Distribution for random errors

Can use binomial distribution to model random errors.

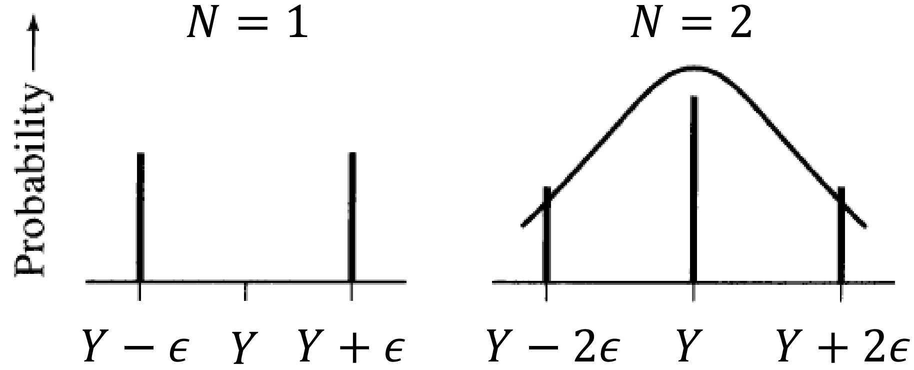
Assumptions:

- Let there be N sources of random error (e.g. reaction time, temperature fluctuations, vibrations etc.).
- Each error introduces a *fixed deviation* $\pm\epsilon$ to our measurement.
- There is a 50% chance that ϵ is positive or negative.

Distribution of random errors

- Let Y be the true mean. If there is just one source of error, the possible outcomes y will be $Y+\epsilon$ and $Y-\epsilon$, both of them equally likely.
- If there are two source of error, the there are three possible outcomes:

$$Y+\epsilon-\epsilon = Y$$
$$Y+\epsilon+\epsilon = Y+2\epsilon$$
$$Y-\epsilon-\epsilon = Y-2\epsilon$$



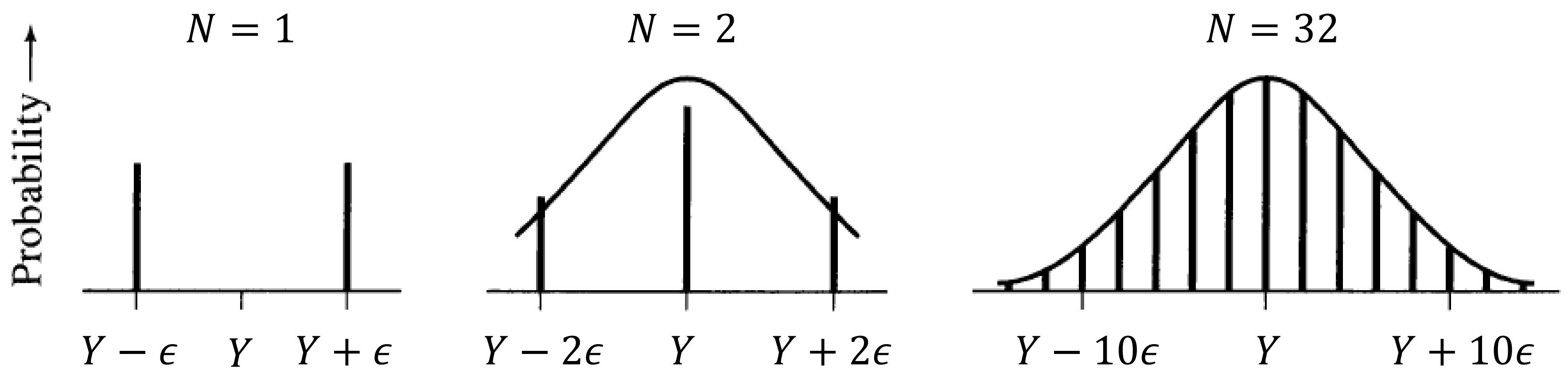
- In general, if there are N sources of error, the outcome could range between $Y+N\epsilon$ and $Y-N\epsilon$. If there are x events that give $+\epsilon$ deviations, then our answer will be

$$\begin{aligned} y &= Y + x\epsilon - (N-x)\epsilon \\ &= Y + (2x-N)\epsilon \end{aligned}$$

