Advanced Calculus
MA1132
Exercises 4 Solutions

1. (a) Use the chain rule to find $\dfrac{df}{dt}$ if

$$f(x,y) = \cosh^2(xy), \quad x(t) = \frac{t}{2}, \quad y(t) = e^t.$$

(b) Use the chain rule to find $\dfrac{df}{dt}$ if

$$f(x,y,z) = \ln(3x^2 - 2y + 4z^3), \quad x(t) = t^{\frac{1}{2}}, \quad y(t) = t^{\frac{2}{3}}, \quad z(t) = t^{-2}.$$

Solution:

(a) We have

$$\begin{aligned}
\frac{df}{dt} &= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \\
&= 2y \sinh(xy)\cosh(xy) \cdot \frac{1}{2} + 2x \sinh(xy)\cosh(xy) \cdot e^t \\
&= \sinh(xy)\cosh(xy)(y + 2xe^t) \\
&= \sinh\left(\frac{te^t}{2}\right)\cosh\left(\frac{te^t}{2}\right)(e^t + te^t) \\
&= e^t \sinh\left(\frac{te^t}{2}\right)\cosh\left(\frac{te^t}{2}\right)(1+t).
\end{aligned}$$

(b) We have

$$\begin{aligned}
\frac{df}{dt} &= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} \\
&= \frac{6x}{3x^2 - 2y + 4z^3} \cdot \frac{1}{2}t^{-\frac{1}{2}} + \frac{-2}{3x^2 - 2y + 4z^3} \cdot \frac{2}{3}t^{-\frac{1}{3}} + \frac{12z^2}{3x^2 - 2y + 4z^3} \cdot (-2t^{-3}) \\
&= \frac{3xt^{-\frac{1}{2}} - \frac{4}{3}t^{-\frac{1}{3}} - 24z^2t^{-3}}{3x^2 - 2y + 4z^3} \\
&= \frac{3t^{\frac{1}{2}}t^{-\frac{1}{2}} - \frac{4}{3}t^{-\frac{1}{3}} - 24(t^{-2})^2t^{-3}}{3(t^{\frac{1}{2}})^2 - 2t^{\frac{2}{3}} + 4(t^{-2})^3} \\
&= \frac{3 - \frac{4}{3}t^{-\frac{1}{3}} - 24t^{-7}}{3t - 2t^{\frac{2}{3}} + 4t^{-6}}.
\end{aligned}$$

2. Use appropriate forms of the chain rule to find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ where

$$z = \sin\frac{x}{2}\,\cos 2y\,; \quad x = 2u + 3v\,, \quad y = u^3 - 2v^2\,.$$

Solution: We have

$$\begin{aligned}
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad &= \quad \frac{1}{2}\cos\frac{x}{2}\,\cos 2y \times 2 - 2\sin\frac{x}{2}\,\sin 2y \times 3u^2 \\
&= \quad \cos\frac{x}{2}\,\cos 2y - 6u^2\,\sin\frac{x}{2}\,\sin 2y \\
&= \quad \cos\frac{2u+3v}{2}\,\cos 2(u^3 - 2v^2) - 6u^2\,\sin\frac{2u+3v}{2}\,\sin 2(u^3 - 2v^2)\,,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \quad &= \quad \frac{1}{2}\cos\frac{x}{2}\,\cos 2y \times 3 - 2\sin\frac{x}{2}\,\sin 2y \times (-4v) \\
&= \quad \frac{3}{2}\cos\frac{x}{2}\,\cos 2y + 8v\,\sin\frac{x}{2}\,\sin 2y \\
&= \quad \frac{3}{2}\cos\frac{2u+3v}{2}\,\cos 2(u^3 - 2v^2) + 8v\,\sin\frac{2u+3v}{2}\,\sin 2(u^3 - 2v^2)\,.
\end{aligned}$$

3. A function $f(x_1, \ldots, x_n)$ is said to be homogeneous of degree k if $f(tx_1, \ldots tx_n) = t^k f(x_1, \ldots, x_n)$ for $t > 0$. Show that it satisfies

