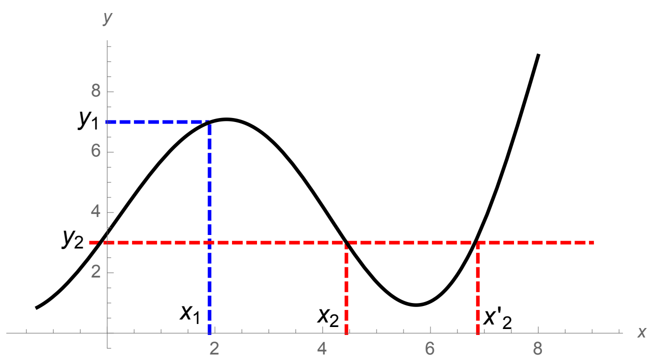
Vector-Valued Functions

1 Parametric curves



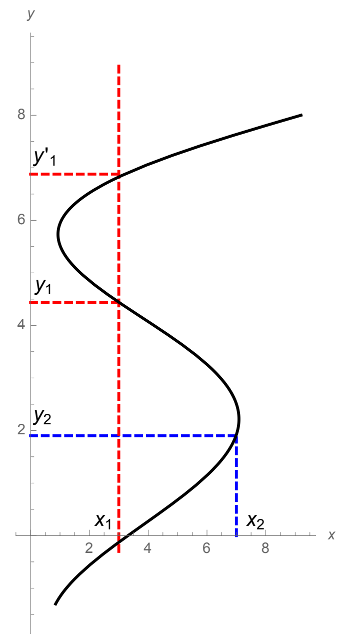


Figure 1: Which curve is a graph of a function?

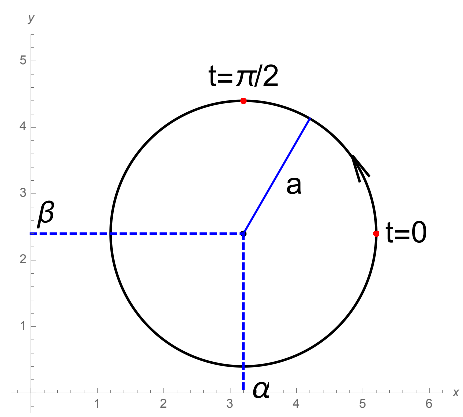


Figure 2: Is it a graph of a function?

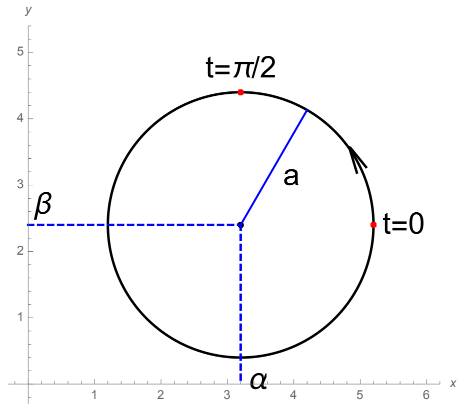


Figure 3: This curve is *not* a graph of a function.

This curve is a circle of radius a centred at (α, β)

$$(x-\alpha)^2 + (y-\beta)^2 = a^2$$

It is a **parametric curve** which can be represented by the eqs

$$x = \alpha + a\cos t\,, \quad y = \beta + a\sin t\,, \quad 0 \le t \le 2\pi\,.$$

In general in 2-space a parametric curve is

$$x = f(t)\,, \quad y = g(t)\,, \quad t_0 \le t \le t_1\,.$$

The graph of a function $y = f(x)$ is a parametric curve

$$x = t\,, \quad y = f(t)\,.$$

The graph of a function $x = f(y)$ is a a parametric curve

$$x = f(t)\,, \quad y = t\,.$$

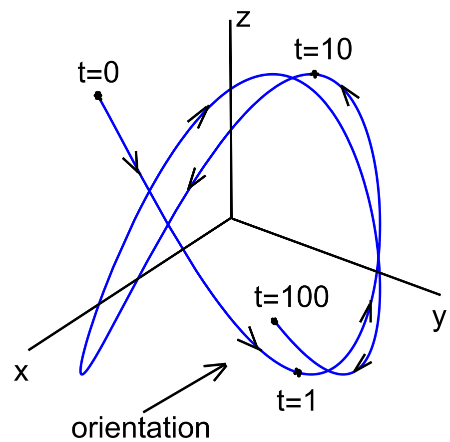


Figure 4: A parametric curve in 3-space

$$x = f(t)\,, \quad y = g(t)\,, \quad z = h(t)\,, \quad t_0 \le t \le t_1\,.$$

These parametric eqs generate a curve in 3-space called the **parametric curve** represented by the eqs, and as t increases the point moves in a specific direction which defines the curve **orientation**.

A parametric curve can be considered as a path of a point particle with the parameter t identified with time.

A parametric curve in n-space is represented by n eqs

$$x_1 = f_1(t)\,, \quad x_2 = f_2(t)\,, \quad \cdots \quad , \quad x_n = f_n(t)\,, \quad t_0 \le t \le t_1\,.$$

Example 1.

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1\,.$$

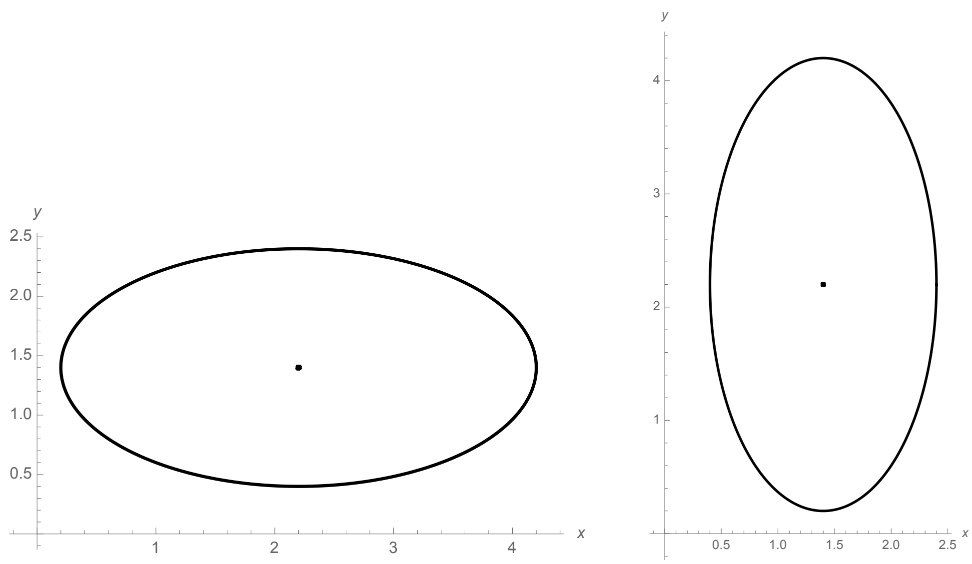


Figure 5: These are ellipses.

