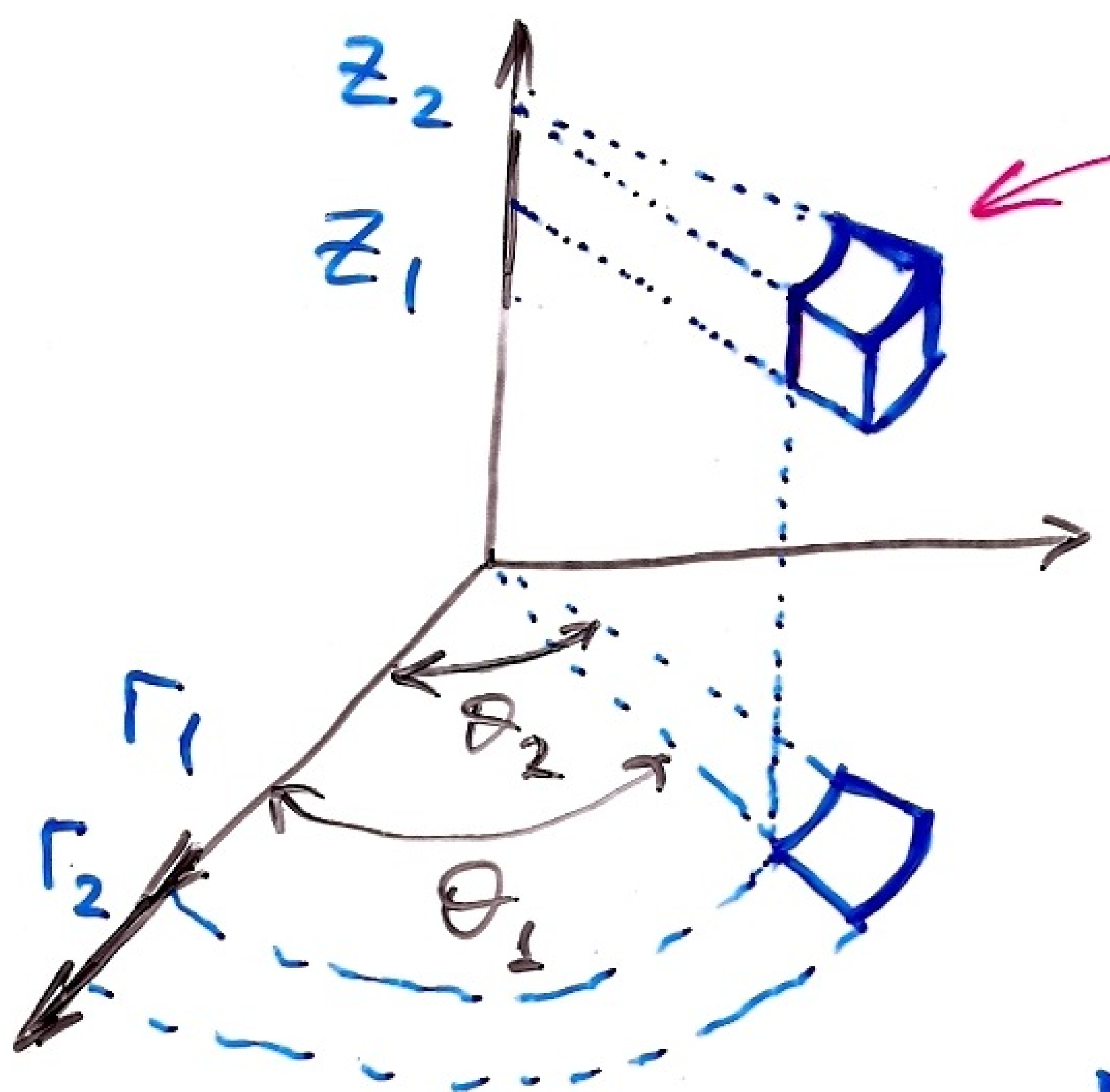
Triple Integrals in cylindrical and spherical coordinates

Cylindrical coordinates



cylindrical wedge or cylindrical element of volume is interior of intersection of $r = \text{const}$, $\theta = \text{const}$, $z = \text{const}$

So, two cylinders : $r = r_1$, $r = r_2$
two half-planes: $\theta = \theta_1$, $\theta = \theta_2$
two planes : $z = z_1$, $z = z_2$

The dimensions: $\theta_2 - \theta_1$, $r_2 - r_1$, $z_2 - z_1$ are called the central angle, thickness and height of the wedge.

