Problem 1

SOLUTION

0) Exact Solution

$$x^2\frac{d^2u}{dx^2} + 3x\frac{du}{dx} - u = 0$$

$$x = e^z$$

Substituting in,

$$(\frac{-du}{dz} + \frac{d^2u}{dz^2}) + 3(\frac{du}{dz}) - u = 0$$

$$\frac{d^2u}{dz^2} + 2\frac{du}{dz} - u = 0$$

$$u = e^{kz} \rightarrow k^2 + 2k - 1 = 0$$

$$k = 1 \pm \sqrt{2}$$

$$u(z) = c_1 e^{(-1+\sqrt{2})z} + c_2 e^{(-1-\sqrt{2})z}$$

$$u(x) = c_1 x^{(-1+\sqrt{2})} + c_2 x^{(-1-\sqrt{2})}$$

$$u(1) = c_1 + c_2 = 0$$

$$u(2) = c_1 2^{(-1+\sqrt{2})} + (-c_1)2^{(-1-\sqrt{2})} = 1$$

$$c_1(2^{(-1+\sqrt{2})} - 2^{(-1-\sqrt{2})}) = 1 \rightarrow c_1 = \frac{1}{2^{(-1+\sqrt{2})} - 2^{(-1-\sqrt{2})}} = 0.8734$$

$$c_2 = -0.8734$$

$$u(x) = (0.8734)x^{(-1+\sqrt{2})} + (-0.8734)x^{(-1-\sqrt{2})}$$

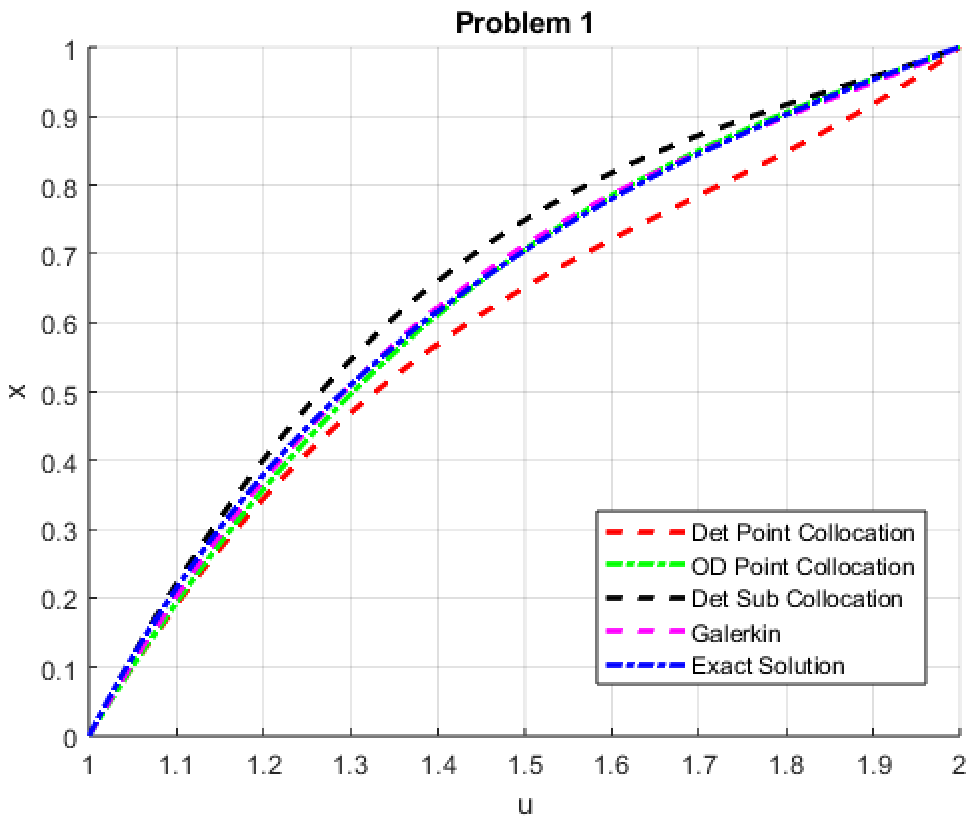

Figure 1: Problem 1 Solution(s)