This ellipse can be represented by the eqs

$$x=\alpha+a\cos t\,,\quad y=\beta+b\sin t\,,\quad 0\le t\le 2\pi\,.$$

Example 1. Ellipse

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1\,.$$

Example 2.

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1\,.$$

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = -1\,.$$

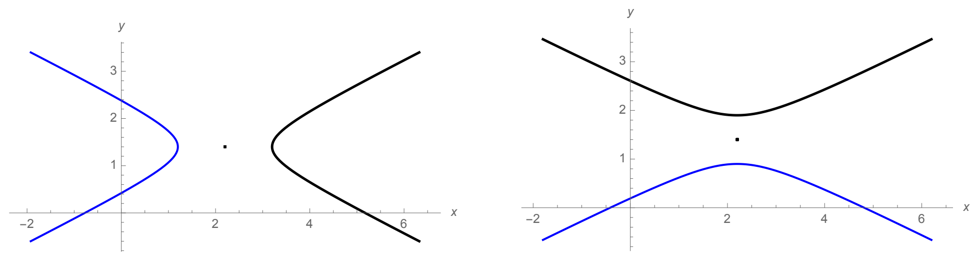


Figure 6: These are hyperbolas.

The right (black) branch of the first hyperbola can be represented by the eqs

$$x = \alpha + a\cosh t\,, \quad y = \beta + b\sinh t\,, \quad -\infty < t < \infty\,.$$

because

$$\cosh t = \frac{e^t + e^{-t}}{2} > 0\,, \quad \sinh t = \frac{e^t - e^{-t}}{2}\,, \quad \cosh^2 t - \sinh^2 t = 1\,.$$

HW: Find a representation of the left (blue) branch of the first hyperbola, and similar representations of both branches of the second hyperbola.

Example 1. Ellipse

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} = 1\,.$$

Example 2. Hyperbolas

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1\,.$$

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = -1\,.$$

Example 3.

$$y = ax^2\,.$$

$$y^2 = ax\,.$$

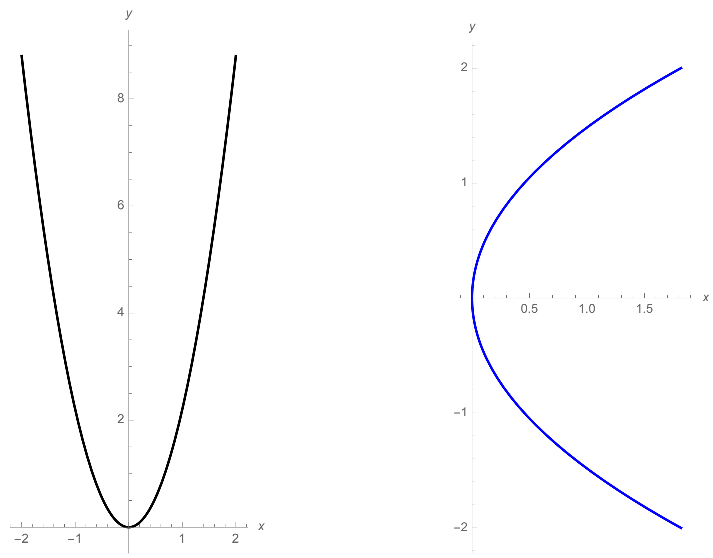


Figure 7: These are parabolas.

They can be represented by the eqs

$$x = t\,, \quad y = at^2\,, \qquad x = \frac{1}{a}t^2\,, \quad y = t\,, \quad -\infty < t < \infty\,.$$

Ellipses, hyperbolas and parabolas are **conic sections**.

Example 4.

$$x = a\cos t\,, \quad y = a\sin t\,, \quad z = vt\,, \quad -\infty < t < \infty\,.$$
$$x = a\sin t\,, \quad y = a\cos t\,, \quad z = vt\,, \quad -\infty < t < \infty\,.$$

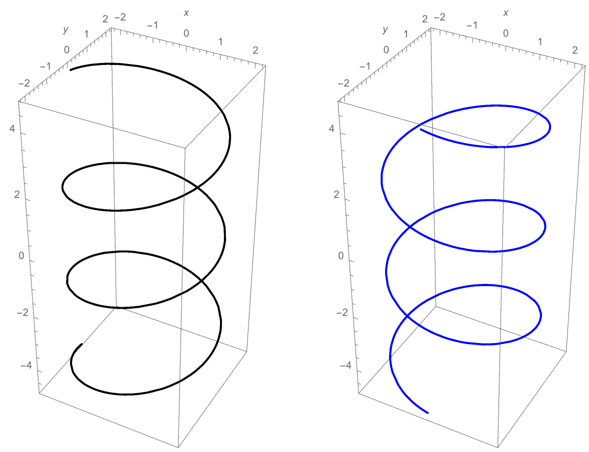


Figure 8: These are circular helixes.

Example 5. Line through $\vec{r}_0 = (x_0, y_0, z_0)$ in the direction of the vector $\vec{v} = (v_x, v_y, v_z)$

$$x = x_0 + v_x t\,, \quad y = y_0 + v_y t\,, \quad z = z_0 + v_z t\,, \quad -\infty < t < \infty\,.$$

Example 5b. Find parametric equations for the line through $(2, 0, 0)$ and $(0, 4, 3)$.

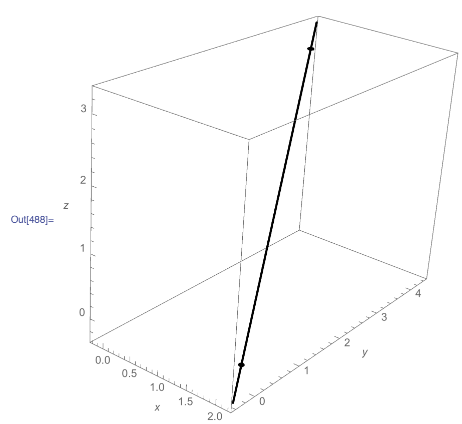


Figure 9: This is the line through $(2, 0, 0)$ and $(0, 4, 3)$.

