Advanced Calculus
MA1132
Exercises 1 Solutions

1. For each of the following, either find the limit or show it doesn't exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-2xy+y^2}{x-y}$.

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2-x^3-xy^2}{x^2+y^2}$.

   (d) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x\sin(y)}{y^2-1}$.

   (e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$.

   (f) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^3-2x^2-xy^2+2y^2}{x^3-2x^2+xy^2-2y^2+x-2}$.

   (g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{|x|+|y|}$.

   (h) $\displaystyle\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1}$.

   (i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$.

   Solution:

   (a) Here we will show the limit doesn't exist by showing the limits along two paths approaching $(0,0)$ are different.
   Along $x=0$:
   $$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{y\to 0} \frac{0}{0+y^2} = \lim_{y\to 0} 0 = 0.$$
   Along $y=0$:
   $$\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2} = \lim_{x\to 0} \frac{x^4}{x^4+0} = \lim_{x\to 0} 1 = 1.$$
   Since these limits are different, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+y^2}$ does not exist.

   (b) Here we will use a factorization.
   $$\lim_{(x,y)\to(0,0)} \frac{x^2-2xy+y^2}{x-y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x-y} = \lim_{(x,y)\to(0,0)} x-y = 0.$$

(c) Here we will use a factorization.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2-x^3-xy^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(1-x)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} 1-x = 1.$$

(d) Here, the denominator is not zero at $(1,0)$, so we can use direct substitution:

$$\lim_{(x,y)\to(1,0)} \frac{x\sin(y)}{y^2-1} = \frac{1\cdot\sin(0)}{0-1} = 0.$$

(e) Here we will use a factorization.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{(\sqrt{x}+\sqrt{y}+2)(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}} \\ &= \lim_{(x,y)\to(0,0)} \sqrt{x}+\sqrt{y}+2 \\ &= 2.\end{aligned}$$

(f) Here we will use two factorizations.

$$\begin{aligned}\lim_{(x,y)\to(2,1)} \frac{x^3-2x^2-xy^2+2y^2}{x^3-2x^2+xy^2-2y^2+x-2} &= \lim_{(x,y)\to(2,1)} \frac{(x-2)(x^2-y^2)}{(x-2)(x^2+y^2+1)} \\ &= \lim_{(x,y)\to(2,1)} \frac{x^2-y^2}{x^2+y^2+1} \\ &= \frac{4-1}{4+1+1} \\ &= \frac{1}{2}.\end{aligned}$$

(g) Here we will show the limit doesn't exist by showing the limits along two paths approaching $(0,0)$ are different.

Along $y=0$, $y>0$:

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{|x|+|y|} = \lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} 1 = 1.$$

Along $y=0$, $y<0$:

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{|x|+|y|} = \lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^+} -1 = -1.$$

Since these limits are different, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{|x|+|y|}$ does not exist.

(h) Here we can use direct substitution:

$$\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{9+16-1} = \sqrt{24} = 2\sqrt{6}.$$

(i) Here we will show the limit doesn't exist by showing the limits along two paths approaching $(0,0)$ are different.

Along $y = 0$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{0}{x^4} = \lim_{x\to 0} 0 = 0.$$

Along $y = x^2$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2x^2}{x^4+x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since these limits are different, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exist.

Note that in this case the limit along $y = x$ (and indeed all straight lines approaching the origin) is also zero, so we have to consider curve paths approaching the origin.

2. For each of the limits in Question 1 that exist at the given point, using the expression you evaluated as a rule, define a function that is continuous at that point.

Solution

For the function in Part (b), we define

$$f(x,y) = \begin{cases} \dfrac{x^2-2xy+y^2}{x-y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

For the function in Part (c), we define

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2-x^3-xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}.$$

Since the function in Part (d) is defined at $(1,0)$, we can just define

$$f(x,y) = \frac{x\sin(y)}{y^2-1}.$$

Note that this function is not continuous (or indeed defined) at points of the form $(x,\pm 1)$, but it is continuous on its domain.

For the function in Part (e), we define

$$f(x,y) = \begin{cases} \dfrac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0) \end{cases}.$$

Similarly to Part (d), this function is not defined everywhere, but it is continuous on its domain.