Divide G by cylindrical wedges

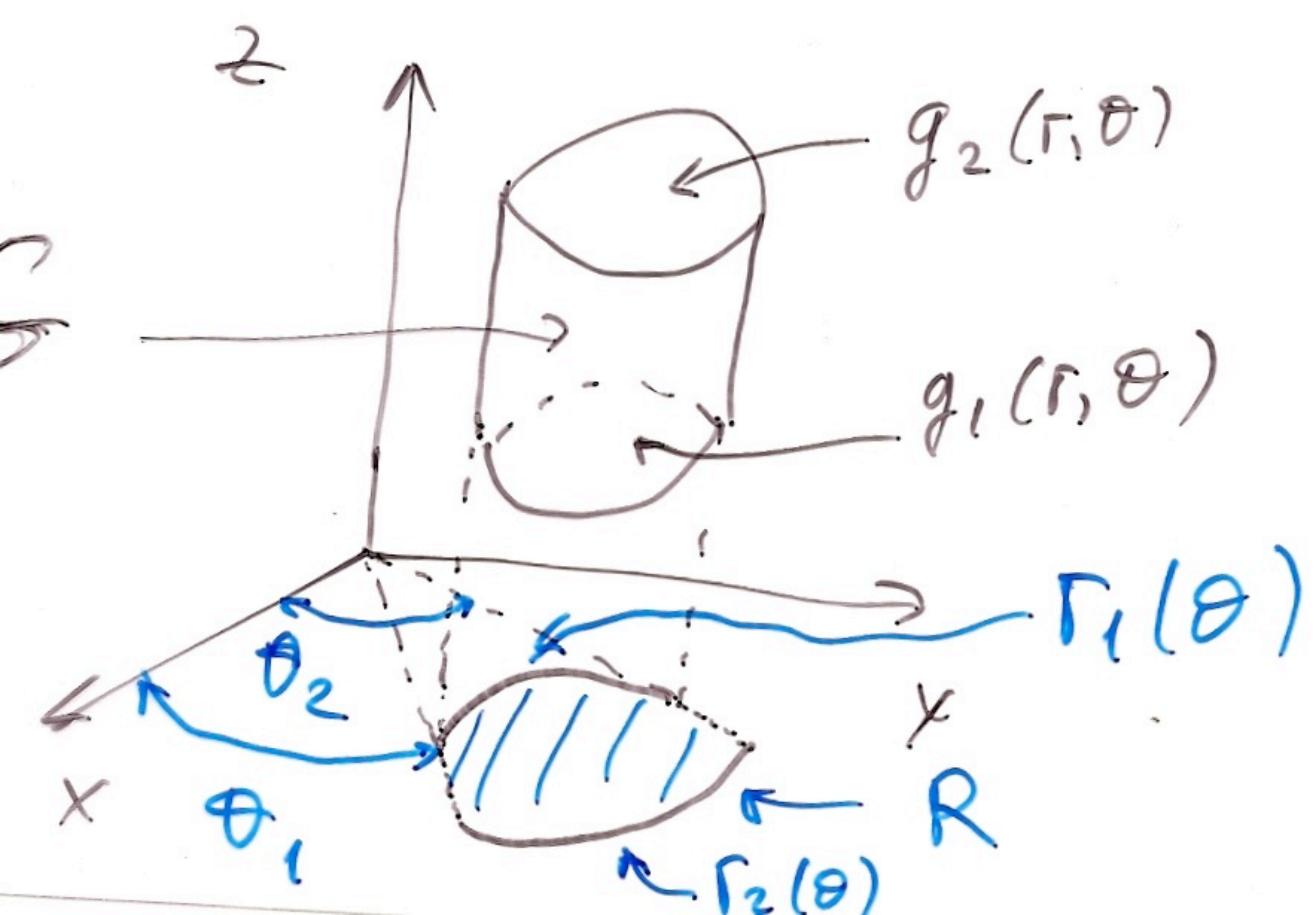
$$\iiint\limits_G f(r,\theta,z)\,dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*)\Delta V_k$$

$$\Delta V_k = [\text{area of base}]\cdot[\text{height}] = r_k^* \Delta r_k \Delta \theta_k \Delta z_k$$

