Lecture swap:

Astrophysics/Vidotto (Friday 3pm)
Electromagnetism/Groh (Monday, 11/Feb, 2pm)

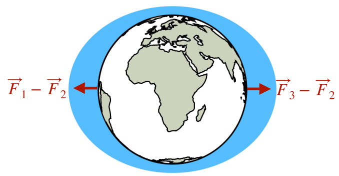
Lecture 4: The laws of planetary motion (part 2)

Read: Ch 22.4 of "Astronomy: a Physical Perspective" (M. Kutner)
Ch. 13.5 of "University Physics" (Young & Freedman)

Prof Aline Vidotto

Quick recap of last lecture

Tidal force



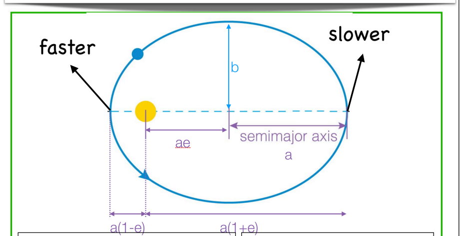
Tidal effects are caused by the difference between the gravitational forces on opposite sides of an object

$$\Delta F_{\rm tid} = \frac{-2GmM}{r^3}\Delta r$$

The Sun and the Moon raise tidal bulges on the Earth, so the orientation Moon-Earth-Sun (lunar phases) affects our tides

Kepler's laws

1. Planetary orbits are **ellipses**, with the Sun at one focus.
2. As a planet moves around its orbit, it sweeps out equal areas in equal times
3. Square of period of planet's orbital motion is proportional to cube of semi-major axis.



For the solar system:

$$\left(\frac{P}{1\rm yr}\right)^2 = \left(\frac{a}{1\rm AU}\right)^3$$

In general:

$$P^2 = ka^3$$

$$k = 4\pi^2/(GM_{\rm central})$$

What we will cover today…

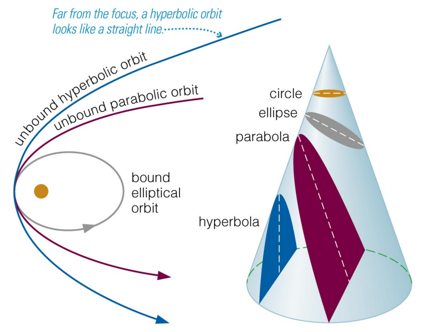
Goal: understand how the laws of "planetary motion" can be extended to other astronomical objects

1. How Newton's law of gravity extend Kepler's laws?
2. Orbital energy
3. Exploring the solar system (interplanetary travel)

1. How Newton's law of gravity extend Kepler's laws?

Extending Kepler's laws to other objects/orbital paths

1. Kepler's laws apply to all orbiting objects, not just planets.
2. Ellipses are not the only orbital paths. Orbits can be:
   - bound
     - ➡ ellipses: 0<e<1
     - ➡ including circles: e=0
   - unbound (example: some comets that only loop the Sun once and never return)
     - ➡ parabola: e=1
     - ➡ hyperbola: e>1



a Orbits allowed by the law of gravity.

b Ellipses (which include circles), parabolas, and hyperbolas are conic sections, made by slicing a cone at different angles.

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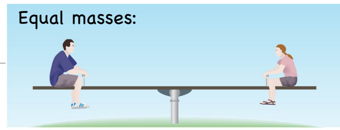
The centre of mass

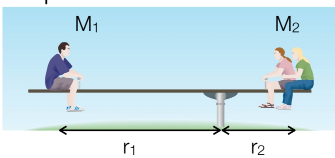
3. Objects orbit their common centre of mass

- We have assumed that as a planet or comet orbits the sun, the sun remains absolutely stationary. In fact, both the sun and the planet orbit around their common centre of mass.
- Note: the two orbiting objects are always on opposite sides of the centre of mass