Distribution of random errors

The probability of x positive deviations is just given by the binomial distribution $B_{N,0.5}(x)$, therefore $y = Y + (2x - N)\epsilon$ is governed by the same distribution:

$$P(y) = B_{N,0.5}\left(\frac{y - Y}{2\epsilon} + \frac{N}{2}\right)$$



The standard deviation of $B_{N,0.5}(x)$ is just $\sigma_x = \sqrt{N/4}$.
Therefore, $\sigma_y = 2\epsilon\sigma_x = \epsilon\sqrt{N}$.
<u>As N increases, a bell shaped curve emerges.</u>

The Normal (Gaussian) Distribution

Can show that in the limit $Np \to \infty$ and $\varepsilon \to 0$, the binomial distribution approaches the continuous Gaussian or Normal distribution:

$$B_{N,p}(x) = \binom{N}{x} p^x q^{N-x} \approx \frac{1}{\sqrt{2\pi Npq}} e^{-\frac{(x-Np)^2}{2Npq}}$$

<u>This limit is independent of the success probability p.</u>

In general, for a given mean μ and variance σ^2, the normal distribution is given by

$$\mathrm{G}_{\mu,\sigma}(\mathrm{x}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Random walk

The summation of random steps is not just a model for random errors but also corresponds to an important physical process:

Diffusion

Molecules/particles in thermal equilibrium get kicked around by neighbouring molecules and perform a random walk (Brownian motion).

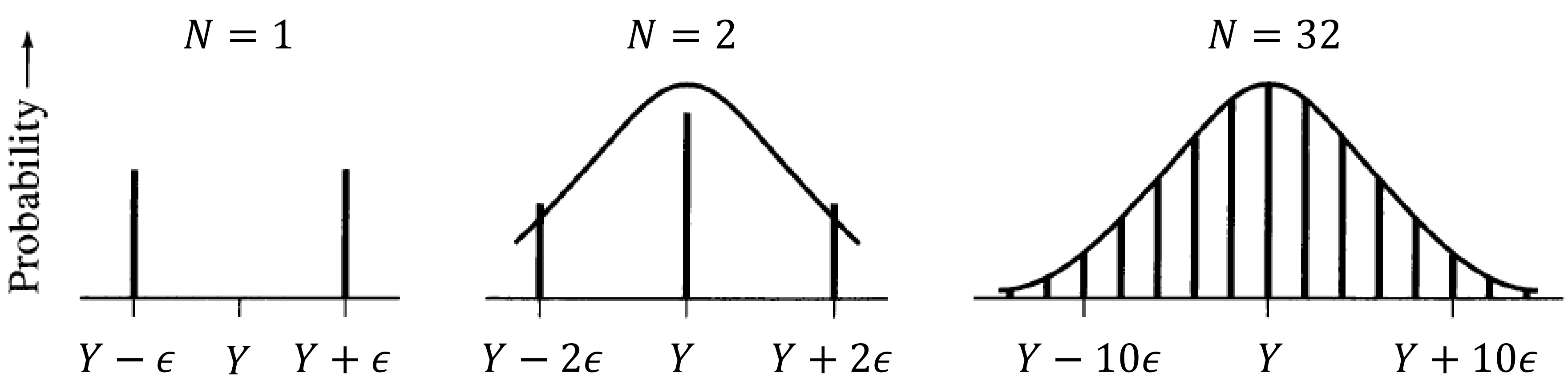
For simplicity let’s look at the 1D case of a random walk along a line: If the typical step size is ϵ in time Δt, then after many time steps (i.e. trials) the distribution will become Gaussian.

Random walk

Same plot as before when we added random errors. Here it shows the spatial distribution of random walks with fixed step size ϵ after N steps.

Each trial corresponds to a time step Δt, so on average distance travelled $\propto \sigma \propto \epsilon\sqrt{N} \propto \sqrt{t}$

Diffusion equation : $\sqrt{\langle y^2 \rangle} \sim \sqrt{Dt}$, where D is the diffusion constant (units m^2/s). Ballistic motion $y = vt$.



