Problem 1

SOLUTION

$$\frac{\delta^2 u}{\delta t^2} + 2h\frac{\delta u}{\delta t} = c^2\frac{\delta^2 u}{\delta t^2}$$

Assuming $u(x,t) = X(x)T(t)$ and substituting,

$$T''X + 2hT'X = c^2X''T = -\lambda$$

$$\frac{T''}{T} + 2h\frac{T'}{T} = c^2\frac{X''}{X} = -\lambda$$

$$T'' + 2hT' + \lambda T = 0; X'' + \frac{\lambda}{c^2}X = 0$$

Spatial problem is of S-L form $\rightarrow$ $p(x) = 1 > 0$, $q(x) = 0$, $r(x) = \frac{1}{c^2} > 0$, which are all continuous. Considering $\lambda = 0$ first,

$$X''(x) = 0 \rightarrow X(x) = Ax + B$$

From BC's

$$X(0) = B = 0$$

$$X(L) = 0 = A(L) \rightarrow A = 0$$

$\lambda = 0$ is trivial. Considering $\lambda < 0$,

$$X''(x) - \frac{\lambda}{c^2}X(x) = 0$$

$$X(x) = Acosh(\frac{\sqrt{\lambda}x}{c}) + Bsin(\frac{\sqrt{\lambda}x}{c})$$

$$X(0) = Acosh(0) + Bsinh(0) = A = 0$$

$$X(L) = Bsinh(\frac{\sqrt{\lambda}L}{c}) = 0 \rightarrow B = 0$$

$\lambda < 0$ is also trivial. Finally, for $\lambda > 0$,

$$X''(x) + \lambda X(x) = 0$$

$$X(x) = Acos(\frac{\sqrt{\lambda}x}{c}) + Bsin(\frac{\sqrt{\lambda}x}{c})$$

$$X(0) = Acos(0) + Bsin(0) = 0 \rightarrow A = 0$$

$$X(L) = Bsin(\frac{\sqrt{\lambda}L}{c}) = 0 \rightarrow \frac{\sqrt{\lambda}L}{c} = n\pi; n\epsilon\mathbb{Z}$$

$$\lambda = (\frac{nc\pi}{L})^2 \rightarrow n\epsilon\mathbb{Z}$$

$$X_n(x) = B_n sin(\frac{nc\pi x}{L})$$

Now for the temporal problem, note that this problem is a constant coefficient problem.

$$T'' + 2hT' + \lambda T = 0$$

Assuming $T(t) = Ce^{kt}$, we have

$$k^2 + 2hk + \lambda = 0 \rightarrow k = -h \pm \sqrt{h^2 - \lambda}$$

$$T(t) = c_1 e^{(-h+\sqrt{h^2-\lambda})t} + c_2 e^{(-h-\sqrt{h^2-\lambda})t}$$

Where $\lambda = (\frac{nc\pi}{L})^2$. Absorbing all constants into C_n and D_n,

$$u(x,t) = \Sigma_{n=1}^{\infty} sin(\frac{nc\pi x}{L})(C_n e^{(-h+\sqrt{h^2-\lambda})t} + D_n e^{(-h-\sqrt{h^2-\lambda})t})$$

$$u(x,0) = f(x) = \Sigma_{n=1}^{\infty} sin(\frac{nc\pi x}{L})(C_n + D_n)$$

Since we know this is an S-L problem, from orthogonality we can say,

$$C_n + D_n = \frac{2}{L}\int_0^L f(x) sin(\frac{nc\pi x}{L})dx = \frac{2}{L}\int_0^L \frac{1}{c^2} sin^2(\frac{nc\pi x}{L})dx = 1$$

Similarly,

$$\frac{\delta u}{\delta t}(x,0) = g(x) = \Sigma_{n=1}^{\infty}(\frac{n\pi}{L})cos(\frac{n\pi x}{L})(C_n + D_n)$$

$$C_n + D_n = \frac{\int_0^L \frac{g(x)}{c^2}\sqrt{h^2-\lambda_n} sin(\frac{nc\pi x}{L})dx}{\int_0^L \sqrt{h^2-\lambda_n} sin^2(\frac{nc\pi x}{L})dx}$$

Problem 2

SOLUTION
Given,

$$\nabla^2 u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

The boundary conditions and initial conditions will be broken up into two different cases. For the first case,

$$u(x,0) = 0$$

$$u(x,H) = 0$$

$$u(0,y) = g(y)$$

$$u(L,y) = 0$$

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = -\lambda$$

$$X''(x) + \lambda X(x) = 0$$

$$Y''(y) - \lambda Y(y) = 0$$

Solving for $Y(y)$, starting with $\lambda = 0$,

$$Y''(y) = 0 \rightarrow Y(y) = Ay + B$$

$$Y(0) = 0 \rightarrow B = 0$$

$$Y(H) = 0 = A(H) \rightarrow A = 0$$

Now for $\lambda < 0$,

$$Y''(y) + \lambda Y(y) = 0 \rightarrow Y(y) = e^{ky}$$

$$Y(y) = Acos(\sqrt{\lambda}y) + Bsin(\sqrt{\lambda}y)$$

$$Y(0) = 0 = A \rightarrow A = 0$$

$$Y(H) = 0 = Bsin(\sqrt{\lambda}H) \rightarrow \sqrt{\lambda}H = n\pi$$

$$\lambda = (\frac{n\pi}{H})^2 \rightarrow n\epsilon\mathbb{Z}$$

I know this is a S-L problem because I can write $p(y) = 1$, $q(y) = 0$ and $r(y) = 1$ which are all continuous and p and r are always positive.

For $\lambda > 0$,

$$Y''(y) = 0 \rightarrow Y(y) = Acosh(\sqrt{\lambda}y) + Bsinh(\sqrt{\lambda}y)$$

$$Y(0) = 0 = Acosh(0) \rightarrow A = 0$$

Because $cosh(y)$ can never be zero.

$$Y(H) = 0 = Bsinh(\sqrt{\lambda}H) \rightarrow B = 0$$

$$Y(y) = \Sigma_{n=1}^{\infty} B_n sin(\frac{n\pi y}{H})$$

Now for the $X(x)$ solution,where $\lambda = 0$,

$$X''(x) = 0 \rightarrow X(x) = Ax + B$$

$$X(0) = g(y) = BY(y)$$

$$X(L) = 0 = AL + B \rightarrow Y(y) \neq 1$$

So this case is trivial because this system cannot be true. For $\lambda < 0$,

$$X''(x) - \lambda X(x) = 0$$

$$X(x) = Acosh(\sqrt{\lambda}x) + Bsinh(\sqrt{\lambda}x)$$

$$X(0) = g(y) = AY(y)$$

$$X(L) = 0 = Acosh(\sqrt{\lambda}L) + Bsinh(\sqrt{\lambda}L)$$

$$A = -B\frac{sinh(\sqrt{\lambda}L)}{cosh(\sqrt{\lambda}L)} = -Btanh(\sqrt{\lambda}L)$$

$$X(x) = -Btanh(\frac{n\pi L}{H})cosh(\frac{n\pi x}{H}) + Bsinh(\frac{n\pi x}{H})$$

Still need to check $\lambda > 0$,

$$X''(x) + \lambda X(x) = 0 \to X(x) = Acos(\sqrt{\lambda}x) + Bsin(\sqrt{\lambda}x)$$

$$X(0) = g(y) = A \to AY(y) = g(y)$$

$$X(L) = 0 \to A = 0$$

Now for the total solution, combining all coefficients into one,

$$u(x, y) = X(x)Y(y) = \Sigma_{n=1}^{\infty} A_n sin(\frac{n\pi y}{H})[-tanh(\frac{n\pi L}{H})cosh(\frac{n\pi x}{H}) + sinh(\frac{n\pi x}{H})]$$

Repeating this process for Case 2 where the BC's and IC's are as follows,

$$u(x, 0) = 0$$

$$u(x, H) = f(x)$$

$$u(0, y) = 0$$

$$u(L, y) = 0$$

Starting with the $X(x)$ solution because of it's homogeneous BC's, starting with $\lambda = 0$,

$$X''(x) = 0 \to X(x) = Ax + B$$

$$X(0) = 0 = B$$

$$X(L) = 0 = A(L) \to A = 0$$

Trivial. Try $\lambda < 0$,

$$X''(x) - \lambda X(x) = 0 \to X(x) = Acosh(\sqrt{\lambda}x) + Bsinh(\sqrt{\lambda}x)$$

$$X(0) = 0 = Acosh(0) \to A = 0$$

Because $cosh(y)$ can never be zero.

$$X(L) = 0 = Bsinh(\sqrt{\lambda}L) \rightarrow B = 0$$

Because $\sqrt{\lambda}L > 0$, so $sin(\sqrt{\lambda}L) \neq 0$. $\lambda < 0$ is gives a trivial solution. Try $\lambda > 0$,

$$X''(x) + \lambda X(x) = 0 \rightarrow X(x) = Acos(\sqrt{\lambda}x) + Bsin(\sqrt{\lambda}x)$$

$$X(0) = 0 = A$$

$$X(L) = 0 = Bsin(\sqrt{\lambda}x) \rightarrow \sqrt{\lambda}L = n\pi$$

$$\lambda = (\frac{n\pi}{L})^2 \rightarrow n\epsilon\mathbb{Z}$$

$$X(x) = \Sigma_{n=1}^{\infty} B_n sin(\frac{n\pi x}{L})$$

Now for $Y(y)$, starting with $\lambda = 0$

$$Y''(y) = 0 \rightarrow Y(y) = Ay + B$$

$$Y(0) = 0 = B$$

$$Y(L) = f(x) = A(L) \rightarrow A = 0$$

Now for $\lambda < 0$,

$$Y''(y) - \lambda Y(y) = 0 \rightarrow Y(y) = Acosh(\sqrt{\lambda}y) + Bsinh(\sqrt{\lambda}y)$$

$$Y(y) = 0 = Acosh(0) \rightarrow A = 0$$

Because $cosh(0) \neq 0$ and $cosh(y)$ is never zero.

$$Y(L) = f(x) = Bsinh((\sqrt{\lambda}H))X(x) \rightarrow B = \frac{f(x)}{X(x)sinh(\sqrt{\lambda}H)}$$

$$Y(y) = \frac{f(x)}{X(x)sinh(\sqrt{\lambda}H)} sinh(\sqrt{\lambda}H) = constant$$

So the total solution of $u(x, y)$ for case two is

$$u(x, y) = CX(x) = \Sigma_{n=1}^{\infty} B_n sin(\frac{n\pi x}{L})$$

Superposing the two solutions, I get

$$u_f(x, y) = \Sigma_{n=1}^{\infty} A_n sin(\frac{n\pi y}{H})[-tanh(\frac{n\pi L}{H})cosh(\frac{n\pi x}{H}) + sinh(\frac{n\pi x}{H})] + \Sigma_{n=1}^{\infty} B_n sin(\frac{n\pi x}{L})$$

Problem 3

SOLUTION

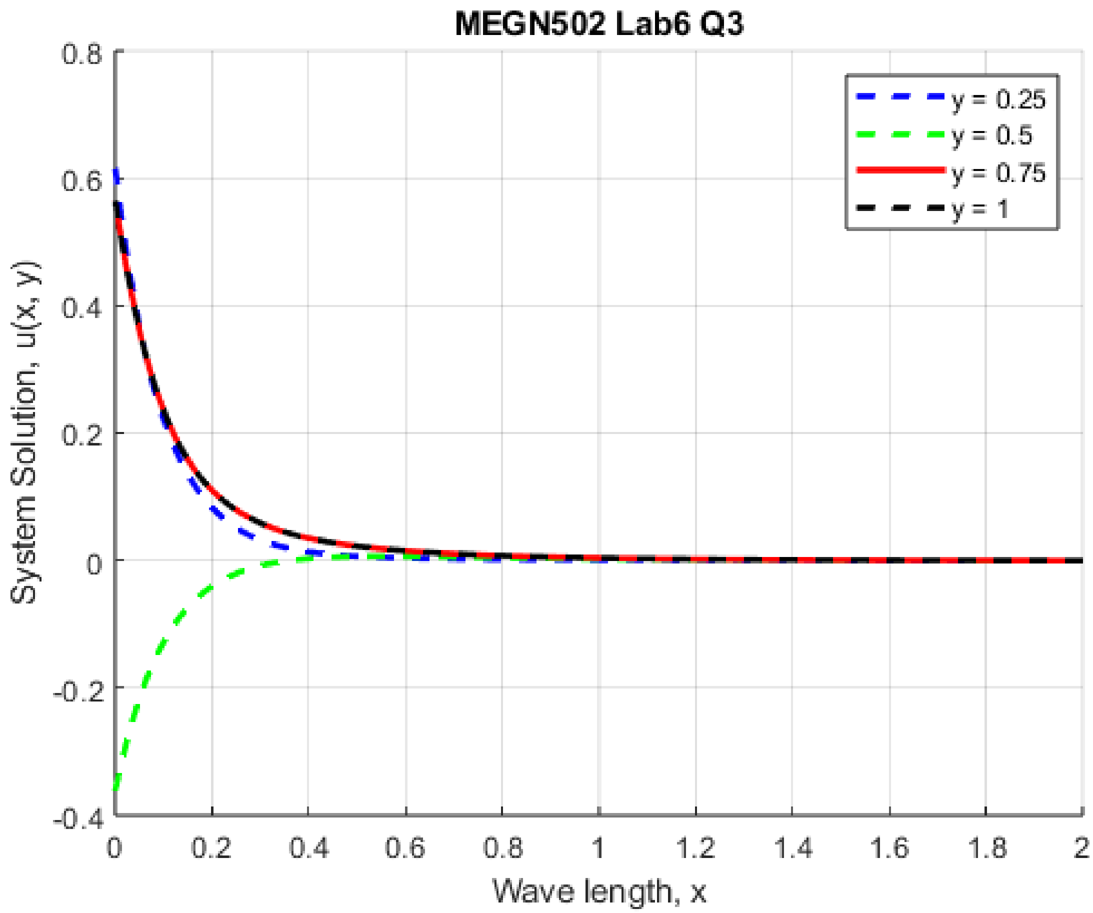

Figure 1: Problem 3

Problem 4

SOLUTION

$$\frac{\delta^2 u}{\delta t^2} = \frac{\delta^2 u}{\delta x^2}$$

Assuming $u(x,t) = X(x)T(t)$

$$X''(x)T(t) = T''(t)X(x)$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = -\lambda$$

$$X''(x) + \lambda X(x) = 0$$

$$T''(t) + \lambda T(t) = 0$$

With homogeneous boundary conditions on x, both of these equations are S-L problems because p, q, and r and all continuous over $0 < x < L$ and p and q are both greater than zero for x and t.

$$p(x) = 1, q(x) = 0, r(x) = 1$$

$$p(t) = 1, q(t) = 0, r(t) = 1$$

Starting with $\lambda = 0$ for the spatial problem,

$$X''(x) = 0 \rightarrow X(x) = Ax + B$$

$$X(0) = B = 0$$

$$X(L) = A(L) = 0 \rightarrow A = 0$$

Trivial. For $\lambda < 0$,

$$X''(x) - \lambda X(x) = 0 \rightarrow X(x) = Acosh(\sqrt{\lambda}x) + Bsinh(\sqrt{\lambda}x)$$

$$X(0) = Acosh(0) = 0 \rightarrow A = 0$$

$$X(L) = Bsinh(\sqrt{\lambda}L) = 0 \rightarrow B = 0$$

Also trivial. For $\lambda > 0$,

$$X''(x) + \lambda X(x) = 0$$

$$X(x) = Acos(\sqrt{\lambda}x) + Bsin(\sqrt{\lambda}x)$$

$$X(0) = A = 0$$

$$X(L) = Bsin(\sqrt{\lambda}L) = 0$$

$$\sqrt{\lambda}L = n\pi \rightarrow \lambda = (\frac{n\pi}{L})^2; n\epsilon\mathbb{Z}$$

$$X_n(x) = B_n sin(\frac{n\pi x}{L})$$

Now that λ is fixed,

$$T(t) = Ccos(\sqrt{\lambda}t) + Dsin(\sqrt{\lambda}t)$$

$$u(x,t) = X(x)T(t) = \Sigma_{n=1}^{\infty} sin(\frac{n\pi x}{L})[E_n cos(\sqrt{\lambda}t) + F_n sin(\sqrt{\lambda}t)]$$

$$u(x,0) = f(x) = \Sigma_{n=1}^{\infty} E_n sin(\frac{n\pi x}{L})$$

$$E_n = \frac{2}{L}\int_0^L f(x) sin(\frac{n\pi x}{L})dx$$

$$E_{n1} = \frac{2}{L}\int_0^a \frac{hx}{a} sin(\frac{n\pi x}{L})dx$$

$$E_{n2} = \frac{2}{L}\int_a^L \frac{h(L-x)}{(L-a)} sin(\frac{n\pi x}{L})dx$$

From MATLAB,

$$E_{n1} = \frac{2h(Lsin(\frac{an\pi}{L}) - an\pi cos(\frac{an\pi}{L}))}{an^2\pi^2}$$

$$E_{n2} = -(2(Lh(Lsin(n\pi) - Lsin(\frac{an\pi}{L})))) - Lsin(\frac{an\pi}{L}) - \frac{Lhn(L\pi cos(\frac{an\pi}{L}) - a\pi cos(\frac{an\pi}{L}))}{Ln^2\pi^2(L-a)}$$

$$\frac{\delta u}{\delta t}(x,0) = 0 = \Sigma_{n=1}^{\infty} F_n \sqrt{\lambda} sin(\frac{n\pi x}{L}) \rightarrow F_n = 0$$

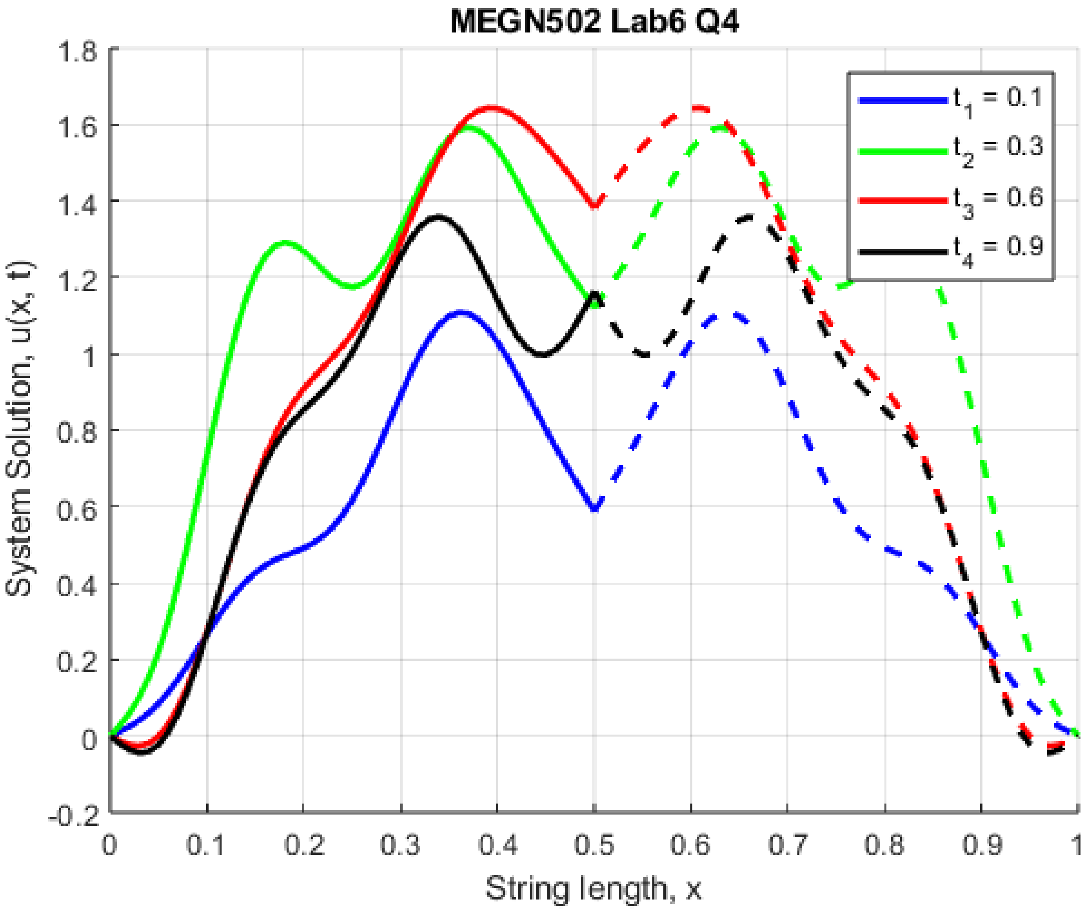

Figure 2: Problem 4