For the function in Part (f), we define

$$f(x,y) = \begin{cases} \dfrac{x^3 - 2x^2 - xy^2 + 2y^2}{x^3 - 2x^2 + xy^2 - 2y^2 + x - 2} & \text{if } (x,y) \neq (2,1) \\ \frac{1}{2} & \text{if } (x,y) = (2,1) \end{cases}.$$

Since the function in Part (h) is defined at $(3,4)$, we can just define

$$f(x,y) = \sqrt{x^2 + y^2 - 1}.$$

Similarly to Part (d), this function is not defined everywhere, but it is continuous on its domain.

3. For the following functions

   i. sketch the domain of f (you may use/check with Mathematica). Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included. Determine whether the domain is an open set, closed set, or neither;

   ii. give the sets where each of the following functions are continuous.

(a) $f(x,y) = \sin(x+y)$.

(b) $f(x,y) = \dfrac{x^2 + y^2}{x^2 - 3x + 2}$.

(c) $f(x,y) = \dfrac{xy}{xy}$.

(d) $f(x,y,z) = \dfrac{1}{x^2 + z^2 - 10}$.

Solution

(a) The function $f(x,y) = \sin(x+y)$ is continuous on $\mathbb{R}^2$.

(b) The function $f(x,y) = \dfrac{x^2 + y^2}{x^2 - 3x + 2}$ is continuous whenever $x^2 - 3x + 2 \neq 0$. That is whenever $(x-2)(x-1) \neq 0$. So it is continuous on $\mathbb{R}^2 \backslash \{(2,y),(1,y) \in \mathbb{R}^2 \colon y \in \mathbb{R}\}$.

(c) The function $f(x,y) = \dfrac{xy}{xy}$ is continuous whenever $xy \neq 0$. So it is continuous on $\{(x,y) \in \mathbb{R}^2 \colon x \neq 0 \text{ or } y \neq 0\}$.

Note that we **CANNOT** cancel to get $f(x,y) = 1$ and then say the function is continuous on $\mathbb{R}^2$. The original function is not defined when $xy = 0$, and if we change the domain, we change the function!

(d) The function $f(x,y,z) = \dfrac{1}{x^2 + z^2 - 10}$ is continuous whenever $x^2 + z^2 - 10 \neq 0$, that is whenever $x^2 + z^2 \neq 10$. So its domain is $\{(x,y,z) \in \mathbb{R}^3 \colon x^2 + z^2 \neq 10\}$. That is everywhere in $\mathbb{R}^3$ except the cylinder (parallel to the y-axis) with equation $x^2 + z^2 = 10$.

Note that since we are in $\mathbb{R}^3$, the equation $x^2 + z^2 = 10$ represents a cylinder rather than a circle.

4. For each of the following functions, calculate all the first and second order partial derivatives, and verify that the corresponding mixed second order partial derivatives are the same.

   (a) $f(x,y) = \ln(1 + y^2 e^{2x})$.

   (b) $f(x,y) = \dfrac{x^2 + y^2}{x + y}$.

   (c) $f(x,y) = \ln(x^2 y \sin x)$.

   (d) $f(x,y,z) = x^2 y \cos z$.

   Solution:

   (a) Using the Chain Rule (note the Product Rule is not needed since we regard y as a constant),

$$\frac{\partial f}{\partial x} = \frac{2y^2 e^{2x}}{1 + y^2 e^{2x}}. \tag{1}$$

   Using the Chain Rule,

$$\frac{\partial f}{\partial y} = \frac{2y e^{2x}}{1 + y^2 e^{2x}}. \tag{2}$$

   Differentiating (1) w.r.t. x, using the Quotient Rule,

$$\frac{\partial^2 f}{\partial x^2} = \frac{4y^2 e^{2x}\,(1 + y^2 e^{2x}) - 2y^2 e^{2x} \cdot 2y^2 e^{2x}}{(1 + y^2 e^{2x})^2} = \frac{4y^2 e^{2x}}{(1 + y^2 e^{2x})^2}.$$

