Mechanics 2341, PS 7, 2018

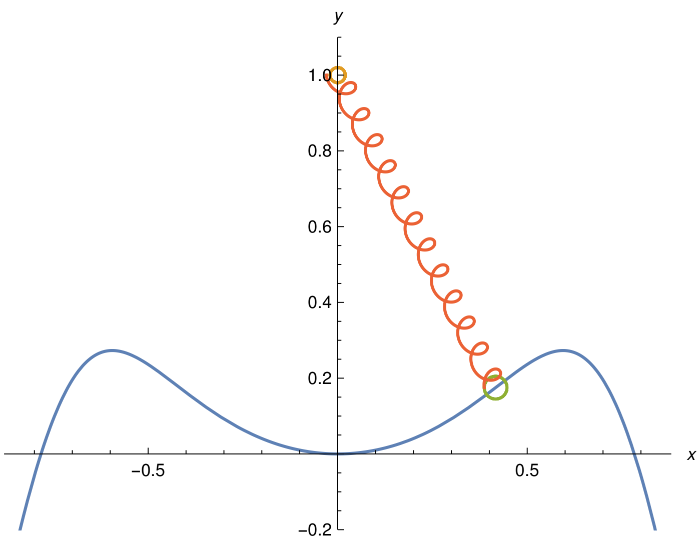
```
FS = FullSimplify;
```

Problem 1.

In[∘]:=

```
Clear[l, g]
l = 1; g = l/2;
f[t_] = l - (l^2 - 2 t^2 + t^6/(3 g^4))^(1/2);
ParametricPlot[{{t, f[t]}, {0.02 Cos[8 t], l + 0.02 Sin[8 t]},
   {g/1.2 + 0.03 Cos[6 t], f[g/1.2] + 0.03 Sin[6 t]},
   {(t + 1) g/2/1.2 - 0.03 Cos[12 Pi t], (f[g/1.2] - l) (t + 1)/2 + l - 0.03 Sin[12 Pi t]}},
 {t, -1, 1}, AxesLabel → {x, y}, PlotRange → {{-0.88, 0.88}, {-1/5, 1.1}}]
```

Out[∘]=



Consider a particle of mass m which is free to move along the curve

$y = l - (l^2 - 2x^2 + x^6/(3g^4))^{(1/2)}$

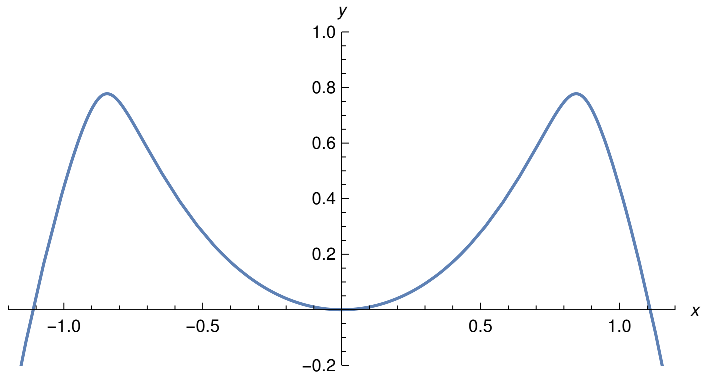
in the xy - plane,
and is attached to an ideal spring whose other end is fixed at a point
with coordinates (0, l). The potential
energy of the spring extended to length L is $kL^2/2$.

1. Use Mathematica to plot the curve for l=1, and g=0.71 l, 0.72 l, 0.73 l, 0.74 l.
Explain why the curve is discontinuous for g=0.73 l, 0.74 l, and find the exact
value of g at which the transition occurs.

Here are the pictures

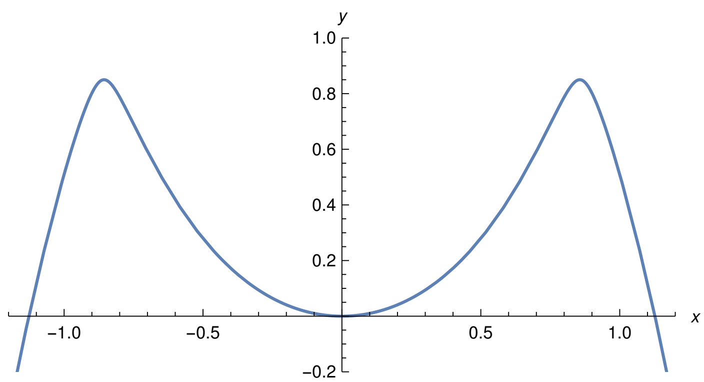
In[∘]:=
```
l = 1; g = 0.71 l;
f[t_] = l - (l^2 - 2 t^2 + t^6 / (3 g^4)) ^ (1 / 2);
ParametricPlot[{{t, f[t]}}, {t, -1.5, 1.5},
 AxesLabel → {x, y}, PlotRange → {{-1.2, 1.2}, {-1 / 5, 1}}]
```

Out[∘]=



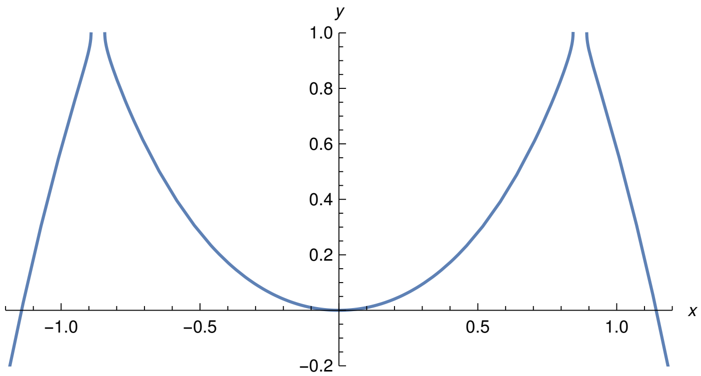
In[∘]:=
```
l = 1; g = 0.72 l;
f[t_] = l - (l^2 - 2 t^2 + t^6 / (3 g^4)) ^ (1 / 2);
ParametricPlot[{{t, f[t]}}, {t, -1.5, 1.5},
 AxesLabel → {x, y}, PlotRange → {{-1.2, 1.2}, {-1 / 5, 1}}]
```

