation may improve optimization loops in PSO algorithm

## Appendix A

# Standard Quadrotor Dynamics

Following 6-DOF derivations in Section:3.1.1, the common reductions typically applied those equations for a generic "+" configuration quadrotor are now presented. Reiterating those four differential equations, Eq:3.9, which describe a rigid body's motion (using rotation matrices and not quaternions):

$$\dot{\vec{\mathcal{E}}} = \mathbb{R}_b^I(-\eta)\vec{v} \quad \in \mathcal{F}^I \quad (\text{A.1a})$$

$$\dot{\vec{v}} = \frac{1}{m} \left[ -\vec{\omega}_b \times m\vec{v} + m\mathbb{R}_I^b(-\eta)\vec{G}_I + \vec{F}_{net} \right] \quad \in \mathcal{F}^b \quad (\text{A.1b})$$

$$\dot{\vec{\eta}} = \Psi(\eta)\vec{\omega}_b \quad \in \mathcal{F}^{v^2}, \mathcal{F}^{v^1}, \mathcal{F}^I \quad (\text{A.1c})$$

$$\dot{\vec{\omega}}_b = \mathbb{I}_b^{-1} \left[ -\vec{\omega}_b \times \mathbb{I}_b\vec{\omega}_b + \vec{\tau}_{net} \right] \quad \in \mathcal{F}^b \quad (\text{A.1d})$$

The net heave thrust produced by rotors  $i = [1 : 4]$  is given by:

$$\vec{T} = \sum_{i=1}^4 \vec{F}_i \quad (\text{A.2a})$$

The simplified relationship between the thrust force  $\vec{F}_i$  and the propellers rotational speed  $\Omega_i$  is approximately quadratic:

$$\vec{F}_i = k_1\Omega_i^2 \quad (\text{A.2b})$$

Similarly the aerodynamic torque opposing each rotating propeller is:

$$\tau_{aero} = k_2\Omega_i^2 \quad (\text{A.3})$$

The control pitch and roll torques are generated by opposing differential lift forces and finally the yaw torque is a net response to the rotational aerodynamic propeller torques. The control torque inputs are then defined as:

$$\tau_\phi = \vec{L}_{arm}(\vec{F}_1 - \vec{F}_3) \quad (\text{A.4a})$$

$$\tau_\theta = \vec{L}_{arm}(\vec{F}_2 - \vec{F}_4) \quad (\text{A.4b})$$

$$\tau_\psi = \sum_{i=1}^4 (-1)^i k_2 \Omega_i \quad (\text{A.4c})$$

Simplifying Eq:?? to component form:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} + \begin{pmatrix} -g\sin(\theta) \\ g\cos(\theta)\sin(\phi) \\ g\cos(\theta)\cos(\phi) \end{pmatrix} + \frac{1}{m} \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix} \quad \in \mathcal{F}^b \quad (\text{A.5})$$

Considering the size of the average angular velocity  $\omega_b$ , the gyroscopic effects on the body (namely the cross product terms) are negligible. Assuming too that the body has a (*roughly*) diagonal inertial matrix:

$$\begin{pmatrix} \frac{\mathbb{I}_y - \mathbb{I}_z}{\mathbb{I}_x} qr \\ \frac{\mathbb{I}_z - \mathbb{I}_x}{\mathbb{I}_y} pr \\ \frac{\mathbb{I}_x - \mathbb{I}_y}{\mathbb{I}_z} pq \end{pmatrix} \approx \vec{0} \quad (\text{A.6})$$

As a result, Eq:?? then simplifies to:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \frac{1}{\mathbb{I}_x} \tau_\phi \\ \frac{1}{\mathbb{I}_y} \tau_\theta \\ \frac{1}{\mathbb{I}_z} \tau_\psi \end{pmatrix} \quad (\text{A.7})$$

