AbsorptionRank

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Introduction

Epidemiology is a subject of much contemporary relevance. Quarantine and case isolation have been shown to be effective policies to counteract pandemics (Auranen et al. 2023). Moreover, the successful trial of the "Covid-Card", for detecting potential transmission events, suggests that prophylactic identification and isolation of high-risk individuals could be practical in the future. The sociosphere of an individual can be modeled as a series of contact times, each paired with a measure of proximity. We abstractly represent each individual as a node, and each contact event by an undirected edge. This model is called a temporal network. The Susceptible-Infected model, abbreviated as SI, is an epidemiological model in which susceptible individuals can become infected due to a contact with an infectious individual. In the SI paradigm, we assume that infected individuals stay infected indefinitely. To date, research on the SI model has largely been focused on computational simulations of many epidemics on a temporal network. In this paper, I propose an alternative method for the determination of high-risk individuals in a social network, and I compare it to existing centrality measures.

Temporal Centrality

Generalization of conventional centrality measures to temporal networks requires a high-granularity representation of the network as a graph. Kostakos et al. [2009] proposed a graph where it is assumed that two nodes are connected if they are reachable within n hops, however this model fails to account for differential rates of transmission. Kim and Anderson [2011] proposed a more general solution. Without loss of generality, consider a set of contact times, $t = (1, 2, 3, \cdots, T)$. We can construct a time-ordered graph, G, with vertices V and edges E, where for all $v_t, u_k \in V$, an edge $e(v_t, u_k) \in E$ can only exist if k = t + 1. We can construct this graph for any temporal network without loss of information. In practice, computational constraints may mandate aggregation of contact times and thus loss of information. Rocha and Masuda [2014] proposed a centrality measure which considers the stationary distribution of a random walk at time t, however this measure does not generalize well to the Susceptible-Infected (SI) model, where infection is a permanent state. Define the distance of the temporal shortest path over time interval [i, j], denoted by $d_{i,j}(v, u)$, as the smallest $d = k - i$, where $i \leq m \leq j$ and there is a path from v_i to u_k. We can now generalize conventional measures of prestige and centrality to temporal networks.

Temporal Prestige

In this study, I am primarily concerned with the likelihood of infection, which is closely related to the more general notion of prestige. In a directional network, a prestigious node is the object of many ties i.e. has many incoming connections. This is a distinct concept from centrality, which is also concerned with outgoing connections. Many conventional measures of centrality are ill-defined in directional graphs, owing to the fact that directional graphs are not necessarily strongly-connected. In sight of this limitation, we generally only consider nodes in the influence domain of node i i.e. the set of all nodes from whom n_i is reachable. Lin (1976) proposed the following measure of prestige for directional relations:

$$P_p(n_i) = \frac{I_i/(g-1)}{\sum d(n_j, n_i)/I_i}$$

Where I_i is the size of the influence domain of node i, and g is the total number of nodes in the network. For temporal networks, I propose a modified version

$$P_p(u_i) = \frac{I_{u_i}/(g-1)}{\sum_{v \in N} d_{0,i}(v, u_i)/I_{u_i}}$$

where N is the influence domain of node u at time i. When the influence domain is empty, the temporal prestige is 0. Kim and Anderson [2011] proposed temporal closeness, a similar metric which considers all time intervals $[t, i], t \in [0, i-1]$.

$$C_{i,j} = \sum_{i \leq t < j} \sum_{u \in V} \frac{1}{d_{t,j}(v, u)}$$

When u is unreachable from v over [t, j], $d_{t,j}(v, u) = \infty$. We cover cases where the denominator is infinite by assuming that $\frac{1}{\infty} = 0$. Note that as we are considering a directional graph, $d_{t,j}(v, u)$ is not equivalent to $d_{t,j}(u, v)$ To turn this formula into a prestige measure, we simply reverse the direction of the paths to get:

$$C^P_{i,j} = \sum_{i \leq t < j} \sum_{u \in V} \frac{1}{d_{t,j}(u, v)}$$

Absorption Prestige

In this section, I propose an ad-hoc prestige measure for epidemiological study. Without loss of generality, we will consider a temporal network consisting of a set of nodes, N, and positive integer contact times, $t = (1, 2, 3, \cdots, T)$. Let $p_{ijk}(t)$ denote the probability of transmission for the k'th contact between individual i and individual j, at time t. Let $n_{ij}(t)$ denote the number of contacts between i and j at time t. Define the transition probability matrix for each contact time as:

$$\mathbf{B}_{ij}(t) = \begin{cases} 1 & i = j, s_i(t) = 0 \\ 0 & i \neq j, s_{ij}(t) = 0 \\ \prod_{m \in N, m \neq i} \prod_{k=1}^{n_{im}(t)} (1 - p_{imk}(t)) & i = j, s_i(t) > 0 \\ (1 - \mathbf{B}_{ii})(s_{ij}(t)/s_i(t)) & i \neq j, s_{ij}(t) > 0 \end{cases}$$

where $s_{ij}(t)$ and $s_i(t)$ are defined as:

$$s_{ij}(t) = 1 - \prod_{k=1}^{n_{ij}(t)} (1 - p_{ijk}(t))$$

$$s_i(t) = \sum_{j \in N, j \neq i} s_{ij}(t)$$

Denote by $\mathbf{B}_i(t)$ the transition matrix obtained by taking $\mathbf{B}(t)$, and setting all entries in the i'th row to zero, except the diagonal entry (which is necessarily one). The walk $\mathbf{B}_i = (\mathbf{B}_i(1), \mathbf{B}_i(2), \cdots, \mathbf{B}_i(T))$ is an absorbing random walk, and the product $C^A_i(t) = \mathbf{B}_i(1)\mathbf{B}_i(2)\cdots\mathbf{B}_i(t)$ will be called the temporal absorption prestige for individual i, at time t.

Example

A simple example network of five nodes is shown in Example_Network.csv. For simplicity, a constant transmission probability was used for all contact events. In total, 500 simulations were run on this network. The absorption centrality and the number of times infected were calculated for the final contact time. Table 1 shows the correlation between the absorption centrality and the probability of infection by the end of the simulation.

Table 1: Pearson correlation between temporal absorption centrality and infected proportion over 500 simulations

	Correlation
p = 0.3	0.6649671
p = 0.2	0.6614322
p = 0.1	0.5206837