Out[∘]=



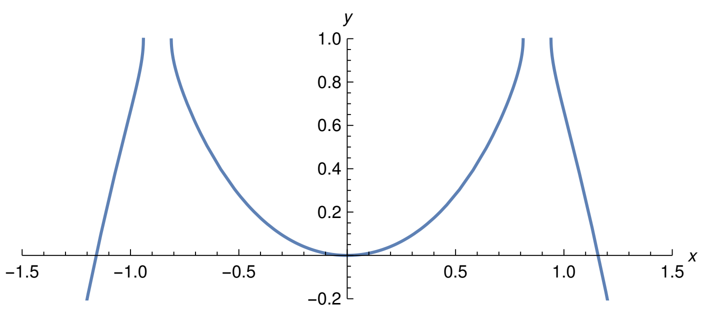
In[∘]:=
```
l = 1; g = 0.73 l;
f[t_] = l - (l^2 - 2 t^2 + t^6 / (3 g^4)) ^ (1 / 2);
ParametricPlot[{{t, f[t]}}, {t, -1.5, 1.5},
 AxesLabel → {x, y}, PlotRange → {{-1.2, 1.2}, {-1 / 5, 1}}]
```

Out[∘]=



In[∘]:=
```
l = 1; g = 0.74 l;
f[t_] = l - (l^2 - 2 t^2 + t^6 / (3 g^4)) ^ (1 / 2);
ParametricPlot[{{t, f[t]}}, {t, -1.5, 1.5},
 AxesLabel → {x, y}, PlotRange → {{-1.5, 1.5}, {-1 / 5, 1}}]
```

Out[∘]=



The curve is discontinuous for g = 0.73 l, 0.74 l because for values of g greater than g_{cr} the function under the square root is negative for some values of x. To find g_{cr} let us consider the function

In[∘]:=
```
Clear[l, g]
g[t_] = l^2 - 2 t^2 + t^6 / (3 g^4)
```

Out[∘]= $l^2 - 2 t^2 + \frac{t^6}{3 g^4}$

Computing the derivative

In[∘]:=
```
Dg[t_] = D[g[t], t]
```

Out[∘]= $-4 t + \frac{2 t^5}{g^4}$

we see that g[t] is decreasing for $|t| \leq 2^{1/4}$ g, and therefore its minimum value is

In[∘]:=
```
Dg[2^(1/4) g] // FS
```
Out[∘]= 0

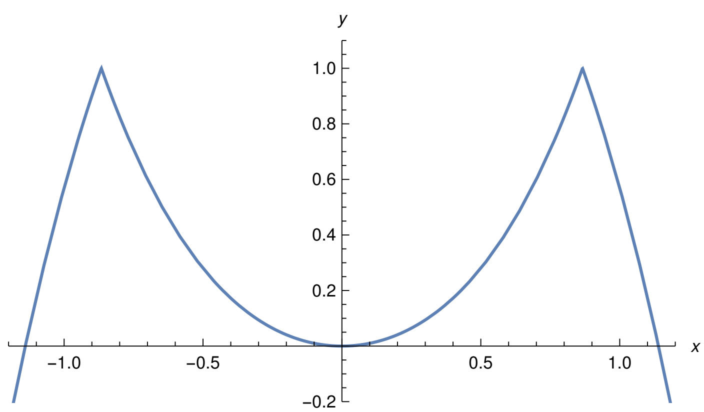
In[∘]:=
```
gmin = g[2^(1/4) g] // FS
```
Out[∘]= $-\frac{4}{3}\sqrt{2}\, g^2 + l^2$

Thus, g_{cr} is equal to

In[∘]:=
```
gcr = g /. Assuming[{g > 0, l > 0}, Solve[-4/3 Sqrt[2] g^2 + l^2 == 0, g]][[2]]
```
Out[∘]= $\frac{\sqrt{3}\, l}{2 \times 2^{1/4}}$

In[∘]:=
```
gcr // N
```
Out[∘]= 0.728238 l

Here is the plot of the curve for $g = g_{cr}$

In[∘]:=
```
l = 1; g = Sqrt[3]/(2 × 2^(1/4)) l;
f[t_] = l - (l^2 - 2 t^2 + t^6 / (3 g^4)) ^ (1/2);
ParametricPlot[{{t, f[t]}}, {t, -1.5, 1.5},
 AxesLabel → {x, y}, PlotRange → {{-1.2, 1.2}, {-1/5, 1.1}}]
```
Out[∘]=


2. Lagrangian

In[∘]:=
```
$Assumptions = {m > 0, k > 0, l > 0};
```
Functions

In[∘]:=
```
Clear[l, g, x, y, z, r, ϕ, L, t, ΔX]
x[t_]; y[t_]; z[t_]; r[t_]; ϕ[t_];
```
Lagrangian

In[•]:=
```
L = m/2 (D[x[t], t]^2 + D[y[t], t]^2) - k/2 ΔX^2
```

Out[•]= $-\frac{k\,\Delta X^2}{2}+\frac{1}{2}m\left(x'[t]^2+y'[t]^2\right)$

where ΔX is the length of the ideal spring whose potential energy is k/2 ΔX^2 for any ΔX

In[•]:=
```
ΔX = (x[t]^2 + (y[t] - l)^2)^(1/2)
```

Out[•]= $\sqrt{x[t]^2+(-l+y[t])^2}$

Constraint

In[•]:=
```
y[t_] = l - (l^2 - 2 x[t]^2 + x[t]^6/(3 g^4))^(1/2)
```

Out[•]= $l-\sqrt{l^2-2\,x[t]^2+\frac{x[t]^6}{3\,g^4}}$

Lagrangian becomes

In[•]:=
```
L = L // FS
```

Out[•]= $\frac{1}{2}\left(-k\left(l^2-x[t]^2+\frac{x[t]^6}{3\,g^4}\right)+m\left(1+\frac{3\left(-2\,g^4\,x[t]+x[t]^5\right)^2}{g^4\left(3\,g^4\,l^2-6\,g^4\,x[t]^2+x[t]^6\right)}\right)x'[t]^2\right)$