Similarly, around the origin and hovering conditions,  $\Psi(\eta) \approx \vec{1}$  for  $\eta \approx \vec{0}$  then  $\dot{\eta} \approx \omega_b$ . Or in component form:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} \approx \begin{pmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{pmatrix} \quad (\text{A.8})$$

Because  $\omega_b$  is small, the Coriolis cross-product term in Eq:?? is depreciated.

$$\begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} \approx \vec{0} \quad (\text{A.9})$$

As such, the differential equations Eq:?? are then simplified to the following six SISO controllable plants:

$$\ddot{x} = (-\cos(\phi)\sin(\theta)\cos(\psi) - \sin(\phi)\sin(\psi)) \frac{1}{m} F \quad (\text{A.10a})$$

$$\ddot{y} = (-\cos(\phi)\sin(\theta)\sin(\psi) + \sin(\phi)\cos(\psi)) \frac{1}{m} F \quad (\text{A.10b})$$

$$\ddot{z} = g - (\cos(\phi)\cos(\theta)) \frac{1}{m} F \quad (\text{A.10c})$$

$$\ddot{\phi} = \frac{1}{\mathbb{I}_x} \tau_\phi \quad (\text{A.10d})$$

$$\ddot{\theta} = \frac{1}{\mathbb{I}_y} \tau_\theta \quad (\text{A.10e})$$

$$\ddot{\psi} = \frac{1}{\mathbb{I}_z} \tau_\psi \quad (\text{A.10f})$$

# Appendix B

## Design Bill of Materials

### B.1 Parts List

Part Name	No. Used	Unit Weight[g]
Electronics		
SPRacing F3 Deluxe Flight Controller	1	8
OrangeRx 615X 2.4 GHz 6CH Receiver	1	9.8
Signal Converter SBUS-PPM-PWM	1	5.0
STLink-V2 Debugger	1	3
RotorStar Super Mini S-BEC 10A	1	30
128x96" OLED Display	1	7
XBee-Pro S1	2	4
HobbyWing XRotor 20A Opto ESC	4	15
OrangeRX RPM Sensor	4	2
HobbyKing Multi-Rotor Power Distribution Board	1	49
Motors		
Corona DS-339MG	8	32
Cobra 2208 2000KV Brushless DC	4	44.2
Frame Components		
APM Flight Controller Damping Platform	1	7
HobbyKing SK450 Replacement Arm (2 pcs)	2	51
SK450 Extended Landing Skid	1	23.25
Alloy Servo Arm (FUTABA)	8	4
10X18X6 Radial Ball Bearing	8	5
80g Damping Ball	32	$\approx 0$
Plastic Retainers for Damping Balls	32	$\approx 0$
3/5mm Aluminum Prop Adapter	4	$\approx 1$
6x4.5 Gemfam 3-Blade Propeller	4	6
M3 6mm Hex Nylon Spacer	8	$\approx 0$
M3 16mm Hex Nylon Spacer	32	$\approx 0$
M3 25mm Nylon Screw	128	$\approx 0.08$
M2.5x10mm Socket Head Cap Screw	36	$\approx 0.2$
M2.5x25mm Socket Head Cap Screw	20	$\approx 0.6$
M2.5 A-Lok Nut	16	$\approx 0$

