Problem 1

SOLUTION

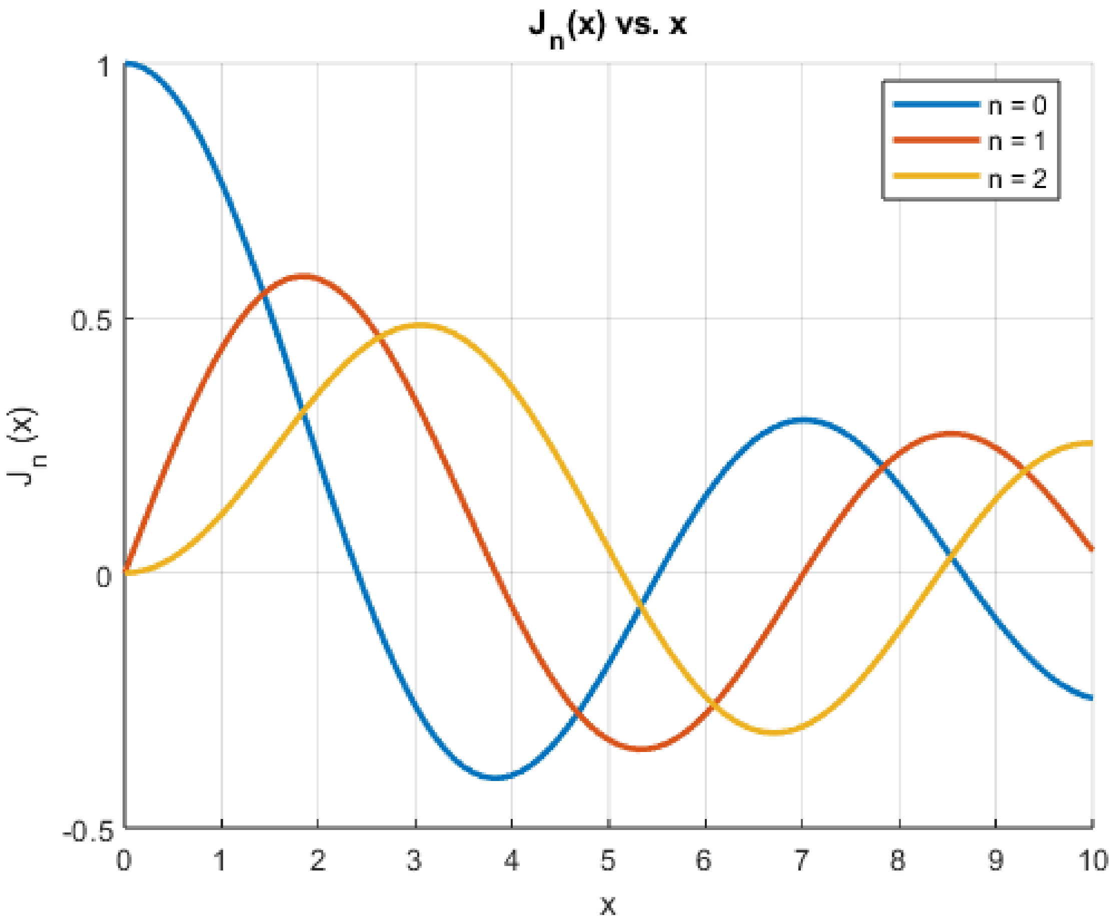

Figure 1: Problem 1

$$\lim_{x\to 0} J_0(x) = 1$$

$$\lim_{x\to 0} J_1(x) = 0$$

$$\lim_{x\to 0} J_2(x) = 0$$

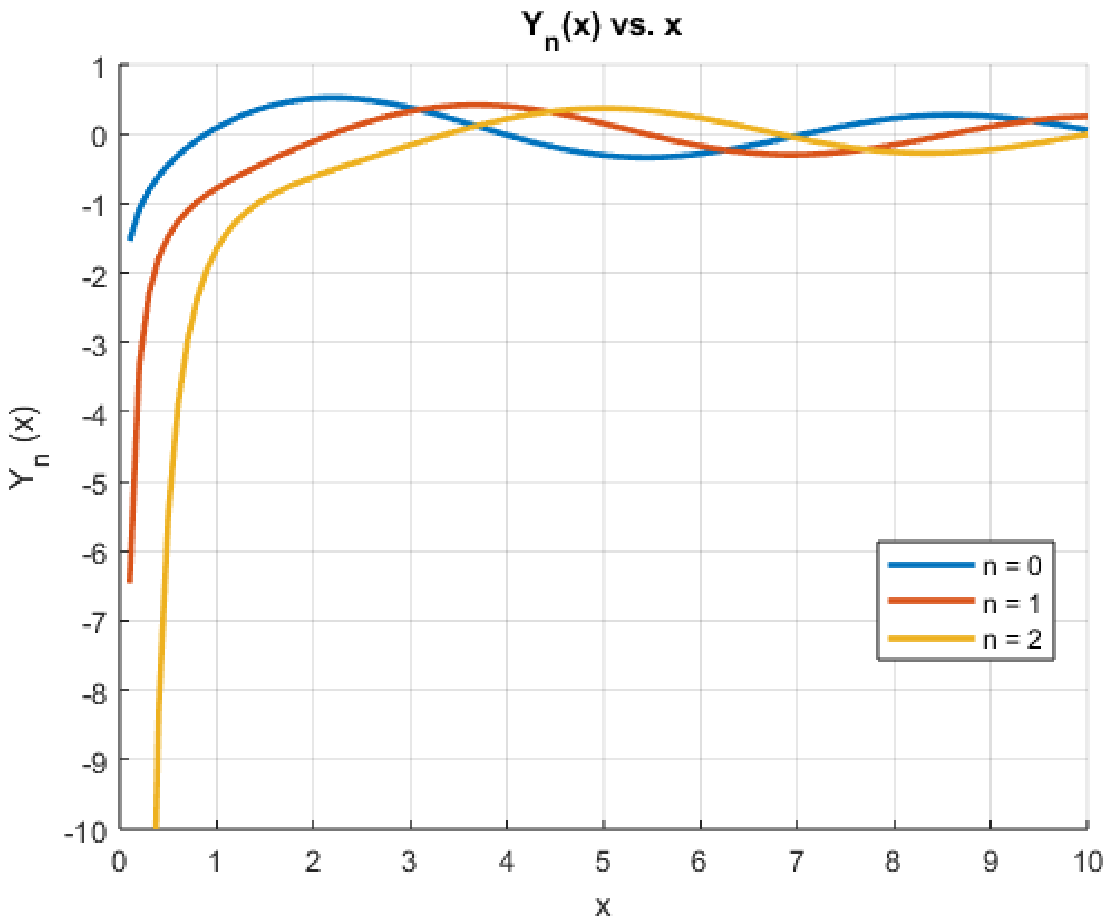

Figure 2: Problem 1

$$\lim_{x\to 0} Y_0(x) < \lim_{x\to 0} Y_1(x) < \lim_{x\to 0} Y_2(x) = -\infty$$

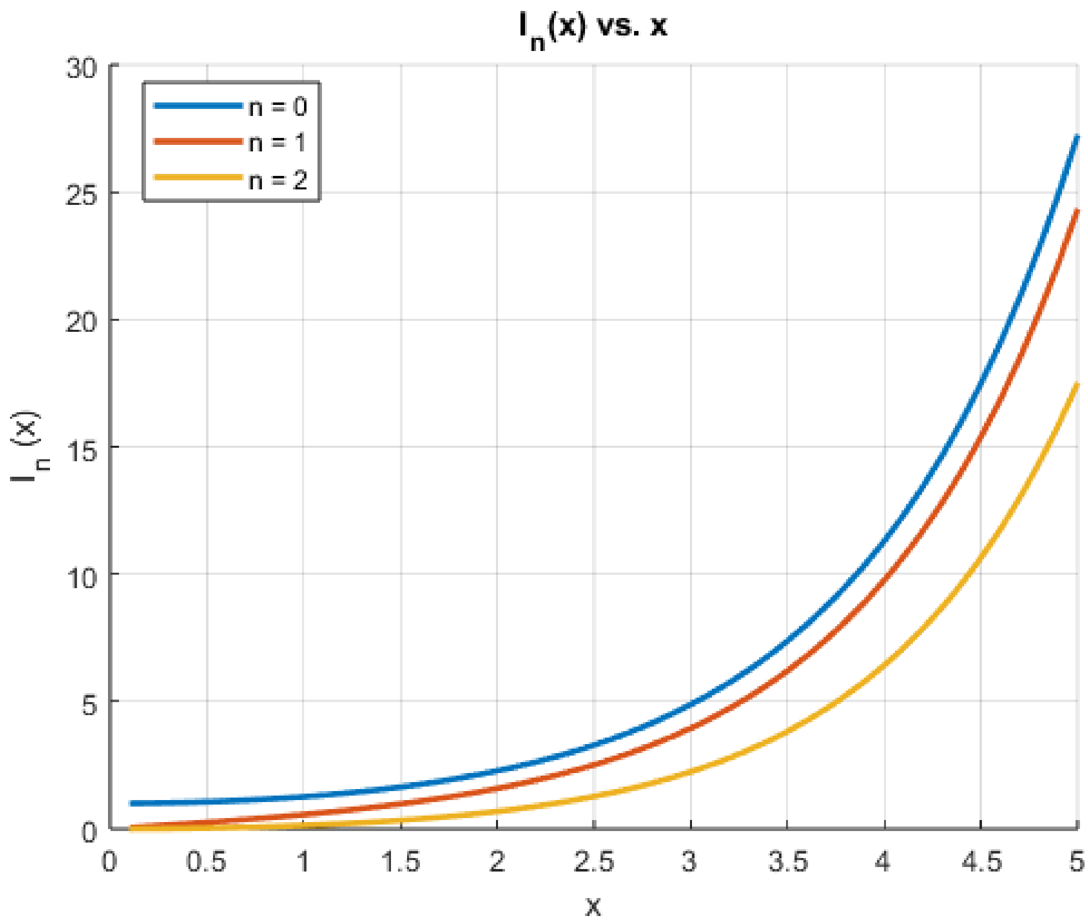

Figure 3: Problem 1

$$\lim_{x\to 0} I_0(x) = 1$$

$$\lim_{x\to 0} I_1(x) = 0$$

