WAVES

PY1T10

SECTION 1: MECHANICAL WAVES

Mechanical Wave = a disturbance that travels/propagates through some material from one region to another

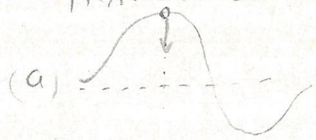
WAVE PULSE = a disturbance that moves through a medium

Waves CAN HAVE 3 FORMS:

(a) TRANSVERSE (eg: on a string)
(b) Longtitudinal (eg: in a FLUID)
(c) ON THE surface of a FLUID (BOTH)

TRANSVERSE

(a)



LONGTITUDINAL

(b)

(c) BOTH (ON SURFACE OF FLUID)

Main Observations on Waves:

- Every element of the wave is oscillating
- Medium is not flowing in transverse direction
  - ONLY THE DISTURBANCE IS
- To describe a wave you must describe an oscillation

Describing OSCILLATIONS:

- need a relationship between net force and displacement in the body
- The wave has a <u>restoring force</u> towards an equilibrium position.

The simplest kind of OSCILLATION OCCURS when the restoring force is <u>proportional</u> to displacement from the equilibrium position

- This kind of OSCILLATION IS CALLED *Simple HARMONIC MOTION*

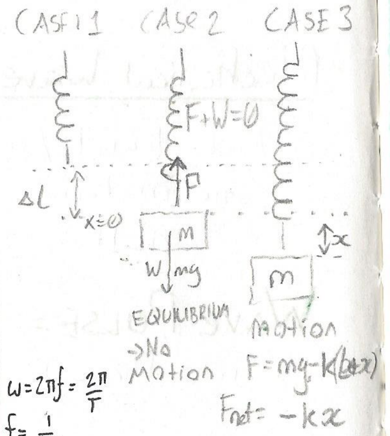
Describing Simple Harmonic Motion

Hooke's Law: $F = -kx \Rightarrow \boxed{a = -\frac{kx}{m}}$

this is a differential equation: $\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$

- Let us DEFINE some terms:

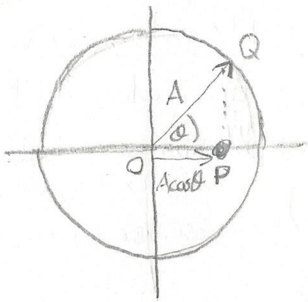
A = Amplitude = maximum magnitude of displacement
T = Period = time to complete 1 cycle
ω = Angular frequency = 2π times the frequency
f = frequency = number of cycles in a unit of time



$\omega = 2\pi f = \frac{2\pi}{T}$

$f = \frac{1}{T}$

- To explain properties of S.H.M. we must express displacement x as a function of time t
- We can find this by noting that $x(t)$ is related to uniform circular motion:
  - SHM is THE PROJECTION OF U.C.M. onto a diameter.



- We call the rotating vector $\cancel{O}Q$ a PHASOR
- We note that it can also be described using the Euler Identity: $Ae^{i\theta} = A\cos\theta + iA\sin\theta$
  - The position p has $x = A\cos\theta$
  - The body feels a centripetal force $F = \frac{mv^2}{r}$
  - $-a_x = -a\cos\theta$

$-ma_x = -ma\cos\theta$

$F_x = -\frac{mv^2}{r}\cos\theta$ but $v = \frac{2\pi A}{T}$

$F_x = -\frac{m}{r}\frac{4\pi r^2}{T^2}\cos\theta$ but $x = r\cos\theta$

$\boxed{F_x = -\frac{m}{T^2}4\pi^2 x = -kx}$

$k = \frac{m4\pi^2}{T^2}$

$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$

- One can also look at the differential equation. $\ddot{x} = -\frac{k}{m}x$

IF we choose/guess $x = A\cos\omega t$

$\dot{x} = -A\omega\sin\omega t$

$\ddot{x} = -A\omega^2\cos\omega t$

THIS gives us that $\ddot{x} = -\omega^2 x$

And so, $\omega^2 = \frac{k}{m}$ $\boxed{\omega = \sqrt{\frac{k}{m}}}$

But $\omega = \frac{2\pi}{T} \Rightarrow T = 2\pi\omega = 2\pi\sqrt{\frac{m}{k}}$ as before

- THIS SHOWS that period does not depend on A
- to account ~~for~~ when the body does not begin at it's maximum, one can add in $\boxed{+\phi}$, the phase difference/angle
- WE CAN FIND ϕ when $t=0$ IF we HAVE the INITIAL DISPLACEMENT & VELOCITY

$x_0 = A\cos(\omega(0) + \varphi)$

$\dot{x}_0 = -A\sin(\omega(0) + \varphi)$

$\frac{\dot{x}_0}{x_0} = \frac{-A\sin(\varphi)}{-A\cos(\varphi)} = \tan\varphi$

so $\varphi = \tan^{-1}\left(\frac{v_0}{x_0}\right)$

$x = A\cos(\omega t + \phi)$

Energy in S.H.M.

- KINETIC ENERGY: $\frac{1}{2}mv^2$
- POTENTIAL ENERGY: $-\int_0^x F\,dx = \int_0^x kx\,dx = \frac{1}{2}kx^2$

$E = K.E + P.E. =$ constant

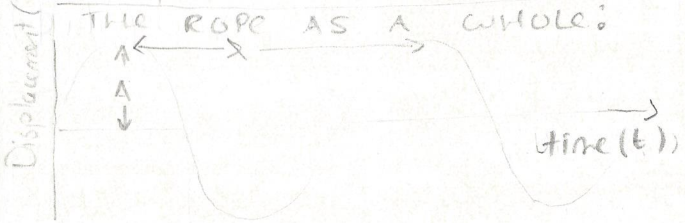
$= \frac{1}{2}m\left[-A\omega\sin(\omega t+\varphi)\right]^2 + \frac{1}{2}k\left[A\cos(\omega t+\varphi)\right]^2$

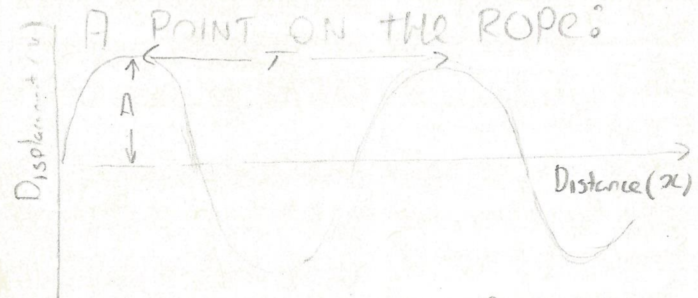
$= \frac{1}{2}mA^2\left(\frac{k}{m}\right)\sin^2(\omega t+\varphi) + \frac{1}{2}kA^2\cos^2(\omega t+\varphi) = \frac{1}{2}A^2k$

$=$

Describing Periodic Transverse Waves

Compare two GRAPHS:





1) $y = A\cos\theta$ but $\frac{\theta}{2\pi} = \frac{t}{T}$

$y(t) = A\cos\frac{2\pi t}{T} = A\cos\omega t$

2) $y = A\cos\theta$ but $\frac{\theta}{2\pi} = \frac{x}{\lambda}$ .. let $k = \frac{2\pi}{\lambda}$

$y(x) = A\cos\left(\frac{2\pi}{\lambda}x\right) = A\cos(kx)$

VELOCITY $= \frac{\lambda}{T} = f\lambda$

MATHEMATICALLY, we get

$y(x,0) = A\cos\left(\frac{2\pi}{\lambda}x\right)$

but $x(t) = vt$

so choosing any $x = x - vt$

$y(x,t) = A\cos\left(\frac{2\pi}{\lambda}(x - vt)\right)$

$$\boxed{y(x,t) = A\cos(kx - \omega t)}$$

FROM this we get

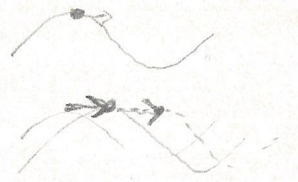
$v_y = \frac{\delta(y(x,t))}{dt} = -A\omega\cos(kx - \omega t)$

$a_y = \frac{\delta(\cancel{v}(x,t))}{\delta t} = -A\omega^2\sin(kx - \omega t)$

What about Phase Velocity?