Central limit theorem

Addition of random, fixed deviations ϵ just one example of the **Central Limit Theorem:**

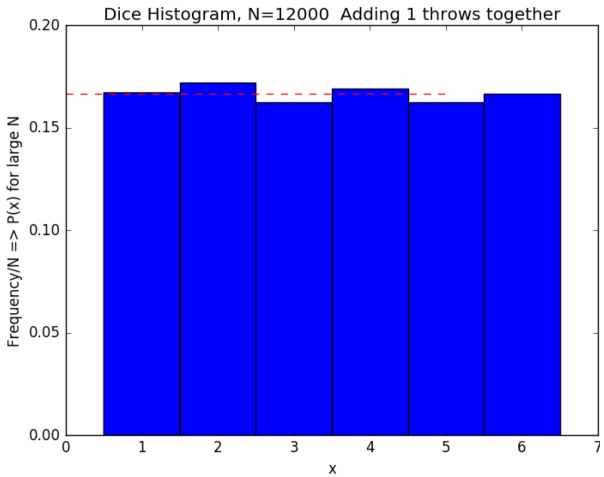
Let x be a random variable governed by an arbitrary probability distribution $P(x)$ with mean μ_x and a finite standard deviation σ_x.

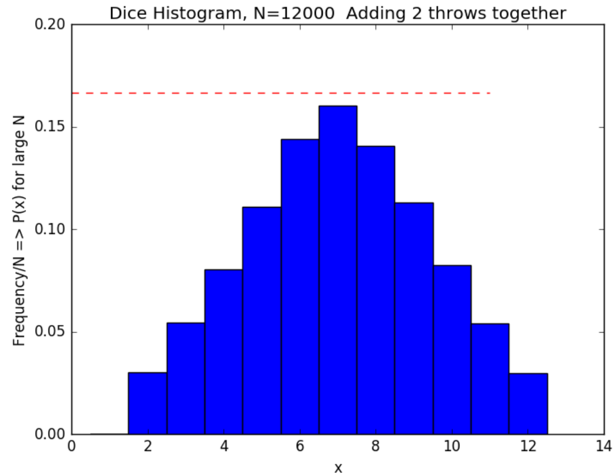
Then the distribution $P(y)$ of the sum of x over N *independent* trials, $y = \sum_{i=1}^{N} x_i$, will converge to a Gaussian distribution with mean $\mu_y = N\mu_x$ and a standard deviation $\sigma_y = \sqrt{N}\sigma_x$

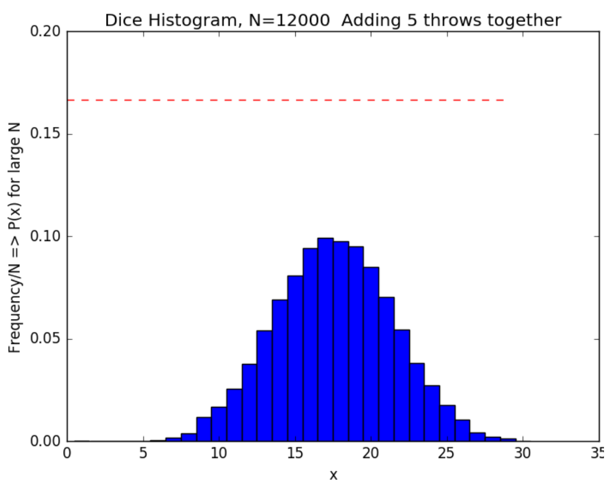
(Formal proof exists.)

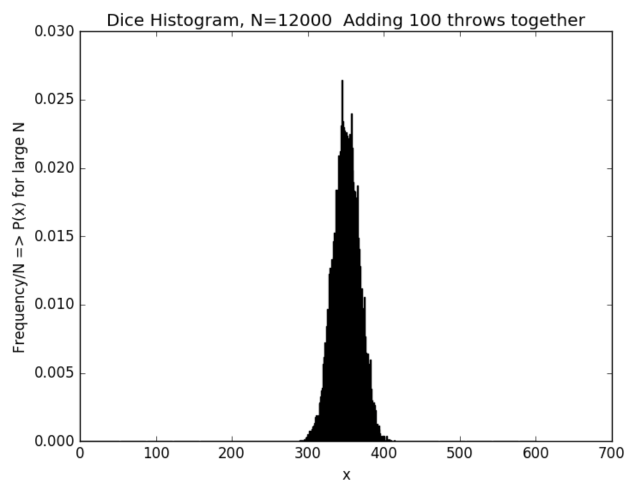
Central limit theorem - example

Throw a dice M times and add the numbers. Do this N times. Here, the sample size is N=12000.