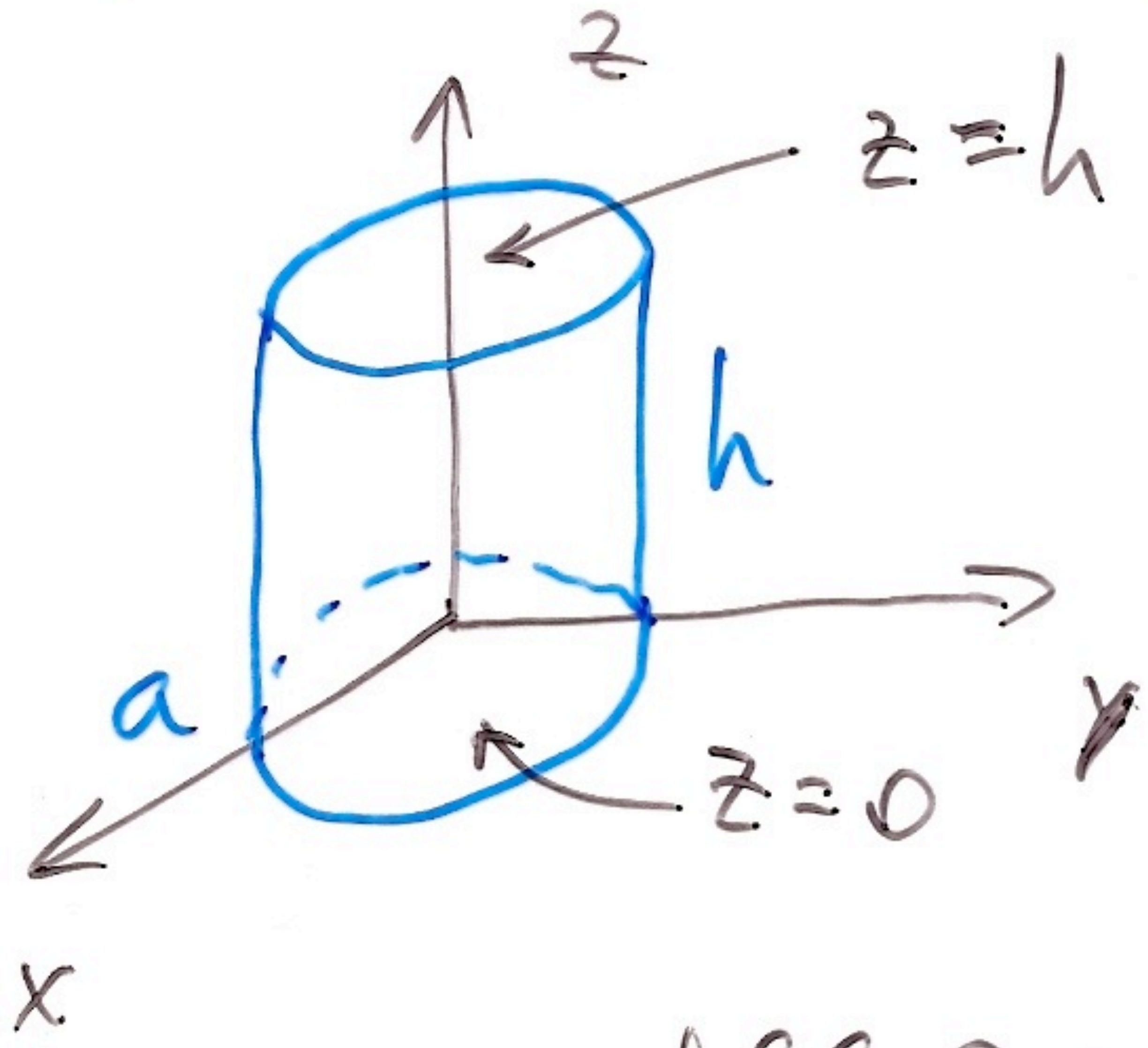
Theorem Let G be a solid whose upper surface is $z = g_2(r,\theta)$ and whose lower surface is $z = g_1(r,\theta)$ in cylindrical coordinates.
If projection of G on the xy-plane is a simple polar region R, and if $f(r,\theta,z)$ is continuous on G, then

$$\iiint_G f(r,\theta,z)\,dV = \iint_R \left[\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,dz\right] dA =$$

$$= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$



Ex. Find the mass and centre of gravity of a cylinder of height h and radius a assuming the density is proportional to the distance between the point and the base 3



$\delta(x,y,z) = K \cdot z$; $K > 0$ is const.

In cylindrical coordinates

$\delta(r,\theta,z) = K \cdot z$

$$M = \iiint_G \delta(r,\theta,z)\, dV =$$

$$= \int_0^{2\pi}\left[\int_0^a \left[\int_0^h K z r \, dz\right] dr\right] d\theta =$$

