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QUANTUM MECHANICS I: WAVE FUNCTIONS

40.1. **IDENTIFY:** Using the momentum of the free electron, we can calculate k and ω and use these to express its wave function.

SET UP: $\Psi(x,t) = Ae^{ikx}e^{-i\omega t}$, $k = p/\hbar$, and $\omega = \hbar k^2/2m$.

EXECUTE: $k = \dfrac{p}{\hbar} = -\dfrac{4.50 \times 10^{-24} \text{ kg} \cdot \text{m/s}}{1.055 \times 10^{-34} \text{ J} \cdot \text{s}} = -4.27 \times 10^{10} \text{ m}^{-1}$.

$\omega = \dfrac{\hbar k^2}{2m} = \dfrac{(1.055 \times 10^{-34} \text{ J} \cdot \text{s})(4.27 \times 10^{10} \text{ m}^{-1})^2}{2(9.108 \times 10^{-31} \text{ kg})} = 1.05 \times 10^{17} \text{ s}^{-1}$.

$\Psi(x,t) = Ae^{-i(4.27 \times 10^{10} \text{ m}^{-1})x}e^{-i(1.05 \times 10^{17} \text{ s}^{-1})t}$.

EVALUATE: The wave function depends on position and time.

40.2. **IDENTIFY:** Using the known wave function for the particle, we want to find where its probability function is a maximum.

SET UP: $|\Psi(x,t)|^2 = |A|^2 (e^{ikx}e^{-i\omega t} - e^{2ikx}e^{-4i\omega t})(e^{-ikx}e^{+i\omega t} - e^{-2ikx}e^{+4i\omega t})$.

$|\Psi(x,t)|^2 = |A|^2 \left[2 - (e^{-i(kx-3\omega t)} + e^{+i(kx-3\omega t)})\right] = 2|A|^2 [1 - \cos(kx - 3\omega t)]$.

EXECUTE: **(a)** For $t = 0$, $|\Psi(x,t)|^2 = 2|A|^2 (1 - \cos(kx))$. $|\Psi(x,t)|^2$ is a maximum when $\cos(kx) = -1$ and this happens when $kx = (2n+1)\pi$, $n = 0, 1, \dots$. $|\Psi(x,t)|^2$ is a maximum for $x = \dfrac{\pi}{k}, \dfrac{3\pi}{k}$, etc.

(b) $t = \dfrac{2\pi}{\omega}$ and $3\omega t = 6\pi$. $|\Psi(x,t)|^2 = 2|A|^2 [1 - \cos(kx - 6\pi)]$. Maximum for $kx - 6\pi = \pi, 3\pi, \dots$, which gives maxima when $x = \dfrac{7\pi}{k}, \dfrac{9\pi}{k}$.

(c) From the results for parts (a) and (b), $v_{av} = \dfrac{7\pi/k - \pi/k}{2\pi/\omega} = \dfrac{3\omega}{k}$. $v_{av} = \dfrac{\omega_2 - \omega_1}{k_2 - k_1}$ with $\omega_2 = 4\omega$, $\omega_1 = \omega$, $k_2 = 2k$ and $k_1 = k$ gives $v_{av} = \dfrac{3\omega}{k}$.

EVALUATE: The expressions in part (c) agree.

40.3. **IDENTIFY:** Use the wave function from Example 40.1.

SET UP: $|\Psi(x,t)|^2 = 2|A|^2 \{1 + \cos[(k_2 - k_1)x - (\omega_2 - \omega_1)t]\}$. $k_2 = 3k_1 = 3k$. $\omega = \dfrac{\hbar k^2}{2m}$, so $\omega_2 = 9\omega_1 = 9\omega$.

$|\Psi(x,t)|^2 = 2|A|^2 [1 + \cos(2kx - 8\omega t)]$.

EXECUTE: **(a)** At $t = 2\pi/\omega$, $|\Psi(x,t)|^2 = 2|A|^2[1 + \cos(2kx - 16\pi)]$. $|\Psi(x,t)|^2$ is maximum for

$\cos(2kx - 16\pi) = 1$. This happens for $2kx - 16\pi = 0, 2\pi,\dots$. Smallest positive x where $|\Psi(x,t)|^2$ is a

maximum is $x = \dfrac{8\pi}{k}$.

(b) From the result of part (a), $v_{av} = \dfrac{8\pi/k}{2\pi/\omega} = \dfrac{4\omega}{k}$. $v_{av} = \dfrac{\omega_2 - \omega_1}{k_2 - k_1} = \dfrac{8\omega}{2k} = \dfrac{4\omega}{k}$.

EVALUATE: The two expressions agree.

40.4. IDENTIFY: Apply the Heisenberg uncertainty principle in the form $\Delta x \Delta p_x \geq \hbar/2$.

SET UP: The uncertainty in the particle position is proportional to the width of $\psi(x)$.

EXECUTE: **(a)** The width of $\psi(x)$ is inversely proportional to $\sqrt{\alpha}$. This can be seen by either plotting the function for different values of α or by finding the full width at half-maximum. The particle's uncertainty in position decreases with increasing α.

(b) Since the uncertainty in position decreases, the uncertainty in momentum must increase.

EVALUATE: As α increases, the function $A(k)$ in Eq. (40.19) must become broader.

40.5. IDENTIFY and SET UP: $\psi(x) = A \sin kx$. The position probability density is given by $|\psi(x)|^2 = A^2 \sin^2 kx$.

EXECUTE: **(a)** The probability is highest where $\sin kx = 1$ so $kx = 2\pi x/\lambda = n\pi/2$, $n = 1, 3, 5, \dots$

$x = n\lambda/4$, $n = 1, 3, 5, \dots$ so $x = \lambda/4, 3\lambda/4, 5\lambda/4, \dots$

(b) The probability of finding the particle is zero where $|\psi|^2 = 0$, which occurs where $\sin kx = 0$ and

$kx = 2\pi x/\lambda = n\pi$, $n = 0, 1, 2, \dots$

$x = n\lambda/2$, $n = 0, 1, 2, \dots$, so $x = 0, \lambda/2, \lambda, 3\lambda/2, \dots$

EVALUATE: The situation is analogous to a standing wave, with the probability analogous to the square of the amplitude of the standing wave.

40.6. IDENTIFY and SET UP: $|\Psi|^2 = \Psi^* \Psi$.

EXECUTE: $\Psi^* = \psi^* \sin \omega t$, so $|\Psi|^2 = \Psi^* \Psi = \psi^* \psi \sin^2 \omega t = |\psi|^2 \sin^2 \omega t$. $|\Psi|^2$ is not time-independent, so Ψ is not the wave function for a stationary state.

EVALUATE: $\Psi = \psi e^{i\omega\phi} = \psi(\cos \omega t + i \sin \omega t)$ is a wave function for a stationary state, since for it

$|\Psi|^2 = |\psi|^2$, which is time independent.

40.7. IDENTIFY: Determine whether or not $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi$ is equal to $E\psi$, for some value of E.

SET UP: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_1}{dx^2} + U\psi_1 = E_1\psi_1$ and $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi_2}{dx^2} + U\psi_2 = E_2\psi_2$.

EXECUTE: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi = BE_1\psi_1 + CE_2\psi_2$. If ψ were a solution with energy E, then

$BE_1\psi_1 + CE_2\psi_2 = BE\psi_1 + CE\psi_2$ or $B(E_1 - E)\psi_1 = C(E - E_2)\psi_2$. This would mean that ψ_1 is a constant multiple of ψ_2, and ψ_1 and ψ_2 would be wave functions with the same energy. However, $E_1 \neq E_2$, so this is

not possible, and ψ cannot be a solution to the equation $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi = E\psi$.

EVALUATE: ψ is a solution if $E_1 = E_2$.

40.8. IDENTIFY: To describe a real situation, a wave function must be normalizable.

SET UP: $|\psi|^2 dV$ is the probability that the particle is found in volume dV. Since the particle must be

somewhere, ψ must have the property that $\int |\psi|^2 dV = 1$ when the integral is taken over all space.

EXECUTE: **(a)** For normalization of the one-dimensional wave function, we have

$$1 = \int_{-\infty}^{\infty} |\psi|^2 \, dx = \int_{-\infty}^{0} (Ae^{bx})^2 \, dx + \int_{0}^{\infty} (Ae^{-bx})^2 \, dx = \int_{-\infty}^{0} A^2 e^{2bx} \, dx + \int_{0}^{\infty} A^2 e^{-2bx} \, dx.$$

$$1 = A^2 \left\{ \frac{e^{2bx}}{2b} \bigg|_{-\infty}^{0} + \frac{e^{-2bx}}{-2b} \bigg|_{0}^{\infty} \right\} = \frac{A^2}{b}, \text{ which gives } A = \sqrt{b} = \sqrt{2.00 \text{ m}^{-1}} = 1.41 \text{ m}^{-1/2}.$$

(b) The graph of the wavefunction versus x is given in Figure 40.8.

(c) (i) $P = \int_{-0.500 \text{ m}}^{+0.500 \text{ m}} |\psi|^2 \, dx = 2 \int_{0}^{+0.500 \text{ m}} A^2 e^{-2bx} \, dx$, where we have used the fact that the wave function is an

even function of x. Evaluating the integral gives

$$P = \frac{-A^2}{b} (e^{-2b(0.500 \text{ m})} - 1) = \frac{-(2.00 \text{ m}^{-1})}{2.00 \text{ m}^{-1}} (e^{-2.00} - 1) = 0.865.$$

There is a little more than an 86% probability that the particle will be found within 50 cm of the origin.

(ii) $P = \int_{-\infty}^{0} (Ae^{bx})^2 \, dx = \int_{-\infty}^{0} A^2 e^{2bx} \, dx = \frac{A^2}{2b} = \frac{2.00 \text{ m}^{-1}}{2(2.00 \text{ m}^{-1})} = \frac{1}{2} = 0.500.$

There is a 50-50 chance that the particle will be found to the left of the origin, which agrees with the fact that the wave function is symmetric about the y-axis.

(iii) $P = \int_{0.500 \text{ m}}^{1.00 \text{ m}} A^2 e^{-2bx} \, dx$

$$= \frac{A^2}{-2b} (e^{-2(2.00 \text{ m}^{-1})(1.00 \text{ m})} - e^{-2(2.00 \text{ m}^{-1})(0.500 \text{ m})}) = -\frac{1}{2}(e^{-4} - e^{-2}) = 0.0585.$$

EVALUATE: There is little chance of finding the particle in regions where the wave function is small.
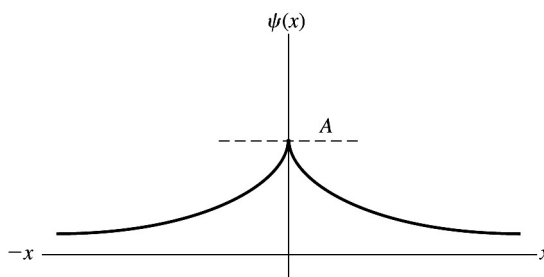

Figure 40.8

40.9. **IDENTIFY** and **SET UP:** The energy levels for a particle in a box are given by $E_n = \frac{n^2 h^2}{8mL^2}$.

EXECUTE: **(a)** The lowest level is for $n = 1$, and $E_1 = \frac{(1)(6.626 \times 10^{-34} \text{ J} \cdot \text{s})^2}{8(0.20 \text{ kg})(1.3 \text{ m})^2} = 1.6 \times 10^{-67}$ J.

(b) $E = \frac{1}{2} mv^2$ so $v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2(1.2 \times 10^{-67} \text{ J})}{0.20 \text{ kg}}} = 1.3 \times 10^{-33}$ m/s. If the ball has this speed the time it would

take it to travel from one side of the table to the other is

$$t = \frac{1.3 \text{ m}}{1.3 \times 10^{-33} \text{ m/s}} = 1.0 \times 10^{33} \text{ s}.$$

(c) $E_1 = \frac{h^2}{8mL^2}$, $E_2 = 4E_1$, so $\Delta E = E_2 - E_1 = 3E_1 = 3(1.6 \times 10^{-67} \text{ J}) = 4.9 \times 10^{-67}$ J.

EVALUATE: **(d)** No, quantum mechanical effects are not important for the game of billiards. The discrete, quantized nature of the energy levels is completely unobservable.

40.10. **IDENTIFY:** Solve the energy-level equation $E_n = \dfrac{n^2 h^2}{8mL^2}$ for L.

SET UP: The ground state has $n = 1$.

EXECUTE: $L = \dfrac{h}{\sqrt{8mE_1}} = \dfrac{(6.626\times 10^{-34} \text{ J}\cdot\text{s})}{\sqrt{8(1.673\times 10^{-27} \text{ kg})(5.0\times 10^6 \text{ eV})(1.602\times 10^{-19} \text{ J/eV})}} = 6.4\times 10^{-15}$ m

EVALUATE: The value of L we calculated is on the order of the diameter of a nucleus.

40.11. **IDENTIFY:** An electron in the lowest energy state in this box must have the same energy as it would in the ground state of hydrogen.

SET UP: The energy of the nth level of an electron in a box is $E_n = \dfrac{nh^2}{8mL^2}$.