**Table B.1:** Parts List

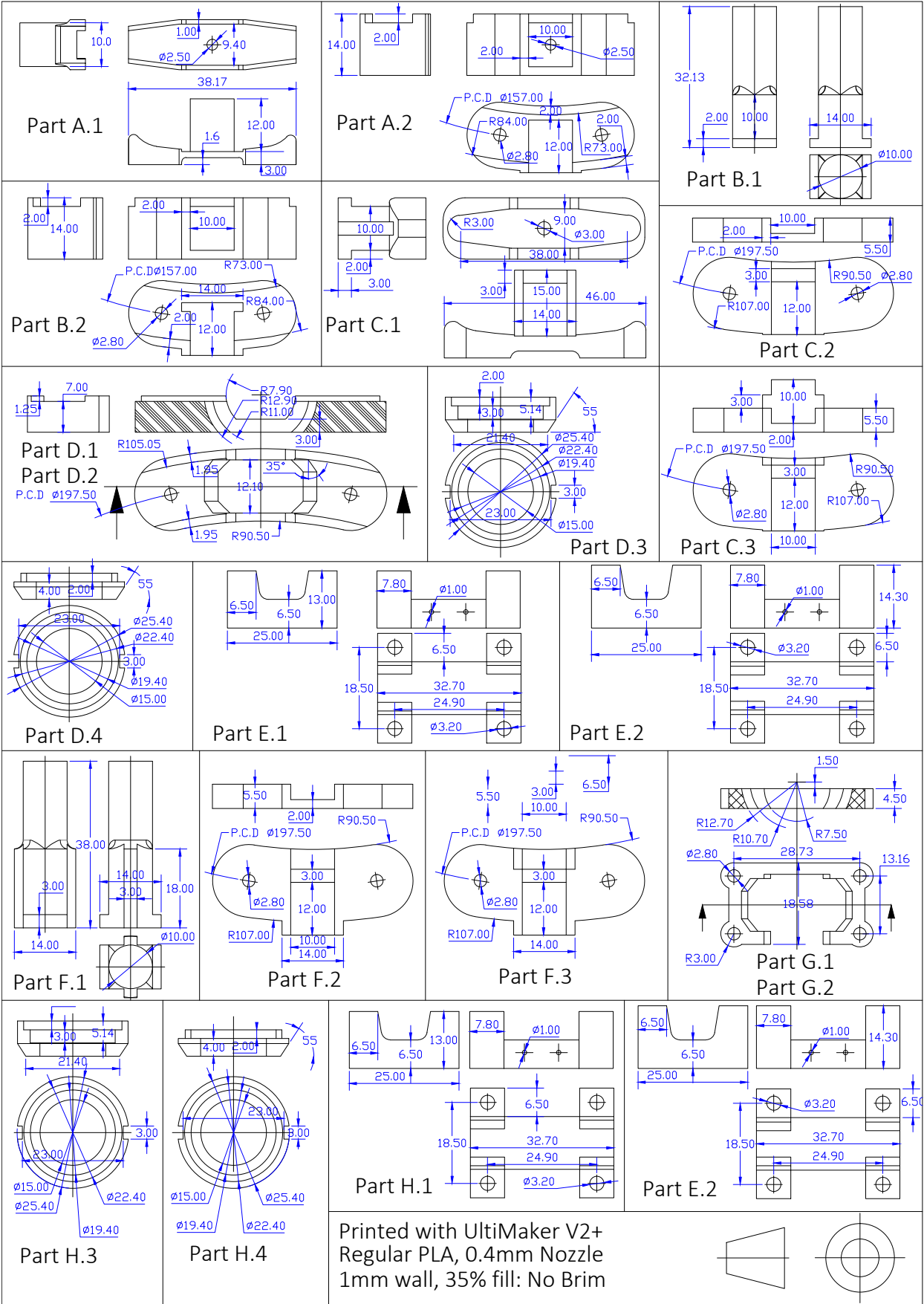


Table B.2: 3D Printed Parts

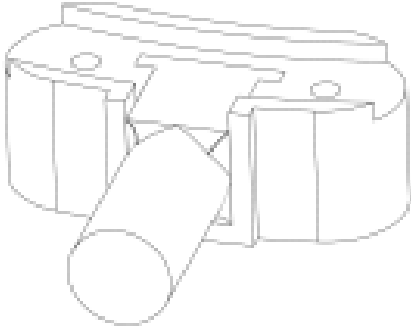
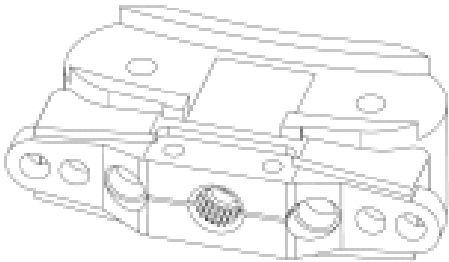
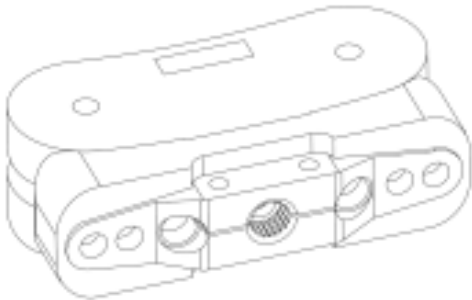

