Advanced Calculus
MA1132
Exercises 9 Solutions

1. Consider the solid G bounded by the surface $x^2 + y^2 + z^2 = a^2$.

   (a) What is the surface $x^2 + y^2 + z^2 = a^2$?

   (b) Find the volume V of the solid G.

   (c) Find the moments of inertia of the solid G.

   (d) Find the gravitational force $\mathbf{F}(\xi, \eta, \zeta)$ exerted on a point particle located at (ξ, η, ζ) by the solid G.

   (e) Find the gravitational potential field $U(\xi, \eta, \zeta)$ of the solid G.

Solution:

(a) It is a sphere of radius a centred at the origin, and the solid is a ball of radius a.

(b) We use spherical coordinates $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$.

The volume V of G is

$$V = \iiint_G dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\phi \, dr d\phi d\theta = \frac{4\pi a^3}{3}\,. \tag{1}$$

(c) Due to the rotational symmetry $I_x = I_y = I_z = \frac{1}{3}(I_x + I_y + I_z)$, so

$$I_x = \frac{2}{3}\iiint_G (x^2 + y^2 + z^2)\, dV = \frac{2}{3}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^4 \sin\phi \, dr d\phi d\theta = \frac{8\pi a^5}{15} = \frac{2}{5}Va^2 = \frac{2}{5}Ma^2\,, \tag{2}$$

where V is equal to the mass of the ball because we set $\delta(x, y, z) = 1$.

(d) Due to the rotational symmetry the force $\mathbf{F}$ points towards the origin, and therefore has the form

$$\mathbf{F}(\xi, \eta, \zeta) = -f(\rho)\frac{\vec{\rho}}{\rho}\,, \quad \vec{\rho} = \xi\,\mathbf{i} + \eta\,\mathbf{j} + \zeta\,\mathbf{k}\,, \quad \rho = |\vec{\rho}| = \sqrt{\xi^2 + \eta^2 + \zeta^2}\,. \tag{3}$$

Thus to compute $f(\rho)$ it is sufficient to choose $(\xi, \eta, \zeta) = (0, 0, \rho)$, $\rho \geq 0$, and compute

$F_z(0,0,\rho)$

$$\begin{aligned}
f(\rho) &= -F_z(0,0,\rho) = -\iiint_G \frac{z-\rho}{(x^2+y^2+(z-\rho)^2)^{3/2}}dV \\
&= -\int_0^{2\pi}\int_0^a\int_0^\pi \frac{r\cos\phi-\rho}{(r^2-2\rho r\cos\phi+\rho^2)^{3/2}}r^2\sin\phi d\phi dr d\theta \\
&= -2\pi\int_0^a\int_{-1}^1 \frac{-rt-\rho}{(r^2+2\rho rt+\rho^2)^{3/2}}r^2 dt dr \\
&= \frac{\pi}{\rho}\int_0^a\int_{-1}^1 \frac{r^2}{\sqrt{r^2+2\rho rt+\rho^2}}dt dr + \frac{\pi}{\rho}\int_0^a\int_{-1}^1 \frac{\rho^2-r^2}{(r^2+2\rho rt+\rho^2)^{3/2}}r^2 dt dr \\
&= \frac{\pi}{\rho^2}\int_0^a r\sqrt{r^2+2\rho rt+\rho^2}|_{-1}^1 dr - \frac{\pi}{\rho^2}\int_0^a \frac{\rho^2-r^2}{\sqrt{r^2+2\rho rt+\rho^2}}|_{-1}^1 r dr \\
&= \frac{\pi}{\rho^2}\int_0^a (r(r+\rho)-r|r-\rho|)dr - \frac{\pi}{\rho^2}\int_0^a (\rho - r - \frac{\rho^2-r^2}{|r-\rho|})r dr \\
&= \frac{\pi}{\rho^2}\int_0^a \left(2r^2 - \frac{2r^2}{r-\rho}|r-\rho|\right)dr = \frac{2\pi}{\rho^2}\int_0^a (1-\operatorname{sign}(r-\rho))\, r^2 dr \\
&= \begin{cases} \frac{4\pi a^3}{3\rho^2} = \frac{M}{\rho^2} & \text{if} \quad \rho \ge a \\ \frac{4\pi\rho}{3} = \frac{M_\rho}{\rho^2} & \text{if} \quad \rho \le a \end{cases},
\end{aligned} \tag{4}$$

where $M_\rho = \frac{4\pi\rho^3}{3}$ is the mass of a ball of radius ρ and density 1. Thus if the particle is outside of the ball then the gravitational force due to the sphere coincides with the force of a point particle of the same mass, while if it is inside the ball only the part of the ball of radius ρ acts on it. The outer shell makes no contribution.

