Mastering physics

- Rounding tolerance for numerical answers is 2% by default.

- Default precision: Always state the result with **3 significant digits**, e.g. 15.15789=15.2 .

- If units are required, a separate answer box for it will be there. Single box means numerical answer only. **In my questions I only ask you for the numerical answer (in units that are stated in the question)**

- Introduction to Mastering Physics assignment available (not graded) to get used to it

Statistics assignment available: Monday 15.10, 9:00

Due date: Monday 5.11, 9:00

Problem set solutions

Question 1

Let x be a continuous random variable governed by the following probability density function:

$$f(x) = \begin{cases} C(x^2 - 1) \; for \; 1 \leq x \leq 3 \\ 0 \qquad elsewhere \end{cases}$$

(a) Use the normalisation condition to find the value of C.

(b) Find the mean of $f(x)$.

(c) Find the second moment of $f(x)$.

(d) Use the definition of the variance to show, that in general, it is given by $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$. Use this formula to calculate the variance of $f(x)$.

Question 1

(a) Normalization requires

$$\int_1^3 C(x^2 - 1)dx = 1$$

$$C\left[\frac{1}{3}x^3 - x\right]_1^3 = 1$$

$$C\left(9 - 3 - \frac{1}{3} + 1\right) = 1$$

$$C = \frac{3}{20}$$

Question 1

(b) Find the mean

$$\langle x \rangle = \int_1^3 xP(x)dx = C\int_1^3 x(x^2-1)dx$$

$$= C\left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_1^3 = C\left(\frac{81}{4} - \frac{9}{2} - \frac{1}{4} + \frac{1}{2}\right)$$

$$C\frac{64}{4} = \frac{3}{20}16 = \frac{12}{5} = 2.4$$

Question 1

(c) Find the second moment:

$$\langle x^2 \rangle = \int_1^3 x^2 P(x) dx = C \int_1^3 x^2 (x^2 - 1) dx$$

$$= C \left[\frac{1}{5} x^5 - \frac{1}{3} x^3 \right]_1^3 = \frac{3}{20} \left(\frac{243}{5} - \frac{27}{3} - \frac{1}{5} + \frac{1}{3} \right)$$

$$= 5.96$$

Question 1

Prove $\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$

$$\sigma^2 = \int (x - \langle x \rangle)^2 P(x) dx$$

$$\sigma^2 = \int (x^2 - 2x\langle x \rangle + \langle x \rangle^2) \quad P(x) dx$$

$$\sigma^2 = \int x^2 P(x) dx - 2\langle x \rangle \int x P(x) dx + \langle x \rangle^2 \int P(x) dx$$

$$= \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= 5.96 - (2.4)^2 = 0.2$$

Question 2

A student monitors the activity of a radioactive source with a detector. The number of counts measured in a given time window obey the Poisson distribution $P_{\mu}(x) = e^{-\mu}\frac{\mu^{x}}{x!}$.

(a) Show explicitly that the mean of this distribution is μ. [Hint: Start with the normalisation condition.]

(b) In the course of 10 minutes, the detector registers a total of 2540 counts. What is the corresponding rate R_{tot} (in counts per minute) and its uncertainty?

(c) The student removes the source in order to record counts due to the background radiation. The detector registers 95 counts in the 3 minutes. What is the corresponding rate of the background,R_{bkg} , and its uncertainty ?

(d) What are the rate and its uncertainty due to the radioactive source alone?

Question 2

(a) Start from normalization: $\sum_{x=0}^{\infty} P_{\mu}(x) = \sum_{x=0}^{\infty} e^{-\mu}\, \frac{\mu^x}{x!} = 1$

Differentiate w.r.t μ (use product rule):

$$\sum_{x=0}^{\infty} -e^{-\mu}\, \frac{\mu^x}{x!} + x e^{-\mu}\, \frac{\mu^{x-1}}{x!} = 0$$

$$-\sum_{x=0}^{\infty} e^{-\mu}\, \frac{\mu^x}{x!} + \sum_{x=0}^{\infty} x e^{-\mu}\, \frac{\mu^{x-1}}{x!} = 0$$

$$-1 + \sum_{x=0}^{\infty} x e^{-\mu}\, \frac{\mu^{x-1}}{x!} = 0$$

$$\sum_{x=0}^{\infty} x e^{-\mu}\, \frac{\mu^{x-1}}{x!} = 1$$

Multiplying both sides with μ, we get

$$\sum_{x=0}^{\infty} x e^{-\mu}\, \frac{\mu^x}{x!} = \sum_{x=0}^{\infty} x P_{\mu}(x) = \langle x \rangle = \mu$$

Question 2

(b)

$$R_{tot} = \frac{2540 \pm \sqrt{2540}}{10min} = 254 \pm 5 \text{ min}^{-1}$$

(c)

$$R_{bg} = \frac{95 \pm \sqrt{95}}{3min} = 32 \pm 3 \text{ min}^{-1}$$

(d)

$$R_{sr} = R_{tot} - R_{bg} = (254 \pm 5 \text{ min}^{-1}) - (32 \pm 3 \text{ min}^{-1})$$
$$= 222 \pm 6$$

Where error has been found through $\sqrt{5^2 + 3^2}$

(subtraction: absolute errors add in quadrature)

Question 3

a) What is the name of the distribution that governs the number of non-overlapping events that have a low probability of occurring during a given time window?

b) A new medicine A is suspected of causing a rare disease. In a clinical trial with 1200 patients taking this medicine, 2 people contract this rare disease. What is the uncertainty in this number of incidents?

c) The clinical trial ran over a period of 2 years. Based on your answer in (b), state the number of incidents of this rare disease per 1000 people per year taking medicine A including its uncertainty.

d) In Ireland, 4870 people have contracted this disease in the last 3 years. The population in Ireland is 4.6 million. Calculate the incidence rate including its uncertainty per 1000 people per year.

e) Assuming that nobody in Ireland has taken medicine A, is there enough evidence to conclude that medicine A is causing this rare disease? Justify your answer.

Question 3

a) Poisson distribution

b) $\sqrt{2} = 1.41$

c) Incidence rate per year per 1000 people: $\frac{2}{1200} 1000 \frac{1}{2} = 0.83$, uncertainty = $\frac{\sqrt{2}}{1200} 1000 \frac{1}{2} = 0.59$.
$\text{Answer } 0.83 \pm 0.59$

d) Incidence rate per year per 1000 people: $\frac{4870}{4.6 \cdot 10^6} 1000 \frac{1}{3} = 0.353$, uncertainty = $\frac{\sqrt{4870}}{4.6 \cdot 10^6} 1000 \frac{1}{3} = 0.005$.
$\text{Answer } 0.353 \pm 0.005$

e) No, error bars of the two incidence rates overlap.
Alternatively, compute difference in incidence rate:
$0.83 - 0.35 = 0.48 \pm \sqrt{0.59^2 + 0.005^2} = 0.48 \pm 0.59$
Error bar in difference is larger than the difference itself.

Question 4

a) Given two independent variables p and q, with an associated uncertainty Δp and Δq, state the general error propagation formula that yields the uncertainty of a function $f(p, q)$ of these two variables.

b) The lens formula relates the focal length f to the image and object distance given by p and q, respectively: $f = \frac{pq}{p+q}$

c) Show that the error in f is given by $\Delta f = \frac{\sqrt{q^4 \Delta p^2 + p^4 \Delta q^2}}{(p+q)^2}$

d) For $p = (3.1 \pm 0.1)cm$ and $q = (6.5 \pm 0.1)cm$, calculate the focal length including its uncertainty.

Question 4

a) $$\Delta f = \sqrt{\left(\frac{\partial f}{\partial p}\Delta p\right)^2 + \left(\frac{\partial f}{\partial q}\Delta q\right)^2}$$

b) $$\frac{\partial f}{\partial p} = \frac{(p+q)\cdot q - pq\cdot 1}{(p+q)^2} = \frac{q^2}{(p+q)^2}$$

$$\frac{\partial f}{\partial q} = \frac{(p+q)\cdot p - pq\cdot 1}{(p+q)^2} = \frac{p^2}{(p+q)^2}$$

So $\Delta f = \frac{1}{(p+q)^2}\sqrt{(q^2\Delta p)^2 + (p^2\Delta q)^2}$

c) Substituting the numbers gives $f = (2.10 \pm 0.05)cm$.

Note: Uncertainty only needs to be stated with one (or at most two) significant figure(s). Last significant figure of result should be of the same order of magnitude as the uncertainty.