Problem 1

SOLUTION

PART A

Given the governing equations for this RLC circuit, the system can be written as,

$$\vec{i'} = A\vec{i} + \vec{F}(t)$$

$$\frac{dq_1}{dt} = i_1$$

$$\frac{dq_2}{dt} = i_2$$

$$\frac{di_1}{dt} = \frac{i_2}{R_1C} - \frac{i_1}{R_1C} + \frac{E'(t)}{R_1}$$

$$\frac{di_2}{dt} = \frac{q_1 - q_2}{LC} - \frac{R_2 i_2}{L}$$

$$A = \begin{bmatrix} \frac{-1}{R_1C} & \frac{1}{R_1C} & 0 & 0 \\ 0 & \frac{-R_2}{L} & \frac{1}{LC} & \frac{-1}{LC} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} \frac{-1}{R_1C} - \lambda & \frac{1}{R_1C} & 0 & 0 \\ 0 & \frac{-R_2}{L} - \lambda & \frac{1}{LC} & \frac{-1}{LC} \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{bmatrix}$$

From $(A - \lambda I)$, I can solve for the λ values by setting the determinant of $(A - \lambda I)$ equal to 0. IE

$$det(A - \lambda I) = 0$$

After I have the corresponding eigen values, I can solve for the respective eigen vectors by setting $(A-\lambda I)\vec{l}$ equal to $\vec{0}$. IE

$$(A - \lambda I)\vec{l} = \vec{0}$$

PART B

$$det(A - \lambda I) = 1 * det(\begin{bmatrix} \frac{1}{R_1C} & 0 & 0 \\ \frac{-R_2}{L} - \lambda & \frac{1}{LC} & \frac{1}{LC} \\ 1 & 0 & -\lambda \end{bmatrix}) - 0 + \lambda * det(\begin{bmatrix} \frac{-1}{R_1C} - \lambda & \frac{1}{R_1C} & 0 \\ 0 & \frac{-R_2}{L} - \lambda & \frac{-1}{LC} \\ 0 & 1 & -\lambda \end{bmatrix}) - 0 = 0$$

$$\frac{-\lambda}{R_1LC} - \lambda((\frac{-1}{R_1C} - \lambda)((\frac{-R_2}{L} - \lambda)(-\lambda) - (\frac{-1}{LC}(1))) = 0$$

$$\lambda^4 + (\frac{1}{R_1C} + \frac{R_2}{L})\lambda^3 + (\frac{R_2}{R_1LC} + \frac{1}{LC})\lambda^2 + (\frac{-1}{R_1LC} + \frac{1}{R_1LC^2})\lambda = 0$$

$$\lambda^4 + 3.1\lambda^3 + 1.3\lambda^2 = 0$$

$$\lambda^2(\lambda^2 + 3.1\lambda + 1.3) = 0$$

$$\lambda = \begin{bmatrix} 0 \\ 0 \\ -2.6 \\ -0.5 \end{bmatrix}$$

Now solving $(A - \lambda_i I)\vec{l_i} = \vec{0}$. Note $\lambda_1 = \lambda_2 = 0$

$$\lambda_{1,2} \rightarrow (A - \lambda_{1,2}I)\vec{l_{1,2}} = A\vec{l_{1,2}} = 0$$

$$-0.1l_a + 0.1l_b = 0$$

$$-3l_b + l_c - l_d = 0$$

$$l_a = 0$$

$$l_b = 0$$

$$\rightarrow l_c = l_d$$

$$\vec{l_{1,2}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_3 \rightarrow (A - \lambda_3 I) = \begin{bmatrix} -0.1373 & 0.1 & 0 & 0 \\ 0 & -3.0373 & 1 & -1 \\ 1 & 0 & -0.0373 & 0 \\ 0 & 1 & 0 & -0.0373 \end{bmatrix}$$

$$(A - \lambda_3 I)\vec{l_3} = \vec{0} \rightarrow 0.1l_b - 0.1373l_a = 0$$

$$-3.0733l_b + l_c - l_d = 0$$

$$l_a - 0.0373l_c = 0$$

$$l_b - 0.0373l_d = 0$$

Let $l_b = 1$.