3. Potential energy and Equilibrium positions

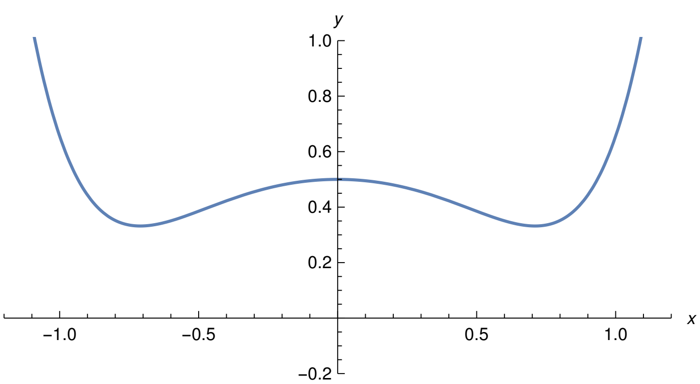
Potential

In[•]:=
```
Clear[U]
U[z_] = -L /. x'[t] → 0 /. x[t] → z
```

Out[•]= $\frac{1}{2}k\left(l^2-z^2+\frac{z^6}{3\,g^4}\right)$

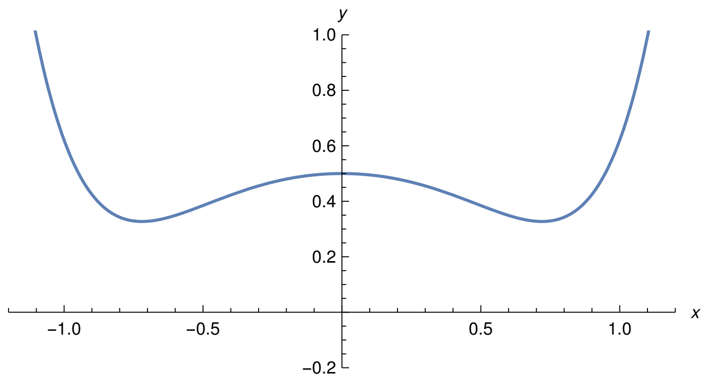
In[•]:=
```
k = 1; l = 1; g = 0.71 l;
U[z_] = 1/2 k (l^2 - z^2 + z^6/(3 g^4));
ParametricPlot[{{t, U[t]}}, {t, -1.5, 1.5},
 AxesLabel → {x, y}, PlotRange → {{-1.2, 1.2}, {-1/5, 1}}]
```

Out[•]=



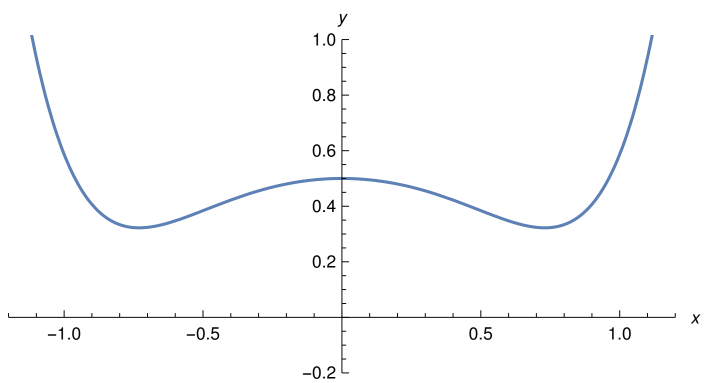
In[•]:=
```
k = 1; l = 1; g = 0.72 l;
U[z_] = 1/2 k (l^2 - z^2 + z^6/(3 g^4));
ParametricPlot[{{t, U[t]}}, {t, -1.5, 1.5},
 AxesLabel → {x, y}, PlotRange → {{-1.2, 1.2}, {-1 / 5, 1}}]
```

Out[•]=

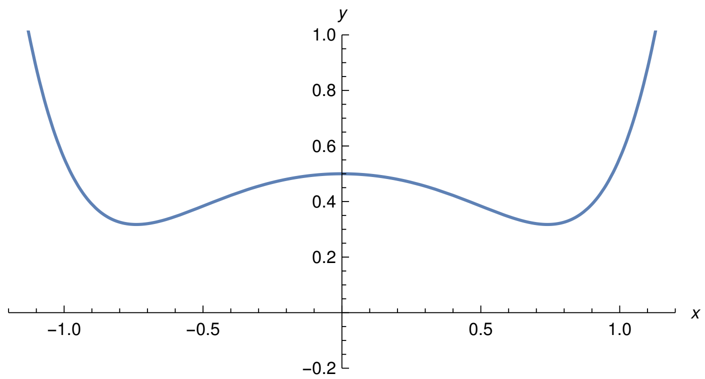


In[•]:=
```
k = 1; l = 1; g = 0.73 l;
U[z_] = 1/2 k (l^2 - z^2 + z^6/(3 g^4));
ParametricPlot[{{t, U[t]}}, {t, -1.5, 1.5},
 AxesLabel → {x, y}, PlotRange → {{-1.2, 1.2}, {-1 / 5, 1}}]
```

Out[•]=



In[∘]:=
```
k = 1; l = 1; g = 0.74 l;
```
$$U[z\_] = \frac{1}{2} k \left(l^2 - z^2 + \frac{z^6}{3 g^4}\right);$$
```
ParametricPlot[{{t, U[t]}}, {t, -1.5, 1.5},
 AxesLabel → {x, y}, PlotRange → {{-1.2, 1.2}, {-1 / 5, 1}}]
```

Out[∘]=


Equilibrium positions have to satisfy D[U,x] = 0

In[∘]:=
```
Clear[l, g, k]
```
$$U[z\_] = \frac{1}{2} k \left(l^2 - z^2 + \frac{z^6}{3 g^4}\right);$$
```
DU[x_] = D[U[x], x] // FS
```

Out[∘]= $k\, x \left(-1 + \frac{x^4}{g^4}\right)$

In[∘]:=
```
Solve[DU[x] == 0, x]
```

Out[∘]= {{x → 0}, {x → -g}, {x → -ⅈ g}, {x → ⅈ g}, {x → g}}

Thus, there are two inequivalent solutions $x_1 = 0$ and $x_2 = g$. Let’s analyse their stability

First equilibrium position

In[∘]:=
```
x1 = 0;
```

Computing the second derivative of U, one gets

In[∘]:=
```
DDU = D[DU[x], x] // FS
```

Out[∘]= $k \left(-1 + \frac{5 x^4}{g^4}\right)$

In[∘]:=
```
DDU1 = DDU /. x → x1
```

Out[∘]= $-k$

Thus, $x_1 = 0$ is unstable

Second equilibrium position

In[•]:= `x2 = g;`

In[•]:= `DDU2 = DDU /. x → x2 // FS`

Out[•]= $4\,k$

and it is positive. Thus, $x_2 = g$ is stable.

4. Quadratic Lagrangian and frequency

Expanding L up to quadratic order in x − x_2 and x', one gets

In[•]:= `L`

Out[•]= $$\frac{1}{2}\left(-k\left(l^2 - x[t]^2 + \frac{x[t]^6}{3\,g^4}\right) + m\left(1 + \frac{3\left(-2\,g^4\,x[t] + x[t]^5\right)^2}{g^4\left(3\,g^4\,l^2 - 6\,g^4\,x[t]^2 + x[t]^6\right)}\right)x'[t]^2\right)$$

In[•]:= `L2 = Collect[(Series[(L /. {x[t] → x2 + eps x[t], x′[t] → eps x′[t]}), {eps, 0, 2}] // Normal) /. eps → 1, {x[t], x'[t]}, FS]`

Out[•]= $$\frac{1}{6}k\left(2\,g^2 - 3\,l^2\right) - 2\,k\,x[t]^2 + \frac{1}{2}\left(1 + \frac{3\,g^2}{-5\,g^2 + 3\,l^2}\right)m\,x'[t]^2$$

The frequency of oscillations about the second equilibrium position is

In[•]:= `ω2 = (- Coefficient[L2, x[t]^2] / Coefficient[L2, x'[t]^2]) ^ (1/2) // FS`

Out[•]= $$2\sqrt{\frac{k}{m + \frac{3\,g^2\,m}{-5\,g^2 + 3\,l^2}}}$$

In[•]:= $$\frac{4\,k}{m + \frac{3\,g^2\,m}{-5\,g^2+3\,l^2}}\ \texttt{// Factor}$$

Out[•]= $$\frac{4\,k\left(5\,g^2 - 3\,l^2\right)}{\left(2\,g^2 - 3\,l^2\right)m}$$

In[•]:= $$\frac{4\,k\left(1 - 5\,g^2/3/\,l^2\right)}{\left(1 - 2\,g^2/3/\,l^2\right)m}$$

Out[•]= $$\frac{4\,k\left(1 - \frac{5\,g^2}{3\,l^2}\right)}{\left(1 - \frac{2\,g^2}{3\,l^2}\right)m}$$

In[•]:= $$\omega_2\text{^}2 - \frac{4\,k\left(1 - \frac{5\,g^2}{3\,l^2}\right)}{m\left(1 - \frac{2\,g^2}{3\,l^2}\right)}\ \texttt{// FS}$$

Out[•]= 0

Note that $\omega_2{}^2 > 0$ for $g < g_{cr}$

Problem 2.

Determine the forced oscillations and find the energy acquired by an oscillator experiencing a force
$F(t) = F_0\, e^{\alpha t} \cos\beta t$, $t < 0$,
$F(t) = F_0\, e^{-\alpha t} \cos\beta t$, $t > 0$,
$\alpha > 0$, $\beta > 0$.
The initial energy as t -> -∞ is E_0 =0.
Analyze the limits (a) α -> 0, β fixed, and (b) β -> 0, α fixed.

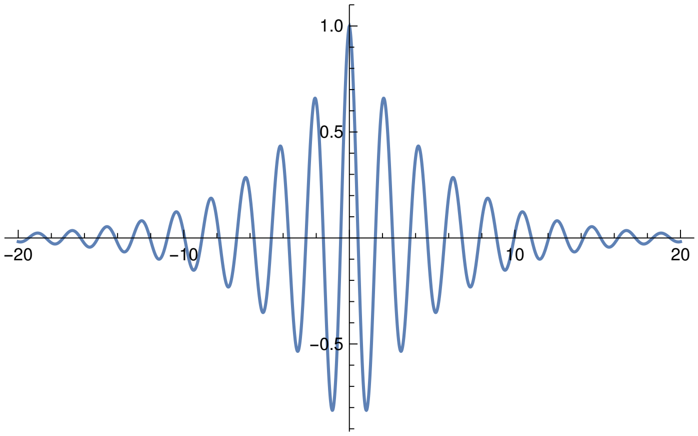
```
Clear[x, y, z, r, ϕ, L, t, ω, F]
x[t_]; y[t_]; z[t_]; r[t_]; ϕ[t_];
$Assumptions = {m > 0, ω > 0, α > 0, β > 0};
```

It is convenient to use the complex force
$F(t) = F_0\, e^{(\alpha + i\beta) t}$, $t < 0$,
$F(t) = F_0\, e^{(-\alpha + i\beta) t}$, $t > 0$,
and then the real part of the solution of the eom is what we need

Eom

```
F[t_] = If[t ≤ 0, F0 E^((α + I β) t), F0 E^((-α + I β) t)];
```

```
Plot[Re[(F[t] /. {F0 → 1, α → 1/5, β → 3})], {t, -20, 20}, PlotRange → All]
```




```
f[t_] = F[t] / m;
```

```
Eom = D[x[t], {t, 2}] + ω^2 x[t] - f0 e^(γ t)
```

$-e^{t\gamma} f_0 + \omega^2 x[t] + x''[t]$

Solution

t < 0

```
x[t_] = B e^(γ t);
```

```
Solve[Eom == 0, B]
```

$$\left\{\left\{B \to \frac{f_0}{\gamma^2 + \omega^2}\right\}\right\}$$

So, for $t \le 0$ the solution is the real part of

```
xm[t_] = x[t] /. {B → f0/(γ^2 + ω^2)} /. f0 → F0/m /. γ -> α + I β
```

$$\frac{e^{t(\alpha + i\beta)} F_0}{m\left((\alpha + i\beta)^2 + \omega^2\right)}$$

The real part of the solution is

```
Xm[t_] = 1/2 (xm[t] + (xm[t] /. {i → -i})) // ExpToTrig // FS // Factor // FS
```

$$\left(e^{t\alpha}\left((\alpha^2 - \beta^2 + \omega^2)\,\mathrm{Cos}[t\beta] + 2\alpha\beta\,\mathrm{Sin}[t\beta]\right)F_0\right) \Big/ \left(m\left(\alpha^2 + (\beta - \omega)^2\right)\left(\alpha^2 + (\beta + \omega)^2\right)\right)$$

At $t = 0$ we find

```
x0 = xm[0]
```

$$\frac{F_0}{m\left((\alpha + i\beta)^2 + \omega^2\right)}$$

```
v0 = D[xm[t], t] /. t → 0
```

$$\frac{(\alpha + i\beta) F_0}{m\left((\alpha + i\beta)^2 + \omega^2\right)}$$

These are the initial conditions for $t > 0$

t > 0

```
x[t_] = Ap e^(I ω t) + Am e^(-I ω t) + B e^(γ t);
```

```
Eom // FS
```

$$e^{t\gamma}\left(B\left(\gamma^2 + \omega^2\right) - f_0\right)$$

The first term is the general solution of the homogeneous equation $\omega^2\, x[t] + x''[t] = 0$, and the second term is the same as before but with different γ. Ap and Am are found from the initial conditions

```
xp[t_] = x[t] /. {B → f₀/(γ² + ω²)} /. f₀ → F₀/m /. γ → -α + I β
```

$$\text{Am}\, e^{-i t \omega} + \text{Ap}\, e^{i t \omega} + \frac{e^{t(-\alpha + i \beta)} F_0}{m\left(\left(-\alpha + i \beta\right)^2 + \omega^2\right)}$$

At t = 0 we find

```
xp0 = xp[0]
```

$$\text{Am} + \text{Ap} + \frac{F_0}{m\left(\left(-\alpha + i \beta\right)^2 + \omega^2\right)}$$

```
vp0 = D[xp[t], t] /. t → 0
```

$$-i\, \text{Am}\, \omega + i\, \text{Ap}\, \omega + \frac{\left(-\alpha + i \beta\right) F_0}{m\left(\left(-\alpha + i \beta\right)^2 + \omega^2\right)}$$

```
Assuming[{α > 0, β > 0, m > 0, F₀ > 0, ω > 0, {Ap, Am} ∈ Complexes},
  Solve[{xp0 == x0, vp0 == v0}, {Ap, Am}]] // FS
```

$$\left\{\left\{\text{Ap} \to -\frac{i \alpha F_0}{m\left(\alpha^2 + (\beta - \omega)^2\right)\omega},\ \text{Am} \to \frac{i \alpha F_0}{m \omega \left(\alpha^2 + (\beta + \omega)^2\right)}\right\}\right\}$$

Thus the solution is

```
xxp[t_] = xp[t] /. {Ap → -(i α F₀)/(m (α² + (β - ω)²) ω), Am → (i α F₀)/(m ω (α² + (β + ω)²))}
```

$$-\frac{i e^{i t \omega} \alpha F_0}{m\left(\alpha^2 + (\beta - \omega)^2\right)\omega} + \frac{e^{t(-\alpha + i \beta)} F_0}{m\left(\left(-\alpha + i \beta\right)^2 + \omega^2\right)} + \frac{i e^{-i t \omega} \alpha F_0}{m \omega \left(\alpha^2 + (\beta + \omega)^2\right)}$$

The real part of the solution is

```
Xp[t_] = 1/2 (xxp[t] + (xxp[t] /. {i → -i, -i -> i})) // Expand
```

$$\frac{i e^{-i t \omega} \alpha F_0}{2 m\left(\alpha^2 + (\beta - \omega)^2\right)\omega} - \frac{i e^{i t \omega} \alpha F_0}{2 m\left(\alpha^2 + (\beta - \omega)^2\right)\omega} + \frac{e^{t(-\alpha - i \beta)} F_0}{2 m\left(\left(-\alpha - i \beta\right)^2 + \omega^2\right)} +$$
$$\frac{e^{t(-\alpha + i \beta)} F_0}{2 m\left(\left(-\alpha + i \beta\right)^2 + \omega^2\right)} + \frac{i e^{-i t \omega} \alpha F_0}{2 m \omega \left(\alpha^2 + (\beta + \omega)^2\right)} - \frac{i e^{i t \omega} \alpha F_0}{2 m \omega \left(\alpha^2 + (\beta + \omega)^2\right)}$$

```
Xp[t_] = FS[ (ⅈ ⅇ^(-ⅈ t ω) α F0)/(2 m (α^2 + (β - ω)^2) ω) - (ⅈ ⅇ^(ⅈ t ω) α F0)/(2 m (α^2 + (β - ω)^2) ω) // Expand] +
  FS[ (ⅇ^(t (-α - ⅈ β)) F0)/(2 m ((-α - ⅈ β)^2 + ω^2)) + (ⅇ^(t (-α + ⅈ β)) F0)/(2 m ((-α + ⅈ β)^2 + ω^2)) // ExpandAll // ExpToTrig] +
  FS[ (ⅈ ⅇ^(-ⅈ t ω) α F0)/(2 m ω (α^2 + (β + ω)^2)) - (ⅈ ⅇ^(ⅈ t ω) α F0)/(2 m ω (α^2 + (β + ω)^2)) // Expand]
```

