Problem 1

SOLUTION

Integral transform on the SPATIAL VARIABLE! Only have $f(0)$ for spatial variable $\rightarrow$ need Fourier Sine transform, $\mathscr{S}$

1) Range: match for x
2) Operational Properties on Derivatives (Need function at zero for Fourier Sine): $\rightarrow u(0,t) = 0$.

$$\mathscr{S}_x(c^2\frac{\partial^2 u}{\partial x^2}) = \mathscr{S}_x(\frac{\partial^2 u}{\partial t^2})$$

$$c^2\mathscr{S}_x(\frac{\partial^2 u}{\partial x^2}) = \mathscr{S}_x(\frac{\partial^2 u}{\partial t^2})$$

LHS:

$$\mathscr{S}_x(\frac{\partial^2 u}{\partial x^2}) = -\lambda^2 F(\lambda) + \lambda f(0) \rightarrow f(0) = 0$$

RHS:

$$\mathscr{S}_x(\frac{\partial^2 u}{\partial t^2}) = \frac{d^2\hat{u}(\lambda,t)}{dt^2}$$

Inserting back into PDE,

$$c^2(-\lambda^2\hat{u}(\lambda,t)) = \frac{d^2\hat{u}(\lambda,t)}{dt^2}$$

$$\frac{d^2\hat{u}(\lambda,t)}{dt^2} + c^2\lambda^2\hat{u}(\lambda,t) = 0 \rightarrow ODE$$

$$\hat{u}(\lambda,t) = A_\lambda e^{-c^2\lambda^2 t}$$

$$A_\lambda = \frac{-G(\lambda)}{c^2\lambda^2}$$

$$\hat{u}(\lambda,t) = \frac{-G(\lambda)}{c^2\lambda^2}e^{-c^2\lambda^2 t}$$

$$u(x,t) = \frac{-2}{\pi}\int_0^\infty \frac{G(\lambda)}{c^2\lambda^2}e^{-c^2\lambda^2 t}sin(\lambda t)d\lambda$$

Problem 2

SOLUTION

Noting that only IC's are given for t (no BC's) and both $u(x,0)$ and $\frac{\partial u}{\partial t}$ are known, I'm thinking about a Laplace transform.
1) Range: match for t only
2) OPoD: Need both $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$ for Laplace transform, and that's what is known.

$$\mathscr{L}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

$$\mathscr{L}^{-1}(F(s)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{st}ds$$

$$\mathscr{L}_t(\frac{\partial^2 u}{\partial t^2} - 2\beta\frac{\partial u}{\partial t} + \beta^2 u) = c^2\mathscr{L}_t(\frac{\partial^2 u}{\partial x^2})$$

$$\mathscr{L}_t(\frac{\partial^2 u}{\partial t^2}) - 2\beta\mathscr{L}_t(\frac{\partial u}{\partial t}) + \beta^2\mathscr{L}_t(u) = c^2\mathscr{L}_t(\frac{\partial^2 u}{\partial x^2})$$

$$\mathscr{L}_t(u) = \hat{u}(x,s)$$

$$\mathscr{L}_t(\frac{\partial u}{\partial t}) = s\hat{u}(x,s) - \hat{u}(x,0)$$

$$\mathscr{L}_t(\frac{\partial^2 u}{\partial t^2}) = s^2\hat{u}(x,s) - s\hat{u}(x,0) - \frac{\partial u}{\partial t}(x,0)$$

$$\mathscr{L}_t(\frac{\partial^2 u}{\partial x^2}) = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2}e^{-st}dt = \frac{d^2}{dx^2}\hat{u}(x,s)$$

Substituting everything into the PDE,

$$(s^2\hat{u}(x,s) - s\hat{u}(x,0) - \frac{\partial u}{\partial t}(x,0)) - 2\beta(s\hat{u}(x,s) - \hat{u}(x,0)) + \beta^2(\hat{u}(x,s)) = c^2(\frac{d^2\hat{u}(x,s))}{dx^2})$$

$$s^2\hat{u}(x,s) - sf(x) - g(x) - 2\beta(s\hat{u}(x,s) - f(x)) + \beta^2(\hat{u}(x,s)) = c^2(\frac{d^2}{dx^2}\hat{u}(x,s))$$

$$\frac{d^2\hat{u}(x,s))}{dx^2} + \frac{2\beta s - s^2 - \beta^2}{c^2}\hat{u}(x,s) = g(x) + (s-2\beta)f(x)$$