$$l_a = \frac{0.1}{0.1373}l_b = 0.7283l_b = 0.7283$$

$$l_d = \frac{1}{0.0373}l_b = 26.8097$$

$$l_c = l_d + 3.0373l_b = 29.8470$$

$$\vec{l_3} = \begin{bmatrix} 0.7283 \\ 1 \\ 29.8740 \\ 26.8097 \end{bmatrix}$$

$$\lambda_4 \to A - \lambda_4 I = \begin{bmatrix} -0.9677 & 0.1 & 0 & 0 \\ 0 & -3.8677 & 1 & -1 \\ 1 & 0 & -0.8677 & 0 \\ 0 & 1 & 0 & -0.8677 \end{bmatrix}$$

Again, let $l_b = 1$

$$l_a = \frac{0.1}{0.9677} l_b = 0.1033$$

$$l_d = \frac{1}{0.8677} l_b = 1.1525$$

$$l_c = l_d + 3.8677 l_b = 5.0202$$

$$\vec{l_4} = \begin{bmatrix} 0.1033 \\ 1 \\ 1.1525 \\ 5.0202 \end{bmatrix}$$

Final Solution,

$$\vec{i} = c_1 \vec{l_1} e^{\lambda_1 t} + c_2 \vec{l_2} t e^{\lambda_2 t} + c_3 \vec{l_3} e^{\lambda_3 t} + c_4 \vec{l_4} e^{\lambda_4 t}$$

$$\vec{i} = c_1 \vec{l_1} + c_2 \vec{l_2} t + c_3 \vec{l_3} e^{\lambda_3 t} + c_4 \vec{l_4} e^{\lambda_4 t}$$

$$\lambda = \begin{bmatrix} 0 \\ 0 \\ -2.6 \\ -0.5 \end{bmatrix}$$

$$\vec{l_{1,2}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{l_3} = \begin{bmatrix} 0.7283 \\ 1 \\ 29.8740 \\ 26.8097 \end{bmatrix}$$

$$\vec{l_4} = \begin{bmatrix} 0.1033 \\ 1 \\ 1.1525 \\ 5.0202 \end{bmatrix}$$

$$i_1 = \frac{dq_1}{dt} = c_1 e^{\lambda_1 t} l_{1c} + c_2 t e^{\lambda_2 t} l_{2c} + c_3 e^{\lambda_3 t} l_{3c} + c_4 e^{\lambda_4 t} l_{4c}$$

$$i_2 = \frac{dq_2}{dt} = c_1 e^{\lambda_1 t} l_{1d} + c_2 t e^{\lambda_2 t} l_{2d} + c_3 e^{\lambda_3 t} l_{3d} + c_4 e^{\lambda_4 t} l_{4d}$$

$$i_1 = c_1 + c_2 t + c_3 e^{-2.6t}(29.8740) + c_4 e^{-0.5t}(1.1525)$$

$$i_2 = c_1 + c_2 t + c_3 e^{-2.6t}(26.8097) + c_4 e^{-0.5t}(5.0202)$$

Problem 2

SOLUTION

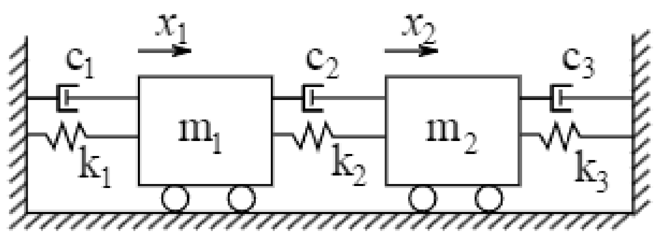

Figure 1: Problem 2

PART A

$$\Sigma_x F_1 = m_1 a_1 = -k_1 x_1 + k_2(x_2 - x_1) - c_1 v_1 + c_2(v_2 - v_1)$$