Distribution becomes progressively bell-shaped.

The mean increases as $N\mu$, Where μ is the mean of the original $P(x) = \frac{1}{6}$

Width increases as $\sqrt{N}$.

Peak value decreases as distribution is spread out over a larger range of x (Area has to be 1)

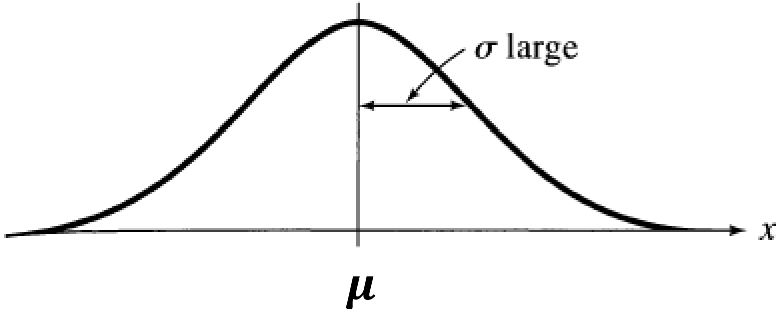
Approaches Gaussian eventually.

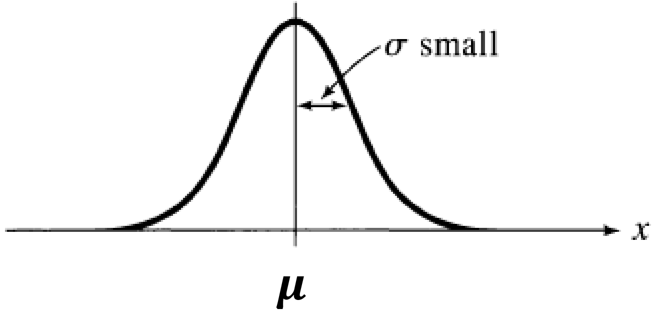
The Normal/Gaussian distribution

- Random errors in measurements are often (not always!) distributed according to the Gaussian

$$G_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- The standard deviation σ reflects the uncertainty in the measurement.
- The Gaussian is symmetric around the mean μ





Normalisation

Gaussian needs to satisfy

$$\int_{-\infty}^{+\infty} G_{\mu,\sigma}(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

Let's check. First, we shift the mean and rescale $x' = (x - \mu)/\sqrt{2}\sigma$, $dx = \sqrt{2}\sigma dx'$

$$\int_{-\infty}^{+\infty} G_{\mu,\sigma}(x)dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x'^2}\,dx' = 1$$

Can show that:

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

So normalisation is satisfied.

The mean and variance of Gaussian distribution

Using change of variables and integration by parts can show that the mean and standard deviation is just

