MA1125 – Calculus
Tutorial solutions #9

1. Compute each of the following indefinite integrals.

$$\int \frac{x^3 - x}{x^2 + 5}\, dx, \qquad \int \frac{x^2 + 5}{x^3 - x}\, dx.$$

When it comes to the first integral, one may use division of polynomials to write

$$\int \frac{x^3 - x}{x^2 + 5}\, dx = \int \left(x - \frac{6x}{x^2 + 5} \right) dx.$$

To integrate the fraction, we let $u = x^2 + 5$. Since $du = 2x\, dx$, we find that

$$\begin{aligned} \int \frac{x^3 - x}{x^2 + 5}\, dx &= \frac{x^2}{2} - \int \frac{6x\, dx}{x^2 + 5} = \frac{x^2}{2} - \int \frac{3\, du}{u} \\ &= \frac{x^2}{2} - 3 \ln u + C = \frac{x^2}{2} - 3 \ln(x^2 + 5) + C. \end{aligned}$$

When it comes to the second integral, one may use partial fractions to write

$$\frac{x^2 + 5}{x^3 - x} = \frac{x^2 + 5}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

for some constants A, B and C. Clearing denominators gives rise to the identity

$$x^2 + 5 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

and this should be valid for all x. Let us then look at some special values of x to get

$$x = -1, 0, 1 \quad \Longrightarrow \quad 6 = 2C, \qquad 5 = -A, \qquad 6 = 2B.$$

This gives $A = -5$ and $B = C = 3$, so the second integral can be expressed in the form

$$\begin{aligned} \int \frac{x^2 + 5}{x^3 - x}\, dx &= \int \left(-\frac{5}{x} + \frac{3}{x-1} + \frac{3}{x+1} \right) dx \\ &= -5 \ln|x| + 3 \ln|x-1| + 3 \ln|x+1| + C. \end{aligned}$$

2. Compute each of the following indefinite integrals.

$$\int \frac{\sqrt{x}}{x+1}\,dx, \qquad \int \frac{\sqrt{x}}{x-1}\,dx.$$

In each case, we let $u = \sqrt{x}$ to simplify. Since $x = u^2$, we have $dx = 2u\,du$ and

$$\int \frac{\sqrt{x}}{x+1}\,dx = \int \frac{u}{u^2+1} \cdot 2u\,du = \int \frac{2u^2}{u^2+1}\,du.$$

This is a rational function that can be simplified using division of polynomials, so

$$\begin{aligned}\int \frac{\sqrt{x}}{x+1}\,dx = \int \frac{2(u^2+1)-2}{u^2+1}\,du &= \int \left(2 - \frac{2}{u^2+1}\right) du \\ &= 2u - 2\tan^{-1} u + C = 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C.\end{aligned}$$

For the second integral, we proceed in a similar fashion to find that

$$\int \frac{\sqrt{x}}{x+1}\,dx = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right) du.$$

In this case, however, one needs to use partial fractions to write

$$\frac{2}{u^2-1} = \frac{2}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1}$$

for some constants A, B that need to be determined. Clearing denominators gives

$$2 = A(u-1) + B(u+1),$$

so we may take $u = \pm 1$ to find that $2B = 2 = -2A$. It easily follows that

$$\begin{aligned}\int \frac{\sqrt{x}}{x-1}\,dx = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right) du &= 2u + \ln|u-1| - \ln|u+1| + C \\ &= 2\sqrt{x} + \ln|\sqrt{x}-1| - \ln|\sqrt{x}+1| + C.\end{aligned}$$

3. Compute each of the following indefinite integrals.

$$\int e^{2x}\cos(e^x)\,dx, \qquad \int \frac{\sin^3 x}{\cos^6 x}\,dx.$$

For the first integral, we let $u = e^x$. Since $du = e^x\,dx$, one finds that

$$\int e^{2x}\cos(e^x)\,dx = \int e^x\cos(e^x)\cdot e^x\,dx = \int u\cos u\,du.$$