$$\Sigma_x F_2 = m_2 a_2 = k_2(x_1 - x_2) - k_3 x_2 + c_2(v_1 - v_2) - c_3 v_2$$

$$\Sigma_x F_1 = m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1) - c_1 \frac{dx_1}{dt} + c_2(\frac{dx_2}{dt} - \frac{dx_1}{dt})$$

$$\Sigma_x F_2 = m_2 \frac{d^2 x_2}{dt^2} = k_2(x_1 - x_2) - k_3 x_2 + c_2(\frac{dx_1}{dt} - \frac{dx_2}{dt}) - c_3 \frac{dx_2}{dt}$$

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} m_1 v_1' \\ m_2 v_2' \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -c_1 - c_2 & c_2 & -k_1 - k_2 & k_2 \\ c_2 & -c_2 - c_3 & k_2 & -k_2 - k_3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ x_1 \\ x_2 \end{bmatrix}$$

$$a = -c_1 - c_2; b = c_2; d = -k_1 - k_2; e = k_2; f = -c_2 - c_3; g = -k_2 - k_3$$

PART B

$$A - \lambda I = \begin{bmatrix} \frac{-c_1 - c_2}{m_1} - \lambda & \frac{c_2}{m_1} & \frac{-k_1 - k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{c_2}{m_2} & \frac{-c_2 - c_3}{m_2} - \lambda & \frac{k_2}{m_2} & \frac{-k_2 - k_3}{m_2} \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{bmatrix}$$

PART C

SEE CODE APPENDED TO THE END OF THIS ASSIGNMENT.
I expect the solution to take the following form

$$\vec{x_H} = c_1 e^{\alpha_1 t}[\vec{a_1} cos(\beta_1 t) - \vec{b_1} sin(\beta_1 t)] + c_2 e^{\alpha_2 t}[\vec{a_2} cos(\beta_2 t) - \vec{b_2} sin(\beta_2 t)] + ...$$

$$... + c_3 e^{\alpha_3 t}[\vec{a_3} cos(\beta_3 t) - \vec{b_3} sin(\beta_3 t)] + c_4 e^{\alpha_4 t}[\vec{a_4} cos(\beta_4 t) - \vec{b_4} sin(\beta_4 t)]$$

But it'd be easier to write the solution in the following form instead

$$\vec{x_H} = \Sigma_j^{n=4}(C_j(\vec{a_j} + i\vec{b_j})e^{(\alpha_j + i\beta_j)t})$$

Note that for all λ, $\lambda_{real} = 0$. Therefore, $\alpha = 0$ for all α and we can write $\lambda_j = \alpha_j + i\beta_j = i\beta_j$. From my MATLAB code

$$\lambda = \vec{\alpha} + i\vec{\beta} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 2.2361 \\ -2.2361 \\ 1 \\ -1 \end{bmatrix} = i \begin{bmatrix} \sqrt{5} \\ -\sqrt{5} \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{l} = \begin{bmatrix} -0.6455 & -0.6455 & 0.5 & 0.5 \\ 0.6455 & 0.6455 & 0.5 & 0.5 \\ 0.2887i & -0.2887i & -0.5i & 0.5i \\ -0.2887i & 0.2887i & -0.5i & 0.5i \end{bmatrix}$$

Where each column is an individual eigen vector. If I divide the first two eigen vectors by 0.2887 and then multiply the second two eigen vectors by 2, I get the following set of eigen vectors

$$\vec{l} = \begin{bmatrix} -2.236 & -2.236 & 1 & 1 \\ 2.236 & 2.236 & 1 & 1 \\ i & -i & -i & i \\ -i & i & -i & i \end{bmatrix}$$

Where $2.236 = \sqrt{5}$. Now I can write the general solution as

$$\vec{x} = c_1 e^{i\sqrt{5}t} \begin{bmatrix} -\sqrt{5} \\ \sqrt{5} \\ i \\ -i \end{bmatrix} + c_2 e^{-i\sqrt{5}t} \begin{bmatrix} -\sqrt{5} \\ \sqrt{5} \\ -i \\ i \end{bmatrix} + c_3 e^{it} \begin{bmatrix} 1 \\ 1 \\ -i \\ -i \end{bmatrix} + c_4 e^{-it} \begin{bmatrix} 1 \\ 1 \\ i \\ i \end{bmatrix}$$

