Problem 1

SOLUTION
Given,

$$A = \begin{bmatrix} 120 & 15 & 0 \\ 15 & 35 & -20 \\ 0 & -20 & -55 \end{bmatrix}$$

From MATLAB, my analytical answers should match the computational answers obtained below. Note that the solutions obtained by MATLAB will show a prime in their definition to distinguish between analytical and computational solutions.

$$\begin{bmatrix} \lambda_1' = -59.30 \\ \lambda_2' = 36.67 \\ \lambda_3' = 122.65 \end{bmatrix}$$

Accordingly,

$$\vec{l_1}' = \begin{bmatrix} -0.0176 \\ 0.2101 \\ 0.9775 \end{bmatrix}$$

$$\vec{l_2}' = \begin{bmatrix} 0.1732 \\ -0.9623 \\ 0.2100 \end{bmatrix}$$

$$\vec{l_3}' = \begin{bmatrix} -0.9847 \\ -0.1730 \\ 0.0195 \end{bmatrix}$$

Analytically, eigen values come from

$$det(A - \lambda I) = 0$$

$$(A - \lambda I) = \begin{bmatrix} 120-\lambda & 15 & 0 \\ 15 & 35-\lambda & -20 \\ 0 & -20 & -55-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = (120-\lambda) * ((35-\lambda)(-55-\lambda) - (-20)(-20)) - (15) * ((15)(-55-\lambda) - 0) + 0 = 0$$

$$... = (120-\lambda) * (-1925 + 20\lambda + \lambda^2 - 400) - (15) * (-825 - 15\lambda)$$

$$... = \lambda^3 - 100\lambda^2 - 490\lambda + 266625 = 0$$

Plotting this characteristic equation on my graphing calculator and inspecting the graph for roots yields the following result, which is consistent with the solution I got in MATLAB. From inspection of this plot, I can see that the eigen values are as follows:

$$\begin{bmatrix} \lambda_1 = -59.30 \\ \lambda_2 = 36.67 \\ \lambda_3 = 122.64 \end{bmatrix}$$

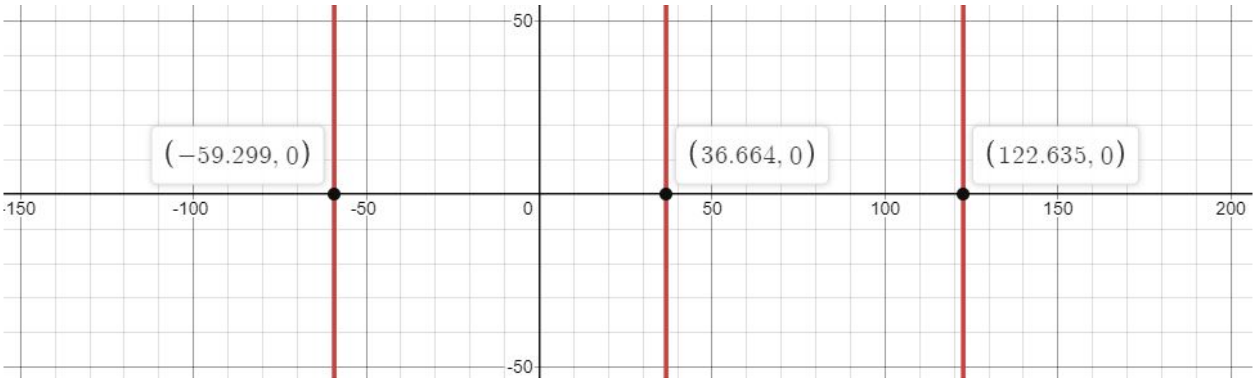

Figure 1: Problem 1

Now that I got the eigen values, I want to get the eigen vectors.

$$(A - \lambda_i I)\vec{l_i} = \vec{0}$$

For $\vec{l_1}$, from row 1,

$$(120 - \lambda_1)l_{1a} + 15l_{1b} = 0 \rightarrow l_{1b} = \frac{(\lambda_1 - 120)l_{1a}}{15}$$