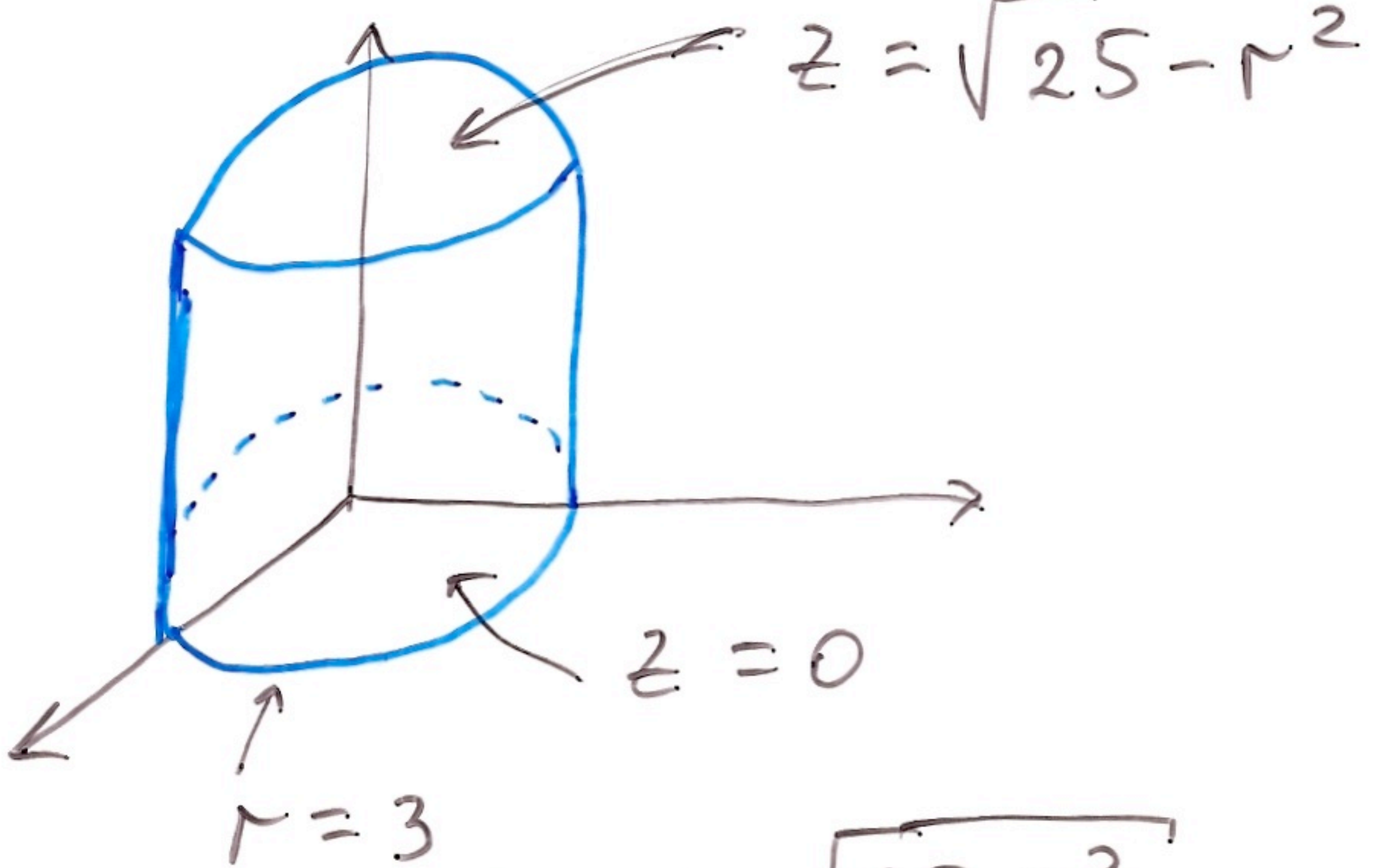
$$= 2\pi \frac{1}{2} a^2 \frac{1}{2} K h^2 = \frac{1}{2} K h^2 \pi a^2$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$\bar{z} = \frac{1}{M} \iiint_G z\, \delta(r,\theta,z)\, dV = \frac{1}{M} \int_0^{2\pi}\int_0^a\int_0^h K z^2 r \, dz\, dr\, d\theta$$

$$= \frac{1}{\frac{1}{2}\cancel{K} h^2 \cancel{\pi} \cancel{a^2}} \cdot \cancel{2\pi} \frac{1}{2} \cancel{a^2} \cdot \frac{1}{3} \cancel{K} h^3 = \frac{2}{3} h$$

Ex. V and centroid of G bounded above by $z=\sqrt{25-x^2-y^2}$, below by the xy-plane, and laterally by $x^2+y^2=9$



$$V=\int_0^{2\pi}\int_0^{3}\int_0^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta =$$

$$= 2\pi\int_0^3 r\sqrt{25-r^2}\,dr =$$

$$= \pi\left(-\frac{2}{3}\left(25-r^2\right)^{\frac{3}{2}}\right)\Big|_0^3 =$$

$$= \pi\left(-\frac{2}{3}\cdot 4^3+\frac{2}{3}\cdot 5^3\right) = \frac{2}{3}\pi(125-64) =$$