PART D

$$\vec{x} = c_1 e^{i\sqrt{5}t} \begin{bmatrix} -\sqrt{5} \\ \sqrt{5} \\ i \\ -i \end{bmatrix} + c_2 e^{-i\sqrt{5}t} \begin{bmatrix} -\sqrt{5} \\ \sqrt{5} \\ -i \\ i \end{bmatrix} + c_3 e^{it} \begin{bmatrix} 1 \\ 1 \\ -i \\ -i \end{bmatrix} + c_4 e^{-it} \begin{bmatrix} 1 \\ 1 \\ i \\ i \end{bmatrix} = ...$$

$$... = c_1 e^{i\sqrt{5}t} (\begin{bmatrix} -\sqrt{5} \\ \sqrt{5} \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}) + c_2 e^{-i\sqrt{5}t} (\begin{bmatrix} -\sqrt{5} \\ \sqrt{5} \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}) + c_3 e^{it} (\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}) + c_4 e^{-it} (\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix})$$

From Euler's formula, looking only at the first two columns of l ($\vec{l_1}$ and $\vec{l_2}$), note that there are no real parts of any eigen values

$$\vec{x_{1,2}} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} sin(\sqrt{5}t) + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} sin(\sqrt{5}t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (c_1 sin(\sqrt{5}t) - c_2 sin(\sqrt{5}t))$$

$$\vec{v_{1,2}} = \sqrt{5}c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} cos(\sqrt{5}t) + \sqrt{5}c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} cos(\sqrt{5}t) = \sqrt{5} \begin{bmatrix} -1 \\ 1 \end{bmatrix} (c_1 cos(\sqrt{5}t) + c_2 cos(\sqrt{5}t))$$

PART E

$$\vec{x_{1,2}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (c_1 sin(\sqrt{5}t) - c_2 sin(\sqrt{5}t)) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (c_1 - c_2) sin(\sqrt{5}t) = ...$$

$$\vec{x_{1,2}} = (c_1 - c_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix} cos(\sqrt{5}t + \phi_1) = B_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} cos(\sqrt{5}t + \phi_1)$$

PART F

Now, looking at the last two eigen vectors/values where $\lambda_{3,4} = \pm 1$

$$\vec{x_{1,2}} = c_3 \begin{bmatrix} -1 \\ -1 \end{bmatrix} sin(t) + c_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} sin(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (-c_3 + c_4) sin(t)$$

$$x_{1,2} = (-c_3 + c_4) \begin{bmatrix} 1 \\ 1 \end{bmatrix} sin(t) = B_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} cos(t + \phi_2)$$

$$\vec{x_{1,2}} = B_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} cos(\sqrt{5}t + \phi_1) + B_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} cos(t + \phi_2)$$

Problem 3

SOLUTION
When $m_1 = m_2 = k_1 = k_3 = 1$, $k_2 = 2$ and $c_1 = c_2 = c_3 = 0.2$,

$$\omega_1 = 2.2159$$

$$\omega_2 = 0.9950$$

PART A

Doing a sensitivity analysis on each individual c_i while letting all other damping coefficients remain constant at 0.2, I get the following results. These results show that the smaller the respective damping coefficients are, the closer the natural frequency of the system is to the original natural frequencies noted above. As the damping coefficients grow, they damp out the system reactions and the eigen values are no longer imaginary after the damping coefficient gets sufficiently large. This happens whenever any c_i values are bigger than 2.