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf\,.$$

Solution: Consider the function

$$F(t, x_1, \ldots, x_n) = f(tx_1, \ldots tx_n) - t^k f(x_1, \ldots, x_n)\,. \tag{1}$$

Compute

$$\begin{aligned}
\frac{\partial}{\partial t}F(t, x_1, \ldots, x_n) &= \frac{\partial}{\partial t}f(tx_1, \ldots tx_n) - \frac{\partial}{\partial t}t^k f(x_1, \ldots, x_n) \\
&= \sum_{i=1}^{n} x_i \frac{\partial f(tx_1, \ldots tx_n)}{\partial x_i} - kt^{k-1} f(x_1, \ldots x_n)\,.
\end{aligned} \tag{2}$$

If f is homogeneous of degree k, then $F = 0$ for any $t > 0$, and setting $t = 1$ in the formula, one gets

$$0 = \sum_{i=1}^{n} x_i \frac{\partial f(x_1, \ldots x_n)}{\partial x_i} - kf(x_1, \ldots x_n)\,. \tag{3}$$

4. Consider the function

$$z = 3e^{y - \frac{\pi}{4}}\cos x - 2e^{\frac{\pi}{2} - x}\sin y$$

(a) Find

$$iii)\ \frac{\partial^2 z}{\partial x\partial y}(\frac{\pi}{2},\frac{\pi}{4})\,,\qquad iv)\ \frac{\partial^2 z}{\partial y\partial x}(\frac{\pi}{2},\frac{\pi}{4})\,.$$

Solution:

$$iii)\ \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\frac{\partial z}{\partial y}=\frac{\partial}{\partial x}\left(3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\cos y\right)=-3e^{y-\frac{\pi}{4}}\sin x+2e^{\frac{\pi}{2}-x}\cos y\ \Rightarrow\ \frac{\partial^2 z}{\partial x\partial y}(\frac{\pi}{2},\frac{\pi}{4})=-3+\sqrt{2}\,.$$

$$iv)\ \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\frac{\partial z}{\partial x}=\frac{\partial}{\partial y}\left(-3e^{y-\frac{\pi}{4}}\sin x+2e^{\frac{\pi}{2}-x}\sin y\right)=-3e^{y-\frac{\pi}{4}}\sin x+2e^{\frac{\pi}{2}-x}\cos y\ \Rightarrow\ \frac{\partial^2 z}{\partial x\partial y}(\frac{\pi}{2},\frac{\pi}{4})=-3+\sqrt{2}\,.$$

(b) Find the slope of the surface $z=3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\sin y$ in the y-direction at the point $(\frac{\pi}{3},\frac{\pi}{6})$.

Solution: The slope k_y is equal to

$$k_y=\frac{\partial z}{\partial y}(\frac{\pi}{3},\frac{\pi}{6})=3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\cos y|_{x=\frac{\pi}{3},y=\frac{\pi}{6}}=\frac{3}{2}e^{-\frac{\pi}{12}}-\sqrt{3}e^{\frac{\pi}{6}}\approx-1.76936\,.$$

(c) Show that the function $z=3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\sin y$ satisfies Laplace's equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0\,.$$

Solution: To this end we compute the following derivatives

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(-3e^{y-\frac{\pi}{4}}\sin x+2e^{\frac{\pi}{2}-x}\sin y\right)=-3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\sin y\,,$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\cos y\right)=3e^{y-\frac{\pi}{4}}\cos x+2e^{\frac{\pi}{2}-x}\sin y\,.$$

The sum of these two expressions is obviously 0.

5. The equations of motion of a system of n particles are given by

$$m_i\ddot{x}_i=-\frac{\partial U(x_1,\dots,x_n)}{\partial x_i}\,,\quad \ddot{x}_i=\frac{d^2x_i}{dt^2}\,,\quad i=1,2,\dots,n\,,$$

where m_i is the mass and x_i is the coordinate of the i-th particle, and $U(x_1,\dots,x_n)$ is the potential energy of the system.