Let $t = 0$ correspond to $(2, 0, 0)$, and let $t = 1$ corresponds to $(0, 4, 3)$. Then,

$$x(0) = x_0 = 2\,, \quad x(1) = x_0 + v_x = 0 \quad \Longrightarrow \quad v_x = -2\,,$$
$$y(0) = y_0 = 0\,, \quad y(1) = y_0 + v_y = 4 \quad \Longrightarrow \quad v_y = 4\,,$$
$$z(0) = z_0 = 0\,, \quad z(1) = z_0 + v_z = 3 \quad \Longrightarrow \quad v_z = 3\,.$$

Thus

$$x = 2 - 2t\,, \quad y = 4t\,, \quad z = 3t\,, \quad -\infty < t < \infty\,.$$

Example 5. Line through $\vec{r}_0 = (x_0, y_0, z_0)$ in the direction of the vector $\vec{v} = (v_x, v_y, v_z)$

$$x = x_0 + v_x t\,, \quad y = y_0 + v_y t\,, \quad z = z_0 + v_z t\,, \quad -\infty < t < \infty\,.$$

Example 6. The same line as in Example 5

$$x = x_0 - \frac{1}{2} v_x t\,, \quad y = y_0 - \frac{1}{2} v_y t\,, \quad z = z_0 - \frac{1}{2} v_z t\,, \quad -\infty < t < \infty\,.$$

Example 7. The same line as in Examples 5 and 6

$$x = x_0 - \frac{1}{2} v_x (t - t_0)\,, \quad y = y_0 - \frac{1}{2} v_y (t - t_0)\,, \quad z = z_0 - \frac{1}{2} v_z (t - t_0)\,.$$

Example 8. This is a segment of the same line as in Examples 5, 6 and 7

$$x = x_0 + v_x t^3\,, \quad y = y_0 + v_y t^3\,, \quad z = z_0 + v_z t^3\,, \quad -1 \le t \le 1\,.$$

Example 9. This is a segment of the same line as in Examples 5, 6 and 7 but it is run twice!

$$x = x_0 + v_x (t-1)^2\,, \quad y = y_0 + v_y (t-1)^2\,, \quad z = z_0 + v_z (t-1)^2\,, \quad 0 \le t \le 2\,.$$

$\exists\ \infty$ many parametric eqs representing the same curve.

Make a change $t = \gamma(\tau)$ where $\gamma(\tau)$ is a one-to-one function on the interval $\tau_0 \le \tau \le \tau_1$ of interest.

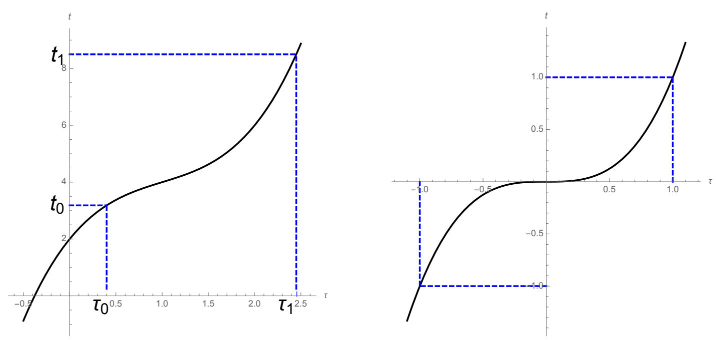


Figure 10: $\gamma(\tau_0) = t_0\,,\ \gamma(\tau_1) = t_1$; $\qquad t = \tau^3$;

The parameterisations are considered as equivalent if $\gamma'(\tau) \neq 0$ for $\tau_0 \le \tau \le \tau_1$. If $\gamma'(\tau) > 0$ the curves have the same orientation.

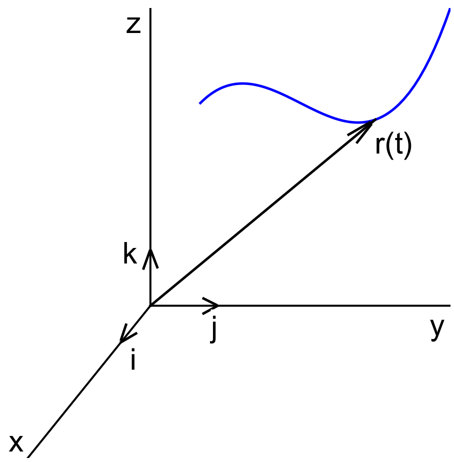
Example.

$$x = \cos t\,, \quad y = \sin t\,,$$

$$t = -\tau \quad \Longrightarrow \quad x = \cos\tau\,, \quad y = -\sin\tau\,,$$

$$t = \tau - \frac{\pi}{2} \quad \Longrightarrow \quad x = \sin\tau\,, \quad y = -\cos\tau\,.$$

2 Vector-Valued Functions



$x = f(t),\ y = g(t),\ z = h(t)$

$\vec{r}(t) = \big(x(t), y(t), z(t)\big)$

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

Thus, $\vec{r}$ is a function of t:

$\vec{r} = \vec{r}(t)$.

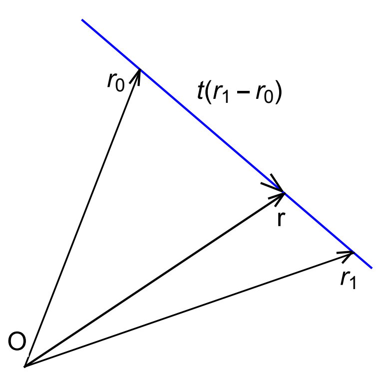
This is a **vector-valued** function of a real variable t.
$x(t), y(t), z(t)$ are **components** of $\vec{r}(t)$.
The graph of $\vec{r}(t)$ is the parametric curve $\mathcal{C}$ described by the components of $\vec{r}(t)$.
$\vec{r}(t)$ is called the radius vector for the curve $\mathcal{C}$.
The domain of $\vec{r}(t)$ is the intersection of domains of $x(t), y(t), z(t)$.
It is called the **natural domain** of $\vec{r}(t)$.