A) Deterministic Point Collocation

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -5.2160 \\ 9.3283 \\ -5.1145 \\ 1.0022 \end{bmatrix}$$

Deterministic point collocation is an underestimation for this problem, and one of the less accurate possible solutions. Which makes sense, because this method only uses 2 weighted residuals.

B) Overdetermined Point Collocation

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -4.2873 \\ 7.0210 \\ -3.2787 \\ 0.5450 \end{bmatrix}$$

Overdetermined Point Collocation is one of the most accurate methods, matching the exact solution very well. That makes sense as well, because this problem makes use of 200 weighted residuals, so it's obviously going to be much more accurate than any other collocation method that only uses two.

C) Deterministic Subdomain Collocation

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -5.6291 \\ 9.7003 \\ -4.9496 \\ 0.8783 \end{bmatrix}$$

The deterministic subdomain collocation method proved to be an overestimation for this problem, and is less accurate than the overdetermined point collocation and Galerkin methods.

D) Galerkin

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -4.9226 \\ 9.3661 \\ -4.1845 \\ 0.7411 \end{bmatrix}$$

Galerkin proved to be one of the most accurate solutions, along with the overdetermined point collocation method.

Problem 2

SOLUTION
See Problem 1 for exact solution derivation.

$$u(x) = 0.8734x^{-1+\sqrt{2}} - 0.8734x^{-1-\sqrt{2}}$$

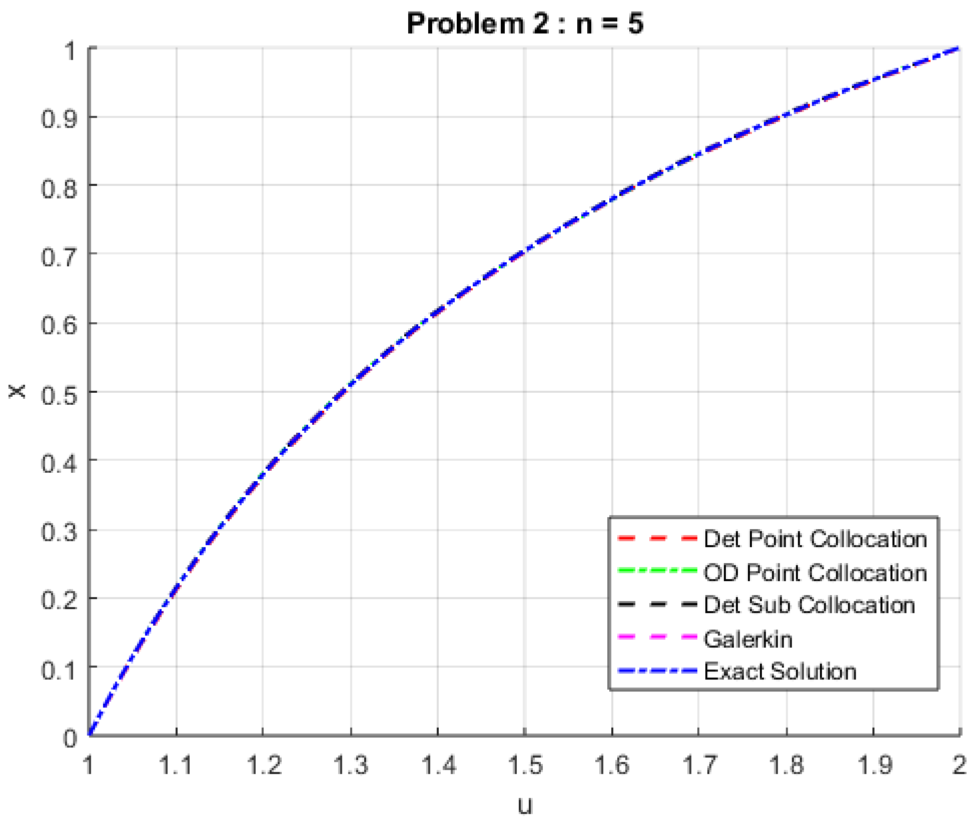

Figure 2: Problem 2 Solution(s)