(a) Find the equations of motion of a system of n particles moving in a Coulomb field

$$U(x_1,\dots,x_n)=\frac{\alpha}{r}\,,\quad r=\big|\sum_{i=1}^{n}x_i\mathbf{e}_i\big|\,.$$

Solution: We have

$$m_i\ddot{x}_i=-\frac{\partial U(x_1,\dots,x_n)}{\partial x_i}=-\frac{\partial}{\partial x_i}\frac{\alpha}{r}=\frac{\alpha}{r^2}\frac{\partial r}{\partial x_i}=\frac{\alpha}{r^2}\frac{x_i}{r}\,.$$

(b) Find the equations of motion of a system of n coupled harmonic oscillators

$$U(x_1, \dots, x_n) = \sum_{i=1}^{n-1} \frac{\kappa}{2}(x_{i+1} - x_i)^2 \,,$$

Solution: We have

$$\begin{aligned}
m_i \ddot{x}_i &= -\frac{\partial U(x_1, \dots, x_n)}{\partial x_i} = -\frac{\partial}{\partial x_i} \sum_{j=1}^{n-1} \frac{\kappa}{2}(x_{j+1} - x_j)^2 = -\kappa \sum_{j=1}^{n-1}(x_{j+1} - x_j)(\delta_{i,j+1} - \delta_{ij}) \\
&= -\kappa \sum_{j=1}^{n-1} \big((x_{j+1} - x_j)\delta_{i,j+1} - (x_{j+1} - x_j)\delta_{ij}\big) \\
&= -\kappa \sum_{j=1}^{n-1} \big((x_i - x_{i-1})\delta_{i,j+1} - (x_{i+1} - x_i)\delta_{ij}\big) \\
&= -\kappa(x_i - x_{i-1}) \sum_{j=1}^{n-1} \delta_{i,j+1} + \kappa(x_{i+1} - x_i) \sum_{j=1}^{n-1} \delta_{ij} \\
&= -\kappa(x_i - x_{i-1}) + \kappa(x_i - x_{i-1})\delta_{i1} + \kappa(x_{i+1} - x_i) - \kappa(x_{i+1} - x_i)\delta_{in} \\
&= -\kappa(2x_i - x_{i-1} - x_{i+1}) + \kappa x_1 \delta_{i1} + \kappa x_n \delta_{in} \,,
\end{aligned} \tag{4}$$

where δ_{ij} is Kronecker's delta $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$, and $x_0 = x_{n+1} = 0$. Thus, more explicitly we get

$$\begin{aligned}
m_1 \ddot{x}_1 &= -\kappa(x_1 - x_2) \,, \\
m_i \ddot{x}_i &= -\kappa(2x_i - x_{i-1} - x_{i+1}) \,, \quad \text{if} \quad i = 2, 3, \dots, n-1 \,, \\
m_n \ddot{x}_n &= -\kappa(x_n - x_{n-1}) \,.
\end{aligned} \tag{5}$$

(c) Find the equations of motion of a system of n particles with pairwise interaction

$$U(x_1, \dots, x_n) = \sum_{i,j=1, i\neq j}^{n} V(x_i - x_j) \,.$$

Here V is an even function of a single variable, and we use the notation

$$\sum_{i,j=1,i\neq j}^{n} a_{ij} \equiv \sum_{j=1}^{n} \sum_{i=1,i\neq j}^{n} a_{ij} = \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} a_{ij} \,.$$

Solution: We have

$$\begin{aligned}
m_i \ddot{x}_i &= -\frac{\partial U(x_1, \dots, x_n)}{\partial x_i} = -\frac{\partial}{\partial x_i} \sum_{j,k=1,j\neq k}^{n} V(x_j - x_k) = -\sum_{j,k=1,j\neq k}^{n} V'(x_j - x_k)(\delta_{ij} - \delta_{ik}) \\
&= -\sum_{k=1,k\neq i}^{n} V'(x_i - x_k) + \sum_{j=1,j\neq i}^{n} V'(x_j - x_i) \\
&= -2 \sum_{j=1,j\neq i}^{n} V'(x_i - x_j) \,.
\end{aligned} \tag{6}$$

