A NOTE TO THE GRADER: My assignment is not very good. Even worse, LaTeX has given me an issue formatting my images. I had to append all images containing numerical results for problem 1 to the end of this document. In order to see results/plots for question 1, please see the last section of this document starting on page 12. The captions of all images should indicate which problem each figure belongs to. Thank you.

Problem 1

SOLUTION

Executive Summary

From inspection of figure 14, my numerical solution significantly differs from my analytical solution. However, comparing figure's 12 and 13, there appears to be an issue with my analytical solution, rather than my numerical solution. Inspection of the u, t projection of figure 13 is the reason I believe my analytical solution is incorrect. My results take the general expected shape, but figure 14 expresses the discrepancy the best.

Analytical Solution

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}$$

$$u(x,t) = X(x)T(t) \rightarrow X(x)T'(t) = X''(x)T(t)$$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda$$

$$T' + \lambda T = 0; X'' + \lambda X = 0$$

Considering the temporal ODE first, $T' + \lambda T = 0$. For $\lambda = 0$,

$$T'(t) = 0 \rightarrow T(t) = A$$

$$T(0) = sin(\pi x) = A$$

For $\lambda < 0$,

$$T' - \lambda T = 0 \rightarrow T = Ae^{\lambda t}$$

$$T(0) = sin(\pi x) = A$$

For $\lambda > 0$,

$$T' + \lambda T = 0 \rightarrow T(t) = Ae^{-\lambda t}$$

$$T(0) = sin(\pi x) = A$$

Now for the spatial variable,

$$X'' + \lambda X = 0$$

. Starting with $\lambda = 0$,

$$X'' = 0 \rightarrow X(x) = Ax + B$$

$$X(0) = 0 = B$$

$$X(1) = 0 = A$$

So $\lambda = 0$ yields a trivial solution for problem. Now looking at $\lambda < 0$,

$$X'' - \lambda X = 0 \rightarrow X(x) = Acosh(\sqrt{\lambda}x) + sinh(\sqrt{\lambda}x)$$

$$X(0) = 0 = Acosh(0) \rightarrow A = 0$$

$$X(1) = 0 = Bsinh(\sqrt{\lambda}) \rightarrow B = 0$$

$\lambda < 0$ also yields the trivial solution. Now try $\lambda > 0$,

$$X'' + \lambda X = 0 \rightarrow X(x) = Acos(\sqrt{\lambda}x) + Bsin(\sqrt{\lambda}x)$$

$$X(0) = 0 = A$$

$$X(1) = 0 = Bsin(\sqrt{\lambda}) \rightarrow \sqrt{\lambda} = \pi$$

$$X(x) = sin(\pi x)$$

$$u(x,t) = X(x)T(t) = sin^2(\pi x)e^{-\pi^2 t}$$

Problem 2

SOLUTION

Executive Summary

Because I know my analytical solution is definitely incorrect from problem 1, and my numerical solution is definitely incorrect for this problem, my work for this problem is not very good. I know my numerical solution is incorrect from inspection of figure 1. However, the boundary conditions are still satisfied.

Numerical Results and comparison to Analytical Solution

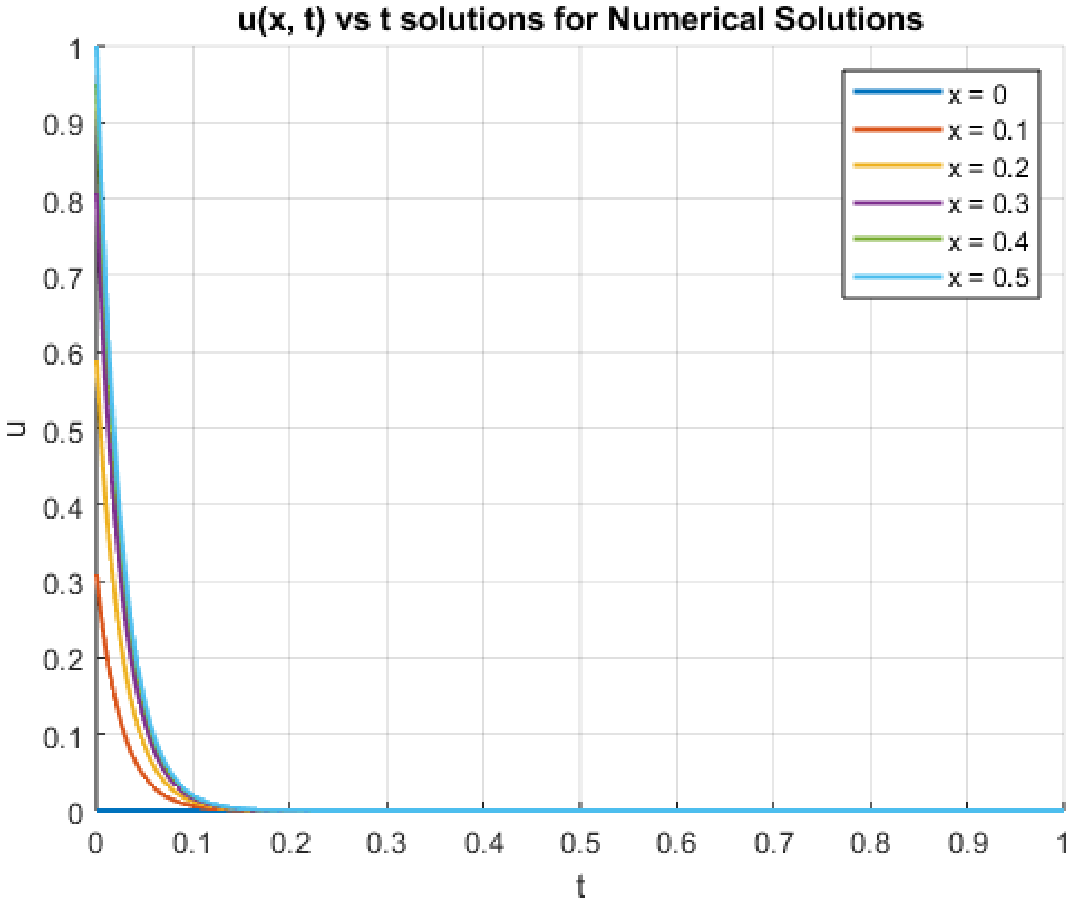


Figure 1: Problem 2 2D numerical results

Problem 3

SOLUTION

Executive Summary

My solution for this problem is also wrong. I provided an explanation of how I derived my numerical solution below. An image of one CM_p matrix is also shown.

Part A

My solution for this implicit Crank-Nicholson using central differences consists of solving a linear system of equations for each step in t. Let CM_p be the square matrix that stores the linear system, u be the desired solution and b be the right hand side of the matrix equation.

$$CM_p\vec{u} = \vec{b}$$

CM_p is tridiagonal and square, being the size of the number of points in the x direction. For this problem, there are 10 x points for every point in t. However, the boundary conditions demands that the first row of the CM_p matrix is determined differently than the rest of the matrix. Rows 1 and 10 of each CM_p matrix are determined by the following relations.