From row 3,

$$-20l_{1b} + (-55 - \lambda_1)l_{1c} = 0 \rightarrow l_{1c} = \frac{20l_{1b}}{(-55 - \lambda_1)} = \frac{20}{-55 - \lambda_1} * \frac{(\lambda_1 - 120)l_{1a}}{15}$$

Now, if I let $l_{1a} = 0$, then I get the following result for $\vec{l_1}$

$$\vec{l_1} = \begin{bmatrix} 1 \\ -11.9533 \\ -55.6091 \end{bmatrix}$$

Note

$$|\vec{l_1}| = \sqrt{1^2 + -11.9533^2 + -55.6091^2} = 56.89$$

I can check this solution by using the second row of the A matrix and plugging in the $\vec{l_1}$ solution I derived by hand. Plugging $\vec{l_1}$ into row 2 of A and checking to see if it equals zero, I get:

$$15l_{1a} + (35 - \lambda_1)l_{1b} - 20l_{1c} = 15*(1) + (35 - (-59.3))*(-11.9533) - 20*(-55.6091) = 15 - 1127.2 + 1112.2 = 0.0\checkmark$$

Normalizing $\vec{l_1}$, I get

$$\vec{l_1} = \begin{bmatrix} 0.01758 \\ -0.2101 \\ -0.9775 \end{bmatrix}$$

Now, for $\vec{l_2}$, I can reuse the equations derived above and let $l_{2a} = 1$.

$$l_{2b} = \frac{(\lambda_2 - 120)l_{2a}}{15} = -5.56$$

$$l_{2c} = \frac{20}{-55 - \lambda_2} * \frac{(\lambda_2 - 120)l_{2a}}{15} = 1.21$$

$$15l_{2a} + (35 - \lambda_2)l_{2b} - 20l_{2c} = 15 * (1) + (35 - (36.67)) * (-5.56) - 20 * (1.21) = 15 + 9.3 - 24.3 = 0.0\checkmark$$

$$\vec{l_2} = \begin{bmatrix} 1 \\ -5.56 \\ 1.21 \end{bmatrix}$$

$$|\vec{l_2}| = \sqrt{1^2 + -5.56^2 + 1.21^2} = 5.78$$

$$\frac{\vec{l_2}}{|\vec{l_2}|} = \begin{bmatrix} 0.1731 \\ -0.9624 \\ 0.2094 \end{bmatrix}$$

Finally for $\vec{l_3}$, I will do the same process as I did for $\vec{l_2}$ above. First let $l_{3a} = 1$:

$$l_{3b} = \frac{(\lambda_3 - 120)l_{3a}}{15} = 0.1760$$

$$l_{3c} = \frac{20}{-55 - \lambda_3} * \frac{(\lambda_3 - 120)l_{3a}}{15} = -0.0198$$

$$15l_{3a} + (35 - \lambda_3)l_{2b} - 20l_{3c} = 15 * (1) + (35 - (122.64)) * (0.1760) - 20 * (-0.0198) = 0.0\checkmark$$

$$|\vec{l_3}| = \sqrt{(1)^2 + (0.18)^2 + (-0.020)^2}$$

Collecting the normalized eigen vector solutions I obtained by hand,

$$\vec{l_1} = \begin{bmatrix} 0.01758 \\ -0.2101 \\ -0.9775 \end{bmatrix}$$

$$\frac{\vec{l_2}}{|\vec{l_2}|} = \begin{bmatrix} 0.1731 \\ -0.9624 \\ 0.2094 \end{bmatrix}$$

$$\vec{l_3} = \begin{bmatrix} 0.9840 \\ 0.1771 \\ -0.01968 \end{bmatrix}$$

From inspection, these solutions match what I got from MATLAB only to an extent because of rounding errors.

Problem 2

SOLUTION

PART A

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 6 & 2 \\ 9 & 9 & 1 \end{bmatrix}$$

$$det(A) = 1(6 - 18) - 3(7 - 18) + 5(63 - 54) = -12 - 33 + 45 = 0$$

The determinant of this matrix is zero, so the rows/columns are linearly dependent, the rank is less than 3 and the system has a trivial solution.