6. The Taylor series is given by

$$f(\vec{x}) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\partial_1^{k_1}\cdots\partial_n^{k_n} f(\vec{x^o})}{k_1!\cdots k_n!} \Delta x_1^{k_1}\cdots\Delta x_n^{k_n}\,, \tag{7}$$

where we denote

$$f(x_1,\ldots,x_n) \equiv f(\vec{x})\,,\quad f(x_1^o,\ldots,x_n^o) \equiv f(\vec{x^o})\,,\quad x_i - x_i^o \equiv \Delta x_i \tag{8}$$

and $\partial_i^0 f \equiv f$; $\partial_i^k f \equiv \frac{\partial^k f}{\partial x_i^k}$ is the k-th partial derivative of f with respect to x_i.

The Taylor series can be equivalently written as

$$f(\vec{x}) = \sum_{q=0}^{\infty} \frac{1}{q!} \sum_{i_1,\ldots,i_q=1}^{n} \frac{\partial^q f(\vec{x^o})}{\partial x_{i_1}\cdots\partial x_{i_q}} \Delta x_{i_1}\cdots\Delta x_{i_q}\,. \tag{9}$$

(a) Check the equality for functions of three variables by computing the Taylor series expansion up to the third order.

Solution: We have ($n = 3$)

$$\begin{aligned}
f(\vec{x}) &= \sum_{k_1,k_2,k_3=0}^{\infty} \frac{\partial_1^{k_1}\partial_2^{k_2}\partial_2^{k_2} f(\vec{x^o})}{k_1!k_2!k_3!} \Delta x_1^{k_1}\Delta x_2^{k_2}\Delta x_3^{k_3} = f(\vec{x^o}) + \sum_{i=1}^{3} \frac{\partial f(\vec{x^o})}{\partial x_i}\Delta x_i \\
&+ \frac{1}{2}\sum_{i=1}^{3} \frac{\partial^2 f(\vec{x^o})}{\partial x_i^2}\Delta x_i^2 + \sum_{1=i<j}^{3} \frac{\partial^2 f(\vec{x^o})}{\partial x_i \partial x_j}\Delta x_i \Delta x_j \\
&+ \frac{1}{3!}\sum_{i=1}^{3} \frac{\partial^3 f(\vec{x^o})}{\partial x_i^3}\Delta x_i^3 + \frac{1}{2}\sum_{1=i<j}^{3} \frac{\partial^3 f(\vec{x^o})}{\partial x_i^2 \partial x_j}\Delta x_i^2 \Delta x_j + \frac{1}{2}\sum_{1=i<j}^{3} \frac{\partial^3 f(\vec{x^o})}{\partial x_i \partial x_j^2}\Delta x_i \Delta x_j^2 \\
&+ \frac{\partial^3 f(\vec{x^o})}{\partial x_1 \partial x_2 \partial x_3}\Delta x_1 \Delta x_2 \Delta x_3 + \mathcal{O}(\Delta x^4)\,,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
f(\vec{x}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i_1,\ldots,i_k=1}^{3} \frac{\partial^k f(\vec{x^o})}{\partial x_{i_1}\cdots\partial x_{i_k}} \Delta x_{i_1}\cdots\Delta x_{i_k} = f(\vec{x^o}) + \sum_{i=1}^{3} \frac{\partial f(\vec{x^o})}{\partial x_i}\Delta x_i \\
&+ \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 f(\vec{x^o})}{\partial x_i \partial x_j}\Delta x_i \Delta x_j + \frac{1}{3!}\sum_{i,j,k=1}^{3} \frac{\partial^3 f(\vec{x^o})}{\partial x_i \partial x_j \partial x_j}\Delta x_i \Delta x_j \Delta x_k + \mathcal{O}(\Delta x^4) \\
&= f(\vec{x^o}) + \sum_{i=1}^{n} \frac{\partial f(\vec{x^o})}{\partial x_i}\Delta x_i + \frac{1}{2}\sum_{i=1}^{3} \frac{\partial^2 f(\vec{x^o})}{\partial x_i^2}\Delta x_i^2 + \sum_{1=i<j}^{3} \frac{\partial^2 f(\vec{x^o})}{\partial x_i \partial x_j}\Delta x_i \Delta x_j \\
&+ \frac{1}{3!}\sum_{i=1}^{3} \frac{\partial^3 f(\vec{x^o})}{\partial x_i^3}\Delta x_i^3 + \frac{1}{2}\sum_{1=i<j}^{3} \frac{\partial^3 f(\vec{x^o})}{\partial x_i^2 \partial x_j}\Delta x_i^2 \Delta x_j + \frac{1}{2}\sum_{1=i<j}^{3} \frac{\partial^3 f(\vec{x^o})}{\partial x_i \partial x_j^2}\Delta x_i \Delta x_j^2 \\
&+ \frac{\partial^3 f(\vec{x^o})}{\partial x_1 \partial x_2 \partial x_3}\Delta x_1 \Delta x_2 \Delta x_3 + \mathcal{O}(\Delta x^4)\,,
\end{aligned} \tag{11}$$