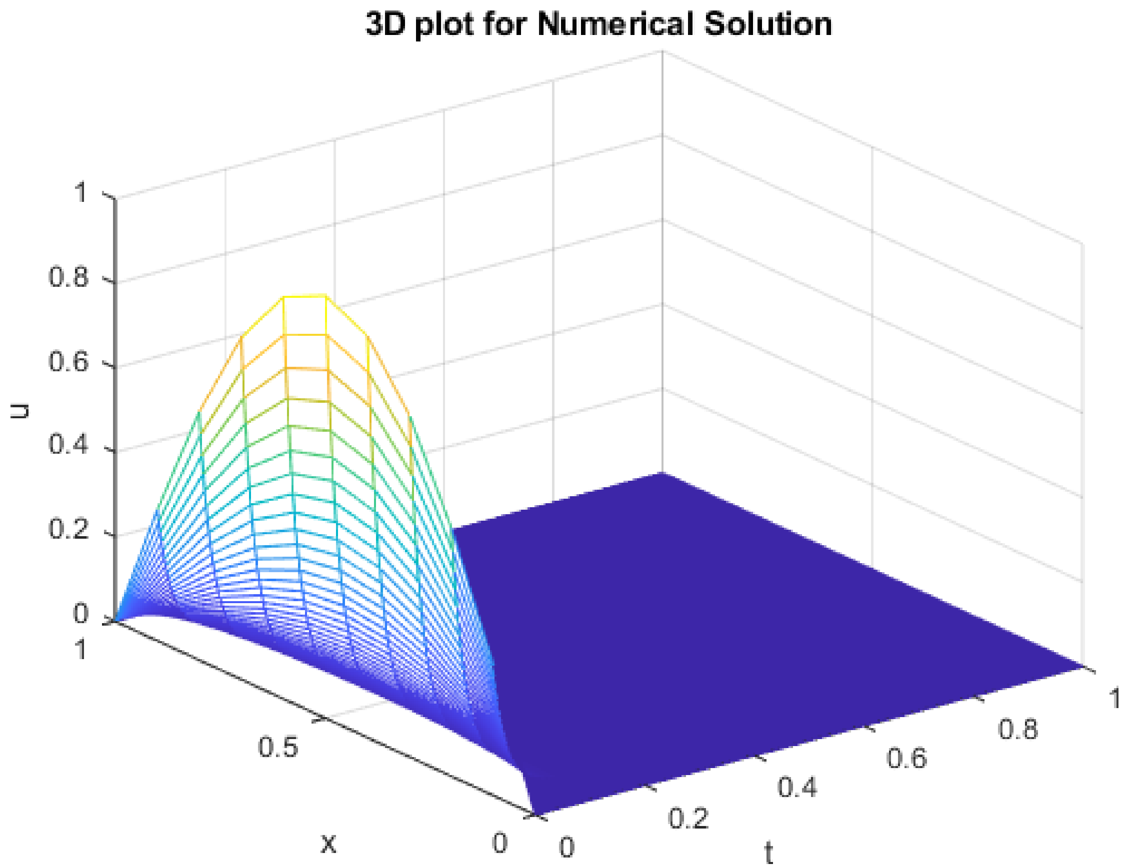


Figure 2: Problem 2 3D numerical results

$$2(1 + r + hr)u_{0,j+1} - 2ru_{1,j+1} = 2(1 - r - hr)u_{0,j} + 2ru_{1,j}$$

$$-2ru_{9,j+1} + 2(1 + r - hr)u_{10,j+1} = 2ru_{9,j} + 2(1 - r + hr)u_{10,j}$$

The right hand side is known and makes up the first entry in the $\vec{b}$ vector. The left hand side of the above relation determines the first two entries of the first row of each CM_p matrix for every point in t. The rest of the first row is all zeros. The image below shows the last CM_p matrix corresponding to when $t = 1$.

The $\vec{b}$ vector is built using the following set of equations, given the boundary conditions.