(e) Due to the rotational symmetry the gravitational potential field depends only on the distance from the particle to the centre of the ball

$$U(\xi,\eta,\zeta) = U(\rho)\,. \tag{5}$$

Thus to compute U it is sufficient to choose $(\xi,\eta,\zeta) = (0,0,\rho)$, $\rho \ge 0$

$$\begin{aligned}
U(\rho) &= -\iiint_G \frac{1}{\sqrt{x^2+y^2+(z-\rho)^2}}dV \\
&= -\int_0^{2\pi}\int_0^a\int_0^\pi \frac{1}{\sqrt{r^2-2\rho r\cos\phi+\rho^2}}r^2\sin\phi d\phi dr d\theta \\
&= -2\pi\int_0^a\int_{-1}^1 \frac{1}{\sqrt{r^2+2\rho rt+\rho^2}}r^2 dt dr = -\frac{2\pi}{\rho}\int_0^a r\sqrt{r^2+2\rho rt+\rho^2}|_{-1}^1 dr \\
&= -\frac{2\pi}{\rho}\int_0^a (r(r+\rho)-r|r-\rho|)dr \\
&= \begin{cases} -\frac{4\pi}{\rho}\int_0^a r^2 dr & \text{if} \quad \rho \ge a \\ -\frac{4\pi}{\rho}\int_0^\rho r^2 dr - 4\pi\int_\rho^a r dr & \text{if} \quad \rho \le a \end{cases} \\
&= \begin{cases} -\frac{4\pi a^3}{3\rho} = -\frac{M}{\rho} & \text{if} \quad \rho \ge a \\ -\frac{4\pi\rho^2}{3} - 2\pi a^2 + 2\pi\rho^2 = \frac{2\pi\rho^2}{3} - 2\pi a^2 & \text{if} \quad \rho \le a \end{cases}.
\end{aligned} \tag{6}$$

Thus if the particle is outside of the ball then the gravitational potential field due to the ball coincides with the force of a point particle of the same mass, while if it is inside the ball then the potential is a sum of a term which is independent of a and a term which is independent of ρ.

2. Use spherical coordinates $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$. Consider the solid G bounded above by the surface $r = a$ and below by the surface $\phi = \gamma$.

   (a) What is the surface $r = a$?

   (b) What is the surface $\phi = \gamma$?

   (c) Sketch the solid G.

   (d) Sketch the projection of the solid G onto the xy-plane.

   (e) Find the volume V of the solid G. Specify your answer for $\gamma = \pi/2$ and $\gamma = \pi$ and explain the result.

   (f) Find the centroid of the solid G. Specify your answer for $\gamma = \pi/2$ and $\gamma = \pi$.

   (g) Find the moments of inertia of the solid G. Specify your answer for $\gamma = \pi/2$ and $\gamma = \pi$.

   (h) Set $\gamma = \pi/2$, and find the gravitational force exerted on a point particle by the solid G if the point particle is located at $(0, 0, \zeta)$.

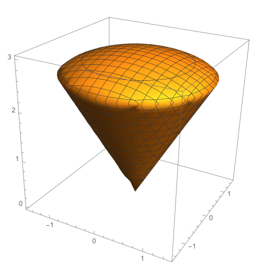
   (i) Set $\gamma = \pi/2$, and find the gravitational potential field $U(0, 0, \zeta)$ of the solid G.

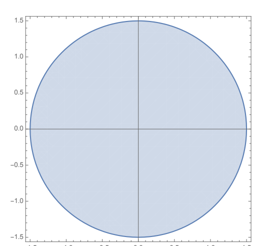
Solution:

(a) It is a sphere of radius a centred at the origin

(b) It is a circular cone

(c),(d) The solid and its projection R onto the xy-plane are shown below. R is a circle of radius a.