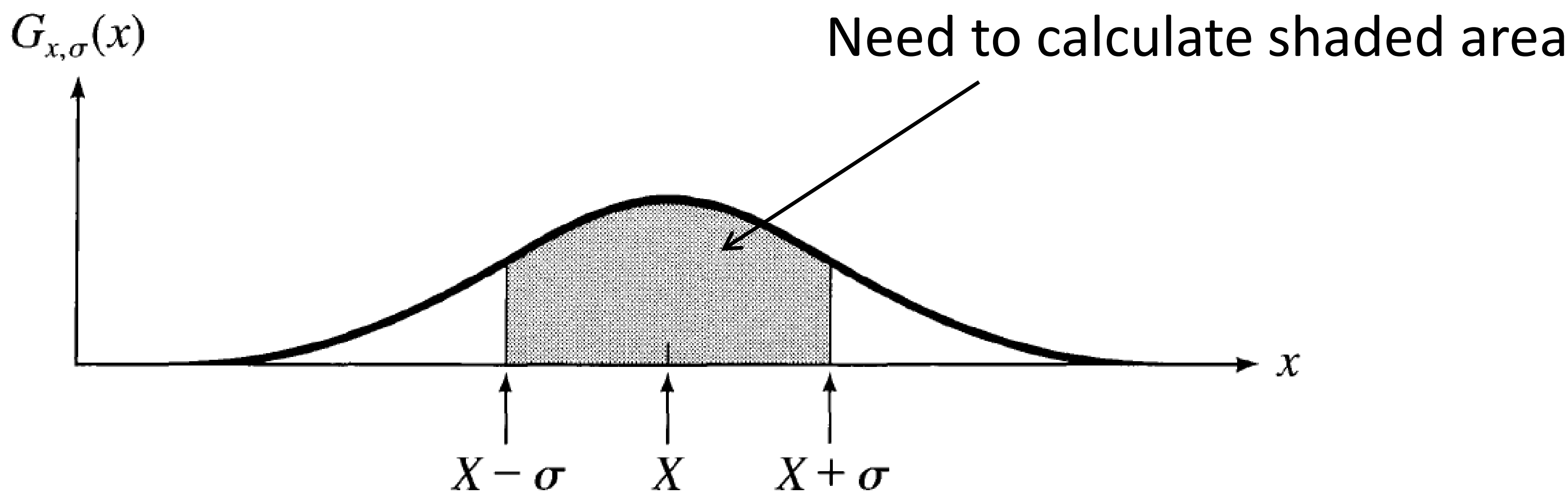
$$\langle x \rangle = \int_{-\infty}^{+\infty} x \cdot G_{\mu,\sigma}(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \boldsymbol{\mu}$$

$$\begin{aligned} Var(x) &= \int_{-\infty}^{+\infty} (x-\mu)^2 \cdot G_{\mu,\sigma}(x) dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} (x-\mu)^2 \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \boldsymbol{\sigma^2} \end{aligned}$$

The 68% confidence limit

In general, the probability of obtaining any given value x in the range $a \leq x \leq b$ is just $\int_a^b P(x)dx$, where $P(x)$ is the probability distribution.

For the Gaussian, what is the probability that x falls within one standard deviation from the mean?



The 68% confidence limit

The probability is given by

$$\int_{\mu-\sigma}^{\mu+\sigma} G_{\mu,\sigma}(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-\sigma}^{\mu+\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = ?$$

While the definite Gaussian integral $\int_{-\infty}^{+\infty} G_{\mu,\sigma}(x)dx$ can be analytically solved, the indefinite integral $\int G_{\mu,\sigma}(x)dx$ cannot.

Due to the importance of integrating Gaussians within particular limits, a special function is defined.

The Error function

Let's do a variable substitution: $z = (x - \mu)/\sigma$.

The mean of x is shifted to zero and x is rescaled by the standard deviation.

$dz = dx/\sigma$ and the limits of the integrals change from $\mu + \sigma$ to 1 and $\mu - \sigma$ to -1

Thus

$$\int_{\mu-\sigma}^{\mu+\sigma} G_{\mu,\sigma}(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} e^{-z^2/2}\, dz$$

In general, for arbitrary limits t , the integral above is called the **error function erf(t):**

$$\mathrm{erf}(t) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{+t} e^{-z^2/2}\, dz$$