$$b_1 = 2(1 - r - hr)u_{1,j} - 2ru_{2,j}$$

$$b_{2:9} = u_{i+1,j} + u_{i-1,j}$$

$$b_{10} = 2ru_{9,j} + 2(1 - r + hr)u_{10,j}$$

Finally, for each point in t, we can obtain a solution for $u(x, t)$

$$u_{:,j} = CM_p^{-1}b$$

Part B

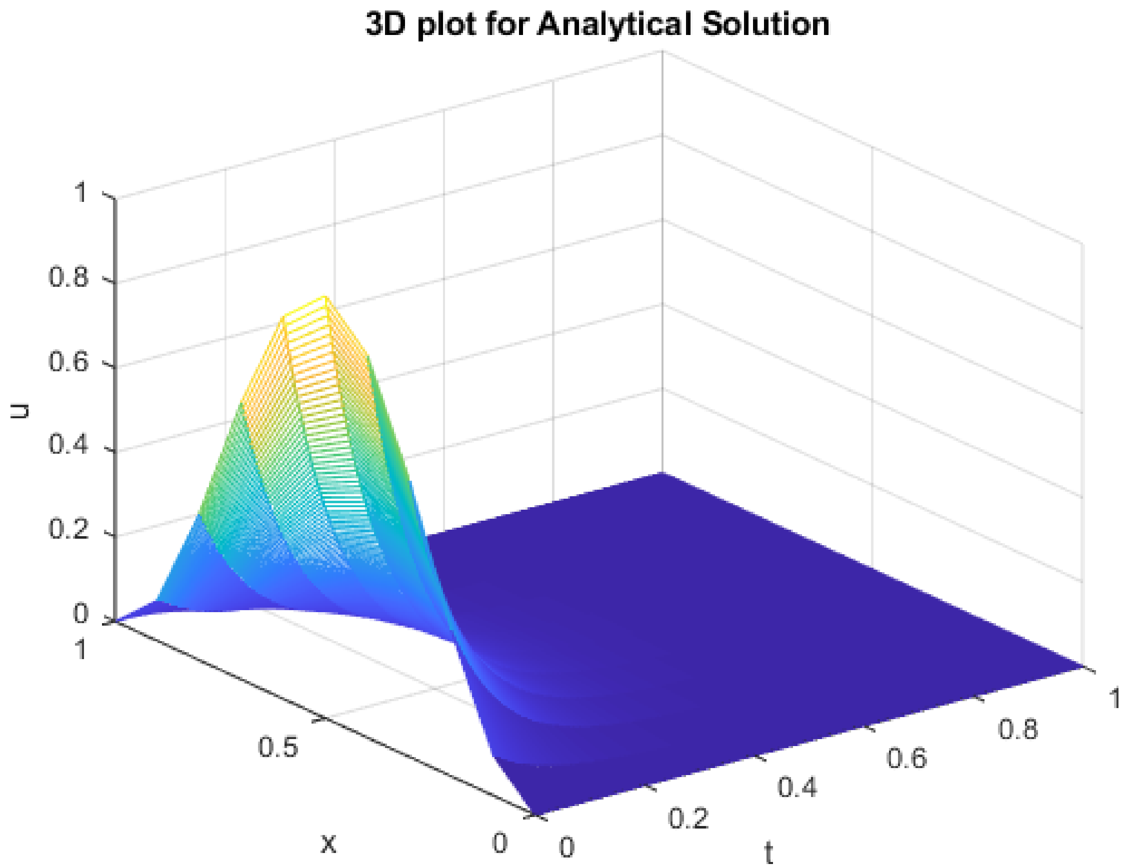


Figure 3: Problem 2 3D analytical results

For my numerical solution,

$$u(x = 0:9, t = 0.2) = \begin{bmatrix} 0.3174 \\ 0.3412 \\ 0.3607 \\ 0.3757 \\ 0.3860 \\ 0.3915 \\ 0.3920 \\ 0.3876 \\ 0.3784 \\ 0.3644 \end{bmatrix}$$