PART B

$$B = \begin{bmatrix} 1 & 5 & 5 \\ 2 & 2 & 1 \\ 1 & 5 & 1 \end{bmatrix}$$

$$det(B) = 1(2-5) - 5(2-1) + 5(10-2) = -3 - 5 + 40 = 32$$

The determinant of matrix B is non-zero, so the rank of the matrix is 3, the rows/columns are linearly independent and it's solution is non-trivial.

Problem 3

SOLUTION

PART A

$$\frac{dx_i}{dt} = \frac{d}{dt}[SaltEnteringTank] - \frac{d}{dt}[SaltLeavingTank]$$

$$\frac{dx_1}{dt} = (2*0) + (1*\frac{x_2}{100}) - (1*\frac{x_1}{100}) - (2*\frac{x_1}{100})$$

$$\frac{dx_1}{dt} = \frac{-3x_1}{100} + \frac{x_2}{100}$$

$$\frac{dx_2}{dt} = (2*\frac{x_1}{100}) - (1*\frac{x_2}{100}) - (1*\frac{x_2}{100})$$

$$\frac{dx_2}{dt} = \frac{2x_1}{100} - \frac{2x_2}{100}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-3}{100} & \frac{1}{100} \\ \frac{2}{100} & \frac{-2}{100} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

PART B

$$|A - \lambda I| = (\frac{-3}{100} - \lambda)(\frac{-1}{50} - \lambda) - (\frac{1}{50})(\frac{1}{100}) = 0$$

$$... = \frac{3}{5000} + \frac{3}{100}\lambda + \frac{1}{50}\lambda + \lambda^2 - \frac{1}{5000} = 0$$

$$\lambda^2 + \frac{5}{100}\lambda + \frac{1}{2500} = 0$$

$$(\lambda + \frac{1}{25})(\lambda + \frac{1}{100}) = 0$$

$$\lambda = \frac{-1}{25}, \frac{-1}{100}$$

Now for eigen vectors, $(A - \lambda I)\vec{l} = \vec{0}$. For $\lambda_1 = \frac{-1}{25}$

$$\begin{bmatrix} \frac{-3}{100} + \frac{1}{25} & \frac{1}{100} \\ \frac{1}{50} & \frac{-1}{50} + \frac{1}{25} \end{bmatrix} \vec{l_1} = \vec{0}$$

$$\frac{1}{100}l_{1a} + \frac{1}{100}l_{1b} = 0$$

$$l_{1a} + l_{1b} = 0 \rightarrow l_{1a} = -l_{1b}$$

$$\vec{l_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now for λ_2, $(A - \lambda_2 I)\vec{l_2} = \vec{0}$.

$$\begin{bmatrix} \frac{-3}{100} + \frac{1}{100} & \frac{1}{100} \\ \frac{1}{50} & \frac{-1}{50} + \frac{1}{100} \end{bmatrix} \vec{l_2} = \vec{0}$$

$$\frac{-2}{100}l_{2a} + \frac{1}{100}l_{2b} = 0 \rightarrow l_{2b} = 2l_{2a}$$

$$\vec{l_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vec{x}(t) = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{\frac{-t}{25}} + C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{\frac{-t}{100}}$$

Letting $t = 0$ and inserting IC's

$$\begin{bmatrix} 20 \\ 5 \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{0} + C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{0}$$

$$\begin{bmatrix} 20 \\ 5 \end{bmatrix} = \begin{bmatrix} -C_1 + C_2 \\ C_1 + 2C_2 \end{bmatrix}$$

$$C_2 + 2C_2 = 20 + 5 \rightarrow C_2 = \frac{25}{3} \rightarrow C_1 = \frac{-35}{3}$$

$$\vec{x}(t) = \frac{-35}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{\frac{-t}{25}} + \frac{25}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{\frac{-t}{100}}$$