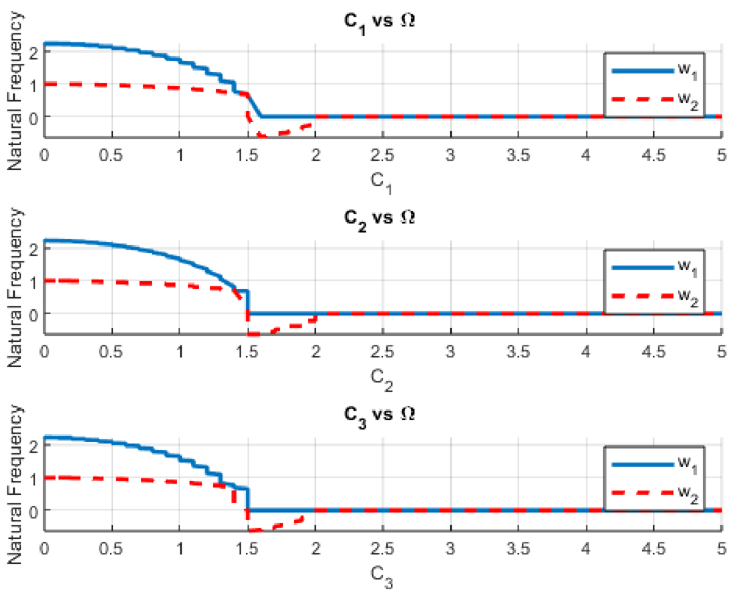

Figure 2: Problem 3b

PART B

The plot below shows the system reaction in time. Upon inspection, you can see that mode 1, where $\omega = 2.2159$ decays faster than mode 2. As time grows, mode 2 dominates the total homogeneous solution. This system is underdamped.

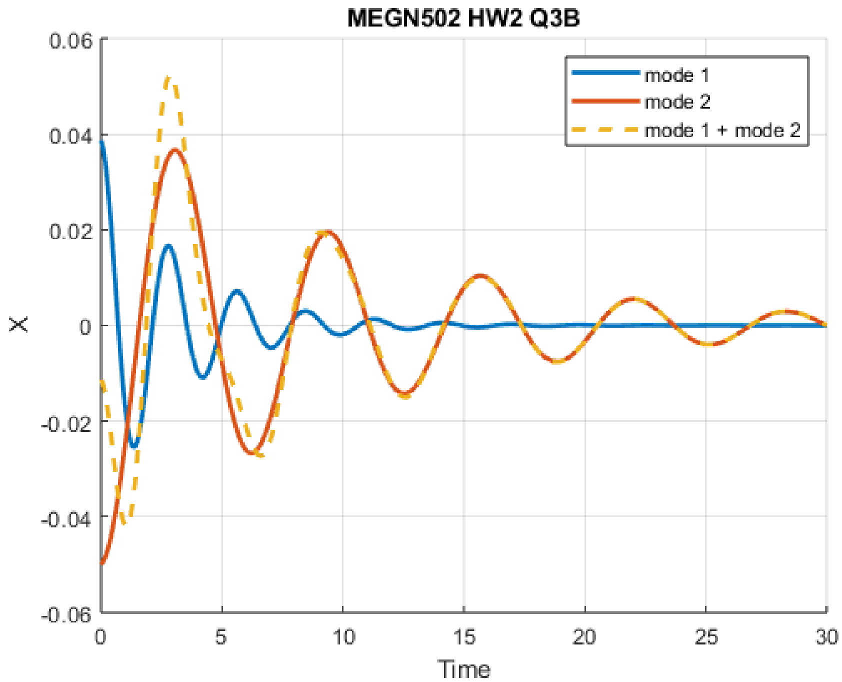

Figure 3: Problem 3b

PART C

As the damping coefficients go up, the system gets stronger and oscillation damps out quickly. Again, mode 1 damps out fastest and the total system solution follows mode 2 closely. This mode is critically damped.