Part C

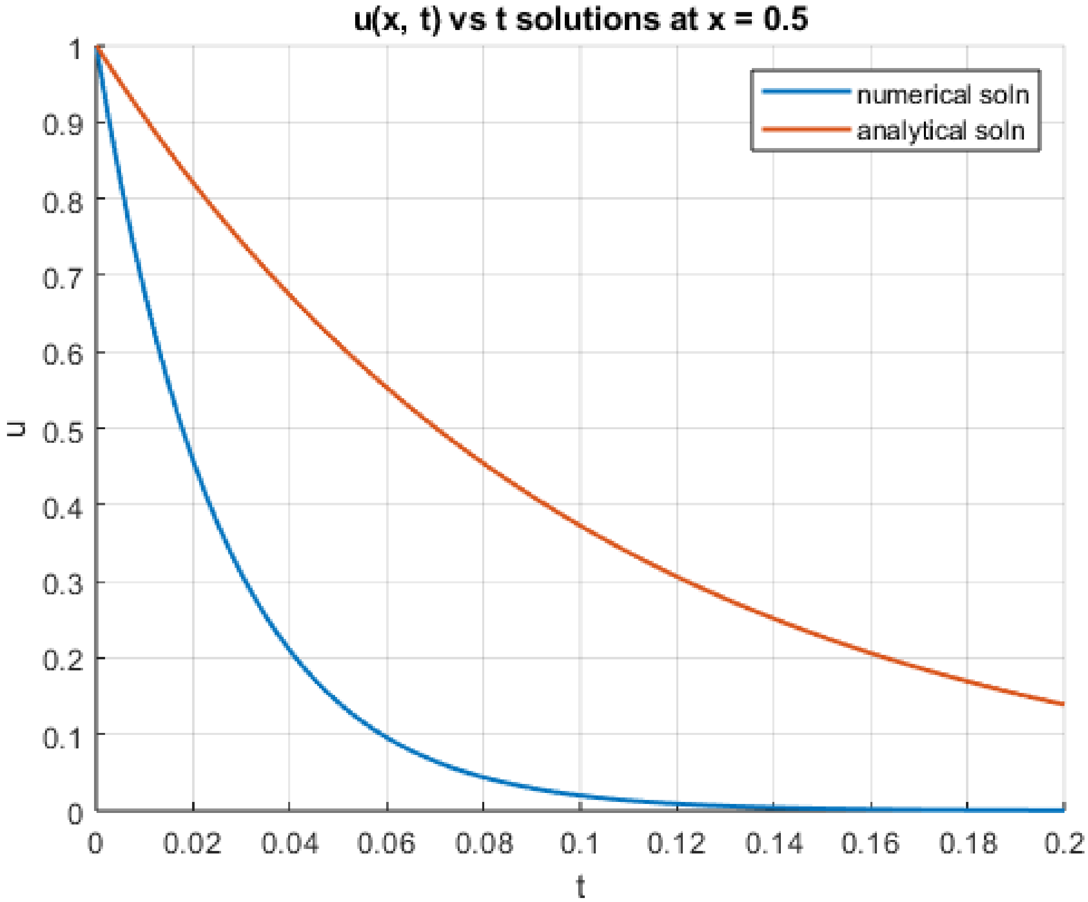


Figure 4: Problem 2 solution comparisons

For the exact solution,

$$u(x = 0 : 9, t = 0.2) \neq [$$

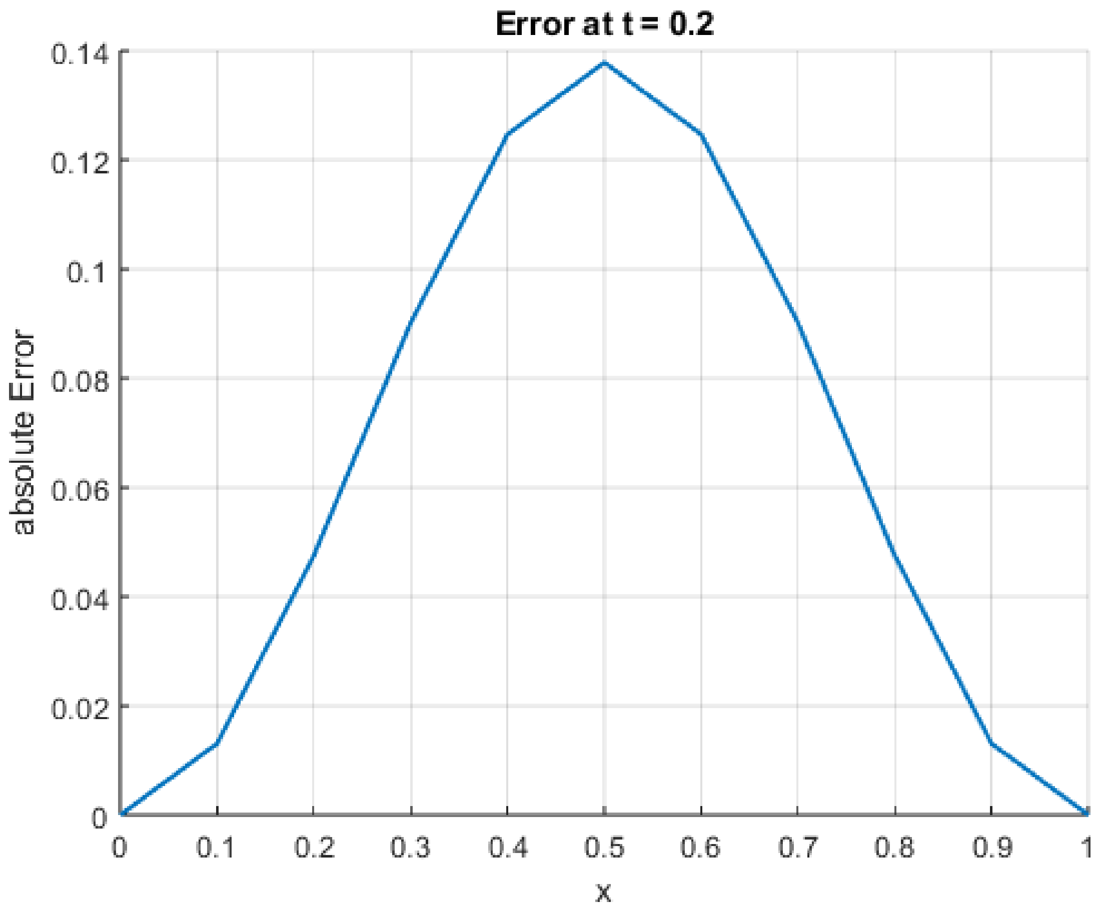
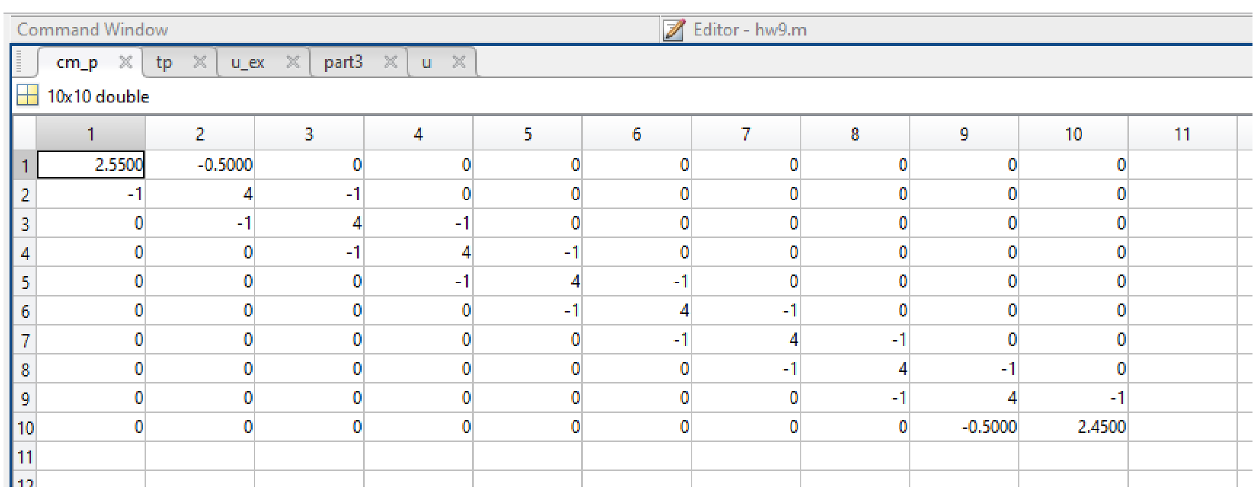


Figure 5: Problem 2 error


Command Window | Editor - hw9.m

cm_p | tp | u_ex | part3 | u

10x10 double

	1	2	3	4	5	6	7	8	9	10	11
1	2.5500	-0.5000	0	0	0	0	0	0	0	0	
2	-1	4	-1	0	0	0	0	0	0	0	
3	0	-1	4	-1	0	0	0	0	0	0	
4	0	0	-1	4	-1	0	0	0	0	0	
5	0	0	0	-1	4	-1	0	0	0	0	
6	0	0	0	0	-1	4	-1	0	0	0	
7	0	0	0	0	0	-1	4	-1	0	0	
8	0	0	0	0	0	0	-1	4	-1	0	
9	0	0	0	0	0	0	0	-1	4	-1	
10	0	0	0	0	0	0	0	0	-0.5000	2.4500	
11											