- assume the phase is constant (track a point of the wave)

$y(x,t) = A\cos(\underbrace{kx - \omega t}_{\text{CONSTANT}})$



Then $kx - \omega t = \text{constant}$

$x = \frac{\text{constant} + \omega t}{k}$

$\dot{x} = \frac{\omega}{k} = \frac{\omega}{\frac{2\pi}{\lambda}} = \frac{2\pi f \cdot \lambda}{2\pi} = f\lambda$

- IN General:

$V = \sqrt{\frac{\text{restoring force}}{\text{resisting inertia}}}$

Wave Speed in a String

Method A

$\underbrace{I = F_y t}_{\text{impulse}} = mv_y$

$\frac{F_y t}{F t} = \frac{mv_y}{mv}$

$F_y t = F\frac{v_y}{v}t = mv_y$

$m = \mu l = \mu vt$

$\Rightarrow \frac{F\cancel{t}}{v} = \mu v\cancel{t}$

$\frac{F}{\mu} = v^2$

$v = \sqrt{\frac{F}{\mu}}$

Method B

$F_{y_1} = -\left(\frac{\delta y}{\delta x}\right)_x F$

$F_{y_2} = \left(\frac{\delta y}{\delta x}\right)_{x+\Delta x} F$

$F_{net_y} = F_{y_1} + F_{y_2}$

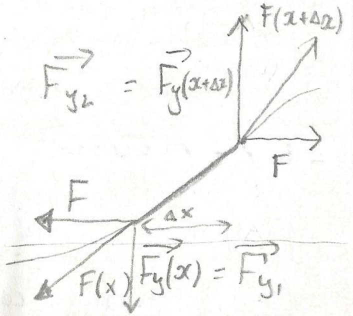
$F_y = F\left[\left(\frac{\delta y}{\delta x}\right)_{x+\Delta x} - \left(\frac{\delta y}{\delta x}\right)_x\right]$

But $F_y = ma = (\mu\Delta x)\frac{\delta^2 y}{\delta t^2}$

$\Rightarrow \mu\Delta x\left(\frac{\delta^2 y}{\delta t^2}\right) = F\left[\left(\frac{\delta y}{\delta x}\right)_1 - \left(\frac{\delta y}{\delta x}\right)_2\right]$

$\frac{\mu}{F}\left(\frac{\delta^2 y}{\delta t^2}\right) = \frac{\left(\frac{\delta y}{\delta x}\right)_1 - \left(\frac{\delta y}{\delta x}\right)_2}{\Delta x}$

$\vec{F}_{y_2} = \vec{F}_{y(x+\Delta x)}$



SEGMENT OF LENGTH Δx

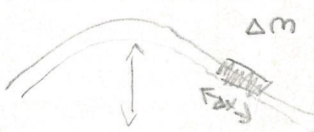
TAKING Lim $\Delta x \to 0$

$\frac{\mu}{F}\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 y}{\delta x^2}$

$\Rightarrow \frac{\delta x^2}{\delta t^2} = \frac{\mu}{F}\frac{\delta^2 y}{\delta^2 y}$

$v^2 = \frac{\mu}{F} \Rightarrow v = \sqrt{\frac{\mu}{F}}$

Energy in Wave Motion



- Consider a small "bit" of an ossilating string

○ $\Delta E_K = \frac{1}{2}(\Delta m)(V_y)^2$ but $\Delta m = \mu \Delta x$

$\Delta E_K = \frac{1}{2}\mu V_y^2 \Delta x$

○ $V_y = \frac{\delta y}{\delta t} = \omega A \sin[kx - \omega t]$

So we get

$\Delta E_K = \frac{\mu}{2}\omega^2 A^2 \sin^2[kx - \omega t]\Delta x$ W.L.O.G let $t = 0$ As 1

$E_K = \int_0^\lambda \frac{\mu}{2}\omega^2 A^2 \sin^2[kx]$ but $\sin^2 A = \frac{1}{2}[1 - \cos 2A]$

$E_K = \int_0^\lambda \mu \frac{1}{2}\omega^2 A^2 \left(\frac{1}{2} - \frac{\cos 2A}{2}\right) = \frac{1}{4}\omega^2 A^2 \mu [1 - \cos 2kx]_0^\lambda$

$\boxed{E_K = \frac{\omega^2 A^2 \mu \lambda}{4}}$ Kinetic Energy

○ similarly, $E_p = \frac{1}{2}ky^2$ but $T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow \frac{k}{m} = \left(\frac{2\pi}{T}\right)^2 = \omega^2$

$k = m\omega^2$

$\omega = \sqrt{\frac{k}{m}}$

$\Delta E_p = \frac{1}{2}\Delta m \omega^2 y^2$ but $\Delta m = \mu \Delta x$

$\Delta E_p = \frac{1}{2}\cancel{\Delta m}\,\omega^2 A^2 \cos^2[kx - \omega t]\mu \Delta x$

$E_p = \int_0^\lambda \frac{1}{2}\omega^2 A^2 \mu^2 \cos^2[kx - \omega t]$

$\boxed{E_p = \frac{\mu \omega^2 A^2 \lambda}{4}}$ Potential Energy

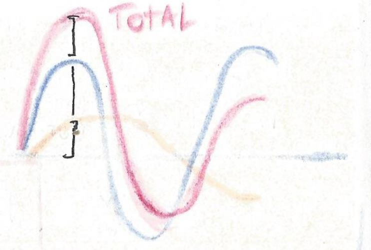
$\boxed{E_T = \frac{\mu \omega^2 A^2 \lambda}{2}}$ Total Energy $(E_p + E_k)$

$\boxed{\text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{1}{2}\mu \omega^2 A^2 v}$ as $\frac{\lambda}{T} = v$

Principle of Superposition & Standing Waves

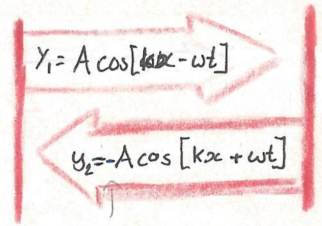
Principle of Superposition:

- The **total displacement** of a string at any point is the algebraic sum of the displacements due to the individual pulses at that point



Standing Wave

○ Forms when two waves of the same frequency & amplitude meet, going in opposite directions



- negative because the wave is inverted at a hard boundary

- Adding $y_1(x,t)$ and $y_2(x,t)$ gives:

$y = A\cos[kx - \omega t] - A\cos[kx - \omega t]$

$y = A(\cos kx \cos \omega t + \sin kx \sin \omega t) - A(\cos kx \cos \omega t - \sin kx \sin \omega t)$