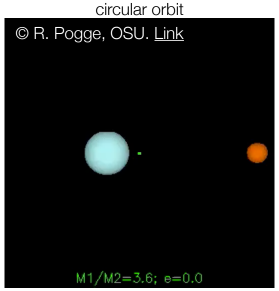
Equal masses:



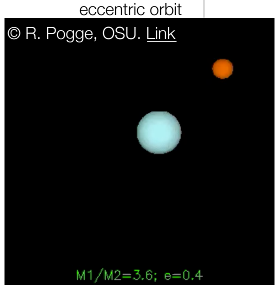
Unequal masses:



circular orbit



eccentric orbit



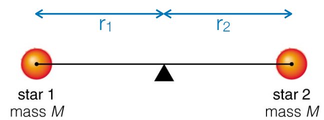
Property of the centre of mass:

$$M_1 r_1 = M_2 r_2$$

This "wobble" is key for finding exoplanets!

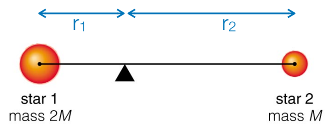
Examples: centre of mass

$$M_1 r_1 = M_2 r_2 \quad \rightarrow \quad r_1 = \frac{M_2}{M_1} r_2$$



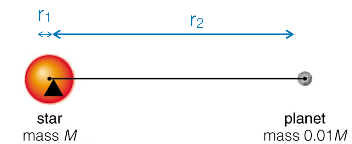
$$r_1 = \frac{M}{M} r_2 = r_2$$

halfway between two objects of same mass



$$r_1 = \frac{M}{2M} r_2 = \frac{1}{2} r_2$$

closer to the object that is more massive



$$r_1 = \frac{0.01M}{M} r_2 = \frac{1}{100} r_2$$

$M_\star \gg M_p$: centre of mass lies inside the star!

Kepler's 3rd law and the mass of the system

4. the relationship between the orbital period and average orbital distance of a system tells us the total mass of the system.

$$P^2 = ka^3 \quad \text{with} \quad k = \frac{4\pi^2}{GM_\odot}$$

- semi-major axis: a
- period: P
- Solar mass: $M_\odot$

For a planet-star system, this equation works out well: $$P^2 = \frac{4\pi^2}{GM_\odot}a^3$$

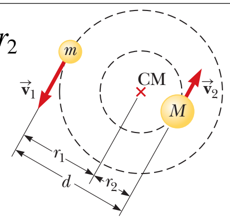
However, the more general form is: $$P^2 = \frac{4\pi^2}{G(M_1 + M_2)}a^3$$

Masses of two bodies (e.g. two stars, a star and a planet, or a planet and a satellite, etc)

$$P^2 = ka^3 \quad \text{with} \quad k = \frac{4\pi^2}{G(M_1 + M_2)}$$

Demonstration

$$d = r_1 + r_2$$



- Consider 2 stars, separated by a distance d, revolving in circular orbits about their centre of mass.
- The centripetal force in star #1

$$F_{\text{cent}} = \frac{m}{r_1}v_1^2 = \frac{m}{r_1}\frac{(2\pi r_1)^2}{P^2} = mr_1\frac{(2\pi)^2}{P^2}$$

- The centripetal force in star #2

$$F_{\text{cent}} = \frac{M}{r_2}v_2^2 = \frac{M}{r_2}\frac{(2\pi r_2)^2}{P^2} = Mr_2\frac{(2\pi)^2}{P^2}$$

- For each star, we have $F_g = F_{\text{cent}}$, with

$$F_g = \frac{GmM}{d^2} \xrightarrow{\text{star\#1}} \frac{GmM}{d^2} = mr_1\frac{(2\pi)^2}{P^2} \rightarrow r_1 = \frac{GM}{d^2}\frac{P^2}{(2\pi)^2}$$

$$F_g = \frac{GmM}{d^2} \xrightarrow{\text{star\#2}} \frac{GmM}{d^2} = Mr_2\frac{(2\pi)^2}{P^2} \rightarrow r_2 = \frac{Gm}{d^2}\frac{P^2}{(2\pi)^2}$$