Figure 6: CM_p matrix when $t = 1$

Problem 1 Appendix

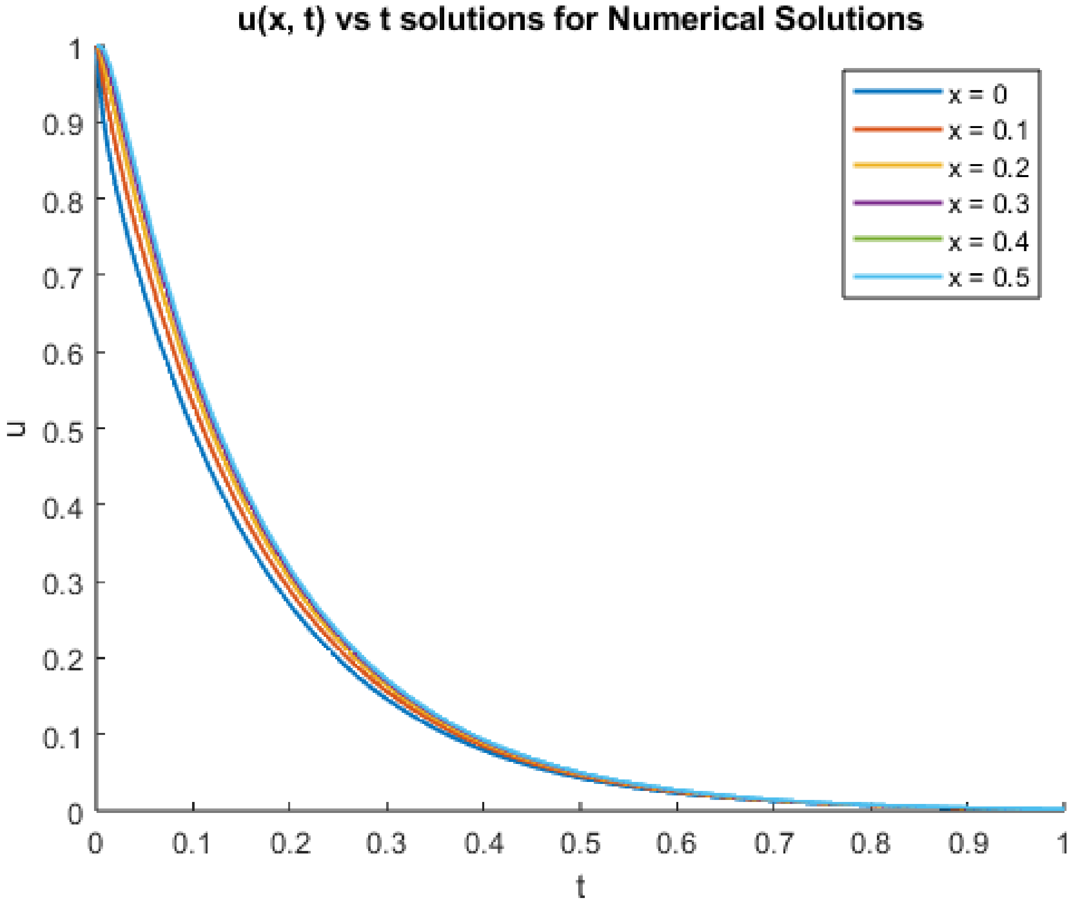


Figure 7: 3b

SOLUTION

Numerical Results and comparison to Analytical Solution for Problem 1

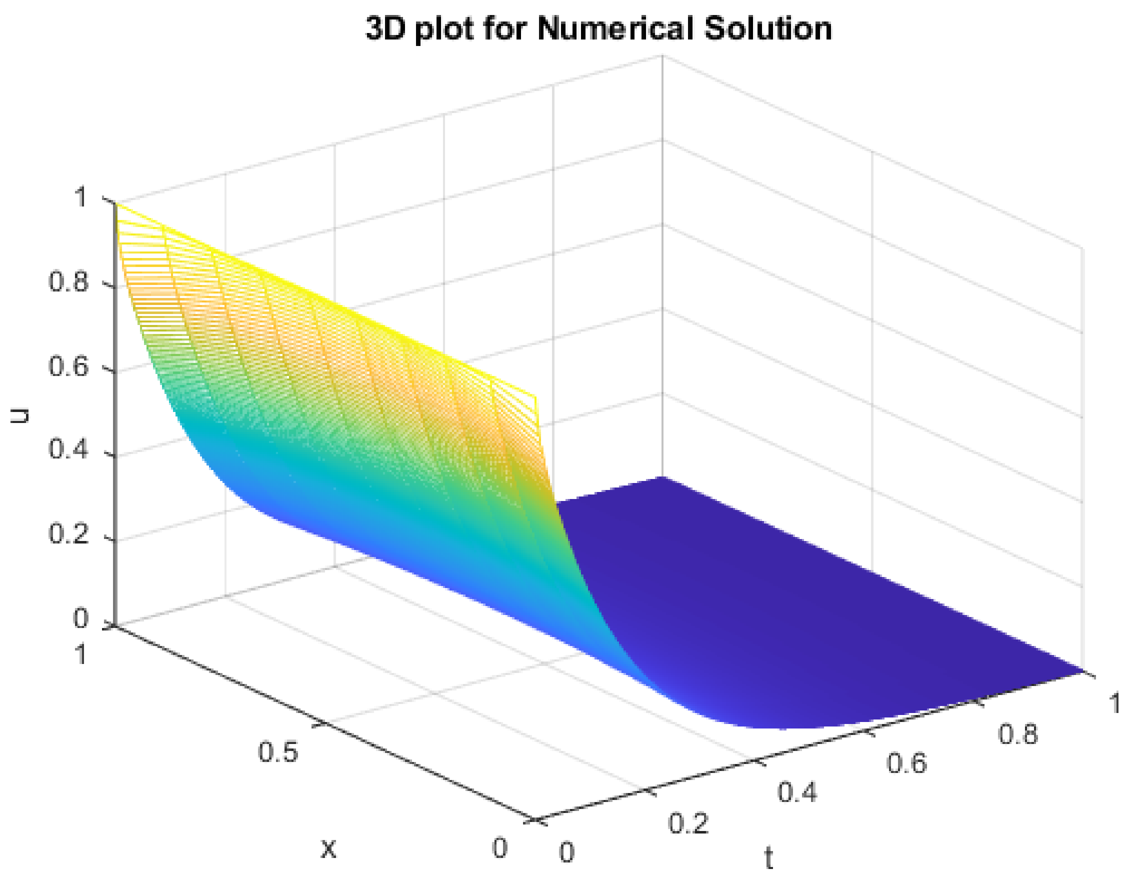

3D plot for Numerical Solution
u
1
0.8
0.6
0.4
0.2
0
x
1
0.5
0
t
0
0.2
0.4
0.6
0.8
1

Figure 8: 3b

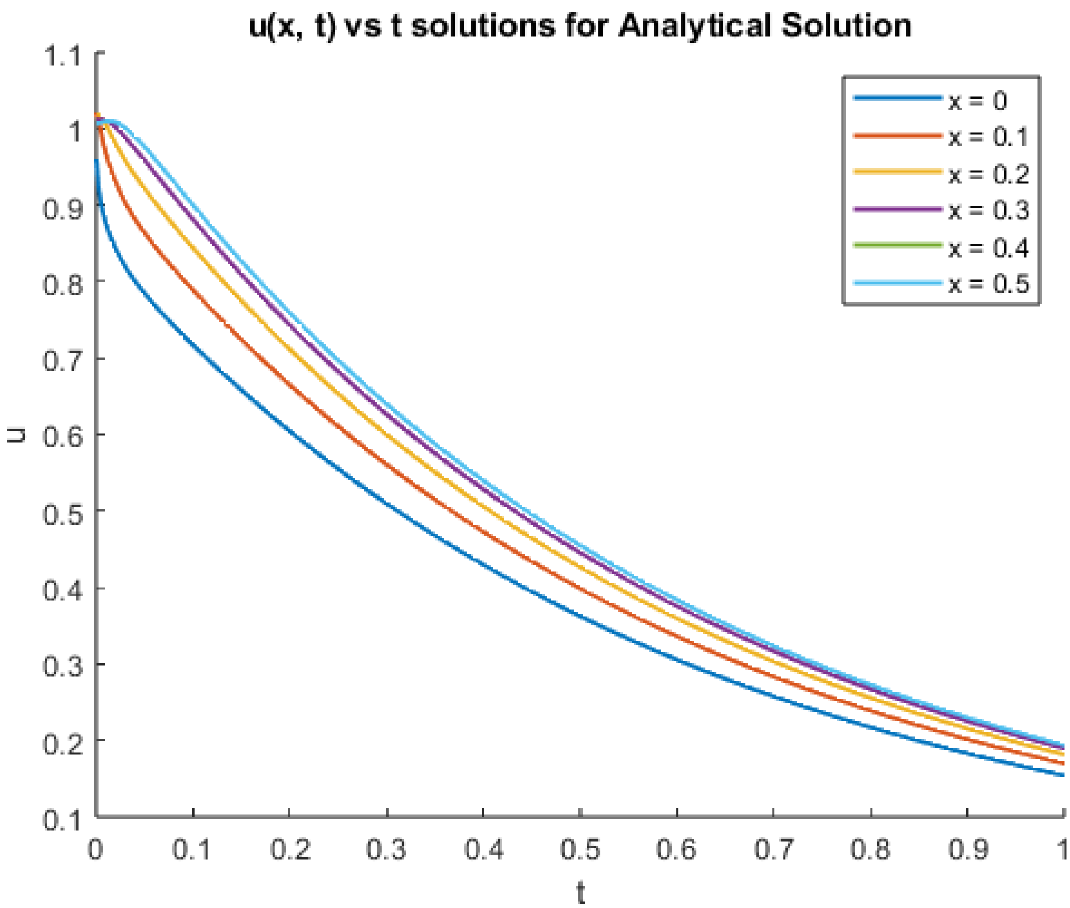

u(x, t) vs t solutions for Analytical Solution
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
u
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
x = 0
x = 0.1
x = 0.2
x = 0.3
x = 0.4
x = 0.5

