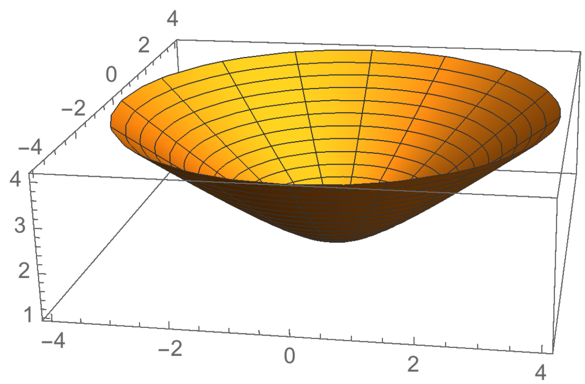
Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 2
Each set of homework questions is worth 100 marks

Problem 1. Consider a particle of mass m moving on the surface
$z = k\sqrt{x^2 + y^2 + c^2}$, $k > 0$, $c > 0$ in a uniform gravitational field $\vec{F} = \{0, 0, -mg\}$.

(a) What is the surface $z = k\sqrt{x^2 + y^2 + c^2}$, $k > 0$?

Use Mathematica to plot the surface for $k = c = 1$.

Answer: It is the upper sheet of a hyperboloid of two sheets obtained by revolving the hyperbola $z^2 = k^2(x^2 + c^2)$ in the xz-plane about the z-axis.



(b) Find the Lagrangian of the particle by using the polar coordinates r, ϕ.

Answer: Without the constraint the Lagrangian would be

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz\,. \tag{0.1}$$

Due to the rotational symmetry about the z-axis it is convenient to use the polar coordinates

$$x = r\cos\phi\,, \quad y = r\sin\phi\,, \tag{0.2}$$

in terms of which the constraint becomes $z = k\sqrt{r^2 + c^2}$, and L is

$$\begin{aligned} L &= \frac{m}{2}\Big(\big(1 + \frac{k^2 r^2}{r^2 + c^2}\big)\dot{r}^2 + r^2\dot{\phi}^2\Big) - mgk\sqrt{r^2 + c^2} \\ &= \frac{m}{2}\Big(\big(1 + k^2 - \frac{k^2 c^2}{r^2 + c^2}\big)\dot{r}^2 + r^2\dot{\phi}^2\Big) - mgk\sqrt{r^2 + c^2}\,. \end{aligned} \tag{0.3}$$

(c) Find the equations of motion of the particle.

Answer: One finds

$$
\begin{aligned}
&\frac{\partial L}{\partial \dot r} = m\big(1+k^2-\frac{k^2c^2}{r^2+c^2}\big)\dot r\,, \quad \frac{\partial L}{\partial r} = \frac{mk^2c^2r}{(r^2+c^2)^2}\dot r^2 + mr\dot\phi^2 - \frac{mgkr}{\sqrt{r^2+c^2}}\,,\\
&\frac{\partial L}{\partial \dot\phi} = mr^2\dot\phi\,, \quad \frac{\partial L}{\partial \phi} = 0\,,
\end{aligned} \tag{0.4}
$$

and therefore the eom are

$$
\begin{aligned}
&\big(1+k^2-\frac{k^2c^2}{r^2+c^2}\big)\ddot r + \frac{2k^2c^2r}{(r^2+c^2)^2}\dot r^2 = \frac{k^2c^2r}{(r^2+c^2)^2}\dot r^2 + r\dot\phi^2 - \frac{gkr}{\sqrt{r^2+c^2}} \;\Rightarrow\\
&\big(1+k^2-\frac{k^2c^2}{r^2+c^2}\big)\ddot r = -\frac{k^2c^2r}{(r^2+c^2)^2}\dot r^2 + r\dot\phi^2 - \frac{gkr}{\sqrt{r^2+c^2}}\,,\\
&\frac{d}{dt}r^2\dot\phi = 0 \;\Rightarrow\; r^2\dot\phi = const\,.
\end{aligned} \tag{0.5}
$$