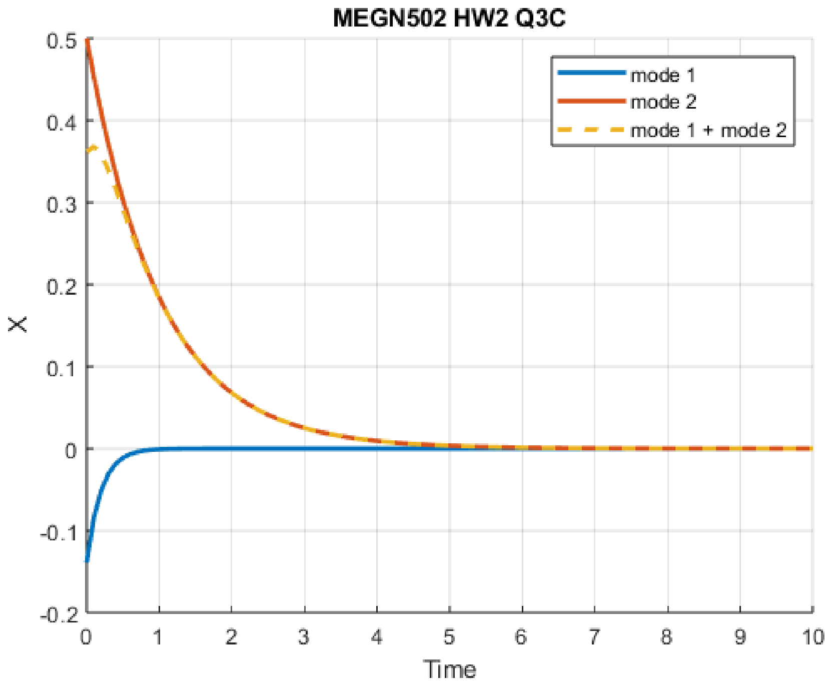

Figure 4: Problem 3c

PART D

Looking at the results where all damping coefficients are equal to 5, I expected the system to damp out much faster. Mode 2 decays much slower than I expected. This mode is overdamped.

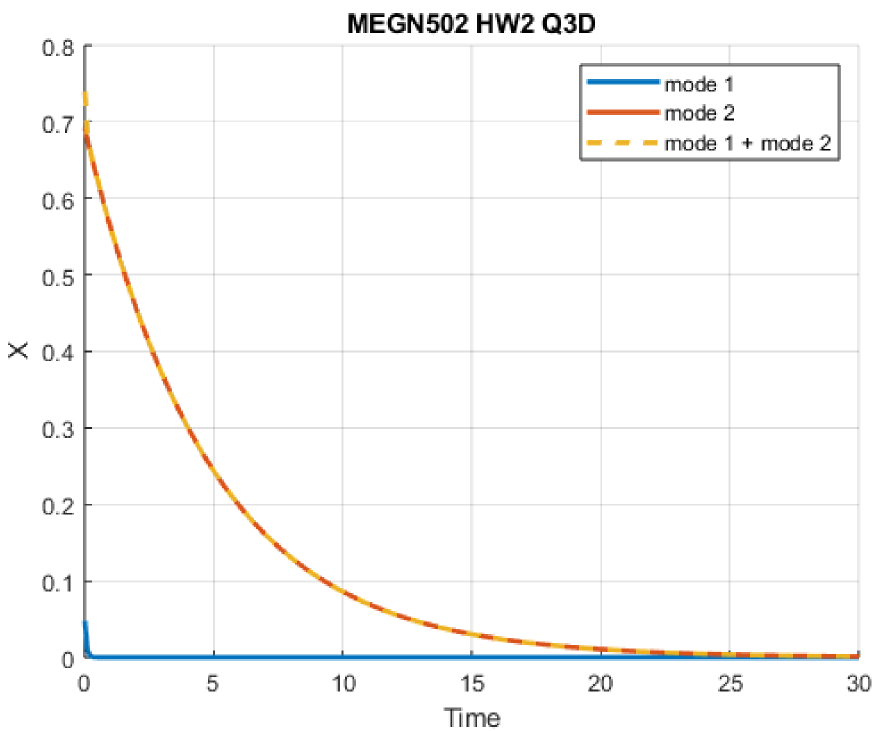

Figure 5: Problem 3d