EXECUTE: An electron in the ground state of hydrogen has an energy of -13.6 eV, so find the width corresponding to an energy of $E_1 = 13.6$ eV. Solving for L gives

$L = \dfrac{h}{\sqrt{8mE_1}} = \dfrac{(6.626\times 10^{-34} \text{ J}\cdot\text{s})}{\sqrt{8(9.11\times 10^{-31} \text{ kg})(13.6 \text{ eV})(1.602\times 10^{-19} \text{ J/eV})}} = 1.66\times 10^{-10}$ m.

EVALUATE: This width is of the same order of magnitude as the diameter of a Bohr atom with the electron in the K shell.

40.12. **IDENTIFY** and **SET UP:** The energy of a photon is $E = hf = h\dfrac{c}{\lambda}$. The energy levels of a particle in a box are given by $E_n = \dfrac{nh^2}{8mL^2}$.

EXECUTE: **(a)** $E = (6.63\times 10^{-34} \text{ J}\cdot\text{s})\dfrac{(3.00\times 10^8 \text{ m/s})}{(122\times 10^{-9} \text{ m})} = 1.63\times 10^{-18}$ J. $\Delta E = \dfrac{h^2}{8mL^2}(n_1^2 - n_2^2)$.

$L = \sqrt{\dfrac{h^2(n_1^2 - n_2^2)}{8m\Delta E}} = \sqrt{\dfrac{(6.63\times 10^{-34} \text{ J}\cdot\text{s})^2(2^2 - 1^2)}{8(9.11\times 10^{-31} \text{ kg})(1.63\times 10^{-18} \text{ J})}} = 3.33\times 10^{-10}$ m.

(b) The ground state energy for an electron in a box of the calculated dimensions is

$E = \dfrac{h^2}{8mL^2} = \dfrac{(6.63\times 10^{-34} \text{ J}\cdot\text{s})^2}{8(9.11\times 10^{-31} \text{ kg})(3.33\times 10^{-10} \text{ m})^2} = 5.43\times 10^{-19}$ J $= 3.40$ eV (one-third of the original photon energy), which does not correspond to the -13.6 eV ground state energy of the hydrogen atom.

EVALUATE: **(c)** Note that the energy levels for a particle in a box are proportional to n^2, whereas the energy levels for the hydrogen atom are proportional to $-\dfrac{1}{n^2}$. A one-dimensional box is not a good model for a hydrogen atom.

40.13. **IDENTIFY** and **SET UP:** The equation $E_n = \dfrac{n^2 h^2}{8mL^2}$ gives the energy levels. Use this to obtain an expression for $E_2 - E_1$ and use the value given for this energy difference to solve for L.

EXECUTE: Ground state energy is $E_1 = \dfrac{h^2}{8mL^2}$; first excited state energy is $E_2 = \dfrac{4h^2}{8mL^2}$. The energy separation between these two levels is $\Delta E = E_2 - E_1 = \dfrac{3h^2}{8mL^2}$. This gives

$L = h\sqrt{\dfrac{3}{8m\Delta E}} = L = 6.626\times 10^{-34} \text{ J}\cdot\text{s}\sqrt{\dfrac{3}{8(9.109\times 10^{-31} \text{ kg})(3.0 \text{ eV})(1.602\times 10^{-19} \text{ J/1 eV})}}$

$= 6.1\times 10^{-10}$ m $= 0.61$ nm.

EVALUATE: This energy difference is typical for an atom and L is comparable to the size of an atom.

40.14. **IDENTIFY:**   The energy of the absorbed photon must be equal to the energy difference between the two states.

SET UP and EXECUTE: The second excited state energy is $E_3 = \dfrac{9\pi^2\hbar^2}{2mL^2}$. The ground state energy is

$E_1 = \dfrac{\pi^2\hbar^2}{2mL^2}$. $E_1 = 2.00$ eV, so $E_3 = 18.0$ eV. For the transition $\Delta E = \dfrac{4\pi^2\hbar^2}{mL^2}$. $\dfrac{hc}{\lambda} = \Delta E$.

$\lambda = \dfrac{hc}{\Delta E} = \dfrac{(4.136\times10^{-15}\text{ eV}\cdot\text{s})(2.998\times10^8\text{ m/s})}{16.0\text{ eV}} = 7.75\times10^{-8}$ m $= 77.5$ nm.

EVALUATE: This wavelength is much shorter than those of visible light.

40.15. **IDENTIFY:**   To describe a real situation, a wave function must be normalizable.

SET UP: $|\psi|^2\ dV$ is the probability that the particle is found in volume dV. Since the particle must be

somewhere, ψ must have the property that $\int|\psi|^2 dV = 1$ when the integral is taken over all space.

EXECUTE: **(a)** In one dimension, as we have here, the integral discussed above is of the form

$\int_{-\infty}^{\infty}|\psi(x)|^2 dx = 1$.

(b) Using the result from part (a), we have $\int_{-\infty}^{\infty}(e^{ax})^2 dx = \int_{-\infty}^{\infty}e^{2ax}dx = \dfrac{e^{2ax}}{2a}\Big|_{-\infty}^{\infty} = \infty$. Hence this wave

function cannot be normalized and therefore cannot be a valid wave function.

(c) We only need to integrate this wave function of 0 to ∞ because it is zero for $x < 0$. For normalization we

have $1 = \int_{-\infty}^{\infty}|\psi|^2 dx = \int_0^{\infty}(Ae^{-bx})^2 dx = \int_0^{\infty}A^2 e^{-2bx}dx = \dfrac{A^2 e^{-2bx}}{-2b}\Big|_0^{\infty} = \dfrac{A^2}{2b}$, which gives $\dfrac{A^2}{2b} = 1$, so $A = \sqrt{2b}$.

EVALUATE: If b were negative, the given wave function could not be normalized, so it would not be allowable.

40.16. **IDENTIFY:**   Find x where ψ_1 is zero and where it is a maximum.

SET UP: $\psi_1 = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{\pi x}{L}\right)$.

EXECUTE: **(a)** The wave function for $n = 1$ vanishes only at $x = 0$ and $x = L$ in the range $0 \le x \le L$.

(b) In the range for x, the sine term is a maximum only at the middle of the box, $x = L/2$.

EVALUATE: **(c)** The answers to parts (a) and (b) are consistent with the figure.

40.17. **IDENTIFY and SET UP:**   For the $n = 2$ first excited state the normalized wave function is given by the

equation $\psi_2(x) = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{2\pi x}{L}\right)$. $|\psi_2(x)|^2 dx = \dfrac{2}{L}\sin^2\left(\dfrac{2\pi x}{L}\right)dx$. Examine $|\psi_2(x)|^2 dx$ and find where it is

zero and where it is maximum.

EXECUTE: **(a)** $|\psi_2|^2 dx = 0$ implies $\sin\left(\dfrac{2\pi x}{L}\right) = 0$.

$\dfrac{2\pi x}{L} = m\pi$, $m = 0, 1, 2, \ldots$; $x = m(L/2)$.

For $m = 0$, $x = 0$; for $m = 1$, $x = L/2$; for $m = 2$, $x = L$.

The probability of finding the particle is zero at $x = 0$, $L/2$, and L.

(b) $|\psi_2|^2 dx$ is maximum when $\sin\left(\dfrac{2\pi x}{L}\right) = \pm 1$.

$\dfrac{2\pi x}{L} = m(\pi/2)$, $m = 1, 3, 5, \ldots$; $x = m(L/4)$.

For $m = 1$, $x = L/4$; for $m = 3$, $x = 3L/4$.

The probability of finding the particle is largest at $x = L/4$ and $3L/4$.

EVALUATE: (c) The answers to part (a) correspond to the zeros of $|\psi|^2$ shown in Figure 40.12 in the textbook and the answers to part (b) correspond to the two values of x where $|\psi|^2$ in the figure is maximum.

40.18. **IDENTIFY:** The energy levels are given by $E_n = \dfrac{n^2 h^2}{8mL^2}$. The wavelength λ of the photon absorbed in an atomic transition is related to the transition energy ΔE by $\lambda = \dfrac{hc}{\Delta E}$.

SET UP: For the ground state $n = 1$ and for the third excited state $n = 4$.

EXECUTE: (a) The third excited state is $n = 4$, so

$$\Delta E = (4^2 - 1)\frac{h^2}{8mL^2} = \frac{15(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11\times10^{-31}\ \text{kg})(0.360\times10^{-9}\ \text{m})^2} = 6.973\times10^{-18}\ \text{J} = 43.5\ \text{eV}.$$

(b) $\lambda = \dfrac{hc}{\Delta E} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.0\times10^8\ \text{m/s})}{6.973\times10^{-18}\ \text{J}} = 28.5\ \text{nm}.$

EVALUATE: This photon is an x ray. As the width of the box increases the transition energy for this transition decreases and the wavelength of the photon increases.

40.19. **IDENTIFY and SET UP:** $\lambda = \dfrac{h}{p} = \dfrac{h}{\sqrt{2mE}}$. The energy of the electron in level n is given By the equation

$$E_n = \frac{n^2 h^2}{8mL^2}.$$

EXECUTE: (a) $E_1 = \dfrac{h^2}{8mL^2} \Rightarrow \lambda_1 = \dfrac{h}{\sqrt{2mh^2/8mL^2}} = 2L = 2(3.0\times10^{-10}\ \text{m}) = 6.0\times10^{-10}\ \text{m}$. The wavelength

is twice the width of the box. $p_1 = \dfrac{h}{\lambda_1} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})}{6.0\times10^{-10}\ \text{m}} = 1.1\times10^{-24}\ \text{kg}\cdot\text{m/s}.$

(b) $E_2 = \dfrac{4h^2}{8mL^2} \Rightarrow \lambda_2 = L = 3.0\times10^{-10}\ \text{m}$. The wavelength is the same as the width of the box.

$p_2 = \dfrac{h}{\lambda_2} = 2p_1 = 2.2\times10^{-24}\ \text{kg}\cdot\text{m/s}.$

(c) $E_3 = \dfrac{9h^2}{8mL^2} \Rightarrow \lambda_3 = \dfrac{2}{3}L = 2.0\times10^{-10}\ \text{m}$. The wavelength is two-thirds the width of the box.

$p_3 = 3p_1 = 3.3\times10^{-24}\ \text{kg}\cdot\text{m/s}.$

EVALUATE: In each case the wavelength is an integer multiple of $\lambda/2$. In the nth state, $p_n = np_1$.

40.20. **IDENTIFY:** The energy of the photon is equal to the energy difference ΔE between the energy levels of the electron.

SET UP: The energy levels of an electron in a one-dimensional box are $E_n = \dfrac{n^2 h^2}{8mL^2}$. The energy of the absorbed photon is $\Delta E = \dfrac{hc}{\lambda}$.

EXECUTE: (a) $\Delta E_{1\to2} = (h^2/8mL^2)(2^2 - 1^2) = 3(h^2/8mL^2) = hc/\lambda_{1\to2}.$

$\Delta E_{2\to3} = (h^2/8mL^2)(3^2 - 2^2) = 5(h^2/8mL^2) = hc/\lambda_{2\to3}.$ Take ratios of these two equations, giving

$\dfrac{3}{5} = \dfrac{hc/\lambda_{1\to2}}{hc/\lambda_{2\to3}} = \dfrac{\lambda_{2\to3}}{\lambda_{1\to2}} \qquad \to \qquad \lambda_{2\to3} = (3/5)\lambda_{1\to2} = (3/5)(426\ \text{nm}) = 256\ \text{nm}.$

(b) Follow the same procedure as in part (a), giving

$\lambda_{1\to3} = (3/8)\lambda_{1\to2} = (3/8)(426\ \text{nm}) = 160\ \text{nm}.$

(c) From part (a), we know that $\Delta E_{1\to 2} = \dfrac{3h^2}{8mL^2} = \dfrac{hc}{\lambda_{1\to 2}}$. Solving for L gives

$$L = \sqrt{\frac{3h\lambda_{1\to 2}}{8\,mc}} = \sqrt{\frac{3(6.626\times 10^{-34}\ \text{J}\cdot\text{s})(426\times 10^{-9}\ \text{m})}{8(9.109\times 10^{-31}\ \text{kg})(2.998\times 10^{8}\ \text{m/s})}} = 6.23\times 10^{-10}\ \text{m} = 0.623\ \text{nm}.$$

EVALUATE: The width L of this box is about 6 times the diameter of a hydrogen atom.

40.21.   IDENTIFY: Figure 40.15b in the textbook gives values for the bound state energy of a square well for which $U_0 = 6E_{1\text{-IDW}}$.

SET UP: $E_{1\text{-IDW}} = \dfrac{\pi^2\hbar^2}{2mL^2}$.

EXECUTE: $E_1 = 0.625E_{1\text{-IDW}} = 0.625\dfrac{\pi^2\hbar^2}{2mL^2}$; $E_1 = 2.00\ \text{eV} = 3.20\times 10^{-19}\ \text{J}$.

$$L = \pi\hbar\left(\frac{0.625}{2(9.109\times 10^{-31}\ \text{kg})(3.20\times 10^{-19}\ \text{J})}\right)^{1/2} = 3.43\times 10^{-10}\ \text{m}.$$

EVALUATE: As L increases the ground state energy decreases.