Figure 9: 3c

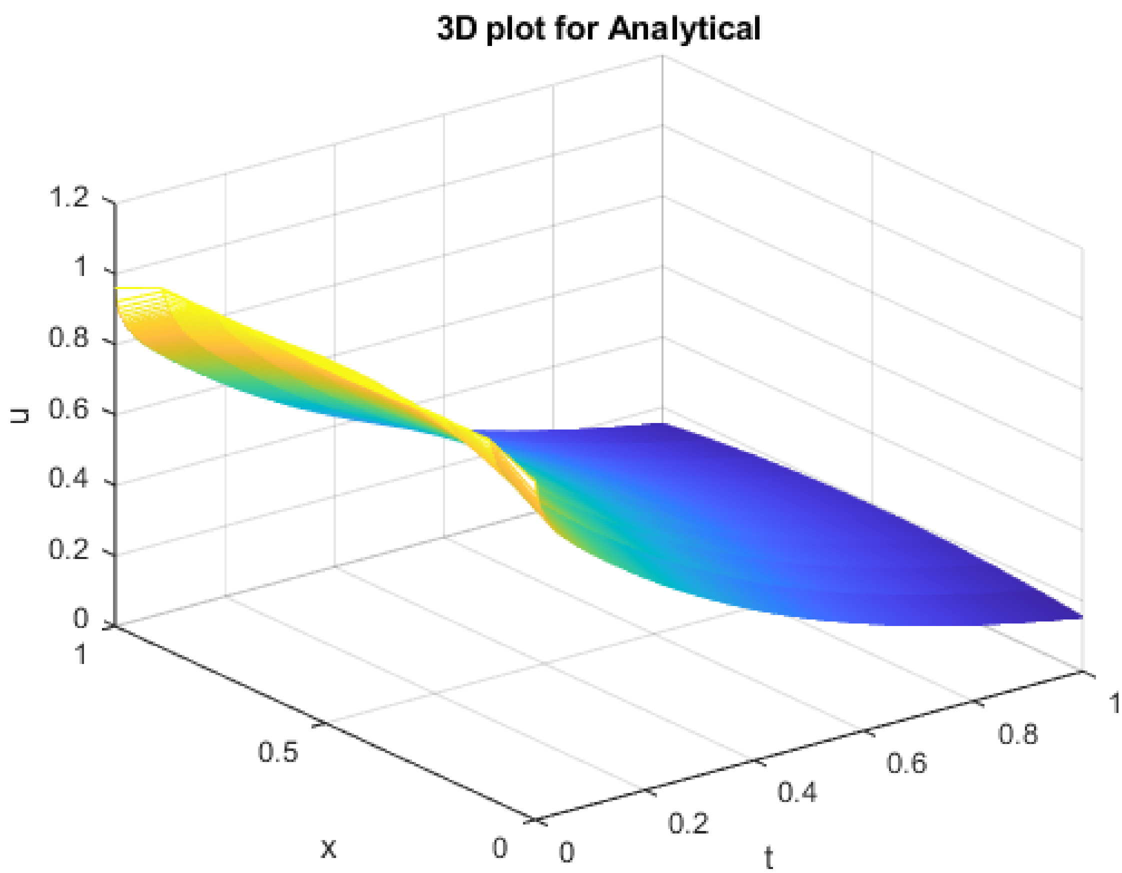


Figure 10: 3c

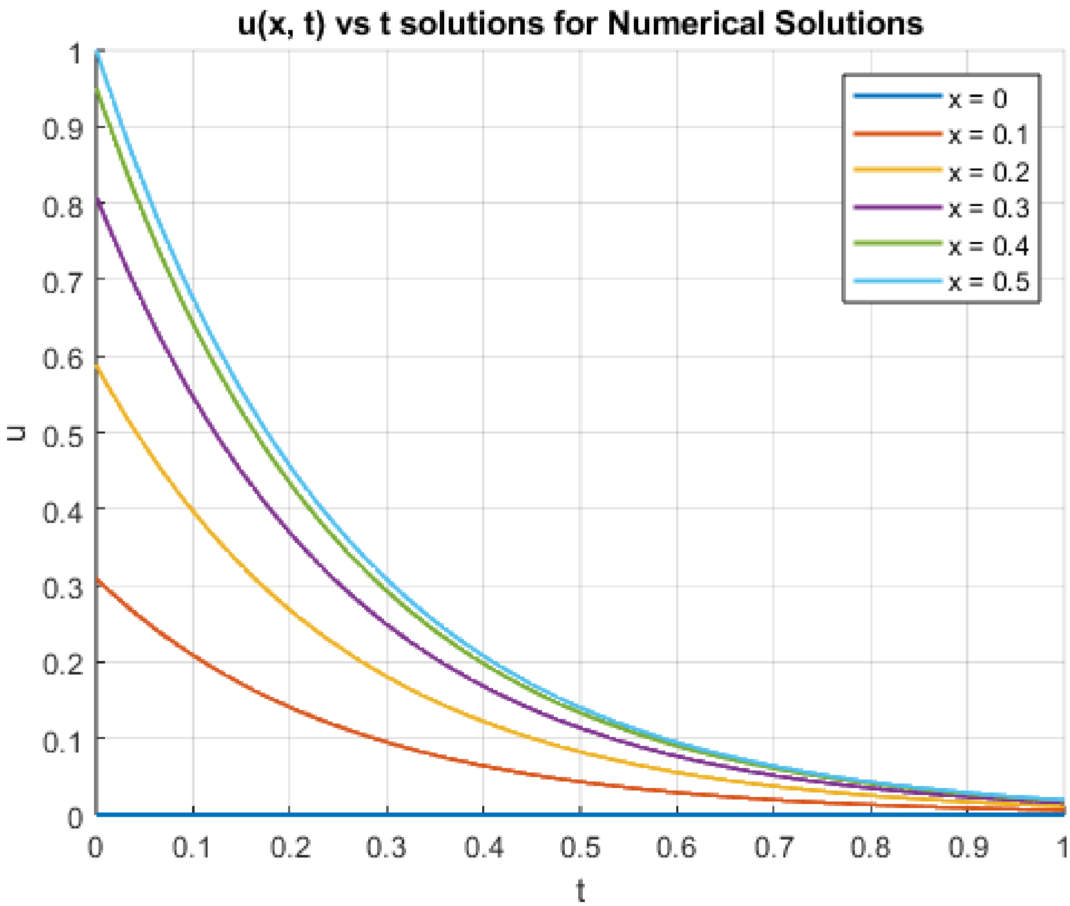


Figure 11: Problem 1 2D numerical results