   Differentiating (1) w.r.t. y, using the Quotient Rule,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{4y e^{2x}\,(1 + y^2 e^{2x}) - 2y^2 e^{2x} \cdot 2y e^{2x}}{(1 + y^2 e^{2x})^2} = \frac{4y e^{2x}}{(1 + y^2 e^{2x})^2}.$$

   Differentiating (2) w.r.t. x, using the Quotient Rule,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{4y e^{2x}\,(1 + y^2 e^{2x}) - 2y e^{2x} \cdot 2y^2 e^{2x}}{(1 + y^2 e^{2x})^2} = \frac{4y e^{2x}}{(1 + y^2 e^{2x})^2} = \frac{\partial^2 f}{\partial y \partial x}.$$

   Differentiating (2) w.r.t. y, using the Quotient Rule,

$$\frac{\partial^2 f}{\partial y^2} = \frac{2e^{2x}\,(1 + y^2 e^{2x}) - 2y e^{2x} \cdot 2y e^{2x}}{(1 + y^2 e^{2x})^2} = \frac{2e^{2x}(1 - y^2 e^{2x})}{(1 + y^2 e^{2x})^2}.$$

   (b) Using the Quotient Rule,

$$\frac{\partial f}{\partial x} = \frac{2x(x+y) - (x^2 + y^2)(1)}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}. \tag{3}$$

   Using the Quotient Rule,

$$\frac{\partial f}{\partial y} = \frac{2y(x+y) - (x^2 + y^2)(1)}{(x+y)^2} = \frac{y^2 + 2xy - x^2}{(x+y)^2}. \tag{4}$$

Differentiating (3) w.r.t. x, using the Chain and Quotient Rules,

$$\frac{\partial^2 f}{\partial x^2} = \frac{(2x+2y)(x+y)^2 - (x^2+2xy-y^2)2(x+y)}{(x+y)^4} = \frac{4y^2}{(x+y)^3}.$$

Differentiating (3) w.r.t. y, using the Chain and Quotient Rules,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{(2x-2y)(x+y)^2 - (x^2+2xy-y^2)2(x+y)}{(x+y)^4} = -\frac{4xy}{(x+y)^3}.$$

Differentiating (4) w.r.t. x, using the Chain and Quotient Rules,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(2y-2x)(x+y)^2 - (y^2+2xy-x^2)2(x+y)}{(x+y)^4} = -\frac{4xy}{(x+y)^3} = \frac{\partial^2 f}{\partial y \partial x}.$$

Differentiating (4) w.r.t. y, using the Chain and Quotient Rules,

$$\frac{\partial^2 f}{\partial y^2} = \frac{(2y+2x)(x+y)^2 - (y^2+2xy-x^2)2(x+y)}{(x+y)^4} = \frac{4x^2}{(x+y)^3}.$$

(c) Using the Product and Chain Rules,

$$\frac{\partial f}{\partial x} = \frac{2xy\sin x + x^2 y \cos x}{x^2 y \sin x} = \frac{2}{x} + \frac{\cos x}{\sin x}. \tag{5}$$

Using the Chain Rule,

$$\frac{\partial f}{\partial y} = \frac{x^2 \sin x}{x^2 y \sin x}. = \frac{1}{y}. \tag{6}$$

Differentiating (5) w.r.t. x, using the Quotient Rule,

$$\frac{\partial^2 f}{\partial x^2} = -\frac{2}{x^2} + \frac{-\sin x(\sin x) - \cos x(\cos x)}{\sin^2 x} = -\frac{2}{x^2} - \frac{1}{\sin^2 x} = -\frac{2}{x^2} - \operatorname{cosec}^2 x$$

Differentiating (5) w.r.t. y,

$$\frac{\partial^2 f}{\partial y \partial x} = 0.$$

Differentiating (6) w.r.t. x,

$$\frac{\partial^2 f}{\partial x \partial y} = 0 = \frac{\partial^2 f}{\partial y \partial x}.$$