$$\left(e^{-t\alpha}\left(\left(\alpha^2-\beta^2+\omega^2\right)\mathrm{Cos}[t\,\beta]-2\,\alpha\,\beta\,\mathrm{Sin}[t\,\beta]\right)F_0\right)\Big/\left(m\left(\alpha^2+(\beta-\omega)^2\right)\left(\alpha^2+(\beta+\omega)^2\right)\right)+\frac{\alpha\,\mathrm{Sin}[t\,\omega]\,F_0}{m\left(\alpha^2+(\beta-\omega)^2\right)\omega}+\frac{\alpha\,\mathrm{Sin}[t\,\omega]\,F_0}{m\,\omega\left(\alpha^2+(\beta+\omega)^2\right)}$$

```
Xm[t]
```

$$\left(e^{t\alpha}\left(\left(\alpha^2-\beta^2+\omega^2\right)\mathrm{Cos}[t\,\beta]+2\,\alpha\,\beta\,\mathrm{Sin}[t\,\beta]\right)F_0\right)\Big/\left(m\left(\alpha^2+(\beta-\omega)^2\right)\left(\alpha^2+(\beta+\omega)^2\right)\right)$$

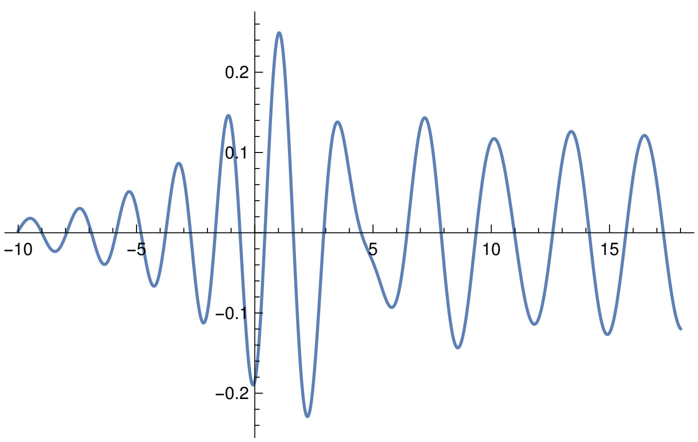
The solution

```
X[t_] = If[t ≤ 0, Xm[t], Xp[t]]
```

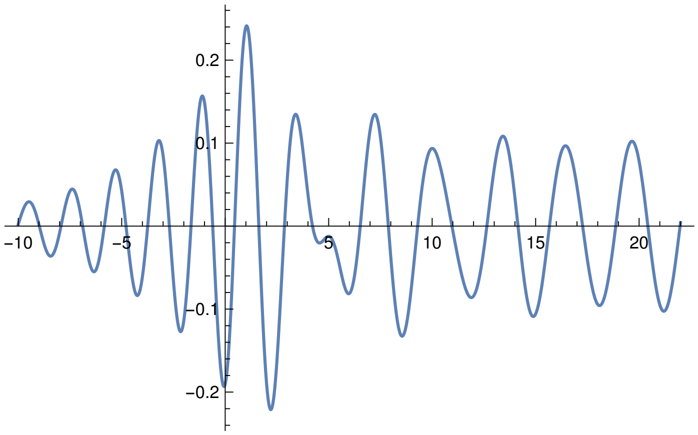
If[t ≤ 0, Xm[t], Xp[t]]

Plots of the solution

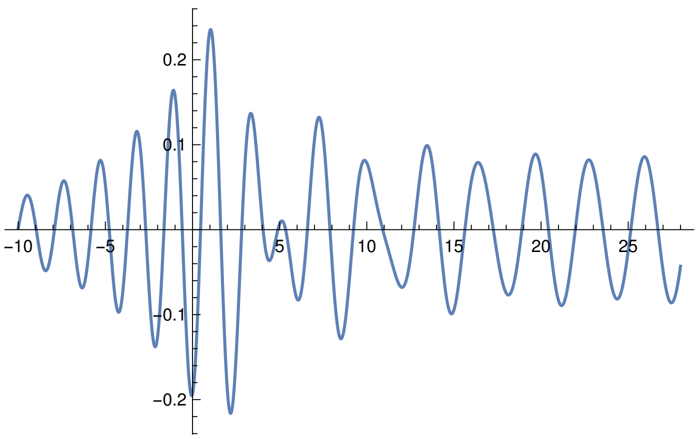
```
Plot[(X[t] /. {F0 → 1, α → 1 / 4, β → 3, ω → 2, m → 1}), {t, -10, 18}, PlotRange → All]
```



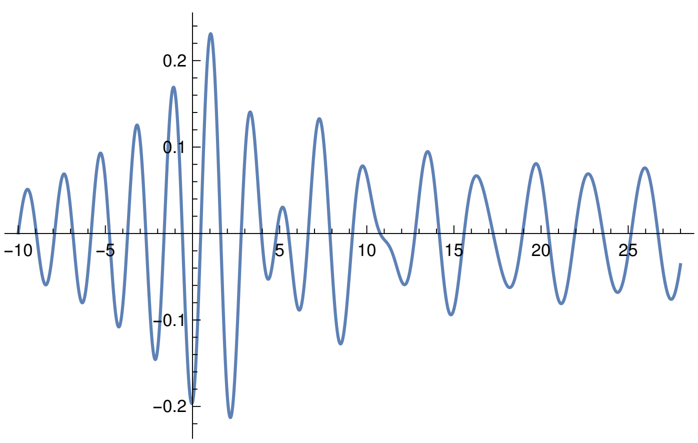
```
Plot[(X[t] /. {F0 → 1, α → 1 / 5, β → 3, ω → 2, m → 1}), {t, -10, 22}, PlotRange → All]
```




```
Plot[(X[t] /. {F0 → 1, α → 1/6, β → 3, ω → 2, m → 1}), {t, -10, 28}, PlotRange → All]
```




```
Plot[(X[t] /. {F0 → 1, α → 1 / 7, β → 3, ω → 2, m → 1}), {t, -10, 28}, PlotRange → All]
```