(e) The volume V of G is

$$V = \iiint_G dV = \int_0^{2\pi}\int_0^{\gamma}\int_0^{a} r^2\sin\phi\, dr\, d\phi\, d\theta = 2\pi\frac{a^3}{3}(-\cos\phi|_0^{\gamma}) = 2\pi\frac{a^3}{3}(1-\cos\gamma)$$
$$= \frac{4\pi a^3}{3}\sin^2\frac{\gamma}{2}\,. \tag{7}$$

$$V_{\gamma=\pi/2} = \frac{2\pi a^3}{3}\,, \quad V_{\gamma=\pi} = \frac{4\pi a^3}{3}\,, \tag{8}$$

because G is a half-ball at $\gamma = \pi/2$, and a ball at $\gamma = \pi$.

(f) The x and y coordinates of the centroid are 0 by the symmetry, and the z coordinate is

$$\begin{aligned} z_c &= \frac{1}{V}\iiint_G z\,dV = \frac{1}{V}\int_0^{2\pi}\int_0^{\gamma}\int_0^{a} r\cos\phi\, r^2\sin\phi\, dr d\phi d\theta = \frac{\pi a^4}{4V}\int_0^{\gamma}\sin 2\phi d\phi \\ &= \frac{\pi a^4}{8V}(1-\cos 2\gamma) = \frac{\pi a^4}{4V}\sin^2\gamma = \frac{3}{4}a\cos^2\frac{\gamma}{2} = \frac{3}{8}a(1+\cos\gamma)\,. \end{aligned} \tag{9}$$

$$z_{c,\gamma=\pi/2} = \frac{3a}{8}\,, \quad z_{c,\gamma=\pi} = 0\,. \tag{10}$$

(g) We first find I_z

$$\begin{aligned} I_z &= \iiint_G (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\gamma}\int_0^{a} r^2\sin^2\phi\, r^2\sin\phi\, dr d\phi d\theta = \frac{2\pi a^5}{5}\int_0^{\gamma}\sin^3\phi d\phi \\ &= \frac{2\pi a^5}{5}\int_0^{\gamma}(1-\cos^2\phi)d(-\cos\phi) = \frac{2\pi a^5}{5}(1-\cos\gamma-\frac{1}{3}(1-\cos^3\gamma)) \\ &= \frac{4\pi a^5}{15}\sin^2\frac{\gamma}{2}(2-\cos\gamma-\cos^2\gamma) = \frac{Ma^2}{5}(2-\cos\gamma-\cos^2\gamma) \\ &= \frac{2Ma^2}{5}(2+\cos\gamma)\sin^2\frac{\gamma}{2})\,. \end{aligned} \tag{11}$$

$$I_{z,\gamma=\pi/2} = \frac{2M_{\gamma=\pi/2}}{5}a^2\,, \quad I_{z,\gamma=\pi} = \frac{2M_{\gamma=\pi}}{5}a^2\,. \tag{12}$$

where V is equal to the mass of G because we set $\delta(x,y,z) = 1$. Then $I_x = I_y = \frac{1}{2}(I_x+I_y)$ by the rotational symmetry, so

$$\begin{aligned} I_x &= \iiint_G (\frac{1}{2}(x^2+y^2)+z^2)\,dV = \frac{1}{2}I_z + \int_0^{2\pi}\int_0^{\gamma}\int_0^{a} r^2\cos^2\phi\, r^2\sin\phi\, dr d\phi d\theta \\ &= \frac{1}{2}I_z + \frac{2\pi a^5}{5}\int_0^{\gamma}\cos^2\phi d(-\cos\phi) = \frac{1}{2}I_z + \frac{2\pi a^5}{15}(1-\cos^3\gamma) \\ &= \frac{1}{2}I_z + \frac{4\pi a^5}{15}\sin^2\frac{\gamma}{2}(1+\cos\gamma+\cos^2\gamma) \\ &= \frac{Ma^2}{10}(2-\cos\gamma-\cos^2\gamma) + \frac{Ma^2}{5}(1+\cos\gamma+\cos^2\gamma) = \frac{Ma^2}{10}(4+\cos\gamma+\cos^2\gamma)\,. \end{aligned} \tag{13}$$

$$I_{x,\gamma=\pi/2} = \frac{2M_{\gamma=\pi/2}}{5}a^2\,, \quad I_{x,\gamma=\pi} = \frac{2M_{\gamma=\pi}}{5}a^2\,. \tag{14}$$