Problem 2. Consider a particle of mass m moving on the surface

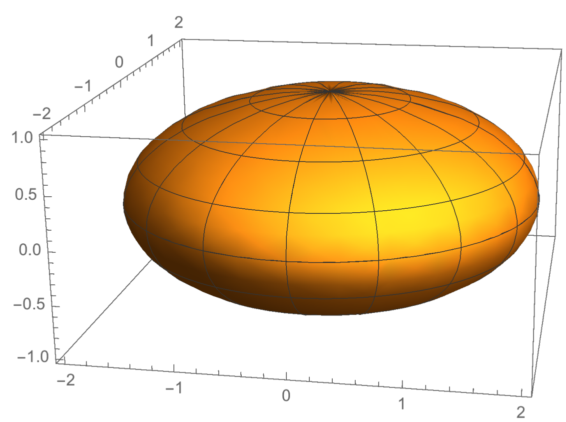
$$x^2+y^2+\frac{z^2}{\kappa^2} = a^2\,, \quad a>0\,,\ \kappa>0$$

in a uniform gravitational field $\vec F = \{0,0,-mg\}$.

(a) What is the surface $x^2+y^2+\frac{z^2}{\kappa^2} = a^2\,, \quad a>0\,,\ \kappa>0$?

Use Mathematica to plot the surface for $a=2\,,\ \kappa=1/2$.

Answer: It is the ellipsoid obtained by revolving the ellipse $x^2+\frac{z^2}{\kappa^2}=a^2$ with the semi-axis a and $a\kappa$ in the xz-plane about the z-axis.

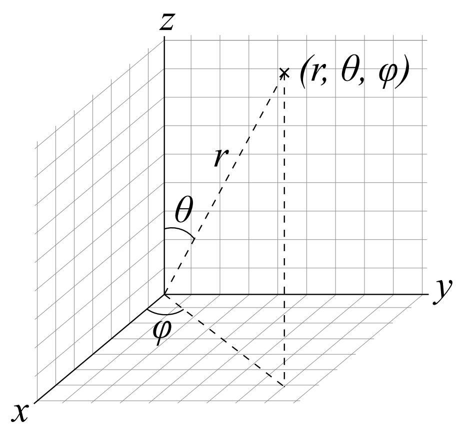


(b) Introduce the spherical coordinates by using the physics conventions $(r\,,\theta\,,\varphi)$ (radial, polar, azimuthal), and draw the corresponding picture.

Answer: The spherical coordinates are

$$x = r\cos\varphi\sin\theta\,, \quad y = r\sin\varphi\sin\theta\,, \quad z = r\cos\theta\,, \tag{0.6}$$

see the picture



(c) Introduce coordinates $(\rho\,,\,\theta\,,\,\varphi)$ similar to the spherical ones so that the equation of the surface $x^2 + y^2 + \frac{z^2}{\kappa^2} = a^2\,, \quad a > 0\,,\ \kappa > 0$ in terms of these coordinates becomes $\rho = a$, and derive an expression for the Lagrangian of the particle in term of these coordinates.

Answer: It is convenient to use the coordinates

$$x = \rho\cos\varphi\sin\theta\,, \quad y = \rho\sin\varphi\sin\theta\,, \quad z = \kappa\,\rho\cos\theta\,. \tag{0.7}$$

In terms of these coordinates the constraint becomes $\rho = a$.

Without the constraint the Lagrangian would be

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz\,, \tag{0.8}$$

and in terms of the coordinates $(\theta\,,\,\varphi)$, L becomes

$$\begin{aligned} L &= \frac{ma^2}{2}\big((\cos^2\theta + \kappa^2\sin^2\theta)\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\big) - mg\kappa a\cos\theta \\ &= \frac{ma^2}{2}\big((1 + (\kappa^2 - 1)\sin^2\theta)\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\big) - mg\kappa a\cos\theta\,. \end{aligned} \tag{0.9}$$

(d) Find the equations of motion of the particle.