The Error function

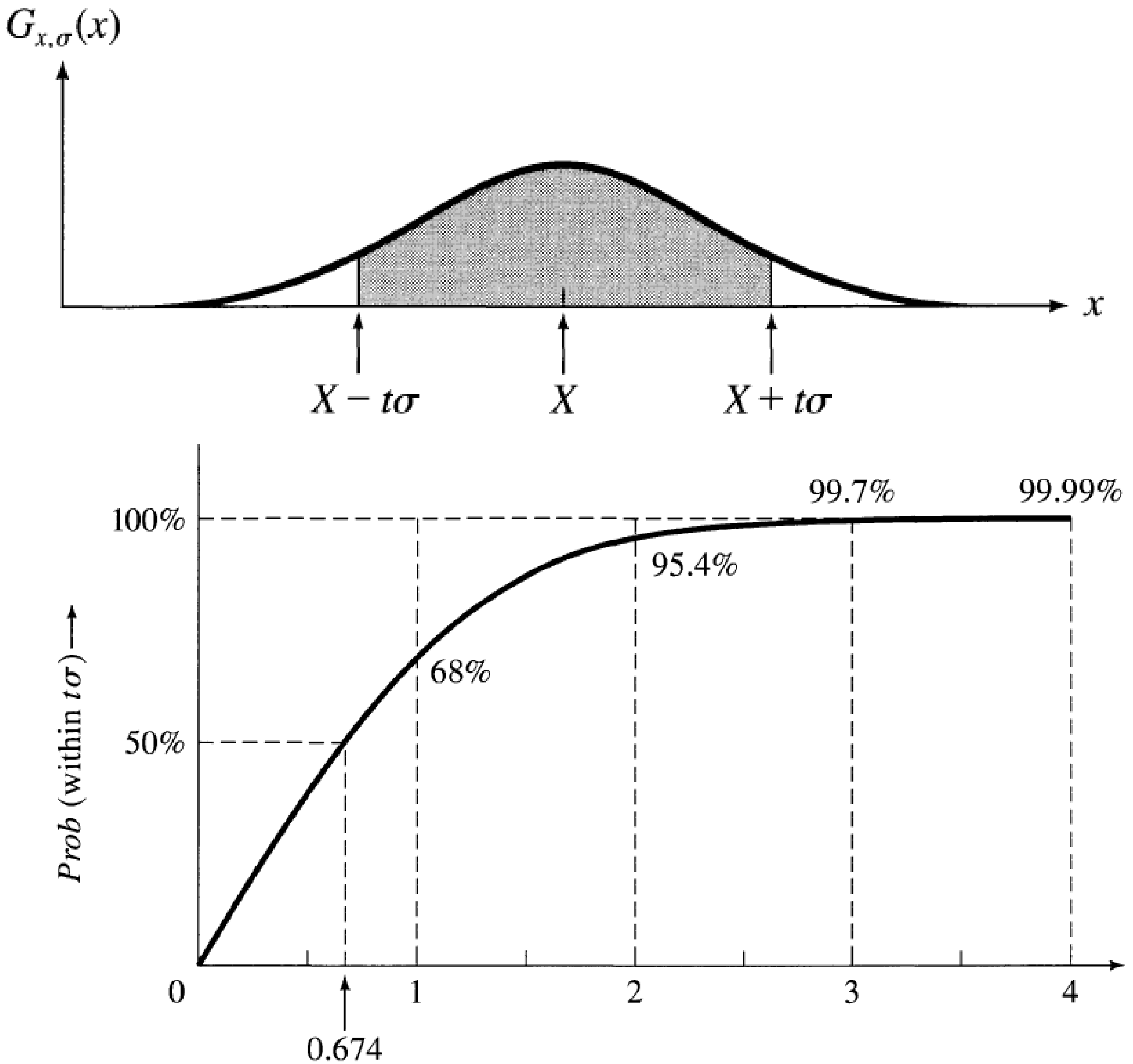
The probability for a random variable (governed by a Gaussian) to fall within $t \cdot \sigma$ of the mean is given by the error function (or normal error integral)

$$\mathrm{erf}(t) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{+t} e^{-z^2/2}\, dz$$

