1 CRR model

A binomial tree model is given as below:

- Let X be a random walk with transition probability

$$p(X_n = (j+1)\Delta x | X_{n-1} = j\Delta x) = p, \quad p(X_n = (j-1)\Delta x | X_{n-1} = j\Delta x) = 1 - p.$$

A CRR$(N; s, r, \sigma, T)$ model assumes Stock price S satisfies with the following additional conditions:

- S starts from s;
- $X_t = \ln S_t$ follows the random walk with transition probability to upper leg probability

$$\hat{p} = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

  and down leg probability $1 - \hat{p}$.

[Q.]

- Is CRR arbitrage free model?
- What is the limit behavior of X as $N \to \infty$?

2 Computation of CRR vanilla options

2.1 Vanilla call/put option computing

A pricing engine for vanilla call/put can be designed by either Backward iteration or Monte-carlo method. One shall compare two methods by using the following example.

We consider an option with

$$\text{type} = \text{call, maturity } T = 1 \text{ and strike } K = 100.$$

Its underlying stock has

$$\text{spot price } S_0 = 100 \text{ and volatility } \sigma = 20\%.$$

Moreover, the interest rate is 5%.

You are expected to try the following tasks:

1. Compute BSM call value.
2. Compute the value of CRR(N = 2000) and observe if it is sufficiently close to BSM value?
3. Next, we want to demonstrate the convergence

$$\text{CRR(N)} \to \text{BSM, as } N \to \infty.$$

To do that, we take the number of steps $N = 10 + 20k$, where k ranges over all integers in $\{0, 1, \ldots, 49\}$.

pseudocode

- plot a graph of the mapping, for $k \in range(50)$

$$k \mapsto CRR(10 + 20k).$$

- on the same figure, plot a horizontal line with the level of BSM value.
- observe if the CRR curve actually converging to the BSM line?

4. Repeat the above convergence demonstration by replacing the number of steps by

$$N = 10 + 25k.$$

Observe the CRR curve converging to BSM line in the same manner?

2.2 Extension

One can try to extend the function of the pricing engine by Monte-Carlo method to some other options:

- Digital option
- American option
- Barrier option
- Option with default risk
- Any useful exotic option pricing

Some other investigations are possible if the pricing engine is reliable, for instance

- How does option price change as volatility is getting bigger? What do you think of the price if $\sigma \to 0$ or $\sigma \to \infty$?
- How sensitively does the price change as the spot price is getting bigger? More sensitivity questions can be referred to Greeks.
- Any other useful features for discovery?

3 Further investigations

- Can one improve pricing on Digital option and Barrier option? See importance sampling/variance reduction, etc.
- American option is indeed MDP and one can try one can try any of reinforcement learnings. Probably one can start with Q-learning.
- Try Q-learning to American option with barrier/default risk.
- Try deep BSDE to American option with barrier/default risk.