Non-homogenous constant coefficient ODE. Starting with homogenous solution, $\hat{u}_h = e^{\omega x}$

$$\omega^2 + \frac{2\beta s - s^2 - \beta^2}{c^2} = 0 \rightarrow \omega(s) = \pm i\sqrt{\frac{2\beta s - s^2 - \beta^2}{c^2}}$$

$$\hat{u}_h(x,s) = A_s cos(\omega_s x) + B_s sin(\omega_s x)$$

Let $C_f(h(x))$ represent a linearly independent family of functions from $g(x) + (s-2\beta)f(x)$.

$$\hat{u}_p(x,s) = C_f[g(x) + (s-2\beta)f(x)]$$

$$\hat{u}(x,s) = \hat{u}_h(x,s) + \hat{u}_p(x,s)$$

$$\hat{u}(x,s) = A_s cos(\omega_s x) + B_s sin(\omega_s x) + C_f[g(x) + (s-2\beta)f(x)]$$

$$u(x,t) = \mathscr{L}^{-1}(\hat{u}(x,s)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(x,s)e^{st}ds$$

Problem 3

SOLUTION

$$\mathscr{F}(sin\omega t) = \int_{-\infty}^{\infty} sin(\omega t)e^{-i\lambda t}dt$$

$$\mathscr{F}(sin\omega t) = \frac{1}{2i}\int_{-\infty}^{\infty} (e^{i\omega t} - e^{-i\omega t}))e^{-i\lambda t}dt$$

$$\mathscr{F}(sin\omega t) = \frac{1}{2i}\int_{-\infty}^{\infty} (e^{-it(\lambda-\omega)} - e^{-it(\lambda+\omega)}))dt$$

From lecture,

$$\mathscr{F}(1) = 2\pi\delta(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}dt$$

$$\int_{-\infty}^{\infty} e^{-it(\lambda-\omega)}dt = 2\pi\delta(\lambda-\omega)$$

$$-\int_{-\infty}^{\infty} e^{-it(\lambda+\omega)}dt = -2\pi\delta(\lambda+\omega)$$

$$\mathscr{F}(sin\omega t) = \frac{1}{2i}(2\pi\delta(\lambda-\omega) - 2\pi\delta(\lambda+\omega)) = \frac{\pi}{i}\delta(\lambda-\omega) - \delta(\lambda+\omega)$$

$$\mathscr{F}(sin\omega t) = \frac{\pi}{i}[\delta(\lambda-\omega) - \delta(\lambda+\omega)]$$

Problem 4

SOLUTION

$$\mathscr{F}(f'''(x)) = \int_{-\infty}^{\infty} f'''(x)e^{i\lambda x}dx$$

$$\int_{-\infty}^{\infty} f'''(x)e^{i\lambda x}dx = [f''(x)e^{i\lambda x}]|_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} f''(x)e^{i\lambda x}dx...$$

$$... = [f''(x)e^{i\lambda x}]|_{-\infty}^{\infty} - i\lambda[(f'(x)e^{i\lambda x})|_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} f'(x)e^{i\lambda x}dx]$$

$$... = [f''(x)e^{i\lambda x}]|_{-\infty}^{\infty} - i\lambda[(f'(x)e^{i\lambda x})|_{-\infty}^{\infty} - i\lambda[(f(x)e^{i\lambda x})|_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx]]]$$

Assuming, $\lim_{|x|\to\infty} f(x) = \lim_{|x|\to\infty} f'(x) = \lim_{|x|\to\infty} f''(x) = 0$, we can write

$$\mathscr{F}(f'''(x)) = 0 - i\lambda[0 - i\lambda[0 - i\lambda\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx]]] = (-i\lambda)^3\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx$$

Note

$$\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx = F(\lambda)$$

So,

$$\mathscr{F}(f'''(x)) = (-i\lambda)^3\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx = (-i\lambda)^3F(\lambda)$$

Problem 5

SOLUTION

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

To solve Laplace's equation, I'll look at Laplace integral transforms.
1) Range: $\rightarrow$ Limits match for both x and y when considering only the first quadrant
2) OPoD: $\rightarrow$ need both function and derivative at zero to use Laplace transforms, which we have for both x and y
I'm choosing Laplace integral transforms because although I could make other choices, Laplace transforms utilizes the most boundary conditions, so that there are less unknowns left at the end of the problem. Also, I think it's logical to use the 'Laplace' integral transforms to solve the 'Laplace' equation.

$$-\mathscr{L}_y(\frac{\partial^2 u}{\partial x^2}) = \mathscr{L}_y(\frac{\partial^2 u}{\partial y^2})$$