which proves the formula up to the third order.

(b) Check the equality by computing the Taylor series expansion up to the third order.

Solution: We have

$$
\begin{aligned}
f(\vec{x}) &= \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\partial_1^{k_1}\cdots\partial_n^{k_n} f(\vec{x^o})}{k_1!\cdots k_n!}\Delta x_1^{k_1}\cdots\Delta x_n^{k_n} = f(\vec{x^o}) + \sum_{i=1}^{n}\frac{\partial f(\vec{x^o})}{\partial x_i}\Delta x_i \\
&+ \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 f(\vec{x^o})}{\partial x_i^2}\Delta x_i^2 + \sum_{1=i<j}^{n}\frac{\partial^2 f(\vec{x^o})}{\partial x_i\partial x_j}\Delta x_i\Delta x_j \\
&+ \frac{1}{3!}\sum_{i=1}^{n}\frac{\partial^3 f(\vec{x^o})}{\partial x_i^3}\Delta x_i^3 + \frac{1}{2}\sum_{1=i<j}^{n}\frac{\partial^3 f(\vec{x^o})}{\partial x_i^2\partial x_j}\Delta x_i^2\Delta x_j + \frac{1}{2}\sum_{1=i<j}^{n}\frac{\partial^3 f(\vec{x^o})}{\partial x_i\partial x_j^2}\Delta x_i\Delta x_j^2 \\
&+ \sum_{1=i<j<k}^{n}\frac{\partial^3 f(\vec{x^o})}{\partial x_i\partial x_j\partial x_k}\Delta x_i\Delta x_j\Delta x_k + \mathcal{O}(\Delta x^4)\,,
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
f(\vec{x}) &= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i_1,\ldots,i_k=1}^{n}\frac{\partial^k f(\vec{x^o})}{\partial x_{i_1}\cdots\partial x_{i_k}}\Delta x_{i_1}\cdots\Delta x_{i_k} = f(\vec{x^o}) + \sum_{i=1}^{n}\frac{\partial f(\vec{x^o})}{\partial x_i}\Delta x_i \\
&+ \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f(\vec{x^o})}{\partial x_i\partial x_j}\Delta x_i\Delta x_j + \frac{1}{3!}\sum_{i,j,k=1}^{n}\frac{\partial^3 f(\vec{x^o})}{\partial x_i\partial x_j\partial x_j}\Delta x_i\Delta x_j\Delta x_k + \mathcal{O}(\Delta x^4) \\
&= f(\vec{x^o}) + \sum_{i=1}^{n}\frac{\partial f(\vec{x^o})}{\partial x_i}\Delta x_i + \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 f(\vec{x^o})}{\partial x_i^2}\Delta x_i^2 + \sum_{1=i<j}^{n}\frac{\partial^2 f(\vec{x^o})}{\partial x_i\partial x_j}\Delta x_i\Delta x_j \\
&+ \frac{1}{3!}\sum_{i=1}^{n}\frac{\partial^3 f(\vec{x^o})}{\partial x_i^3}\Delta x_i^3 + \frac{1}{2}\sum_{1=i<j}^{n}\frac{\partial^3 f(\vec{x^o})}{\partial x_i^2\partial x_j}\Delta x_i^2\Delta x_j + \frac{1}{2}\sum_{1=i<j}^{n}\frac{\partial^3 f(\vec{x^o})}{\partial x_i\partial x_j^2}\Delta x_i\Delta x_j^2 \\
&+ \sum_{1=i<j<k}^{n}\frac{\partial^3 f(\vec{x^o})}{\partial x_i\partial x_j\partial x_k}\Delta x_i\Delta x_j\Delta x_k + \mathcal{O}(\Delta x^4)\,,
\end{aligned}
\tag{13}
$$