Example.

$$\vec{r}(t) = \sqrt{t-1}\,\vec{i} + \ln(3-t)\vec{j} + \frac{1}{t-2}\vec{k}\,.$$

$$D(\vec{r}) = [1,2) \cup (2,3)\,.$$

3 Vector form of a line segment



If $\vec{r}_0$ is a vector with its initial point at O

then the line through the terminal point

$\vec{r}_0$ and parallel to $\vec{v}$ is

$\vec{r} = \vec{r}(t) = \vec{r}_0 + t\,\vec{v}$.

So, $\vec{r}(0) = \vec{r}_0$, $\vec{r}(1) = \vec{r}_1 = \vec{r}_0 + \vec{v}$

$\Rightarrow\ \vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = (1-t)\vec{r}_0 + t\,\vec{r}_1$

This is a **two-point form** of a line.

If $0 \le t \le 1$, then $\vec{r} = (1-t)\vec{r}_0 + t\,\vec{r}_1$ represents the line segment from $\vec{r}_0$ to $\vec{r}_1$.

4 Calculus of Vector-valued Functions

$$\lim_{t\to a}\vec{r}(t) = \Big(\lim_{t\to a}x(t)\Big)\vec{i} + \Big(\lim_{t\to a}y(t)\Big)\vec{j} + \Big(\lim_{t\to a}z(t)\Big)\vec{k}\,,$$

where all the three limits must exist.

A vector-valued function is continuous at $t = a$ (or on an interval I) if all its components are. Then

$$\lim_{t\to a}\vec{r}(t) = \vec{r}(a) \quad \forall a \in I\,.$$

Derivative of $\vec{r}$ with respect to t is the following vector

$$\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}\,.$$

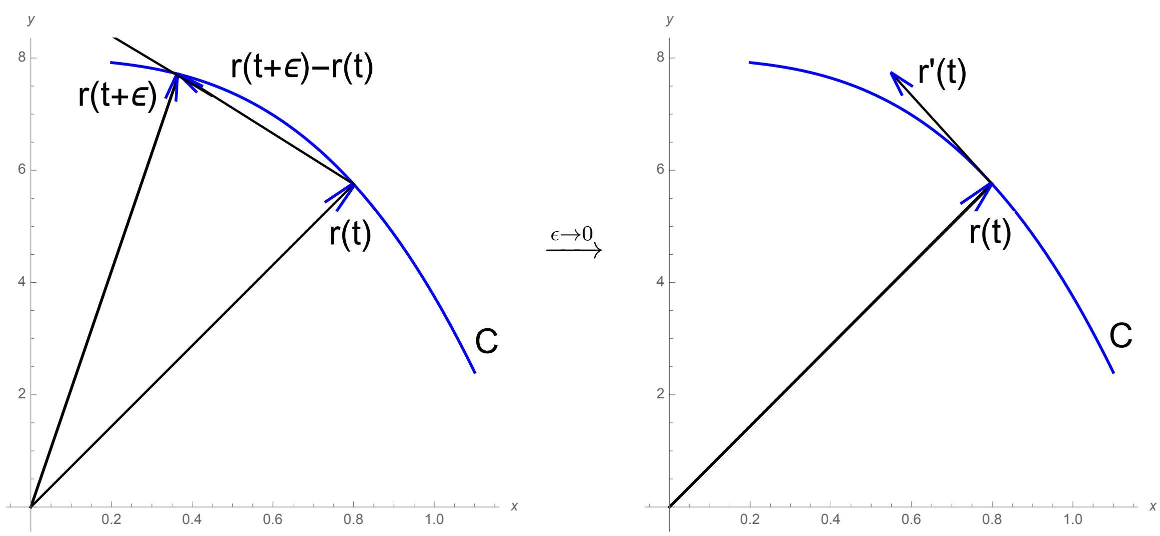
$\vec{r}(t)$ is differentiable if all its components are.

Notations

$$\frac{d}{dt}\vec{r}(t)\,, \quad \frac{d\vec{r}}{dt}\,, \quad \vec{r}\,'(t)\,, \quad \vec{r}\,'\,, \quad \dot{\vec{r}}(t)\,, \quad \dot{\vec{r}}\,, \quad \frac{d}{dt}[\vec{r}(t)]\,.$$

Derivative Rules

1. $\frac{d}{dt}\big[a\,\vec{r}_1(t) + b\,\vec{r}_2(t)\big] = a\,\vec{r}_1\,'(t) + b\,\vec{r}_2\,'(t)\,, \quad a, b$ are const
2. $\frac{d}{dt}\big[f(t)\,\vec{r}(t)\big] = f'(t)\,\vec{r}(t) + f(t)\,\vec{r}\,'(t)\,,$



If $\vec{r}\,'(t) \neq 0$ and it is positioned with its initial point at the terminal point of $\vec{r}$ then $\vec{r}\,'$ is tangent to $\mathcal{C}$ and points in the direction of increasing parameter.

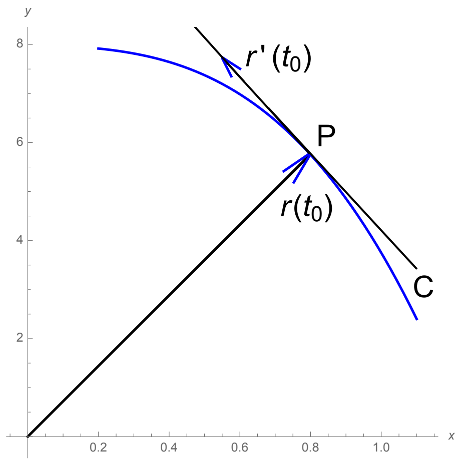
In mechanics it is velocity of a particle moving along $\mathcal{C}$.

Example.

$$\vec{r}(t) = 2\cos\frac{\pi}{2}t\,\vec{i} + \sqrt[3]{1+3e^{2t}}\,\vec{j} + \int_1^{\ln t}\frac{e^{3u}}{u}du\,\vec{k}\,.$$

$$\vec{r}\,'(t) = ?$$

5 Tangent lines to graphs of vector-valued functions



Let P be a point on the graph $\mathcal{C}$ of a VV function $\vec{r}(t)$, and let $\vec{r}\,'(t_0) \neq 0$ where $\vec{r}(t_0)$ is the radius vector from O to P.

Then, $\vec{r}\,'(t_0)$ is a **tangent vector** to $\mathcal{C}$ at P, and the line through P that is parallel to $\vec{r}\,'(t_0)$ is the tangent line to C at P.