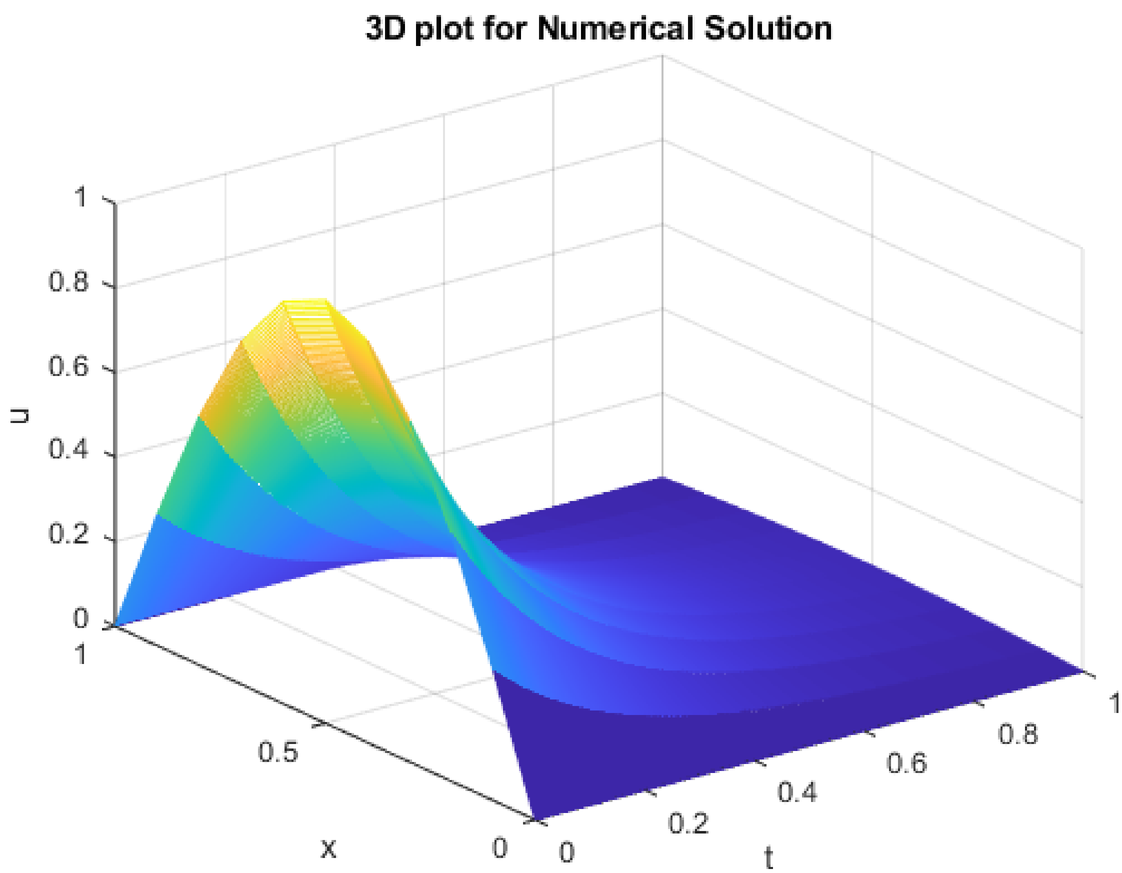


Figure 12: Problem 1 3D numerical results

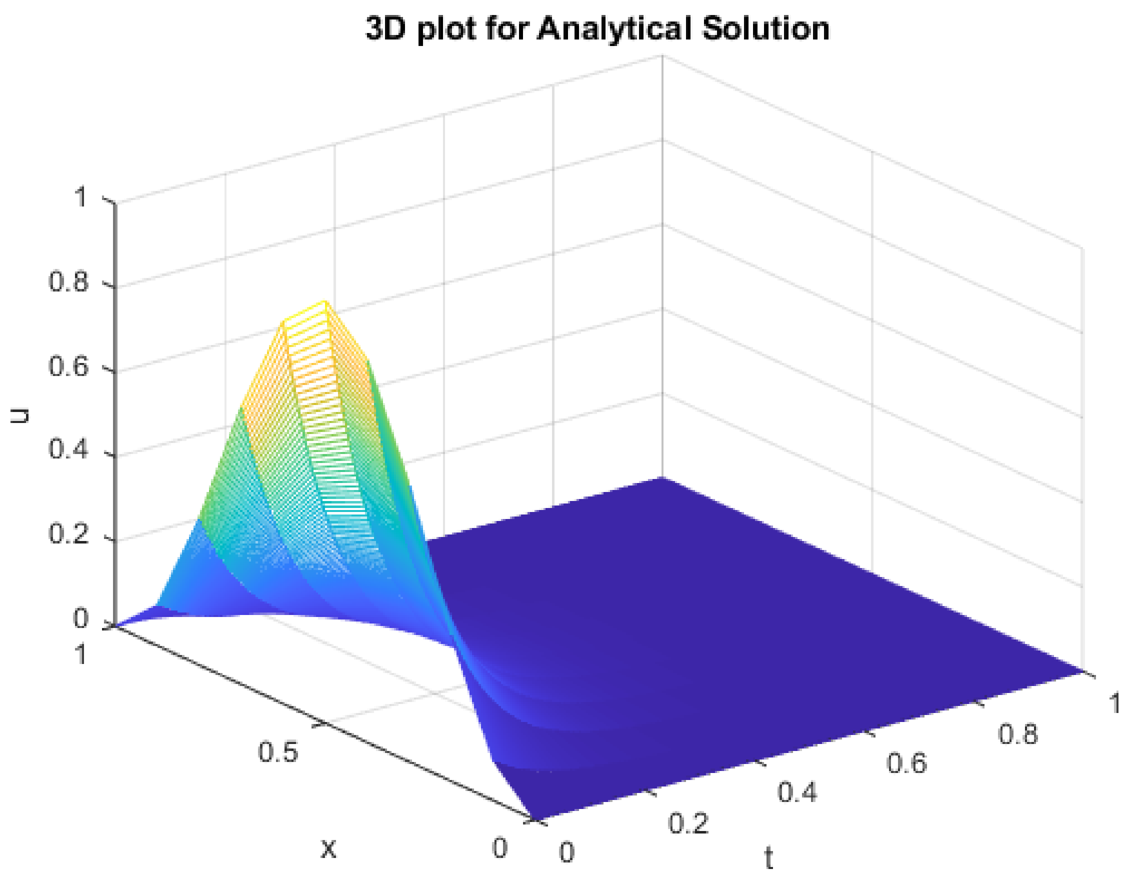


Figure 13: Problem 1 3D analytical results