which proves the formula up to the third order.

(c) Show that the Taylor series can be equivalently written as in (9) for functions of n variables

Solution: We have

$$\begin{aligned}
f(\vec{x}) &= \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{\partial_1^{k_1}\cdots\partial_n^{k_n} f(\vec{x^o})}{k_1!\cdots k_n!} \Delta x_1^{k_1}\cdots\Delta x_n^{k_n} \\
&= \sum_{q=0}^{\infty} \frac{1}{q!} \sum_{k_1,\ldots,k_n=0}^{k_1+\cdots+k_n=q} \frac{q!}{k_1!\cdots k_n!} \frac{\partial^q f(\vec{x^o})}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}} \Delta x_1^{k_1}\cdots\Delta x_n^{k_n} \\
&= \sum_{q=0}^{\infty} \frac{1}{q!} \Big(\sum_{i=1}^{n} \frac{\partial^k f(\vec{x^o})}{\partial x_i^q} \Delta x_i^q + q \sum_{i\neq j=1}^{n} \frac{\partial^q f(\vec{x^o})}{\partial x_i^{q-1}\partial x_j} \Delta x_i^{q-1}\Delta x_j \\
&+ \frac{q(q-1)}{2} \sum_{i\neq j=1}^{n} \frac{\partial^q f(\vec{x^o})}{\partial x_i^{q-2}\partial x_j^2} \Delta x_i^{q-2}\Delta x_j^2 + q(q-1) \sum_{i\neq j\neq k=1}^{n} \frac{\partial^q f(\vec{x^o})}{\partial x_i^{q-2}\partial x_j \partial x_k} \Delta x_i^{q-2}\Delta x_j \Delta x_k \\
&+ \frac{q!}{(q-3)!3!} \sum_{i\neq j=1}^{n} \frac{\partial^q f(\vec{x^o})}{\partial x_i^{q-3}\partial x_j} \Delta x_i^{q-3}\Delta x_j^3 + \frac{q!}{(q-3)!2!} \sum_{i\neq j\neq k=1}^{n} \frac{\partial^q f(\vec{x^o})}{\partial x_i^{q-3}\partial x_j^2 \partial x_k} \Delta x_i^{q-3}\Delta x_j^2 \Delta x_k \\
&+ \frac{q!}{(q-3)!} \sum_{i\neq j\neq k\neq l=1}^{n} \frac{\partial^q f(\vec{x^o})}{\partial x_i^{q-3}\partial x_j \partial x_k \partial x_l} \Delta x_i^{q-3}\Delta x_j \Delta x_k \Delta x_l + \cdots \Big) \\
&= \sum_{q=0}^{\infty} \frac{1}{q!} \sum_{i_1,\ldots,i_q=1}^{n} \frac{\partial^q f(\vec{x^o})}{\partial x_{i_1}\cdots\partial x_{i_q}} \Delta x_{i_1}\cdots\Delta x_{i_q} \,.
\end{aligned} \tag{14}$$

The last step has to be proven.