The tangent line is given by the vector eq

$$\vec{R}(t) = \vec{r}_0 + t\,\vec{v}_0\,, \quad \vec{v}_0 = \vec{r}\,'(t_0)\,, \quad \vec{r}_0 = \vec{r}(t_0)\,.$$

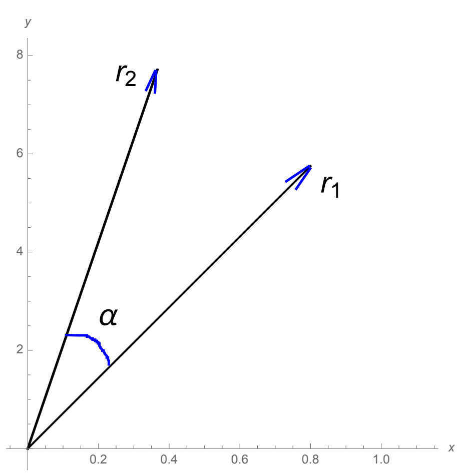
The unit tangent vector is $\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{r}\,'}{|\vec{r}\,'|}$.

Example.

$$\vec{r}(t) = a\sin t\,\vec{i} + a\cos t\,\vec{j} + v\,t\,\vec{k}\,,$$

$$\vec{r}\,'(t) = ?\,, \quad |\vec{r}\,'(t)| = ?\,, \quad \vec{T}(t) = ?\,, \quad \vec{R}(t) \text{ at } t = \frac{\pi}{3}?\,,$$

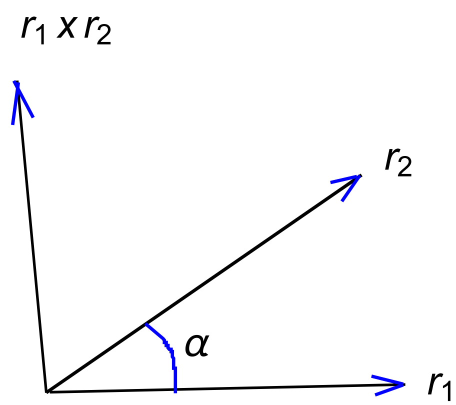
6 Derivatives of dot and cross products



$$\vec{r}_1 = (x_1, y_1, z_1), \quad \vec{r}_2 = (x_2, y_2, z_2)\,.$$

$$\vec{r}_1 \cdot \vec{r}_2 = x_1x_2 + y_1y_2 + z_1z_2$$

$$= |\vec{r}_1|\,|\vec{r}_2|\cos\alpha = \vec{r}_2 \cdot \vec{r}_1$$



$$\vec r_1 \times \vec r_2 = +(y_1 z_2 - y_2 z_1)\,\vec i \\ -(x_1 z_2 - x_2 z_1)\,\vec j \\ +(x_1 y_2 - x_2 y_1)\,\vec k$$

$$= \begin{vmatrix} \vec i & \vec j & \vec k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = -\vec r_2 \times \vec r_1$$

$$|\vec r_1 \times \vec r_2| = |\vec r_1|\,|\vec r_2| \sin\alpha\,,$$

$$\vec r_1 \times \vec r_2 \ \perp\ \vec r_1 \text{ and } \vec r_2\,.$$

$$\big(\vec r_1 \times \vec r_2\big)_i = \sum_{j,k=1}^{3} \epsilon_{ijk}\, x_j\, y_k = \epsilon_{ijk}\, x_j\, y_k\,,$$

where $\epsilon_{123} = 1$, and ϵ_{ijk} is skew-symmetric with respect to i, j, k:

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj} = \epsilon_{jki} = \epsilon_{kij}\,.$$

We used Einstein's summation rule: sum over repeated indices.

$$\frac{d}{dt}(\vec r_1 \cdot \vec r_2) = \frac{d\vec r_1}{dt} \cdot \vec r_2 + \vec r_1 \cdot \frac{d\vec r_2}{dt}\,,$$

$$\frac{d}{dt}(\vec r_1 \times \vec r_2) = \frac{d\vec r_1}{dt} \times \vec r_2 + \vec r_1 \times \frac{d\vec r_2}{dt}\,.$$

Theorem. If $|\vec r(t)|$ is a constant then $\vec r(t) \cdot \vec r\,'(t) = 0$.

Proof:

$$\frac{d}{dt}(\vec r \cdot \vec r) = \frac{d}{dt}|\vec r|^2 = 0\,,$$

$$\frac{d}{dt}(\vec r \cdot \vec r) = \frac{d\vec r}{dt} \cdot \vec r + \vec r \cdot \frac{d\vec r}{dt} = 2\vec r \cdot \frac{d\vec r}{dt}\,,$$

Thus, $\vec r(t) \perp \vec r\,'(t)$, and $\vec T(t) \perp \vec T\,'(t)$.

Example. A curve on the surface of a sphere that is centred at the origin has $|\vec r(t)|$=const, thus $\vec r(t) \perp \vec r\,'(t)$.

7 Integrals of Vector-valued Functions

$$\int_a^b \vec{r}(t)dt = \Big(\int_a^b x(t)dt\Big)\vec{i} + \Big(\int_a^b y(t)dt\Big)\vec{j} + \Big(\int_a^b z(t)dt\Big)\vec{k}$$

Example 1.

$$\int_0^1 (\sqrt{3t+1}\,\vec{i} + e^{2t}\vec{j} + 3\sin(\pi t)\vec{k}\,)dt \ = ?$$

Rules of integration

$$\int_a^b (c_1\vec{r}_1(t) + c_2\vec{r}_2(t))dt = c_1\int_a^b \vec{r}_1(t)dt + c_2\int_a^b \vec{r}_2(t)dt$$

An **antiderivative** for a v-v function $\vec{r}(t)$ is a v-v function $\vec{R}(t)$ such that

$$\vec{R}\,'(t) = \vec{r}(t) \quad \Longrightarrow \quad \int \vec{r}(t)dt = \vec{R}(t) + \vec{C}$$

Example 2.

$$\int (\frac{1}{t+1}\vec{i} + t\cos(t^2-1)\vec{j}\,)dt \ = ?$$

Integration properties

$$\frac{d}{dt}\int \vec{r}(t)dt = \vec{r}(t) \quad \text{and} \quad \int \vec{r}\,'(t)dt = \vec{r}(t) + \vec{C}$$

Fundamental Theorem of Calculus

$$\int_a^b \vec{r}(t)dt \ = \ \vec{R}(t)\,\Big|_a^b \ = \ \vec{R}(b) - \vec{R}(a)$$

Example 3.

$$\int_0^1 (\frac{1}{t+1}\vec{i} + t\cos(t^2-1)\vec{j}\,)t \ = ?$$

Example 4. Find $\vec{r}(t)$ given that

$\vec{r}\,'(t) = (\,\frac{1}{t+1}, t\cos(t^2-1)\,)$, $\vec{r}(1) = (1,3)$

8 Arc Length Parametrisation

We say that $\vec{r}(t)$ is **smoothly parameterised** or that $\vec{r}(t)$ is a **smooth function** of t if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq 0$ for any allowed value of t.

Geometrically it implies that the tangent vector $\vec{r}\,'(t)$ varies continuously along the curve. For this reason a smoothly parameterised function is said to have a **continuously turning tangent vector**.

Example 1.

$$\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j} + vt\,\vec{k}\,, \quad \vec{r}\,'(t) \ = ?$$

Example 2.

$$\vec{r}(t) = t^2\,\vec{i} + t^3\,\vec{j}\,, \quad \vec{r}\,'(t) \ = ?$$

Change of Parameter

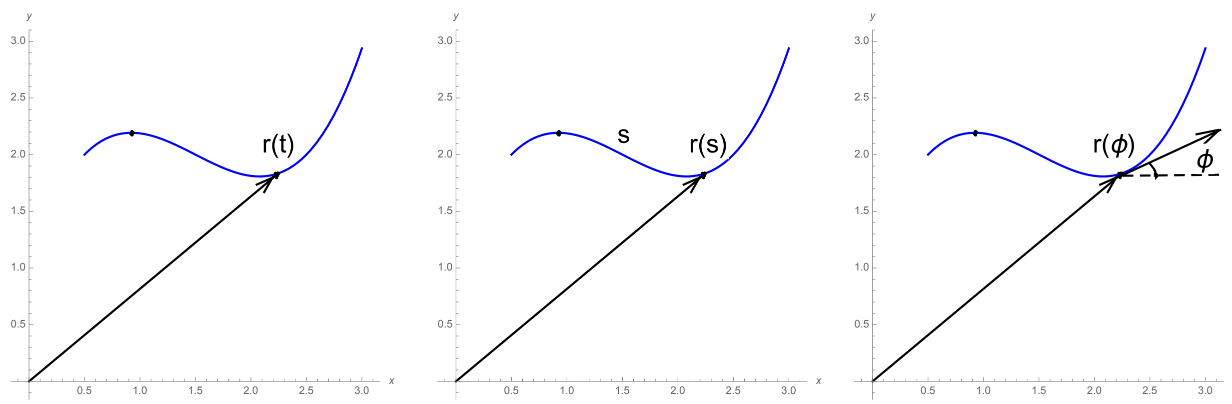


Figure 11: Parameters: time; distance travelled; angle ϕ

A change of parameter in a v-v function $\vec{r}(t)$ is a substitution $t = g(\tau)$ that produces a new v-v function $\vec{r}(g(\tau))$ having the same graph as $\vec{r}(t)$.

Example. Find a change of parameter $t = g(\tau)$ for the circle $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}, \quad 0 \leq t \leq 2\pi$ such that the circle is traced (a) counterclockwise; (b) clockwise; as τ increases over the interval $[0, 1]$.

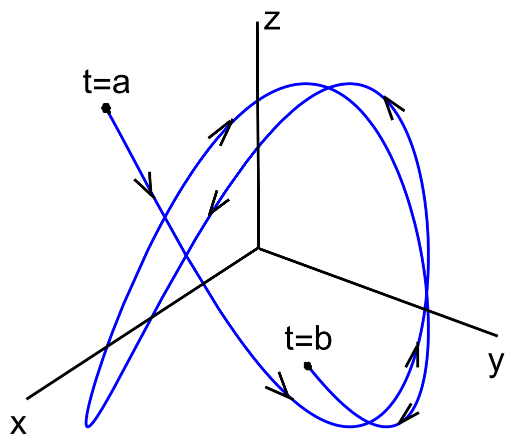
Chain rule:

$$\frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{dt}\frac{dt}{d\tau}\,, \quad t = g(\tau)\,.$$

A change of parameter $t = g(\tau)$ in which $\vec{r}(g(\tau))$ is smooth if $\vec{r}(t)$ is smooth is called a smooth change of parameter.

It requires $dt/d\tau \neq 0 \ \forall \tau$ and $dt/d\tau$ continuous. If $dt/d\tau > 0$ it is positive change of parameter, if $dt/d\tau < 0$ it is negative.

Arc Length as a Parameter



The arc length L of a parametric curve

$x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$:

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

In a vector form $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

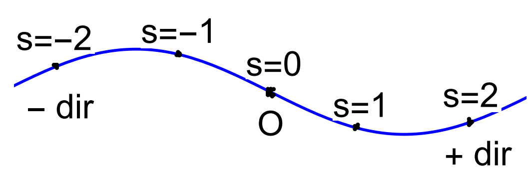
$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \quad \Longrightarrow \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\Longrightarrow \quad L = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b |\vec{r}\,'(t)|\, dt\,.$$

Example.

$$x = a\cos t \quad y = a\sin t \quad z = vt \quad 0 \le t \le \pi \quad L = ?$$

The length of arc measured along the curve from some fixed reference point can serve as a parameter.

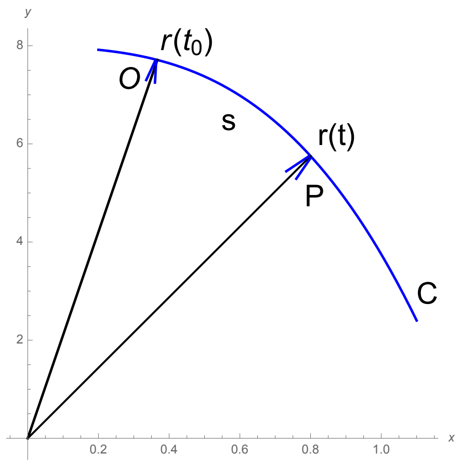


1. Select an arbitrary point on the curve $\mathcal{C}$ to serve as a **reference point** O.
2. Choose one direction along $\mathcal{C}$ to be **positive**, and the other to be **negative**.
3. If P is a point on $\mathcal{C}$, let s be the "signed" arc length along $\mathcal{C}$ from O to P, where s is positive/negative if P is in +/- parts of $\mathcal{C}$.
4. $x = x(s)$, $y = y(s)$, $z = z(s)$ is called an arc length parametrisation of the curve. It depends on the reference point and direction.