$\boxed{y = 2A \sin kx \sin \omega t}$ new amplitude is $\boxed{2A}$ (Not A)

$y = 2A \sin kx \sin \omega t$

- This gives us nodes when $\sin kx = 0$ and antinodes when $\sin kx = 1$

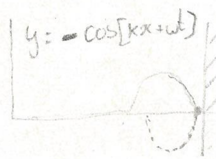
Antinodes: $kx = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$ but $k = \frac{2\pi}{\lambda}$

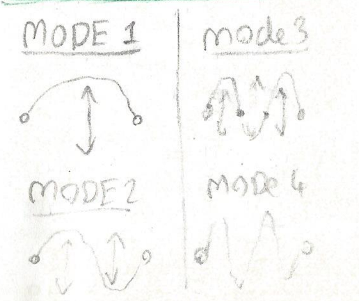
$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots \quad \frac{n\lambda}{2} + \frac{\lambda}{4}$

Nodes: $kx = 0, \pi, 2\pi \ldots$

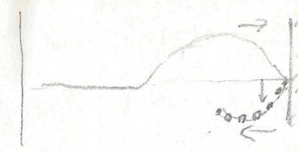
$x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2} \ldots \quad \frac{n\lambda}{2}$

- Note: This was for a hard boundary.



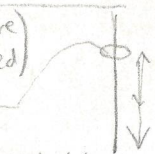


Hard Boundary



Soft Boundary

(Free to move - not inverted)



(Derivation equivalent)

Complex Standing Waves

- we know that the fundemental frequency is when the string vibrates on it's 1^{st} mode.

i.e: $\boxed{L = \frac{\lambda}{2}}$ BUT, $v = f\lambda$ AND $v = \sqrt{\frac{F_T}{M}}$

so $L = \frac{1}{2}\left(\frac{v}{f}\right) \Rightarrow \boxed{f_F = \frac{1}{2L}\sqrt{\frac{F_T}{M}}}$

- The other modes are an integer multiple of this frequency

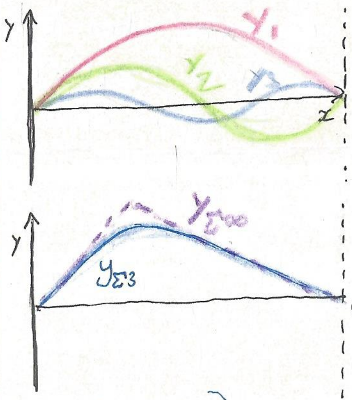
- HOWEVER, IN REAL-LIFE SITUATIONS, it is rare to get exactly one mode at once only. OFTEN, the higher modes interfering WITH EACHOTHER.

- This gives us a complex wave, given by a fourier series, THE SUM OF MANY SINUSOIDAL FUNCTIONS

- AN EXAMPLE OF THIS IS SHOWN ON THE RIGHT.

we get $y(x,t) = y_1(x,t) + y_2(x,t) + y_3(x,t) + \dots$

- THE EXAMPLE SHOWS AN APPROXIMATION OF THE PLUCKING OF A GUITAR STRING



$$\left.\begin{aligned} y_1 &= A \sin k_1 x \\ y_2 &= \frac{A}{2\sqrt{2}} \sin 2k_1 x \\ y_3 &= \frac{A}{9} \sin 3k_1 x \end{aligned}\right\} = y_{\Sigma 3} \approx y_{\Sigma}$$

SECTION 2: Sound Waves

- Sound is a longtitudinal wave in a medium. Frequencies of sound within the range of 20Hz to 20 000Hz are called in the "audible range", because the human ear is sensitive to them

$$y(x,t) = A\cos(kx - \omega t)$$

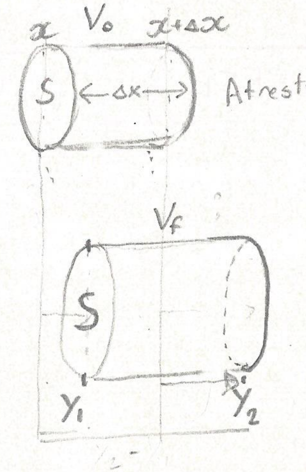
CAN ALSO BE USED TO DESCRIBE sound waves

- Sound can also be described in terms of *PREASSURE VARIATIONS*.

  - Let $p(x,t)$ be the instantaneous preasure variation (ΔP)

    i.e: TOTAL Preassure = NORMAL Preassure + $p(x,t)$

  - CONSIDER A CYLINDER w/ length Δx and area S FILLED W/ A FLUID:



THEN: $V_i = S\Delta x$ The initial volume/Avg Volume

- THE SOUND molecules who Average at x have a displacement $y_1 = y(x,t)$
- THE SOUND molecules who average at $x+\Delta x$ have a displacement $y_2 = y(x+\Delta x, t)$

Then ~~$V_f = S(y_2 - y_1 \Delta x)$~~

$$V_f = S \cdot (y_2 + \Delta x - y_1)$$

$$V_f = S \cdot (y_2 - y_1) + S\Delta x \qquad \text{But } \Delta V = V_f - V_i$$

$$\boxed{\Delta V = S(y_2 - y_1)} \qquad \text{Devide BOTH SIDES BY } V_i$$

$$\frac{\Delta V}{V_i} = \frac{\cancel{S} \cdot (y(x,t) - y(x+\Delta x, t))}{\cancel{S}\,\Delta x}$$

- TAKE THE LIMIT 4 $\Delta x \to 0$

$$\frac{dV}{V_i} = \frac{\delta(y(x,t))}{\delta x} \qquad \text{BUT } \Delta P = -B\frac{\Delta V}{V}$$

$$\Delta p = -B\frac{\delta y}{\delta x} = -B\frac{\delta(A\cos(kx - \omega t))}{\delta x}$$

$\Rightarrow$

$$\Rightarrow \boxed{p(x,t) = BkA\sin(kx - \omega t)}$$

$P \cdot V = \text{constant}$

$P(t) \cdot V(t) = PV$

$P(t) = PV \cdot \left(\frac{1}{V(t)}\right)$

$\frac{\delta P}{\delta V} = PV\left(\frac{-1}{V(t)^2}\right)$

$\Delta P = \frac{-PV}{V(t)^2}\Delta V$

Becomes $\rightarrow \boxed{\Delta P = \frac{-B\Delta V}{V}}$

where B = "BULK MODULUS" — a constant

Perception of a sound wave

we shall define

$$\text{Intensity} = \frac{\text{Power}}{\text{Area}} = \frac{P}{4\pi^2 r^2}$$ FOR SPHERICAL SOURCES

THEN we PERCIEVE:

PITCH as the frequency of the wave

Loudness as the preassure amplitude increases

timbre/QUALITY - depends on harmonics

ANOTHER FACTOR IS HOW IT begins (attack) and How it ends (decay)

noise - combination of all frequencies - not just HARMONICS

SPEED OF SOUND WAVES

V in general is $= \sqrt{\frac{\text{RESTORING FORCE}}{\text{INERTIA RESISTING FORCE}}}$