Answer: One finds

$$\begin{aligned} &\frac{\partial L}{\partial\dot{\theta}} = ma^2(1 + (\kappa^2 - 1)\sin^2\theta)\dot{\theta}\,, \quad \frac{\partial L}{\partial\theta} = ma^2\sin\theta\cos\theta\dot{\varphi}^2 + mg\kappa a\sin\theta\,, \\ &\frac{\partial L}{\partial\dot{\varphi}} = ma^2\sin^2\theta\dot{\varphi}\,, \quad \frac{\partial L}{\partial\varphi} = 0\,, \end{aligned} \tag{0.10}$$

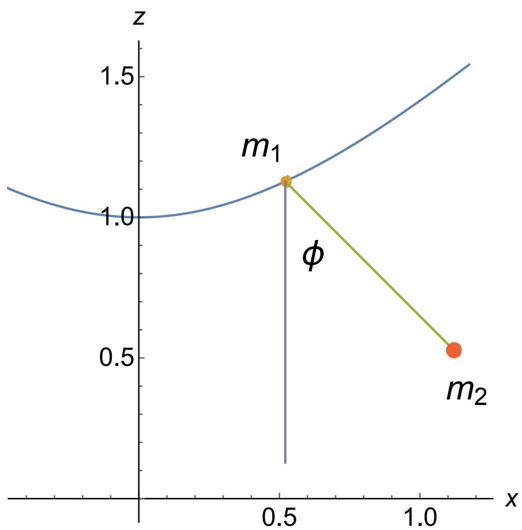
and therefore the eom are

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\varphi}^2 + \frac{g}{a}\sin\theta\,, \quad \frac{d}{dt}\sin^2\theta\,\dot{\varphi} = 0 \;\Rightarrow\; \sin^2\theta\,\dot{\varphi} = const\,. \tag{0.11}$$

Problem 3. Consider a pendulum of mass m_2, with a mass m_1 at the point of support which can move on a curve in the vertical xz-plane defined parametrically by the equations $x = f(q)\,,\ z = h(q)$, where q is a parameter of the curve. Assume that the motion takes place only in the vertical xz-plane. The potential energy of the system is

$$U = m_1 g z_1 + m_2 g z_2\,,$$

where x_1 and x_2 are the coordinates of the particles.



(a) Find the Lagrangian of the system.

Answer: Without the constraints the Lagrangian would be

$$L = \frac{m_1}{2}(\dot{x}_1^2 + \dot{z}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{z}_2^2) - m_1 g z_1 - m_2 g z_2\,. \tag{0.12}$$

The constraints are

$$C_1 = x_1 - f(q) = 0\,, \quad C_2 = z_1 - h(q) = 0\,, \quad C_3 = (x_2 - x_1)^2 + (z_2 - z_1)^2 - l^2 = 0\,. \tag{0.13}$$

Using q and the angle ϕ between the rod connecting m_1 and m_2 and the vertical as generalised coordinates, the solution of the constraints is

$$x_1 = f(q)\,, \quad z_1 = h(q)\,, \quad x_2 = f(q) + l\sin\phi\,, \quad z_2 = h(q) - l\cos\phi\,. \tag{0.14}$$

Substituting the solution into L, one gets

$$\begin{aligned} L = {} & \frac{m_1}{2}(f'^2 + h'^2)\dot{q}^2 + \frac{m_2}{2}((f'\dot{q} + l\dot{\phi}\cos\phi)^2 + (h'\dot{q} - l\dot{\phi}\sin\phi)^2) \\ & - (m_1 + m_2) g h(q) + m_2 g l \cos\phi\,, \end{aligned} \tag{0.15}$$

where $f' = \frac{df}{dq}$, $h' = \frac{dh}{dq}$.

(b) Assume that $q = s$ where s is an arc length parameter, simplify the Lagrangian and find the eom.

Answer: If $q = s$ is an arc length parameter then $f'^2 + h'^2 = 1$, and L takes the form

$$\begin{aligned} L = &\frac{m_1 + m_2}{2}\dot{s}^2 + \frac{m_2}{2} l^2 \dot{\phi}^2 + m_2 l (f' \cos\phi - h' \sin\phi)\dot{s}\dot{\phi} \\ &- (m_1 + m_2) g h(s) + m_2 g l \cos\phi \,. \end{aligned} \tag{0.16}$$

One then finds

$$\begin{aligned} &\frac{\partial L}{\partial \dot{s}} = (m_1 + m_2)\dot{s} + m_2 l (f' \cos\phi - h' \sin\phi)\dot{\phi}\,, \quad \frac{\partial L}{\partial s} = m_2 l (f'' \cos\phi - h'' \sin\phi)\dot{s}\dot{\phi} \\ &\qquad - (m_1 + m_2) g h'(s)\,, \\ &\frac{\partial L}{\partial \dot{\phi}} = m_2 l^2 \dot{\phi} + m_2 l (f' \cos\phi - h' \sin\phi)\dot{s}\,, \quad \frac{\partial L}{\partial \phi} = m_2 l(-f' \sin\phi - h' \cos\phi)\dot{s}\dot{\phi} - m_2 g l \sin\phi\,, \end{aligned} \tag{0.17}$$

and therefore the eom are

$$\begin{aligned} &(m_1 + m_2)\ddot{s} + m_2 l \frac{d}{dt}\big((f' \cos\phi - h' \sin\phi)\dot{\phi}\big) = m_2 l (f'' \cos\phi - h'' \sin\phi)\dot{s}\dot{\phi} - (m_1 + m_2) g h'(s)\,, \\ &l^2 \ddot{\phi} + l \frac{d}{dt}\big((f' \cos\phi - h' \sin\phi)\dot{s}\big) = -l\big((f' \sin\phi + h' \cos\phi)\dot{s}\dot{\phi} + g \sin\phi\big)\,. \end{aligned} \tag{0.18}$$

(c) Let the curve be a hyperbola:

$$-\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1\,, \quad z > 0\,.$$

Introduce any parametrisation of the hyperbola, identify $f(q)$ and $g(q)$, and write the Lagrangian.

Use Mathematica to find an arc length parameter of the hyperbola as a function of your parameter.

Answer:

This hyperbola can be parametrised as

$$x = f(q) = a \sinh q\,, \quad z = g(q) = b \cosh q\,, \tag{0.19}$$

where q is a parameter which plays the role of the generalised coordinate for the first particle. The Lagrangian then becomes

$$\begin{aligned} L = &\frac{m_1 + m_2}{2}(a^2 \cosh^2 q + b^2 \sinh^2 q)\dot{q}^2 + \frac{m_2}{2} l^2 \dot{\phi}^2 + m_2 l (a \cosh q \cos\phi - b \sinh q \sin\phi)\dot{q}\dot{\phi} \\ &- (m_1 + m_2) g a \cosh q + m_2 g l \cos\phi\,, \end{aligned} \tag{0.20}$$

An arc length parameter can be found by solving the equation

$$ds^2 = dx^2 + dz^2 = (a^2 \cosh^2 q + b^2 \sinh^2 q) dq^2 \,, \tag{0.21}$$

and therefore choosing $s = 0$ at $q = 0$, one gets

$$s = \int_0^q \sqrt{(a^2 \cosh^2 w + b^2 \sinh^2 w)}\, dw = -iaE\big(iq \big| 1 + \frac{b^2}{a^2}\big) \,, \tag{0.22}$$

where $E(q|m)$ is the incomplete elliptic integral of the second kind.