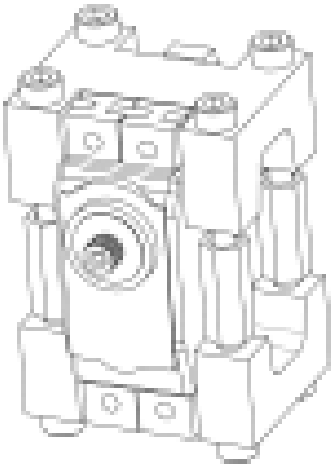
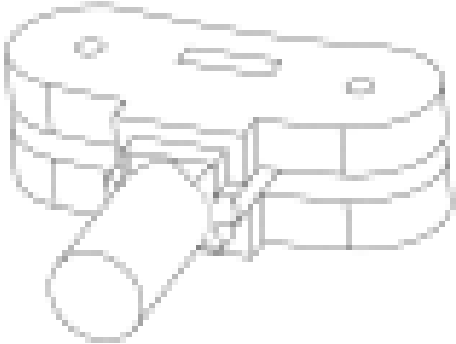
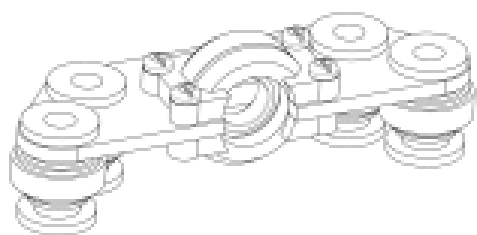
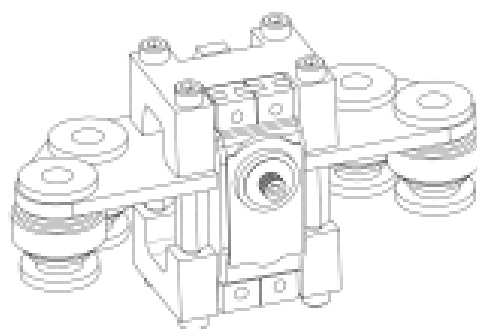
Bracket Assemblies 2	
 <p><b>Figure B.1:</b> Bearing Bracket Inner Ring Assembly Parts: A.1, A.2</p>	 <p><b>Figure B.2:</b> Servo Bracket Inner Ring Assembly Parts: B.1, B.2, M3 Servo Horn</p>
 <p><b>Figure B.3:</b> Servo Bracket Middle Ring Assembly Parts: C.1, C.2, C.3, M3 Servo Horn</p>	 <p><b>Figure B.4:</b> Bearing Holder Middle Ring Assembly Parts: D.1, D.2, D.3, D.4, 18-10 Bearing</p>
 <p><b>Figure B.5:</b> Servo Mount Middle Ring Assembly Parts: E.1, E.2, Corona Servo &amp; Fasteners</p>	 <p><b>Figure B.6:</b> Bearing Shaft Middle Ring Assembly Parts: F.1, F.2, F.3</p>

Table B.3: Inner & Middle Ring Assemblies

## Bracket Assemblies 2



**Figure B.7:** Bearing Holder Damping Assembly  
Parts: G.1, G.2, G.3, G.4, 18-10 Bearing, 80g  
Damping Balls, Bearing Holder Damping Bracket