Differentiating (6) w.r.t. y,

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y^2}.$$

(d)

$$\frac{\partial f}{\partial x} = 2xy\cos z. \tag{7}$$

$$\frac{\partial f}{\partial y} = x^2\cos z. \tag{8}$$

$$\frac{\partial f}{\partial z} = -x^2 y\sin z. \tag{9}$$

Differentiating (7) w.r.t. x,

$$\frac{\partial^2 f}{\partial x^2} = 2y\cos z.$$

Differentiating (7) w.r.t. y,

$$\frac{\partial^2 f}{\partial y\partial x} = 2x\cos z.$$

Differentiating (7) w.r.t. z,

$$\frac{\partial^2 f}{\partial z\partial x} = -2xy\sin z.$$

Differentiating (8) w.r.t. x,

$$\frac{\partial^2 f}{\partial x\partial y} = 2x\cos z.$$

Differentiating (8) w.r.t. y,

$$\frac{\partial^2 f}{\partial y^2} = 0.$$

Differentiating (8) w.r.t. z,

$$\frac{\partial^2 f}{\partial z\partial y} = -x^2\sin z.$$

Differentiating (9) w.r.t. x,

$$\frac{\partial^2 f}{\partial x\partial z} = -2xy\sin z.$$

Differentiating (9) w.r.t. y,

$$\frac{\partial^2 f}{\partial y\partial z} = -x^2\sin z.$$

Differentiating (9) w.r.t. z,

$$\frac{\partial^2 f}{\partial z^2} = -x^2 y \cos z.$$

Note we have

$$\frac{\partial^2 f}{\partial y \partial x} = 2x \cos z = \frac{\partial^2 f}{\partial x \partial y},$$

$$\frac{\partial^2 f}{\partial z \partial x} = -2xy \sin z = \frac{\partial^2 f}{\partial x \partial z}$$

and

$$\frac{\partial^2 f}{\partial z \partial y} = -x^2 \sin z = \frac{\partial^2 f}{\partial y \partial z},$$

as expected.

5. For each of the following functions, by differentiating implicitly, find each of the first order partial derivatives. You should regard z (or w if it appears) as the dependent variable, and you should leave your answer in terms of the independent and the dependent variables.

(a) $\ln(2x^2 + y - z^3) = x$.

(b) $y^2 + z \cos(xyz) = 0$.

(c) $e^{xy} \sinh z - z^2 x + y = 0$.

(d) $\ln(2x^2 + yz^3 + 3w) = z$.

Solution

(a) Differentiating w.r.t. x, we obtain

$$\begin{aligned}
\frac{4x - 3z^2 \dfrac{\partial z}{\partial x}}{2x^2 + y - z^3} = 1 \quad &\Rightarrow \quad 4x - 3z^2 \frac{\partial z}{\partial x} = 2x^2 + y - z^3 \\
&\Rightarrow \quad -3z^2 \frac{\partial z}{\partial x} = 2x^2 - 4x + y - z^3 \\
&\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-2x^2 + 4x - y + z^3}{3z^2}.
\end{aligned}$$

Differentiating w.r.t. y, we obtain

$$\begin{aligned}
\frac{1 - 3z^2 \dfrac{\partial z}{\partial y}}{2x^2 + y - z^3} = 0 \quad &\Rightarrow \quad 1 - 3z^2 \frac{\partial z}{\partial y} = 0 \\
&\Rightarrow \quad -3z^2 \frac{\partial z}{\partial y} = -1 \\
&\Rightarrow \quad \frac{\partial z}{\partial y} = \frac{1}{3z^2}.
\end{aligned}$$

(b) Differentiating w.r.t. x, we obtain

$$\begin{aligned}
&\frac{\partial z}{\partial x}\cos(xyz) + z\left(yz + xy\frac{\partial z}{\partial x}\right)(-\sin(xyz)) = 0 \\
&\qquad \Rightarrow \quad \frac{\partial z}{\partial x}(\cos(xyz) - xyz\sin(xyz)) = yz^2\sin(xyz) \\
&\qquad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{yz^2\sin(xyz)}{\cos(xyz) - xyz\sin(xyz)}.
\end{aligned}$$

Differentiating w.r.t. y, we obtain

$$\begin{aligned}
&2y + \frac{\partial z}{\partial y}\cos(xyz) + z\left(xz + xy\frac{\partial z}{\partial y}\right)(-\sin(xyz)) = 0 \\
&\qquad \Rightarrow \quad \frac{\partial z}{\partial y}(\cos(xyz) - xyz\sin(xyz)) = xz^2\sin(xyz) - 2y \\
&\qquad \Rightarrow \quad \frac{\partial z}{\partial y} = \frac{xz^2\sin(xyz) - 2y}{\cos(xyz) - xyz\sin(xyz)}.
\end{aligned}$$

(c) Differentiating w.r.t. x, we obtain

$$\begin{aligned}
&ye^{xy}\sinh(z) + e^{xy}\cosh(z)\frac{\partial z}{\partial x} - z^2 - 2xz\frac{\partial z}{\partial x} = 0 \\
&\qquad \Rightarrow \quad \frac{\partial z}{\partial x}(e^{xy}\cosh(z) - 2xz) = z^2 - ye^{xy}\sinh(z) \\
&\qquad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{z^2 - ye^{xy}\sinh(z)}{e^{xy}\cosh(z) - 2xz}.
\end{aligned}$$

Differentiating w.r.t. y, we obtain

$$\begin{aligned}
&ye^{xy}\sinh(z) + e^{xy}\cosh(z)\frac{\partial z}{\partial x} - 2xz\frac{\partial z}{\partial y} + 1 = 0 \\
&\qquad \Rightarrow \quad \frac{\partial z}{\partial y}(e^{xy}\cosh(z) - 2xz) = -1 - ye^{xy}\sinh(z) \\
&\qquad \Rightarrow \quad \frac{\partial z}{\partial y} = \frac{1 + ye^{xy}\sinh(z)}{2xz - e^{xy}\cosh(z)}.
\end{aligned}$$

(d) Differentiating w.r.t. x, we obtain

$$\frac{4x + 3\frac{\partial w}{\partial x}}{2x^2 + yz^3 + 3w} = 0 \quad \Rightarrow \quad 4x + 3\frac{\partial w}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial x} = -\frac{4x}{3}.$$

Differentiating w.r.t. y, we obtain

$$\frac{z^3 + 3\frac{\partial w}{\partial y}}{2x^2 + yz^3 + 3w} = 0 \quad \Rightarrow \quad z^3 + 3\frac{\partial w}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial w}{\partial y} = -\frac{z^3}{3}.$$

Differentiating w.r.t. z, we obtain

$$\frac{3yz^2 + 3\dfrac{\partial w}{\partial z}}{2x^2 + yz^3 + 3w} = 1 \quad \Rightarrow \quad 3yz^2 + 3\frac{\partial w}{\partial z} = 2x^2 + yz^3 + 3w$$
$$\Rightarrow \quad 3\frac{\partial w}{\partial z} = 2x^2 + yz^3 + 3w - y - 3yz^2$$
$$\Rightarrow \quad \frac{\partial w}{\partial z} = \frac{2x^2 + yz^3 + 3w - y - 3yz^2}{3}.$$

6. Sketch the level curve $z = k$ for the specified values of k

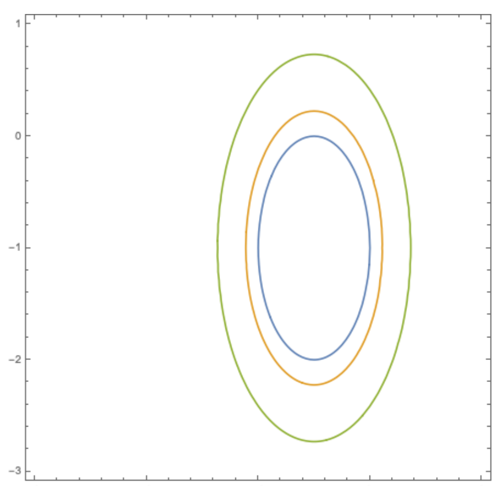
$$z = 8x^2 - 8x + 2y^2 + 4y\,, \quad k = -2, -1, 2\,.$$

Solution

The level curve equation $8x^2 - 8x + 2y^2 + 4y = k$ can be written as

$$8(x + \frac{1}{2})^2 + 2(y+1)^2 = k + 4\,,$$

and, therefore, the level curves are ellipses with the centre located at $(-1/2, -1)$. For $k = -2$ the level curve is just the point $(-1/2, -1)$.