PART C

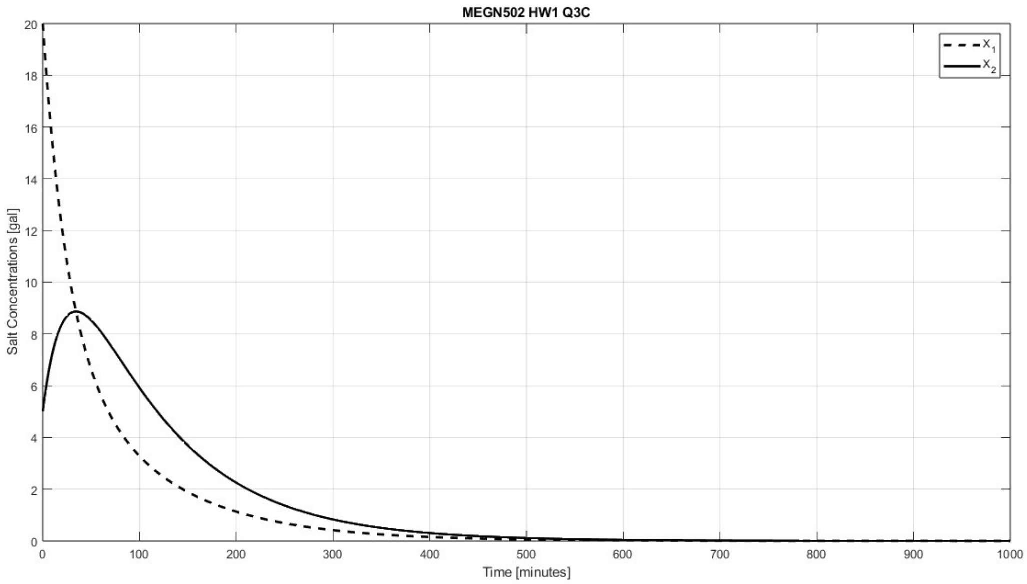

Figure 2: Problem 3C

PART D

From inspection of the figure above, you can see that both $x_1(t)$ and $x_2(t)$ converge to zero as time gets large. The steady state solution is as follows:

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$$

Problem 4

SOLUTION
From the given information, we can say that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -a & 0 & 0 \\ a & -b & 0 \\ 0 & b & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Note

$$A = \begin{bmatrix} -a & 0 & 0 \\ a & -b & 0 \\ 0 & b & 0 \end{bmatrix}$$

- Solve for eigen values

$$det(A - \lambda I) = 0 = (-a-\lambda)((-b-\lambda)(-\lambda) - 0) + 0 + 0$$

$$\lambda^3 + (a+b)\lambda^2 + ab\lambda = \lambda[\lambda^2 + (a+b)\lambda + ab] = \lambda(\lambda+a)(\lambda+b) = 0$$

$$\begin{bmatrix} \lambda_1 = 0 \\ \lambda_2 = -a \\ \lambda_3 = -b \end{bmatrix}$$

- Solve for eigen vectors

$$\lambda_1 = 0 \rightarrow (A - \lambda_1 I)\vec{l_1} = A\vec{l_1} = \vec{0}$$

$$-al_{1a} = 0 \rightarrow l_{1a} = 0$$

$$al_{1a} - bl_{1b} = 0 - bl_{1b} = 0 \rightarrow l_{1b} = 0$$

It must be that $l_{1c} = 1$ so that $\vec{l_1}$ is non-trivial. Let $l_{1c} = 1$

$$\vec{l_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -a \rightarrow (A - \lambda_2 I)\vec{l_2} = \vec{0}$$

$$A - \lambda_2 I = \begin{bmatrix} 0 & 0 & 0 \\ a & -b+a & 0 \\ 0 & b & a \end{bmatrix}$$