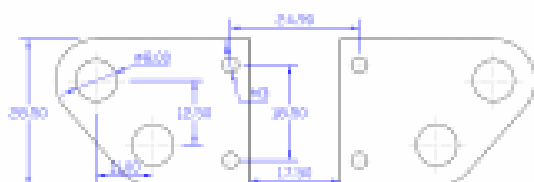


**Figure B.8:** Servo Mount Damping Assembly  
Parts: H.1, H.2, Corona Servo & Fasteners, 80g  
Damping Balls, Servo Mount Damping Bracket

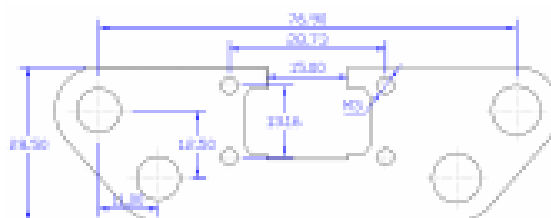
Table B.4: Damping Assemblies

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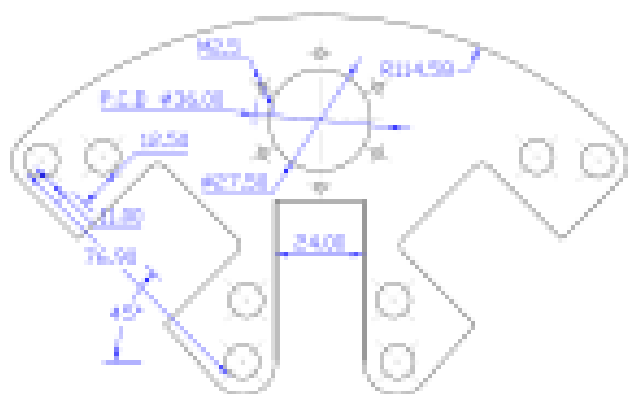
## Laser Cut Brackets



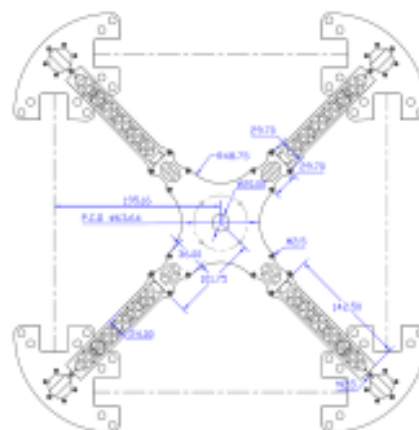
**Figure B.9:** Servo Mount Damping Bracket