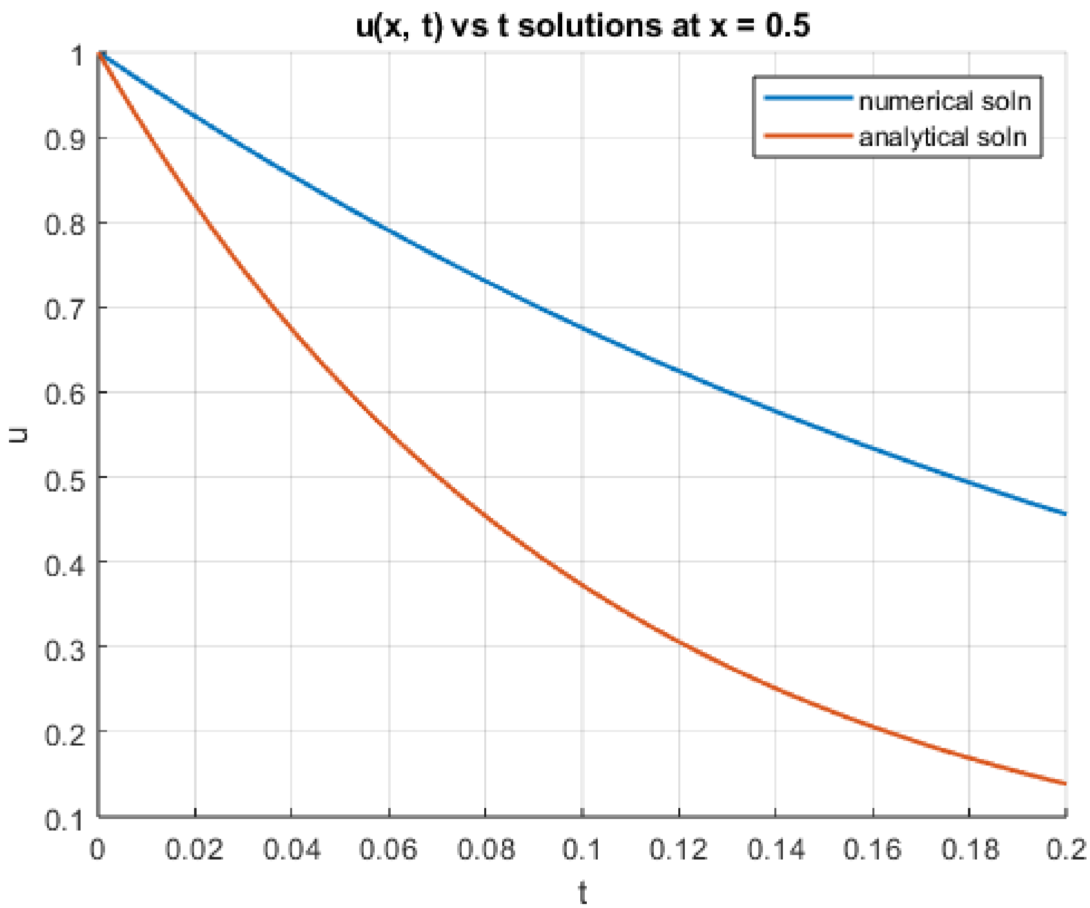


Figure 14: Problem 1 solution comparisons

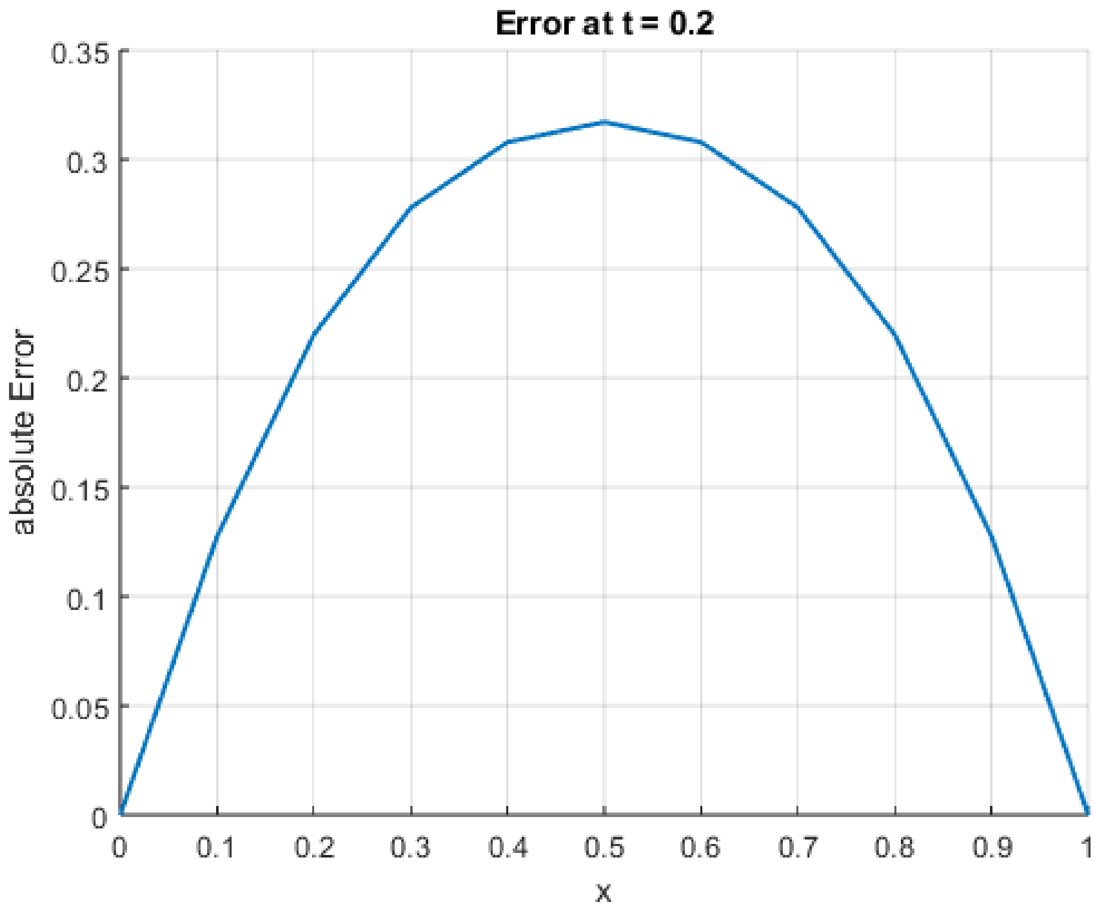


Figure 15: Problem 1 error