$$= \frac{122}{3}\pi$$

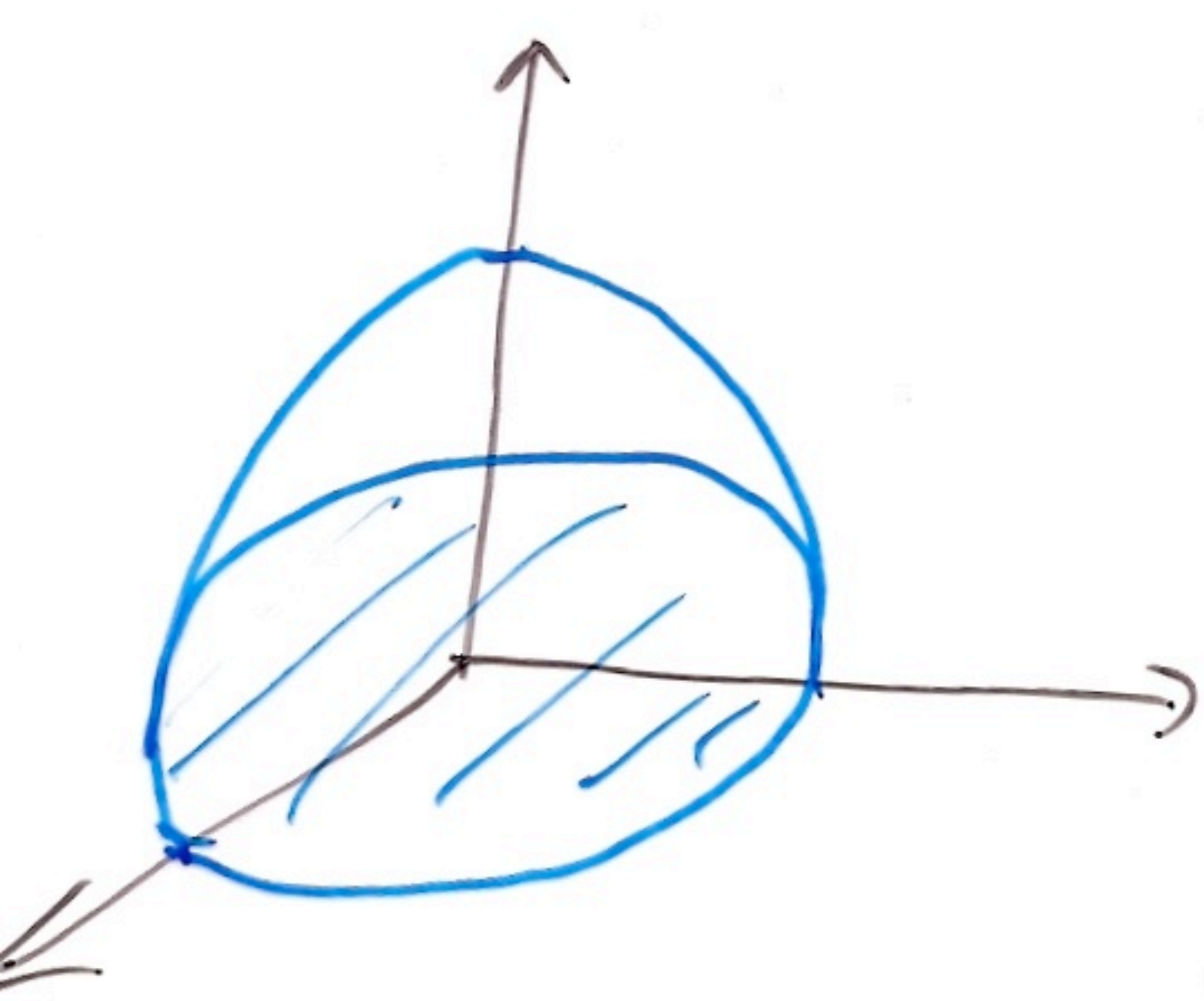
$$\bar{z} = \frac{1107}{488} \; ; \; \bar{x}=\bar{y}=0$$

Converting triple integrals from rectangular to cylindrical coordinates

$$\boxed{\iiint_G f(x,y,z)\,dV = \iiint_G f(r\cos\theta, r\sin\theta, z)\, r \times dz\,dr\,d\theta}$$

Ex.

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2 \underbrace{dz\,dy\,dx}_{dV} \overset{*}{=}$$



From $z = 9 - x^2 - y^2 \Rightarrow$

$\Rightarrow$ the upper surface is a paraboloid.