A) Deterministic Point Collocation

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -11.4876 \\ 29.5314 \\ -30.6885 \\ 16.8662 \\ -4.7710 \\ 0.4595 \end{bmatrix}$$

Increasing n from 3 to 5 helped the Deterministic Point Collocation solution out a lot. It matches the exact solution very well and far improved from question 1.

B) Overdetermined Point Collocation

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -12.0553 \\ 31.0434 \\ -32.1265 \\ 17.4230 \\ -4.8249 \\ 0.5403 \end{bmatrix}$$

The Overdetermined Point Collocation solution worked well with $n = 3$, so it didn't really have much room for improvement. It still matches the exact solution very well.

C) Deterministic Subdomain Collocation

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -12.6349 \\ 33.1344 \\ -35.0972 \\ 19.4982 \\ -5.5364 \\ 0.6360 \end{bmatrix}$$

The Deterministic Subdomain Collocation solution improved a lot from question 1, and now matches the exact solution very well.

D) Galerkin

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -12.1525 \\ 31.5764 \\ -33.1202 \\ 18.2623 \\ -5.1560 \\ 0.5899 \end{bmatrix}$$

The Galerkin solution matched the exact solution very well in question 1, so it also did not have much room for improvement. It still matches the exact solution very well when $n = 5$ now.

Problem 3

SOLUTION

$$T(x) = \Sigma_{j=0}^{6} a_j x^j$$

From BC's

$$a_0 = T_f$$

$$a_1 = -(2a_2L + 3a_3L^2 + 4a_4L^3 + 5a_5L^4 6a_6L^5)$$

When $L = 1$,

$$T(x) = T_f + -(2a_2L + 3a_3L^2 + 4a_4L^3 + 5a_5L^4 6a_6L^5)x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$$

$$T_f + a_2(x^2 - 2x) + a_3(x^3 - 3x) + a_4(x^4 - 4x) + a_5(x^5 - 5x) + a_6(x^6 - 6x)$$

$$\frac{dT}{dx} = -(2a_2L + 3a_3L^2 + 4a_4L^3 + 5a_5L^4 6a_6L^5) + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5$$

$$\frac{d^2T}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4$$

Sub into ODE

$$R_x = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 - ...$$

$$...\alpha((T_f + a_2(x^2 - 2x) + a_3(x^3 - 3x) + a_4(x^4 - 4x) + a_5(x^5 - 5x) + a_6(x^6 - 6x))^4 - T_\infty^4)$$

$$\phi_j(x) = (x^j - jx)$$

$$\int_0^L \phi_j(x)R(x)dx = \int_0^L (x^j - jx)R(x)d = 0$$

Which yields the following system

```
Command Window
>> double(A)

ans =

   -1.3333   -2.5000   -3.6000   -4.6667   -5.7143
   -2.5000   -4.8000   -7.0000   -9.1429  -11.2500
   -3.6000   -7.0000  -10.2857  -13.5000  -16.6667
   -4.6667   -9.1429  -13.5000  -17.7778  -22.0000
   -5.7143  -11.2500  -16.6667  -22.0000  -27.2727

>> double(b)

ans =

    0.1828
    0.3427
    0.4936
    0.6398
    0.7834
```

Figure 3: Problem 3 Linear System

And the following solution

$$\begin{bmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} -0.1371 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a_1 = -2a_2 * L = 0.2742 \rightarrow a_0 = T_f = 300$$