40.22.   IDENTIFY: $\lambda = \dfrac{h}{p}$. p is related to E by $E = \dfrac{p^2}{2m} + U$.

SET UP: For $x > L$, $U = U_0$. For $0 < x < L$, $U = 0$.

EXECUTE: For $0 < x < L$, $p = \sqrt{2mE} = \sqrt{2m(3U_0)}$ and $\lambda_{\text{in}} = \dfrac{h}{\sqrt{2m(3U_0)}}$. For $x > L$,

$p = \sqrt{2m(E - U_0)} = \sqrt{2m(2U_0)}$ and $\lambda_{\text{out}} = \dfrac{h}{\sqrt{2m(E - U_0)}} = \dfrac{h}{\sqrt{2m(2U_0)}}$. Thus, the ratio of the

wavelengths is $\dfrac{\lambda_{\text{out}}}{\lambda_{\text{in}}} = \dfrac{\sqrt{2m(3U_0)}}{\sqrt{2m(2U_0)}} = \sqrt{\dfrac{3}{2}}$.

EVALUATE: For $x > L$ some of the energy is potential and the kinetic energy is less than it is for $0 < x < L$, where $U = 0$. Therefore, outside the box p is less and λ is greater than inside the box.

40.23.   IDENTIFY: The energy of the photon is the energy given to the electron.

SET UP: Since $U_0 = 6E_{1\text{-IDW}}$ we can use the result $E_1 = 0.625E_{1\text{-IDW}}$ from Section 40.4. When the electron is outside the well it has potential energy U_0, so the minimum energy that must be given to the electron is $U_0 - E_1 = 5.375E_{1\text{-IDW}}$.

EXECUTE: The maximum wavelength of the photon would be

$$\lambda = \frac{hc}{U_0 - E_1} = \frac{hc}{(5.375)(h^2/8mL^2)} = \frac{8mL^2c}{(5.375)h} = \frac{8(9.11\times 10^{-31}\ \text{kg})(1.50\times 10^{-9}\ \text{m})^2(3.00\times 10^{8}\ \text{m/s})}{(5.375)(6.63\times 10^{-34}\ \text{J}\cdot\text{s})}$$

$$= 1.38\times 10^{-6}\ \text{m}.$$

EVALUATE: This photon is in the infrared. The wavelength of the photon decreases when the width of the well decreases.

40.24.   IDENTIFY: In a finite potential well, the energy levels are lowered compared to the energy levels in an infinite well. The energy of the photon removes the electron from its energy state in the well and any left-over energy is the kinetic energy K of the electron.

SET UP: The energy levels for an infinitely deep well are $E_{n\text{-IDW}} = \dfrac{n^2 h^2}{8mL^2}$, and $n = 1$ is the ground state.
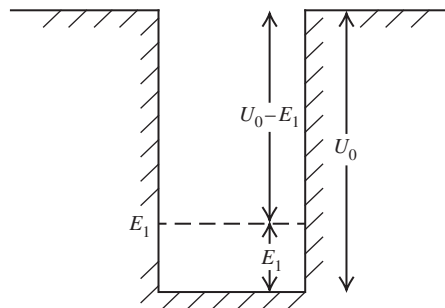
The energy of a photon is $E = hc/\lambda$.

Figure 40.24

EXECUTE: Figure 40.24 illustrates the various energies involved. In this case, $U_0 = 6E_{1\text{-IDW}}$. Figure 40.15(b) in the textbook shows that the ground state energy E_1 in the finite well is $E_1 = 0.625E_{1\text{-IDW}}$. The electron already has energy E_1 in the well, so the energy just to remove it from the well is $U_0 - E_1$. Conservation of energy gives $E_{\text{photon}} = E_{\text{remove el}} + K$, which we can write as

$$\frac{hc}{\lambda} = (U_0 - E_1) + K = 6E_{1\text{-IDW}} - 0.625E_{1\text{-IDW}} + K = 5.375E_{1\text{-IDW}} + K.$$

Solving for K and using $E_{1\text{-IDW}} = \dfrac{h^2}{8mL^2}$ gives

$$K = \frac{hc}{\lambda} - 5.375E_{1\text{-IDW}} = \frac{hc}{\lambda} - \frac{5.735h^2}{8mL^2}.$$

Using $\lambda = 72\times10^{-9}$ m and $L = 4.00\times10^{-10}$ m, plus the usual values of the constants h, c, and m, we get $K = 2.76\times10^{-18}$ J $- 2.02\times10^{-18}$ J $= 7.4\times10^{-19}$ J, which we can express in electron-volts as $K = 17.2$ eV $- 12.6$ eV $= 4.6$ eV.

EVALUATE: The photon has 17.2 eV and it takes 12.6 eV just to remove the electron from the well, so the remaining 4.6 eV is the kinetic energy of the electron.

40.25. **IDENTIFY:** Find the transition energy ΔE and set it equal to the energy of the absorbed photon. Use $E = hc/\lambda$, to find the wavelength of the photon.

SET UP: $U_0 = 6E_{1\text{-IDW}}$, as in Figure 40.15 in the textbook, so $E_1 = 0.625E_{1\text{-IDW}}$ and $E_3 = 5.09E_{1\text{-IDW}}$ with $E_{1\text{-IDW}} = \dfrac{\pi^2\hbar^2}{2mL^2}$. In this problem the particle bound in the well is a proton, so $m = 1.673\times10^{-27}$ kg.

EXECUTE: $E_{1\text{-IDW}} = \dfrac{\pi^2\hbar^2}{2mL^2} = \dfrac{\pi^2(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{2(1.673\times10^{-27}\text{ kg})(4.0\times10^{-15}\text{ m})^2} = 2.052\times10^{-12}$ J. The transition energy is $\Delta E = E_3 - E_1 = (5.09 - 0.625)E_{1\text{-IDW}} = 4.465E_{1\text{-IDW}}$. $\Delta E = 4.465(2.052\times10^{-12}\text{ J}) = 9.162\times10^{-12}$ J. The wavelength of the photon that is absorbed is related to the transition energy by $\Delta E = hc/\lambda$, so

$$\lambda = \frac{hc}{\Delta E} = \frac{(6.626\times10^{-34}\text{ J}\cdot\text{s})(2.998\times10^8\text{ m/s})}{9.162\times10^{-12}\text{ J}} = 2.2\times10^{-14}\text{ m} = 22\text{ fm}.$$

EVALUATE: The wavelength of the photon is comparable to the size of the well.

40.26. **IDENTIFY:** The longest wavelength corresponds to the smallest energy change.

SET UP: The ground level energy level of the infinite well is $E_{1\text{-IDW}} = \dfrac{h^2}{8mL^2}$, and the energy of the photon must be equal to the energy difference between the two levels.

EXECUTE: The 582-nm photon must correspond to the $n = 1$ to $n = 2$ transition. Since $U_0 = 6E_{1\text{-IDW}}$, we have $E_2 = 2.43E_{1\text{-IDW}}$ and $E_1 = 0.625E_{1\text{-IDW}}$. The energy of the photon is equal to the energy difference between the two levels, and $E_{1\text{-IDW}} = \dfrac{h^2}{8mL^2}$, which gives

$E_\gamma = E_2 - E_1 \Rightarrow \dfrac{hc}{\lambda} = (2.43 - 0.625)E_{\text{1-1DW}} = \dfrac{1.805\, h^2}{8mL^2}$. Solving for L gives

$L = \sqrt{\dfrac{(1.805)h\lambda}{8mc}} = \sqrt{\dfrac{(1.805)(6.626\times 10^{-34}\ \text{J}\cdot\text{s})(5.82\times 10^{-7}\ \text{m})}{8(9.11\times 10^{-31}\ \text{kg})(3.00\times 10^8\ \text{m/s})}} = 5.64\times 10^{-10}\ \text{m} = 0.564\ \text{nm}.$

EVALUATE: This width is slightly more than half that of a Bohr hydrogen atom.

40.27.   **IDENTIFY and SET UP:**   The probability is $T = Ge^{-2\kappa L}$, with $G = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$.

$E = 32$ eV, $U_0 = 41$ eV, $L = 0.25\times 10^{-9}$ m. Calculate T.

EXECUTE: **(a)** $G = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right) = 16\dfrac{32}{41}\left(1 - \dfrac{32}{41}\right) = 2.741.$

$\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}.$

$\kappa = \dfrac{\sqrt{2(9.109\times 10^{-31}\ \text{kg})(41\ \text{eV} - 32\ \text{eV})(1.602\times 10^{-19}\ \text{J/eV})}}{1.055\times 10^{-34}\ \text{J}\cdot\text{s}} = 1.536\times 10^{10}\ \text{m}^{-1}.$

$T = Ge^{-2\kappa L} = (2.741)e^{-2(1.536\times 10^{10}\ \text{m}^{-1})(0.25\times 10^{-9}\ \text{m})} = 2.741e^{-7.68} = 0.0013.$

(b) The only change is the mass m, which appears in κ.

$\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}.$

$\kappa = \dfrac{\sqrt{2(1.673\times 10^{-27}\ \text{kg})(41\ \text{eV} - 32\ \text{eV})(1.602\times 10^{-19}\ \text{J/eV})}}{1.055\times 10^{-34}\ \text{J}\cdot\text{s}} = 6.584\times 10^{11}\ \text{m}^{-1}.$

Then $T = Ge^{-2\kappa L} = (2.741)e^{-2(6.584\times 10^{11}\ \text{m}^{-1})(0.25\times 10^{-9}\ \text{m})} = 2.741e^{-392.2} = 10^{-143}.$

EVALUATE: The more massive proton has a much smaller probability of tunneling than the electron does.

40.28.   **IDENTIFY:**   The tunneling probability is $T = Ge^{-2\kappa L}$, with $G = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$. so

$T = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)e^{\frac{-2\sqrt{2m(U_0 - E)}}{\hbar}L}.$

SET UP: $U_0 = 30.0\times 10^6$ eV, $L = 2.0\times 10^{-15}$ m, $m = 6.64\times 10^{-27}$ kg.

EXECUTE: **(a)** $U_0 - E = 1.0\times 10^6$ eV $(E = 29.0\times 10^6$ eV), $T = 0.090.$

(b) If $U_0 - E = 10.0\times 10^6$ eV $(E = 20.0\times 10^6$ eV), $T = 0.014.$

EVALUATE: T is less when $U_0 - E$ is 10.0 MeV than when $U_0 - E$ is 1.0 MeV.

40.29.   **IDENTIFY:**   The tunneling probability is $T = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0 - E)}/\hbar}.$

SET UP: $\dfrac{E}{U_0} = \dfrac{6.0\ \text{eV}}{11.0\ \text{eV}}$ and $E - U_0 = 5\ \text{eV} = 8.0\times 10^{-19}\ \text{J}.$

EXECUTE: **(a)** $L = 0.80\times 10^{-9}$ m:

$T = 16\left(\dfrac{6.0\ \text{eV}}{11.0\ \text{eV}}\right)\left(1 - \dfrac{6.0\ \text{ev}}{11.0\ \text{eV}}\right)e^{-2(0.80\times 10^{-9}\ \text{m})\sqrt{2(9.11\times 10^{-31}\ \text{kg})(8.0\times 10^{-19}\ \text{J})}/1.055\times 10^{-34}\ \text{J}\cdot\text{s}} = 4.4\times 10^{-8}.$

(b) $L = 0.40\times 10^{-9}$ m: $T = 4.2\times 10^{-4}.$

EVALUATE: The tunneling probability is less when the barrier is wider.

40.30. **IDENTIFY:** The transmission coefficient is $T = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)e^{-2\sqrt{2m(U_0 - E)}L/\hbar}$.

SET UP: $E = 5.0$ eV, $L = 0.60 \times 10^{-9}$ m, and $m = 9.11 \times 10^{-31}$ kg.

EXECUTE: **(a)** $U_0 = 7.0$ eV $\Rightarrow T = 5.5 \times 10^{-4}$.

(b) $U_0 = 9.0$ eV $\Rightarrow T = 1.8 \times 10^{-5}$.

(c) $U_0 = 13.0$ eV $\Rightarrow T = 1.1 \times 10^{-7}$.

EVALUATE: T decreases when the height of the barrier increases.

40.31. **IDENTIFY and SET UP:** Use $\lambda = h/p$, where $K = p^2/2m$ and $E = K + U$.

EXECUTE: $\lambda = h/p = h/\sqrt{2mK}$, so $\lambda\sqrt{K}$ is constant. $\lambda_1\sqrt{K_1} = \lambda_2\sqrt{K_2}$; λ_1 and K_1 are for $x > L$ where $K_1 = 2U_0$ and λ_2 and K_2 are for $0 < x < L$ where $K_2 = E - U_0 = U_0$.

$\dfrac{\lambda_1}{\lambda_2} = \sqrt{\dfrac{K_2}{K_1}} = \sqrt{\dfrac{U_0}{2U_0}} = \dfrac{1}{\sqrt{2}}$.

EVALUATE: When the particle is passing over the barrier its kinetic energy is less and its wavelength is larger.

40.32. **IDENTIFY:** The probability of tunneling depends on the energy of the particle and the width of the barrier.

SET UP: The probability of tunneling is approximately $T = Ge^{-2\kappa L}$, where $G = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)$ and

$\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$.