$$bl_{2b} + al_{2c} = 0 \rightarrow l_{2c} = \frac{-b}{a} l_{2b}$$

$$bl_{2a} + (-b+a)l_{2b} = 0 \rightarrow l_{2a} = \frac{b-a}{b} l_{2b}$$

Let $l_{2b} = 1$

$$\vec{l_2} = \begin{bmatrix} \frac{b-a}{b} l_{2b} \\ l_{2b} \\ \frac{-b}{a} l_{2b} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\lambda_3 = -b \rightarrow (A - \lambda_3 I)\vec{l_3} = \vec{0}$$

$$A - \lambda_3 I = \begin{bmatrix} -a+b & 0 & 0 \\ a & 0 & 0 \\ 0 & b & b \end{bmatrix}$$

$$bl_{3b} + bl_{3c} = 0 \rightarrow l_{3b} = -l_{3c}$$

From inspection, $l_{3a} = 0$. Let $l_{3b} = 1$.

$$\vec{l_3} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{X}(t) = c_1 \vec{l_1} e^{\lambda_1 t} + c_2 \vec{l_2} e^{\lambda_2 t} + c_3 \vec{l_3} e^{\lambda_3 t}$$

$$\vec{X}(t) = \begin{bmatrix} c_2 e^{-at} \\ c_2 e^{-at} + c_3 e^{-bt} \\ c_1 - 2c_2 e^{-at} - c_3 e^{-bt} \end{bmatrix}$$

$$\vec{X}(0) = \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}c_2e^{-at}\\c_2e^{-at}+c_3e^{-bt}\\c_1-2c_2e^{-at}-c_3e^{-bt}\end{bmatrix} \rightarrow \vec{c} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$$

$$\vec{X}(t) = \begin{bmatrix}e^{-at}\\e^{-at}-e^{-bt}\\1-2e^{-at}+e^{-bt}\end{bmatrix}$$

Below is the specified plot for problem 4, along with the steady state solutions for all 3 variables of the system.

$$\lim_{t\to\infty} x(t) = 0 \rightarrow BLUE$$

$$\lim_{t\to\infty} y(t) = 0 \rightarrow RED$$

$$\lim_{t\to\infty} z(t) = 1 \rightarrow YELLOW$$

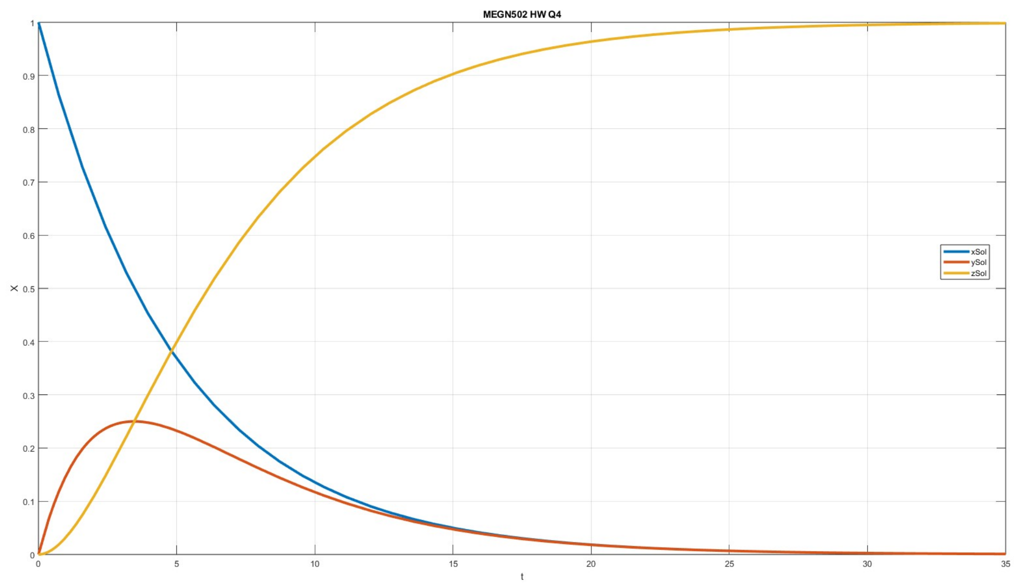

Figure 3: Problem 4