Example.

$$x = a\cos t\,, \quad y = a\sin t\,, \quad 0 \le t \le 2\pi$$

Finding arc length parameterisation



$$s = \int_{t_0}^{t} \left| \frac{d\vec{r}}{du} \right| du$$

Since $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| > 0$ it is a positive change of parameter from t to s.

Example 1.

$$\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$$

Example 2. A line through $\vec{r}_0$ and parallel to $\vec{v}$

Properties of arc length parameterisation

1. $\forall t$, $\left|\frac{ds}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right| > 0$ (it is speed).
2. $\forall s$, the tangent vector has length 1: $\left|\frac{d\vec{r}}{ds}\right| = \left|\frac{ds}{ds}\right| = 1$
3. If $\left|\frac{d\vec{r}}{dt}\right| = 1$ $\forall t$, then $\forall t_0$, $s = t - t_0$ is the arc length parameter that as the reference point at $t = t_0$.

9 Unit Tangent, Normal and Binormal Vectors

The unit tangent vector:

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\,.$$

The unit normal vector to $\mathcal{C}$ at t:

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}\,, \qquad \vec{N}(t) \perp \vec{T}(t)\,.$$

Example.

$$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + vt\,\vec{k}$$

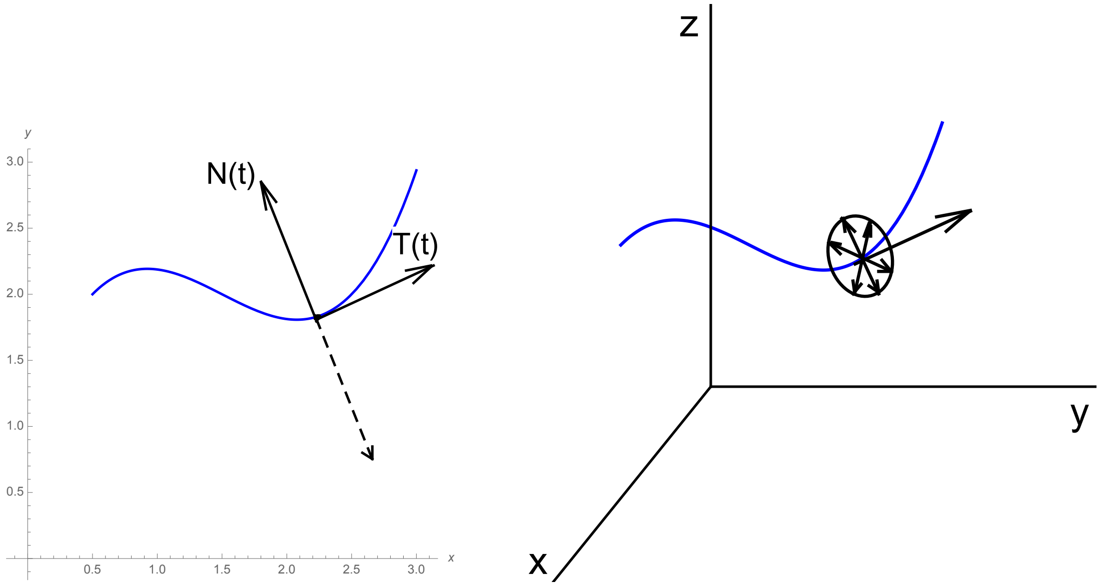


Figure 12: In 2-d $\exists$ two unit vectors $\perp \vec{T}$. In 3-d $\exists$ ∞ vectors $\perp \vec{T}$.

Inward unit normal vectors in 2-space

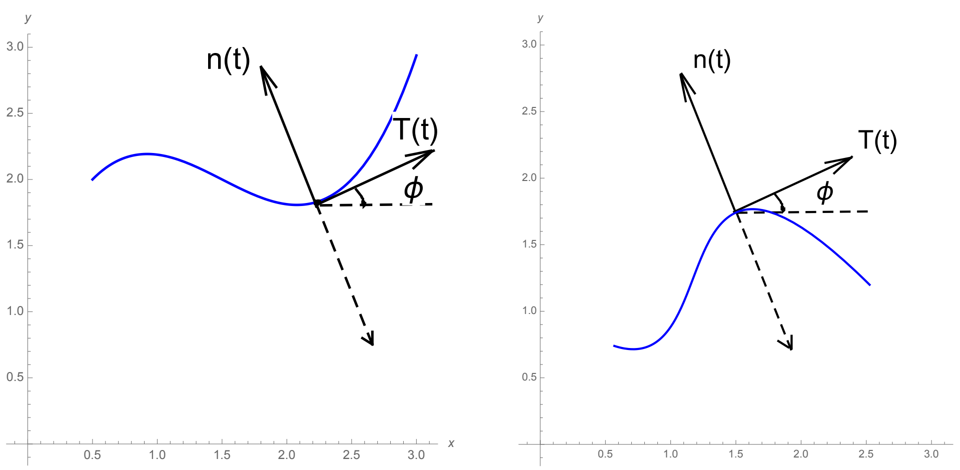


Figure 13: $\phi(t)$ increases: $d\phi/dt > 0$. $\phi(t)$ decreases: $d\phi/dt < 0$.

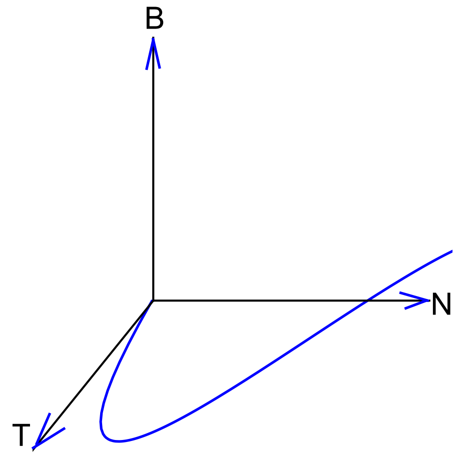
Let $\phi(t)$ be the angle from the positive x-axis to $\vec{T}(t)$, and let $\vec{n}(t)$ be the unit vector that results when $\vec{T}(t)$ is rotated counterclockwise through an angle of $\pi/2$:

$$\vec{T}(t) = \cos\phi\,\vec{i} + \sin\phi\,\vec{j}\,, \qquad \vec{n}(t) = -\sin\phi\,\vec{i} + \cos\phi\,\vec{j}\,,$$

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{d\phi}\,\frac{d\phi}{dt} = (-\sin\phi\,\vec{i} + \cos\phi\,\vec{j})\,\frac{d\phi}{dt} = \vec{n}(t)\,\frac{d\phi}{dt}\,.$$

Thus, $\frac{d\vec{T}}{dt}$ and $\vec{N}$ always point towards the concave side of $\mathcal{C}$. $\vec{N}$ is also called the **inward** unit normal if it is in 2-space.