EXECUTE: $G = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right) = 16\dfrac{50.0 \text{ eV}}{70.0 \text{ eV}}\left(1 - \dfrac{50.0 \text{ eV}}{70.0 \text{ eV}}\right) = 3.27$.

$\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar} = \dfrac{\sqrt{2(1.67 \times 10^{-27} \text{ kg})(70.0 \text{ eV} - 50.0 \text{ eV})(1.60 \times 10^{-19} \text{ J/eV})}}{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})/2\pi} = 9.8 \times 10^{11} \text{ m}^{-1}$.

Solving $T = Ge^{-2\kappa L}$ for L gives

$L = \dfrac{1}{2\kappa}\ln(G/T) = \dfrac{1}{2(9.8 \times 10^{11} \text{ m}^{-1})}\ln\left(\dfrac{3.27}{0.0080}\right) = 3.1 \times 10^{-12} \text{ m} = 3.1 \text{ pm}$.

If the proton were replaced with an electron, we note that κ is proportional to $\sqrt{m}$. So for an electron under similar conditions we have $\kappa_e = \kappa_p\sqrt{\dfrac{m_e}{m_p}}$. Therefore the width of the barrier for an electron is

$L_e = L_p\sqrt{\dfrac{m_p}{m_e}} = (3.1 \text{ pm})\sqrt{\dfrac{1.67 \times 10^{-27} \text{ kg}}{9.11 \times 10^{-31} \text{ kg}}} = 0.13 \text{ nm}$, which is much larger than for a proton.

EVALUATE: An electron can tunnel through a much wider barrier than a proton of the same energy.

40.33. **IDENTIFY and SET UP:** The energy levels are given by $E_n = (n + \tfrac{1}{2})\hbar\omega$, where $\omega = \sqrt{\dfrac{k'}{m}}$.

EXECUTE: $\omega = \sqrt{\dfrac{k'}{m}} = \sqrt{\dfrac{110 \text{ N/m}}{0.250 \text{ kg}}} = 21.0 \text{ rad/s}$.

The ground state energy is given by $E_n = (n + \tfrac{1}{2})\hbar\omega$, where $n = 0$.

$E_0 = \tfrac{1}{2}\hbar\omega = \tfrac{1}{2}(1.055 \times 10^{-34} \text{ J} \cdot \text{s})(21.0 \text{ rad/s}) = 1.11 \times 10^{-33} \text{ J}(1 \text{ eV}/1.602 \times 10^{-19} \text{ J}) = 6.93 \times 10^{-15} \text{ eV}$.

$E_n = (n + \tfrac{1}{2})\hbar\omega$, $E_{(n+1)} = (n + 1 + \tfrac{1}{2})\hbar\omega$.

The energy separation between these adjacent levels is

$\Delta E = E_{n+1} - E_n = \hbar\omega = 2E_0 = 2(1.11 \times 10^{-33} \text{ J}) = 2.22 \times 10^{-33} \text{ J} = 1.39 \times 10^{-14} \text{ eV}$.

EVALUATE: These energies are extremely small; quantum effects are not important for this oscillator.

40.34.   IDENTIFY: Calculate $\dfrac{d^2\psi}{dx^2}$ and insert the result into Eq. (40.44).

SET UP: $\dfrac{d}{dx}e^{-\delta x^2} = -2\delta x e^{-\delta x^2}$ and $\dfrac{d^2}{dx^2}e^{-\delta x^2} = (4\delta^2 x^2 - 2\delta)e^{-\delta x^2}$.

EXECUTE: Let $\sqrt{mk'}/2\hbar = \delta$, and so $\dfrac{d\psi}{dx} = -2x\delta\psi$ and $\dfrac{d^2\psi}{dx^2} = (4x^2\delta^2 - 2\delta)\psi$, and ψ is a solution of

Eq. (40.44) if $E = \dfrac{\hbar^2}{m}\delta = \tfrac{1}{2}\hbar\sqrt{k'/m} = \tfrac{1}{2}\hbar\omega$.

EVALUATE: $E = \tfrac{1}{2}\hbar\omega$ agrees with the equation $E_n = (n + \tfrac{1}{2})\hbar\omega$ for $n = 0$.

40.35.   IDENTIFY: We can model the molecule as a harmonic oscillator. The energy of the photon is equal to the energy difference between the two levels of the oscillator.

SET UP: The energy of a photon is $E_\gamma = hf = hc/\lambda$, and the energy levels of a harmonic oscillator are

given by $E_n = (n + \tfrac{1}{2})\hbar\sqrt{\dfrac{k'}{m}} = (n + \tfrac{1}{2})\hbar\omega$.

EXECUTE: (a) The photon's energy is $E_\gamma = \dfrac{hc}{\lambda} = \dfrac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^8\text{ m/s})}{5.8\times10^{-6}\text{ m}} = 0.21$ eV.

(b) The transition energy is $\Delta E = E_{n+1} - E_n = \hbar\omega = \hbar\sqrt{\dfrac{k'}{m}}$, which gives $\dfrac{2\pi\hbar c}{\lambda} = \hbar\sqrt{\dfrac{k'}{m}}$. Solving for k', we

get $k' = \dfrac{4\pi^2 c^2 m}{\lambda^2} = \dfrac{4\pi^2(3.00\times10^8\text{ m/s})^2(5.6\times10^{-26}\text{ kg})}{(5.8\times10^{-6}\text{ m})^2} = 5{,}900$ N/m.

EVALUATE: This would be a rather strong spring in the physics lab.

40.36.   IDENTIFY: The energy of the absorbed photon must be equal to the energy difference between the two states.

SET UP and EXECUTE: $\Delta E = \dfrac{hc}{\lambda} = \dfrac{(4.136\times10^{-15}\text{ eV}\cdot\text{s})(2.998\times10^8\text{ m/s})}{6.35\times10^{-6}\text{ m}} = 0.1953$ eV $= \hbar\omega$. $\Delta E = \hbar\omega$.

$E_0 = \dfrac{\hbar\omega}{2} = \dfrac{0.1953\text{ eV}}{2} = 0.0976$ eV.

EVALUATE: The energy of the photon is not equal to the energy of the ground state, but rather it is the energy *difference* between the two states.

40.37.   IDENTIFY: The photon energy equals the transition energy for the atom.

SET UP: According to the energy level equation $E_n = (n + \tfrac{1}{2})\hbar\omega$, the energy released during the transition

between two adjacent levels is twice the ground state energy $E_3 - E_2 = \hbar\omega = 2E_0 = 11.2$ eV.

EXECUTE: For a photon of energy E,

$E = hf \Rightarrow \lambda = \dfrac{c}{f} = \dfrac{hc}{E} = \dfrac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^8\text{ m/s})}{(11.2\text{ eV})(1.60\times10^{-19}\text{ J/eV})} = 111$ nm.

EVALUATE: This photon is in the ultraviolet.

40.38.   IDENTIFY: The energy of the absorbed (or emitted) photon energy is equal to the energy difference between the levels of the oscillator.

SET UP: The energy levels for a harmonic oscillator are $E_n = (n + \tfrac{1}{2})\hbar\omega$, where $\omega = \sqrt{k'/m}$.

EXECUTE: (a) The energy difference between *any* two adjacent levels is

$\Delta E_{n+1} - \Delta E_n = (n + \tfrac{3}{2})\hbar\omega - (n + \tfrac{1}{2})\hbar\omega = \hbar\omega$. Therefore transitions between *any* adjacent levels will emit (or absorb) photons of the same energy and hence the same wavelength. So the $2 \to 3$ transition absorbs a photon of the same wavelength as the $1 \to 2$ transition, which is $\lambda = 6.50\ \mu$m.

(b) Since transitions between adjacent levels emit photons of the same energy, transitions between levels for which n differs by 2 will emit energy $\hbar\omega + \hbar\omega = 2\hbar\omega$. So the photon absorbed in the $1 \to 3$

transition will have twice the energy (and therefore half the wavelength) as the photon in the $1 \to 2$ transition, so its wavelength will be $\frac{1}{2}(6.50 \ \mu\text{m}) = 3.25 \ \mu\text{m}$.

(c) For the $1 \to 2$ transition, the photon energy is $\hbar\omega$ and $\omega = \sqrt{k'/m}$, so
$\hbar\omega$ = energy of photon = hc/λ. This gives

$$\omega = \sqrt{k'/m} \ = \frac{hc}{\hbar\lambda} = \frac{2\pi c}{\lambda} = \frac{2\pi(3.00\times10^8 \text{ m/s})}{6.50\times10^{-6} \text{ m}} = 2.90\times10^{14} \text{ rad/s}.$$

EVALUATE: The frequency of this oscillator would be $f = \omega/2\pi = 4.62\times10^{14}$ Hz, *much* higher than typical classical oscillators.

40.39. IDENTIFY and **SET UP:** Use the energies given in $E_n = (n+\frac{1}{2})\hbar\omega$ to solve for the amplitude A and maximum speed v_{max} of the oscillator. Use these to estimate Δx and Δp_x and compute the uncertainty product $\Delta x \Delta p_x$.

EXECUTE: The total energy of a Newtonian oscillator is given by $E = \frac{1}{2}k'A^2$ where k' is the force constant and A is the amplitude of the oscillator. Set this equal to the energy $E = (n+\frac{1}{2})\hbar\omega$ of an excited level that has quantum number n, where $\omega = \sqrt{\frac{k'}{m}}$, and solve for A: $\frac{1}{2}k'A^2 = (n+\frac{1}{2})\hbar\omega$. $A = \sqrt{\frac{(2n+1)\hbar\omega}{k'}}$.

The total energy of the Newtonian oscillator can also be written as $E = \frac{1}{2}mv_{\text{max}}^2$. Set this equal to

$E = (n+\frac{1}{2})\hbar\omega$ and solve for v_{max}: $\frac{1}{2}mv_{\text{max}}^2 = (n+\frac{1}{2})\hbar\omega$. $v_{\text{max}} = \sqrt{\frac{(2n+1)\hbar\omega}{m}}$. Thus the maximum linear momentum of the oscillator is $p_{\text{max}} = mv_{\text{max}} = \sqrt{(2n+1)\hbar m\omega}$. Now assume that $A/\sqrt{2}$ represents the uncertainty Δx in position and that $p_{\text{max}}/\sqrt{2}$ is the corresponding uncertainty Δp_x in momentum. Then the uncertainty product is

$$\Delta x \Delta p_x = \left(\frac{1}{\sqrt{2}}\sqrt{\frac{(2n+1)\hbar\omega}{k'}}\right)\left(\frac{1}{\sqrt{2}}\sqrt{(2n+1)\hbar m\omega}\right) = \frac{(2n+1)\hbar\omega}{2}\sqrt{\frac{m}{k'}} = \frac{(2n+1)\hbar\omega}{2}\left(\frac{1}{\omega}\right) = (2n+1)\frac{\hbar}{2}.$$

EVALUATE: For $n = 0$ this gives $\Delta x \Delta p_x = \hbar/2$, in agreement with the result derived in Section 40.5. The uncertainty product $\Delta x \Delta p_x$ increases with n.

40.40. IDENTIFY: Compute the ratio specified in the problem.

SET UP: For $n = 0$, $A = \sqrt{\frac{\hbar\omega}{k'}}$. $\omega = \sqrt{\frac{k'}{m}}$.

EXECUTE: **(a)** $\frac{|\psi(A)|^2}{|\psi(0)|^2} = \exp\left(-\frac{\sqrt{mk'}}{\hbar}A^2\right) = \exp\left(-\sqrt{mk'}\frac{\omega}{k'}\right) = e^{-1} = 0.368$. This is consistent with what is shown in Figure 40.27 in the textbook.

(b) $\frac{|\psi(2A)^2|}{|\psi(0)|^2} = \exp\left(-\frac{\sqrt{mk'}}{\hbar}(2A)^2\right) = \exp\left(-\sqrt{mk'}4\frac{\omega}{k'}\right) = e^{-4} = 1.83\times10^{-2}$. This figure cannot be read this precisely, but the qualitative decrease in amplitude with distance is clear.

EVALUATE: The wave function decays exponentially as x increases beyond $x = A$.

40.41. IDENTIFY: We model the atomic vibration in the crystal as a harmonic oscillator.

SET UP: The energy levels of a harmonic oscillator are given by $E_n = (n+\frac{1}{2})\hbar\sqrt{\frac{k'}{m}} = (n+\frac{1}{2})\hbar\omega$.

EXECUTE: **(a)** The ground state energy of a simple harmonic oscillator is

$$E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{\frac{k'}{m}} = \frac{(1.055\times10^{-34} \text{ J}\cdot\text{s})}{2}\sqrt{\frac{12.2 \text{ N/m}}{3.82\times10^{-26} \text{ kg}}} = 9.43\times10^{-22} \text{ J} = 5.89\times10^{-3} \text{ eV}.$$

(b) $E_4 - E_3 = \hbar\omega = 2E_0 = 0.0118$ eV, so $\lambda = \dfrac{hc}{E} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.00\times10^8\ \text{m/s})}{1.88\times10^{-21}\ \text{J}} = 106\ \mu\text{m}.$

(c) $E_{n+1} - E_n = \hbar\omega = 2E_0 = 0.0118$ eV.

EVALUATE: These energy differences are much smaller than those due to electron transitions in the hydrogen atom.