**Figure B.10:** Bearing Holder Damping Bracket



**Figure B.11:** Arm Mount Damping Bracket



**Figure B.12:** Frame Brackets

**Table B.5:** Laser Cut Damping Brackets



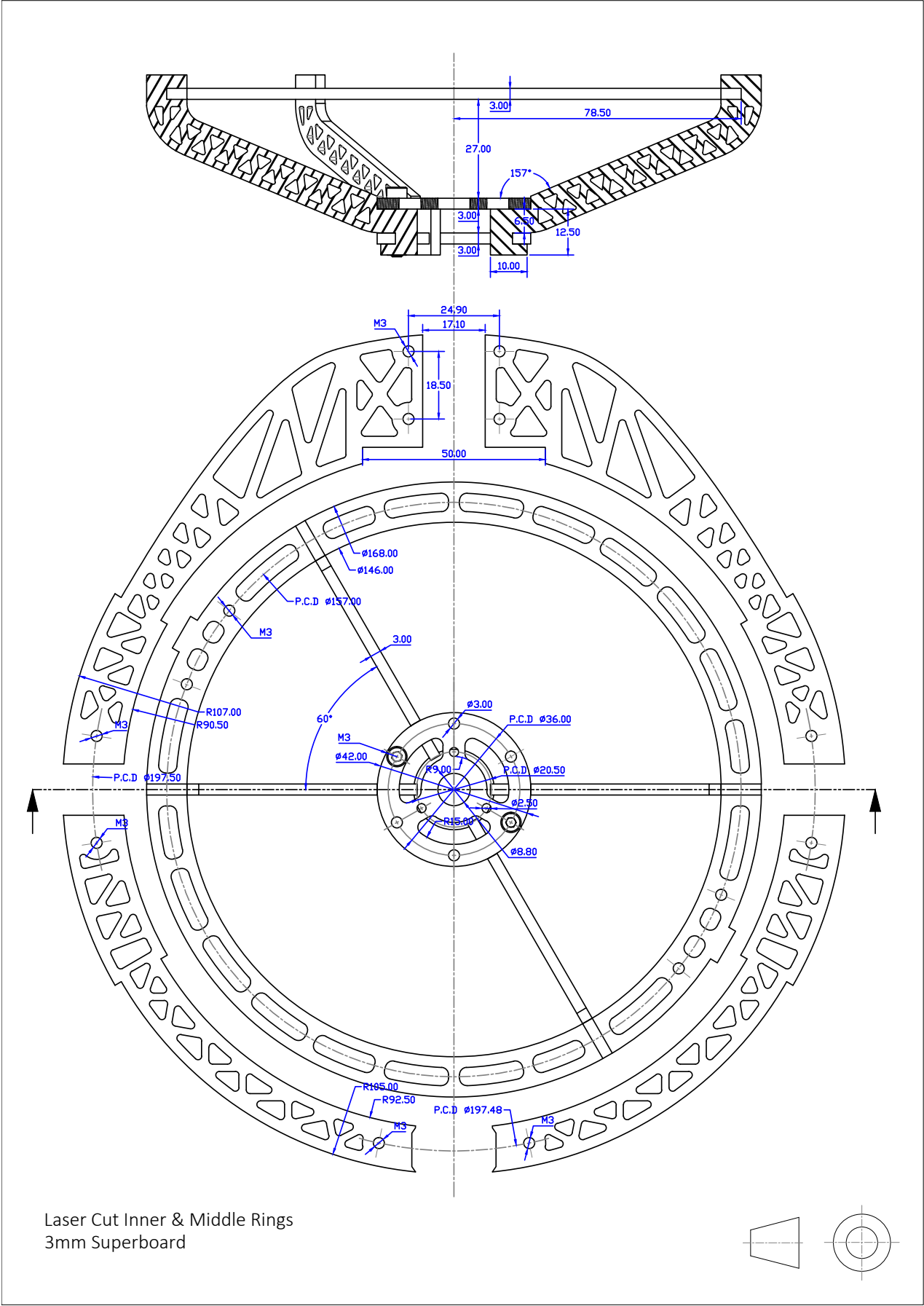


Table B.6: Laser Cut Parts

## B.2 F3 Deluxe Schematic Diagram

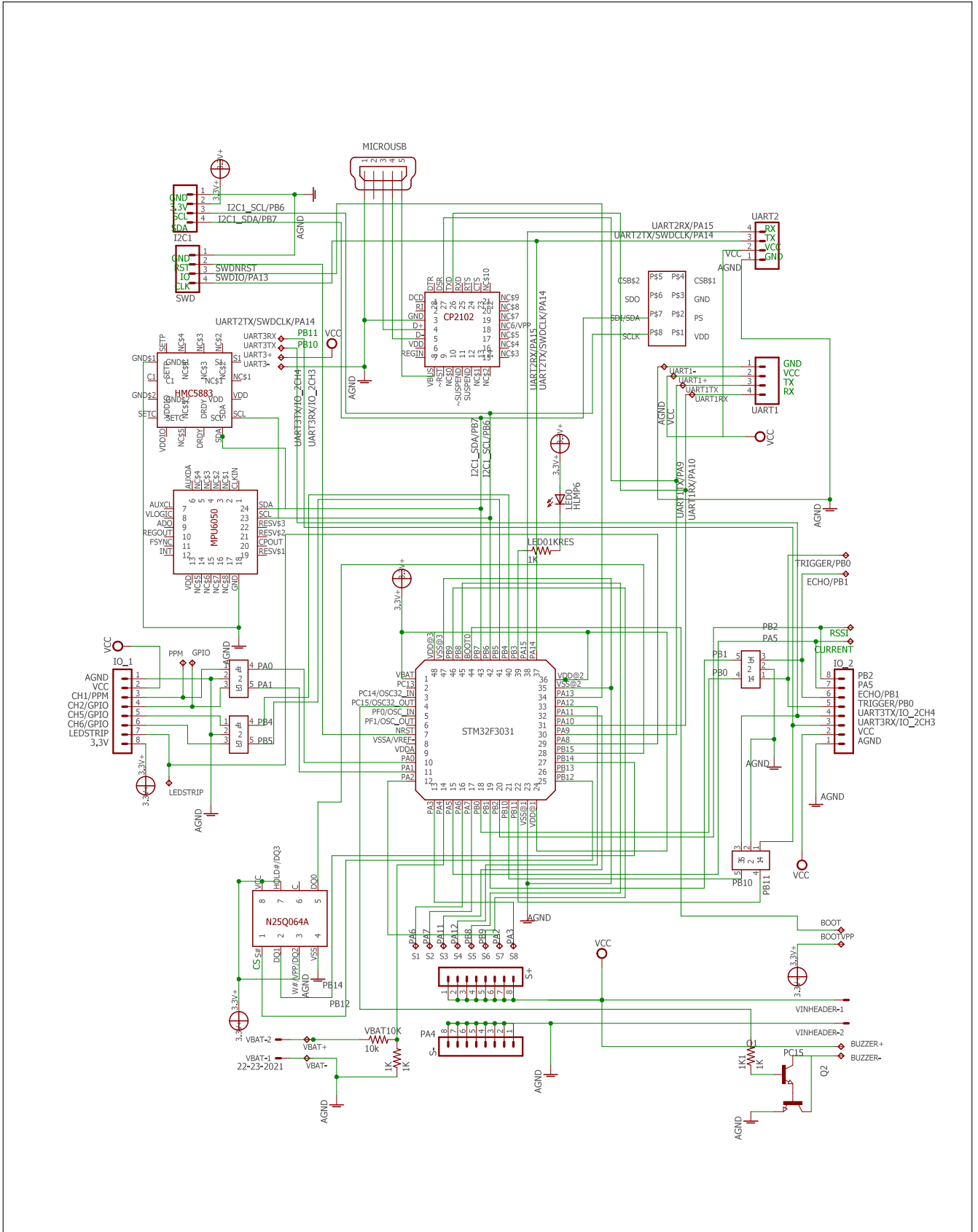


Figure B.13: F3 Deluxe Flight Controller Hardware Schematic

## Appendix C

# System ID Test Data

### C.1 Servo Data

## C.2 Cobra CM2208-200KV Thrust Data

Cobra CM-2208/20 Motor Propeller Data										
Magnets 14-Pole	Motor Wind 20-Turn Delta	Motor Kv 2000 RPM/Volt		No-Load Current Io = 0.77 Amps @ 10v		Motor Resistance Rm = 0.076 Ohms		I Max 20 Amps	P Max (3S) 220 W	
Stator 12-Slot	Outside Diameter 27.7 mm, 1.091 in.	Body Length 24.0 mm, 0.945 in.		Total Shaft Length 45.2 mm, 1.780 in.		Shaft Diameter 3.17 mm, 0.125 in.		Motor Weight 44.2 gm, 1.56 oz		
Test Data From Sample Motor		Input Io Value	6.0 V 0.59 A	8.0 V 0.67 A	10.0V 0.77 A	12.0V 0.87 A	Measured Kv value 1988 RPM/Volt @ 10v	Measured Rm Value 0.076 Ohms		