Next, we integrate by parts with $dv = \cos u\, du$. This gives $v = \sin u$ and so

$$\begin{aligned}\int e^{2x}\cos(e^x)\,dx = u\sin u - \int \sin u\,du &= u\sin u + \cos u + C\\ &= e^x\sin(e^x) + \cos(e^x) + C.\end{aligned}$$

For the second integral, it is better to simplify the given expression and write

$$\int \frac{\sin^3 x}{\cos^6 x}\,dx = \int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \sec^3 x\cdot\tan^3 x\,dx.$$

To compute this integral, we let $u = \sec x$. Then $du = \sec x\tan x\,dx$ and we get

$$\begin{aligned}\int \frac{\sin^3 x}{\cos^6 x}\,dx &= \int \sec^2 x\cdot\tan^2 x\cdot\sec x\tan x\,dx = \int u^2(u^2-1)\,du\\ &= \int (u^4-u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.\end{aligned}$$

4. Show that each of the following sequences converges.

$$a_n = \sqrt{\frac{n^2+1}{n^3+2}}, \qquad b_n = \frac{\sin n}{n^2}, \qquad c_n = n^{1/n}.$$

Since the limit of a square root is the square root of the limit, it should be clear that

$$\lim_{n\to\infty}\frac{n^2+1}{n^3+2} = \lim_{n\to\infty}\frac{n^2}{n^3} = \lim_{n\to\infty}\frac{1}{n} = 0 \quad\Longrightarrow\quad \lim_{n\to\infty} a_n = \sqrt{0} = 0.$$

The limit of the second sequence is also zero because $-1/n^2 \leq b_n \leq 1/n^2$ for each $n \geq 1$. This means that b_n lies between two sequences that converge to zero. Finally, one has

$$c_n = n^{1/n} \quad\Longrightarrow\quad \ln c_n = \ln n^{1/n} = \frac{\ln n}{n}.$$

Since $\ln n \to \infty$ as $n \to \infty$, one may use L'Hôpital's rule to conclude that

$$\lim_{n\to\infty}\ln c_n = \lim_{n\to\infty}\frac{1/n}{1} = 0 \quad\Longrightarrow\quad \lim_{n\to\infty} c_n = e^0 = 1.$$

5. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$ for each $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 3$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.

Since the first two terms are $a_1 = 1$ and $a_2 = \sqrt{7}$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 3$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 7 \leq 6 + a_n \leq 6 + a_{n+1} \leq 9 \quad &\Longrightarrow \quad \sqrt{7} \leq a_{n+1} \leq a_{n+2} \leq 3 \\ &\Longrightarrow \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 3. \end{aligned}$$

In particular, the statement holds for $n + 1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = \sqrt{6 + a_n} \quad \Longrightarrow \quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6 + a_n} \quad \Longrightarrow \quad L = \sqrt{6 + L}.$$

This leads to the quadratic equation $L^2 = 6 + L$ which implies that $L = -2, 3$. Since the terms of the sequence satisfy $1 \leq a_n \leq 3$, however, the limit must be $L = 3$.

6. Use the formula for a geometric series to compute each of the following sums.

$$\sum_{n=0}^{\infty} \frac{2^n}{7^n}, \qquad \sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{3n+1}}, \qquad \sum_{n=2}^{\infty} \frac{3^{n+1}}{2^{4n-3}}.$$

The first sum is the sum of a geometric series with $x = 2/7$ and one easily finds that

$$\sum_{n=0}^{\infty} \frac{2^n}{7^n} = \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n = \frac{1}{1 - 2/7} = \frac{7}{5}.$$

The second sum is the sum of a geometric series with $x = 3/8$ and we similarly get

$$\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{3n+1}} = \frac{3^2}{2} \sum_{n=1}^{\infty} \left(\frac{3}{8}\right)^n = \frac{9}{2} \cdot \frac{3/8}{1 - 3/8} = \frac{27}{10}.$$

To compute the third sum, we shift the index of summation to conclude that

$$\sum_{n=2}^{\infty} \frac{3^{n+1}}{2^{4n-3}} = \sum_{n=0}^{\infty} \frac{3^{n+3}}{2^{4(n+2)-3}} = \frac{27}{32} \sum_{n=0}^{\infty} \left(\frac{3}{16}\right)^n = \frac{27}{32} \cdot \frac{1}{1 - 3/16} = \frac{27}{26}.$$

7. Infinite sums of continuous functions need not be continuous. In fact, show that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

is a geometric series that satisfies $f(x) = 1 + x^2$ for all $x \neq 0$, while $f(0) = 0$.