So we get $V = \sqrt{\frac{B}{\rho}}$ LEFT TO DERIVE

RATE OF ENERGY TRANSFER

① $E_K = \frac{1}{2}mv^2$ so, $\Delta E_K = \frac{1}{2}\Delta m\, v^2$

BUT $v = \frac{\delta\, A\cos(kx-\omega t)}{\delta t} = -A\omega\sin(kx-\omega t)$

$$v^2 = (A\omega\sin(kx-\omega t))^2$$

AND $\rho = \frac{\Delta m}{\Delta V} = \frac{\Delta m}{\underbrace{\text{Area}}_{S}\,\Delta x} \Rightarrow \Delta m = \rho S \Delta x$ Let Area = S

$\Rightarrow \Delta E_K = \frac{1}{2}\rho S \Delta x\, (\omega A\sin(kx-\omega t))^2$

$\Rightarrow$ Limit as $\Delta x \to 0$

$$\int dE_K = \int_0^{\lambda} \frac{1}{2}\rho A^2\omega^2 S \sin^2(kx-\omega t)\, dx$$

$$= \frac{1}{2}\rho A^2 S\omega^2 \int_0^{\lambda} \sin^2(kx)\, dx$$

$$\boxed{E_K = \frac{1}{4}\rho A^2\omega^2 S\lambda}$$ KINETIC ENERGY

SIMILARLY, we get

$$\boxed{E_P = \frac{1}{4}\rho S(\omega A)^2\lambda}$$ POTENTIAL ENERGY

$$\Rightarrow \boxed{E = \frac{1}{2}\rho S(\omega A)^2\lambda}$$ TOTAL ENERGY

$$\Rightarrow P = \frac{\Delta E}{\Delta t} = \frac{E_{TOTAL}}{T} = \frac{\frac{1}{2}\rho S(\omega A)^2\lambda}{T}$$

$$\boxed{P = \frac{1}{2}\rho S(\omega A)^2 v}$$ POWER

$$\Rightarrow I = \frac{\text{Power}}{\text{Area}} = \frac{P}{S} = \boxed{\frac{1}{2}\rho(\omega A)^2 V}$$ INTENSITY

Time-Averaged sound intensity gives similar result.

② ALTERNATIVELY, one could use that

$$I = \frac{\text{Power}}{\text{AREA}} = \frac{\text{Work}}{\text{Area} \times \text{Time}} = \frac{\text{Force} \times \text{Distance}}{\text{Area} \times \text{Time}} = \text{Preassure} \times \text{Velocity}$$

$$I = \frac{\Delta P}{\Delta S} = (\underline{BkA\sin(kx-\omega t)})(\omega A\sin(kx-\omega t))$$

$I(x,t) = \omega BkA^2\sin^2(kx-\omega t) \to$ Integral gives same result

Sound Intensity

Beats & Interference

- Like all waves, the principle of superposition applies, and we can hear them interfere.

  - when there are two waves of very close frequency, we can hear their interference as **BEATS**. suppose we have 2 waves:

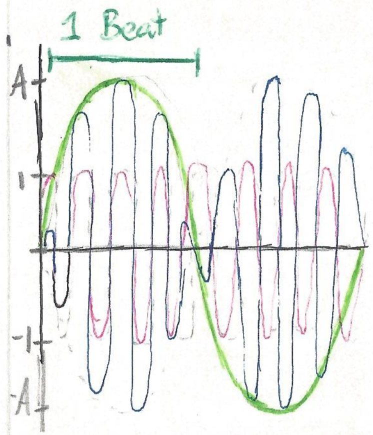
$$S_1 = A\cos\omega_1 t = A\cos(2\pi f_1 t)$$
$$S_2 = A\cos\omega_2 t = A\cos(2\pi f_2 t)$$

then $S(t) = S_1 + S_2 = A\left[\cos(2\pi f_1 t) + \cos(2\pi f_2 t)\right]$

BUT $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

THEN: $S(t) = 2A\cos\left(2\pi\left(\frac{f_1 t + f_2 t}{2}\right)\right)\cos\left(2\pi\left(\frac{f_1 t - f_2 t}{2}\right)\right)$

$$St = 2A\cos\underbrace{\left[2\pi\left(\frac{f_1+f_2}{2}\right)t\right]}_{\text{AVERAGE OF } f_1 \text{ and } f_2}\cos\underbrace{\left[2\pi\left(\frac{f_1-f_2}{2}\right)t\right]}_{\frac{1}{2}\text{ the difference}}$$



- BUT LOOKING AT THE DIAGRAM, 2 beats occur every PERIOD

$$T_{BEAT} = 2\left(\frac{f_1 - f_2}{2}\right) = |f_1 - f_2|$$

The Doppler Effect

Section 3: Light

- From maxwell's theory, we get the wave equation:

$$\boxed{\frac{\delta^2 E(x,t)}{\delta x^2} = \frac{1}{V^2}\frac{\delta^2 E(x,t)}{\delta t^2}}$$

From the theory we get $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3\times10^8 ms^{-1}$

This gives us that the equations:

- $E(x,t) = E_0 \cos(kx - \omega t)$ ELECTRIC PART
- $B(x,t) = B_0 \cos(kx - \omega t)$ MAGNETIC PART

ARE SOLUTIONS TO THE WAVE EQUATION

- The electromagnetic spectrum is very wide, and all the different wavelengths/frequencies can be used for different things:

- There are many models of light, with different levels of DIFFICULTY.
  - It is reccommended to use the easiest one that can get the job done out of
    - GEOMETRIC OPTICS for large lengths
      - DESCRIBES A RAY (for reflection, refraction, lenses)
    - PHYSICAL OPTICS
      - DESCRIBES A WAVE
    - PARTICLE APPROACH
      - LIGHT = Energy PACKAged in discrete bundles
    - QUANTUM ELECTRODYNAMICS
      - COMPREHENSIVE THEORY OF BOTH WAVE & PARTICLE

History

1600s - Newton
- PARTICLES (corpuscles)

1670 - Huygens
- waves

1770 - Young
- interference (2 slits)

1870 - Maxwell
- Electromagnetism

1900s - Einstein
- wave-PARTICLE DUALITY

EM SPECTRUM

10^{-12} m GAMMA

10^{-10} m X RAY

10^{-8} m ULTRA VIOLET

10^{-6} m VISIBLE

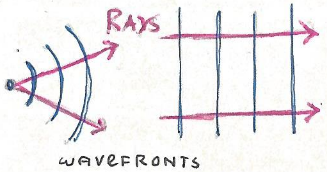
10^{-5} m INFRA RED

10^{-2} m MICROWAVE

10^{3} m RADIO

RAY THEORY

RAY = Imaginary line along the direction of travel of the wave perpendicular to the wave front

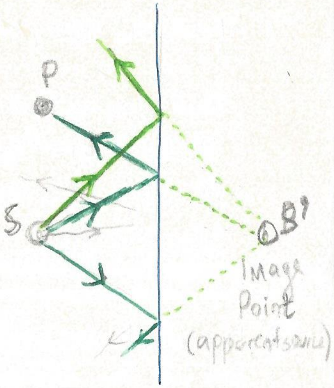


- USING THIS THEORY, we can discover:

The LAWS OF REFLECTION: $\boxed{\theta_i = \theta_r}$

① The incident ray, the reflected ray, and the normal are all in the same plane

② The angle of incidence = the angle of reflection



∘ why does this rule work?