Prop Manf.	Prop Size	Li-Po Cells	Input Voltage	Motor Amps	Input Watts	Prop RPM	Pitch Speed in MPH	Thrust Grams	Thrust Ounces	Thrust Eff. Grams/W
APC	5.25x4.75-E	3	11.1	13.34	148.1	17,507	78.7	451	15.91	3.05
APC	5.5x4.5-E	3	11.1	13.67	151.7	17,388	74.1	456	16.08	3.01
APC	6x4-E	3	11.1	14.87	165.1	17,003	64.4	630	22.22	3.82
APC	7x4-SF	3	11.1	21.82	242.2	13,985	53.0	840	29.63	3.47
APC	7x5-E	3	11.1	24.02	266.6	13,272	62.8	797	28.11	2.99
FC	5x4.5	3	11.1	8.66	96.1	19,061	81.2	428	15.10	4.45
FC	5x4.5x3	3	11.1	12.38	137.4	17,825	76.0	534	18.84	3.89
FC	6x4.5	3	11.1	15.47	171.7	16,792	71.6	721	25.43	4.20
GemFan	5x3	3	11.1	6.67	74.0	19,801	56.3	374	13.19	5.05
HQ	5x4	3	11.1	7.13	79.1	18,182	68.9	373	13.16	4.71
HQ	5x4x3	3	11.1	9.25	102.7	17,401	65.9	449	15.84	4.37
HQ	5x4.5-BN	3	11.1	11.17	124.0	16,902	72.0	487	17.18	3.93
HQ	6x3	3	11.1	7.34	81.5	18,128	51.5	419	14.78	5.14
HQ	6x4.5	3	11.1	13.53	150.2	16,206	69.1	645	22.75	4.29
HQ	6x4.5x3	3	11.1	17.60	195.4	15,137	64.5	762	26.88	3.90
HQ	7x4	3	11.1	20.71	229.9	14,250	54.0	850	29.98	3.70
HQ	7x4.5	3	11.1	20.31	225.4	14,351	61.2	865	30.51	3.84