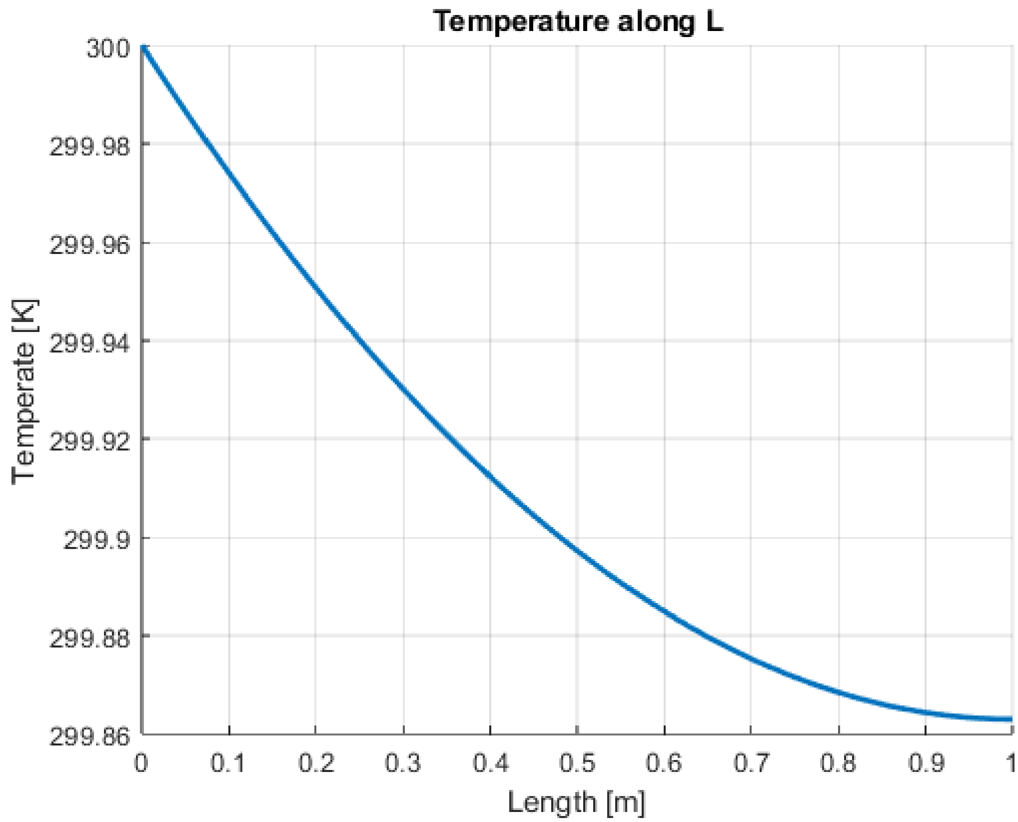

Figure 4: Problem 3 Solution

Problem 4

SOLUTION

$$u(x,y) = A_y x(1-x)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2A_y + A_y'' x(1-x)$$

$$\frac{d^2 A_y}{dy^2} - \frac{2}{x(1-x)} A_y = 0$$

$$A_y = c_1 e^{\sqrt{x^2-1}y} + c_2 e^{-\sqrt{x^2-1}y}$$

$$u(x,y) = (c_1 e^{\sqrt{x^2-1}y} + c_2 e^{-\sqrt{x^2-1}y}) x(1-x)$$

Plugging in BC's

$$u(x,\infty) = 0 \rightarrow c_1 e^{\sqrt{x^2-1}y} + c_2 e^{-\sqrt{x^2-1}y} \rightarrow c_1 = 0$$