IN 150.B.C. HERO OF ALEXANDRA theorised:

- *The path taken by the light going from the point S to P via the reflecting surface is the shortest one possible.*

∘ WHILE this is true, it doesn't explain refraction:

- As light enters a new medium, the speed changes. It is always ≤ to the speed in a pure vaccuum, ~~and given~~

THIS ratio is a constant: $n = \frac{c}{v}$ (c - speed in vaccum, v - speed in material)

- From this we can observe an *Empirical LAW*:

SNELL - DECARTES LAW OF REFRACTION:

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_t}{n_i} \iff \sin\theta_i\, n_i = \sin\theta_t\, n_t$$

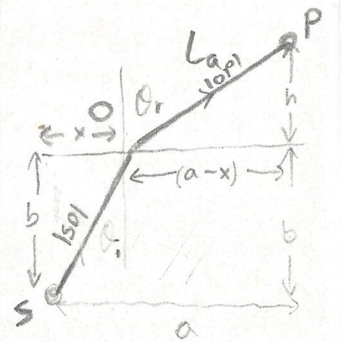
- This no longer is the shortest distance, but gives us that the Light takes the path of *LEAST TIME* (THE FERMAT PRINCIPLE)

∘ This can be further generalised to be the PATH OF LEAST ENERGY

TOTAL INTERNAL REFLECTION & FERMAT PRINCIPLE VERIFICATION

FERMAT PRINCIPLE: PATH TAKEN BY LIGHT IS OF least time.

∘ Suppose we have a point S and point P as shown on the right separated by a "barrier" between two mediums of different density – the path ~~takes~~ takes through the barrier at a point O



∘ time taken $t = \frac{|SO|}{V_g} + \frac{|OP|}{V_a}$

BUT $|SO| = \sqrt{x^2+b^2}$ & $|OP| = \sqrt{(a-x)^2 + h^2}$

so $t = \frac{\sqrt{x^2+b^2}}{V_g} + \frac{\sqrt{(a-x)^2+h^2}}{V_a}$

$$\Rightarrow \frac{dt}{dx} = \frac{\cancel{\frac{1}{2}}(x^2+b^2)^{-\frac{1}{2}}(\cancel{2}x)}{V_g} + \frac{\cancel{\frac{1}{2}}((a-x)^2+h^2)^{-\frac{1}{2}}\cancel{2}(a-x)(-1)}{V_a}$$

$$\frac{dt}{dx} = \frac{x}{V_g\sqrt{x^2+b^2}} + \frac{x-a}{V_a\sqrt{(a-x)^2+h^2}}$$

But: $\sin\theta_i = \frac{x}{|SO|} = \frac{x}{\sqrt{x^2+b^2}}$ & $\sin\theta_r = \frac{a-x}{|OP|} = \frac{a-x}{\sqrt{(x-a)^2+h^2}}$

$$\Rightarrow \frac{dt}{dx} = \frac{\sin\theta_i}{V_i} - \frac{\sin\theta_r}{V_r} = 0 \text{ when minimum}$$

$$\Rightarrow (c)\frac{\sin\theta_i}{V_i} = \frac{\sin\theta_r}{V_r}(c) \text{ times by } c$$

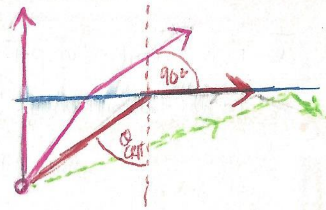
$$\Rightarrow n_i \sin\theta_i = n_r \sin\theta_r$$

- This shows us that the fermat principle is a good hypothesis

TOTAL INTERNAL REFLECTION

- When light is travelling from a denser to rarer medium, there is a point at which it cannot be refracted out, and instead, it gets reflected back in: this occurs at the *CRITICAL ANGLE*

when $\theta_r = 90°$ and $\theta_i = \theta_{CRIT}$



$n_i \sin 90° = n_r \sin\theta_{CRIT}$

$\sin^{-1}\left(\frac{n_i}{n_r}\right) = \theta_{CRIT}$

- Uses in FIBRE-OPTIC CABLES etc...

Dispersion & Polarisation

- WHITE LIGHT is a superposition of waves of ALL the different colours.

We know $n = \frac{c}{v}$ and $c = f\lambda$.

As Frequency does not change when changing medium:

$$n = \frac{f\lambda_0}{f\lambda} \Rightarrow \boxed{\lambda = \frac{\lambda_0}{n}} \text{ OR } \boxed{n = \frac{\lambda_0}{\lambda}}$$

- The speed of light in a substance also depends on wave length. IN general n decreases w/ increasing λ.
- DISPERSION depends on this difference, and causes WHITE LIGHT to SEPARATE INTO COLOURS OF LIGHT

POLARISATION = The restriction of movement of a transverse wave along a single PLANE

- when the electric field of a wave has only y-displacements, it is said to be y polarised
- UNPOLARISED "natural" light can be polarised using a polariser such as a POLAROID

MALUS' LAW: $I = I_0 \cos^2\phi$

- shows that $I \propto$ ~~amp~~ I_0 and depends squarely on the cosine of the angle between the polarised wave and polariser

- This is because $E_{TRANSVERSE} = E_0 \cos\phi$

and $I \propto E_0^2$ (FROM BEFORE, Power $\propto$ Amplitude^2)

so $I = I_0 \cos^2\phi$

- FOR UNIFORM UNPOLARISED LIGHT: we have I_0

To get the new intensity: $I_{Avg} = \frac{1}{\pi}\int_0^{\pi} I_0 \cos^2\phi \, d\phi = \frac{1}{2} I_0$

so, $\boxed{I = \frac{1}{2} I_0}$

SCATTERING

$I_{SCATTERING} \propto \frac{1}{\lambda^4}$

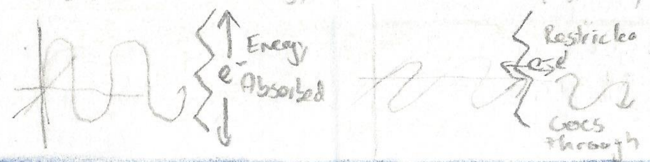
$\lambda_{BLUE} < \lambda_{RED}$

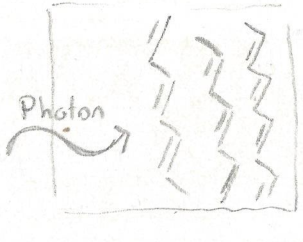
POLARIZATION Continued

How POLAROID WORKS: LONG POLYMERS IN STRANDS

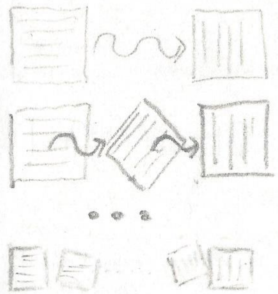
(I) PARALLEL to strands
- electrons can oscillate along the chains & absorb the energy

(II) PERPENDICULAR
- ELECTRONS cannot move very much
- RESTRICTED Energy Absorpt





PUTTING POLARIZERS Between:



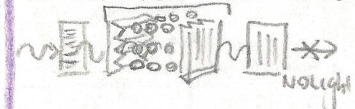
$I = I_0 \cos^2(90°) = 0$

$I = I_0(\cos^2 45)(\cos^2 45) = 25\%$

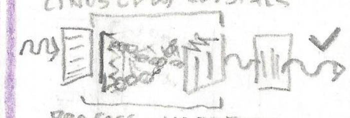
$I = I_0(\cos^2(1))^{90} = 97.3\%$

USED IN L.C.Ds

ON Electric field - Restricted electrons



NO ELECTRIC FIELD, LINES UP W/ CRYSTALS



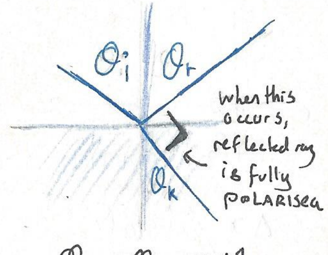
PROCESS ON LEFT FOLLOWS

POLARISATION BY REFLECTION

Brewster's Law: there exists a certain angle at which only polarised light will reflect

this happens $\theta_{reflected} + \theta_{refracted} = 90°$

- This is because light becomes partially or linearly polarised when reflected/scattered/refracted etc.
- Brewster's LAW CAN BE DERIVED w/ maxwell's equations.