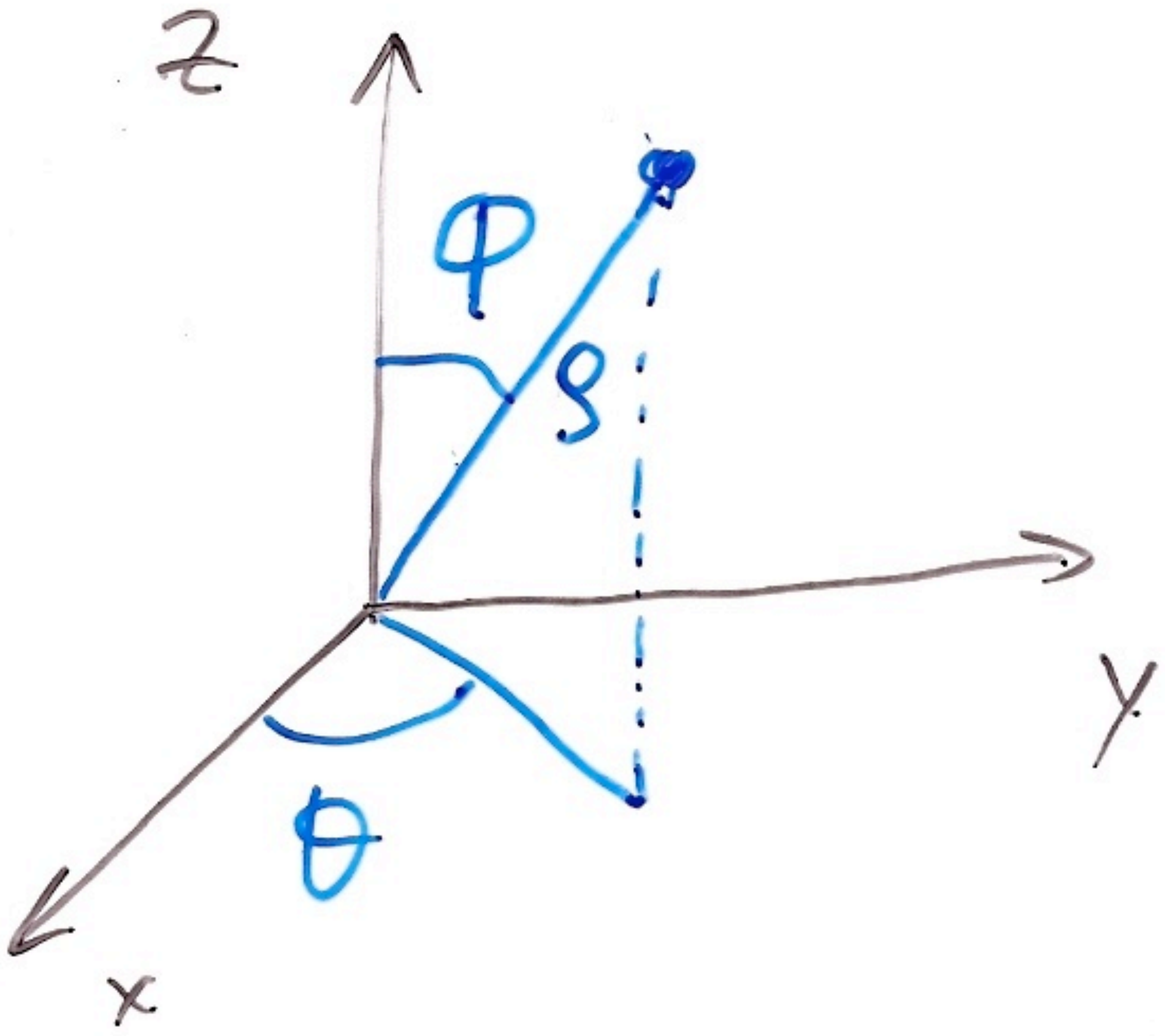
From $z = 0 \Rightarrow$ the lower surface is the xy-plane

Thus, R is the circle $x^2+y^2=9$

$$\overset{*}{=} \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2\cos^2\theta\, r\, dz\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^3 r^3\cos^2\theta\,(9-r^2)\,dr\,d\theta = \frac{243}{4}\pi$$

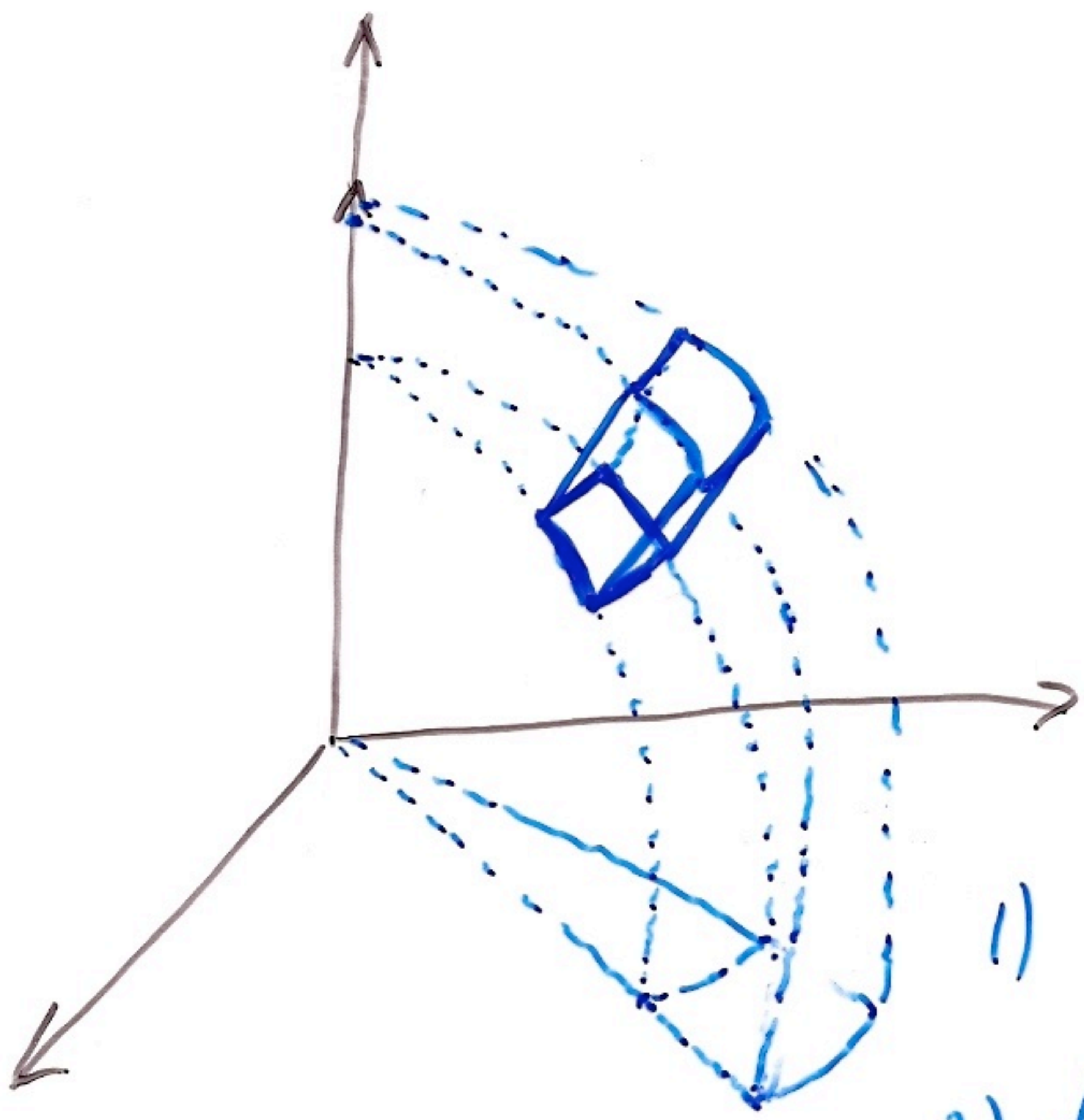
Triple integrals in spherical coordinates



