MA1125 – Calculus
Tutorial solutions #5

1. Show that the polynomial $f(x) = x^3 - 5x^2 - 8x + 1$ has exactly one root in $(0, 1)$.

Being a polynomial, f is continuous on the interval $[0, 1]$ and we also have

$$f(0) = 1, \qquad f(1) = 1 - 5 - 8 + 1 = -11.$$

Since $f(0)$ and $f(1)$ have opposite signs, f must have a root that lies in $(0, 1)$. To show it is unique, suppose that f has two roots in $(0, 1)$. Then f' must have a root in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 3x^2 - 10x - 8 = (3x + 2)(x - 4).$$

Since f' has no roots in $(0, 1)$, we conclude that f has exactly one root in $(0, 1)$.

2. Let $b > 1$ be a given constant. Use the mean value theorem to show that

$$1 - \frac{1}{b} < \ln b < b - 1.$$

Since $f(x) = \ln x$ is differentiable with $f'(x) = 1/x$, the mean value theorem gives

$$\frac{f(b) - f(1)}{b - 1} = f'(c) = \frac{1}{c}$$

for some point $1 < c < b$. Using the fact that $\frac{1}{b} < \frac{1}{c} < 1$, one may thus conclude that

$$\frac{1}{b} < \frac{\ln b - \ln 1}{b - 1} < 1 \quad \Longrightarrow \quad 1 - \frac{1}{b} < \ln b < b - 1.$$

3. Compute each of the following limits.

$$L_1 = \lim_{x \to 2} \frac{2x^3 - 5x^2 + 5x - 6}{3x^3 - 5x^2 - 4}, \qquad L_2 = \lim_{x \to \infty} \frac{\ln x}{x^2}, \qquad L_3 = \lim_{x \to 0} (x + \cos x)^{1/x}.$$

The first limit has the form $0/0$, so one may use L'Hôpital's rule to get

$$L_1 = \lim_{x \to 2} \frac{6x^2 - 10x + 5}{9x^2 - 10x} = \frac{24 - 20 + 5}{36 - 20} = \frac{9}{16}.$$

The second limit has the form ∞/∞, so L'Hôpital's rule is still applicable and

$$L_2 = \lim_{x \to \infty} \frac{1/x}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0.$$

The third limit involves a non-constant exponent which can be eliminated by writing

$$\ln L_3 = \ln \lim_{x\to 0} (x+\cos x)^{1/x} = \lim_{x\to 0} \ln(x+\cos x)^{1/x} = \lim_{x\to 0} \frac{\ln(x+\cos x)}{x}.$$

This gives a limit of the form 0/0, so one may use L'Hôpital's rule to find that

$$\ln L_3 = \lim_{x\to 0} \frac{1-\sin x}{x+\cos x} = \frac{1-0}{0+1} = 1.$$

Since $\ln L_3 = 1$, the original limit L_3 is then equal to $L_3 = e^{\ln L_3} = e$.

4. For which values of x is $f(x) = (\ln x)^2$ increasing? For which values is it concave up?

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = 2\ln x \cdot (\ln x)' = \frac{2\ln x}{x}.$$

Since the given function is only defined at points $x > 0$, it is increasing if and only if

$$\ln x > 0 \quad \Longleftrightarrow \quad x > e^0 \quad \Longleftrightarrow \quad x > 1.$$

To say that $f(x)$ is concave up is to say that $f''(x) > 0$. According to the quotient rule,

$$f''(x) = \frac{(2/x)\cdot x - 2\ln x}{x^2} = \frac{2(1-\ln x)}{x^2}.$$

Since the denominator is always positive, $f(x)$ is then concave up if and only if

$$1 - \ln x > 0 \quad \Longleftrightarrow \quad \ln x < 1 \quad \Longleftrightarrow \quad 0 < x < e.$$

5. Find the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down. Use this information to sketch the graph of f.

$$f(x) = \frac{x^2}{x^2+3}.$$

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. In this case, we have

$$f'(x) = \frac{2x\cdot(x^2+3) - 2x\cdot x^2}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2},$$

so it is clear that $f(x)$ is increasing if and only if $x > 0$. To say that $f(x)$ is concave up is to say that $f''(x) > 0$. Using both the quotient rule and the chain rule, we get

$$f''(x) = \frac{6(x^2+3)^2 - 2(x^2+3)\cdot 2x\cdot 6x}{(x^2+3)^4} = \frac{6(x^2+3) - 24x^2}{(x^2+3)^3} = \frac{18(1-x^2)}{(x^2+3)^3}.$$

Since the denominator is always positive, $f(x)$ is then concave up if and only if

$$1 - x^2 > 0 \quad \Longleftrightarrow \quad x^2 < 1 \quad \Longleftrightarrow \quad -1 < x < 1.$$

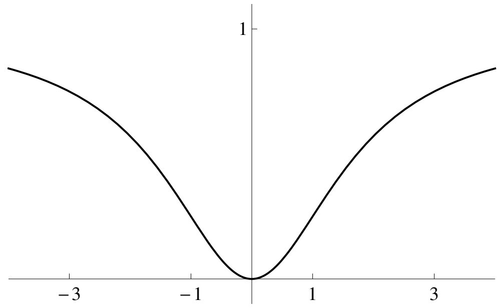

Figure 1: The graph of $f(x) = \dfrac{x^2}{x^2+3}$.