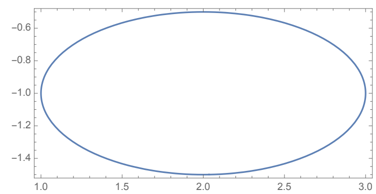


7. A cone is the union of a set of half-lines that start at a common apex point and go through a base which can be any parametric curve. Show that the graph of $z = \sqrt{x^2 - 4x + 4y^2 + 8y + 8}$ is a cone. Where is its apex? What can be chosen as its base? Sketch the base.

Solution: We write $x^2 - 4x + 4y^2 + 8y + 8 = (x-2)^2 + 4(y+1)^2$. Then the apex is obviously at the point $\mathcal{A} = (2, -1, 0)$. Let (x_0, y_0, z_0) be any point but the apex of the surface $z = \sqrt{(x-2)^2 + 4(y+1)^2}$. We need to show that the half-line that start at the apex and go through the point belongs to the surface. The half-line has the parametric equations

$$x = 2+t(x_0-2)\,, \quad y = -1+t(y_0+1)\,, \quad z = tz_0\,, \quad t \geq 0\,, \quad z_0 = \sqrt{(x_0-2)^2+4(y_0+1)^2}\,, \tag{10}$$

where $t = 0$ corresponds to the apex, and $t = 1$ corresponds to the point (x_0, y_0, z_0). Substituting these equations into $z = \sqrt{(x-2)^2+4(y+1)^2}$ one finds that the equation $z = \sqrt{(x-2)^2+4(y+1)^2}$ is satisfied for any $t \geq 0$ which means that this half-line belongs to the surface. As a base of the cone one can choose the intersection of the cone with the plane $z = 1$ which gives the curve $(x-2)^2 + 4(y+1)^2 = 1$. This curve is an ellipse.



8. Show that the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ is a connected surface, that is any two points of the hyperboloid can be connected by a curve which lies on the hyperboloid. Sketch the hyperboloid.

Solution: Let (x_0, y_0, z_0) and (x_1, y_1, z_1) be any two points on the hyperboloid. It is clear that if (a, b, c) is a point on the hyperboloid then all points satisfying $x^2 + y^2 = a^2 + b^2\,,\ z = c$ are also on the hyperboloid. These points obviously form a circle. Thus, the points (x_0, y_0, z_0) and (x_1, y_1, z_1) can be connected by arcs of their circles to the points $(x_0', 0, z_0)$ and $(x_1', 0, z_1)$ in the xz-plane where $x_0' > 0$ and $x_1' > 0$, and $x_0'^2 - z_0^2 = 1$, $x_1'^2 - z_1^2 = 1$. The equation $x^2 - z^2 = 1\,,\ x > 0$ gives obviously one branch of the hyperbola $x = \sqrt{1+z^2}$ and all points on this branch are connected to each other. The hyperboloid is a surface of revolution, and it can be obtained from the hyperbola $x = \sqrt{1+z^2}$ by rotating it about the z-axis.

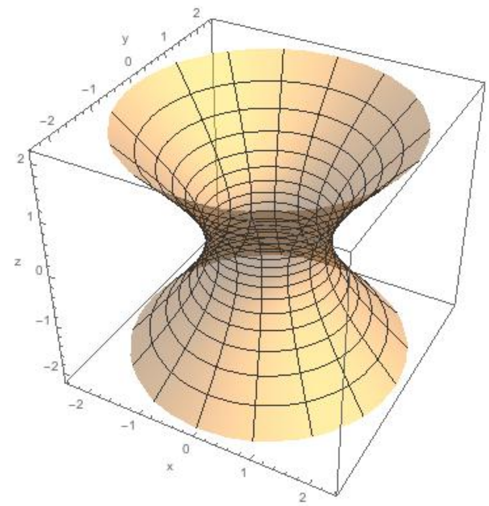
9. Sketch the graph of the function and identify it

   (a) $z = -\sqrt{2x - x^2 - y^2}$

   Solution: It is a semi-sphere centred at $(1, 0, 0)$.

   (b) $z = \sqrt{2y - y^2}$

   Solution: It is half of a cylinder of radius 1 with the axis of symmetry parallel to the y-axis and through $(1, 0, 0)$.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