For $t = 1, \mathrm{erf}(\mathrm{t} = 1) \approx 0.68 = 68\%$

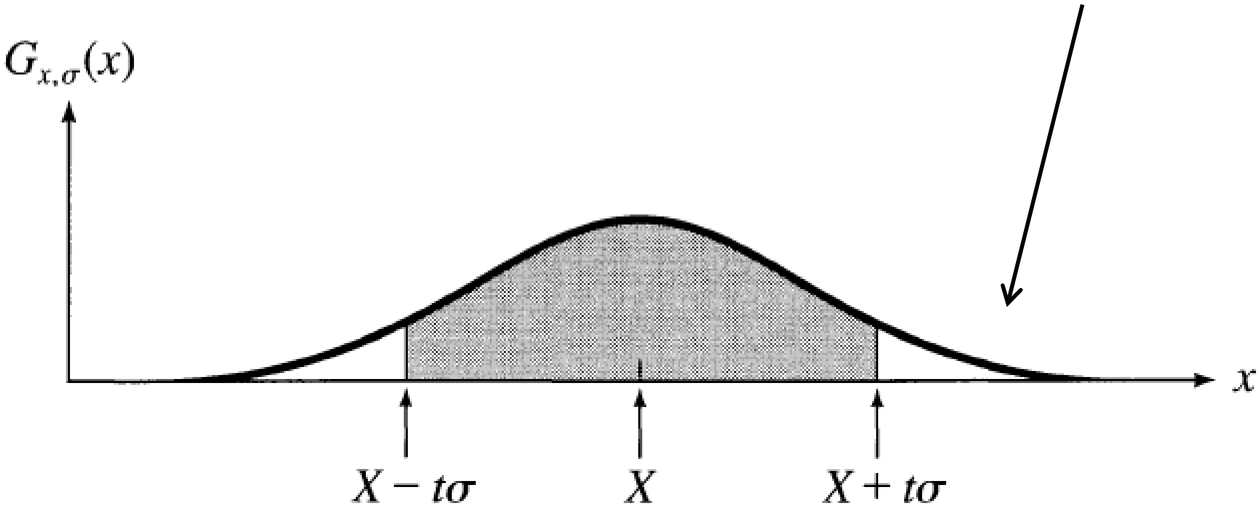
Note that the definition of $\mathrm{erf}(t)$ can differ slightly depending on the book/source (different integral limits), but it is always a definite integral of a Gaussian.

The Error function



The Error function

- Can use the error function to calculate probabilities that x falls outside some bound.
- E.g. what is the probability that $x \geq \mu + 3\sigma$?



- $G_{\mu,\sigma}(x \geq \mu + 3\sigma) = \frac{1-\text{erf}(t=3)}{2} = \frac{1-0.997}{2} = 0.0015 = 0.15\%$

Limited sample: Best estimate of the mean

In practice, number of measurements are limited.

Is the average of N measurements $(x_1, x_2, .., x_N)$,

$$\langle x \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i$$

the **best estimate for the true mean μ** of the limiting normal distribution?

Now that we know the limiting distribution of random errors, we can quantify this.

Limited sample: Best estimate of the mean

Let's calculate the probability of obtaining these N measurements:

Probability of measuring x_1:

$$P_1 = P(x_1 \le x \le x_1 + dx) = G_{\mu,\sigma}(x_1)dx \propto \frac{1}{\sigma} e^{-(x_1-\mu)^2/2\sigma^2}$$

Probability of measuring x_2:

$$P_2 \propto \frac{1}{\sigma} e^{-(x_2-\mu)^2/2\sigma^2}$$

etc.

Principle of maximum likelihood

The probability that we obtain the whole set of N readings $(x_1, x_2.., x_N)$ is

$$P_{\mu,\sigma}(x_1, x_2.., x_N) \; = \; P_1 \cdot P_2 \cdot .. \cdot P_N \quad \propto \; \frac{1}{\sigma^N} e^{-\sum_{i=1}^{N}(x_i-\mu)^2/2\sigma^2}$$

The best estimate of μ (and σ) is the one which maximises the above probability of measuring these N values – this is the **principle of maximum likelihood.**

The probability above is maximised when the exponent $\sum_{i=1}^{N}(x_i - \mu)^2/2\sigma^2$ is at a minimum.

Best estimate of the mean

Now differentiate with respect to μ and set to 0 to find the minimum of the exponent $\sum_{i=1}^{N}(x_i-\mu)^2/2\sigma^2$

$$\sum_{i=1}^{N}(x_i-\mu)=0$$

$$\sum_{i=1}^{N}x_i-\sum_{i=1}^{N}\mu=0$$

$$\sum_{i=1}^{N}x_i-N\mu=0$$

Best estimate of μ is just the average $\frac{1}{N}\sum_{i=1}^{N}x_i$

Best estimate of the standard deviation

Similar calculation yields (differentiating w.r.t. σ to find maximum probability) best estimate of standard deviation.

Best estimate for standard deviation (so called sample standard deviation, see Appendix E in Taylor book for derivation):

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x - \langle x \rangle)^2}$$

Measured, not true mean

Note that $\frac{1}{N-1}$ prefactor instead of the usual $\frac{1}{N}$. This correction is useful when you only have a few measurements (< 5 or so). For large N, this difference becomes negligible.

For the remainder of the course will use ordinary standard deviation with $\frac{1}{N}$ prefactor, unless explicitly stated otherwise.