- Summing both eqs:

$$r_1 + r_2 = \frac{G(M+m)}{d^2}\frac{P^2}{(2\pi)^2}$$

$$\boxed{d^3 = \frac{G(M+m)}{4\pi^2}P^2}$$

Example: masses of solar system objects

$$P^2 = \frac{4\pi^2}{G(M_1 + M_2)}a^3 \quad \rightarrow \quad \boxed{(M_1 + M_2) = \frac{4\pi^2}{G}\frac{a^3}{P^2}}$$

- Earth's orbital period (1 yr) and average distance (1 AU) tell us the Sun's mass.

$$M_1 = M_\odot \qquad M_2 = M_\oplus \qquad M_\odot + M_\oplus \simeq M_\odot \longrightarrow M_\odot = \frac{4\pi^2}{G}\frac{a^3_{\text{orb},\oplus}}{P^2_{\text{orb},\oplus}}$$

- Orbital period and distance of a satellite from Earth tell us Earth's mass.

$$M_1 = M_\oplus \qquad M_2 = M_{\text{sat}} \qquad M_\oplus + M_{\text{sat}} \simeq M_\oplus \longrightarrow M_\oplus = \frac{4\pi^2}{G}\frac{a^3_{\text{orb,sat}}}{P^2_{\text{orb,sat}}}$$

- Orbital period and distance of a moon of Jupiter tell us Jupiter's mass.

$$M_1 = M_{\text{jup}} \qquad M_2 = M_{\text{moon}} \qquad M_{\text{jup}} + M_{\text{moon}} \simeq M_{\text{jup}} \longrightarrow M_{\text{jup}} = \frac{4\pi^2}{G}\frac{a^3_{\text{orb,moon}}}{P^2_{\text{orb,moon}}}$$

Example: mass of Jupiter

- The Jovian satellite Europa has an orbital (sidereal) period of 3.551 days and a semi-major axis of 671,100 km. Using this information, determine the mass of Jupiter.

$$(M_1 + M_2) = \frac{4\pi^2}{G}\frac{a^3}{P^2}$$

$$M_1 = M_{\text{jup}} \qquad M_2 = M_{\text{moon}}$$

$$M_{\text{jup}} + M_{\text{moon}} \simeq M_{\text{jup}}$$

$$M_{\text{jup}} = \frac{4\pi^2}{G}\frac{a^3_{\text{orb,moon}}}{P^2_{\text{orb,moon}}}$$

$$= \frac{4\pi^2}{G}\frac{(671100 \times 10^3\text{m})^3}{(3.551 \times 24 \times 3600\text{s})^2}$$

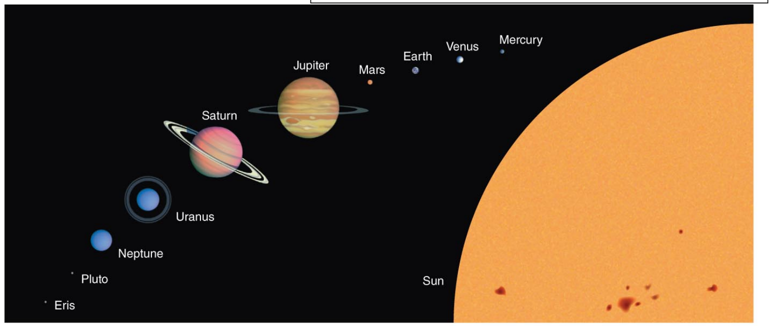
$$= 1.9 \times 10^{27}\ \text{kg}$$

Mass distribution of the solar system

$M_{\text{jup}} = 1.9 \times 10^{27}$ kg

$M_{\text{sun}} = 2 \times 10^{30}$ kg

Object	% of total mass
Sun	99.85
Planets	0.135 of which: Jupiter 71% Saturn 21%
Satellites	5x10^{-5}
Comets	~0.01
Asteroids	remainder



a The scaled sizes (but not distances) of the Sun, the planets, and the two largest known dwarf planets.

Example: masses of other objects

$$P^2 = \frac{4\pi^2}{G(M_1+M_2)} a^3 \quad \rightarrow \quad \boxed{(M_1+M_2) = \frac{4\pi^2}{G}\frac{a^3}{P^2}}$$

- Note: if the objects have comparable masses (eg, two stars), the approximation $M_1 + M_2 \simeq M_1$ cannot be used. All you know then in the sum of both masses!
- Example: Two stars orbit each other with a period of 8 years at an orbital distance of 10 AU. Find the mass of the system.

$$(M_1+M_2) = \frac{4\pi^2}{(6.67\times10^{-11})}\frac{(10\times1.5\times10^{11}\text{m})^3}{(8\times365\times24\times3600\text{s})^2} = 3.1\times10^{31}\text{kg}$$
$$= 15.6 M_\odot$$

Useful trick:

$$\boxed{\frac{(M_1+M_2)}{M_\odot} = \frac{(a/1\text{au})^3}{(P/1\text{yr})^2}} \quad \rightarrow \quad \frac{(M_1+M_2)}{M_\odot} = \frac{(10\cancel{\text{au}}/1\cancel{\text{au}})^3}{(8\cancel{\text{yr}}/1\cancel{\text{yr}})^2} = 15.6$$

Conceptual question

Imagine another solar system, with a star of the same mass as the Sun. Suppose a planet with a mass twice that of Earth ($2M_{\text{Earth}}$) orbits at a distance of 1 AU from the star. What is the orbital period of this planet?

(a) 2 years
(b) 1 year
(c) 6 months
(d) It cannot be determined from the information given.

Conceptual question

Imagine another solar system, with a star of the same mass as the Sun. Suppose a planet with a mass twice that of Earth ($2M_{\text{Earth}}$) orbits at a distance of 1 AU from the star. What is the orbital period of this planet?

(a) 2 years
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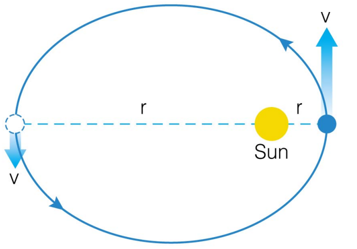
2. Orbital energy (mechanical energy)

kinetic + potential

$$E = \frac{mv_{\text{orb}}^2}{2} - \frac{GmM}{r} = \text{const}$$

Larger orbital distances, gravitational energy is less negative (more potential energy), thus lower kinetic energy: smaller speeds



Smaller orbital distances, gravitational energy is more negative (less potential energy), thus higher kinetic energy: larger speeds

Energies in elliptical orbits

For circular orbits:

$$v_{\text{orb}}^2 = \frac{GM}{r}$$

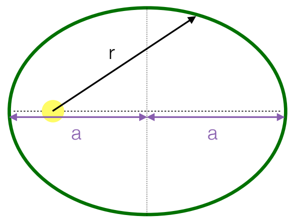
$$\downarrow$$

$$E = \frac{mv_{\text{orb}}^2}{2} - \frac{GmM}{r}$$

$$\downarrow$$

$$E_{\text{circ}} = -\frac{GmM}{2r}$$

For elliptical orbits:



- semi-major axis: a
- radial coordinate: r

$$v_{\text{orb}}^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right]$$

$$\downarrow$$

$$E = \frac{mv_{\text{orb}}^2}{2} - \frac{GmM}{r}$$

$$\downarrow$$

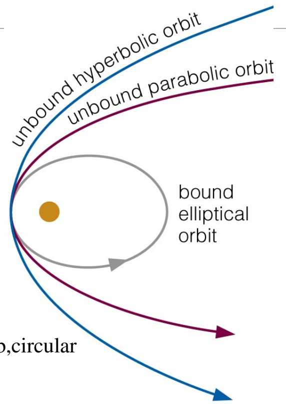
$$E_{\text{ell}} = -\frac{GmM}{2a}$$

Energy of different orbits

- E<0: orbit is bound (elliptical orbit, including circular orbit)
- E=0: orbit is unbound and parabolic (e.g., a comet with E=0 would escape the solar system)
  - Note that, by setting E=0, we found the escape speed in past lectures.

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r}} \quad \rightarrow \quad v_{\text{esc}} = \sqrt{2}\, v_{\text{orb,circular}}$$

- E>0: orbit is unbound and hyperbolic



Ways to alter the orbital energy

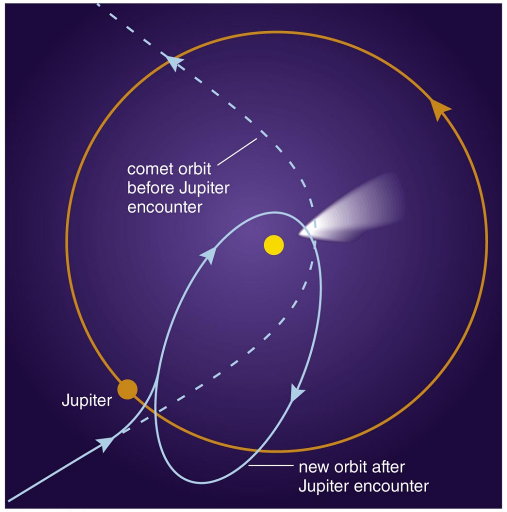
- The total orbital energy is conserved <u>as long as</u> no other object causes the planet (or other object in orbit) to gain or lose orbital energy

$$E = \frac{mv_{\text{orb}}^2}{2} - \frac{GmM}{r}$$

- Note
  - Orbits do not change spontaneously
  - Orbits can only change through exchange of energy

Ways to lose orbital energy: Gravitational encounter

- Consider a comet in an unbound orbit
- Comet passes by Jupiter & exchange energy: comet loses energy and its orbit becomes bound (elliptical)
- Jupiter gains exactly as much energy, but the effects in Jupiter is unnoticeable due to its much greater mass
- This can also work on reverse: a gravitationally bound comet can be kicked out into more distant orbits (even be ejected from the solar system)



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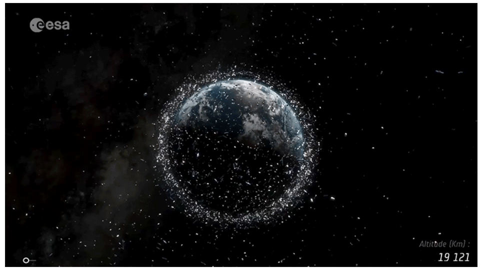
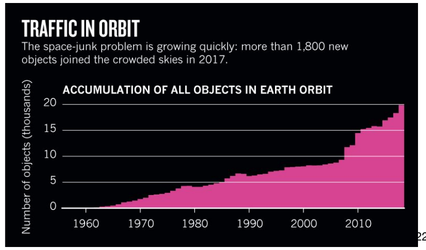
Ways to lose orbital energy: Atmospheric drag

- Consider a satellite in orbit around the Earth
  - If the orbit is low, then satellite experiences drag from Earth's atmosphere
  - Satellite loses orbital energy. At lower orbit, atmosphere is denser, causing more drag
  - Eventually satellite plummets to Earth
- Orbital energy is converted into thermal energy: satellite burns up.
- Remember the Chinese Space Station Tiangong-1?

The space junk problem





Read more: https://www.nature.com/articles/d41586-018-06170-1

How to gain orbital energy?

- A space transportation vehicle releases a 470-kg communications satellite while in a circular orbit 280 km above the surface of the Earth. A rocket engine on the satellite boosts it into a higher circular orbit at 36,000 km. How much energy does the engine have to provide?

$$r_i = R_E + 280\text{km}$$
$$r_f = R_E + 36000\text{km}$$
$$R_E = 6400\text{km}$$

initial orbital energy: $$E_{\text{circ}} = -\frac{GmM}{2r_i}$$

final orbital energy: $$E_{\text{circ}} = -\frac{GmM}{2r_f}$$

energy gained: $$\Delta E = -\frac{GmM}{2r_f} - (-1)\frac{GmM}{2r_i} = \frac{-GmM}{2}\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

$$\Delta E = \frac{-(6.67\times 10^{-11})(470)(6\times 10^{24})}{2}\left(\frac{1}{42{,}400\times 10^3} - \frac{1}{6{,}680\times 10^3}\right)$$

$$= 1.1\times 10^{10}\text{J}$$

Needs to supply the equivalent energy in fuel to boost the satellite to such a higher orbit!

Conceptual question

What happens to the energy of an object while it follows an unbound orbit around the Sun?

(a) The total remains constant while gravitational potential energy is converted to kinetic energy as it approaches the Sun.

(b) The total increases when the object approaches the Sun and decreases when it recedes from the Sun.

(c) The total, kinetic, and potential energy remain constant.

(d) The total remains constant while kinetic energy is converted to gravitational potential energy as it approaches the Sun.

Conceptual question

What happens to the energy of an object while it follows an unbound orbit around the Sun?

(a) **The total remains constant while gravitational potential energy is converted to kinetic energy as it approaches the Sun.**

(b) The total increases when the object approaches the Sun and decreases when it recedes from the Sun.

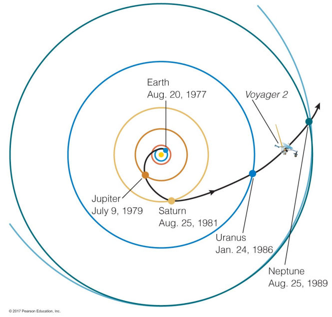
(c) The total, kinetic, and potential energy remain constant.

(d) The total remains constant while kinetic energy is converted to gravitational potential energy as it approaches the Sun.

3. Traveling through the solar system

Types of spacecrafts

- Flyby: flies by another world only once: cheaper but less time to gather data
- Orbiter: goes into orbit around another world
- Probe/lander: lands on surface
- Sample return mission: returns a sample of another world's surface to Earth (e.g., Apollo missions to the moon)
  - trip to the moon takes a few days



How do robotic spacecraft work?

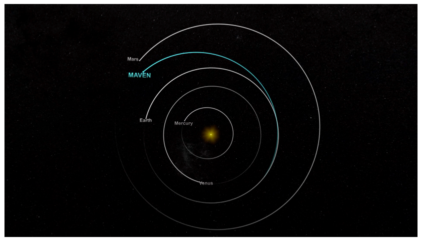
- In considering travel to other planets, we use elliptical orbits
  - An object touching both the orbits of Earth and Mars cannot be circular about the Sun
- Recall: total energy of an elliptical orbit:

$$E_{\rm ell} = -\frac{GmM}{2a}$$

  - energy does not depend on the eccentricity, but only on its semi-major axis
  - the orbiter is orbiting the Sun, hence

$$E_{\rm ell} = -\frac{Gm_{\rm orb}M_\odot}{2a}$$

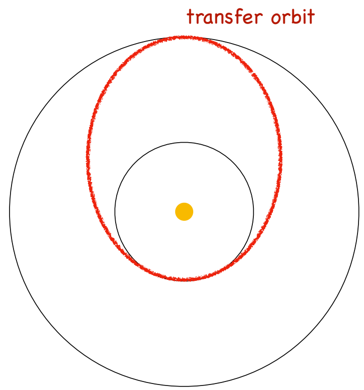
The trick in launching a planetary probe is to place it in an orbit that intersects the planet's orbit. This is called the transfer orbit:



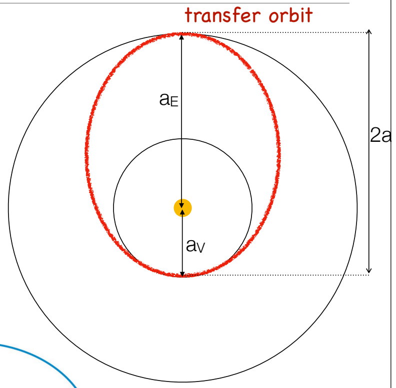
Watch the entire film (4min): https://www.youtube.com/watch?v=1Hm8b-L62y4

Traveling to other solar system planets

- There are many orbits that could be used for the transfer. However, the most economical one is the one that has a minimum energy
  - for inner planets: this happens when the Earth is at aphelion and planet at perihelion
  - for outer planets: this happens when the Earth is at perihelion and planet at aphelion



Example: minimum energy orbit to Venus



- Orbit of Venus is inside the orbit of Earth:
  - for inner planets: this happens when the Earth is at aphelion and planet at perihelion
- The major axis of the orbit is

$$2a = a_E + a_V$$

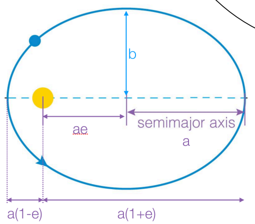
$$a = \frac{1+0.72}{2} = 0.86\mathrm{AU}$$

- The Sun is at one focus, so the distance at perihelion is a_V. From the properties of the ellipse, we derive eccentricity:

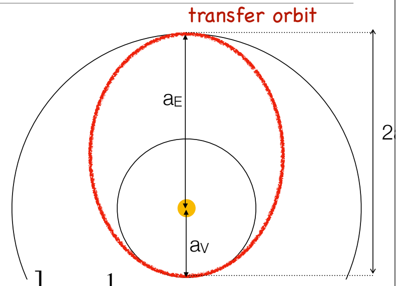
$$a(1-e) = a_V$$

$$e = \frac{a - a_V}{a} =$$

$$= \frac{0.86 - 0.72}{0.86} = 0.16$$



Example: minimum energy orbit to Venus



- At launch (when $r=a_E$), the velocity is

$$v^2_{\mathrm{orb}} = GM_\odot \left[\frac{2}{r} - \frac{1}{a}\right]$$

Recall: the probe is orbiting the Sun!

$$= GM_\odot \left[\frac{2}{a_E} - \frac{1}{(a_E + a_V)/2}\right]$$

$$= (6.67\times10^{-11})(2\times10^{30}\mathrm{kg})\left[\frac{2}{1\mathrm{AU}} - \frac{1}{(0.86\mathrm{AU})}\right]\frac{1}{1.5\times10^{11}\mathrm{m}}$$

$$v_{\mathrm{orb}} = 2.73\times10^4\mathrm{m/s} = 27.3\mathrm{km/s}$$

Because the orbital speed of the Earth is 30km/s, the probe must be going 2.7km/s slower than the Earth → one needs to launch the probe in the opposite direction of Earth's orbital motion!

$$\left(\frac{P}{1\mathrm{yr}}\right)^2 = \left(\frac{a}{1\mathrm{AU}}\right)^3$$

$$P = \left(\frac{0.86\mathrm{AU}}{1\mathrm{AU}}\right)^{3/2} = 0.8\mathrm{yr}$$

The trip to Venus takes P/2 = 0.4 yr, nearly 5 months.

Conceptual question

Sputnik I was launched into orbit around Earth in 1957. It had a perigee (the closest approach to Earth, measured from Earth's centre) of 6.81×10^6 m and an apogee (the furthest point from Earth's centre) of 7.53×10^6 m. What was its speed when it was at its perigee?

(a) 8230 m/s
(b) 11,000 m/s
(c) 13,400 m/s
(d) 7840 m/s
(e) 7180 m/s

The mass of Earth is 5.97×10^{24} kg and $G = 6.67 \times 10^{-11}$ N m²/kg².

Extra slides - read at home

Elliptical orbits

- Some of our derivations assumed circular orbits (orbital energy, derivation of Kepler's 3rd law).
- However, the same quantities can be derived by relaxing this assumption and generalising to elliptical orbits.
  - Read Ch. 5.4 (Kutner) for simple & beautiful derivations of elliptical orbits