40.42. **IDENTIFY:** Carry out the calculations specified in the problem.

SET UP: A standard integral is $\int_0^\infty e^{-\alpha^2 k^2}\cos(kx)dk = \dfrac{\sqrt{\pi}}{2\alpha}e^{-x^2/4\alpha^2}.$

EXECUTE: **(a)** $B(k) = e^{-\alpha^2 k^2}.$ $B(0) = B_{\max} = 1.$ $B(k_h) = \frac{1}{2} = e^{-\alpha^2 k_h^2} \Rightarrow \ln(1/2) = -\alpha^2 k_h^2$

$\Rightarrow k_h = \dfrac{1}{\alpha}\sqrt{\ln(2)} = w_k.$

(b) $\psi(x) = \int_0^\infty e^{-\alpha^2 k^2}\cos kx\, dk = \dfrac{\sqrt{\pi}}{2\alpha}(e^{-x^2/4\alpha^2}).$ $\psi(x)$ is a maximum when $x = 0.$

(c) $\psi(x_h) = \dfrac{\sqrt{\pi}}{4\alpha}$ when $e^{-x_h^2/4\alpha^2} = \frac{1}{2} \Rightarrow \dfrac{-x_h^2}{4\alpha^2} = \ln(1/2) \Rightarrow x_h = 2\alpha\sqrt{\ln 2} = w_x.$

(d) $w_p w_x = \left(\dfrac{hw_k}{2\pi}\right)w_x = \dfrac{h}{2\pi}\left(\dfrac{1}{\alpha}\sqrt{\ln 2}\right)(2\alpha\sqrt{\ln 2}) = \dfrac{h}{2\pi}(2\ln 2) = \dfrac{h\ln 2}{\pi} = (2\ln 2)\hbar.$

EVALUATE: The Heisenberg Uncertainty Principle says that $\Delta x\Delta p_x \geq \hbar/2.$ If $\Delta x = w_x$ and $\Delta p_x = w_p,$ then the uncertainty principle says $w_x w_p \geq \hbar/2.$ So our result is consistent with the uncertainty principle since $(2\ln 2)\hbar > \hbar/2.$

40.43. **IDENTIFY:** We know the wave function of a particle in a box.

SET UP and EXECUTE: **(a)** $\Psi(x,t) = \dfrac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1 t/\hbar} + \dfrac{1}{\sqrt{2}}\psi_3(x)e^{-iE_3 t/\hbar}.$

$\Psi^*(x,t) = \dfrac{1}{\sqrt{2}}\psi_1(x)e^{+iE_1 t/\hbar} + \dfrac{1}{\sqrt{2}}\psi_3(x)e^{+iE_3 t/\hbar}.$

$|\Psi(x,t)|^2 = \frac{1}{2}[\psi_1^2 + \psi_3^2 + \psi_1\psi_3(e^{i(E_3-E_1)t/\hbar} + e^{-i(E_3-E_1)t/\hbar})] = \frac{1}{2}\left[\psi_1^2 + \psi_3^2 + 2\psi_1\psi_3\cos\left(\dfrac{[E_3-E_1]t}{\hbar}\right)\right].$

$\psi_1 = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{\pi x}{L}\right).$ $\psi_3 = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{3\pi x}{L}\right).$ $E_3 = \dfrac{9\pi^2\hbar^2}{2mL^2}$ and $E_1 = \dfrac{\pi^2\hbar^2}{2mL^2},$ so $E_3 - E_1 = \dfrac{4\pi^2\hbar^2}{mL^2}.$

$|\Psi(x,t)|^2 = \dfrac{1}{L}\left[\sin^2\left(\dfrac{\pi x}{L}\right) + \sin^2\left(\dfrac{3\pi x}{L}\right) + 2\sin\left(\dfrac{\pi x}{L}\right)\sin\left(\dfrac{3\pi x}{L}\right)\cos\left(\dfrac{4\pi^2\hbar t}{mL^2}\right)\right].$ At $x = L/2,$

$\sin\left(\dfrac{\pi x}{L}\right) = \sin\left(\dfrac{\pi}{2}\right) = 1.$ $\sin\left(\dfrac{3\pi x}{L}\right) = \sin\left(\dfrac{3\pi}{2}\right) = -1.$ $|\Psi(x,t)|^2 = \dfrac{2}{L}\left[1 - \cos\left(\dfrac{4\pi^2\hbar t}{mL^2}\right)\right].$

(b) $\omega_{\text{osc}} = \dfrac{E_3 - E_1}{\hbar} = \dfrac{4\pi^2\hbar}{mL^2}.$

EVALUATE: Note that $\Delta E = \hbar\omega.$

40.44. **IDENTIFY:** Evaluate $\psi(x) = \int_0^\infty B(k)\cos kx\, dk$ for the function $B(k)$ specified in the problem.

SET UP: $\int\cos kx\, dk = \dfrac{1}{x}\sin kx.$

EXECUTE: **(a)** $\psi(x) = \int_0^\infty B(k)\cos kx\, dk = \int_0^{k_0}\left(\dfrac{1}{k_0}\right)\cos kx\, dk = \dfrac{\sin kx}{k_0 x}\Big|_0^{k_0} = \dfrac{\sin k_0 x}{k_0 x}.$

(b) $\psi(x)$ has a maximum value at the origin $x = 0$. $\psi(x_0) = 0$ when $k_0 x_0 = \pi$ so $x_0 = \dfrac{\pi}{k_0}$. Thus the width of

this function $w_x = 2x_0 = \dfrac{2\pi}{k_0}$. If $k_0 = \dfrac{2\pi}{L}$, $w_x = L$. $B(k)$ versus k is graphed in Figure 40.44a. The graph of

$\psi(x)$ versus x is in Figure 40.44b.

(c) If $k_0 = \dfrac{\pi}{L}$, $w_x = 2L$.

EVALUATE: **(d)** $w_p w_x = \left(\dfrac{hw_k}{2\pi}\right)\left(\dfrac{2\pi}{k_0}\right) = \dfrac{hw_k}{k_0} = \dfrac{hk_0}{k_0} = h$. If $\Delta x = w_x$ and $\Delta p_x = w_p$, then the uncertainty

principle states that $w_p w_x \geq \dfrac{\hbar}{2}$. For us, no matter what k_0 is, $w_p w_x = h$, which is greater than $\hbar/2$.
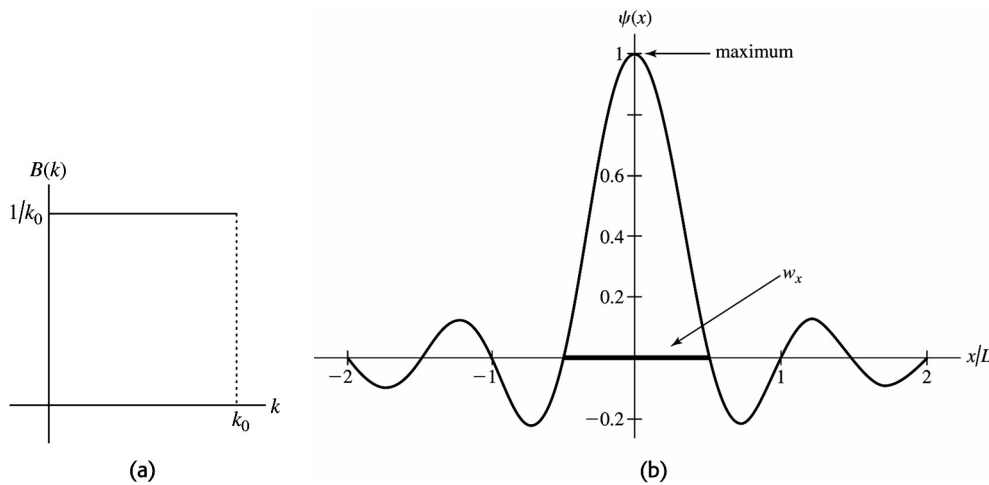

Figure 40.44

40.45. **IDENTIFY:** Let I refer to the region $x < 0$ and let II refer to the region $x > 0$, so $\psi_I(x) = Ae^{ik_1 x} + Be^{-ik_1 x}$

and $\psi_{II}(x) = Ce^{ik_2 x}$. Set $\psi_I(0) = \psi_{II}(0)$ and $\dfrac{d\psi_I}{dx} = \dfrac{d\psi_{II}}{dx}$ at $x = 0$.

SET UP: $\dfrac{d}{dx}(e^{ikx}) = ike^{ikx}$.

EXECUTE: $\psi_I(0) = \psi_{II}(0)$ gives $A + B = C$. $\dfrac{d\psi_I}{dx} = \dfrac{d\psi_{II}}{dx}$ at $x = 0$ gives $ik_1 A - ik_1 B = ik_2 C$. Solving this

pair of equations for B and C gives $B = \left(\dfrac{k_1 - k_2}{k_1 + k_2}\right)A$ and $C = \left(\dfrac{2k_2}{k_1 + k_2}\right)A$.

EVALUATE: The probability of reflection is $R = \dfrac{B^2}{A^2} = \dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$. The probability of transmission is

$T = \dfrac{C^2}{A^2} = \dfrac{4k_1^2}{(k_1 + k_2)^2}$. Note that $R + T = 1$.

40.46. **IDENTIFY:** The probability of finding the particle between x_1 and x_2 is $\int_{x_1}^{x_2} |\psi|^2 \, dx$.

SET UP: For the ground state $\psi_1 = \sqrt{\dfrac{2}{L}} \sin\dfrac{\pi x}{L}$. $\sin^2 \theta = \tfrac{1}{2}(1 - \cos 2\theta)$. $\int \cos \alpha x \, dx = \dfrac{1}{\alpha} \sin \alpha x$.

EXECUTE: **(a)** $\dfrac{2}{L}\int_0^{L/4}\sin^2\dfrac{\pi x}{L}dx=\dfrac{2}{L}\int_0^{L/4}\dfrac{1}{2}\left(1-\cos\dfrac{2\pi x}{L}\right)dx=\dfrac{1}{L}\left(x-\dfrac{L}{2\pi}\sin\dfrac{2\pi x}{L}\right)\Big|_0^{L/4}=\dfrac{1}{4}-\dfrac{1}{2\pi}$, which is

about 0.0908.

(b) Repeating with limits of $L/4$ and $L/2$ gives $\dfrac{1}{L}\left(x-\dfrac{L}{2\pi}\sin\dfrac{2\pi x}{L}\right)\Big|_{L/4}^{L/2}=\dfrac{1}{4}+\dfrac{1}{2\pi}$, about 0.409.

(c) The particle is much likely to be nearer the middle of the box than the edge.

EVALUATE: **(d)** The results sum to exactly $\frac{1}{2}$. Since the probability of the particle being anywhere in the

box is unity, the probability of the particle being found between $x=L/2$ and $x=L$ is also $\frac{1}{2}$. This means

that the particle is as likely to be between $x=0$ and $L/2$ as it is to be between $x=L/2$ and $x=L$.

(e) These results are consistent with Figure 40.12b in the textbook. This figure shows a greater probability

near the center of the box. It also shows symmetry of $|\psi|^2$ about the center of the box.

40.47. **IDENTIFY** and **SET UP:** The energy levels are given by the equation $E_n=\dfrac{n^2h^2}{8mL^2}$. Calculate ΔE for the

transition and set $\Delta E=hc/\lambda$, the energy of the photon.

EXECUTE: **(a)** Ground level, $n=1$, $E_1=\dfrac{h^2}{8mL^2}$. First excited level, $n=2$, $E_2=\dfrac{4h^2}{8mL^2}$. The transition

energy is $\Delta E=E_2-E_1=\dfrac{3h^2}{8mL^2}$. Set the transition energy equal to the energy hc/λ of the emitted photon.

This gives $\dfrac{hc}{\lambda}=\dfrac{3h^2}{8mL^2}$. $\lambda=\dfrac{8mcL^2}{3h}=\dfrac{8(9.109\times10^{-31}\text{ kg})(2.998\times10^8\text{ m/s})(4.18\times10^{-9}\text{ m})^2}{3(6.626\times10^{-34}\text{ J}\cdot\text{s})}$.

$\lambda=1.92\times10^{-5}\text{ m}=19.2\ \mu\text{m}$.

(b) Second excited level has $n=3$ and $E_3=\dfrac{9h^2}{8mL^2}$ The transition energy is

$\Delta E=E_3-E_2=\dfrac{9h^2}{8mL^2}-\dfrac{4h^2}{8mL^2}=\dfrac{5h^2}{8mL^2}$. $\dfrac{hc}{\lambda}=\dfrac{5h^2}{8mL^2}$ so $\lambda=\dfrac{8mcL^2}{5h}=\dfrac{3}{5}(19.2\ \mu\text{m})=11.5\ \mu\text{m}$.

EVALUATE: The energy spacing between adjacent levels increases with n, and this corresponds to a

shorter wavelength and more energetic photon in part (b) than in part (a).

40.48. **IDENTIFY:** The probability is $|\psi|^2\,dx$, with ψ evaluated at the specified value of x.

SET UP: For the ground state, the normalized wave function is $\psi_1=\sqrt{2/L}\,\sin(\pi x/L)$.

EXECUTE: **(a)** $(2/L)\sin^2(\pi/4)dx=dx/L$.

