MA1125 – Calculus
Homework #8 solutions

1. Compute each of the following indefinite integrals.

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx, \qquad \int x\sqrt{1-x}\,dx.$$

For the first integral, we let $u=\sqrt{x}$. This gives $x=u^2$ and $dx=2u\,du$, so

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\sin u}{u}\cdot 2u\,du = \int 2\sin u\,du = -2\cos u + C = -2\cos\sqrt{x} + C.$$

For the second integral, we let $u=1-x$. This gives $x=1-u$ and $dx=-du$, so

$$\begin{aligned}\int x\sqrt{1-x}\,dx &= -\int (1-u)\sqrt{u}\,du = \int (u^{3/2}-u^{1/2})\,du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C.\end{aligned}$$

2. Compute each of the following indefinite integrals.

$$\int \sin^3 x\cdot\cos^4 x\,dx, \qquad \int \tan^2 x\cdot\sec^6 x\,dx.$$

For the first integral, we use the substitution $u=\cos x$. Since $du=-\sin x\,dx$, we get

$$\begin{aligned}\int \sin^3 x\cdot\cos^4 x\,dx &= \int \sin^2 x\cdot\cos^4 x\cdot\sin x\,dx = -\int (1-u^2)\cdot u^4\,du \\ &= \int (u^6-u^4)\,du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C.\end{aligned}$$

For the second integral, we use the substitution $u=\tan x$. Since $du=\sec^2 x\,dx$, we get

$$\begin{aligned}\int \tan^2 x\cdot\sec^6 x\,dx &= \int \tan^2 x\cdot\sec^4 x\cdot\sec^2 x\,dx = \int u^2(1+u^2)^2\,du \\ &= \int (u^2+u^6+2u^4)\,du = \frac{1}{3}u^3 + \frac{1}{7}u^7 + \frac{2}{5}u^5 + C \\ &= \frac{\tan^3 x}{3} + \frac{\tan^7 x}{7} + \frac{2\tan^5 x}{5} + C.\end{aligned}$$

3. Compute each of the following indefinite integrals.

$$\int x^3(\ln x)^2\,dx, \qquad \int x^3\sqrt{4-x^2}\,dx.$$

For the first integral, let $u=(\ln x)^2$ and $dv=x^3\,dx$. Then $du=\frac{2\ln x}{x}\,dx$ and $v=\frac{x^4}{4}$, so

$$\int x^3(\ln x)^2\,dx=\frac{x^4}{4}(\ln x)^2-\int\frac{2\ln x}{x}\cdot\frac{x^4}{4}\,dx=\frac{x^4}{4}(\ln x)^2-\frac{1}{2}\int x^3(\ln x)\,dx.$$

Next, we take $u=\ln x$ and $dv=x^3\,dx$. Since $du=\frac{1}{x}\,dx$ and $v=\frac{x^4}{4}$, we conclude that

$$\int x^3(\ln x)^2\,dx=\frac{x^4}{4}(\ln x)^2-\frac{x^4}{8}\ln x+\int\frac{x^3}{8}\,dx=\frac{x^4}{4}(\ln x)^2-\frac{x^4}{8}\ln x+\frac{x^4}{32}+C.$$

For the second integral, let $x=2\sin\theta$ for some angle $-\pi/2\leq\theta\leq\pi/2$. Then

$$\int x^3\sqrt{4-x^2}\,dx=\int 8\sin^3\theta\cdot\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta=32\int\sin^3\theta\cdot\cos^2\theta\,d\theta.$$

This can be further simplified by letting $u=\cos\theta$, in which case $du=-\sin\theta\,d\theta$ and

$$\begin{aligned}\int x^3\sqrt{4-x^2}\,dx&=-32\int(1-u^2)\cdot u^2\,du=32\int(u^4-u^2)\,du\\&=32\left(\frac{1}{5}u^5-\frac{1}{3}u^3\right)=32\left(\frac{\cos^5\theta}{5}-\frac{\cos^3\theta}{3}\right)+C.\end{aligned}$$

Since $4\cos^2\theta=4-4\sin^2\theta=4-x^2$, we also have $\cos\theta=\frac{1}{2}(4-x^2)^{1/2}$ and so

$$\int x^3\sqrt{4-x^2}\,dx=\frac{(4-x^2)^{5/2}}{5}-\frac{4(4-x^2)^{3/2}}{3}+C.$$

4. Find the area of the region enclosed by the graphs of $f(x)=e^{2x}$ and $g(x)=3e^x-2$.

Letting $z=e^x$ for simplicity, we get $f(x)=z^2$ and $g(x)=3z-2$. It easily follows that

$$f(x)\leq g(x)\quad\Longleftrightarrow\quad z^2\leq 3z-2\quad\Longleftrightarrow\quad(z-2)(z-1)\leq 0\quad\Longleftrightarrow\quad 1\leq z\leq 2.$$

In other words, $f(x)\leq g(x)$ if and only if $0\leq x\leq\ln 2$, so the area of the region is

$$\begin{aligned}\text{Area}&=\int_0^{\ln 2}[g(x)-f(x)]\,dx=\int_0^{\ln 2}(3e^x-2-e^{2x})\,dx\\&=\left[3e^x-2x-\frac{1}{2}e^{2x}\right]_0^{\ln 2}=\frac{3}{2}-2\ln 2.\end{aligned}$$

5. Find the volume of the solid that is obtained by rotating the graph of $f(x) = xe^x$ around the x-axis over the interval $[0, 1]$.

The volume of the solid is the integral of $\pi f(x)^2$ and this is given by

$$\text{Volume} = \pi \int_0^1 x^2 e^{2x}\, dx.$$

To simplify this expression, let $u = x^2$ and $dv = e^{2x}\, dx$. Then $du = 2x\, dx$ and $v = \frac{1}{2}e^{2x}$, so

$$\int_0^1 x^2 e^{2x}\, dx = \left[\frac{x^2}{2}e^{2x}\right]_0^1 - \int_0^1 xe^{2x}\, dx.$$

Once again, we take $u = x$ and $dv = e^{2x}\, dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$, so

$$\begin{aligned}\int_0^1 x^2 e^{2x}\, dx &= \left[\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x}\right]_0^1 + \frac{1}{2}\int_0^1 e^{2x}\, dx \\ &= \left[\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x}\right]_0^1.\end{aligned}$$

The volume of the solid is given by π times the last integral, so it is given by

$$\text{Volume} = \pi\left[\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x}\right]_0^1 = \frac{\pi(e^2 - 1)}{4}.$$