Energy acquired by the oscillator

At t -> ∞ the solution becomes the harmonic one

```
X∞[t_] = α Sin[t ω] F0 / (m (α^2 + (β - ω)^2) ω) + α Sin[t ω] F0 / (m ω (α^2 + (β + ω)^2));
```

$$X_\infty[t\_] = \frac{\alpha \,\mathrm{Sin}[t\,\omega]\, F_0}{m \left(\alpha^2 + (\beta-\omega)^2\right) \omega} + \frac{\alpha \,\mathrm{Sin}[t\,\omega]\, F_0}{m\,\omega \left(\alpha^2 + (\beta+\omega)^2\right)};$$

whose energy is

```
En = m/2 D[X∞[t], t]^2 + m/2 ω^2 X∞[t]^2 // FS
```

$$\frac{2\,\alpha^2 \left(\alpha^2+\beta^2+\omega^2\right)^2 F_0^2}{m \left(\alpha^2+(\beta-\omega)^2\right)^2 \left(\alpha^2+(\beta+\omega)^2\right)^2}$$

Limit α -> 0, β fixed

```
Series[Xm[t], {α, 0, 1}] // FS
```

$$-\frac{\mathrm{Cos}[t\,\beta]\,F_0}{m\,\beta^2 - m\,\omega^2} + \frac{\left(t\left(-\beta^2+\omega^2\right)\mathrm{Cos}[t\,\beta] + 2\,\beta\,\mathrm{Sin}[t\,\beta]\right) F_0\,\alpha}{m\left(\beta^2-\omega^2\right)^2} + O[\alpha]^2$$

```
Series[Xp[t], {α, 0, 1}] // FS
```

$$-\frac{\mathrm{Cos}[t\,\beta]\,F_0}{m\,\beta^2 - m\,\omega^2} + \left(\left(t\,(\beta-\omega)\,\omega\,(\beta+\omega)\,\mathrm{Cos}[t\,\beta] - 2\,\beta\,\omega\,\mathrm{Sin}[t\,\beta] + 2\left(\beta^2+\omega^2\right)\mathrm{Sin}[t\,\omega]\right) F_0\,\alpha\right) \Big/ \left(m\,\omega\left(\beta^2-\omega^2\right)^2\right) + O[\alpha]^2$$

```
Series[En, {α, 0, 1}] // FS
```

$$O[\alpha]^2$$

Limit β -> 0, α fixed

```
Series[Xm[t], {β, 0, 1}] // FS
```

$$\frac{e^{t\,\alpha}\,F_0}{m\left(\alpha^2+\omega^2\right)} + O[\beta]^2$$

```
Series[Xp[t], {β, 0, 1}] // FS
```

$$\frac{e^{-t\,\alpha}\,\omega\,F_0 + 2\,\alpha\,\mathrm{Sin}[t\,\omega]\,F_0}{m\,\alpha^2\,\omega + m\,\omega^3} + O[\beta]^2$$

```
Series[En, {β, 0, 1}] // FS
```

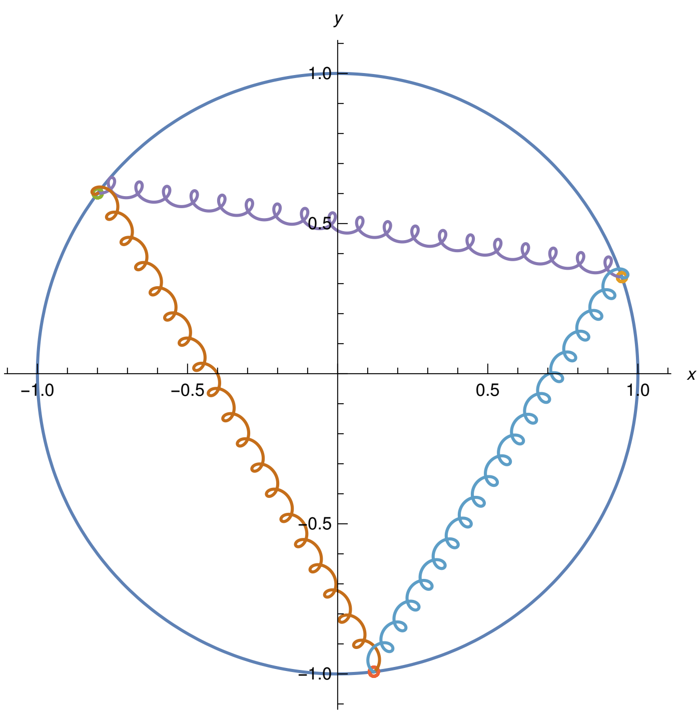
$$\frac{2\,\alpha^2\,F_0^2}{m\left(\alpha^2+\omega^2\right)^2} + O[\beta]^2$$

Problem 3.

Find the normal coordinates and frequencies of a system of three equal masses connected to each other by identical springs and constrained to move on a circle.

In[•]:=
```
t1 = Pi / 6 - Pi / 16;
t2 = Pi / 6 + 2 Pi / 3 - Pi / 26;
t3 = Pi / 6 + 4 Pi / 3 + Pi / 26;
ParametricPlot[{{Cos[t], Sin[t]}, {Cos[t1] + 0.015 Cos[14 t],
    Sin[t1] + 0.015 Sin[14 t]}, {Cos[t2] + 0.015 Cos[14 t], Sin[t2] + 0.015 Sin[14 t]},
   {Cos[t3] + 0.015 Cos[14 t], Sin[t3] + 0.015 Sin[14 t]}, {Cos[t1] +
     (Cos[t2] - Cos[t1]) / 2 / Pi t - 0.03 Sin[19 t], Sin[t1] + (Sin[t2] - Sin[t1]) / 2 / Pi
      t + 0.03 - 0.03 Cos[19 t]}, {Cos[t2] + (Cos[t3] - Cos[t2]) / 2 / Pi t - 0.03 Sin[19 t],
    Sin[t2] + (Sin[t3] - Sin[t2]) / 2 / Pi t + 0.03 - 0.03 Cos[19 t]}, {Cos[t3] +
     (Cos[t1] - Cos[t3]) / 2 / Pi t - 0.03 Sin[19 t], Sin[t3] + (Sin[t1] - Sin[t3]) / 2 / Pi
      t + 0.03 - 0.03 Cos[19 t]}}, {t, 0, 2 Pi}, AxesLabel → {x, y}]
```