Prop Manf.	Prop Size	Li-Po Cells	Input Voltage	Motor Amps	Input Watts	Prop RPM	Pitch Speed in MPH	Thrust Grams	Thrust Ounces	Thrust Eff. Grams/W
APC	5.25x4.75-E	4	14.8	17.29	255.9	20,560	92.5	603	21.27	2.36
APC	5.5x4.5-E	4	14.8	17.87	264.5	20,436	87.1	635	22.40	2.40
APC	6x4-E	4	14.8	20.15	298.2	19,829	75.1	837	29.52	2.81
FC	5x4.5	4	14.8	10.89	161.2	22,511	95.9	588	20.74	3.65
FC	5x4.5x3	4	14.8	16.43	243.2	20,828	88.8	718	25.33	2.95
FC	6x4.5	4	14.8	20.09	297.3	19,809	84.4	998	35.20	3.36
HQ	4x4.5-BN	4	14.8	10.45	154.7	22,661	96.6	477	16.83	3.08
HQ	5x3	4	14.8	6.88	101.8	23,580	67.0	442	15.59	4.34
HQ	5x4	4	14.8	10.22	151.3	22,739	86.1	589	20.78	3.89
HQ	5x4x3	4	14.8	13.26	196.2	21,763	82.4	710	25.04	3.62
HQ	5x4.5-BN	4	14.8	16.10	238.3	20,899	89.1	744	26.24	3.12
HQ	6x3	4	14.8	11.06	163.7	22,512	64.0	679	23.95	4.15
HQ	6x4.5	4	14.8	19.62	290.4	19,948	85.0	982	34.64	3.38

Figure C.1: Official Test Results for Cobra Motors

## Appendix D

### Inertias

$$\mathbb{I}_{inner} \tag{D.1a}$$

$$\mathbb{I}_{middle} \tag{D.1b}$$

$$\mathbb{I}_{body} \tag{D.1c}$$

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