# TRINITY COLLEGE DUBLIN
# THE UNIVERSITY OF DUBLIN

## School of Mathematics

**JF Mathematics**
**JF Theoretical Physics**
**JF TSM Mathematics**

**Trinity Term 2015**

**MA1132 — Advanced Calculus**

**Thursday, May 14** **Luce Upper** **14.00 — 16.00**

**Dr. Sergey Frolov**

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**Instructions to Candidates:**

ATTEMPT FOUR QUESTIONS

**Materials Permitted for this Examination:**

Formulae and Tables tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

**You may not start this examination until you are instructed to do so by the Invigilator.**

Each question is worth 20 marks. Show the details of your work.

1. The equations of motion of a system of $n$ particles are given by

$$m_i\ddot{x}_i = -\frac{\partial U(x_1,\dots,x_n)}{\partial x_i}\,, \quad \ddot{x}_i = \frac{d^2x_i}{dt^2}\,, \quad i = 1,2,\dots,n\,,$$

where $m_i$ is the mass and $x_i$ is the coordinate of the $i$-th particle, and $U(x_1,\dots,x_n)$ is the potential energy of the system.

Consider a system of $n$ coupled anharmonic oscillators with the potential

$$U(x_1,\dots,x_n) = \sum_{i=1}^{n-1}\frac{\kappa}{2}(x_{i+1}-x_i)^2 + \sum_{i=1}^{n}\frac{\lambda}{4}x_i^4\,.$$

(a) 3 marks. Find the equations of motion of the first particle ($x_1$).

(b) 3 marks. Find the equations of motion of the second particle ($x_2$).

(c) 3 marks. Find the equations of motion of the last particle ($x_n$).

(d) 8 marks. Find the equations of motion of the $i$-th particle ($x_i$) for $1 < i < n$.

(e) 3 marks. Write the equations of motion of the $i$-th particle ($x_i$) for $1 \le i \le n$ by using the Kronecker delta $\delta_{ij}$.

2. Consider the surface

$$z = f(x,y) = \ln\left(\frac{1}{2}e^{2/3}\sqrt[3]{12\sin(x-2y)+8y^2-x^3-6x^2y+32}\right)\,.$$

(a) 9 marks. Find an equation for the tangent plane to the surface at the point $P = (2,1,z_0)$ where $z_0 = f(2,1)$.

(b) 3 marks. Find points of intersection of the tangent plane with the $x$-, $y$- and $z$-axes.

(c) 2 marks. Sketch the tangent plane, and show the point $P = (2,1,z_0)$ on it.

(d) 4 marks. Find parametric equations for the normal line to the surface at the point $P = (2,1,z_0)$.

(e) 2 marks. Sketch the normal line to the surface at the point $P = (2,1,z_0)$.

3. Consider the intersection of the surfaces

$$z = \lambda x + \mu y + h\,, \quad \lambda > 0\,,\ \mu > 0\,,\ h > 0\,, \quad \text{and} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

   (a) 1 mark. What is the surface $z = \lambda x + \mu y + h$ ?

   (b) 1 mark. What is the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ?

   (c) 2 mark. Sketch the surfaces for $a = b = \lambda = \mu = h = 1$.

   (d) 12 mark. Find the coordinates of the point on the intersection which has the maximum $z$-coordinate.

   (e) 4 mark. Find the coordinates of the point on the intersection which has the minimum $z$-coordinate.

4. Consider the integral

$$\int_{-5}^{3} \int_{2-\sqrt{15-2y-y^2}}^{2+\sqrt{15-2y-y^2}} f(x,y)\,dx\,dy\,.$$

   (a) 6 marks. Determine and sketch the integration region $R$.

   (b) 3 marks. Reverse the order of integration.

   (c) 11 marks. Compute the integral if

$$f(x,y) = x\,y\,.$$

5. (a) 1 mark. Express rectangular coordinates in terms of cylindrical coordinates, and draw the corresponding picture.

   (b) Consider the solid $G$ bounded above by the surface $x^2 + y^2 + 2z = 15$ and below by the surface $z = \sqrt{x^2 + y^2}$.

      i. 1 mark. What is the surface $x^2 + y^2 + 2z = 15$ ?

      ii. 1 mark. What is the surface $z = \sqrt{x^2 + y^2}$ ?

      iii. 2 marks. Sketch the solid $G$, and its projection onto the $xy$-plane.

      iv. 5 marks. Use triple integral and cylindrical coordinates to compute the volume $V$ of the solid $G$.

v. 7 marks. Use triple integral and cylindrical coordinates to find the mass $M$ of the solid $G$ if its density is

$$\delta(x,y,z) = \frac{1 - e^{-\sqrt{x^2+y^2}}}{x^2+y^2+5\sqrt{x^2+y^2}}\,.$$

vi. 3 marks. What is the density of the solid $G$ at the origin: $\delta(0,0,0) = ?$

# Useful Formulae

1. Let $\mathbf{r}(t)$ be a vector function with values in $\mathbb{R}^3$: $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$.
   (a) The unit normal vector is $\mathbf{N}(t) = \frac{\frac{d\mathbf{T}}{dt}}{|\frac{d\mathbf{T}}{dt}|}$.
   (b) The unit binormal vector is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.
   (c) The curvature of $C$ is $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.

2. Let $\sigma$ be a surface in $\mathbb{R}^3$: $z = f(x, y)$
   (a) The slope $k_x$ of the surface in the $x$-direction at $(x_0, y_0)$ is $k_x = \frac{\partial z}{\partial x}(x_0, y_0)$.
   (b) The slope $k_y$ of the surface in the $y$-direction at $(x_0, y_0)$ is $k_y = \frac{\partial z}{\partial y}(x_0, y_0)$.
   (c) The equation for the tangent plane to the surface at the point $P = (x_0, y_0, z_0)$ is
   $z = z_0 + k_x(x - x_0) + k_y(y - y_0)$.
   (d) Parametric equations for the normal line to the surface at $P = (x_0, y_0, z_0)$ are
   $\mathbf{r}(t) = \mathbf{r}_0 + t(-k_x\mathbf{i} - k_y\mathbf{j} + \mathbf{k})$, $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$.
   (e) The mass of the lamina with the density $\delta(x, y, z)$ that is the portion of the surface that is above a region $R$ in the $xy$-plane is
   $M = \iint_\sigma \delta(x, y, z)\, dS = \iint_R \delta(x, y, z) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$.

3. Let $R$ be a plain lamina with density $\delta(x, y)$.
   (a) Its mass is equal to $M = \iint_R \delta(x, y)\, dA$.
   (b) The $x$-coordinate of its centre of gravity is equal to $x_{cg} = \frac{1}{M} \iint_R x\, \delta(x, y)\, dA$.
   (c) The $y$-coordinate of its centre of gravity is equal to $y_{cg} = \frac{1}{M} \iint_R y\, \delta(x, y)\, dA$.

4. The volume element in spherical coordinates is $dV = r^2 \sin\phi\, dr\, d\phi\, d\theta$.

5. Let a region $R_{xy}$ in the $xy$-plane be mapped to a region $R_{uv}$ in the $uv$-plane under the change of variables $u = u(x, y)$, $v = v(x, y)$.
   (a) The magnitude of the Jacobian of the change is $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right|$.
   (b) The integral over $R_{xy}$ is $\iint_{R_{xy}} f(x, y)\, dA_{xy} = \iint_{R_{uv}} f\left(x(u, v), y(u, v)\right) \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} dA_{uv}$.