Binormal vector in 3-space is



$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)\,,$$

$$\vec{T} \perp \vec{N}\,, \quad \vec{T} \perp \vec{B}\,, \quad \vec{N} \perp \vec{B}\,,$$

$$|\vec{T}| = |\vec{N}| = |\vec{B}| = 1\,.$$

Thus, $\vec{T}, \vec{N}, \vec{B}$ are three mutually orthogonal vectors, and they determine right-handed coordinate system in 3-space, which is called the TNB-frame or Frenet frame.

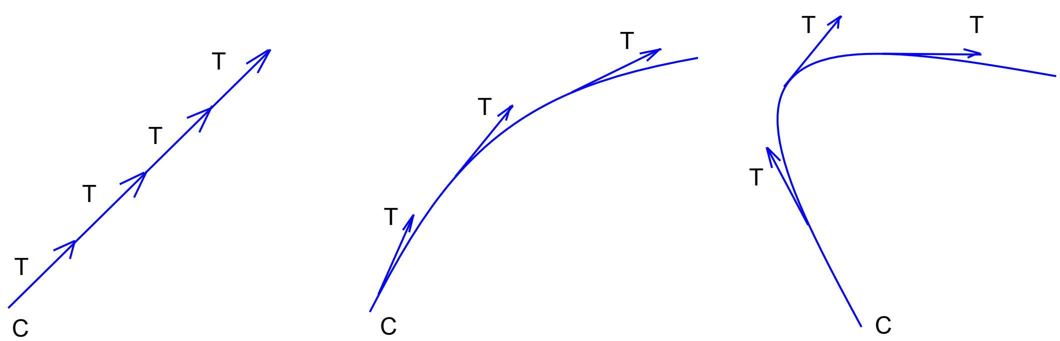
$\vec{T}, \vec{N}, \vec{B}$ in arc length parametrisation

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \vec{r}\,'(s)\,,$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \frac{\vec{r}\,''(s)}{|\vec{r}\,''(s)|}\,,$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{|\vec{r}\,'(t) \times \vec{r}\,''(t)|} = \frac{\vec{r}\,'(s) \times \vec{r}\,''(s)}{|\vec{r}\,''(s)|}\,.$$

Curvature



For a line $\frac{d\vec{T}}{ds} = 0$. If $\mathcal{C}$ bends slightly then $\vec{T}$ undergoes a gradual change of direction; if $\mathcal{C}$ bends sharply then $\vec{T}$ undergoes a rapid change of direction. Thus, $\frac{d\vec{T}}{ds}$ is a measure of "sharpness" of $\mathcal{C}$. In 3-space one should study $\frac{d\vec{T}}{ds}$, $\frac{d\vec{N}}{ds}$, $\frac{d\vec{B}}{ds}$.

Definition. If $\mathcal{C}$ is a smooth curve that is parameterised by arc length, then the **curvature** of $\mathcal{C}$ is

$$\kappa = \kappa(s) = \left|\frac{d\vec{T}}{ds}\right| = |\vec{r}\,''(s)|\,.$$

Thus, $\vec{T}\,'(s) = \kappa \vec{N}$.

Example 1.

$$\vec{r} = \vec{r}_0 + s\,\vec{u}$$

Example 2.

$$\vec{r} = a\cos\frac{s}{a}\,\vec{i} + a\sin\frac{s}{a}\,\vec{i}\,, \quad 0 \le s \le 2\pi a$$

Curvature for arbitrary parametrisation

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\right|\left|\frac{ds}{dt}\right| = \frac{\left|\vec{T}\,'(t)\right|}{|\vec{r}\,'(t)|}\,,$$

$$\vec{r}\,'(t) = |\vec{r}\,'(t)|\,\vec{T}(t) \quad \Longrightarrow \quad \vec{r}\,''(t) = |\vec{r}\,'(t)|'\,\vec{T}(t) + |\vec{r}\,'(t)|\,\vec{T}\,'(t)\,.$$

Then, $\vec{T}\,'(t) = \left|\vec{T}\,'(t)\right|\vec{N}(t)$ and $\left|\vec{T}\,'(t)\right| = \kappa(t)\,|\vec{r}\,'(t)|$

$$\Longrightarrow \quad \vec{T}\,'(t) = \kappa(t)\,|\vec{r}\,'(t)|\,\vec{N}(t)$$

$$\Longrightarrow \quad \vec{r}\,''(t) = |\vec{r}\,'(t)|'\,\vec{T}(t) + \kappa(t)\,|\vec{r}\,'(t)|^2\,\vec{N}(t)$$

$$\vec{T}(t) \times \vec{r}\,''(t) = \frac{1}{|\vec{r}\,'(t)|}\vec{r}\,'(t) \times \vec{r}\,''(t) = \kappa(t)\,|\vec{r}\,'(t)|^2\,\vec{B}(t)$$

$$\Longrightarrow \quad \kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

Example.

$$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + vt\,\vec{k}$$

Radius of curvature

If a curve $\mathcal{C}$ in 2-space has nonzero curvature κ at P then the circle of radius $\rho = 1/\kappa$ sharing a common tangent with $\mathcal{C}$ at P, and centred on the concave side of the curve at P, is called the **osculating** circle or circle of curvature at P. The osculating circle and $\mathcal{C}$ have equal curvatures at P. The radius ρ is called the radius of curvature at P, and the centre of the circle is the centre of curvature at P.

In 2-space we saw that

$$\vec{T}(\phi) = \cos\phi\,\vec{i} + \sin\phi\,\vec{j} \quad \Longrightarrow \quad \frac{d\vec{T}}{ds} = \frac{d\vec{T}}{d\phi}\,\frac{d\phi}{ds}$$

$$\Longrightarrow \quad \kappa(s) = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{d\phi}\right|\left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{ds}\right|.$$

Thus, the curvature in 2-space is the magnitude of the rate of change of ϕ with respect to s.