$$\lim_{x\to 0} I_2(x) = 0$$

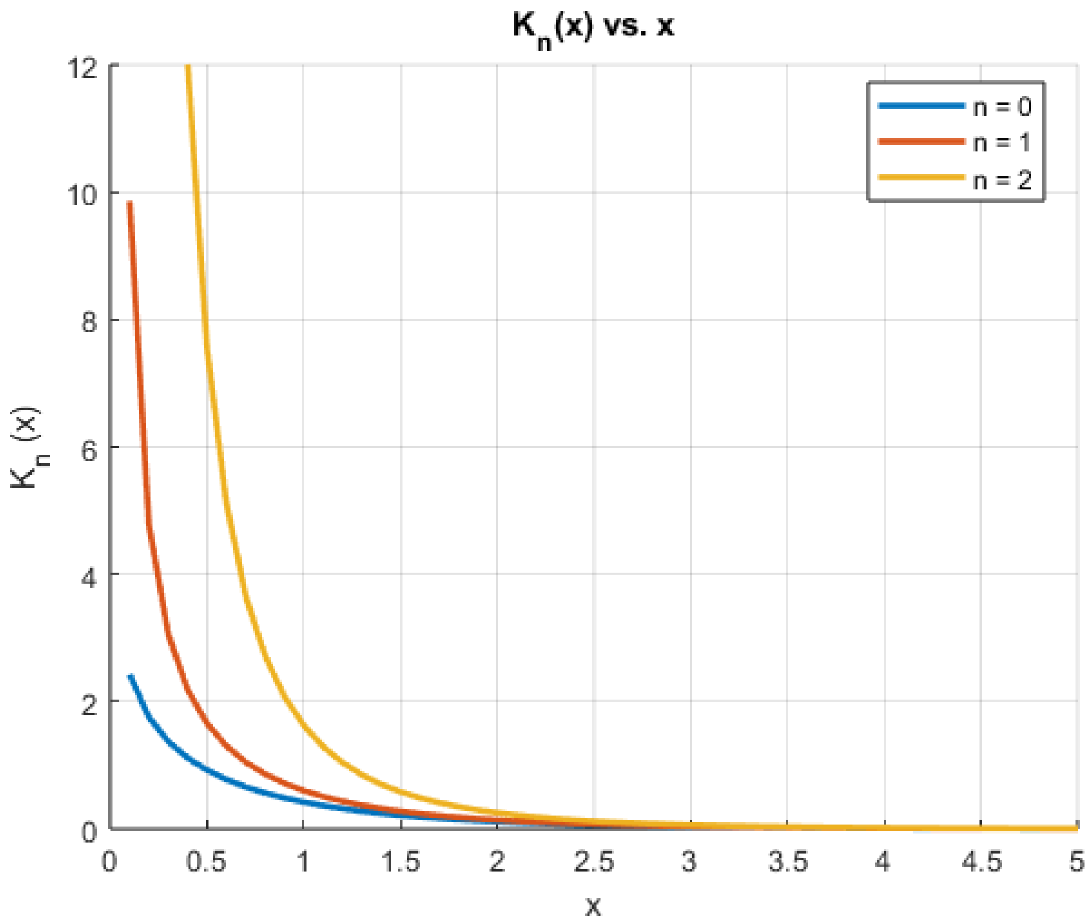

Figure 4: Problem 1

$$\lim_{x\to 0} K_0(x) < \lim_{x\to 0} K_1(x) < \lim_{x\to 0} K_2(x) = \infty$$

Problem 2

SOLUTION

PART 2A

$$a = -1, b = 0, c = 0, d = 4, s = 2, r = 1$$

$$p = \frac{1}{s}\sqrt{(\frac{1-a}{2})^2 - c} = \frac{1}{2}$$

$$\mu = \frac{\sqrt{d}}{s} = \frac{\sqrt{4}}{2} = 1$$

$$y(x) = x(C_1 J_p(x^2) + C_2 J_{-p}(x^2))$$

PART 2B

