Numerical PDEs Homework #0

Problem 1: Finite differences Write a matlab code that uses finite differences to compute the first derivative of

$$f(x) = \exp(\sin(x))$$

on the interval $0 \leq x \leq 2\pi$. Use second-order centered differences on internal grid points and first-order one-sided differences at endpoints.

1. Show a graph of the exact analytic solution $f'(x)$ and your numerical solution $f_n'(x)$.
2. Show a relative error convergence plot by plotting the relative error on the y-axis and the grid size N on the x-axis. Measure relative error as

   $$Err = \frac{\|f'(x) - f_n'(x)\|_p}{\|f'(x)\|_p}.$$

   Plot the error using both the $p = \infty, 2$ norms in a log-log scale. The relative errors should be lines with slopes 1 and 3/2 respectively, i.e. it scales with $\frac{1}{N}, \frac{1}{N^{3/2}}$. Extra Credit: The 3/2 slope is strange, why is that happening?

Problem 2: Second-order derivatives Repeat problem 1 but compute the second derivative $f''(x)$. Assume periodicity and wrap around the grid instead of using one-sided finite differences.

Problem 3: ODE Consider the ODE

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \sin(x)\frac{\mathrm{d}u}{\mathrm{d}x} + u(x) = f(x)$$

defined on $0 \leq x \leq 5$. Solve the ODE using second-order finite difference methods with the Dirichlet conditions. Use $u(x) = \sin(x)$ to test your solution by finding boundary conditions and the necessary forcing function $f(x)$. Show an error plot that proves your method converges with second order.

Problem 4: Poisson Equation Use finite difference methods to solve the Poisson equation $\nabla^2 u = f(x, y)$ in the domain $x \in [0, 5], y \in [0, 5]$. Use the test solution $u(x, y) = \sin(x)\cos(y)$.

1. Apply Dirichlet conditions on all boundaries so that $u(x, y)$ satisfies the PDE and the boundary conditions, and show a plot of your solution and a second order convergence plot (similar to the first three problems).
2. Modify your code to apply Neumann conditions on all boundaries. **This won't work.** Why not? – both in terms of the linear algebra and in terms of the actual PDE we want to solve. Suggest a solution to the issue.