$\rho = \text{const} \rightarrow$ sphere

$\theta = \text{const} \rightarrow$ a half-plane

$\varphi = \text{const} \rightarrow$ right circular cone or for $\varphi = \frac{\pi}{2}$ the xy-plane



A spherical wedge or spherical element of volume is between

1) two spheres: $\rho = \rho_1$, $\rho = \rho_2$
2) two half-planes: $\theta = \theta_1$, $\theta = \theta_2$
3) nappes of two right circular cones: $\varphi = \varphi_1$, $\varphi = \varphi_2$

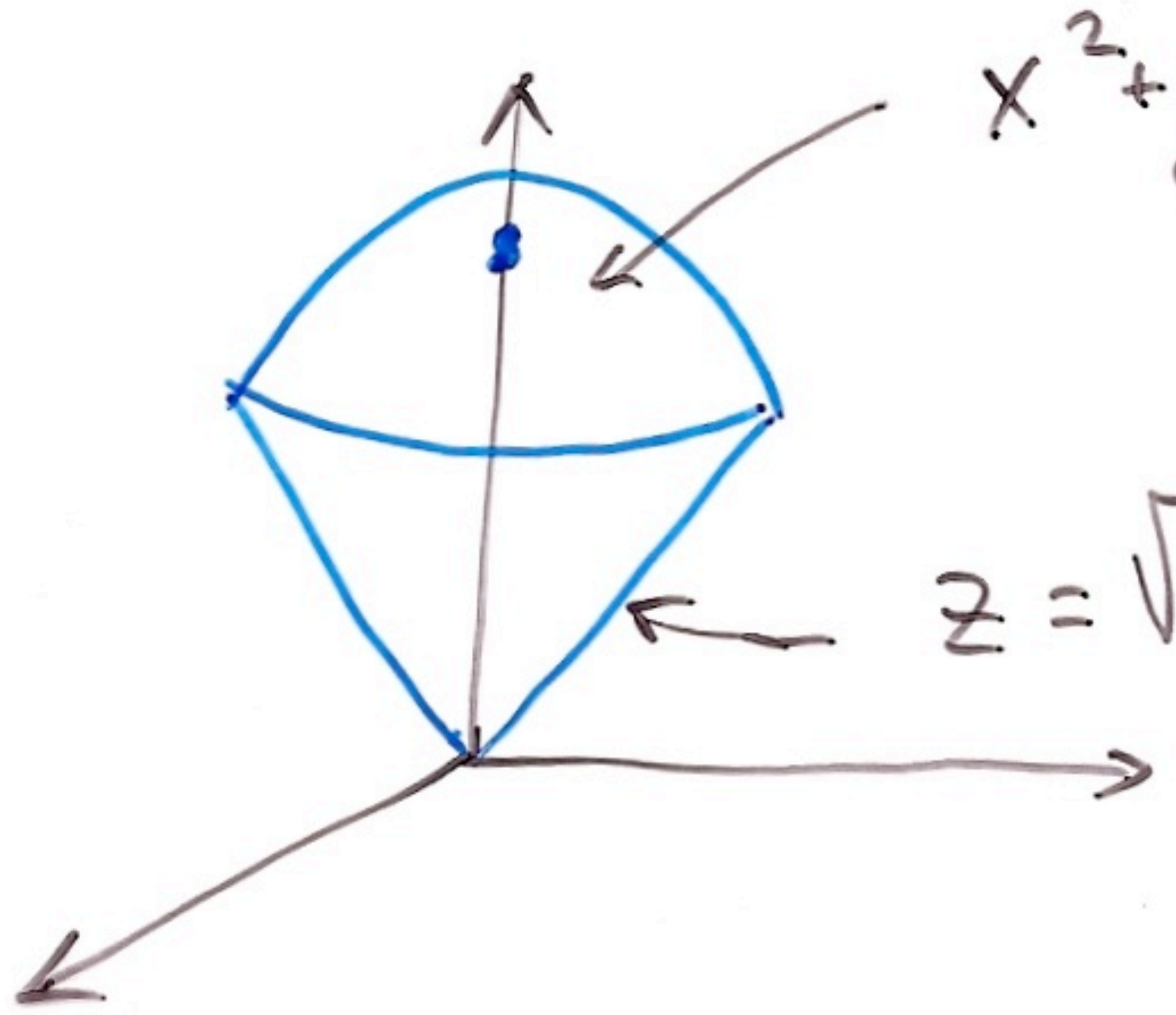
Numbers: $\rho_2 - \rho_1$, $\theta_2 - \theta_1$, $\varphi_2 - \varphi_1$ are the dimensions of a spherical wedge.