(b) $(2/L)\sin^2(\pi/2)dx=2dx/L$.

(c) $(2/L)\sin^2(3\pi/4)=dx/L$.

EVALUATE: Our results agree with Figure 40.12b in the textbook. $|\psi|^2$ is largest at the center of the box,

at $x=L/2$. $|\psi|^2$ is symmetric about the center of the box, so is the same at $x=L/4$ as at $x=3L/4$.

40.49. **IDENTIFY** and **SET UP:** The normalized wave function for the $n=2$ first excited level is

$\psi_2=\sqrt{\dfrac{2}{L}}\sin\left(\dfrac{2\pi x}{L}\right)$. $P=|\psi(x)|^2\,dx$ is the probability that the particle will be found in the interval x to $x+dx$.

EXECUTE: **(a)** $x=L/4$.

$\psi(x)=\sqrt{\dfrac{2}{L}}\sin\left(\left(\dfrac{2\pi}{L}\right)\left(\dfrac{L}{4}\right)\right)=\sqrt{\dfrac{2}{L}}\sin\left(\dfrac{\pi}{2}\right)=\sqrt{\dfrac{2}{L}}$.

$P=(2/L)dx$.

(b) $x = L/2$.

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{L}{2}\right)\right) = \sqrt{\frac{2}{L}}\sin(\pi) = 0.$$

$P = 0$.

(c) $x = 3L/4$.

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{3L}{4}\right)\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi}{2}\right) = -\sqrt{\frac{2}{L}}.$$

$P = (2/L)dx$.

EVALUATE: Our results are consistent with the $n = 2$ part of Figure 40.12 in the textbook. $|\psi|^2$ is zero at the center of the box and is symmetric about this point.

40.50.   IDENTIFY: The impulse applied to a particle equals its change in momentum.

SET UP: For a particle in a box, the magnitude of its momentum is $p = \hbar k = \dfrac{nh}{2L}$.

EXECUTE: $\Delta\vec{p} = \vec{p}_{\text{final}} - \vec{p}_{\text{initial}}$. $|\vec{p}| = \hbar k = \dfrac{\hbar n\pi}{L} = \dfrac{hn}{2L}$. At $x = 0$ the initial momentum at the wall is

$\vec{p}_{\text{initial}} = -\dfrac{hn}{2L}\hat{i}$ and the final momentum, after turning around, is $\vec{p}_{\text{final}} = +\dfrac{hn}{2L}\hat{i}$. So,

$\Delta\vec{p} = +\dfrac{hn}{2L}\hat{i} - \left(-\dfrac{hn}{2L}\hat{i}\right) = +\dfrac{hn}{L}\hat{i}$. At $x = L$ the initial momentum is $\vec{p}_{\text{initial}} = +\dfrac{hn}{2L}\hat{i}$ and the final

momentum, after turning around, is $\vec{p}_{\text{final}} = -\dfrac{hn}{2L}\hat{i}$. So, $\Delta\vec{p} = -\dfrac{hn}{2L}\hat{i} - \dfrac{hn}{2L}\hat{i} = -\dfrac{hn}{L}\hat{i}$.

EVALUATE: The impulse increases with n.

40.51.   IDENTIFY: The probability of the particle being between x_1 and x_2 is $\int_{x_1}^{x_2}|\psi|^2\,dx$, where ψ is the normalized wave function for the particle.

(a) SET UP: The normalized wave function for the ground state is $\psi_1 = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{\pi x}{L}\right)$.

EXECUTE: The probability P of the particle being between $x = L/4$ and $x = 3L/4$ is

$P = \int_{L/4}^{3L/4}|\psi_1|^2\,dx = \dfrac{2}{L}\int_{L/4}^{3L/4}\sin^2\left(\dfrac{\pi x}{L}\right)dx$. Let $y = \pi x/L$; $dx = (L/\pi)dy$ and the integration limits become

$\pi/4$ and $3\pi/4$.

$P = \dfrac{2}{L}\left(\dfrac{L}{\pi}\right)\int_{\pi/4}^{3\pi/4}\sin^2 y\,dy = \dfrac{2}{\pi}\left[\dfrac{1}{2}y - \dfrac{1}{4}\sin 2y\right]_{\pi/4}^{3\pi/4}$.

$P = \dfrac{2}{\pi}\left[\dfrac{3\pi}{8} - \dfrac{\pi}{8} - \dfrac{1}{4}\sin\left(\dfrac{3\pi}{2}\right) + \dfrac{1}{4}\sin\left(\dfrac{\pi}{2}\right)\right]$.

$P = \dfrac{2}{\pi}\left(\dfrac{\pi}{4} - \dfrac{1}{4}(-1) + \dfrac{1}{4}(1)\right) = \dfrac{1}{2} + \dfrac{1}{\pi} = 0.818$. (Note: The integral formula $\int \sin^2 y\,dy = \dfrac{1}{2}y - \dfrac{1}{4}\sin 2y$ was used.)

(b) SET UP: The normalized wave function for the first excited state is $\psi_2 = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{2\pi x}{L}\right)$.

EXECUTE: $P = \int_{L/4}^{3L/4}|\psi_2|^2\,dx = \dfrac{2}{L}\int_{L/4}^{3L/4}\sin^2\left(\dfrac{2\pi x}{L}\right)dx$. Let $y = 2\pi x/L$; $dx = (L/2\pi)dy$ and the integration limits become $\pi/2$ and $3\pi/2$.

$P = \dfrac{2}{L}\left(\dfrac{L}{2\pi}\right)\int_{\pi/2}^{3\pi/2}\sin^2 y\,dy = \dfrac{1}{\pi}\left[\dfrac{1}{2}y - \dfrac{1}{4}\sin 2y\right]_{\pi/2}^{3\pi/2} = \dfrac{1}{\pi}\left(\dfrac{3\pi}{4} - \dfrac{\pi}{4}\right) = 0.500$.

EVALUATE: **(c)** These results are consistent with Figure 40.11b in the textbook. That figure shows that $|\psi|^2$ is more concentrated near the center of the box for the ground state than for the first excited state; this is consistent with the answer to part (a) being larger than the answer to part (b). Also, this figure shows that for the first excited state half the area under $|\psi|^2$ curve lies between $L/4$ and $3L/4$, consistent with our answer to part (b).

40.52.   **IDENTIFY:**   We start with the penetration distance formula given in the problem.

SET UP: The given formula is $\eta = \dfrac{\hbar}{\sqrt{2m(U_0 - E)}}$.

EXECUTE: **(a)** Substitute the given numbers into the formula:

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = \frac{1.055 \times 10^{-34} \text{ J} \cdot \text{s}}{\sqrt{2(9.11 \times 10^{-31} \text{ kg})(20 \text{ eV} - 13 \text{ eV})(1.602 \times 10^{-19} \text{ J/eV})}} = 7.4 \times 10^{-11} \text{ m}.$$

(b) $\eta = \dfrac{1.055 \times 10^{-34} \text{ J} \cdot \text{s}}{\sqrt{2(1.67 \times 10^{-27} \text{ kg})(30 \text{ MeV} - 20 \text{ MeV})(1.602 \times 10^{-13} \text{ J/MeV})}} = 1.44 \times 10^{-15} \text{ m}.$

EVALUATE: The penetration depth varies widely depending on the mass and energy of the particle.

40.53.   **IDENTIFY:**   Carry out the calculations that are specified in the problem.

SET UP: For a free particle, $U(x) = 0$ so Schrödinger's equation becomes $\dfrac{d^2\psi(x)}{dx^2} = -\dfrac{2m}{\hbar^2} E\psi(x)$.

EXECUTE: **(a)** The graph is given in Figure 40.53.

(b) For $x < 0$: $\psi(x) = e^{+\kappa x}$. $\dfrac{d\psi(x)}{dx} = \kappa e^{+\kappa x}$. $\dfrac{d^2\psi(x)}{dx} = \kappa^2 e^{+\kappa x}$. So $\kappa^2 = -\dfrac{2m}{\hbar^2} E \Rightarrow E = -\dfrac{\hbar^2 \kappa^2}{2m}$.

(c) For $x > 0$: $\psi(x) = e^{-\kappa x}$. $\dfrac{d\psi(x)}{dx} = -\kappa e^{-\kappa x}$. $\dfrac{d^2\psi(x)}{dx} = \kappa^2 e^{-\kappa x}$. So again $\kappa^2 = -\dfrac{2m}{\hbar^2} E \Rightarrow E = \dfrac{-\hbar^2 \kappa^2}{2m}$.

Parts (b) and (c) show $\psi(x)$ satisfies the Schrödinger's equation, provided $E = \dfrac{-\hbar^2 \kappa^2}{2m}$.

EVALUATE: **(d)** $\dfrac{d\psi(x)}{dx}$ is discontinuous at $x = 0$. (That is, it is negative for $x > 0$ and positive for $x < 0$.)

Therefore, this ψ is not an acceptable wave function; $d\psi/dx$ must be continuous everywhere, except where $U \to \infty$.
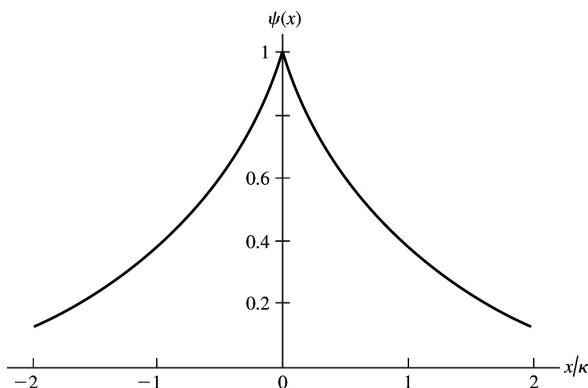

Figure 40.53

40.54.   **IDENTIFY:** $T = Ge^{-2\kappa L}$ with $G = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$, so $L = -\dfrac{1}{2\kappa}\ln\left(\dfrac{T}{G}\right)$.

SET UP: $E = 5.5 \text{ eV}, U_0 = 10.0 \text{ eV}, m = 9.11 \times 10^{-31} \text{ kg, and } T = 0.0050.$

EXECUTE: $\kappa = \dfrac{\sqrt{2(9.11\times10^{-31}\text{ kg})(4.5\text{ eV})(1.60\times10^{-19}\text{ J/eV})}}{(1.054\times10^{-34}\text{ J}\cdot\text{s})} = 1.09\times10^{10}\text{ m}^{-1}$ and

$G = 16\dfrac{5.5\text{ eV}}{10.0\text{ eV}}\left(1 - \dfrac{5.5\text{ eV}}{10.0\text{ eV}}\right) = 3.96$. Therefore the barrier width L is

$L = -\dfrac{1}{2\kappa}\ln\left(\dfrac{T}{G}\right) = -\dfrac{1}{2(1.09\times10^{10}\text{ m}^{-1})}\ln\left(\dfrac{0.0050}{3.96}\right) = 3.1\times10^{-10}\text{ m} = 0.31\text{ nm}.$

EVALUATE: The energies here are comparable to those of electrons in atoms, and the barrier width we calculated is on the order of the diameter of an atom.

40.55. **IDENTIFY:** Eq. (40.38) applies for $0 \le x \le L$. Eq. (40.40) applies for $x < 0$ and $x > L$. $D = 0$ for $x < 0$ and $C = 0$ for $x > L$.

SET UP: Let $k = \dfrac{\sqrt{2mE}}{\hbar}$. $\dfrac{d}{dx}\sin kx = k\cos kx$. $\dfrac{d}{dx}\cos kx = -k\sin kx$. $\dfrac{d}{dx}e^{\kappa x} = \kappa e^{\kappa x}$. $\dfrac{d}{dx}e^{-\kappa x} = -\kappa e^{-\kappa x}$.

EXECUTE: **(a)** We set the solutions for inside and outside the well equal to each other at the well boundaries, $x = 0$ and L.

$x = 0$: $B\sin(0) + A = C \Rightarrow A = C$, since we must have $D = 0$ for $x < 0$.

$x = L$: $B\sin\dfrac{\sqrt{2mE}L}{\hbar} + A\cos\dfrac{\sqrt{2mE}L}{\hbar} = +De^{-\kappa L}$ since $C = 0$ for $x > L$.

This gives $B\sin kL + A\cos kL = De^{-\kappa L}$, where $k = \dfrac{\sqrt{2mE}}{\hbar}$.

(b) Requiring continuous derivatives at the boundaries yields

$x = 0$: $\dfrac{d\psi}{dx} = kB\cos(k\cdot 0) - kA\sin(k\cdot 0) = kB = \kappa Ce^{k\cdot 0} \Rightarrow kB = \kappa C.$

$x = L$: $kB\cos kL - kA\sin kL = -\kappa De^{-\kappa L}$.

EVALUATE: These boundary conditions allow for B, C, and D to be expressed in terms of an overall normalization constant A.

40.56. **IDENTIFY:** Compare the energy E of the oscillator to the equation $E_n = (n + \tfrac{1}{2})\hbar\omega$ in order to determine n.

SET UP: At the equilibrium position the potential energy is zero and the kinetic energy equals the total energy.