$$\theta_r + \theta_k = 90°$$

we CALL $\theta_B = \theta_i = \theta_r$ the angle at which this occurs

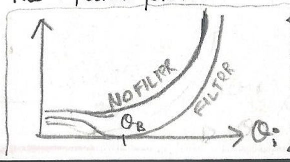
$n_i \sin\theta_i = n_r \sin\theta_r$

$n_i \sin\theta_B = n_r \sin(90 - \theta_B) = n_r \cos\theta_B$

$$\Rightarrow \frac{\sin\theta_B}{\cos\theta_B} = \boxed{\frac{n_r}{n_i} = \tan\theta_B}$$

- This means reflections off a reflective surface (For example, a lake) can be tamed by using a POLARIZER $\perp$ to the reflection plane

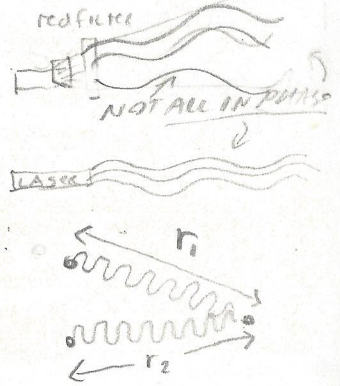
- ONE CAN GRAPH $I_{reflected}$ vs I_0 W/ FILTER:



ON EXAM!

SECTION 4: DIFFRACTION & INTERFERENCE

- IT is easiest to describe interference with Coherent sources of Light.
  ie: Same Frequency & Same Phase
- PRINCIPLE OF SUPERPOSITION APPLIES TO LIGHT ALSO



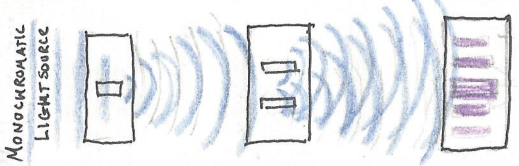
- Suppose we have 2 sources of light.
  The distance to a point from S_1 is r_1 and from S_2 is r_2

Then we get:

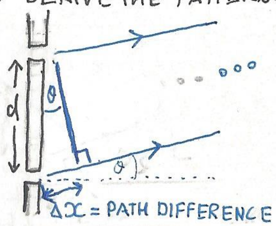
CONSTRUCTIVE INTERFERENCE
when $\Delta r = r_2 - r_1 = n\lambda$
SO THEY ARE IN PHASE
Then $A = 2A_0$

DESTRUCTIVE INTERFERENCE
when $\Delta r = r_2 - r_1 = (n+0.5)\lambda$
So they are out of phase
$A = 0$

2 SOURCE INTERFERENCE (YOUNG'S SLITS)



TO DERIVE THE PATTERN:



- As the distance is v large compared to d, we approximate the two beams to be parallel

$\boxed{\Delta x = d\sin\theta}$ THIS GIVES US the PHASE DIFFERENCE $\boxed{\phi = k\Delta x = \frac{2\pi}{\lambda} \cdot d\sin\theta}$

Describing the two waves gives us: $E_1 = E_0\cos(\omega t)$
$E_2 = E_0\cos(\omega t + \phi)$

$\Rightarrow E_p = E_1 + E_2 = E_0\cos(\omega t) + E_0\cos(\omega t + \phi)$

$\Rightarrow \boxed{E_p = 2E_0\cos\left(\frac{2\omega t + \phi}{2}\right)\cos\left(-\frac{\phi}{2}\right)}$

$\frac{\phi}{2} = n\pi \Rightarrow$ constructive $\Rightarrow d\sin\theta = n\lambda$ DESTRUCTIVE $\Rightarrow d\sin\theta = (n+0.5)\lambda$
$\Rightarrow \cancel{2\pi}\, d\sin\theta/\lambda = \cancel{2\pi}\, n \Rightarrow$

$\boxed{E_p = 2E_0\cos\left(\omega t + \frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)}$

where $\phi = \frac{2\pi}{\lambda} d\sin\theta$

INTENSITY = time averaged rate at which energy is transfered

For 1 wave: $E_1 = E_0\cos\omega t$

$I_0 = \text{const} \times \langle E_0^2\cos^2\omega t\rangle$
$= \text{const} \times E_0^2 \times \frac{1}{2}$

For 2 SLITS: $E_p = 2E_0\cos\left(\omega t + \frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)$

$I_p = \text{const} \times \langle 4E_0^2\cos^2\left(\omega t + \frac{\phi}{2}\right)\cos^2\left(\frac{\phi}{2}\right)\rangle$
$= \text{const} \times 4E_0^2 \times \frac{1}{2} \times \cos^2\left(\frac{\phi}{2}\right)$

- BUT $\text{const} \times E_0^2 \times \frac{1}{2} = I_0$

$\boxed{I_p = 4I_0\cos^2\left(\frac{\phi}{2}\right)}$

constructive: $d\sin\theta = m\lambda$
destructive: $d\sin\theta = (m+\frac{1}{2})\lambda$

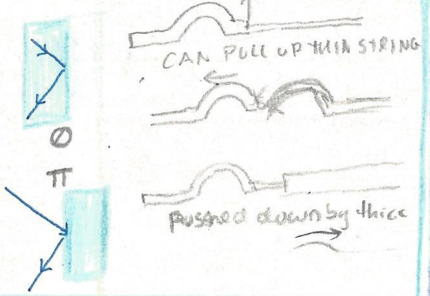
THIN-FILM INTERFERENCE

- A way of producing colours w/ only reflection & different path lengths.
- WHAT WE SEE DEPENDS ON PHASE DIFFERENCE

1) PHASE CHANGES ON REFLECTION
2) PATH DIFFERENCE due to EXTRA THICCNESS

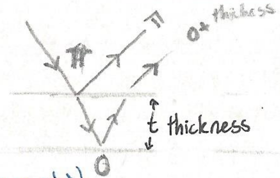
(a) IF HIGH n → LOW n
- No phase change

(b) IF LOW n → HIGH n
- π phase change



[2] Extra distance is covered by one of the photons:

- we want this distance to be a multiple of the wavelength (in this case $+\frac{1}{2}\lambda$)

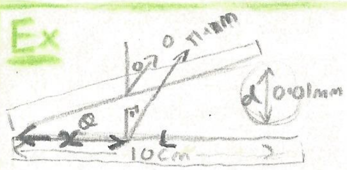


so: $2t = (m+\frac{1}{2})\lambda_f$ but $\lambda_f = \frac{\lambda_0}{n_f}$

BUT we must take into account the change in λ in the other material

$\rightarrow 2t = (m+\frac{1}{2})\frac{\lambda_0}{n_f}$

Ex



2 PLATES 10cm long TOUCH at one end and SEPARATED by 0.01mm diameter wire on other
Angle between 1st 2 reflections?
Light of λ 420nm shone about ⊥