$$a = 1, b = 1, c = -2, d = 2, s = 1, r = 2$$

$$p = \frac{1}{s}\sqrt{(\frac{1-a}{2})^2 - c} = \sqrt{2}$$

$$\mu = \frac{\sqrt{d}}{s} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$y(x) = e^{\frac{-x^2}{2}}(C_1 J_p(\mu x) + C_2 J_{-p}(\mu x))$$

PART 2C

$$a = -1, b = 0, c = 0, d = -1, s = 1, r = 1$$

$$p = \frac{1}{s}\sqrt{(\frac{1-a}{2})^2 - c} = 1$$

$$\mu = \frac{\sqrt{d}}{s} = \frac{\sqrt{-1}}{1} = i$$

$$y(x) = x(C_1 I_n(x) + C_2 K_n(x))$$

PART 2D

$$a = 0, b = 0, c = 0, d = -1, s = 2, r = 1$$

$$p = \frac{1}{s}\sqrt{(\frac{1-a}{2})^2 - c} = \frac{1}{4}$$

$$\mu = \frac{\sqrt{d}}{s} = \frac{\sqrt{-1}}{2} = \frac{i}{2}$$

$$y(x) = \sqrt{x}(C_1 I_p(\frac{x^2}{2}) + C_2 I_{-p}(\frac{x^2}{2}))$$

Problem 3

SOLUTION
Given

$$\frac{d}{dr}[(2\pi t_d k)\frac{dT}{d}] - (4\pi r h_T)(T - T_\infty) = 0$$

$$2\pi r t_d k\frac{d^2T}{d^2} + 2\pi t_d k\frac{dT}{dr} - 4\pi r h_T(T - T_\infty) = 0$$

Let $\theta = T - T_\infty$. Note $\frac{dT}{dr} = \frac{d\theta}{dr}$ and $\frac{d^2T}{dr^2} = \frac{d^2\theta}{dr^2}$.

$$r\frac{d^2\theta}{dr^2} + \frac{d\theta}{dr} - \frac{2h_T r}{t_d k}\theta = 0$$

$$r^2\frac{d^2\theta}{dr^2} + r\frac{d\theta}{dr} - \frac{2h_T r^2}{t_d k}\theta = 0$$

Bessel Equation! From the formulas used in question 2 above,

$$a = 1, b = 0, c = 0, d = \frac{-2h_T}{t_d k}, r = 1, s = 1$$

$$p = \frac{1}{s}\sqrt{(\frac{1-a}{2})^2 - c} \rightarrow p = 0$$

$$\mu = \frac{\sqrt{d}}{s} = \frac{\sqrt{\frac{-2h_T}{t_d k}}}{s} = \pm i3.0928 \times 10^3$$

$$T(r) = \sqrt{r}Z_0(\mu r)$$

$$Z_0(\mu r) = C_1 I_0(\mu r) + C_2 K_0(\mu r)$$

$$T(r) = \sqrt{r}(C_1 I_0(\mu r) + C_2 K_0(\mu r))$$

$$\frac{dT}{dr} = \frac{1}{2\sqrt{r}}(C_1 I_0(\mu r) + C_2 K_0(\mu r)) + \sqrt{r}(\mu C_1 I_1(\mu r) - \mu C_2 K_1(\mu r))$$

See Wolfram alpha website for derivative formulae for Bessel Functions: https://functions.wolfram.com/Bessel-TypeFunctions/BesselI/20/01/02/

From the given boundary conditions at $r = r_l$, we know

$$\frac{dT}{dr}|_{r_l} = \frac{1.4}{-2\pi r_l t_d k} = -7.1784 \times 10^4 \frac{K}{m}$$

$$\frac{dT}{dr}|_{r_l} = \frac{1}{2\sqrt{r_l}}(C_1 I_0(\mu r_l) + C_2 K_0(\mu r_l)) + \sqrt{r_l}(\mu C_1 I_1(\mu r_l) - \mu C_2 K_1(\mu r_l))$$

$$\frac{dT}{dr}|_{r_l} = C_1(\frac{1}{2\sqrt{r_l}} I_0(\mu r_l) + \sqrt{r_l}\mu I_1(\mu r_l)) + C_2(\frac{1}{2\sqrt{r_l}} K_0(\mu r_l) - \sqrt{r_l}\mu K_1(\mu r_l))$$

$$C_2 = \frac{\frac{dT}{dr}|_{r_l} - C_1(\frac{1}{2\sqrt{r_l}} I_0(\mu r_l) + \sqrt{r_l}\mu I_1(\mu r_l))}{(\frac{1}{2\sqrt{r_l}} K_0(\mu r_l) - \sqrt{r_l}\mu K_1(\mu r_l))}$$

Considering boundary conditions at $r = r_d$, $T - T_\infty = 0 \rightarrow \theta = 0$. Therefore,

$$\frac{d}{dr}(2\pi r t_d k \frac{dT}{dr}) - 0 = 0 \rightarrow \frac{dT}{dr}|_{r_d} = 0$$

$$\frac{dT}{dr}|_{r_d} = \frac{1}{2\sqrt{r_d}}(C_1 I_0(\mu r_d) + C_2 K_0(\mu r_d)) + \sqrt{r_d}(\mu C_1 I_1(\mu r_d) - \mu C_2 K_1(\mu r_d)) = 0$$

$$\frac{1}{2\sqrt{r_d}}(C_1 I_0(\mu r_d) + (\frac{\frac{dT}{dr}|_{r_l} - C_1(\frac{1}{2\sqrt{r_l}} I_0(\mu r_l) + \sqrt{r_l}\mu I_1(\mu r_l))}{(\frac{1}{2\sqrt{r_l}} K_0(\mu r_l) - \sqrt{r_l}\mu K_1(\mu r_l))}) K_0(\mu r_d)) + ...$$

$$... + \sqrt{r_d}(\mu C_1 I_1(\mu r_d) - (\frac{\frac{dT}{dr}|_{r_l} - C_1(\frac{1}{2\sqrt{r_l}} I_0(\mu r_l) + \sqrt{r_l}\mu I_1(\mu r_l))}{(\frac{1}{2\sqrt{r_l}} K_0(\mu r_l) - \sqrt{r_l}\mu K_1(\mu r_l))}) \mu K_1(\mu r_d)) = 0$$

Yikes...

OK, new plan $\rightarrow$ let $C_1 = -T_\infty = -300$, for sanity. Then $C_2 = 24$.

$$T(r) = \sqrt{r}(C_1 I_0(\mu r) + C_2 K_0(\mu r))$$

This plot is provided below.

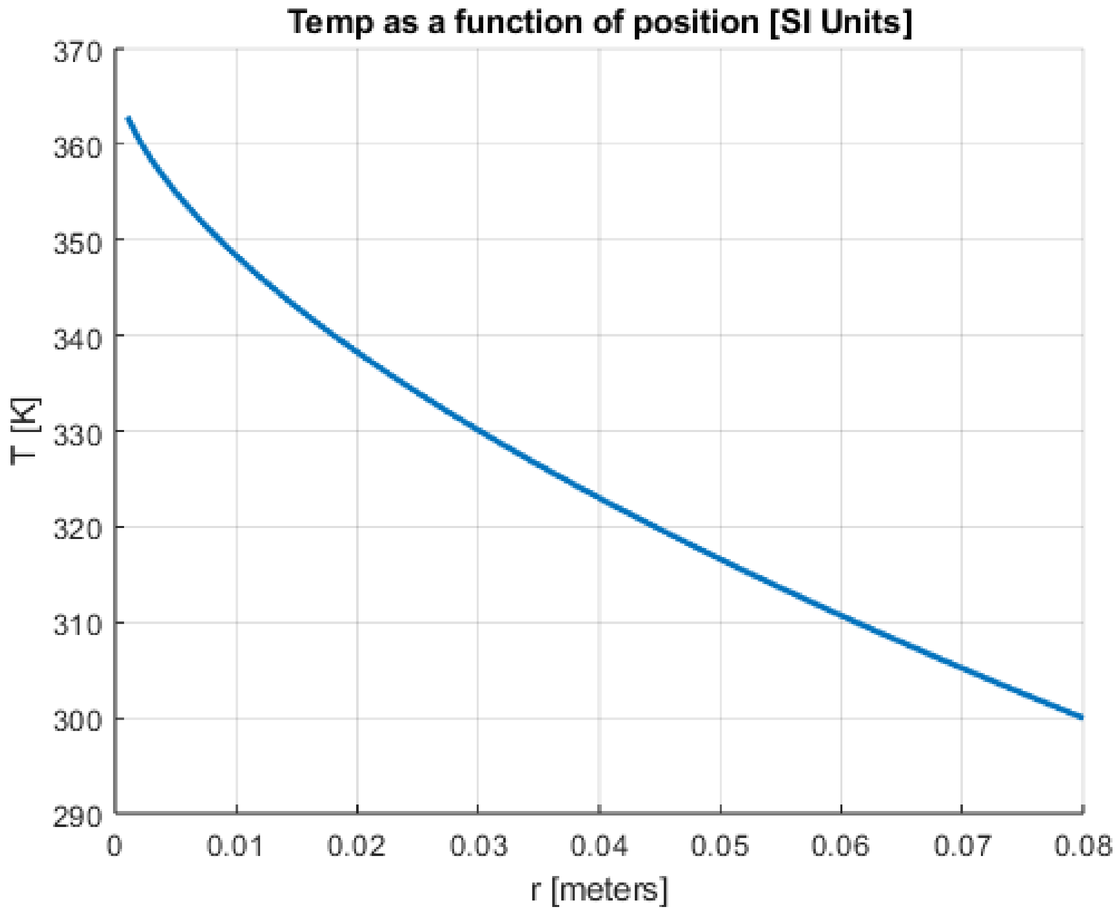

Figure 5: Problem 3

Problem 4

SOLUTION
Given

$$J_p(x) = \Sigma_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{k!(k+p)!}$$

WTS

$$\frac{d}{dx}(J_p(\alpha x)) = \alpha J_{p-1}(\alpha x) - \frac{p}{x} J_p(\alpha x)$$

Looking at the LHS,

$$\frac{d}{dx} J_p(\alpha x) = \Sigma_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (2k+p) x^{2k+p-1}$$

Now looking at the first term of the RHS,

$$\alpha J_{p-1}(\alpha x) = \alpha \Sigma_{k=0}^{\infty} \frac{(-1)^k (\frac{\alpha x}{2})^{2k+p-1}}{k!(k+p-1)!} = \Sigma_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p-1)!} (\alpha)^{2k+p} (\frac{x}{2})^{2k+p-1}$$

Note that $(k+p-1)! = \frac{(k+p)!}{(k+p)}$. Substituting in, we have

$$\alpha J_{p-1}(\alpha x) = \alpha \Sigma_{k=0}^{\infty} \frac{(-1)^k (\frac{\alpha x}{2})^{2k+p-1}}{k!(k+p-1)!} = \Sigma_{k=0}^{\infty} \frac{(-1)^k (k+p)}{k!(k+p)!} (\alpha^{2k+p}) (\frac{1}{2})^{2k+p-1} (x^{2k+p-1})$$

$$\alpha J_{p-1}(\alpha x) = \Sigma_{k=0}^{\infty} \frac{(-1)^k (k+p)}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (\frac{1}{2})^{-1} (x^{2k+p-1})$$

$$\alpha J_{p-1}(\alpha x) = (2) \Sigma_{k=0}^{\infty} \frac{(-1)^k (k+p)}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (x^{2k+p-1})$$

$$\alpha J_{p-1}(\alpha x) = \Sigma_{k=0}^{\infty} (2k+2p) [\frac{(-1)^k}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (x^{2k+p-1})]$$

For the second term of the RHS,

$$\frac{p}{x} J_p(\alpha x) = \frac{p}{x} \Sigma_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!} (\frac{\alpha x}{2})^{2k+p}$$

$$\frac{p}{x} J_p(\alpha x) = p \Sigma_{k=0}^{\infty} \frac{(-1)^k}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (x)^{2k+p-1}$$

$$\frac{p}{x} J_p(\alpha x) = \Sigma_{k=0}^{\infty} p [\frac{(-1)^k}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (x)^{2k+p-1}]$$

Now, combining terms,

$$\alpha J_{p-1}(\alpha x) - \frac{p}{x} J_p(\alpha x) = \Sigma_{k=0}^{\infty} (2k+2p) [\frac{(-1)^k}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (x^{2k+p-1})] - \Sigma_{k=0}^{\infty} p [\frac{(-1)^k}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (x)^{2k+p-1}]$$

$$\alpha J_{p-1}(\alpha x) - \frac{p}{x} J_p(\alpha x) = \Sigma_{k=0}^{\infty} (2k+2p-p) [\frac{(-1)^k}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (x)^{2k+p-1}]$$

$$\alpha J_{p-1}(\alpha x) - \frac{p}{x} J_p(\alpha x) = \Sigma_{k=0}^{\infty} (2k+p) [\frac{(-1)^k}{k!(k+p)!} (\frac{\alpha}{2})^{2k+p} (x)^{2k+p-1}]$$

Which is exactly identical to $\frac{d}{dx}(J_p(\alpha x))$ from above. So we can say,

$$\frac{d}{dx}(J_p(\alpha x)) = \alpha J_{p-1}(\alpha x) - \frac{p}{x} J_p(\alpha x)$$