(h) Let $\gamma = \pi/2$. Obviously the only nonvanishing component is F_z

$$
\begin{aligned}
F_z(0,0,\zeta) &= \iiint_G \frac{z-\zeta}{(x^2+y^2+(z-\zeta)^2)^{3/2}} dV \\
&= \int_0^{2\pi}\int_0^a\int_0^{\pi/2} \frac{r\cos\phi - \zeta}{(r^2-2\zeta r\cos\phi+\zeta^2)^{3/2}} r^2 \sin\phi d\phi dr d\theta \\
&= 2\pi \int_0^a \int_{-1}^0 \frac{-rt-\zeta}{(r^2+2\zeta rt+\zeta^2)^{3/2}} r^2 dt dr \\
&= -\frac{\pi}{\zeta}\int_0^a\int_{-1}^0 \frac{r^2}{\sqrt{r^2+2\zeta rt+\zeta^2}} dt dr - \frac{\pi}{\zeta}\int_0^a\int_{-1}^0 \frac{\zeta^2-r^2}{(r^2+2\zeta rt+\zeta^2)^{3/2}} r^2 dt dr \\
&= -\frac{\pi}{\zeta^2}\int_0^a r\sqrt{r^2+2\zeta rt+\zeta^2}|_{-1}^0 dr + \frac{\pi}{\zeta^2}\int_0^a \frac{\zeta^2-r^2}{\sqrt{r^2+2\zeta rt+\zeta^2}}|_{-1}^0 r dr \\
&= -\frac{\pi}{\zeta^2}\int_0^a (\sqrt{r^2+\zeta^2}-|r-\zeta|) r dr + \frac{\pi}{\zeta^2}\int_0^a (\frac{\zeta^2-r^2}{\sqrt{r^2+\zeta^2}} - \frac{\zeta^2-r^2}{|r-\zeta|}) r dr \\
&= \frac{2\pi}{\zeta^2}\int_0^a (\frac{-r}{\sqrt{r^2+\zeta^2}} - \frac{\zeta-r}{|r-\zeta|}) r^2 dr = \frac{2\pi}{\zeta^2}\int_0^a \left(-\frac{r^3}{\sqrt{r^2+\zeta^2}} + r^2 \mathrm{sign}(r-\zeta)\right) dr\,.
\end{aligned}
\tag{15}
$$

We then compute the first integral

$$
\begin{aligned}
I_1 &= -\frac{2\pi}{\zeta^2}\int_0^a \frac{r^3}{\sqrt{r^2+\zeta^2}} dr = -2\pi|\zeta|\int_0^{a/\zeta} \frac{y^3}{\sqrt{y^2+1}} dy \\
&= -2\pi|\zeta|\frac{1}{3}\left(y^2-2\right)\sqrt{y^2+1}\Big|_0^{a/\zeta} \\
&= -\frac{2\pi|\zeta|}{3}\left(\left(\frac{a^2}{\zeta^2}-2\right)\sqrt{\frac{a^2}{\zeta^2}+1}+2\right),
\end{aligned}
\tag{16}
$$

and the second integral

$$
I_2 = \frac{2\pi}{\zeta^2}\int_0^a r^2 \mathrm{sign}(r-\zeta) dr = \begin{cases} -\frac{2\pi}{\zeta^2}\int_0^a r^2 dr = -\frac{2\pi a^3}{3\zeta^2} = -\frac{M}{\zeta^2} & \text{if} \quad \zeta \geq a \\ -\frac{2\pi}{\zeta^2}\int_0^\zeta r^2 dr + \frac{2\pi}{\zeta^2}\int_\zeta^a r^2 dr = -\frac{4\pi}{3}\zeta + \frac{2\pi a^3}{3\zeta^2} & \text{if} \quad 0 \leq \zeta \leq a \\ \frac{2\pi}{\zeta^2}\int_0^a r^2 dr = \frac{2\pi a^3}{3\zeta^2} = \frac{M}{\zeta^2} & \text{if} \quad \zeta \leq 0 \end{cases}.
\tag{17}
$$