RHS:

$$\mathscr{L}_y(\frac{\partial^2 u}{\partial y^2}) = s^2\hat{u}(x,s) - s\hat{u}(x,0) - \frac{\partial u}{\partial y}(x,0) = s^2\hat{u}(x,s) - 2x^2$$

LHS:

$$-\mathscr{L}_y(\frac{\partial^2 u}{\partial x^2}) = -\frac{d^2\hat{u}(x,s)}{dx^2}$$

Substituting back into PDE,

$$\frac{d^2\hat{u}(x,s)}{dx^2} + s^2\hat{u}(x,s) = 2x^2$$

Which is a non-homogenous ODE. First, solving the homogenous equation

$$\frac{d^2\hat{u}}{dy^2} + s^2\hat{u} = 0 \rightarrow \hat{u}_h = e^{kx}$$

$$k^2 + s^2 = 0 \rightarrow k = \pm is \rightarrow \hat{u}_h(x,s) = A_s cos(sx) + B_s sin(sx)$$

$$\hat{u}_p(x,s) = c_1 + c_2 x + c_3 x^2 \rightarrow 2c_3 + s^2(c_1 + c_2 x + c_3 x^2) = 2x^2$$

$$2c_3 + s^2 c_1 = 0; s^2 c_2 = 0; s^2 c_3 = 2$$

$$c_3 = \frac{2}{s^2}$$

$$c_2 = 0$$

$$c_1 = \frac{-4}{s^4}$$

$$\hat{u}(x,s) = \hat{u}_h(x,s) + \hat{u}_p(x,s) = A_s cos(sx) + B_s sin(sx) + \frac{-4}{s^4} + \frac{2}{s^2}x^2$$

$$u(0,y) = 0 \rightarrow \hat{u}(0,s) = 0$$

$$\hat{u}(0,s) = A_s + \frac{-4}{s^4} = 0 \rightarrow A_s = \frac{4}{s^4}$$

$$\frac{\partial u}{\partial y}(0,y) = \delta(y) \rightarrow \mathscr{L}_y(\frac{\partial u}{\partial y}(0,y)) = \mathscr{L}_y(\delta(y)) = 1$$

$$\frac{\partial \hat{u}}{\partial s}(0,s) = 1 = \frac{16}{s^5} - \frac{16}{s^5} + sB_s \rightarrow B_s = \frac{1}{s}$$

$$\hat{u}(x,s) = \frac{4}{s^4}cos(sx) + \frac{1}{s}sin(sx) + \frac{-4}{s^4} + \frac{2}{s^2}x^2$$

$$u(x,y) = \mathscr{L}_y^{-1}(\hat{u}(x,s))$$

Problem 6

SOLUTION

Pair 1

$$\mathcal{F}^{-1}(\mathcal{F}(f(x))) = f(x) \rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f(y)e^{i\lambda y}dy)e^{-i\lambda x}d\lambda$$

$$\mathcal{F}^{-1}(\mathcal{F}(f(x))) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(y)e^{i\lambda(y-x)}dyd\lambda = ...$$

$$... = \int_{-\infty}^{\infty}f(y)[\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda(y-x)}d\lambda]dy = \int_{-\infty}^{\infty}f(y)(-\delta(y-x))dy = \int_{-\infty}^{\infty}f(y)\delta(x-y))dy = f(x)$$

Pair 2

$$\mathcal{F}^{-1}(\mathcal{F}(f(x))) = f(x) \rightarrow \int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}(f(y)e^{i\lambda y}dy)e^{-i\lambda x}d\lambda = \int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)e^{i\lambda(y-x)}dyd\lambda = ...$$

$$... = \int_{-\infty}^{\infty}f(y)(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda(y-x)}d\lambda)dy = \int_{-\infty}^{\infty}f(y)(-\delta(y-x))dy = \int_{-\infty}^{\infty}f(y)\delta(x-y)dy = f(x)$$

Pair 3

$$\mathcal{F}^{-1}(\mathcal{F}(f(x))) = f(x) \rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(f(y)e^{-i\lambda y}dy)e^{i\lambda x}d\lambda = ...$$

$$... = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(y)e^{i\lambda(x-y)}dyd\lambda = \int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)e^{i\lambda(x-y)}d\lambda dy = ...$$

$$... = \int_{-\infty}^{\infty}f(y)(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda(x-y)}d\lambda)dy = \int_{-\infty}^{\infty}f(y)\delta(x-y)dy = f(x)$$