$$u_j = (c_2 e^{-\sqrt{x^2-1}y}) x(1-x)$$

Using point collocation for $0 < y < \infty$ at $x = \frac{1}{2}$

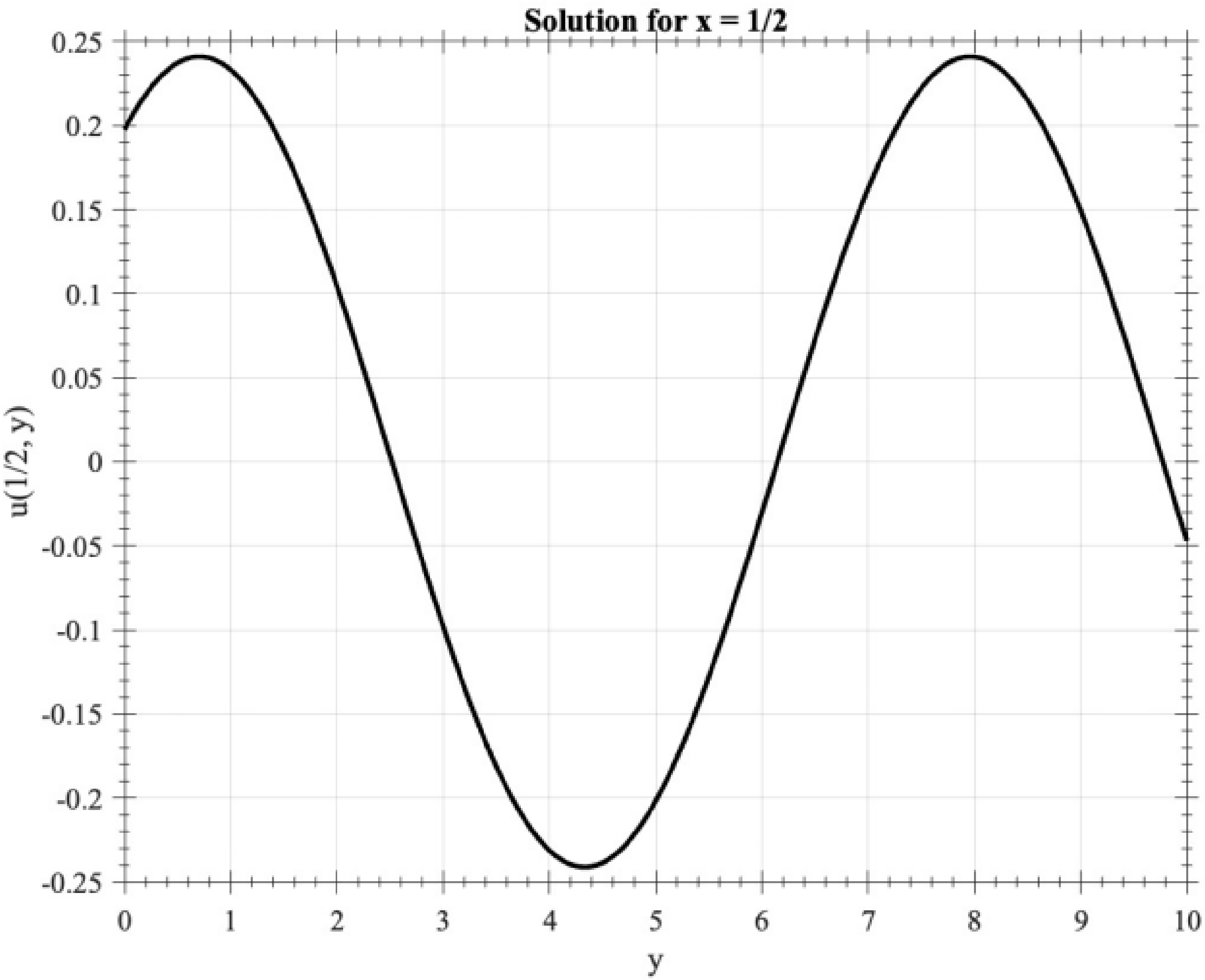

Figure 5: Problem 4