Advanced Calculus
MA1132
Homework Assignment 2

Kirk M. Soodhalter

ksoodha@maths.tcd.ie

To be completed and handed in AT THE BEGINNING of tutorial on Friday, 15. March

NO LATE ASSIGNMENTS WILL BE ACCEPTED. IF YOU CANNOT ATTEND TUTORIALS, PLEASE MAKE ARRANGEMENTS TO EMAIL YOUR SOLUTIONS TO YOUR TUTOR

0. (#13) Determine whether the limit exists. If so, find its value

   (a)
   $$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{e^{x^2+y^2}-1}$$

   (b)
   $$\lim_{(x,y)\to(1,-1)} \frac{1-\cosh(x+y)}{\sin(x^2-y^2)\ln(\frac{2x}{x-y})}$$

   (c)
   $$\lim_{(x,y)\to(0,0)} \frac{3+\cos(2x)-4\cosh(y)}{1-\sqrt[4]{1+x^2+y^2}}$$

1. Find all first and second order partial derivatives of the function
   $$f(x,y) = x\sin(y\ln(x)),$$
   and hence verify that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ for this function.

2. Let $f(x,y,z) = x\cos(x+y+z)$.

   (a) Find the directional derivative of f at the point $\left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\right)$ in the direction $(8,-4,-1)$.

   (b) Find the unit vectors in the directions in which f is increasing/decreasing most rapidly at the point $\left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\right)$, and give the rate of increase and decrease, respectively.

3. Find and classify the critical points of the function $f(x,y) = x^2y - 2xy^2 + 3xy + 4$.

4. Show that if $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\,.$$

5. Consider the surface

$$z = f(x, y) = \ln\left(\frac{1}{2}e^{2/3}\sqrt[3]{8x^2 - 6xy^2 - y^3 + 32 - 12\sin(2x - y)}\right)\,.$$

   (a) Find an equation for the tangent plane to the surface at the point $P = (1, 2, z_0)$ where $z_0 = f(1, 2)$.
   (b) Find points of intersection of the tangent plane with the x-, y- and z-axes.
   (c) Sketch the tangent plane, and show the point $P = (1, 2, z_0)$ on it.
   (d) Find parametric equations for the normal line to the surface at the point $P = (1, 2, z_0)$.
   (e) Sketch the normal line to the surface at the point $P = (1, 2, z_0)$.

6. Show that the equation of the plane that is tangent to the cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

at (x_0, y_0, z_0) can be written in the form

$$\frac{x_0}{a^2}x + \frac{y_0}{b^2}y - \frac{z_0}{c^2}z = 0\,.$$

7. Prove: If the surfaces $z = f(x_1, \ldots, x_n)$ and $z = g(x_1, \ldots, x_n)$ intersect at $P = (x_1^o, \ldots, x_n^o, z^o)$, and if f and g are differentiable at $(x_1^o, \ldots, x_n^o)$, then the normal lines at P are perpendicular if and only if

$$\sum_{i=1}^{n} \frac{\partial f(x_1^o, \ldots, x_n^o)}{\partial x_i}\frac{\partial g(x_1^o, \ldots, x_n^o)}{\partial x_i} = -1\,.$$

8. Consider the function

$$f(x, y) = x^2 - xy^2 - 3x + y^4 + 5$$

Locate all relative maxima, relative minima, and saddle points, if any. Use Mathematica to plot its graph.

9. Consider the function

$$z = 3e^{y - \frac{\pi}{4}}\cos x - 2e^{\frac{\pi}{2} - x}\sin y$$

   (a) Find

$$iii)\quad \frac{\partial^2 z}{\partial x \partial y}\left(\frac{\pi}{2}, \frac{\pi}{4}\right), \qquad iv)\quad \frac{\partial^2 z}{\partial y \partial x}\left(\frac{\pi}{2}, \frac{\pi}{4}\right).$$

(b) Find the slope of the surface $z = 3e^{y-\frac{\pi}{4}}\cos x - 2e^{\frac{\pi}{2}-x}\sin y$ in the y-direction at the point $(\frac{\pi}{3}, \frac{\pi}{6})$.

(c) Show that the function $z = 3e^{y-\frac{\pi}{4}}\cos x - 2e^{\frac{\pi}{2}-x}\sin y$ satisfies Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0\,.$$

10. The equations of motion of a system of n particles are given by

$$m_i\ddot{x}_i = -\frac{\partial U(x_1,\ldots,x_n)}{\partial x_i}\,, \quad \ddot{x}_i = \frac{d^2x_i}{dt^2}\,, \quad i = 1, 2, \ldots, n\,,$$

where m_i is the mass and x_i is the coordinate of the i-th particle, and $U(x_1,\ldots,x_n)$ is the potential energy of the system.

(a) Consider a system of n particles moving in a central field

$$U(x_1,\ldots,x_n) = V(r)\,, \quad r = \Big|\sum_{i=1}^{n} x_i\mathbf{e}_i\Big|\,,$$

where V is a smooth function of a single variable.

i. Find the equations of motion of the first particle (x_1).
ii. Find the equations of motion of the second particle (x_2).
iii. Find the equations of motion of the last particle (x_n).
iv. Find the equations of motion of the i-th particle (x_i) for $1 < i < n$.
v. Write the equations of motion of the i-th particle (x_i) for $1 \le i \le n$ by using the Kronecker delta δ_{ij}.

(b) Find the equations of motion of a system of n particles with the rational Calogero-Moser potential

$$U(x_1,\ldots,x_n) = \sum_{i,j=1,i\neq j}^{n} \frac{\alpha}{(x_i - x_j)^2}\,.$$

11. Compute the differential df of

$$f(x_1, x_2, \ldots, x_n) = (\frac{1}{2} + x_1)^{\alpha_1}(\frac{1}{2} + x_2)^{\alpha_2}\cdots(\frac{1}{2} + x_n)^{\alpha_n}\,,$$

and find its local linear approximation at $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

12. Consider the function $f(x, y, z) = \cos(x^2 + y^2 + z^2)$. Find the Taylor series expansion of $f(x, y, z)$ about the point $\mathbf{x}_0 = (0, 0, 0)$ up to the third order.

13. Consider the "Higgs" potential

$$U(x_1,\ldots,x_n) = -\frac{\kappa^2}{2}r^2 + \frac{\lambda^2}{4}r^4\,, \quad r = \Big|\sum_{i=1}^{n} x_i\mathbf{e}_i\Big|\,, \quad \kappa > 0, \lambda > 0\,.$$

(a) Plot the potential for $n = 1$, and for $\kappa = \lambda = 2$.

(b) Plot the potential for $n = 2$, and for $\kappa = \lambda = 2$.

(c) Find the Taylor series expansion of the "Higgs" potential about the point $x_1^o = \frac{\kappa}{\lambda}$, $x_i^o = 0$, $i = 2, \ldots, n$ up to the fourth order in $y_i \equiv x_i - x_i^o$. Use Mathematica to check your answer.