It is clear that $f(0) = 0$. To verify the other assertion, we note that

$$f(x) = x^2 \sum_{n=0}^{\infty} \left(\frac{1}{1+x^2} \right)^n.$$

This geometric series converges if and only if $\frac{1}{1+x^2} < 1$, hence if and only if $x \neq 0$. Since

$$f(x) = x^2 \cdot \frac{1}{1 - \frac{1}{1+x^2}} = x^2 \cdot \frac{1+x^2}{x^2},$$

we have $f(x) = 1 + x^2$ for all $x \neq 0$ and $f(0) = 0$. Thus, f is not continuous at $x = 0$.

8. Compute each of the following indefinite integrals.

$$\int \frac{x+2}{x^2+4x+8}\, dx, \qquad \int \frac{5x+7}{x^2+4x+8}\, dx.$$

For the first integral, we let $u = x^2 + 4x + 8$. Since $du = (2x+4)\, dx$, this gives

$$\int \frac{x+2}{x^2+4x+8}\, dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 4x + 8| + C.$$

The second integral can be easily related to the first integral by writing

$$\int \frac{5x+7}{x^2+4x+8}\, dx = \int \frac{5(x+2)-3}{x^2+4x+8}\, dx = \frac{5}{2} \ln |x^2+4x+8| - 3 \int \frac{dx}{x^2+4x+8}.$$

Once we now complete the square, we may let $u = (x+2)/2$ to conclude that

$$\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{(x+2)^2+4} = \int \frac{2\, du}{4u^2+4} = \frac{1}{2} \int \frac{du}{u^2+1}.$$

This integral is merely $\frac{1}{2} \tan^{-1} u$, so the original integral is equal to

$$\begin{aligned} \int \frac{5x+7}{x^2+4x+8}\, dx &= \frac{5}{2} \ln |x^2+4x+8| - \frac{3}{2} \tan^{-1} u + C \\ &= \frac{5}{2} \ln |x^2+4x+8| - \frac{3}{2} \tan^{-1} \frac{x+2}{2} + C. \end{aligned}$$

9. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = 3 + \sqrt{a_n}$ for each $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 9$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.

Since the first two terms are $a_1 = 1$ and $a_2 = 3 + 1 = 4$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 9$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 1 \leq \sqrt{a_n} \leq \sqrt{a_{n+1}} \leq 3 \quad &\Longrightarrow \quad 4 \leq a_{n+1} \leq a_{n+2} \leq 6 \\ &\Longrightarrow \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 9. \end{aligned}$$

In particular, the statement holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = 3 + \sqrt{a_n} \quad \Longrightarrow \quad \lim_{n\to\infty} a_{n+1} = 3 + \lim_{n\to\infty} \sqrt{a_n} \quad \Longrightarrow \quad L = 3 + \sqrt{L}.$$

This gives the quadratic equation $(L-3)^2 = L$, which one may easily solve to get

$$L^2 - 6L + 9 = L \quad \Longrightarrow \quad L^2 - 7L + 9 = 0 \quad \Longrightarrow \quad L = \frac{7 \pm \sqrt{13}}{2}.$$

Since $L - 3 = \sqrt{L} \geq 0$, however, we also have $L \geq 3$ and the limit is $L = \frac{1}{2}(7 + \sqrt{13})$.

10. Suppose the series $\sum_{n=1}^{\infty} a_n$ converges. Show that the series $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ diverges.

Since $\sum_{n=1}^{\infty} a_n$ converges, we must have $\lim_{n\to\infty} a_n = 0$ by the nth term test, so

$$\lim_{n\to\infty} \frac{1}{1+a_n} = 1.$$

Using the nth term test once again, we conclude that $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ diverges.