$\theta \approx \sin\theta \approx \tan\theta = \frac{d}{L} = 10^{-4}$

$2t = (m+0.5)\lambda$
$t = \frac{\lambda}{2}(m+\frac{1}{2})$ $\Rightarrow$ $t(x) = \tan\theta\, x$
$x = \frac{t}{\tan\theta} = \frac{(m+0.5)\lambda}{2\tan\theta}$

$\Delta x = \frac{t_a - t_b}{\tan\theta} = \frac{(m_a - m_b)\lambda}{2\tan\theta} = \frac{\lambda}{2\times10^{-4}}$

$\Delta x = 5\times10^{-3}$

WHAT IF FILLED w/ water? $\lambda = \frac{\lambda_0}{n}$ $n\uparrow \Rightarrow \lambda\downarrow \rightarrow \Delta x\downarrow$ CLOSER

PHASOR NOTATION & Huygen's PRINCIPLE

Huygen's Principle:

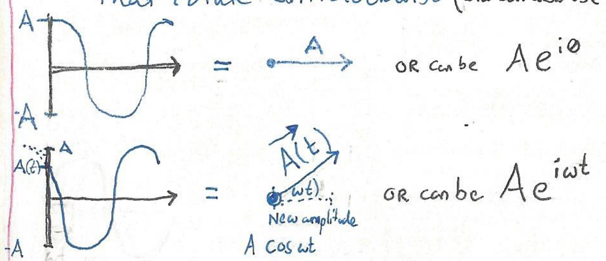
Every point of a wavefront may be considered the source of secondary "wavelets" that spread out in all directions w/ a speed equal to that of the propagation wave

- THIS is a geometrical method for finding
  - The shape of the wavefront at a later time FROM the known shape of a wavefront at an instant

  → Found by constructing surface tangent to secondary wavelets
- EVERY POINT is CONSIDERED A SOURCE OF SECONDARY WAVELET

PHASOR REPRESENTATION

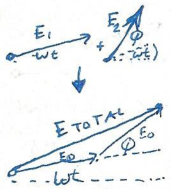
- One can also represent waves as vectors that rotate anticlockwise (one can also use $e^{i\theta}$)



- Then it is the |x axis| or the Real part that gives the amplitude at an instant
- Lets take for example 2 waves:

$\boxed{E_1 = E_0\cos(\omega t) \quad \text{and} \quad E_2 = E_0\cos(\omega t + \phi)}$

THEN we can draw:



- We can use this diagram to find intensity $\propto E^2$

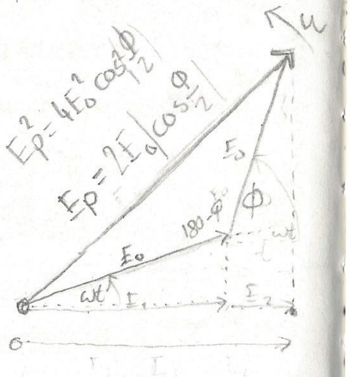
FROM COSINE RULE: (LET $t = 0$)

$E_T^{\,2} = E_1^{\,2} + E_2^{\,2} - 2E_1E_2\cos\phi$

$E_T^{\,2} = E_0^{\,2} + E_0^{\,2} - 2E_0^{\,2}\cos\phi$

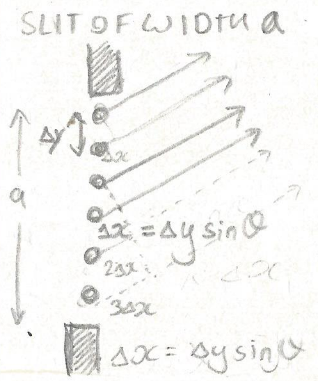
$E_T^{\,2} = 2E_0^{\,2}(1-\cos\phi) = 2E_0^{\,2}\left(2\cos^2\frac{\phi}{2}\right)$

$E_T^{\,2} = 4E_0^{\,2}\cos^2\frac{\phi}{2}$ SAME AS BEFORE



DIFFRACTION – FRAUNHOFER DIFFRACTION ANALYSIS

- IN THE real world, slits cannot be "points", but light has to pass trough a hole of FINITE SIZE.
- AT some/all sizes the light interferes with itself, as due to huygen's Principle. the wavefront at the slit can be taken to be a number of point sources that interact.
- We can represent the wavelets from each of these point-sources with PHASORS



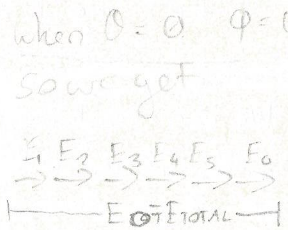
- suppose we have a slit of width a

we split the wavefront into n wavelets of $E = \frac{E_0}{n}$ Amplitude.
- for simplicity, we look at the instant $t = 0$
- The phasors for each wavelet we get are:

(counting E_1 as the top one, and E_n $n-1$ from the top)

$\boxed{E_1 = E,\ E_2 = E\cos\phi,\ E_3 = E\cos 2\phi,\ E_4 = E\cos 3\phi \ldots \text{etc.}}$

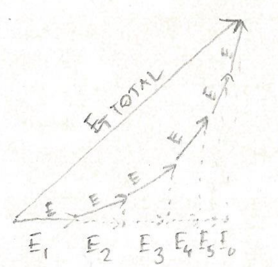
where ϕ is the phase difference of each wavelet



- we take $\boxed{\phi = k\Delta x}$, but $\boxed{\Delta x = \Delta y\sin\theta}$, $\boxed{\Delta y = \frac{a}{n}}$, $\boxed{k = \frac{2\pi}{\lambda}}$

so $\boxed{\phi = \frac{2\pi}{\lambda}\cdot\Delta y\sin\theta}$

- IF we take $\theta = 0$, then adding up all the PHASORS gives us $E + E + \cdots + E = E_0$ as wanted
- IF we take $\theta > 0$, then we get a diagram as shown on the right for E_T total for $n = 6$.

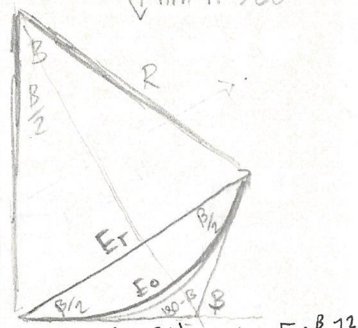


- IF we take the LIMIT AS $n \to \infty$, we get a curve. since all of the sub-pieces are of length E,

$n \times E = n \times \frac{E_0}{n} = E_0 \Rightarrow$ the curve is of length E_0
- we also get that the tangent angle to the final point is $n\phi = n\cdot\frac{2\pi}{\lambda}\sin\theta\cdot\Delta y$ but $\Delta y = \frac{a}{n}$

so $\boxed{\beta = n\phi = \frac{2\pi a\sin\theta}{\lambda}}$

lim n→∞



- now, using geometry, we can find E_P: $\boxed{R\beta = E_0}$

$\sin\left(\frac{\beta}{2}\right) = \frac{E_T}{2}\Big/R \Rightarrow E_T = \sin\left(\frac{\beta}{2}\right)\cdot 2R = \frac{2E_0}{\beta}\cdot\sin\left(\frac{\beta}{2}\right)$

FINALLY $E_T = E_0\sin\left(\frac{\beta}{2}\right)\Big/\left(\frac{\beta}{2}\right)$

