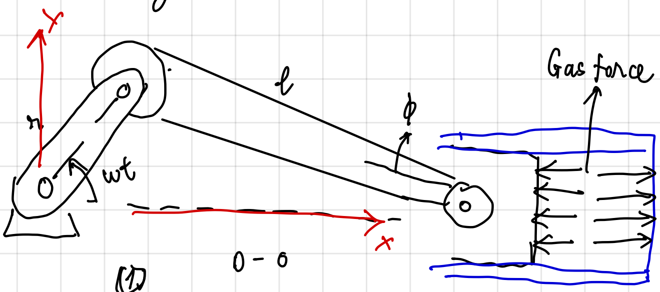
ENGINE DYNAMICS - Analysis- Calculation

1/ Piston - Cylinder Kinematic Analysis

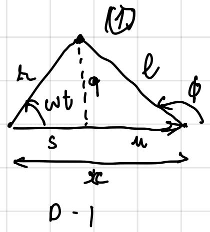
Consider the system below:



Let ω is the angular velocity of crank

Let consider the crank radius be r and the conrod be l. The angle of crank with X is θ and the conrod with X is ϕ. We have Kinematic diagrams (1)

We have: $\begin{cases} q = r\sin\theta = l\sin\phi \\ \theta = \omega t \end{cases}$

$\Rightarrow$ 

$$\Rightarrow \sin\phi = \frac{r}{l}\sin\theta = \frac{r}{l}\sin\omega t$$

We also have: $s = r\cos\theta = r\cos\omega t$; $u = l\cos\phi$

* Position of slider: $x = s + u = r\cos\omega t + l\cos\phi$

We have: $\cos\phi = 1 - \sin^2\phi = \sqrt{1 - \left(\frac{r}{l}\sin\omega t\right)^2}$

$\Rightarrow$ Equation of motion: $x = r\cos\omega t + l\sqrt{1 - \left(\frac{r}{l}\sin\omega t\right)^2}$. (Expression piston position)

Equation of velocity of piston: $\dot{x} = -r\omega\left[\sin\omega t + \frac{r}{2l}\frac{\sin 2\omega t}{\sqrt{1-\left(\frac{r}{l}\sin^2\omega t\right)^2}}\right]$

Acceleration of piston: $\ddot{x} = -r\omega^2\left\{\cos\omega t - \frac{r\left[l^2\left(1-2\cos^2\omega t\right) - r^2\sin^4\omega t\right]}{\left[l^2 - \left(r\sin\omega t\right)^2\right]^{3/2}}\right.$

* Using simplification (APPENDIX)

$$\boxed{\begin{aligned} x &\simeq l - \frac{r^2}{4l} + r\left(\cos\omega t + \frac{r}{4l}\cos 2\omega t\right) \\ \dot{x} &\simeq -r\omega\left(\sin\omega t + \frac{r}{2l}\sin 2\omega t\right) \\ \ddot{x} &\simeq -r\omega^2\left(\cos\omega t + \frac{r}{l}\cos 2\omega t\right) \end{aligned}}$$

* Assumption: Crank is pure rotation and piston is pure translation

2. Dynamic Analysis

The gas force is due to the gas pressure from exploding fuel - air mixture

Let F_g : gas force, A_p : area of piston, B : bore of cylinder (piston diameter) (Diagram 00)

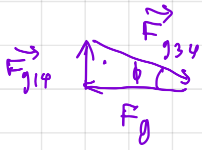
$$F_g = -P_g A_p \hat{i} \quad ; \quad A_p = \frac{\pi}{4} B^2$$

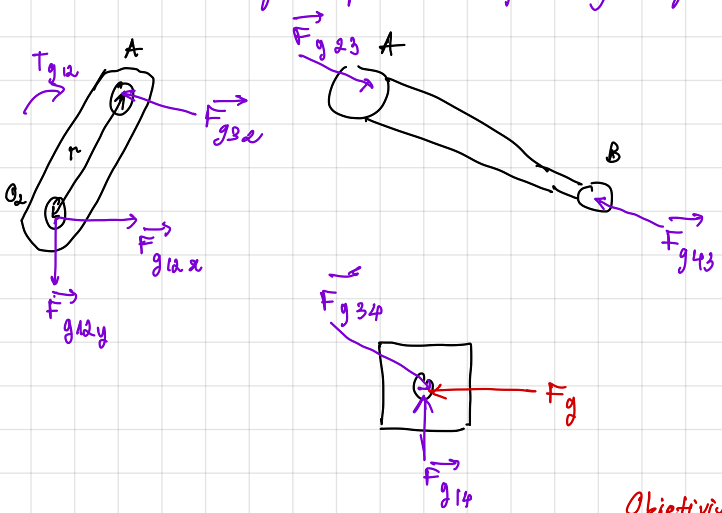
$$\Rightarrow \quad F_g = -\frac{\pi}{4} P_g B^2 \hat{i}$$