$$\iiint_G f(\rho,\theta,\phi)\,dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*)\,\Delta V_k$$

$$\Delta V_k = \rho_k^{*2} \sin\phi_k^* \,\Delta\rho_k\, \Delta\phi_k\, \Delta\theta_k$$

$$\boxed{\iiint_G f(\rho,\theta,\phi)\,dV = \underbrace{\iiint}_{\text{appropriate limits}} f(\rho,\theta,\phi)\,\rho^2 \sin\phi \,d\rho\, d\phi\, d\theta}$$

Ex. V and centroid of G bounded above by $x^2+y^2+z^2=16$ and below by the cone $z=\sqrt{x^2+y^2}$



$x^2+y^2+z^2=16 \Rightarrow \rho = 4$

$z=\sqrt{x^2+y^2} \Rightarrow \phi = \frac{\pi}{4}$

$z = \rho\cos\phi$

$\sqrt{x^2+y^2} = \rho \sin\phi$

$\cos\phi = \sin\phi$

$\Downarrow$

$\phi = \frac{\pi}{4}$

$$V = \iiint_G dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4} \rho^2 \sin\phi \,d\rho\, d\phi\, d\theta =$$

$$= \frac{64}{3}\pi(2-\sqrt{2}) > 0$$

$$\bar{z} = \frac{1}{V}\iiint_G z\,dV = \frac{1}{V}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4} \rho^3 \sin\phi\cos\phi\, d\rho\, d\phi\, d\theta =$$

$$= \frac{32\pi}{V} = \frac{3}{2(2-\sqrt{2})} \approx 2.56$$

Converting triple integrals from rectangular to spherical coordinates

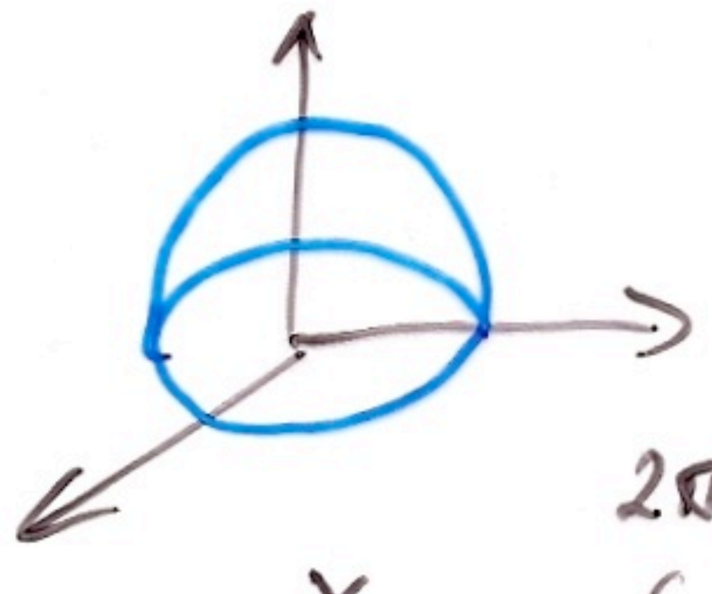
$$\iiint_G f(x,y,z)\,dV =$$

$$= \iiint_{\text{appropriate limits}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \times$$

$$\times\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Ex.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2} \overset{*}{=}$$



$z = \sqrt{4-x^2-y^2} \longrightarrow$ sphere $R=2$

$y = \sqrt{4-x^2} \longrightarrow$ cylinder $R=2$

$$\overset{*}{=} \int_0^{2\pi}\int_0^{2}\int_0^{\pi/2} \rho^2\cos^2\phi\cdot\rho\cdot\rho^2\sin\phi\, d\phi\, d\rho\, d\theta =$$

$$= 2\pi\cdot\frac{1}{6}2^6\cdot\frac{1}{3} = \frac{64\pi}{9}$$