6. Show that the polynomial $f(x) = x^3 + x^2 - 5x + 1$ has exactly two roots in $(0, 2)$.

To prove existence using Bolzano's theorem, we note that f is continuous with

$$f(0) = 1, \qquad f(1) = 1 + 1 - 5 + 1 = -2, \qquad f(2) = 8 + 4 - 10 + 1 = 3.$$

In view of Bolzano's theorem, f must then have a root in $(0, 1)$ and another root in $(1, 2)$, so it has two roots in $(0, 2)$. Suppose that it has three roots in $(0, 2)$. Then f' must have two roots in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 3x^2 + 2x - 5 = (3x + 5)(x - 1).$$

Since f' has only one root in $(0, 2)$, we conclude that f has only two roots in $(0, 2)$.

7. Use the mean value theorem for the case $f(x) = \sqrt{x+4}$ to show that

$$2 + \frac{1}{2} < \sqrt{7} < 2 + \frac{3}{4}.$$

According to the mean value theorem, there exists a point $0 < c < 3$ such that

$$\frac{f(3) - f(0)}{3 - 0} = f'(c) \quad \Longrightarrow \quad \frac{\sqrt{7} - \sqrt{4}}{3} = \frac{1}{2\sqrt{c+4}}.$$

To estimate the square root on the right hand side, we note that

$$0 < c < 3 \quad \Longrightarrow \quad 4 < c + 4 < 7 < 9 \quad \Longrightarrow \quad 2 < \sqrt{c+4} < 3.$$

Once we now combine the last two equations, we may easily conclude that

$$\frac{1}{3} < \frac{1}{\sqrt{c+4}} < \frac{1}{2} \quad \Longrightarrow \quad \frac{1}{2} < \sqrt{7} - 2 < \frac{3}{4}.$$

8. Compute each of the following limits.

$$L_1 = \lim_{x\to 2} \frac{x^3 - 5x^2 + 8x - 4}{x^3 - 3x^2 + 4}, \qquad L_2 = \lim_{x\to 1} \frac{\ln x}{x^4 - 1}, \qquad L_3 = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}.$$

The first limit has the form $0/0$, so one may use L'Hôpital's rule to find that

$$L_1 = \lim_{x\to 2} \frac{3x^2 - 10x + 8}{3x^2 - 6x}.$$

Since the limit on the right hand side is still a limit of the form $0/0$, one has

$$L_1 = \lim_{x\to 2} \frac{6x - 10}{6x - 6} = \frac{12 - 10}{12 - 6} = \frac{1}{3}.$$

The second limit is also of the form $0/0$ and an application of L'Hôpital's rule gives

$$L_2 = \lim_{x\to 1} \frac{1/x}{4x^3} = \frac{1}{4}.$$

The third limit has the form ∞/∞, so it follows by L'Hôpital's rule that

$$L_3 = \lim_{x\to 0^+} \frac{(\sin x)^{-1} \cdot \cos x}{(\tan x)^{-1} \cdot \sec^2 x}.$$

Since both $\cos x$ and $\sec x$ are approaching 1 as x approaches zero, we conclude that

$$L_3 = \lim_{x\to 0^+} \frac{\tan x}{\sin x} = \lim_{x\to 0^+} \frac{1}{\cos x} = 1.$$

9. For which values of x is $f(x) = e^{-2x^2}$ increasing? For which values is it concave up?

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = e^{-2x^2} \cdot (-2x^2)' = -4xe^{-2x^2}.$$

Since the exponential factor is always positive, $f(x)$ is increasing if and only if $x < 0$. To say that $f(x)$ is concave up is to say that $f''(x) > 0$. In this case, we have

$$f''(x) = -4e^{-2x^2} - 4x \cdot (-4x)e^{-2x^2} = (16x^2 - 4)e^{-2x^2} = 4(2x - 1)(2x + 1)e^{-2x^2}.$$

It easily follows that $f(x)$ is concave up if and only if $x \in (-\infty, -1/2) \cup (1/2, +\infty)$.

10. Show that there exists a unique number $1 < x < \pi$ such that $x^3 = 3\sin x + 1$.

It is clear that $f(x) = x^3 - 3\sin x - 1$ is continuous on $[1, \pi]$ and we also have

$$f(1) = -3\sin 1 < 0, \qquad f(\pi) = \pi^3 - 1 > 0.$$

Since $f(1)$ and $f(\pi)$ have opposite signs, f must have a root that lies in $(1, \pi)$. To show it is unique, suppose f has two roots in $(1, \pi)$. Then f' must have a root in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 3x^2 - 3\cos x > 3 - 3\cos x = 3(1 - \cos x) \geq 0$$

for all $x > 1$. In particular, f' has no roots in $(1, \pi)$ and f has exactly one root in $(1, \pi)$.