$E_P = E_0\left(\frac{\sin(\frac{\beta}{2})}{\beta/2}\right) \Rightarrow I_P = I_0\left[\frac{\sin\frac{\beta}{2}}{\beta/2}\right]^2$

DIFFRACTION

- on the previous page, we got

$$E_p = E_0 \frac{\sin(\frac{\beta}{2})}{(\frac{\beta}{2})} \quad w/\ \beta = \frac{2\pi a \sin\theta}{\lambda}$$

- note we do not have to get the "real" part of this as it will rotate anyway.
- IF we let $\alpha = \frac{\beta}{2}$, then we get $\longrightarrow$
- We know Intensity $\propto E^2$, so

$$\boxed{I_p = I_0 \left[\frac{\sin\alpha}{\alpha}\right]^2}$$ IF ALL OTHER factors are the same

$$\alpha = \frac{\pi a \sin\theta}{\lambda}$$

$$E_p = E_0\left[\frac{\sin\alpha}{\alpha}\right]$$

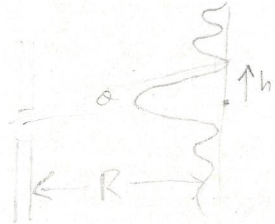
$$I_p = I_0\left[\frac{\sin\alpha}{\alpha}\right]^2$$

- We can now look at this on a screen
  - TO FIND the size of the central fringe we find the distance between the two first dark fringes

$$\Rightarrow E_p = 0 \Rightarrow \sin(\tfrac{\beta}{2}) = 0 \quad \text{BUT } \tfrac{\beta}{2} \neq 0$$

$$\alpha = m\pi = \frac{\pi a \sin\theta}{\lambda} \quad \& \, m=1$$

$$\Rightarrow \boxed{\frac{\lambda}{a} = \sin\theta}$$



looking at the macroscopic scale, the fringe is h up from the zero point and R m away

$$\Rightarrow \boxed{\tan\theta = \frac{h}{R}}$$

- when θ is small, $\tan\theta \approx \sin\theta$

$$\Rightarrow \frac{\lambda}{a} = \frac{h}{R} \Rightarrow \boxed{h = \frac{\lambda}{a}\cdot R}$$

- Now let's find when the fringe is infinitely wide: $(h=\infty)$

$$\tan\theta = \frac{\infty}{R} \Rightarrow \boxed{\theta = \tan^{-1}(\infty) = \frac{\pi}{2}\ (90)}$$

$$\Rightarrow \frac{\lambda}{a} = \sin(\tfrac{\pi}{2}) = 1 \Rightarrow \boxed{\lambda = a}$$

WHEN $a \leq \lambda$ the fringe is 180° wide

DIFFRACTION & INTERFERENCE COMBINED

- FOR 2 SLIT INTERFERENCE, WE HAVE:

$$I_p = I_{MAX} \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

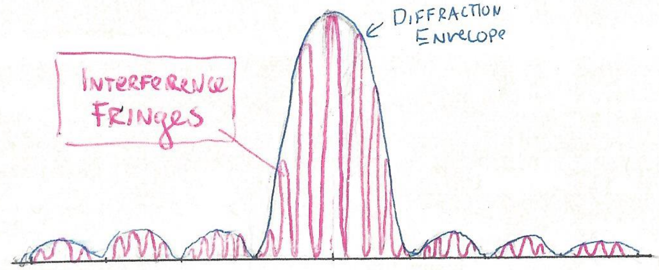
- FOR DIFFRACTION, WE HAVE:

$$I_p = I_0 \left[\frac{\sin\left[\frac{\pi a \sin\theta}{\lambda}\right]}{\frac{\pi a \sin\theta}{\lambda}}\right]^2$$

- COMBINING THE TWO, WE GET THAT

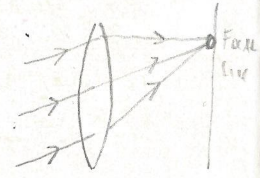
$$\boxed{I_p = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)\cdot\left[\frac{\sin\left[\frac{\pi a \sin\theta}{\lambda}\right]}{\frac{\pi a \sin\theta}{\lambda}}\right]^2}$$

- The PATTERN we GET FROM THIS IS:

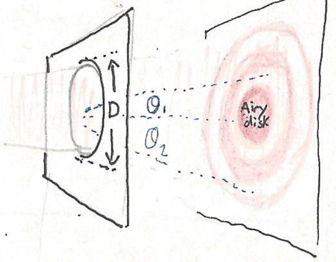


Circular Aperture, Limit to Resolution

SIDE NOTE: up to this point, we have assumed that the PARRALLEL Beams go TO A POINT.
IN REALITY, they must have a Lens to focus those parallel beams onto a point

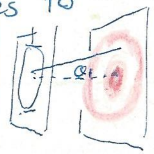


- IF you shine light through a small circular hole (APERTURE), you will see a Bright spot in the middle (AIRY DISK) and a series of dark & bright rings around it.



- one can desire an expression for the radii of the dark circles to be given by

$$\boxed{\begin{aligned} \sin\theta_1 &= 1.22\frac{\lambda}{D} \\ \sin\theta_2 &= 2.23\frac{\lambda}{D} \\ &\vdots \\ \sin\theta_3 &= 3.24\frac{\lambda}{D} \end{aligned}}$$

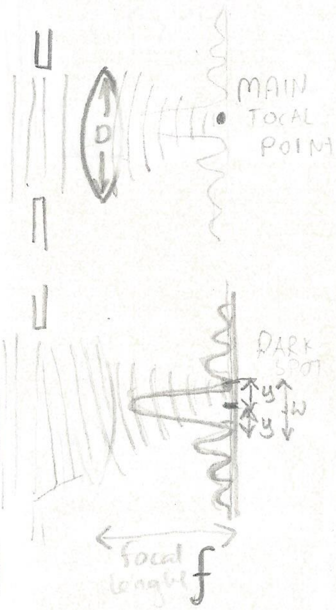


- But of course, to get a POINT, one would have to use a LENS to focus the parallel beams of light
- However, diffraction occurs, so IF one wants an image on a screen, they must account for this
- Let y = the distance of the dark ring from the centre point
  f = the focal length of a lens

Then: $\sin\theta_1 = \frac{y}{f}$, BUT we know $\sin\theta_1 = 1.22\frac{\lambda}{D}$

$\Rightarrow 1.22\frac{\lambda}{D} = \frac{y}{f} \Rightarrow \boxed{y = \frac{\lambda}{D}(1.22)\cdot f}$

→ THIS GIVES IS the MINIMUM RADIUS THAT WE CAN MAKE A DOT/AIRY DISK
→ THE MINIMUM RESOLUTION



Limit to Resolution, Rayleigh Criterion

- The minimum radius $= \frac{\lambda}{D}(1.22)f$

- BY the RAYLEIGH CRITERION,
  IF we have 2 POINT SOURCES of light, separated by an angle α between their centers,
  - THE minimum angular displacement to be able to resolve the two sources is $\alpha = \theta_1$ $\left(= 1.22\frac{\lambda}{D}\right)$
    - FOR $\boxed{\alpha > \theta_1}$, IT is easy to see distinct points
    - For $\boxed{\alpha < \theta_1}$, IT is impossible to tell whether there is 1 source or two As they overlap too much (or almost so)
    - IT Doesn't matter How much you ZOOM IN/MAGNIFY, as you STILL CANNOT see

- NOTE: