MECHANIC

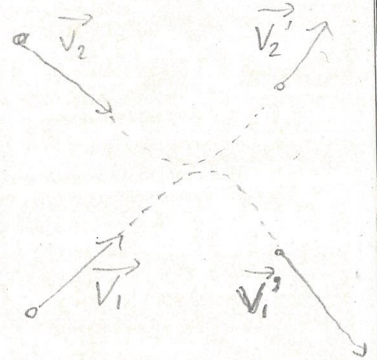
11

MA124

COLLISIONS

COLLISIONS CAN BE DEVIDED INTO 3 REGIMES:

1. INTERACTION Between objects is Negligible
   $\Rightarrow$ move in straight lines
2. OBJECTS interact $\Rightarrow$
   $\Rightarrow$ motion nolonger follows straight lines
3. INTERACTION IS AGAIN NEGLIGIBLE
   $\Rightarrow$ motion in straight lines



Which quantities are conserved during a collision?

- Total momentum is conserved, sind there are no external forces

$$\vec{P} = \vec{p_1} + \vec{p_2} = \text{constant}$$

- KINETIC ENERGY IS NOT conserved in general, since the interaction force may not be conservative $(K = \frac{1}{2}mv^2)$

$$K_{initial} = K_{Final} + Q$$

$$\begin{cases} Q < 0 & \text{inelastic collision} \\ Q = 0 & \text{elastic collision} \\ Q > 0 & \text{superelastic [internal Energy Converted]} \end{cases}$$

Unknowns for an elastic collision in N dimensions:

$$2N - (\underbrace{N}_{\text{cons. of Momentum}} + \underbrace{1}_{\text{cons. of Energy}}) = N - 1$$

- FOR 1 DIMENSION, cons of Momentum + cons. of Energy is enough to solve
- For 2+ Dimensions, more info (eg: angles) are needed to solve it

Collisions in 1 Dimension

Initial: $m_1 \xrightarrow{v_1}$ m_2

Final: $m_1 \xrightarrow{v_1'}$ $m_2 \xrightarrow{v_2'}$

$\boxed{\text{3 CASES:}}$ $\boxed{m_1 < m_2}$ $\boxed{m_1 = m_2}$ $\boxed{m_1 > m_2}$

THM: The relative initial speed $|V_2 - V_1|$ is equal to the final relative speed $|V_2' - V_1'|$ for an elastic collision in 1 dimension

PROOF: set $V_2 = 0$ and $V_1 = V$ w.l.o.g

Cons of Momentum: $m_1 V = m_1 V_1' + m_2 V_2'$ $(*)$

Cons of Energy: $\frac{1}{2} m_1 V^2 = \frac{1}{2} m_1 (V_1')^2 + \frac{1}{2} m_2 (V_2')^2$ $(**)$

$(*)$ Gives $\boxed{\frac{m_2}{m_1} = \frac{V - V_1'}{V_2'}}$

$(**)$ Gives $\boxed{V^2 = (V_1')^2 + \frac{m_2}{m_1}(V_2')^2}$

Now substitute to get

$$V^2 = (V_1')^2 + \left(\frac{V - V_1'}{V_2'}\right)(V_2')^2$$

$$\Rightarrow 0 = (V_1')^2 + V V_2' - V_1' V_2' - V^2$$

$$0 = (V_1' - V)(V_1' + V_2' + V_1')$$

WHICH GIVES 2 SOLUTIONS:

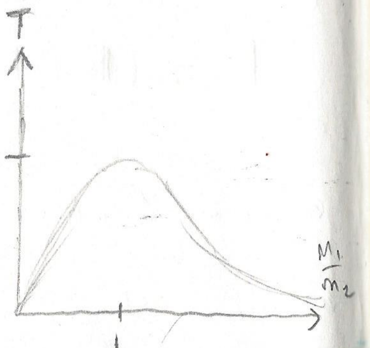
① $V = V_1'$ (INITIAL SOLUTION) ② $V = V_2' - V_1'$ (FINAL SOLUTION)

Solution for Final Velocities:

$$\boxed{V_1' = \frac{m_1 - m_2}{m_1 + m_2} V} \qquad \boxed{V_2' = \frac{2 m_1}{m_1 + m_2} V}$$

Energy Transfer Coefficient (T)

$$T = \left(\frac{K_{2f}}{K_{1i}}\right) = \frac{\frac{1}{2} m_2 (V_2')^2}{\frac{1}{2} m_1 V^2} = \frac{4 m_1 m_2}{(m_1 + m_2)^2} = \frac{4 m_1}{m_2} \cdot \frac{1}{(1 + \frac{m_1}{m_2})^2}$$



Inelastic Collisions

Example - two bodies attatch on collision

→ kinetic Energy is not conserved

what is $Q = K_{initial} - K_{final}$?

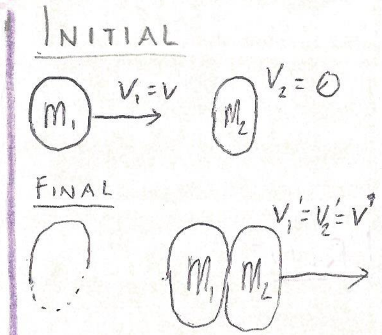
$$K_i = \frac{1}{2} m_1 V^2$$

$$K_f = \frac{1}{2}(m_1 + m_2)(V')^2$$

cons. of momentum: $m_1 V = (m_1 + m_2) V'$

$$\Rightarrow Q = \frac{1}{2} m_1 V^2 - \frac{1}{2} \frac{\cancel{(m_1 + m_2)}}{\cancel{(m_1+m_2)}} \left(\frac{m_1 \quad V}{(m_1 + m_2)}\right)^2$$

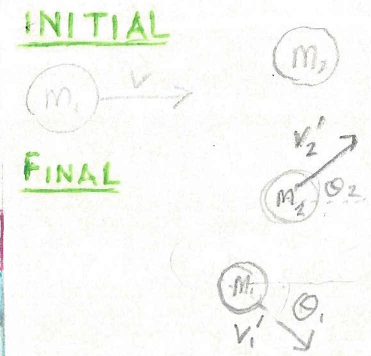
$$Q = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} V^2 > 0$$



Collisions in 2 Dimensions

θ_1, θ_2 are (laboratory) scattering angles

4 unknowns - 3 EQUATIONS
need one more
so we look at



Collisions in C.O.M coordinates

$$\vec{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i, \text{ where } M = \sum_{i=1}^{m} m_i$$

$$\Rightarrow \vec{F}_{ext} = M \ddot{\vec{R}} \quad \text{but } \vec{F}_{ext} = 0 \Rightarrow \ddot{\vec{R}} = 0$$

WE GET:

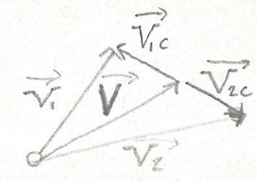
$$\vec{V} = \dot{\vec{R}} = \frac{m_1 \vec{V}_1 + m_2 \vec{V}_2}{m_1 + m_2}$$

AND FIND C.O.M VELOCITIES:

$$\vec{V}_{1c} = \vec{V}_1 - \vec{V} = \frac{m_2 (\vec{V}_1 - \vec{V}_2)}{m_1 + m_2} \rightarrow \vec{P}_{1c} = \frac{m_1 m_2 (\vec{V}_1 - \vec{V}_2)}{m_1 + m_2}$$

$$\vec{V}_{2c} = \vec{V}_2 - \vec{V} = -\frac{m_1 (\vec{V}_1 - \vec{V}_2)}{m_1 + m_2} \rightarrow \vec{P}_{2c} = -\frac{m_1 m_2 (\vec{V}_1 - \vec{V}_2)}{m_1 + m_2}$$

SO $\boxed{\vec{P}_{1c} = \mu \vec{V}_{rel} \quad \vec{P}_{2c} = -\mu \vec{V}_{rel}}$



We let $\frac{m_1 m_2}{m_1 + m_2} = \mu$ for simplicity

$$\vec{V}_1 - \vec{V}_2 = \vec{V}_{rel}$$

Energy & C.O.M. Coordinates

Proposition 1

The total kinetic energy K of a 2-particle system in a lab frame can be expressed as $K = K_c + \frac{1}{2}(m_1+m_2)V^2$

$$K = K_c + \frac{1}{2}(m_1+m_2)V^2$$

Proof: $K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

$$= \frac{1}{2}m_1(\vec{V_{1c}} + \vec{V})^2 + \frac{1}{2}m_2(\vec{V_{2c}} + \vec{V})^2$$

$$K = (m_1\vec{V_{1c}} + m_2\vec{V_{2c}})\cdot\vec{V} + \frac{1}{2}m_1V_{1c}^2 + \frac{1}{2}m_2V_{2c}^2 + \frac{1}{2}(m_1+m_2)V^2$$

But ($m_1\vec{V_{1c}} + m_2\vec{V_{2c}} = \vec{P_c} = 0$ in C.O.M coord system)

$$K = \frac{1}{2}m_1v_{1c}^2 + \frac{1}{2}m_2V_{2c}^2 + \frac{1}{2}(m_1+m_2)V^2 = \boxed{K_c + \frac{1}{2}(m_1+m_2)V^2}$$

Proposition 2

In the C.O.M coordinate system, K_c can be expressed as $K_c = \frac{1}{2}\mu V_{rel}^2$ where $\vec{V_{rel}} = V_{1c} - V_{2c} = V_1 - V_2$ $\quad \mu = m_1m_2/m_1+m_2$

Proof: $K_c = \frac{1}{2}m_1V_{1c}^2 + \frac{1}{2}m_2V_{2c}^2$ but $V_{1c} = \frac{m_2}{m_1+m_2}(V_1 - V_2)$

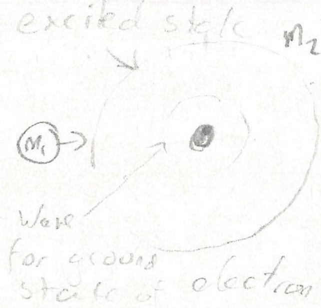
$V_{2c} = \frac{-m_1}{m_1+m_2}(V_1 - V_2)$

so $K_c = \frac{1}{2}\mu V_{rel}^2$

Inelastic Collision w/ an Atom

- Let $\mathcal{E}$ be the difference of energies of the ground state & excited state

Q: what is the minimal energy of the particle w/ mass m_1 s.t. the electron is brought into the excited state?



we need: $\boxed{\frac{1}{2}\mu V_{rel}^2 \geq \mathcal{E}}$

so $\frac{1}{2}\frac{m_1m_2}{m_1+m_2}v^2 \geq \mathcal{E} \Rightarrow \left(\frac{1}{2}m_1v^2\right)\frac{m_2}{m_1+m_2} \geq \mathcal{E}$

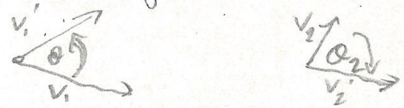
so $\frac{1}{2}m_1v^2 \geq (1 + \frac{m_1}{m_2})\mathcal{E}$

consider two cases:

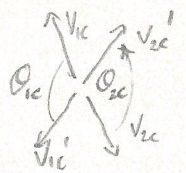
1. $m_1 \ll m_2$ (electron vs mercury)
   $\Rightarrow \frac{1}{2}m_1v^2 \gtrsim \mathcal{E}$
2. $m_1 \approx m_2$ (α particle against Helium)
   $\frac{1}{2}mv^2 \gtrsim 2\mathcal{E}$

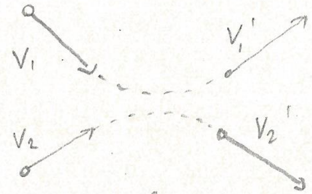
Collisions in C.O.M - Angles

- In lab frame, scattering angles θ_1 & θ_2 are in general different

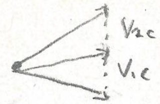


- In C.O.M. frame, both angles are the same.

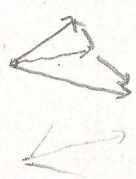




Initial C.O.M velocities



Final C.O.M velocities



Elastic Collisions under C.O.M

THM: For elastic collisions:

$$|V_{1c}| = |V_{1c}'| \qquad |V_{2c}| = |V_{2c}'|$$

Proof: collision is elastic

$$\frac{1}{2} m_1 v_{1c}^2 + \frac{1}{2} m_2 v_{2c}^2 = \frac{1}{2} m_1 v_{1c}'^2 + \frac{1}{2} m_2 v_{2c}'^2$$

$\emptyset$ momentum is conserved $P_c = P_c' = \emptyset$

$$\Rightarrow m_1 V_{1c} - m_2 V_{2c} = \emptyset \qquad \Rightarrow V_{2c} = \frac{m_1 V_{1c}}{m_2}$$

$$m_1 V_{1c}' - m_2 V_{2c}' = \emptyset \qquad \Rightarrow V_{2c}' = \frac{m_1 V_{1c}'}{m_2}$$

THUS:

$$\frac{1}{2} m_1 V_{1c}^2 + \frac{1}{2} \frac{m_1^2}{m_2} V_{1c} = \frac{1}{2} m_1 V_{1c}'^2 + \frac{1}{2} \frac{m_1^2}{m_2} V_{1c}^2$$

$$\Rightarrow \vec{V}_{1c}^2 = \vec{V}_{1c}'^2 \qquad \Rightarrow |\vec{V}_{1c}| = |\vec{V}_{1c}'| \text{ and } |V_{2c}| = |V_{2c}'|$$

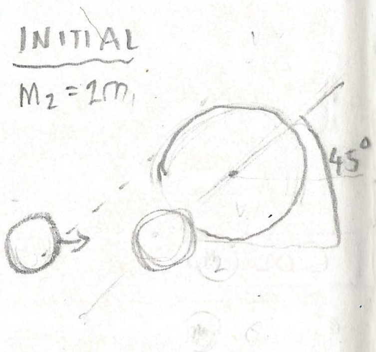
Skew Elastic Collision

Determine θ_2 and θ_1 given $2m_1 = m_2$

SOLUTION: $\vec{V} = \frac{1}{3}\vec{V}_1 \Rightarrow \vec{V}_{1c} = \frac{2}{3}\vec{V}_1$, $\vec{V}_{2c} = -\frac{1}{3}\vec{V}_1$

INITIAL

$M_2 = 2m_1$



Rigid Body Motion

Chasles' Theorem (±1830)

An arbitrary displacement of a rigid object can be represented as a translation of its center of mass $\vec{R}$, plus a rotation around $\vec{R}$

Proof: (2 masses connected by a rigid massless rod of length L)

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad d\vec{R} = \frac{m_1 d\vec{r}_1 + m_2 d\vec{r}_2}{m_1 + m_2}$$

$\emptyset$ WE NEED TO SHOW:

① $d\vec{r}_j' \perp \vec{r}_j'$ OR $\vec{r}_j' \cdot d\vec{r}_j' = \emptyset$ where $j \in \{1,2\}$

② $\theta_1 = \theta_2$ OR $\frac{d\vec{r}_1'}{r_1} = -\frac{d\vec{r}_2'}{r_2}$

FIRSTLY:

$$\vec{r}_1' = \vec{r}_1 - \vec{R} = \left(\frac{m_1 + m_2}{m_1 + m_2}\right)\vec{r}_1 - \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \frac{m_2(\vec{r}_1 - \vec{r}_2)}{m_1 + m_2}$$

$$\vec{r}_2' = \frac{m_1(\vec{r}_2 - \vec{r}_1)}{m_1 + m_2} = -\frac{m_1}{m_2}\vec{r}_1'$$

$$d\vec{r}_1' = \frac{m_2(d\vec{r}_1' - d\vec{r}_2')}{m_1 + m_2}$$

As WE GET

$$L^2 = (\vec{r}_1 - \vec{r}_2)^2$$

$$\Rightarrow (\vec{r}_1 - \vec{r}_2) \cdot (d\vec{r}_1 - d\vec{r}_2) = \emptyset$$

$$\Rightarrow \vec{r}_j' \cdot d\vec{r}_j' = \emptyset \text{ for } j \in \{1,2\}$$

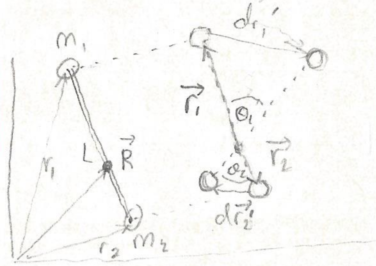
We then GET

$$\frac{dr_1'}{r_1'} = \frac{-\frac{m_2}{m_1} d\vec{r}_2'}{\frac{m_2}{m_1}\vec{r}_2'} = -\frac{dr_2'}{r_2}$$

Angular Momentum: $\vec{L} = \vec{r} \times \vec{p}$

Torque: $\vec{\tau} = \vec{r} \times \vec{F}$

2nd Law of Rotation: $\vec{\tau} = \frac{d\vec{L}}{dt}$



ANGULAR MOMENTUM, TORQUE & K.E IN DIFFERENT FRAMES w.r.t C.O.M

$$L_z = I_o\omega + (\vec{R} \times M\vec{V})_z$$

PROOF: $\vec{L} = \sum_{j=1}^{N} \vec{r}_j \times m_j \dot{\vec{r}}_j$

$$= \sum_{j=1}^{N} (\vec{R} + \vec{r}_j') \times m_j (\dot{\vec{R}} + \dot{\vec{r}}_j')$$

$$\vec{L} = \vec{R} \times \sum_{j=1}^{N} m_j \dot{\vec{R}} + \vec{R} \times \sum m_j \dot{\vec{r}}_j' + \sum m_j \vec{r}_j' \times \dot{\vec{R}} + \sum \vec{r}_j' \times m_j \dot{\vec{r}}_j'$$

TOTAL MOMENTUM

SUM OF MOMENTUMS W.R.T C.O.M IS ZERO

SUM gives ZERO FROM C.O.M $\left(\sum_{j=1}^{N} m_j \vec{r}_j' = \sum_{j=1}^{N} m_j (\vec{r}_j - \vec{R}) = 0\right)$

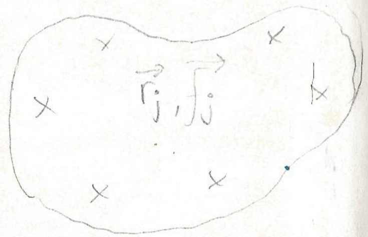
$I_o\omega$

$$L = I_o\omega + 0 + 0 + (\vec{R} \times M\vec{V})_z$$

$$L_z = I_o\omega + (\vec{R} \times M\vec{V})_z$$

TORQUE: $\tau_z = \tau_o + (\vec{R} \times \vec{F})_z$

$$\tau_z = \tau_o + (\vec{R} \times \vec{F})_z$$



PROOF: $\vec{\tau} = \sum \vec{r}_j \times \vec{f}_j = \sum (\vec{r}_j' + \vec{R}) \times \vec{f}_j$

$$= \sum_{j=1}^{N} \vec{r}_j' \times \vec{f}_j + \vec{R} \times \vec{F}$$

Now we just need to see $\tau_z = \frac{d}{dt} L_z$ $\quad \tau_o = \frac{d}{dt} I_o\omega$

$\tau_z = \tau_o + (\vec{R} \times \vec{F})_z$

KINETIC ENERGY: $K = \frac{1}{2} I_o\omega^2 + \frac{1}{2} MV^2$

PROOF: $K = \frac{1}{2} \sum_{j=1}^{n} m_j v_j^2 = \frac{1}{2} \sum_{j=1}^{n} m_j (v_j' + V_j)^2$

$$K = \frac{1}{2} \sum_{j=1}^{n} m_j v_j'^2 + \underbrace{\frac{1}{2} \sum_{j=1}^{n} m_j (2v_j \cdot V)}_{\text{AVG VELOCITY} = 0 \text{ IN C.O.M FRAME}} + \frac{1}{2} \sum_{j=1}^{n} m_j V^2$$

$$\Rightarrow K = \frac{1}{2}$$

Yo-Yo

Does the smaller or larger radius have a greater acceleration? $\boxed{r_1 < r_2}$ $\boxed{a_1 ? a_2}$

0 on the circumference: $\quad V_c = \omega r$

$a_c = \dot{\omega} r$

Equations of motion:

$$ma_c = mg - T$$

$$\left.\begin{aligned} L &= I_y\omega \\ \tau &= rT \end{aligned}\right\} \xrightarrow{\tau = \dot{L}} rT = I_y \dot{\omega}$$

$$\dot{\omega} = \frac{a_c}{r} \Rightarrow \quad rT = I_y \frac{a_c}{r} \Rightarrow T = I_y \frac{a_c}{r^2}$$

$$ma_c = mg - I_y \frac{a_c}{r^2}$$

$$\Rightarrow \frac{ma_c}{m} + \frac{I_y}{r^2}\frac{a_c}{m} = \frac{mg}{m}$$

$$\Rightarrow \boxed{a_c = \frac{g}{1 + \frac{I_y}{mr^2}} \overset{I = \frac{mR^2}{2}}{\approx} \frac{g}{1 + \frac{R^2}{2r^2}}}$$

$\Rightarrow$ for $r_1 < r_2$ we get $a_1 < a_2$ (needs to spin for less)

ENERGY METHOD

$$K = \frac{1}{2}mv^2 + \frac{1}{2} I_y \omega^2 \qquad U = mgh$$

we know:

$$E = K + U = \text{constant}$$

$$\Rightarrow \dot{E} = 0 = \frac{1}{2} m (2v)(\dot{v}) + \frac{1}{2} I_y (2\omega)\dot{\omega} - mg\dot{h}$$

$$0 = mv_c a_c + I_y \frac{v_c}{r} \frac{a_c}{r} - mgv_c$$

$$\Rightarrow a_c \left(mv_c + \frac{I_y v_c}{r^2}\right) = mgv_c$$

$$\Rightarrow \boxed{a_c = \frac{g}{1 + \frac{I_y}{mr^2}}}$$

"TRANSLATIONS ARE commutative"

ROTATION: Can we parameterise the orientation of an object?

as $\vec\Theta = \Theta_x\hat i + \Theta_y\hat j + \Theta_z\hat k$?

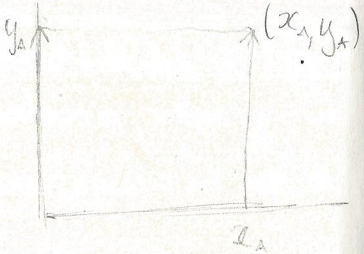
ANS: NO eg: $\Theta = 90^\circ\hat j + 90^\circ\hat k$

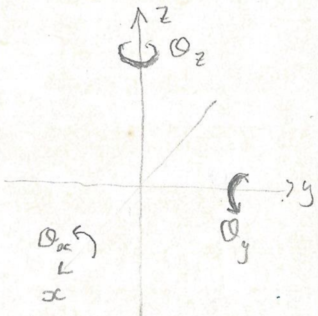
Ambeguity since rotation is not commutative

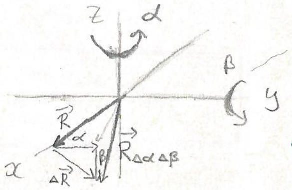
(However the non-commulivity is 2nd order in angles)

We want an angular velocity $\vec\omega$

$$\vec\omega = \frac{d\Theta_x}{dt}\hat i + \frac{d\Theta_y}{dt}\hat j + \frac{d\Theta_z}{dt}\hat k$$







Example of Non-commutivity/Ambiguity

We well look at α then β rotation, and vice versa

|α→β| ① $\vec r_\alpha = r\cos\alpha\,\hat i + r\sin\alpha\,\hat j$

② $\vec r_{\alpha\beta} = r\cos\alpha[\cos\beta\,\hat i - \sin\beta\,\hat k] + r\sin\alpha\,\hat j$

first order taylor expansion $\approx r\hat i - r\beta\hat k + r\alpha\hat j + O(\alpha^2,\beta^2)$

|β→α| ① $\vec r_\beta = r\cos\beta\,\hat i - r\sin\beta\,\hat k$

② $\vec r_{\beta\alpha} = r\cos\beta\,[\cos\alpha\,\hat j + \sin\alpha\,\hat j] - r\sin\beta\,\hat k$

1st Order Taylor Exp. $\approx r\hat i + r\alpha\hat j - r\beta\hat k + O(\alpha^2,\beta^2)$

CONCLUSION: $\vec\omega$ is an ambiguous quantity as we can write $\vec\omega = \frac{d}{dt}(\Theta_x\hat i + \Theta_y\hat j + \Theta_z\hat k)$, and for small angles Δα and Δβ: $\Delta\Theta = \Delta\beta\hat j + \Delta\alpha\hat k$

$$\vec\omega = \lim_{\Delta t\to 0}\frac{\Delta\vec\Theta}{\Delta t} = \frac{d\beta}{dt}\hat j + \frac{d\alpha}{dt}\hat k$$

$$\Delta\vec R = \vec R_{\alpha\beta} - \vec R$$

$$\vec v = \lim_{\Delta t\to 0}\frac{\vec R_{\alpha\beta}-\vec R}{\Delta t} = \lim_{\Delta t\to 0}\frac{R\alpha\hat j - R\beta\hat k}{\Delta t} = R\frac{d\alpha}{dt}\hat j - R\frac{d\beta}{dt}\hat k$$

NOTE: SAME RESULT FOR $\vec v$ IF $\Delta\vec R = \vec R_{\beta\alpha} - \vec R$

- not ok to have angles in general
- IS OK IF Θ is small
- This means we can write ω in terms of the angles parameterised

Thus we can get the velocity is:

$$\vec V = \begin{vmatrix}\hat i & \hat j & \hat k\\ 0 & \frac{d\beta}{dt} & \frac{d\alpha}{dt}\\ R & 0 & 0\end{vmatrix} = 0\hat i - (-R\frac{d\alpha}{dt})\hat j + (-R\frac{d\beta}{dt})\hat k$$

i.e: $\vec V = \vec\omega\times R$

$$|\vec\omega\times\vec R| = \omega R\sin\varphi = \omega\rho$$

But we know $\frac{dR}{dt} = \lim_{\Delta t\to 0}\rho\frac{d\Theta}{dt} = \rho\omega$

$$\Rightarrow \frac{dR}{dt} = \vec\omega\times R$$

DETERMINE $\vec L$ w.r.t. the center

① $\vec L = \vec R_1\times\vec p_1 + \vec R_2\times\vec p_2$

$|\vec R_1| = L$

$|\vec p_1| = mL\sin(\frac{\pi}{2}-\alpha)\omega = mL\cos(\alpha)\omega$ $\Big\}$ $|\vec L| = 2mL^2\omega\cos\alpha$

(N.B: $\vec\omega$ and $\vec L$ are not parallel)

② $\vec L = 2m\vec R_1\times\vec V_1 = 2mR_1\times(\vec\omega\times\vec R_1)$

$|\vec L| = 2m\omega L^2\cos\alpha$

Ex 8.5 Analytic determination of τ

components of $\vec L$:

$L_x = L\sin\alpha\cos\omega t$

$L_y = L\sin\alpha\sin\omega t$

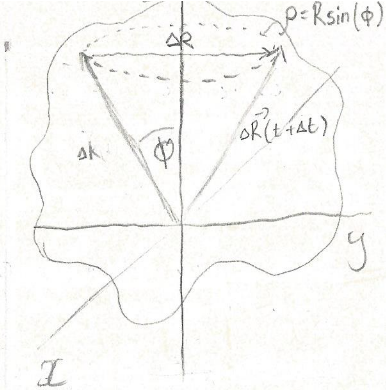
$L_z = L\cos\alpha$

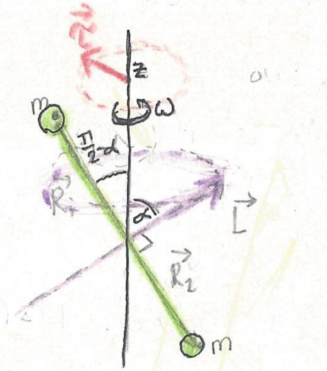
$\tau = \frac{dL}{dt}$

$$\tau = \frac{d}{dt}(L_x\hat i + L_y\hat j + L_z\hat k)$$

$$= \omega(-L\sin\alpha\sin\omega t\,\hat i + L\sin\alpha\cos\omega t\,\hat j)$$

$|\vec\tau| = \omega L\sin\alpha$ (as $\sqrt{\sin^2\omega t+\cos^2\omega t} = 1$)





$L = \vec r\times\vec p = 2m\vec r\times\vec v = 2m\vec r\times(\vec\omega\times\vec r)$

$\tau = \frac{d\vec L}{dt}$ (torque)

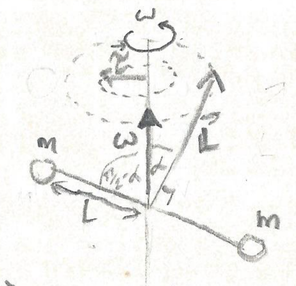
$L = \vec r\times\vec p = 2m\vec r\times(\vec\omega\times\vec r)$

$\tau = \vec F\times\vec r = \dot{\vec L} = \omega L\sin\alpha$

For which α do we get $\vec{\tau} = 0$?

⊗ $\alpha = 0, \pi, 2\pi, \dots \Rightarrow \vec{\omega}$ is parallel to $\vec{L}$

⊗ $\alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \dots \Rightarrow$ then $\vec{L} = 0$



Statically Balanced

C.o.M. is on axis of rotation

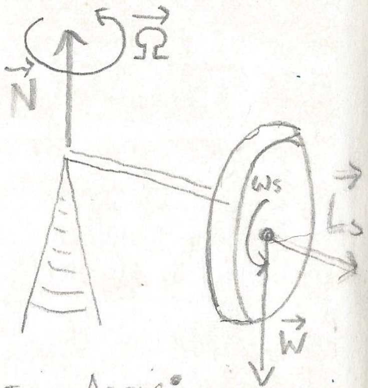
Dynamically Balanced

~~C.o.M.~~ $\vec{\omega}$ and $\vec{L}$ are parallel

$|\vec{L}| = 2m\omega l^2 \cos\alpha$

$|\vec{\tau}| = \omega L \sin\alpha$

Gyroscopic Motion

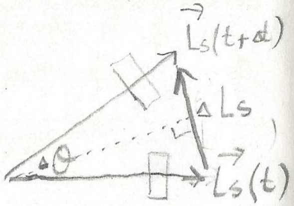


- By looking from above, we can see that: $|\Delta \vec{L}_s| = 2\sin\left(\frac{\Delta\theta}{2}\right) L_s$

$$\left|\frac{d\vec{L}}{dt}\right| = \lim_{\Delta t \to 0} \frac{|\Delta L_s|}{\Delta t} = \frac{2\left(\frac{\Delta\theta}{2}\right) L_s}{\Delta t} = \Omega L_s$$

$\sin\theta \approx \theta$

From Above:



$\vec{L}_s$ changes $\Rightarrow$ there must be a torque

$|\vec{\tau}| = lW = \left|\frac{dL_s}{dt}\right| = \Omega L_s = \Omega I_0 \omega_s$

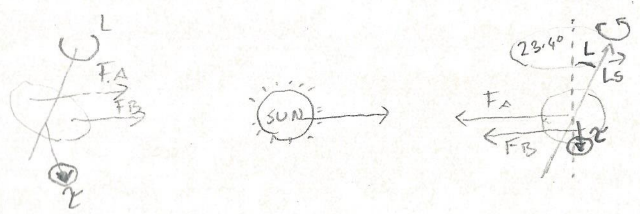
~~but~~ $\left(I_0 \approx \frac{1}{2}MR^2\right)$

$\Rightarrow$ ~~LW = ΩL~~ $\Omega = lW / I_0 \omega_s$

Valid only when $\omega \gg \Omega$

τ is into the page

Precession of the Equinoxes/Solstices



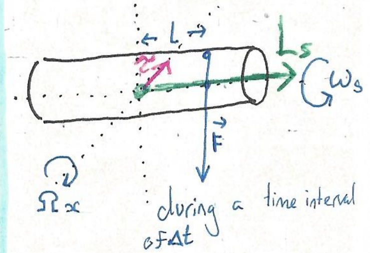
Period of procession 26000 years

Stability of Rotating Objects + Cons Ang Momentum

- What happens when you apply a force on a rotating body/cylinder a length L from it's C.o.M?

$\boxed{1)\ \omega_s = 0}$

$L = \tau \Delta t = F l \Delta t$, $L = I_x \Omega_x$ $\Big\}$ $\Omega_x = \frac{F l \Delta t}{I_x}$ After the force F is applied



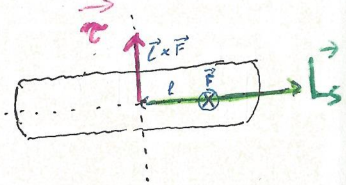
$\boxed{2)\ \omega_s \gg 0}$

$\tau = R \times F$ shown on the diagram

We get a precession of $\vec{L}$, due to the torque, while the force acts

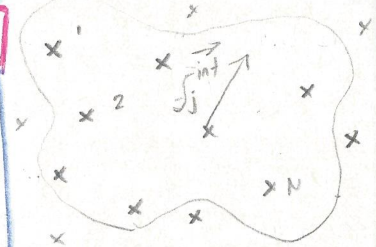
$$\left|\frac{dL_s}{dx}\right| = \Omega_z L_s \Rightarrow \Omega_z = \frac{Fl}{L_s}$$

precession stops ~~on~~ when F stops



Conservation of Angular Momentum

- Suppose N particles have momentum $\vec{p}_j$, position $\vec{r}_j$ where we have $j = 1, 2, \dots N$
- Let us assume $\vec{f}_{j\,ext} = 0 \Rightarrow \vec{P} = \sum_{j=1}^{N} \vec{p}_j$ is conserved
- Then consider torque: $\vec{\tau}_j = \vec{r}_j \times \vec{f}_j^{\,int}$ on particle j



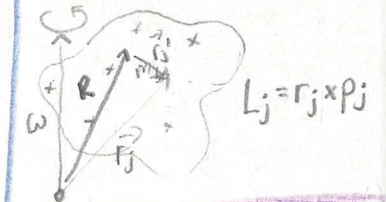
- Let $\vec{\tau}_{jk} = \vec{r}_j \times \vec{f}_{jk}$ be the torque on j due to k

$\vec{\tau}_{jk} + \vec{\tau}_{kj} = (\vec{r}_j - \vec{r}_k) \times \vec{f}_{jk}$

THEN: $\vec{\tau}_{jk} + \vec{\tau}_{kj} = \vec{r}_j \times \vec{f}_{jk} + \vec{r}_k \times \vec{f}_{kj}$ ~~= ...~~ but $\vec{f}_{kj} = -\vec{f}_{jk}$

$= \vec{r}_j \times \vec{f}_{jk} - \vec{r}_k \times \vec{f}_{jk} = (\vec{r}_j - \vec{r}_k) \times (\vec{f}_{jk})$

$\Rightarrow$ vanishes if $\vec{r}_{jk}$ is $\parallel$ to $\vec{f}_{jk}$, but not in general



$L_j = r_j \times p_j$

For Momentum: $\vec{p} = m\vec{v}$: $\vec{p}$ & $\vec{v}$ are parallel

For Angular Momentum: $\vec{L}$ and $\vec{\omega}$ not parallel in general

$$\vec{L} = \sum_{j=1}^{N} r_j \times m_j \dot{\vec{r}}_j = \sum_{j=1}^{N} (\vec{R} + \vec{r}_j{}') \times m_j (\vec{V} + \dot{\vec{r}}_j{}') \quad \text{in C.o.M.}$$

$$= \vec{R} \times M\vec{V} + \vec{R} \times \sum m_j \dot{\vec{r}}_j{}' + \sum m_j \vec{r}_j{}' \times \vec{V} + \sum \vec{r}_j{}' \times m_j \dot{\vec{r}}_j{}'$$

$$\omega \times r = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z\omega_y - y\omega_z \\ x\omega_z - z\omega_x \\ y\omega_x - x\omega_y \end{pmatrix}$$

$$r \times (\omega \times r) = \begin{pmatrix} y^2\omega_x - yx\omega_y - xz\omega_z + z^2\omega_x \\ x^2\omega_y - xy\omega_x - yz\omega_z + z^2\omega_y \\ x^2\omega_z - xz\omega_x - yz\omega_y + y^2\omega_z \end{pmatrix}$$

so we get $\boxed{L_0 = \sum_{j=1}^{N} \vec{r}_j{}' \times m_j \dot{\vec{r}}_j{}' = \sum_{j=1}^{N} \vec{r}_j{}' \times m_j (\vec{\omega} \times \vec{r}_j{}')}$

- multiplying out we get the result on the right
- We define moments of inertia about an axis on the right

$\vec{L}_{0x} = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

$\Rightarrow \vec{L}_{0y} = I_{yy}\omega_y + I_{xy}\omega_x + I_{zy}\omega_z \rightarrow$ Tensor of Inertia

$\vec{L}_{0z} = I_{zz}\omega_z + I_{xz}\omega_x + I_{zy}\omega_y$

Moments of Inertia

$I_{xx} = \sum_{j=1}^{N} m_j (y_j^2 + z_j^2) \dots$

Products of Inertia

$I_{xy} = -\sum_{j=1}^{N} m_j x_j y_j = I_{yx} \dots$

Conservation of Angular Momentum - Tensor of Inertia

$$L_{ox} = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$
$$L_{oy} = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_{oz} = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

$$\tilde{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$ TENSOR OF INERTIA $= \tilde{I}^T$

$\Rightarrow \boxed{\vec{L}_o = \tilde{I}\vec{\omega}}$ $\tilde{I}$ is "the rotational equivalent of mass"

WE DEFINED:

$I_{xx} = \sum m_j(\cancel{x_j^2}\, y_j^2 + z_j^2)$

$I_{yy} = \sum m_j(x_j^2 + z_j^2)$

$I_{zz} = \sum m_j(x_j^2 + y_j^2)$

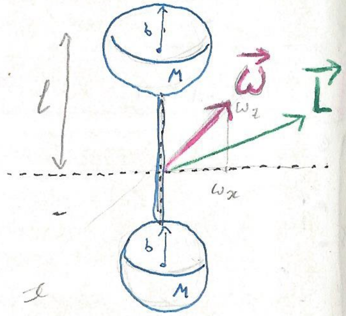
$I_{xy} = I_{yx} = -\sum m_j x_j y_j$

$I_{yz} = I_{zy} = -\sum m_j y_j z_j$

$I_{xz} = I_{zx} = -\sum m_j x_j z_j$

Ex 8.13 - Rotating Dumb Bell

- sphere has radius b and mass M
- $\vec{\omega} = \omega_y \hat{j} + \omega_z \hat{k}$
- Determine the angular momentum $\vec{L}$



$I_{xy} = -\sum_{j=1}^{N} m_j x_j y_j = 0$

o since for any element located at (x_j, y_j), there is an element at $(-x_j, y_j) = 0$

o similarly, $I_{xz} = I_{yz} = 0$

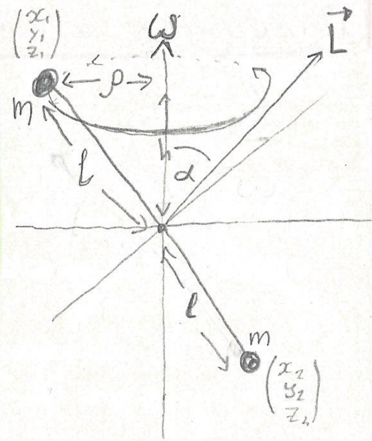
□ I_{xx} NOT NEEDED since $\omega_x = 0$ (but $= I_{yy}$)

□ $I_{yy} = 2\left[ML^2 + \frac{2}{5}mb^2\right]$

□ $I_{zz} = 2\left[\frac{2}{5}mb^2\right]$ depends only on x, y

$$\Rightarrow \vec{L} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ I_{yy}\omega_y \\ I_{zz}\omega_z \end{pmatrix}$$

- Now suppose the dumbell is tilkd w.r.t the axes, and rederive $\vec{L}$ using $\tilde{I}$. $\rho = l\cos\alpha$ $h = l\sin\alpha$



POSITION OF MASS 1: $\begin{pmatrix} -\rho\cos\omega t \\ -\rho\sin\omega t \\ h \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

POSITION OF MASS 2: $(-x_1, -y_1, -z_1)$

$$\Rightarrow I_{xx} = m(y_1^2 + z_1^2) + m(\underbrace{y_2^2}_{=y_1^2} + \underbrace{z_2^2}_{=z_1^2}) = 2m(\rho^2\sin^2\omega t + h^2)$$

$$I_{yx} = I_{xy} = -2m x_1 y_1 = 2m\rho^2 \sin(\omega t)\cos(\omega t)$$

$$I_{zy} = I_{yz} = -2m y_1 z_1 = 2m\rho h\sin(\omega t)$$

$$I_{xz} = I_{zx} = -2m x_1 z_1 = +2m\rho h\cos(\omega t)$$

$$I_{yy} = 2m(x_1^2 + y_1^2) = 2m(\rho^2\cos^2\omega t + h^2)$$

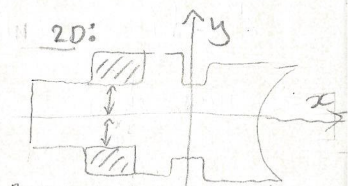
$$I_{zz} = 2m\rho^2$$

$$\vec{L} = 2m\begin{pmatrix} \rho^2\sin^2\omega t + h^2 & \rho^2\sin\omega t\cos\omega t & \rho h\cos\omega t \\ \rho^2\sin\omega t\cos\omega t & \rho^2\cos^2\omega t + h^2 & \rho h\sin\omega t \\ \rho h\cos\omega t & \rho h\sin\omega t & \rho^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \begin{pmatrix} 2m\rho h\omega\cos\omega t \\ 2m\rho h\omega\sin\omega t \\ 2m\rho^2\omega \end{pmatrix}$$

- This shows us that under different coordinate systems, $\tilde{I}$ can be greatly simplified

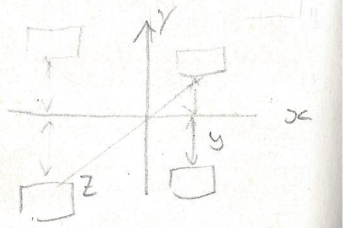
IN 2D:

$$I_{xy} = -\sum_{j=1}^{N} m_j x_j y_j = -\sum_{j, y<0} m_j x_j y_j - \sum_{j, y>0} m_j x_j y_j = 0$$



REFLECTION SYMMETRY IN THE x-AXIS

IN 3D: THE REFLECTION SYMMETRY IN (x, z) plane WILL MEAN: $I_{xy} = 0 = I_{yx}$ AND $I_{yz} = 0 = I_{zy}$



- IF all of the PRODUCTS OF INERTIA VANSSH, $\tilde{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$, the $\hat{i}, \hat{j}, \hat{k}$ are Eigenvectors of $\tilde{I}$

$$L = \tilde{I} \cdot \vec{\omega}$$

Diagonalisation of the Tensor of Inertia

$\vec{L} = \tilde{I}\,\vec{\omega}$

$I\hat{i} = \omega_{xx}\,\hat{i}$
$I\hat{j} = \omega_{yy}\,\hat{j}$
$I\hat{k} = \omega_{zz}\,\hat{k}$

- IF $\vec{\omega}$ is parallel to $\hat{i}, \hat{j}$, OR $\hat{k}$, THEN $\vec{L}$ is parallel to $\vec{\omega}$
- $\tilde{I}$ can be diagonalised using techniques we learn from LINEAR ALGEBRA
- SINCE $\tilde{I} = \tilde{I}^T$, it can be diagonalised using an orthogonal matrix ($A^T = A^{-1}$)
- Eigenvectors of $\tilde{I}$ are the principle axes of the object $\Rightarrow$ SYLVESTER
- $\tilde{I}' = A^{-1} I A$

WITH A IS ORTHOGONAL ($A^T = A^{-1}$)

$$\tilde{I}' = \begin{pmatrix} I_{xx}' & 0 & 0 \\ 0 & I_{yy}' & 0 \\ 0 & 0 & I_{zz}' \end{pmatrix}$$

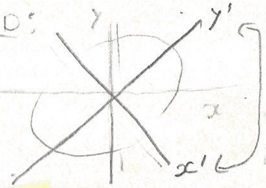
- New basis vectors $\hat{i}', \hat{j}', \hat{k}'$ are given in terms of $\hat{i}, \hat{j}$ and $\hat{k}$ by

$$\boxed{\hat{i}' = A\hat{i}, \quad \hat{j}' = A\hat{j}, \quad \hat{k}' = A\hat{k}}$$

→ SINCE A is orthogonal, $(\hat{i}, \hat{j}, \hat{k})$ is orthogonal
→ $\hat{i}', \hat{j}', \hat{k}'$ span the principle axes of the object

- $\hat{i}', \hat{j}', \hat{k}'$ are the EIGENVECTORS OF $\tilde{I}'$ (and $\tilde{I}$)

IN 2D:



PRINCIPAL AXES OF THE OBJECT

$A = \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}$

Reflection Symetry

$\tilde{I}' = \begin{pmatrix} I_{xx}' & 0 \\ 0 & I_{yy}' \end{pmatrix}$

2D OBJECT

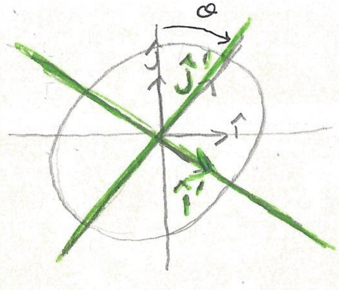
PRINCIPAL AXES found in Green (reflection Symmetry)

$\Rightarrow I_{xy}' = 0$ due to symmetry in y'

$\Rightarrow \tilde{I}' = \begin{pmatrix} I_{xx}' & 0 \\ 0 & I_{yy}' \end{pmatrix}$ diagonal

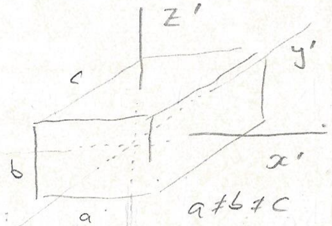
$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\tilde{I}' = A^{-1}\tilde{I}A \qquad \tilde{I} = A\tilde{I}'A^{-1}$



3D OBJECTS

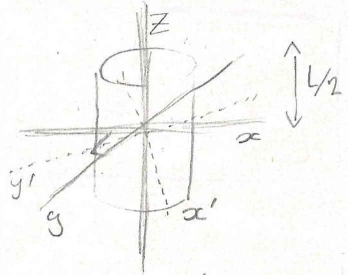
- FOR A CUBOID OF $x \neq y \neq z$, WE must have the axes THROUGH THE CENTER, PERPENDICULAR TO THE FACES



- FOR A CYLENDER: THERE IS MORE SYMMETRY

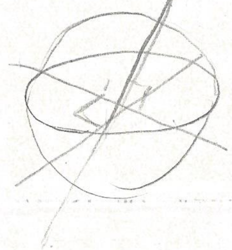
BOTH (x, y, z) & (x', y', z') ARE PRINCIPAL AXES

$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Between them



- FOR A SHERE

ANY SET OF ORTHOGONAL AXES THROUGH THE CENTER ~~ARE A~~ PRINCIPAL AXES FOR THE SPHERE



Rotational Kinetic Energy

FOR FIXED AXIS ROTATION: $K = \frac{1}{2} I \omega^2$

THIS GENERALISES: $K = \frac{1}{2} I \vec{\omega} \cdot \vec{\omega} = \frac{1}{2} \vec{L} \cdot \vec{\omega}$

→ corresponds to $\frac{1}{2} m v^2 = \frac{1}{2} \vec{p} \cdot \vec{v}$

$$\boxed{K = \frac{1}{2} I \vec{\omega} \cdot \vec{\omega} = \frac{1}{2} \vec{L} \cdot \vec{\omega}}$$

Rotational Kinetic Energy

$$K = \frac{1}{2} m v^2 + \frac{1}{2}\omega^T \tilde{I}\omega$$
$\frac{1}{2}(\tilde{I}\omega)\cdot\omega$

PROOF:

$\vec{r}_j{}' = \vec{R} - \vec{r}_j$

$\vec{v}_j{}' = \vec{V} - \vec{v}_j$

$$K = \sum_{j=1}^{N} \frac{1}{2} m v_j^2 + \sum \frac{1}{2} m_j (\vec{V} + \vec{v}_j{}')^2$$

Avg Velocity in C.o.M. 0

$$K = \frac{1}{2} M V^2 + \cancel{\sum m_j \vec{V}\cdot\vec{v}_j} + \boxed{\frac{1}{2}\sum_{j=1}^{N} m_j (v_j{}')^2} K_{ROT}$$

$$K_{ROT} = \frac{1}{2}\sum_{j=1}^{N} m_j (\vec{\omega}\times\vec{r}_j{}')\cdot(\vec{\omega}\times\vec{r}_j{}')$$

IDENTITY: $(\vec{A}\times\vec{B})\cdot\vec{C} = \vec{A}\cdot(\vec{B}\times\vec{C})$ (*)

AS $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \cdot \vec{C} = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

- letting $A = \vec{\omega}$ $B = \vec{r}_j{}'$ $C = \vec{\omega}\times\vec{r}_j{}'$

$$K_{ROT} = \frac{1}{2}\sum_{j=1} m_j \vec{\omega}\cdot[\vec{r}_j{}'\times(\vec{\omega}\times\vec{r}_j{}')],$$ from before

- but we know $L = \sum m_j \vec{r}_j{}'\times(\vec{\omega}\times\vec{r}_j{}')$

$$K_{ROT} = \frac{1}{2}\vec{\omega}\cdot\vec{L}_0 = \frac{1}{2}\vec{\omega}\cdot\tilde{I}\omega$$

ANGULAR MOMENTUM IN C.o.M COORD SYSTEM

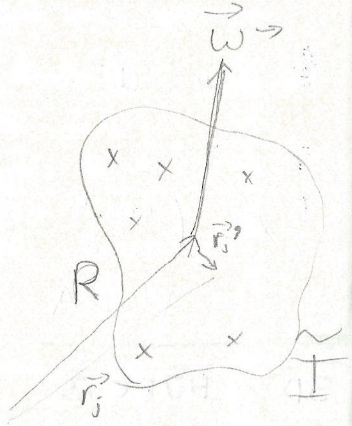
$$L = \sum_{j=1}^{N} m_j \vec{r}_j\times\vec{r}_j{}' = \sum_{j=1}^{N} m_j \vec{r}_j\times(\vec{\omega}\times\vec{r}_j) = \tilde{I}\vec{\omega}$$

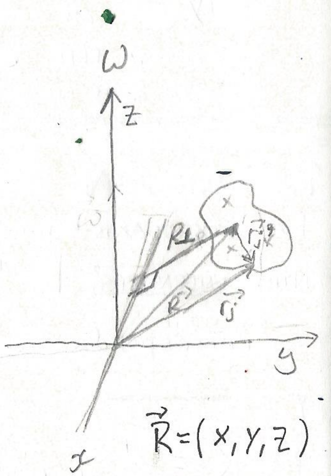
- $\tilde{I}$ is expressed in terms of C.o.M. coordinates

$I_{xx} = \sum m_j (y_j^2 + z_j^2) = \sum m_j \rho_j^2$ ($\vec{\rho}_j$ is $\perp$ to x Axis) $= (0, y_j, z_j)$

$I_{xx} = \sum m_j (\vec{R}_\perp + \vec{\rho}_j{}')$
$= M(Y^2 + Z^2) + I_{0xx}$ (other part avges 0)

$I_{xy} = -\sum m_j (x_j y_j) = -\sum m_j (X + x_j')(Y + y_j') = -MXY - I_{0xy}$





$\vec{R} = (X, Y, Z)$

$\vec{R}_\perp = (0, Y, Z)$

Rigid Body Dynamics

$\vec{\tau} = \frac{d\vec{L}}{dt}$ $\vec{L} = \tilde{I}\vec{\omega}$ | $\vec{F} = \frac{d\vec{p}}{dt}$ $\vec{p} = m\vec{v}$

- Suppose we have a disk spinning with a slight wobble

$\vec{L}_{SPINNING}$ = shown

we determine L_x

1. $\frac{d\theta_x}{dt} \neq 0 \Rightarrow I_{xx}\frac{d\theta_x}{dt}$
2. due to tilt of $\vec{L}_s$: $L_s \sin(\theta_y)$

So $\boxed{L_x = I_{xx}\frac{d\theta_x}{dt} + L_s\sin(\theta_y)}$

SIMILARLY: $\boxed{L_y = I_{yy}\frac{d\theta_y}{dt} - L_s\sin(\theta_x)}$

- small angle approximation (θ_x, θ_y)

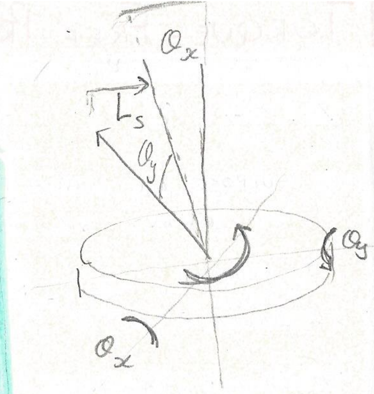
$I_{xx} = I_{yy} = I_\perp$

$L_x = I_\perp\dot{\theta}_x + L_s\theta_y$

$L_y = I_\perp\dot{\theta}_y + L_s\theta_x$ $\tau = 0$

$L_z = L_s = I_s\omega_s$

$\Rightarrow \begin{vmatrix} I_\perp\ddot{\theta}_x + L_s\dot{\theta}_y = 0 \\ I_\perp\ddot{\theta}_y + L_s\dot{\theta}_x = 0 \end{vmatrix}$ $\gamma = \frac{I}{I_\perp}\omega_s$



TORQUE-FREE PRECESSION

$\emptyset = \frac{d\vec{L}}{dt} = \vec{\tau}$

- SUPPOSE WE HAVE A ROTATING DISK W/ NO TORQUE, S.T. WE GET A WOBBLE

IN PRINCIPAL AXES COORDINATES: $\tilde{I}_p = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_s \end{pmatrix}$

ROTATION MATRICES:

$R_{\theta_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix}$ $R_{\theta_y} = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$

USING SMALL ANGLE APPROXIMATION:

$$R_{\theta_x\theta_y} = R_{\theta_x} R_{\theta_y} = \begin{pmatrix} 1 & 0 & \theta_y \\ 0 & 1 & -\theta_x \\ -\theta_y & \theta_x & 1 \end{pmatrix}$$

$$R_{\theta_x\theta_y}^{-1} = \begin{pmatrix} 1 & 0 & -\theta_y \\ 0 & 1 & \theta_x \\ \theta_y & -\theta_x & 1 \end{pmatrix}$$

$\Rightarrow \vec{\omega}_s = R_{\theta_x\theta_y}\begin{pmatrix} 0 \\ 0 \\ \omega_s \end{pmatrix}$ = angular velocity due to spinning

TENSOR OF INERTIA:

$\tilde{I} = R_{\theta_x\theta_y} \tilde{I}_p R_{\theta_x\theta_y}^{-1}$ $\quad \vec{L}_0 = \tilde{I}\vec{\omega}$

$$\vec{\omega} = \underbrace{R_{\theta_x\theta_y}\begin{pmatrix} 0 \\ 0 \\ \omega_s \end{pmatrix}}_{\text{SPIN}} + \underbrace{\begin{pmatrix} d\theta_x/dt \\ d\theta_y/dt \\ 0 \end{pmatrix}}_{\text{WOBBLE}}$$

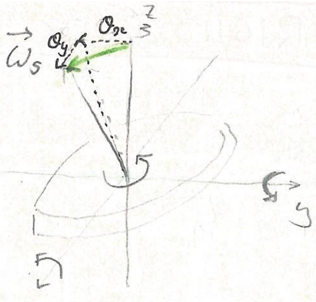
USING THIS INFORMATION, WE CAN GET:

$$\vec{L}_0 = \tilde{I}\vec{\omega} = R_{\theta_x\theta_y}\tilde{I}_p\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} + R_{\theta_x\theta_y}\tilde{I}R_{\theta_x\theta_y}^{-1}\begin{pmatrix} \dot\theta_x \\ \dot\theta_y \\ 0 \end{pmatrix}$$

WHICH CAN BE USED TO OBTAIN

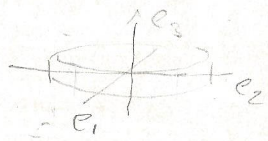
$L = \begin{pmatrix} \theta_y L_s + I_\perp\dot\theta_x \\ -\theta_x L_s + I_\perp\dot\theta_y \\ L_s\omega_s \end{pmatrix}$, but $\vec{\tau} = \frac{d\vec{L}}{dt} = 0$

$$\Rightarrow \begin{array}{l} I_\perp\ddot\theta_x + L_s\dot\theta_y = 0 \\ I_\perp\ddot\theta_y - L_s\dot\theta_x = 0 \end{array}$$



PRINCIPAL AXES:

$\tilde{I}_p = \begin{pmatrix} I_\perp & 0 & 0 \\ 0 & I_\perp & 0 \\ 0 & 0 & I_s \end{pmatrix}$



$I_\perp\ddot\theta_x + L_s\dot\theta_y = 0 \quad (*)$

$I_\perp\ddot\theta_y - L_s\dot\theta_x = 0$

But as $\begin{pmatrix} \theta_y L_s + I_\perp\dot\theta_x \\ -\theta_x L_s + I_\perp\dot\theta_y \\ L_s\omega_s \end{pmatrix} = \begin{pmatrix} L_s \\ L_y \\ L_z \end{pmatrix}$

$$\frac{I_\perp\dot\theta_y}{I_\perp} = \frac{L_y}{I_\perp} + \frac{\theta_x L_s}{I_\perp}$$

$$\boxed{\dot\theta_y = \frac{L_y}{I_\perp} + \theta_x\frac{L_s}{I_\perp}}$$

$$(*)\quad I_\perp\ddot\theta_x + L_s\left(\frac{L_y}{I_\perp} + \theta_x\frac{L_s}{I_\perp}\right) = 0$$

$$I_\perp\ddot\theta_x + \frac{L_s^2}{I_\perp}\left(\theta_x + \frac{L_y}{L_s}\right) = 0$$

$$I_\perp\frac{d^2}{dt^2}\left(\theta_x + \frac{L_y}{L_s}\right) + \frac{L_s^2}{I_\perp}\left(\theta_x + \frac{L_y}{L_s}\right) = 0$$

$$\frac{d^2}{dt^2}\left(\theta_x + \frac{L_y}{L_s}\right) + \frac{L_s^2}{I_\perp^2}\left(\theta_x + \frac{L_y}{L_s}\right) = 0$$

$\Rightarrow$ EQUATIONS FOR SIMPLE HARMONIC MOTION

GENERAL SOLUTION: $\boxed{\theta_x(t) = A\cos(\gamma t + \phi) - \frac{L_y}{L_s}}$

(where γ = angular speed for "wobbling")

$$\gamma = \frac{L_s}{I_\perp} = \frac{I_s}{I_\perp}\omega_s \approx 2\omega_s$$

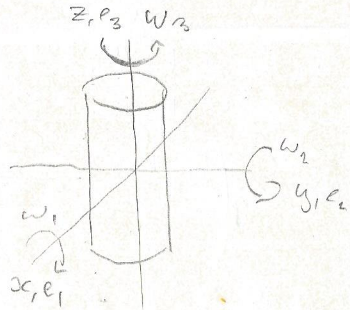
Nutation: phenomenon that the spinning axis precesses on a cone

EULER EQUATIONS

$\vec{\tau} = \frac{d}{dt}\vec{L}$

TENSOR OF INERTIA W.R.T PRINCIPLE AXIS: $\tilde{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tilde{I}$

$\vec{L_1} = I_1\vec{\omega_1} \quad L_2 = I_2\omega_2 \quad L_3 = I_3\omega_3$



After a time Δt the object has rotated over angles:

$\Delta\theta_1 = \omega_1 \Delta t$
$\Delta\theta_2 = \omega_2 \Delta t$
$\Delta\theta_3 = \omega_3 \Delta t$

Assume small angles $\left(\begin{matrix}\cos\theta \to 1 \\ \sin\theta \to \theta\end{matrix}\right)$

$\Delta L_x = L_x(\Delta t) - L_x(0)$

$\Delta L_x = I_1 \Delta\omega_1 + L_3 \Delta\theta_2 - L_2 \Delta\theta_3$

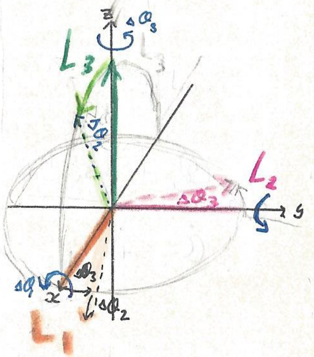
DEVIDING BY Δt and taking LIM $\Delta t \to 0$

$$\frac{dL_x}{dt} = I_1\dot{\omega}_1 + L_3\omega_2 - L_2\omega_3$$

BUT $L_3 = I_3\omega_3$, $L_2 = I_2\omega_2$

$$\frac{dL_x}{dt} = I_1\dot{\omega}_1 + I_3\omega_3\omega_2 - I_2\omega_3\omega_2$$

$$\boxed{\begin{aligned} \frac{dL_x}{dt} &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 \\ \frac{d}{dt}L_y &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 \\ \frac{d}{dt}L_z &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \end{aligned}}$$



x component:

$L_3: 0 \to L_3 \sin\theta_2 \approx \theta_2 L_3$

$L_2: 0 \to -L_2 \sin\theta_3 \approx -\theta_3 L_2$

$L_1: L_1 \to L_1 \cos\theta_2 \cos\theta_3 \approx L_1$

BUT ALSO DUE TO $\vec{\omega}$ changes:

$L_1: I_1\omega_1 \to I_1(\omega_1 + \Delta\omega_1)$

STABILITY OF ROTATIONAL MOTION

Around which principal axis is rotational motion stable?

→ Assume object is spinning around principal axis 1 with angular speed $\omega_1 \gg \omega_2, \omega_3$

→ TORQUE FREE $\Rightarrow \frac{d\vec{L}}{dt} = 0$

TAKE THE DERIVATIVE OF $\frac{d}{dt}(L_y)$:

$$0 = I_2\ddot{\omega}_2 + (\dot{\omega}_1\omega_3 + \omega_1\dot{\omega}_3)(I_1 - I_3)$$

NOW SUBSTITUTE FOR $\dot{\omega}_3$ and $\dot{\omega}_1$ (WITH $I_3\dot{\omega}_3 = -\omega_1\omega_2(I_2 - I_1)$)

$$\boxed{I_2\ddot{\omega}_2 + (I_1 - I_3)\left(\underbrace{\cancel{-\omega_2\omega_3^2\left(\frac{I_3 - I_2}{I_1}\right)}}_{\text{AS } \omega_3 \ll \omega_1 \; 0} - \left(\frac{I_2 - I_1}{I_3}\right)\omega_1^2\omega_2\right)}$$

$$\Rightarrow \boxed{0 = \ddot{\omega}_2 + \underbrace{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}_{\alpha^2}\omega_1^2\omega_2 = 0}$$

FOR $\alpha^2 > 0$, $\omega_2 \sim A\cos(\alpha t + \phi)$

$\alpha^2 < 0$, $\omega_2 \sim Ae^{|\alpha| t} + Be^{-|\alpha| t}$

STABLE IF α^2 positive
1. $I_2, I_3 > I_1$ — I_1 is Smallest
2. $I_1 > I_2$ & $I_1 > I_3$ — I_1 is Biggest

Now consider CYLINDER w/ $I_2 = I_3 = I_\perp$:

$0 = I_\perp\dot{\omega}_2 + (I_1 - I_\perp)\omega_1\omega_3 \qquad 0 = I_\perp\dot{\omega}_3 + (I_\perp - I_1)\omega_1\omega_3$

$\xrightarrow{2^{nd} \text{ Derivative}} \quad 0 = \ddot{\omega}_2 + \frac{(I_1 - I_\perp)^2}{I_\perp^2}\omega_s^2\omega_2$ (FROM BEFORE)

- ω_2 performs SHM w/ Angular velocity:

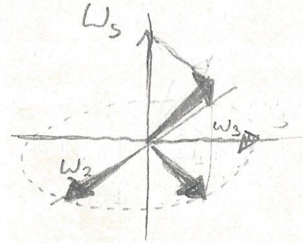
$$\gamma = \left|\frac{I_1 - I_\perp}{I_\perp}\right|\omega_s$$

w.r.t. the object

$$\omega_2 = \omega_\perp \cos(\gamma t + \phi)$$

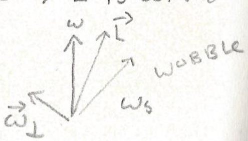
SIMILARLY, LOOKING FROM BEFORE, we get

$$\omega_3 = -\frac{1}{\gamma}\frac{d\omega_2}{dt} = \omega_\perp \sin(\gamma t + \phi)$$



$\omega_\perp$ precesses around $\vec{\omega}_s$ and so does ω

$\vec{\tau} = 0 \Rightarrow \vec{L}$ is constant



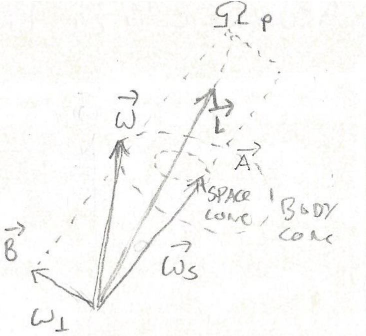
$$\gamma = \left|\frac{I_1 - I_\perp}{I_\perp}\right|\omega \qquad \omega_2 = \omega_\perp \cos(\alpha t + \phi) \qquad \omega_3 = \omega_\perp \sin(\alpha t + \phi)$$

- motion is torque free
  - $\Rightarrow$ $\vec{L}$ is constant in lab frame
  - $\Rightarrow$ In the lab frame, $\vec{\omega}$ processes around $\vec{\omega}_s$ with angular velocity $\boxed{\gamma + \omega_s = \frac{I_1}{I_\perp}\omega_s}$

- What is the angular velocity of $\vec{\omega}$ processing around $\vec{L}$?

$$A = \Omega_p \cos\alpha = \frac{I_1}{I_\perp}\omega_s$$

$$\Omega_p = \frac{I_1}{I_\perp \cos\alpha}\omega_s = \frac{L}{I_\perp}$$

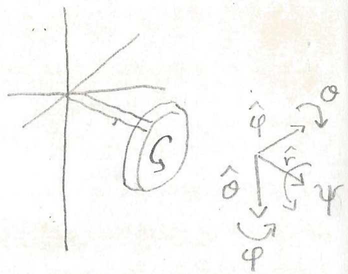


SPACE CONE: TRACED OUT BY $\vec{\omega}_s$ IN PRINCIPAL FRAME

BODY CONE: TRACED OUT BY $\vec{\omega}$ IN LAB FRAME

TORQUE INDUCED PRECESSION

- CONSIDER MASSLESS STICK WITH SPINNING DISK ON END OF IT



- MOMENTS OF INERTIA IN COORD. SYSTEM SHOWN:

$$I_{rr} = I\,\text{disk}$$

$$I_{\theta\theta} = I_{\varphi\varphi} = I\,\text{disk}_\perp + ML^2 = I_p$$

- LOOKING FOR ANGULAR MOMENTUM DERIVATIVES (TORQUE)

$$\frac{d}{dt}(L_r) = I_{rr}\dot{\omega}_s + \overset{0}{\cancel{(I_p - I_p)}}\,\omega_\theta\omega_\varphi$$

$$\frac{d}{dt}(L_\theta) = I_p\dot{\omega}_\theta + (I_{rr} - I_{\varphi\varphi})\omega_s\omega_\varphi = I_p\dot{\omega}_\theta - \omega_\varphi L_s$$

$$\frac{d}{dt}(L_\varphi) = I_p\dot{\omega}_\varphi + \omega_\theta L_s$$

- Looking at torque: $\left.\begin{array}{l}\tau_r = 0\\ \tau_\theta = \ell W\\ \tau_\varphi = 0\end{array}\right\}\ \vec{\tau} = LW\hat{\varphi}$

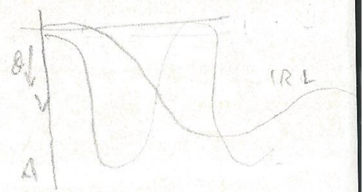
- NOW ANALYZE ω_θ and [illegible]

$$\omega_\theta = A\sin(\gamma t + \phi) \qquad \text{w/ } \gamma = \frac{L_s}{I_p}$$

$$\omega_\varphi = -\frac{LW}{L_s} + A\cos(\gamma t + \phi)$$

$$\theta(t) = -\frac{A}{\gamma}\cos(\gamma t + \alpha) + \theta_0$$

FOR $t = 0$, $\theta = 0$ and $\dot{\theta} = 0$



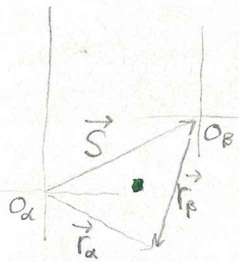
Non-Inertial Systems & Fictitious Forces

- **Inertial System:** A coordinate system in which objects have a constant velocity IF no force is acting on the object

$\vec{F} = 0 \Rightarrow \vec{a} = 0$

IN AN INERTIAL SYSTEM

- **Consider 2 coordinate systems:**
  O_α is an inertial system
  O_β is some other coordinate system.
  $\vec{S}(t)$ is the vector between their origins.



- O_β inertial I.F.F. $\ddot{\vec{S}}(t) = 0$
  $\Rightarrow \quad \vec{S}(t) = \vec{V} \cdot t + \vec{R}$
- Let us consider a uniformly accelerating

$$\ddot{\vec{S}}(t) = \vec{A} \quad \Rightarrow O_\beta \text{ not inertial}$$

$$\vec{r}_\beta = \vec{r}_\alpha - \vec{S}(t)$$

$$\vec{v}_\beta = \vec{v}_\alpha - \dot{\vec{S}}(t)$$

Acceleration: $\vec{a}_\beta = \vec{a}_\alpha - \ddot{\vec{S}}(t) = \vec{a}_\alpha - \vec{A}$

$$m\vec{a}_\beta = m\vec{a}_\alpha - m\vec{A}$$

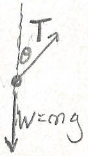
$$\vec{F}_\beta = \underbrace{\vec{F}_\alpha}_{\text{PHYSICAL FORCE}} + \underbrace{\vec{F}_{FICT}}_{\text{FICTITIOUS FORCE, SINCE } \beta \text{ IS NOT AN INERTIAL SYSTEM}}$$

Ex 9.1: Accelerating Vehicle

- Suppose static angle θ, constant acceleration A

Determine θ: $T\cos\theta - W = 0$
$T\sin\theta = mA$

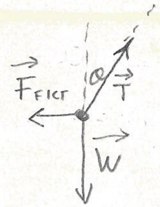
$$\tan\theta = \frac{mA}{W} = \frac{mA}{mg} = \frac{A}{g} \Rightarrow \theta = \operatorname{Arctan}\left(\frac{A}{g}\right)$$



② **Non-Inertial System accelerating with the vehicle:**

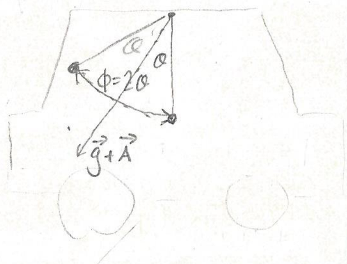
$$T\cos\theta - W = 0$$
$$T\sin\theta - mA = 0$$

$\Rightarrow$ SAME RESULT: $\theta = \arctan\left(\frac{A}{g}\right) \quad T = m(g^2 + A^2)^{\frac{1}{2}}$



Ex 9.2 Car is at rest for $t < 0$, $\theta = 0$

For time $t \geq 0$, car accelerates w/ Acceleration $\vec{A}$.
What is the maximum angle ϕ the mass swings through?

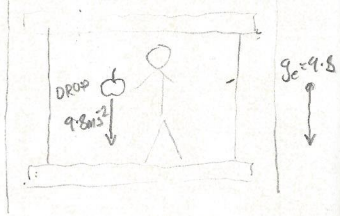


- There is an effective gravitational acceleration of $\vec{g} + \vec{A}$ in the non-inertial system
  IN THE NEW SYSTEM, we can compare it to ~~an~~ pendulum in motion, angle of maximum is $\phi = 2\theta$

- The acceleration due to gravity OR due to an accelerating reference frame are locally indistinguishable

Einstein's Thought Experiment

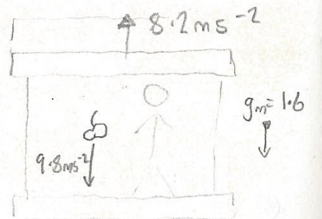
CASE I – Elevator is at rest on the surface of the earth. A person drops an apple.
$\Rightarrow$ Apple falls with $a = g_e = 9.8\,ms^{-2}$ relative to the elevator.



CASE II – Now elevator is at the moon
$g_{moon} = 1.6\,ms^{-2}$

- IF the elevator now accelerates upwards with an acceleration $a = 8.2\,ms^{-2}$

$\Rightarrow$ apple falls with $a = 1.6 + 8.2 = 9.8\,ms^{-2}$ relative to the elevator



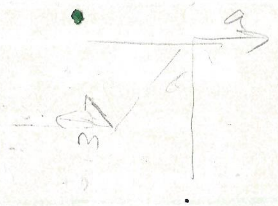
- INDISTINGUISHABILITY hinges at fact that:

$$M_{grav} = M_{inertial} \Rightarrow M_{grav}\vec{g} - m_{inertial}\vec{A} = \vec{F}$$

$\vec{F} = m_{inertial}\,\vec{a}$

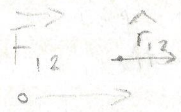
$m_{grav}\, g\,\theta = -\, m_{inertial}\, l\, \ddot{\theta}$

$\Rightarrow$ PERIOD $T = 2\pi\sqrt{\frac{L}{g}}\sqrt{\frac{m_{inertial}}{m_{gravitational}}}$



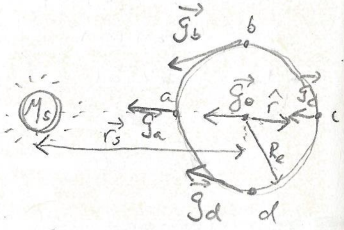
TIDES

$$\vec{F}_{12} = -\frac{GM_1M_2}{r^2}\hat{r}_{12}$$



GRAVITATIONAL FIELD: (ACCELERATION OF PARTICLE 1) $\vec{G}(\vec{r}) = \frac{\vec{F}_{12}}{M_1} = -\frac{GM_2}{r_s^2}\hat{r}_{12}$

- EARTH IS IN FREE FALL TOWARDS THE SUN, WITH ACCELERATION $\vec{g}_o = \frac{GM_{sun}}{r_s^2}\hat{r}$



- WE DETERMINE THE RELATIVE ACCELERATION:

$$\vec{g}'(r) = \vec{g}(r) - \vec{g}_o$$

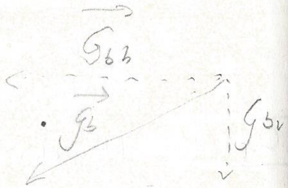
$$\frac{R_e}{R_s} = \frac{6.4\times10^3 km}{1.5\times10^8 km} = 4.3\times10^{-3}$$

$$\Rightarrow g_a' = \frac{GM_s}{(r_s-R_e)^2} - \frac{GM_s}{r_s^2} = \frac{GM_s}{r_s^2}\left(\frac{1}{\left(1-\frac{R_e}{r_s}\right)^2} - 1\right)$$

$$\Rightarrow g_a' = \frac{GM_s}{r_s^2}\left(\qquad \right) \approx 2G_o\frac{R_e}{R_s}$$

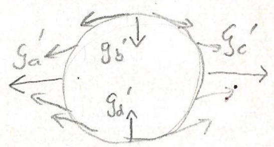
$$g_c' \approx -2G_o'\frac{R_e}{R_s}$$

$$g_b = \frac{GM_s}{(r_s^2+R_e^2)}$$



$$g_{bh} = g_b\cos\alpha = g_b\frac{r_s}{\sqrt{r_s^2+R_e^2}} = \frac{g_b}{\sqrt{1+\left(\frac{R_e}{R_s}\right)^2}} \approx g_b \approx g_o$$

$$g_{bv} = g_b\sin\alpha \approx g_b\frac{R_e}{r_s} \approx g_o\frac{R_e}{R_s}$$



- $\frac{G_oR_e}{r_s} = \frac{GM_sR_e}{r_s^3}$ DEPENDS ON GRADIENT

SUN: $\frac{M_s}{r_s^3} \approx 5.99\times10^{-4} kgm^{-3}$

MOON: $\frac{M_{Moon}}{r_{Moon}^3} \approx 1.34\times10^{-3} kgm^{-3}$

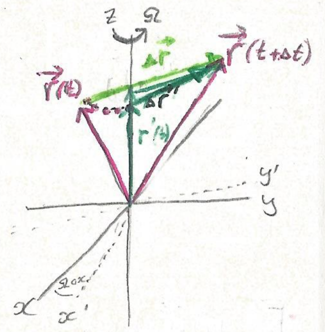
$\Rightarrow$ EFFECT OF MOON IS MUCH GREATER

$\vec{F} = m\vec{a}$ $\quad \vec{F}_{rot} = m\vec{a}_{rot}$

- What are the fictitious forces for a rotating non-inertial system?

$$\vec{a}_{rot} = \vec{a} - \vec{A}$$

- TIME DERIVATIVES IN ROTATING COORD SYSTEMS



- CONSIDER LAB FRAME w/ x, y, z AXES
- OBJECT/FRAME IS ROTATING ABOUT Z axis
- suppose the body has it's own independent motion (FOR EXAMPLE, A FLY IN THE LAB)

$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

- $r'(t)$ is the vector $\vec{r}(t)$ fixed in the rotating coordinate system

$$\Delta\vec{r}' = \vec{r}'(t+\Delta t) - \vec{r}'(t)$$

THUS: $\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$

$$= \underbrace{\vec{r}(t+\Delta t) - r'(t)}_{=\Delta r'} + \underbrace{r'(t) - \vec{r}(t)}_{+\vec{\Omega}\times\vec{r}(t)\Delta t}$$

$$\lim_{\Delta t\to0}\frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta r'}{\Delta t} + \frac{\Omega\times\vec{r}(t)\Delta t}{\Delta t}$$

$$\vec{V}_{INERTIAL} = \vec{V}_{ROT} + \Omega\times\vec{r}(t)$$

$$a_{INTERTIAL} = a_{ROTA} + \Omega\times V_{INT}(t) + \dot{\Omega}\times$$

- THE GENERAL RELATION FOR ANY $\vec{B}$

$$\left(\frac{d\vec{B}}{dt}\right)_{INT} = \left(\frac{d\vec{B}}{dt}\right)_{ROT} + \Omega\times\vec{B}$$

SEEN BEFORE w/ THE EULER EQUATIONS.

$\vec{L} = \hat{I}\vec{\omega} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$ IN TERMS OF THE PRINCIPAL AXES COORDINATE SYSTEM

$$\vec{\omega}\times\begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} = \begin{pmatrix} (I_3-I_2)\omega_3\omega_2 \\ (I_1-I_3)\omega_1\omega_3 \\ (I_2-I_1)\omega_1\omega_2 \end{pmatrix}$$

$$\left(\frac{dL}{dt}\right)_{rat} = \left(\frac{dL}{dt}\right)_{rot} + \vec{\omega}\times\vec{L} \quad \text{Euler Equations}$$

$\frac{d}{dt}(V_{rot} + \vec{\Omega}\times\vec{r})_{rot} + \Omega\times($

$a_{rot} + 2\Omega\times V_{rot} + \Omega\times(\Omega\times r)$

Fictitious Forces & Rotation

$$\left(\frac{dB}{dt}\right)_{INT} = \left(\frac{dB}{dt}\right)_{rot} + \vec{\Omega}\times\vec{B}$$

$$\vec{V}_{INT} = V_{Rot} + \vec{\Omega}\times\vec{r}$$

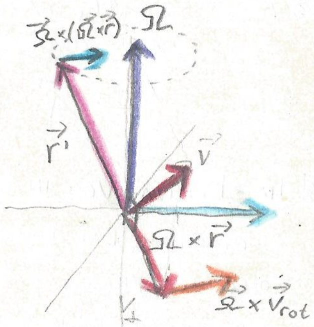
$$\vec{a}_{INT} = \frac{d}{dt}\left(V_{INT}\right) = \frac{d}{dt}\left(V_{rot} + \vec{\Omega}\times\vec{B}\right) +$$

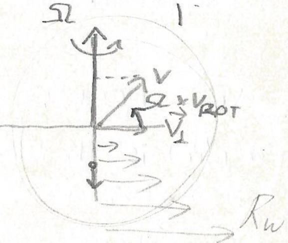
$$\vec{a}_{INT} = \frac{d}{dt}\vec{a}_{rot} + \underbrace{2\vec{\Omega}\times\vec{V}_{rot}}_{\Omega V_\perp} + \underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{CENTRIPETAL FORCE ACCELERATION } \rho\Omega^2}$$

$$\vec{F}_{rot} = m\vec{a}_{rot}$$

$$\vec{F}_{rot} = m\vec{a}_{int} - 2m\,\Omega\times\vec{V}_{rot} - m\Omega\times(\Omega\times\vec{r})$$

$$\vec{F}_{fict} = \underbrace{-2m\vec{\Omega}\times\vec{V}_{rot}}_{\text{CORIOLIS FORCE}} \underbrace{- m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{CENTRIFUGAL FORCE}}$$





Deflection of Falling Mass

- Suppose we have a falling particle towards the center of the earth
  - $\vec{V}_{rot} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$
  - $\vec{a}_{rot} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$
  - $\vec{F}_{INT} = -mg\hat{r}$

LOOK AT <u>CENTRIFUGAL FORCE:</u>

$$-m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = -m\vec{\Omega}\times\hat{\theta}\Omega r = m\Omega^2 r\hat{r}$$

<u>CORIOLIS FORCE:</u>

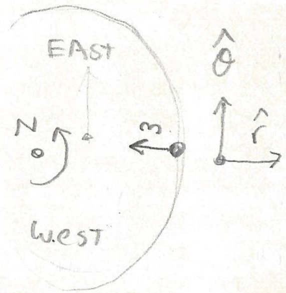
$$-2m\vec{\Omega}\times\vec{V}_{rot} = -2m\vec{\Omega}\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$
$$= -2m(\Omega\dot{r}\hat{\theta} - r\Omega\dot{\theta}\hat{r})$$

$$\vec{F}_{rot} = \vec{F}_{INT} - m\left[2\vec{\Omega}\times\vec{V}_{rot} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})\right]$$
$$= F_r\hat{r} + F_\theta\hat{\theta}$$

<u>EQS OF MOTION</u>

$$F_r = -mg + 2mr\Omega\dot{\theta} + m\Omega^2 r = m(\ddot{r} - r\dot{\theta}^2)$$
$$F_\theta = -2m\dot{r}\Omega = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

WE SOLVE FOR θ



$$F_r = m(\ddot{r} - r\dot{\theta}^2) = -mg + 2mr\Omega\dot{\theta} + m\Omega^2 r$$
$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -2m\dot{r}\Omega$$

- <u>We Assume $\dot{\theta} \ll \Omega$</u>

$$\Rightarrow \ddot{r} \approx g - \Omega^2 r \approx -g \quad (\ddot{r} - \cancel{r\dot{\theta}^2} = -g + \cancel{2r\Omega\dot{\theta}} + \Omega^2 r)$$

$$\Rightarrow r\ddot{\theta} \approx -2\dot{r}\Omega \quad (r\ddot{\theta} + \cancel{2\dot{r}\dot{\theta}} = -2\dot{r}\Omega)$$

- <u>Looking at time $t=0$, $\dot{r}(0)=0$:</u>

$\Rightarrow$ DOWNWARD VELOCITY $\dot{r} = -gt$

$$\Rightarrow r\ddot{\theta} = -2(-gt)\Omega$$

$$\Rightarrow \ddot{\theta} = \frac{2gt\Omega}{r} \underset{r\approx R_e}{=} \frac{2g\Omega}{R_e}t$$

- <u>We solve this w/ initial conditions of $\dot{\theta}(0)=0$ and $\theta(0)=0$:</u>

$$\iint \ddot{\theta}\,dt\,dt = \iint \frac{2g\Omega}{R_e}t\,dt\,dt =$$

$$\boxed{\theta = \frac{2g\Omega}{3R_e}t^3}$$

- <u>THUS, we get:</u>

$$y = R_e\theta \Rightarrow y(h) = R_e\theta\left(\sqrt{\frac{2h}{g}}\right)$$

$$\Rightarrow y(h) = \frac{1}{3}g\Omega\left(\frac{2h}{3}\right)^{\frac{3}{2}}$$

DEFLECTION TO THE EAST IS SEEN

FOR DUBLIN, Poolbeg Chimney (208m)

$y = 4\,cm$

Central Force Motion

$m_1\vec{a}_1 = \boxed{f(\vec{r})\hat{r}}$ Continuous Function of r

$m_2\vec{a}_2 = -f(r)\hat{r}$

we can reduce the number of free variables

- By Conservation of Momentum:
  The Center of Mass $\vec{R}$ is the origin of an inertial coord.

- we study the motion in the center-of-mass coordinate system:
  - $\boxed{\mu\ddot{\vec{r}} = f(r)\hat{r}}$ w/ $\mu = \frac{m_1 m_2}{m_1+m_2}$ called "reduced mass"
  - $\boxed{\vec{L}_c = \mu\vec{r}\times\vec{V} = \text{constant}}$ because $\vec{\tau}_c = 0$

as $\vec{L}$ is constant, this means that $\vec{r}$ is confined to the (x-y)-plane

$$|\vec{L}_c| = \ell = \text{constant} = \mu r^2\dot{\theta}$$

$\Rightarrow$ only $r(t)$ is an unknown variable

$K_c = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)$ but $\dot{\theta} = \frac{\ell}{\mu r^2}$

$= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\cancel{\mu r^2} L^2}{\mu^{\cancel{2}} r^{\cancel{4}2}}$

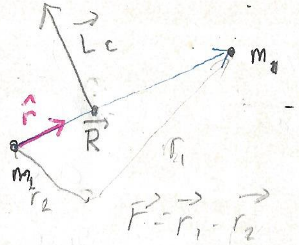
$$\boxed{K_c = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2}$$

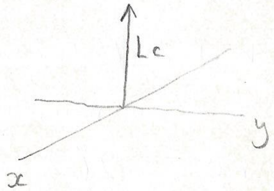
$L = \frac{m_2}{m_1+m_2}\vec{r}\times m_1\vec{V}_1 - \frac{m_1}{m_1+m_2}\vec{r}\times m_2\vec{V}_2$

$= \frac{m_1 m_2}{m_1+m_2}(\vec{r}\times(\vec{V}_1 - V_2))$

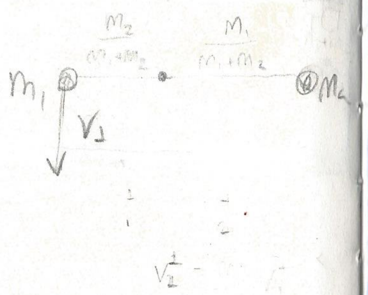
$= \mu\vec{r}\times\dot{\vec{r}} = \mu\vec{r}\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$

$\ell = \mu r(r\dot{\theta}) = \mu r^2\theta$





$K_c = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})^2$



Central Force Motion - Potential Energy

$$W_c = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F}(\vec{r})\cdot d\vec{r}$$

For Conservative Forces:

- W_c is independant of the path c
- only depends on start point $\vec{r}_a$ & end $\vec{r}_b$

THEN: $W_c = \mathcal{U}(\vec{r}_a) - \mathcal{U}(\vec{r}_b)$

where $\mathcal{U}(r) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{r}$

- For Central Forces: $U(\vec{r}) = \int_{\vec{r}_a}^{\vec{r}} f(\vec{r})\hat{r}\cdot d\hat{r}$

  $U(r) = \int_{r_0}^{r} f(r')dr'$

- For Gravitational Force: $U(r) = -\frac{Gm_1m_2}{r}$

- Total Energy:
  $E = K_c + \mathcal{U} = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{L^2}{2\mu r^2} + \mathcal{U}(r)}$

- Effective Potential Energy:
  $$\boxed{\mathcal{U}_{eff}(r) = \frac{L^2}{2\mu r^2} + \mathcal{U}r}$$

Using this, we can solve for $\dot{r}$ from the previous equation

$\frac{1}{2}\mu\dot{r}^2 = E - \mathcal{U}_{eff} = E - \frac{L^2}{2\mu r^2} + \mathcal{U}(r)$

$\frac{dr}{dt} = \dot{r} = \sqrt{2(E - \mathcal{U}_{eff})/\mu}$

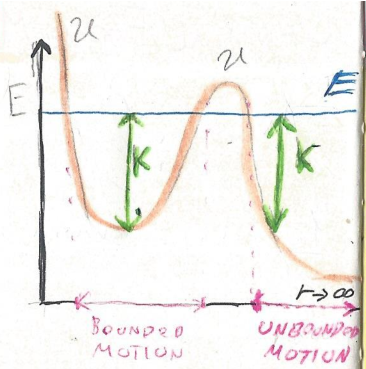
$\int \frac{1}{\sqrt{\dots}}dr = \int dt$

ENERGY DIAGRAMS

- $E = \underbrace{\frac{1}{2}\mu\dot{r}^2}_{\geqslant 0} + U(r)$

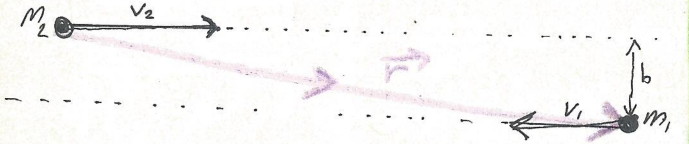
$\Rightarrow$ Motion can ONLY occur if $E - U(r) \geqslant 0$

- CAN HAVE BOUNDED & UNBOUNDED MOTION



Ex 10.1

SUPPOSE WE HAVE 2 NON-INTERACTING PARTICLES:



- WE LOOK AT THE ENERGY DIAGRAM

E is CONSTANT $= \frac{1}{2}\mu v_0^2$

$U_{eff}(r) = \frac{\ell^2}{2\mu r^2}$

$E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}$

$\Rightarrow \frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2}$

BUT $\ell = |L| = \mu b \underbrace{v_0}_{\text{relative velocity}} = \mu|r\times(\vec{v}_1 - \vec{v}_2)|$

THIS MEANS: $v_0 = \frac{\ell}{\mu b}$

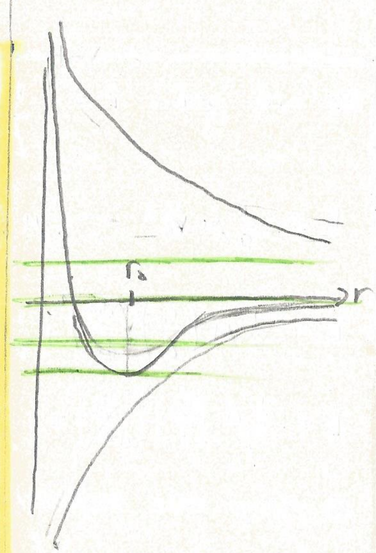
$\Rightarrow E = \frac{1}{2}\mu\left(\frac{\ell^2}{\mu^2 b^2}\right) = \frac{1}{2}\mu\frac{\mu^2 b^2 v_0^2}{\mu^2 b^2}$

GRAVITATIONAL FORCE

$U_g(r) = -\frac{GMm}{r} = \frac{-c}{r}$ $(c = GMm)$

$U_c(r) = \frac{\ell}{2\mu r^2}$

$\Rightarrow \boxed{U_{eff}(r) = \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r}}$



IF $\ell \neq 0$ THEN THERE IS ALWAYS A MINIMUM WHERE $U'_{eff}(r_0) = \frac{-\ell^2}{\mu r_0^3} + \frac{c}{r_0^2} = 0$

$\Rightarrow \frac{\ell^2}{\mu r_0} = c \Rightarrow \underbrace{\boxed{r_0 = \frac{\ell^2}{\mu c}}}_{\text{MINIMUM}}$

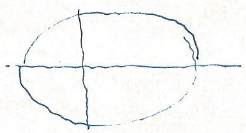
- $U_{eff}(r_0)_{min} = -\frac{1}{2}\frac{\mu c^2}{\ell^2}$ MINIMUM u

- WE CONSIDER 4 CASES OF ENERGY:

1. $E_1 = U_{eff}(r_0) \Rightarrow$ bound motion w/ $r = r_0 =$ CONSTANT
2. $U_{eff}(r_0) < E_2 < 0 \Rightarrow$ bound motion
3. $E_3 = 0$ MARGINALLY BOUND MOTION $\lim_{r\to\infty} V(r) = 0$
4. $E_4 > 0$ unbound motion

TRAJECTORY FOR $U_{eff} \leq E_1 < 0$

QUALATATIVELY:



- $\ell = |L_c| = \mu r^2\dot{\theta}$ $\Rightarrow \dot{\theta} = \frac{\ell}{\mu r^2}$
- $E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$
- FOR SIMPLICITY, WE INTRODUCE $y = \frac{c}{r} \Rightarrow r = \frac{c}{y}$

$\dot{r} = -\frac{c}{y^2}\cdot\frac{dy}{dt} = -\frac{c}{y^2}\cdot\frac{dy}{d\theta}\cdot\frac{d\theta}{dt}$

$\dot{r} = -\frac{c}{y^2}\frac{dy}{d\theta}\cdot\frac{\ell}{\mu r^2} = -\frac{\cancel{c}}{\cancel{y^2}}\cdot\frac{dy}{d\theta}\cdot\frac{\ell\cancel{y^2}}{\mu\cancel{c^2}}$

but $r_0 = \frac{\ell^2}{\mu c}$ $\Rightarrow \dot{r} = -\frac{dy}{d\theta}\frac{\ell^2}{\mu c}\cdot\frac{1}{\ell} = \boxed{-\frac{dy}{d\theta}\frac{r_0}{\ell}}$

So we got $\dot{r} = \frac{r_0}{\ell}\frac{dy}{d\theta}$

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu\left(\frac{\ell}{y}\right)^2 - \frac{c}{r}$$

$$E = \frac{1}{2}\mu\left(\frac{r_0}{\ell}\right)^2\left(\frac{dy}{d\theta}\right)^2 + \frac{1}{2}\mu\left(\frac{c}{y}\right)^2\left(\frac{y^2 \ell}{\mu c^2}\right)^2 - y$$

$$E = \frac{1}{2}\mu\left(\frac{r_0}{\ell}\right)^2\left(\frac{dy}{d\theta}\right)^2 + \frac{1}{2}\frac{y^2 L^2}{\mu c^2} - y$$

TAKE THE DERIVATIVE TO θ OF E

$$\Rightarrow \frac{d}{d\theta}(E) = 0 = \frac{1}{2}\mu\frac{r_0^2}{L^2}\left(2\frac{dy}{d\theta}\frac{d^2y}{d\theta}\right) + \frac{1}{2}\frac{L^2}{\mu c^2}\left(2y\frac{dy}{d\theta}\right) - \frac{dy}{d\theta}$$

$\Rightarrow$

$$\Rightarrow \frac{dy}{d\theta} = 0 \quad \text{OR} \quad \boxed{\frac{d^2y}{d\theta^2} + y - \frac{\ell^2}{\mu r_0^2} = 0}$$

THIS GIVES US EQUATIONS FOR HARMONIC MOTION

$$y(\theta) = \frac{c}{r_0} - A\cos(\theta + B)$$

AND WE TAKE $A > 0$, $B = 0$

$$\Rightarrow r(\theta) = \frac{c}{y(\theta)} = \frac{r_0}{1 - \frac{A r_0}{c}\cos(\theta)}$$

WE NOW TRY TO DETERMINE A:

At r_+ and r_- we have $\dot{r} = 0$

$$E = \frac{1}{2}\mu r^2\left(\frac{L^2}{\mu r^2}\right)^2 - \frac{c}{r}$$

$$E = \left(\frac{r_0}{2r^2}\right)c - \left(\frac{1}{r}\right)c$$

$$\Rightarrow r^2 E + rC - \frac{1}{2}r_0 C = 0$$

FOR $r = r_2$ and $r = r_-$

CENTRAL FORCE MOTION DERIVATION

WE LOOK AT C.O.M. COORDINATES:

$$\boxed{\vec{r} = \vec{r}_1 - \vec{r}_2} \quad \text{THEN:} \quad \boxed{\vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r}} \quad \boxed{\vec{r}_2 = \frac{-m_1}{m_1+m_2}\vec{r}}$$

$$\boxed{\dot{\vec{r}} = \vec{V}_1 - \vec{V}_2} \quad \boxed{V_1 = \frac{m_2}{m_1+m_2}\dot{\vec{r}}} \quad \boxed{V_2 = \frac{-m_1}{m_1+m_2}\dot{\vec{r}}}$$

WE LOOK AT KINETIC ENERGY:

$K = K_1 + K_2 = \frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2$ change to be in terms of $\dot{r}$

$$= \frac{1}{2}m_1\left(\frac{m_2\dot{\vec{r}}}{m_1+m_2}\right)^2 + \frac{1}{2}m_2\left(\frac{-m_1\dot{\vec{r}}}{m_1+m_2}\right)^2$$

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\left(\frac{m_2}{m_1+m_2}\right)\dot{\vec{r}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\left(\frac{m_1}{m_1+m_2}\right)\dot{\vec{r}}^2$$

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\left(\frac{m_1+m_2}{m_1+m_2}\right)\dot{\vec{r}}^2$$

We call $\boxed{\frac{m_1 m_2}{m_1+m_2} = \mu}$ the "relative" or "reduced" mass

We recall $\vec{V} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$$K = \frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})^2 = \boxed{\frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2)}$$

We CONSIDER ANGULAR MOMENTUM L_c

$$\vec{L}_c = L_1 + L_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{r}_1 \times m_1\vec{V}_1 + \vec{r}_2 \times m_2\vec{V}_2$$

now convert $\vec{r}_1$, $\vec{r}_2$ in terms of $\vec{r}$

$$\vec{L}_c = r_1 \times m_1 V_1 + r_2 \times m_2 V_2 = \frac{m_2\vec{r}}{m_1+m_2} \times m_1 V_1 - \frac{m_1 r}{m_1+m_2} \times m_2 V_2$$

$$L_c = \mu\vec{r} \times \vec{V}_1 - \mu\vec{r} \times \vec{V}_2$$

$$= \mu\vec{r} \times (\vec{V}_1 - \vec{V}_2) = \mu\vec{r} \times (\dot{\vec{r}})$$

$= \mu\vec{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$ but $\hat{r} \times \hat{r} = 0$, $|\hat{r} \times \hat{\theta}| = 1$

$$|L_c| = |\mu\vec{r} \times r\dot{\theta}\hat{\theta}|$$

$\boxed{L = \mu r^2\dot{\theta}}$ WHICH IS CONSTANT AS THE ONLY FORCE IS A CENTRAL FORCE $f(r)\hat{r}$

$$\Rightarrow \boxed{\dot{\theta} = \frac{L}{\mu r^2}}$$

RETURNING TO LOOK AT ENERGIES

$$K = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\left(\frac{L^2}{\mu r^2}\right)^2$$

$$E = K + U = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{1}{2}\frac{L^2}{\mu r^2} + U}_{\text{CALLED } U_{\text{effective}}}$$

GRAVITATIONAL

$$E = K + \frac{L}{2\mu r^2} - \frac{c}{r}$$

Eccentricity $\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu c^2}}$

$$r(\theta) = \frac{r_0}{1 - \varepsilon\cos(\theta)}$$

$$\boxed{r_+ = \frac{r_0}{1-\varepsilon}} \quad \boxed{r_- = \frac{r_0}{1+\varepsilon}}$$

$$r - \varepsilon r\cos\theta = r_0$$

$$\sqrt{x^2+y^2} - \varepsilon x = r_0$$

$$x^2+y^2 = r_0^2 + 2r_0\varepsilon x + \varepsilon^2 x^2$$

$$y^2 + x^2 - \varepsilon^2x^2 - 2r_0\varepsilon x = r_0^2$$

$$y^2 + (1-\varepsilon^2)x^2 - 2\varepsilon x r_0 = r_0^2$$

$$\frac{y^2}{1-\varepsilon^2} + x^2 - \frac{2\varepsilon r_0}{1-\varepsilon^2}x + \left(\frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2 = r_0^2 + \left(\frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2$$

$$\frac{y^2}{1-\varepsilon^2} + \left(x - \frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2 = r_0^2\left(1 + \frac{\varepsilon^2}{(1-\varepsilon^2)^2}\right)$$

$$\frac{y^2}{(1-\varepsilon^2)} + \left(x - \frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2 = \frac{r_0^2}{(1-\varepsilon^2)^2}$$

STANDARD FORM FOR ELLIPSE: $(A > B)$

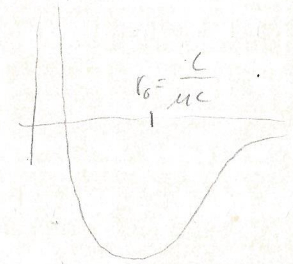
$$\left(\frac{x-c}{A}\right)^2 + \left(\frac{y}{B}\right)^2 = 1$$

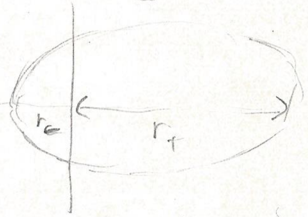
- A = SEMIMAJOR AXIS
  B = SEMIMINOR AXIS
- $\boxed{d_1 + d_2 = 2A}$ for all points on ellipse for x_1, x_2 are 2 focal points on the ellipse

$$\boxed{A^2 = \left(\frac{r_0}{1-\varepsilon^2}\right)^2 \quad B^2 = \left(\frac{r_0^2}{1-\varepsilon^2}\right) \quad C = \left(\frac{\varepsilon r_0}{1-\varepsilon^2}\right)}$$

ALSO: $L = \sqrt{A^2 - B^2}$

IF $C = L$, then the ORIGIN IS A FOCUS POINT



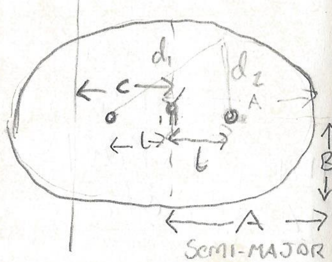


$$r = \frac{r_0}{1 - \varepsilon\cos\theta}$$

$$A = \frac{r_0}{1-\varepsilon^2}$$

$$B = \frac{r_0}{1+\varepsilon^2}$$

$$C = \frac{\varepsilon r_0}{1-\varepsilon^2}$$



KEPLER'S 1ST LAW & 3RD LAW 1609

- THE ORBITS OF PLANETS ARE ELLIPSES, WITH THE SUN AT THE FOCUS POINT (1st Law)

$\varepsilon_{earth} = 0.017$
$\varepsilon_{pluto} = 0.252$

HALLEY'S COMET: $\varepsilon = 0{,}967$
$T = 75$ years → 1705

NEWTON'S COMET: $\varepsilon = 0{,}999986$
$T = 9400$ years

SAME COMET :o = CONFIRMATION OF THEORY

- 2ND LAW: EQUAL AREAS IN EQUAL TIMES
- 3RD LAW: THE PERIOD OF REVOLUTION T OF AN OBJECT AROUND THE SUN IS RELATED TO THE SEMI-MAJOR AXIS A SUCH THAT:

$$T^2 = kA^3$$

WHERE k IS INDEPENDANT OF THE PLANET/COMET

PROOF $\frac{dA}{dt} = \frac{1}{2} r\cdot v_\perp = \frac{1}{2}r^2\dot\theta$ BUT: $\ell = \mu r^2\dot\theta$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\left(\frac{\ell}{\mu r^2}\right) = \frac{\ell}{2\mu}$$

$$\Rightarrow \int dA = \pi AB = \frac{\ell}{2\mu}T$$

ellipse

$$\Rightarrow T = \frac{2\mu}{\ell}\pi AB$$

BUT $A = \frac{1}{2}(r_+ + r_-) = \frac{1}{2}\left(\frac{r_0}{1-\varepsilon} + \frac{r_0}{1+\varepsilon}\right) = \frac{r_0}{1-\varepsilon^2}$

BUT $r_0 = \frac{\ell^2}{\mu c}$ $\quad \varepsilon = 1 + \frac{2E\ell^2}{\mu c^2}$

$$\Rightarrow A = \frac{\left(\frac{\ell^2}{\mu c}\right)}{1 - \left(1 + \frac{2E\ell^2}{\mu c^2}\right)} = \frac{\frac{\ell^2}{\mu c}}{-\frac{2E\ell^2}{\mu c^2}} = \boxed{\frac{-C}{2E}}$$

WE GET A INDEPENDANT OF ℓ

$$\Rightarrow B^2 = A^2 - L^2 \text{ but } L = \frac{\varepsilon r_0}{1-\varepsilon^2} = \varepsilon A$$

$$\Rightarrow B^2 = A^2 - \varepsilon^2A^2$$

$$= A\left(1 - \left(1 + \frac{2E\ell^2}{\mu c^2}\right)\right) = A^2\left(\frac{2E\ell^2}{\mu c^2}\right)$$

$$= -A\cdot\left(-\frac{C}{2E}\right)\cdot\left(\frac{2E\ell^2}{\mu c^2}\right)$$

$$\boxed{B = \frac{AL^2}{\mu c}}$$

So: $T = \frac{2\mu}{l} \pi A B \qquad A = \frac{-C}{2E} \qquad B^2 = \frac{l^2 A}{\mu C}$

$$\Rightarrow T^2 = \frac{4\pi^2 \cancel{\mu^2}}{\cancel{l^2}} \cdot \frac{C}{4E}^2 \cdot \frac{\cancel{l^2} A}{\cancel{\mu} C}$$

$$\boxed{T^2 = \frac{4\pi^2 \mu}{C} A^3}$$

- we get that $k = \frac{4\pi^2 \mu}{C}$

$$k = 4\pi^2 \frac{1}{GMm} \cdot \frac{mM}{m+M} = \frac{4\pi^2}{G(M+m)}$$

$$\boxed{k = \frac{4\pi^2}{G(M+m)}}$$

$$T^2 = \frac{4\pi^2}{G(M+m)} R^3$$

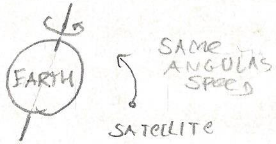
SATELLITE TRANSFER ORBIT - GEOSTATIONARY

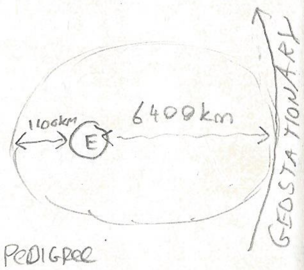
- GEOSTATIONARY $\Rightarrow T = \frac{2\pi}{\omega_e}$

$$\frac{4\pi^2}{\Omega_e^2} = \frac{4\pi^2}{G(M+m)} A^3 \Rightarrow A^3 = \frac{G(M+m)}{\Omega_e^2}$$

BUT $m << M$

$$\Rightarrow A = \left(\frac{GM}{\Omega_e^2}\right)^{\frac{1}{3}} \approx 422 \times 10^3 \text{ km}$$





- We could first move to elliptical orbit, then add energy to make circular

→ WHAT IS THE WORK DONE BY THE ENGINES TO BRING THE SATELLITE FROM ELLIPTIC TO GEOSTATIONARY ORBIT?

- $E = -\frac{C}{2A}$ (A is semimajor axis)

- $(2A)$ ELLIPSE $= 6400 + 1100 = 7500$ km
- $(2A)$ CIRCLE $= 12800$ km

CENTRAL FORCE MOTION

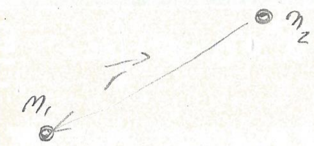
- SUMMARY SO FAR

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 - \frac{C}{r}$$

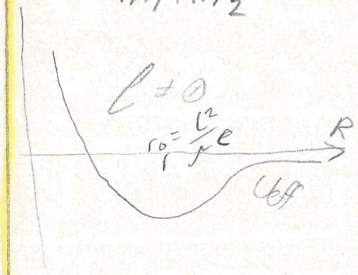
$$= \frac{1}{2}\mu \dot{r}^2 + \underbrace{\frac{l^2}{2\mu r^2} - \frac{C}{r}}_{U_{eff}}$$

$C = G m_1 m_2$ for grav interactions

$l = \mu r^2 \dot{\theta}$



$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

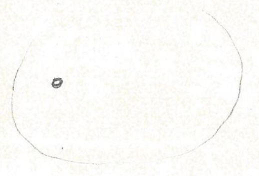


- U_{eff} is minimum at $r_0 = \frac{l^2}{\mu C}$

- FOR $U_{eff} < E$, we get elliptical motion

$$r(\theta) = \frac{r_0}{1 - \varepsilon \cos\theta}$$

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu C^2}}$$



- FOR $E = 0$ we get a parabola

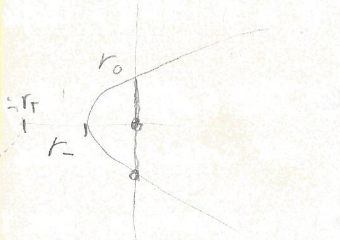
$\varepsilon = 1$ as $y^2 - 2r_0 x = r_0^2$

- FOR $E > 0$ we get an $\varepsilon > 0$: HYPERBOLA

$$y^2 + (1-\varepsilon^2)\left(x - \frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2 = \frac{r_0^2}{1-\varepsilon^2}$$

OF THE FORM: $-\left(\frac{x - C}{A}\right)^2 + \left(\frac{y}{B}\right)^2 = -1$

STANDARD FORM FOR HYPERBOLA



$$\boxed{r_- = \frac{r_0}{1+\varepsilon}} \qquad \boxed{r_+ = \frac{r_0}{1-\varepsilon}} < 1$$

- r_+ SHOWS THE TRAJECTORY IF THE FORCE WERE A REPULSIVE FORCE

LOOKING AT HYPERBOLIC TRAJECTORIES

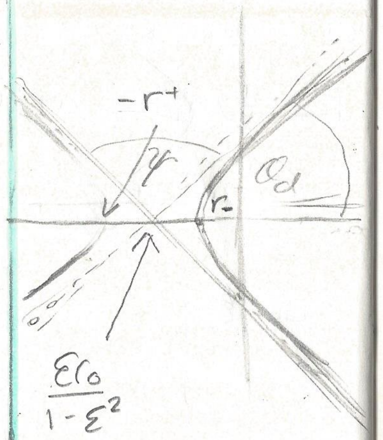
$$y^2 \mp (1-\varepsilon^2)\left(x - \frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2 = \frac{r_0^2}{1-\varepsilon^2}$$

EQUATIONS FOR THE ASYMPTOTES

$\frac{r_0^2}{1-\varepsilon^2}$ becomes neglegable as $x, y \Rightarrow \pm\infty$

$$y^2 = (\varepsilon^2-1)\left(x - \frac{\varepsilon r_0}{1-\varepsilon^2}\right)^2$$

$$\boxed{y = \pm\sqrt{\varepsilon^2-1}\left(x - \frac{\varepsilon r_0}{1-\varepsilon^2}\right)}$$



CONSERVED QUANTITIES

$\ell = \mu v_0 b \qquad b$ = impact parameter

$$r_0 = \frac{\ell^2}{\mu C} = \frac{\mu v_0^2 b^2}{C} = \frac{2E b^2}{C}$$

AS E = K
when $U_{eff} = \infty$
$E = \frac{1}{2}\mu v_0^2$

$$\varepsilon = \sqrt{1 + \left(\frac{2Eb}{C}\right)^2}$$

$$r(\theta) = \frac{r_0}{1-\varepsilon\cos\theta} \Rightarrow \cos\theta_d = \frac{1}{\varepsilon}$$

$\psi = \pi - 2\theta_d$ = deflection angle

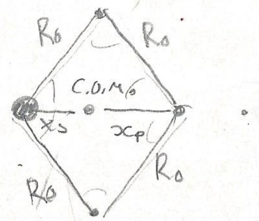
- THE 4 different trajectories can be realised as cross-sections of a cone
- FOR 3 BODIES, THE TRAJECTORIES ARE IN GENERAL CHAOTIC AND UNPREDICTABLE
- THERE ARE SOME EXAMPLES OF SOLUTIONS THOUGH

EG: LAGRANGE POINTS $(m_3 \ll m_1, m_2)$

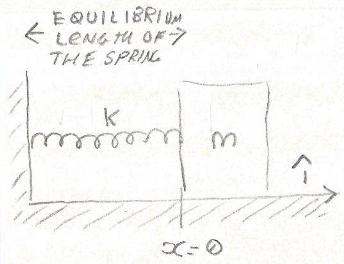
$R_0 = x_s + x_p \qquad x_s = \frac{M_p R_0}{x_s + x_p} \qquad x_p = \frac{M_s R_0}{x_s + x_p}$

ASTEROID

$$\Omega = G(M_s + M_p)/R_0^3$$



HARMONIC OSCILLATOR



Easiest Case: BLOCK OF MASS m ON A FRICTIONLESS TABLE ATTATCHED TO A SPRING s.t:

$$\vec{F} = -k\vec{x}\,\hat{\imath}$$

EQUATIONS OF MOTION:

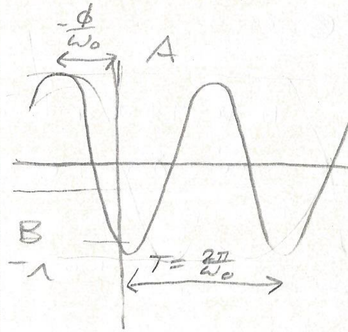
$$m\ddot{x} + kx = 0$$

→ GENERAL SOLUTION

$$\boxed{x(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}}$$

OR ALTERNATIVE FORMS

$$\boxed{x(t) = A\cos(\omega_0 t + \phi)}$$



BUT $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$$\Rightarrow B = A\cos\phi \qquad C = -A\sin\phi$$

$$\boxed{B = x_0} \qquad \boxed{\omega_0 C = V_0}$$

$$A^2 = B^2 + C^2$$

$$\boxed{A = \sqrt{x_0^2 + \left(\frac{V_0}{\omega_0}\right)^2} \qquad \phi = \arctan\left(\frac{-V_0}{\omega_0 x_0}\right)}$$

ENERGY

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\omega_0^2 A^2 \sin^2(\phi + \omega t)\right)$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}k\left(A^2\cos^2(\omega_0 t + \phi)\right)$$

BUT WE KNOW $\omega_0^2 = \frac{k}{m}$

$$E = K + U = \frac{1}{2}kA^2\left(\overbrace{\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)}^{1}\right)$$

DAMPENED HARMONIC OSCILLATOR

- We now include a "viscous" dampening force:

$$\vec{F}_v = -b\vec{v}$$

- b depends on fluid and on shape of block
- In 1 dimension:

$$\vec{F}_v = -b\dot{x}$$

- Equations of motion now:

$$m\ddot{x} + kx + b\dot{x} = 0$$

OR $\ddot{x} + \gamma\dot{x} + \omega_0 x = 0$, with $\gamma = \frac{b}{m}$

TO SOLVE THIS, CONSIDER A SECOND EQUATION

$$\ddot{y} + \gamma\dot{y} + \omega_0 y = 0$$

AND SET $Z(t) = x(t) + i\,y(t) \in \mathbb{C}(t)$

THEN: $\ddot{Z} + \gamma\dot{Z} + \omega_0 Z = 0$

- WE TAKE "ANSATZ" $Z = Z_0 e^{\alpha t}$

$$\Rightarrow \cancel{Z_0}\alpha^2 \cancel{e^{\alpha t}} + \gamma \cancel{Z_0}\alpha \cancel{e^{\alpha t}} + \omega_0^2 \cancel{Z_0} \cancel{e^{\alpha t}} = 0$$

$$\Rightarrow \alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

- IN GENERAL we get 2 SOLUTIONS:

$$\boxed{\alpha_1 = -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \qquad \alpha_2 = -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}}$$

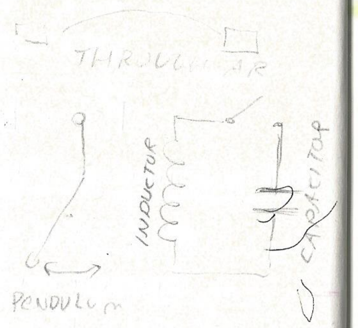
GENERAL SOLUTION FOR $Z(t)$:

$$Z(t) = Z_1 e^{\alpha_1 t} + Z_2 e^{\alpha_2 t}, \quad w/ Z_1, Z_2 \in \mathbb{C}$$

3 CASES:

1. UNDERDAMPENING: $\gamma^2 < 4\omega_0$ $\quad x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)$
2. OVERDAMPENING: $\gamma^2 > 4\omega_0$ $\quad x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$
3. CRITICAL DAMPING: $\gamma^2 = 4\omega$ $\quad x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$

EXAMPLES



$D^2x + \gamma Dx + \omega_0 x = 0$

$$Z(t) = Z_1 e^{\alpha_1 t} + Z_2 e^{\alpha_2 t}, \quad \alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

① UNDERDAMPING: $\gamma^2 < 4\omega_0^2$

$$\alpha_{1,2} = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$$\alpha_{1,2} = -\frac{\gamma}{2} \pm i\omega_1$$

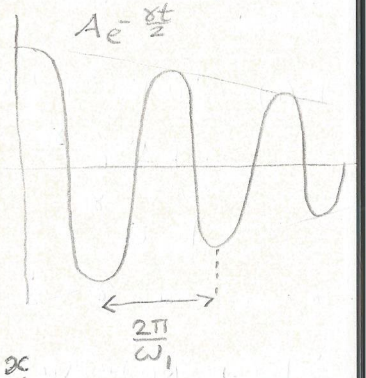
$$Z(t) = Z_1 e^{-\frac{\gamma}{2} + i\omega_1 t} + Z_2 e^{-\frac{\gamma}{2} - i\omega_1 t}$$

- USE EULER'S FORMULA: $e^{i\varphi} = \cos\varphi + i\sin\varphi$

$$x(t) = \mathrm{Re}(Z(t)) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)$$

$$\boxed{x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)}$$

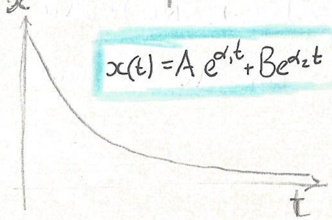
$\omega_1 = \sqrt{\omega_0^2 - \gamma^2/4}$



② OVERDAMPING $\gamma^2 > 4\omega_0^2$

- THEN α_1 and α_2 are REAL AND NEGATIVE (<0)

$$\Rightarrow x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$



$$\boxed{x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}}$$

③ CRITICAL DAMPING $\gamma^2 = 4\omega_0^2$

- THEN $\alpha_1 = \alpha_2 = -\frac{\gamma}{2}$
- THIS GIVES A GENERAL SOLUTION OF

$$x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$$

ENERGY

$E(t) = K + U$

- UNDERDAMPED

ENERGY

UNDERDAMPED: $x(t) = Ae^{-\gamma t/2}\cos(\omega_1 t)$

$$V(t) = -Ae^{-\frac{\gamma t}{2}}\omega_1 \sin(\omega_1 t) - \frac{A\gamma}{2}e^{-\frac{\gamma t}{2}}\cos(\omega_1 t)$$

LIGHT DAMPING $\gamma \ll \omega_1$ $\approx -A\omega_0 e^{-\frac{\gamma t}{2}}\sin \omega t$

$$E(t) = \frac{1}{2}mv(t)^2 + \frac{1}{2}kx(t)^2$$
$$= \frac{1}{2}kA^2e^{-\gamma t}$$
$$= E_0 e^{-\gamma t}$$

- we CALL $\tau = \frac{1}{\gamma}$ THE <u>DAMPING TIME</u>

$$\Rightarrow E(t) = K(t) + \mathcal{U}(t) = E(0) + W_f(t)$$

BUT $W_f(t)$ = WORK DONE BY FRICTION FORCE

$$= \int_{x_0}^{x(t)} f\,dx = \int_a^t f\,v(t)\,dt$$

$$W_f(t) = -\int_0^t f\,v(\sigma)\,d\sigma = \int_0^t bv(\sigma)^2\,d\sigma$$

- <u>OVERDAMPED CASE</u> $x(t) = Ae^{\alpha_1 t}$

$V(t) = A\alpha_1 e^{\alpha_1 t}$

$$E = K + U = \frac{1}{2}m(A^2\alpha_1^2 e^{2\alpha_1 t}) + \frac{1}{2}k(A^2 e^{2\alpha_1 t})$$

- we LOOK AT $W_f(t)$ again

$$W_f(t) = -\int_0^t b\alpha_1^2 A^2 e^{2\alpha_1 \sigma}\,d\sigma$$
$$= -\frac{b\alpha_1}{2}A^2\left(e^{2\alpha_1 t} - 1\right)$$

WE TRY TO VERIFY

$$E_0 = E(t) - W_f(t) =$$
$$= \frac{1}{2}A^2 e^{2\alpha_1 t}\underbrace{(m\alpha_1^2 + k + b\alpha_1)}_{=0 \text{ as } \alpha_1 \cdot} \frac{b\alpha_1}{2}A^2 = E_0$$

QUALITY FACTOR

- THE QUALITY FACTOR Q IS DEFINED AS

$$Q = \frac{\text{ENERGY STORED IN HARMONIC OSCILLATOR}}{\text{ENERGY DISSAPATED PER RADIAN}}$$

- $E(t) = E_0 e^{-\gamma t}$

$\Delta E = E_0(-\gamma)\Delta t$

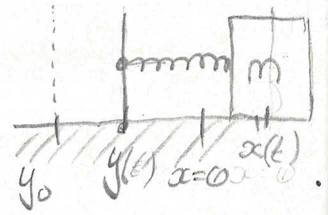
1 radian $\Delta E = E_0 \frac{-\gamma}{\omega_1}$

$$\text{SO } Q = \frac{E}{\gamma\frac{E}{\omega_1}} \approx \frac{\omega_1}{\gamma} \approx \frac{\omega_0}{\gamma}$$

Undamped Forced Harmonic Oscillators

- $y(t) = y_0 \cos(\omega t)$, where we have ω is ARBITRARY (Not related to ω_0)

(WE MOVE THE EQUILIBRIUM POINT HARMONICALLY)



- EQUATIONS OF MOTION:

- Force on m due to the string:

$$F(t) = k(y(t) - x(t))$$

$$m\ddot{x} = F(t)$$

$$\boxed{m\ddot{x} + kx = F_0 \cos \omega t}$$ where $F_0 = y_0 k$

- WE COMPLEXIFY THE EQUATION:

$$m\ddot{z} + kz = F_0 \cos\omega t + i F_0 \sin\omega t$$
$$= F_0 e^{i\omega t}$$

- YET AGAIN, WE CHOOSE AN ANSATZ: $\boxed{z = z_0 e^{i\omega t}}$

$\Rightarrow -m z_0 \omega^2 e^{i\omega t} + k z_0 e^{i\omega t} = F_0 e^{i\omega t}$

$\Rightarrow -m z_0 \omega^2 + k z_0 = F_0$

$\Rightarrow z_0 = \dfrac{F_0}{-m\omega^2 + k} \overset{\omega_0 = \sqrt{k/m}}{=} \dfrac{F_0/m}{\omega_0^2 - \omega^2}$

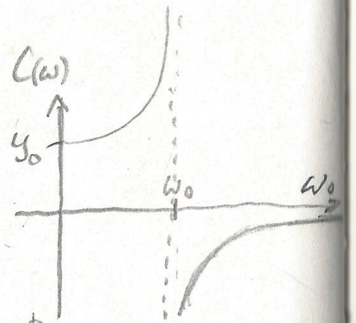
- TAKING THE REAL PART AGAIN:

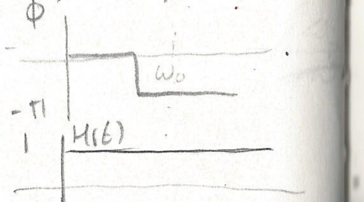
$$x(t) = \frac{F_0}{m}\frac{1}{\omega_0^2 - \omega^2}\cos(\omega t) + B\cos(\omega t + \phi)$$

B, ϕ, INTEGRATION CONSTANTS

- Let $C(\omega) = \dfrac{F_0}{m}\dfrac{1}{\omega_0^2 - \omega^2}$

- THE DIVERGENCE OF $C(\omega)$ for $\omega \to \omega_0$ IS CALLED RESONANCE





Letting $x(t) = A\cos(\omega t + \phi)$ WE GET THAT $A = \|C(\omega)\|$ & $\phi = -\pi H(\omega - \omega_0)$

Driven Damped Harmonic Oscillators

- EQUATIONS OF MOTION:

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t$$

$\xrightarrow{\text{Complexify}}$ $\ddot{z} + \gamma\dot{z} + \omega_0^2 x = \left(\frac{F_0}{m}\right) e^{i\omega t}$

using the ansatz: $z = z_0 e^{i\omega t}$

$$i^2\omega^2 z_0 e^{i\omega t} + \gamma i\omega z_0 e^{i\omega t} + \omega_0^2 z e^{i\omega t} = \left(\frac{F_0}{m}\right)e^{i\omega t}$$

$\Rightarrow z_0 \cancel{e^{i\omega t}}(-\omega^2 + \gamma\omega i + \omega_0^2) = \left(\frac{F_0}{m}\right)\cancel{e^{i\omega t}}$

$\Rightarrow \boxed{z_0 = \dfrac{F_0/m}{\omega_0^2 - \omega^2 + i\omega\gamma}} \in \mathbb{C}$

$\Rightarrow z_0 = \dfrac{F_0}{m}\dfrac{(\omega_0^2 - \omega^2) - i\omega\gamma}{(\omega_0^2 - \omega^2) + (\omega\gamma)^2}$

$\Rightarrow z_0 = z_0 e^{i\omega t} = A(\omega)e^{i(\omega t + \phi(\omega))}$

- WE TAKE THE REAL PART TO FIND $x(t)$

$$A(\omega) = \frac{F_0}{m}\left(\frac{1}{(\omega^2 - \omega_0)^2 + (\gamma\omega)^2}\right)^{\frac{1}{2}}$$

$$\phi(\omega) = \arctan\left(\frac{\omega\gamma}{\omega^2 - \omega_0^2}\right)$$

- PARTICULAR SOLUTION TO EQUATION OF MOTION:

$$x(t) = A(\omega)\cos(\omega t + \phi(\omega))$$

$$x(t) = A(\omega)\cos(\omega t + \phi(\omega))$$

$$A(\omega) = \frac{F_0}{m}\left(\frac{1}{(\omega^2-\omega_0^2)^2-(\gamma\omega)^2}\right)^{\frac{1}{2}}$$

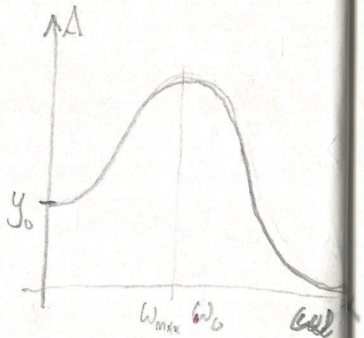
$$\phi(\omega) = \arctan\left(\frac{\gamma\omega}{\omega^2-\omega_0}\right)$$

O MAXIMUM OF $A(\omega)$

$$\Rightarrow \frac{d}{d\omega_g}\left((\omega_0^2-\omega^2)^2 + \gamma\omega^2\right) = 0$$

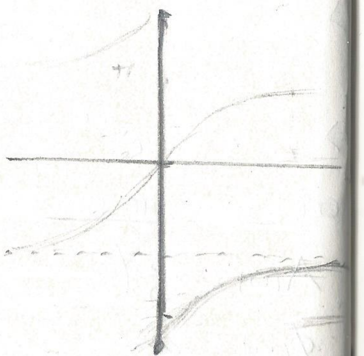
$$\Rightarrow -2\omega_0^2 + 2\omega^2 + \gamma = 0$$

$$\Rightarrow \boxed{\omega_{max} = \omega_0 - \frac{\gamma}{2}}$$



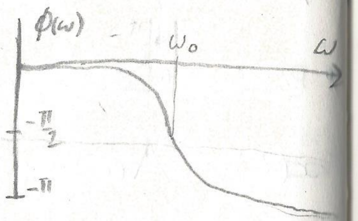
O LIGHT DAMPENING $\omega_{max} \approx \omega_0$

$$\Rightarrow A(\omega) = \frac{F_0}{m}\left(\frac{1}{(\gamma\omega)^2}\right)^{\frac{1}{2}} = \frac{F_0}{m\gamma\omega_0} = \frac{F_0}{b\omega_0}$$



- PHASE ϕ
  - THE USUAL DOMAIN FOR ϕ IS $(-\frac{\pi}{2}, \frac{\pi}{2})$ to define the inverse function (ARCTAN)
  - However, we take the blue arc as the definition for arctan for it to be CONTINUOUS

LOOKING AT ENERGY



$$\mathcal{U} = \frac{1}{2}k(x-y)^2 \quad \text{POTENTIAL ENERGY}$$

$$= \frac{1}{2}k\left(y_0^2\cos^2\omega t - 2y_0A(\omega)\cos(\omega t)\cos(\omega t+\phi) + A(\omega)^2\cos^2(\omega t+\phi)\right)$$

ENERGY - DRIVEN DAMPED HARMONIC OSC.

$$\mathcal{U} = \frac{1}{2}k(x^* - y^*)^2 \quad \text{POTENTIAL ENERGY}$$

$$= \frac{1}{2}k \; y_0^2\cos^2\omega t - 2y_0A(\omega)\cos(\omega t)\cos(\omega t+\phi) + A(\omega)^2\cos^2(\omega t+\phi)$$

RECALL: $\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)\,dt$

IS THE AVERAGE for a periodic function with a period of oscillation of T

$$\langle\cos(\omega t)\rangle = \langle\sin(\omega t)\rangle = 0$$

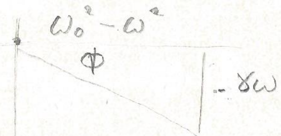
$$\langle\cos^2(\omega t)\rangle = \langle\sin^2(\omega t)\rangle = \frac{1}{2}$$

$$\mathcal{U} = \frac{1}{2}k\left(\frac{y_0^2}{2} - \frac{1}{2}\cancel{k}y_0A(\omega)\cos\phi + \frac{A(\omega)^2}{2}\right)$$

$$= \frac{1}{4}ky_0^2 - \frac{1}{4}k\,y_0A(\omega)\cos\phi + \frac{1}{4}A(\omega)^2k$$

- WE TRY TO SIMPLIFY BY WRITING ϕ IN TERMS of ω_0, ω

$$\cos\phi = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)+\gamma^2\omega^2}}\cdot\frac{F}{m}\cdot\frac{m}{F_0} = \frac{m}{F_0}A(\omega)\cdot(\omega_0^2-\omega^2)$$



$$\langle u\rangle = \frac{1}{4}ky_0^2 - \frac{1}{4}\cancel{ky_0A(\omega)}\frac{m}{F_0}A(\omega)(\omega_0^2-\omega^2) + \frac{1}{4}A(\omega)^2k$$

$$= \frac{1}{4}ky_0^2 + \frac{1}{2}mA(\omega)^2\omega^2 - \frac{1}{4}kA(\omega)^2$$

FOR $\omega \approx \omega_0$

$$\langle u\rangle \approx \frac{1}{4}kA(\omega)^2$$

$$\langle K \rangle = \langle \tfrac{1}{2} m v^2 \rangle$$
$$= \langle \tfrac{1}{2} m [\omega A(\omega)^2 \sin(\omega t + \phi)]^2 \rangle$$
$$= \frac{1}{4} m \omega^2 A(\omega)^2$$
$$= \frac{1}{16} \frac{F_0^2}{m} \cdot \frac{1}{(\omega-\omega_0)^2 + (\frac{\gamma}{2})^2}$$

NOW USING THE APPROXIMATION FOR $\omega_0 \approx \omega$

$$\omega_0^2 - \omega^2 \approx 2\omega_0(\omega_0 - \omega)$$

- WE CAN EXPRESS THIS AS

$$\langle K \rangle = E_0 \, Q^2 g(\omega), \text{ where}$$

① $E_0 = \frac{1}{4} \frac{F_0}{m\omega_0^2} = \frac{1}{4} y_0^2 k$

= AVERAGE KINETIC ENERGY OF A PARTICLE W/ MASS "m" AND

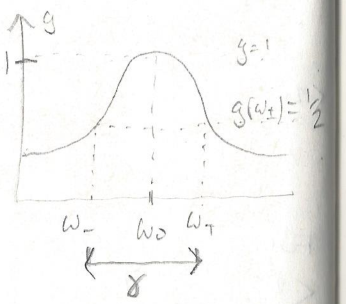
$x(t) = y_0 \cos(\omega_0 t)$

② $Q^2 = \left(\frac{\omega_0}{\gamma}\right)^2 = \left(\frac{\omega_0}{\Delta\omega}\right)^2$ QUALITY FACTOR - measure for selectiveness

③ $g(\omega)$ IS THE LINE SLOPE OF THE FUNCTION

$$g(\omega) = \frac{(\frac{\gamma}{2})^2}{(\omega_0 - \omega)^2 + (\frac{\gamma}{2})^2}$$

WHERE $\Delta\omega = \omega_+ - \omega_- = \gamma$



GENERAL SOLUTION

$$x(t) = x_p(t) + x_{free}(t)$$

PARTICULAR SOLUTION → $x_p(t) = A(\omega)\cos(\omega t + \phi(\omega))$

$$x_{free}(t) = B e^{-\frac{\gamma t}{2}} \cos(\omega_1 t + \beta)$$

→ WITH B & β undetermined parameters

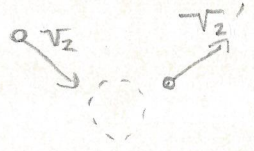
- THE PERIOD TO REACH THE STATIONARY SOLUTION $x_p(t)$ IS THE TRANSIENT

TRANSIENT $\tau = \frac{1}{\gamma}$

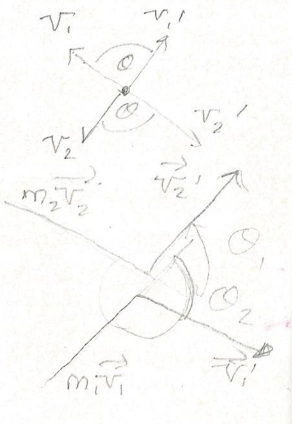
THIS GIVES $\tau \Delta\omega = 1$, reminiscent of uncertainty relation that is found in quantum mechanics

REVIEW OF COLLISIONS

- CONSERVATION LAWS:
  - Momentum is conserved
    $$\vec{p}_1 + \vec{p}_2 = \vec{p}_1{}' + \vec{p}_2{}'$$
  - KINETIC ENERGY is not conserved in general
    FOR ELASTIC COLLISIONS: $K_i = K_f$, $Q = 0$
    OTHERWISE:
    $$K_i = K_f + Q$$

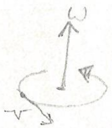


- CENTER OF MASS COORDINATE SYSTEM
  - Origin $O_c = \vec{R}$
    $\Rightarrow$ total momentum vanishes, $\vec{P}_c = 0$
  - Two scattering angles θ_1, θ_2 become a single angle θ



REVIEW OF RIGID BODY MOTION

- ANGULAR VELOCITY $\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$
  $\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$



- ANGULAR MOMENTUM $\vec{L} = m\vec{r} \times \vec{v}$
  $= m\vec{r} \times (\vec{\omega} \times \vec{r})$
- Torque: $\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$
- EXTENDED OBJECTS:
  $$\vec{L} = \sum_{i=1}^{n} m_i \vec{r}_i \times (\omega \times \vec{r}_i) = \tilde{I}\vec{\omega}$$
  - WHERE $\tilde{I}$ IS a 3×3 matrix "TENSOR OF INERTIA"
- TENSOR OF INERTIA
  $$\tilde{I} = \tilde{I}^T = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ \vdots & I_{yy} & I_{yz} \\ \vdots & \cdot & I_{zz} \end{pmatrix}$$

WHERE $I_{xx} = \sum_{i=1}^{n} m_i (y_i^2 + z_i^2)$ MOMENTS ~~PRODUCTS~~ OF INERTIA

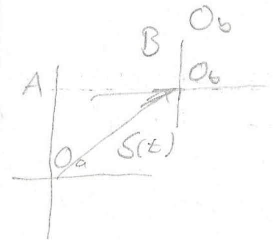
$I_{xy} = -\sum_{i=1}^{n} m_i x_i y_i$ PRODUCTS OF INERTIA

→ EIGENVECTORS GIVE PRINCIPAL AXIS

- TENSOR OF INERTIA: $\tilde{I}$
  - 3 eigenvectors of $\tilde{I}$ are the principal axes of the object.
    $\tilde{I}$ is diagonal if we choose these axes as the basis of our coord system
    $$\vec{L} = \tilde{I}\vec{\omega}, \quad \vec{\tau} = \frac{d}{dt}\vec{L} = \frac{d}{dt}(\tilde{I}\vec{\omega})$$
- EULER EQUATIONS:
  $$\frac{d}{dt}L_x = I_1\frac{d\omega_1}{dt} + (I_3 - I_1)\omega_2\omega_3$$
  $$\frac{d}{dt}L_y = I_2\frac{d\omega_2}{dt} + (I_1 - I_3)\omega_1\omega_3$$
  $$\frac{d}{dt}L_z = I_3\frac{d\omega_3}{dt} + (I_2 - I_3)\omega_2\omega_1$$

REVIEW OF NON-INERTIAL SYSTEMS

Assume A IS AN INERTIAL SYSTEM, then B is an inertial system iff:
$$S(t) = \vec{a} + \vec{v}t$$
FOR some CONSTANT VECTORS $\vec{a}$, $\vec{v}$



- IF $\vec{S}$ IS NOT OF THIS FORM, THEN B IS A NON INERTIAL SYSTEM
  $$\boxed{\vec{F}_B = \vec{F}_a + \vec{F}_{FICT}, \text{ where } F_{fict} = -m\frac{d^2\vec{S}}{dt^2}}$$
- ROTATING
  $$\vec{F}_{rot} = \vec{F}_{IN} + \vec{F}_{cor} + \vec{F}_{centr}$$
  $$\vec{F}_{cor} = -2m\vec{\Omega} \times \vec{v}_{rot}$$
  $$\vec{F}_{centr} = -m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

Review of Central Forces

- For 2 bodies undergoing a central forc

$$\vec{F}_{12} = f(r)\,\hat{r} \qquad \vec{F}_{21} = -\vec{F}_{12} = -f(r)\,\hat{r}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \hat{r} = \vec{r}/\|\vec{r}\|$$

- <u>center of mass coordinate system</u>: $r = \vec{r}_1 - \vec{r}_2$

<u>EQs of Motion</u>: $\mu \ddot{\vec{r}} = f(r)\,\hat{r}$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

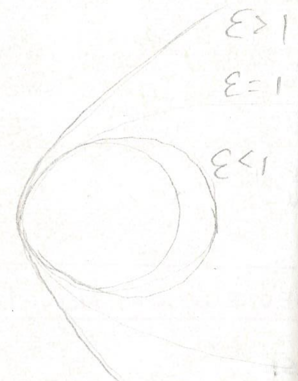
<u>Conservation of Angular Momentum</u>:

$$\frac{d}{dt}\vec{L} = 0 \Rightarrow |\vec{L}| = L = mr^2\dot{\theta}$$

$$\Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{L}{2\mu r^2} + U(r)}_{U_{effective}}$$

<u>For gravitational interactions</u>: $U(r) = -\frac{Gm_1m_2}{r} = -\frac{c}{r}$

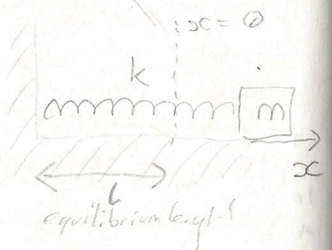
Gives a trajectory: $r(\theta) = \frac{r_0}{1 - \varepsilon\cos\theta}$



Review of Harmonic Oscillators

- <u>$m\ddot{x} = -kx$</u> $\rightarrow x(t) = B\cos(\omega_0 t + \phi)$, $\omega_0 = \sqrt{\frac{k}{m}}$
  B, ϕ arbitrary constants

ENERGY: $K = \frac{1}{2}m\dot{x}^2$, $U = \frac{1}{2}kx^2$, $E = \frac{1}{2}kB^2$



- <u>Viscous Dampening</u>: $F_f = -b\dot{x}$

EQ. OF MOTION: $m\ddot{x} + b\dot{x} + kx = 0$

$\rightarrow$ <u>UNDERDAMPING</u>: $\gamma = \frac{b}{m}$, $\omega_0^2 - \frac{\gamma^2}{4} > 0$

$$x(t) = Be^{-\frac{\gamma}{2}t}\cos(\omega_1 t + \phi), \quad \omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$\rightarrow$ <u>OVERDAMPED</u>:

$$\omega_0^2 - \frac{\gamma^2}{4} < 0 \Rightarrow x(t) = B_1 e^{\alpha_1 t} + B_2 e^{\alpha_2 t}$$

$$\alpha_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} < 0$$

$\rightarrow$ <u>CRITICAL DAMPING</u>

$$\omega_0^2 - \frac{\gamma^2}{4} = 0 \Rightarrow x(t) = B_1 e^{-\frac{\gamma}{2}t} + B_2 t\, e^{-\frac{\gamma}{2}t}$$

<u>DRIVEN HARMONIC OSCILLATORS</u>:

$$m\ddot{x} + b\dot{x} + kx = ky_0\cos\omega t$$

PARTICULAR SOLUTION $x_p(t) = A(\omega)\cos(\omega t + \phi(\omega))$

$$A(\omega) = \frac{ky_0}{m}\sqrt{\frac{1}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

$$\phi(\omega) = \arctan\left(\frac{\omega\gamma}{\omega^2 - \omega_0^2}\right)$$