EXECUTE: **(a)** $E = \tfrac{1}{2}mv^2 = [n + (1/2)]\hbar\omega = [n + (1/2)]hf$, and solving for n,

$n = \dfrac{\tfrac{1}{2}mv^2}{hf} - \dfrac{1}{2} = \dfrac{(1/2)(0.020\text{ kg})(0.480\text{ m/s})^2}{(6.626\times10^{-34}\text{ J}\cdot\text{s})(1.50\text{ Hz})} - \dfrac{1}{2} = 2.3\times10^{30}.$

(b) The difference between energies is $\hbar\omega = hf = (6.63\times10^{-34}\text{ J}\cdot\text{s})(1.50\text{ Hz}) = 9.95\times10^{-34}\text{ J}$. This energy is too small to be detected with current technology.

EVALUATE: This oscillator can be described classically; quantum effects play no measurable role.

40.57. **IDENTIFY** and **SET UP:** Calculate the angular frequency ω of the pendulum and apply $E_n = (n + \tfrac{1}{2})\hbar\omega$ for the energy levels.

EXECUTE: $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.500\text{ s}} = 4\pi\text{ s}^{-1}.$

The ground-state energy is $E_0 = \tfrac{1}{2}\hbar\omega = \tfrac{1}{2}(1.055\times10^{-34}\text{ J}\cdot\text{s})(4\pi\text{ s}^{-1}) = 6.63\times10^{-34}\text{ J}.$

$E_0 = 6.63\times10^{-34}\text{ J}(1\text{ eV}/1.602\times10^{-19}\text{ J}) = 4.14\times10^{-15}\text{ eV}.$

$E_n = (n + \tfrac{1}{2})\hbar\omega.$

$E_{n+1} = (n + 1 + \tfrac{1}{2})\hbar\omega.$

The energy difference between the adjacent energy levels is

$\Delta E = E_{n+1} - E_n = \hbar\omega = 2E_0 = 1.33\times10^{-33}$ J $= 8.30\times10^{-15}$ eV.

EVALUATE: These energies are much too small to detect. Quantum effects are not important for ordinary size objects.

40.58. IDENTIFY: If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: The given wave function is $\psi_1(x) = A_1 x e^{-\alpha^2 x^2/2}$ and the Schrödinger equation is

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{k'x^2}{2}\psi(x) = E\,\psi(x).$$

EXECUTE: **(a)** Start by taking the indicated derivatives: $\psi_1(x) = A_1 x e^{-\alpha^2 x^2/2}$.

$$\frac{d\psi_1(x)}{dx} = -\alpha^2 x^2 A_1 e^{-\alpha^2 x^2/2} + A_1 e^{-\alpha^2 x^2/2}.$$

$$\frac{d^2\psi_1(x)}{dx^2} = -A_1\alpha^2 2x e^{-\alpha^2 x^2/2} - A_1\alpha^2 x^2(-\alpha^2 x)e^{-\alpha^2 x^2/2} + A_1(-\alpha^2 x)e^{-\alpha^2 x^2/2}.$$

$$\frac{d^2\psi_1(x)}{dx^2} = \left[-2\alpha^2 + (\alpha^2)^2 x^2 - \alpha^2\right]\psi_1(x) = \left[-3\alpha^2 + (\alpha^2)^2 x^2\right]\psi_1(x).$$

$$-\frac{\hbar}{2m}\frac{d^2\psi_1(x)}{dx^2} = -\frac{\hbar^2}{2m}\left[-3\alpha^2 + (\alpha^2)^2 x^2\right]\psi_1(x).$$

Equation (40.44) is $-\dfrac{\hbar}{2m}\dfrac{d^2\psi(x)}{dx^2} + \dfrac{k'x^2}{2}\psi(x) = E\,\psi(x).$ Substituting the above result into that equation

gives $-\dfrac{\hbar^2}{2m}\left[-3\alpha^2 + (\alpha^2)^2 x^2\right]\psi_1(x) + \dfrac{k'x^2}{2}\psi_1(x) = E\,\psi_1(x).$ Since $\alpha^2 = \dfrac{m\omega}{\hbar}$ and $\omega = \sqrt{\dfrac{k'}{m}}$, the coefficient

of x^2 is $-\dfrac{\hbar^2}{2m}(\alpha^2)^2 + \dfrac{k'}{2} = -\dfrac{\hbar^2}{2m}\left(\dfrac{m\omega}{\hbar}\right)^2 + \dfrac{m\omega^2}{2} = 0.$

(b) $A_1 = \left(\dfrac{m\omega}{\hbar}\right)^{3/4}\left(\dfrac{4}{\pi}\right)^{1/4}.$

(c) The probability density function $|\psi|^2$ is $|\psi_1(x)|^2 = A_1^2 x^2 e^{-\alpha^2 x^2}$.

At $x = 0$, $|\psi_1|^2 = 0$. $\dfrac{d|\psi_1(x)|^2}{dx} = A_1^2 2x e^{-\alpha^2 x^2} + A_1^2 x^2(-\alpha^2 2x)e^{-\alpha^2 x^2} = A_1^2 2x e^{-\alpha^2 x^2} - A_1^2 2x^3\alpha^2 e^{-\alpha^2 x^2}.$

At $x = 0$, $\dfrac{d|\psi_1(x)|^2}{dx} = 0$. At $x = \pm\dfrac{1}{\alpha}$, $\dfrac{d|\psi_1(x)|^2}{dx} = 0.$

$$\frac{d^2|\psi_1(x)|^2}{dx^2} = A_1^2 2e^{-\alpha^2 x^2} + A_1^2 2x(-\alpha^2 2x)e^{-\alpha^2 x^2} - A_1^2 2(3x^2)\alpha^2 e^{-\alpha^2 x^2} - A_1^2 2x^3\alpha^2(-\alpha^2 2x)e^{-\alpha^2 x^2}.$$

$$\frac{d^2|\psi_1(x)|^2}{dx^2} = A_1^2 2e^{-\alpha^2 x^2} - A_1^2 4x^2\alpha^2 e^{-\alpha^2 x^2} - A_1^2 6x^2\alpha^2 e^{-\alpha^2 x^2} + A_1^2 8x^4(\alpha^2)^2 e^{-\alpha^2 x^2}.$$ At $x = 0$,

$\dfrac{d^2|\psi_1(x)|^2}{dx^2} > 0.$ So at $x = 0$, the first derivative is zero and the second derivative is positive. Therefore,

the probability density function has a minimum at $x = 0$. At $x = \pm\dfrac{1}{\alpha}$, $\dfrac{d^2|\psi_1(x)|^2}{dx^2} < 0.$ So at $x = \pm\dfrac{1}{\alpha}$, the

first derivative is zero and the second derivative is negative. Therefore, the probability density function has

maxima at $x = \pm\dfrac{1}{\alpha}$, corresponding to the classical turning points for $n = 0$ as found in the previous question.

EVALUATE: $\psi_1(x) = A_1 x e^{-\alpha^2 x^2/2}$ is a solution to Eq. (40.44) if $-\dfrac{\hbar^2}{2m}(-3\alpha^2)\psi_1(x) = E\,\psi_1(x)$ or

$E = \dfrac{3\hbar^2\alpha^2}{2m} = \dfrac{3\hbar\omega}{2}$. $E_1 = \dfrac{3\hbar\omega}{2}$ corresponds to $n = 1$ in Equation (40.46).

40.59.   **IDENTIFY:**   For a standing wave in the box, there must be a node at each wall and $n\left(\dfrac{\lambda}{2}\right) = L$.

SET UP: $p = \dfrac{h}{\lambda}$ so $mv = \dfrac{h}{\lambda}$.

EXECUTE: **(a)** For a standing wave, $n\lambda = 2L$, and $E_n = \dfrac{p^2}{2m} = \dfrac{(h/\lambda)^2}{2m} = \dfrac{n^2 h^2}{8mL^2}$.

(b) With $L = a_0 = 0.5292 \times 10^{-10}$ m, $E_1 = 2.15 \times 10^{-17}$ J $= 134$ eV.

EVALUATE: For a hydrogen atom, E_n is proportional to $1/n^2$ so this is a very poor model for a hydrogen atom. In particular, it gives very inaccurate values for the separations between energy levels.

40.60.   **IDENTIFY** and **SET UP:**   Follow the steps specified in the problem.

EXECUTE: **(a)** As with the particle in a box, $\psi(x) = A\sin kx$, where A is a constant and $k^2 = 2mE/\hbar^2$. Unlike the particle in a box, however, k and hence E do not have simple forms.

(b) For $x > L$, the wave function must have the form of $\psi(x) = Ce^{\kappa x} + De^{-\kappa x}$. For the wave function to remain finite as $x \to \infty$, $C = 0$. The constant $\kappa^2 = 2m(U_0 - E)/\hbar$, as in $\psi(x) = Ce^{\kappa x} + De^{-\kappa x}$.

(c) At $x = L$, $A\sin kL = De^{-\kappa L}$ and $kA\cos kL = -\kappa De^{-\kappa L}$. Dividing the second of these by the first gives $k\cot kL = -\kappa$, a transcendental equation that must be solved numerically for different values of the length L and the ratio E/U_0.

EVALUATE: When $U_0 \to \infty$, $\kappa \to \infty$ and $\dfrac{\cos(kL)}{\sin(kL)} \to \infty$. The solutions become $k = \dfrac{n\pi}{L}$, $n = 1, 2, 3, \ldots$, the same as for a particle in a box.

40.61.   **IDENTIFY** and **SET UP:**   The energy levels for an infinite potential well are $E_n = \dfrac{n^2 h^2}{8mL^2} = E_1 n^2$. The energy of the absorbed photon is equal to the energy difference between the levels. The energy of a photon is $E = hf = hc/\lambda$, so $\Delta E = hf = E_{n+1} - E_n$.

EXECUTE: **(a)** For the first transition, we have $hf_1 = E_1(n^2 - 1^2)$, and for the second transition we have $hf_2 = E_1[(n+1)^2 - 1^2]$. Taking the ratio of these two equations gives

$\dfrac{hf_2}{hf_1} = \dfrac{16.9}{9.0} = \dfrac{(n+1)^2 - 1}{n^2 - 1} = \dfrac{n^2 + 2n + 1 - 1}{n^2 - 1} = \dfrac{n^2 + 2n}{n^2 - 1}$.

Rearranging and collecting terms gives the quadratic equation $n^2\left(\dfrac{16.9}{9.0} - 1\right) - 2n - \dfrac{16.9}{9.0} = 0$. Using the

quadratic formula and taking the positive root gives $n = 3.0$, so $n = 3$. Therefore the transitions are from the $n = 3$ and $n = 4$ levels to the $n = 1$ level.

(b) Using the $3 \to 1$ transition with $f_1 = 9.0 \times 10^{14}$ Hz, we have
$hf_1 = (h^2/8mL^2)(3^2 - 1^2) = h^2/mL^2$.

$L = \sqrt{\dfrac{h}{f_1 m}} = \sqrt{\dfrac{6.626 \times 10^{-34} \text{ J} \cdot \text{s}}{(9.0 \times 10^{14} \text{ Hz})(9.109 \times 10^{-31} \text{ kg})}} = 9.0 \times 10^{-10}$ m $= 0.90$ nm.

(c) The longest wavelength is for the smallest energy, and that would be for a transition between $n = 1$ and $n = 2$ levels. Comparing the $1 \to 3$ transition and the $1 \to 2$ transition, we have

$\dfrac{hf_{1\to 2}}{hf_{1\to 3}} = \dfrac{E_1(2^2 - 1^2)}{E_1(3^2 - 1^2)}$ $\to$ $f_{1\to 2} = \dfrac{3}{8} f_{1\to 3} = \dfrac{3}{8}(9.0 \times 10^{14} \text{ Hz})$.

$$\lambda = c/f = \frac{3.00 \times 10^8 \text{ m/s}}{\frac{3}{8}(9.0 \times 10^{14} \text{ Hz})} = 890 \text{ nm}.$$

EVALUATE: This wavelength is too long to be visible light. The wavelength of the 9.0×10^{14} Hz photon is 333 nm, which is too short to be visible, as is the 16.9×10^{14} Hz photon. So none of these photons will be visible.

40.62. **IDENTIFY** and **SET UP:** Provided that $T \ll 1$, the probability T of a particle with energy E and mass m tunneling through a potential barrier of width L and height U_0 is $T = Ge^{-2\kappa L}$, where $G = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)$ and $\kappa = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$.

EXECUTE: **(a)** Using the values of E in the table in the problem, we calculate G and κ. For example, for

$E = 4.0$ eV, we have $G = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right) = 16\frac{4.0 \text{ eV}}{8.0 \text{ eV}}\left(1 - \frac{4.0 \text{ eV}}{8.0 \text{ eV}}\right) = 4.0$.

$U_0 - E = 8.0$ eV $- 4.0$ eV $= 4.0$ eV $= 6.4 \times 10^{-19}$ J.

$$\kappa = \frac{\sqrt{2m(U_0 - E)}}{\hbar} = \frac{\sqrt{2(9.11 \times 10^{-31} \text{ kg})(6.4 \times 10^{-19} \text{ J})}}{1.055 \times 10^{-34} \text{ J} \cdot \text{s}} = 1.02 \times 10^{10} \text{ m}^{-1}.$$

Repeating these calculations for the other values of E, and also calculating $\ln(T/G)$, we get the following:

E(eV)	T	G	κ (m^{-1})	$\ln(T/G)$
4.0	2.40×10^{-6}	4.0	1.02×10^{10}	-14.3
5.0	1.50×10^{-5}	3.75	8.87×10^9	-12.4
6.0	1.20×10^{-4}	3.0	7.25×10^9	-10.1
7.0	1.30×10^{-3}	1.75	5.12×10^9	-7.2
7.6	8.10×10^{-3}	0.76	3.24×10^9	-4.5

Figure 40.62 shows the graph of $\ln(T/G)$ versus κ. For $T \ll 1$, we have $T = Ge^{-2\kappa L}$, which we can write as $T/G = e^{-2\kappa L}$. Taking natural logarithms of both sides of the equation gives $\ln(T/G) = -2L\kappa$. From this last equation, we would expect a graph of $\ln(T/G)$ versus κ to be a straight line with slope equal to $-2L$.
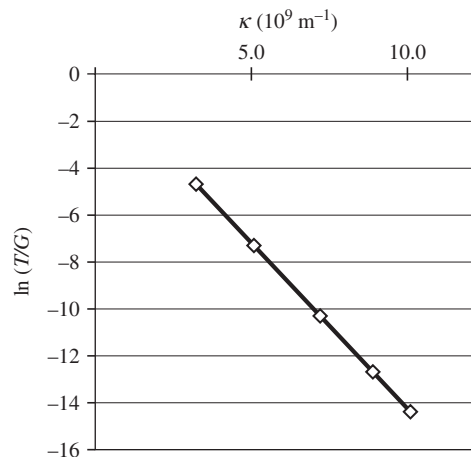

Figure 40.62

(b) The slope of the best-fit straight line for our graph is -1.396×10^{-9} m, so $-2L = -1.396 \times 10^{-9}$ m, which gives $L = 0.698 \times 10^{-9}$ m, which rounds to $L = 0.70$ nm.

EVALUATE: This width is about 7 times the width of a hydrogen atom in the Bohr model.

40.63. **IDENTIFY** and **SET UP:** The transmission coefficient T is equal to 1 when the width L of the barrier is $L = \frac{1}{2}\lambda,\ \lambda,\ \frac{3}{2}\lambda,\ 2\lambda,\ \ldots = n\lambda/2$, where $n = 1, 2, 3, \ldots$, and where λ is the de Broglie wavelength of the electron, given by $\lambda = h/p$. The total energy of the electron is $E = U + K$, and $K = p^2/2m$.

EXECUTE: From the condition on λ, we have $\lambda_n = 2L/n$. Therefore $\lambda = h/p = 2L/n$, which gives

$p = nh/2L$. The kinetic energy is $K = p^2/2m$, so $K_n = \dfrac{p^2}{2m} = \dfrac{\left(\dfrac{nh}{2L}\right)^2}{2m} = n^2\left(\dfrac{h^2}{8mL^2}\right) = n^2 K_1$. The three lowest

values of K are for $n = 1, 2,$ and 3.
$K_1 = h^2/8mL^2 = (6.626\times10^{-34}\ \mathrm{J\cdot s})^2/[8(9.11\times10^{-31}\ \mathrm{kg})(1.8\times10^{-10}\ \mathrm{m})^2] = 1.86\times10^{-18}\ \mathrm{J} = 11.6\ \mathrm{eV}$.
The total energy is $E = K + U$, so $E_1 = K_1 + U = 11.6\ \mathrm{eV} + 10\ \mathrm{eV} = 22\ \mathrm{eV}$.
For the $n = 2$ state, we have
$K_2 = 2^2 K_1 = 4(11.6\ \mathrm{eV}) = 46.4\ \mathrm{eV}$, so $E_2 = 46.4\ \mathrm{eV} + 10\ \mathrm{eV} = 56\ \mathrm{eV}$.
For the $n = 3$ state, we have
$K_3 = 3^2 K_1 = 9(11.6\ \mathrm{eV}) = 104.4\ \mathrm{eV}$, so $E_3 = 104.4\ \mathrm{eV} + 10\ \mathrm{eV} = 114\ \mathrm{eV}$, which rounds to $110\ \mathrm{eV}$.
EVALUATE: We cannot use Eq. (40.42) because T is not small.

40.64. **IDENTIFY** and **SET UP:** Follow the steps specified in the problem.

EXECUTE: **(a)** $E = K + U(x) = \dfrac{p^2}{2m} + U(x) \Rightarrow p = \sqrt{2m(E - U(x))}.$ $\lambda = \dfrac{h}{p} \Rightarrow \lambda(x) = \dfrac{h}{\sqrt{2m(E - U(x))}}.$

(b) As $U(x)$ gets larger (i.e., $U(x)$ approaches E from below—recall $k \geq 0$), $E - U(x)$ gets smaller, so $\lambda(x)$ gets larger.

(c) When $E = U(x)$, $E - U(x) = 0$, so $\lambda(x) \to \infty$.

(d) $\displaystyle\int_a^b \dfrac{dx}{\lambda(x)} = \int_a^b \dfrac{dx}{h/\sqrt{2m(E-U(x))}} = \dfrac{1}{h}\int_a^b \sqrt{2m(E-U(x))}\ dx = \dfrac{n}{2} \Rightarrow \int_a^b \sqrt{2m(E-U(x))}\ dx = \dfrac{hn}{2}.$

(e) $U(x) = 0$ for $0 < x < L$ with classical turning points at $x = 0$ and $x = L$. So,

$\displaystyle\int_a^b \sqrt{2m[E - U(x)]}\ dx = \int_0^L \sqrt{2mE}\ dx = \sqrt{2mE}\int_0^L dx = \sqrt{2mE}\,L.$ So, from part (d),

$\sqrt{2mE}\,L = \dfrac{hn}{2} \Rightarrow E = \dfrac{1}{2m}\left(\dfrac{hn}{2L}\right)^2 = \dfrac{h^2 n^2}{8mL^2}.$

EVALUATE: **(f)** Since $U(x) = 0$ in the region between the turning points at $x = 0$ and $x = L$, the result is the *same* as part (e). The height U_0 never enters the calculation. WKB is best used with *smoothly* varying potentials $U(x)$.

40.65. **IDENTIFY:** Perform the calculations specified in the problem.
SET UP: $U(x) = \frac{1}{2}k'x^2.$

EXECUTE: **(a)** At the turning points $E = \frac{1}{2}k'x_{\mathrm{TP}}^2 \Rightarrow x_{\mathrm{TP}} = \pm\sqrt{\dfrac{2E}{k'}}.$

(b) $\displaystyle\int_{-\sqrt{2E/k'}}^{+\sqrt{2E/k'}} \sqrt{2m(E - \frac{1}{2}k'x^2)}\,dx = \dfrac{nh}{2}.$ To evaluate the integral, we want to get it into a form that matches

the standard integral given. $\sqrt{2m\left(E - \dfrac{1}{2}k'x^2\right)} = \sqrt{2mE - mk'x^2} = \sqrt{mk'}\sqrt{\dfrac{2mE}{mk'} - x^2} = \sqrt{mk'}\sqrt{\dfrac{2E}{k'} - x^2}.$

Letting $A^2 = \dfrac{2E}{k'},\ a = -\sqrt{\dfrac{2E}{k'}},$ and $b = +\sqrt{\dfrac{2E}{k'}}$

$$\Rightarrow \sqrt{mk'}\int_a^b \sqrt{A^2 - x^2}\ dx = 2\frac{\sqrt{mk'}}{2}\left[x\sqrt{A^2 - x^2} + A^2 \arcsin\left(\frac{x}{|A|}\right)\right]\Bigg|_0^b$$

$$= \sqrt{mk'}\left[\sqrt{\frac{2E}{k'}}\sqrt{\frac{2E}{k'} - \frac{2E}{k'}} + \frac{2E}{k'}\arcsin\left(\frac{\sqrt{2E/k'}}{\sqrt{2E/k'}}\right)\right] = \sqrt{mk'}\ \frac{2E}{k'}\arcsin(1) = 2E\sqrt{\frac{m}{k'}}(\tfrac{1}{2}).$$

Using WKB, this is equal to $\frac{hn}{2}$, so $E\sqrt{\frac{m}{k'}}\pi = \frac{hn}{2}$. Recall $\omega = \sqrt{\frac{k'}{m}}$, so $E = \frac{h}{2\pi}\omega n = \hbar\omega n$.

EVALUATE: (c) We are missing the zero-point-energy offset of $\frac{\hbar\omega}{2}$ (recall $E = \hbar\omega(n + \tfrac{1}{2})$). It

underestimates the energy. However, our approximation isn't bad at all!

40.66. **IDENTIFY** and **SET UP:**   Perform the calculations specified in the problem.

EXECUTE: (a) At the turning points $E = A|x_{TP}| \Rightarrow x_{TP} = \pm\frac{E}{A}$.

(b) $\int_{-E/A}^{+E/A} \sqrt{2m(E - A|x|)}dx = 2\int_0^{E/A} \sqrt{2m(E - Ax)}\ dx$. Let $y = 2m(E - Ax) \Rightarrow$

$dy = -2mA\ dx$ when $x = \frac{E}{A}$, $y = 0$, and when $x = 0$, $y = 2mE$. So

$$2\int_0^{\frac{E}{A}} \sqrt{2m(E - Ax)}dx = -\frac{1}{mA}\int_{2mE}^0 y^{1/2}dy = -\frac{2}{3mA}y^{3/2}\Bigg|_{2mE}^0 = \frac{2}{3mA}(2mE)^{3/2}.$$ Using WKB, this is equal to $\frac{hn}{2}$.

So, $\frac{2}{3mA}(2mE)^{3/2} = \frac{hn}{2} \Rightarrow E = \frac{1}{2m}\left(\frac{3mAh}{4}\right)^{2/3}n^{2/3}$.

EVALUATE: (c) The difference in energy decreases between successive levels. For example:
$$1^{2/3} - 0^{2/3} = 1,\ 2^{2/3} - 1^{2/3} = 0.59,\ 3^{3/2} - 2^{3/2} = 0.49,\ldots$$

- A sharp ∞ step gave ever-increasing level differences $(\sim n^2)$.

- A parabola $(\sim x^2)$ gave evenly spaced levels $(\sim n)$.

- Now, a linear potential $(\sim x)$ gives ever-decreasing level differences $(\sim n^{2/3})$.

  Roughly speaking, if the curvature of the potential $(\sim$ second derivative) is bigger than that of a parabola, then the level differences will increase. If the curvature is less than a parabola, the differences will decrease.

40.67. **IDENTIFY** and **SET UP:**   The energy levels are $E_{m,n} = (m^2 + n^2)(\pi^2\hbar^2)/2ML^2$.

EXECUTE: For a fixed value of m, the spacing between adjacent energy levels is
$\Delta E = [(m^2 + (n+1)^2 - (m^2 + n^2)](\pi^2\hbar^2)/2ML^2 = (2n+1)(\pi^2\hbar^2)/2ML^2$. As the value of L is decreased, the spacing between adjacent levels increases. In this model, L is the size of the dot, so choice (c) is correct.
EVALUATE: We could also have kept n fixed and varied m by 1.

40.68. **IDENTIFY** and **SET UP:**   The energy levels are $E_{m,n} = (m^2 + n^2)(\pi^2\hbar^2)/2ML^2$. $\Delta E = \frac{hc}{\lambda}$ for the photon

emitted during a transition through energy difference ΔE.

EXECUTE: Since $E \propto 1/L^2$, it is also the case that $\Delta E \propto 1/L^2$, and we know that $\Delta E = \frac{hc}{\lambda}$.

Therefore
$\lambda \propto L^2$. If $L \rightarrow 1.1L$, $\lambda \rightarrow (1.1)^2 \lambda$. So in this case, $\lambda \rightarrow (1.1)^2(550\ \text{nm}) = 666\ \text{nm} \approx 670\ \text{nm}$, which is choice (b).
EVALUATE: Because λ is proportional to the *square* of L instead of just L, a 10% change in L produces around a 20% change in λ.

40.69.   **IDENTIFY** and **SET UP:**   The energy levels are $E_{m,n} = (m^2 + n^2)(\pi^2 \hbar^2)/2ML^2$. $\Delta E = \dfrac{hc}{\lambda}$ for the photon

emitted during a transition through energy difference ΔE.

EXECUTE: Since $\Delta E \propto 1/M$ and $\Delta E = \dfrac{hc}{\lambda}$, it follows that $\lambda \propto 1/M$. Since $\lambda_1 > \lambda_2$, it is true that

$M_1 > M_2$, which makes choice (a) correct.

EVALUATE: When M is large, the energy difference between states is small, so the energy of an emitted (or absorbed) photon is small, so its wavelength is large.

40.70.   **IDENTIFY:**   Apply the Heisenberg uncertainty principle, stated in terms of energy and time.

SET UP: $\Delta E \Delta t \geq \hbar/2$.

EXECUTE: Solving for ΔE gives $\Delta E \geq \dfrac{\hbar}{2\Delta t}$. Therefore increasing the lifetime of the excited states

results in a smaller energy spread because ΔE is small, meaning that the energies are more well-defined. Therefore choice (a) is correct.

EVALUATE: A long lifetime does not imply that the energy is small; but it does tell us that the *uncertainty* in the energy is small.