Thus

$$
F_z(0,0,\zeta) == \begin{cases} -\frac{2\pi\zeta}{3}\left(\left(\frac{a^2}{\zeta^2}-2\right)\sqrt{\frac{a^2}{\zeta^2}+1}+2\right) - \frac{2\pi a^3}{3\zeta^2} & \text{if} \quad \zeta \geq a \\ -\frac{2\pi\zeta}{3}\left(\left(\frac{a^2}{\zeta^2}-2\right)\sqrt{\frac{a^2}{\zeta^2}+1}+2\right) - \frac{4\pi}{3}\zeta + \frac{2\pi a^3}{3\zeta^2} & \text{if} \quad 0 \leq \zeta \leq a \\ \frac{2\pi\zeta}{3}\left(\left(\frac{a^2}{\zeta^2}-2\right)\sqrt{\frac{a^2}{\zeta^2}+1}+2\right) + \frac{2\pi a^3}{3\zeta^2} & \text{if} \quad \zeta \leq 0 \end{cases}.
\tag{18}
$$

Taking the limit $\zeta \to 0$ one gets

$$
F_z(0,0,0) = \pi a\,. \tag{19}
$$

(i) Let $\gamma = \pi/2$.

$$\begin{aligned}U(0,0,\zeta) &= -\iiint_G \frac{1}{\sqrt{x^2+y^2+(z-\zeta)^2}}dV \\ &= -\int_0^{2\pi}\int_0^a\int_0^{\pi/2} \frac{1}{\sqrt{r^2-2\zeta r\cos\phi+\zeta^2}} r^2 \sin\phi d\phi dr d\theta \\ &= -2\pi \int_0^a \int_{-1}^0 \frac{1}{\sqrt{r^2+2\zeta rt+\zeta^2}} r^2 dt dr = -\frac{2\pi}{\zeta}\int_0^a r\sqrt{r^2+2\zeta rt+\zeta^2}|_{-1}^0 dr \\ &= -\frac{2\pi}{\zeta}\int_0^a (\sqrt{r^2+\zeta^2}-|r-\zeta|) r dr\,.\end{aligned} \tag{20}$$

We then compute the first integral

$$\begin{aligned}I_1 &= -\frac{2\pi}{\zeta}\int_0^a r\sqrt{r^2+\zeta^2}dr = -\frac{2\pi}{\zeta}\frac{1}{3}(r^2+\zeta^2)^{3/2}\,|_0^a \\ &= -\frac{2\pi}{3\zeta}\left((a^2+\zeta^2)^{3/2}-|\zeta|^3\right)\,,\end{aligned} \tag{21}$$

and the second integral

$$I_2 = \frac{2\pi}{\zeta}\int_0^a |r-\zeta| r dr = \begin{cases} -\frac{2\pi}{\zeta}\int_0^a (r-\zeta) r dr = \frac{1}{3}\pi a^2\left(3-\frac{2a}{\zeta}\right) & \text{if} \quad \zeta \geq a \\ -\frac{2\pi}{\zeta}\int_0^\zeta (r-\zeta) r dr + \frac{2\pi}{\zeta}\int_\zeta^a (r-\zeta) r dr = \frac{\pi\left(2a^3-3a^2\zeta+2\zeta^3\right)}{3\zeta} & \text{if} \quad 0 \leq \zeta \leq a \\ \frac{2\pi}{\zeta}\int_0^a (r-\zeta) r dr = -\frac{1}{3}\pi a^2\left(3-\frac{2a}{\zeta}\right) & \text{if} \quad \zeta \leq 0 \end{cases}. \tag{22}$$

Thus

$$U(0,0,\zeta) == \begin{cases} -\frac{2\pi}{3\zeta}\left((a^2+\zeta^2)^{3/2}-\zeta^3\right)+\frac{1}{3}\pi a^2\left(3-\frac{2a}{\zeta}\right) & \text{if} \quad \zeta \geq a \\ -\frac{2\pi}{3\zeta}\left((a^2+\zeta^2)^{3/2}-\zeta^3\right)+\frac{\pi\left(2a^3-3a^2\zeta+2\zeta^3\right)}{3\zeta} & \text{if} \quad 0 \leq \zeta \leq a \\ -\frac{2\pi}{3\zeta}\left((a^2+\zeta^2)^{3/2}+\zeta^3\right)-\frac{1}{3}\pi a^2\left(3-\frac{2a}{\zeta}\right) & \text{if} \quad \zeta \leq 0 \end{cases}. \tag{23}$$

Taking the limit $\zeta \to 0$ one gets

$$U(0,0,0) = -\pi a^2\,. \tag{24}$$