* The gas pressure in this is a function of crank angle ωt and is define by the thermodynamics of engine

* Gas torque is due to gas force acting at the moment arm about the crank center O_2

We can calculat gas torque as the following diagram





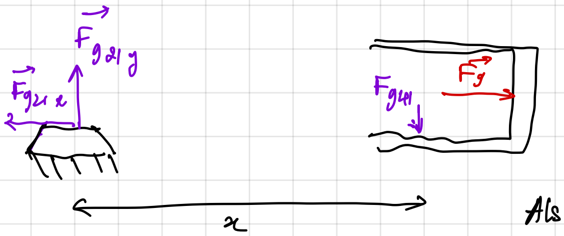
We have : $\vec{F}_{g14} = F_g \tan\phi \, \hat{j}$

$$\vec{F}_{g34} = F_g \hat{i} - F_g \tan\phi \, \hat{j}$$

Also : $\vec{F}_{g41} = -\vec{F}_{g14}$; $\vec{F}_{g43} = -\vec{F}_{g34}$

$\vec{F}_{g23} = -\vec{F}_{g32}$; $\vec{F}_{g23} = -\vec{F}_{g43}$

Objetivive : Find Torque : T_{g21} from $\vec{F}_g$



We have : $T_{g21} = R_A \times \vec{F}_{g32}$

Also $T_{g12} = F_{g41} \, x \, \hat{k}$

$\Rightarrow T_{g21} = F_{g41} \, x \, \hat{k}$: Gas torque expression :

$$\boxed{T_{g21} = F_g \tan\phi \, x \, \hat{k}}$$

We also have : $\tan\phi = \dfrac{r \sin\omega t}{l\cos\phi} = \dfrac{r\sin\omega t}{l\sqrt{1-\left(\frac{r}{l}\sin\omega t\right)^2}}$

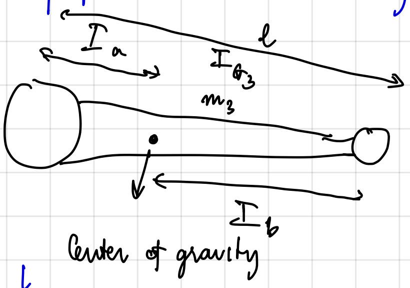
* Gas pressure curve (APPENDIX)

Pressure curve as a function of crank angle θ (rad)

$$P_g(\omega t) = a + b e^{-\left(\frac{\theta - 6.946}{1.068}\right)^2}$$ where $a = 0.1$ MPa and $b = 1.211$ MPa

(This is a typical case of 4stroke engine)

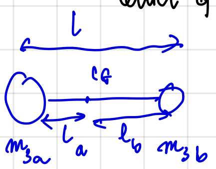
3. Equipvalent masses and dynamics model of conrod



We have: $m_p + m_t = m_3$

$$m_p l_p = m_t l_t$$

$$m_p l_p^2 + m_t l_t^2 = I_{G_3}$$



$$\Rightarrow \quad m_p = m_3 \frac{l_b}{l_p + l_b} \quad ; \quad m_b = m_3 \frac{l_p}{l_p + l_b}$$

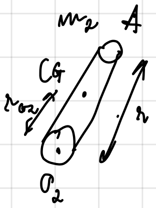
$$\Rightarrow \quad m_3 \frac{l_b}{l_b + l_p} l_p^2 + m_3 \frac{l_p}{l_p + l_b} l_b^2 = I_{G3} = m_3 l_p l_b$$

$$\Rightarrow \quad l_p = \frac{I_{G3}}{m_3 l_b}$$

* Objective : We build a dynamic model of crank and conrod to the simpify version

$\Rightarrow$ Equipvalent masses: $m_{3a} = m_3 \frac{l_b}{l_a + l_b}$; $m_{3b} = m_3 \frac{l_a}{l_a + l_b}$

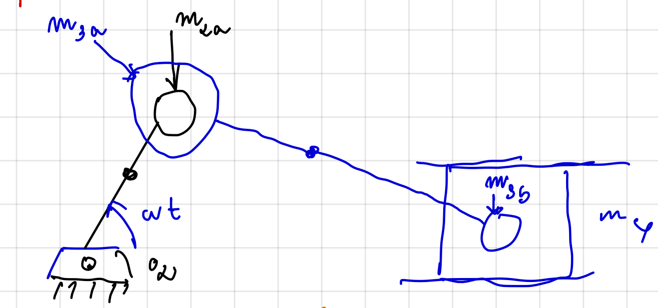
Same with crank:

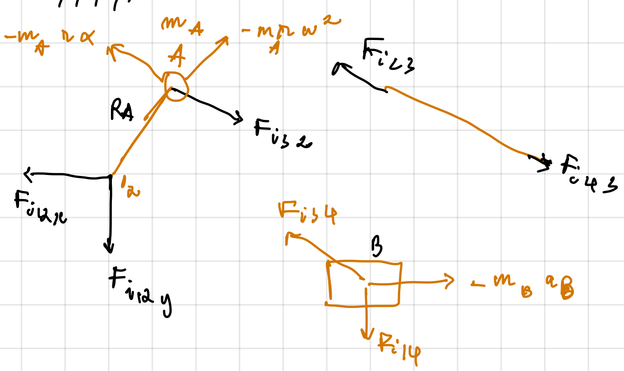


$$m_2 = m_{2a} + m_{2O_2}$$

$$m_{2a} r = m_2 r_{G2} \quad ; \quad m_{2a} = m_2 \frac{r_{G2}}{r}$$

* DYNAMIC MODEL OF ENGINE





* Inertia and shaking force

We have: $R_A = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}$

$$a_A = -r\omega^2 \cos\omega t\,\hat{i} - r\omega^2 \sin\omega t\,\hat{j}$$

Inertia force : $F_i = -m_A a_A - m_B a_B$

$\Rightarrow \quad F_{ix} = -m_A \left(-r\omega^2 \cos\omega t\right) - m_B \ddot{x}$

$F_{iy} = -m_A \left(-r\omega^2 \sin\omega t\right)$

* Shaking force is sum of all force acting on ground plane

$$F_S = F_{21} + F_{41} + F_i$$

$$F_{sx} = m_A \left(r\omega^2 \cos\omega t\right) + \overbrace{m_B \left[r\omega^2 \left(\cos\omega t + \frac{r}{l} \cos 2\omega t\right)\right]}^{m_B \ddot{x}}$$

$$F_{sy} = m_A \left(r\omega^2 \sin\omega t\right)$$

4. Inertia and Shaking Torque

We have : $T_{i21} = -F_{i14}\, x\, \hat{k} \simeq m_B \ddot{x} \tan\phi\, x\, \hat{k}$

Alternate : $T_{i21} = \frac{1}{2} m_B r^2 \omega^2 \left(\frac{r}{2l} \sin\omega t - \sin 2\omega t - \frac{3r}{2l} \sin 3\omega t\right) \hat{k}$

5. Total engine force :

$$T_{total} = T_g + T_i$$

6. Summary :

Equation of motion : $x = r\cos\omega t + l\sqrt{1 - \left(\frac{r}{l} \sin\omega t\right)^2}$

Equations of velocity : $\dot{x}$; acceleration : $\ddot{x}$

Equation of pressure curve : $P_g = a + be^{-\left(\frac{\theta - 6.946}{1.068}\right)^2}$ $(a = 0.1,\ b = 1.211\ MPa)$

Gas force : $\vec{F}_g = -P_g A_p \hat{i}$; $A_p = \frac{\pi}{4} B^2$

Gas torque : $T_g = F_g \tan\phi\, x\, \hat{k}$; $\tan\phi = \frac{r\sin\omega t}{l\cos\phi}$;
(Shaking force)

Inertia force : $F_i = m_A r\omega^2 \cos\omega t + m_B \ddot{x}$ (where m_A, m_B in dynamic model)

Inertia Torque : $\vec{T}_i = m_B \ddot{x} \tan\phi\, x\, \hat{k}$

Engine torque : $T_{total} = T_g + T_i$