Out[•]=



In[•]:=
```
Clear[x, y, z, r, ϕ, t, ω, F, k]
xᵢ_[t_]; yᵢ_[t_]; ϕᵢ_[t_]; ζᵢ_[t_]; ξᵢ_[t_];
$Assumptions = {m > 0, ω > 0, l > 0, k > 0};
```

Lagrangian and constraints

In[•]:=
```
x₄[t_] = x₁[t]; y₄[t_] = y₁[t];
```

In[•]:=
```
Clear[L]
L = Sum[m/2 (D[x_i[t], t]^2 + D[y_i[t], t]^2), {i, 1, 3}] - 1/2 k
   Sum[(((x_i[t] - x_{i+1}[t])^2 + (y_i[t] - y_{i+1}[t])^2)^(1/2) - 3^(1/2) l)^2, {i, 1, 3}]
```

Out[•]=

$$-\frac{1}{2} k \left(\left(-\sqrt{3}\ l + \sqrt{\left((x_1[t] - x_2[t])^2 + (y_1[t] - y_2[t])^2\right)}\right)^2 + \left(-\sqrt{3}\ l + \sqrt{\left((x_2[t] - x_3[t])^2 + (y_2[t] - y_3[t])^2\right)}\right)^2 + \left(-\sqrt{3}\ l + \sqrt{\left((-x_1[t] + x_3[t])^2 + (-y_1[t] + y_3[t])^2\right)}\right)^2\right) + \frac{1}{2} m \left(x_1'[t]^2 + y_1'[t]^2\right) + \frac{1}{2} m \left(x_2'[t]^2 + y_2'[t]^2\right) + \frac{1}{2} m \left(x_3'[t]^2 + y_3'[t]^2\right)$$

It is clear that even without any oscillations the system can rotate with a constant angular velocity about the centre of the circle as a whole with distances between neighboring particles being equal to the equilibrium length of the springs. We assume therefore that in the reference frame which rotates together with the system as a whole, the particles oscillate about points (l,0), (-l/2,$\sqrt{3}$ l/2), (-l/2,-$\sqrt{3}$ l/2), so that their coordinates can be parametrised as

In[•]:=
```
Do[x_i[t_] = l Cos[ϕ_i[t] + 2 Pi/3 (i - 1)];
 y_i[t_] = l Sin[ϕ_i[t] + 2 Pi/3 (i - 1)];, {i, 1, 3}]
```

where ϕ_i are small fluctuations if the system does not rotate as a whole. L in terms of ϕ_i is

In[•]:=
```
L = L // FS
```

Out[•]=

$$\frac{1}{2} l^2 \left(k \left(-15 - 2 \operatorname{Sin}\left[\frac{\pi}{6} - \phi_1[t] + \phi_2[t]\right] + 2\sqrt{6} \sqrt{1 + \operatorname{Sin}\left[\frac{\pi}{6} - \phi_1[t] + \phi_2[t]\right]} - 2 \operatorname{Sin}\left[\frac{\pi}{6} + \phi_1[t] - \phi_3[t]\right] + 2\sqrt{6} \sqrt{1 + \operatorname{Sin}\left[\frac{\pi}{6} + \phi_1[t] - \phi_3[t]\right]} - 2 \operatorname{Sin}\left[\frac{\pi}{6} - \phi_2[t] + \phi_3[t]\right] + 2\sqrt{6} \sqrt{1 + \operatorname{Sin}\left[\frac{\pi}{6} - \phi_2[t] + \phi_3[t]\right]}\right) + m \left(\phi_1'[t]^2 + \phi_2'[t]^2 + \phi_3'[t]^2\right)\right)$$

The Lagrangian is invariant under the shift $\phi_i \to \phi_i + \epsilon$. This is the symmetry that corresponds to the motion of the system as a whole: ``the centre of mass'' with coordinate $\phi_{cm} = 1/3\,(\phi_1 + \phi_2 + \phi_3)$ moves along the circle with constant velocity. This motion is considered as the equilibrium position of the system. To separate this motion we introduce

In[•]:=
```
Do[ϕ_i[t_] = ζ_i[t] + ϕ_cm[t], {i, 1, 3}]
```

where $\zeta_i[t]$ subject to the constraint $\sum_i \zeta_i[t] = 0$, and they are the small fluctuations

In[∘]:= `L = FS[(L // Expand), {Sum[ξi[t], {i, 1, 3}] == 0, Sum[ξi'[t], {i, 1, 3}] == 0}]`

Out[∘]=
$$\frac{1}{2} l^2 \left(k \left(-15 - 2 \mathrm{Sin}\left[\frac{\pi}{6} + \zeta_1[t] - \zeta_3[t]\right] + 2\sqrt{6}\sqrt{1 + \mathrm{Sin}\left[\frac{\pi}{6} + \zeta_1[t] - \zeta_3[t]\right]} - 2 \mathrm{Sin}\left[\frac{\pi}{6} - \zeta_2[t] + \zeta_3[t]\right] + 2\sqrt{6}\sqrt{1 + \mathrm{Sin}\left[\frac{\pi}{6} - \zeta_2[t] + \zeta_3[t]\right]} - 2 \mathrm{Sin}\left[\frac{\pi}{6} + 2\zeta_2[t] + \zeta_3[t]\right] + 2\sqrt{6}\sqrt{1 + \mathrm{Sin}\left[\frac{\pi}{6} + 2\zeta_2[t] + \zeta_3[t]\right]}\right) + 2 m \left(\zeta_2'[t]^2 + \zeta_2'[t]\,\zeta_3'[t] + \zeta_3'[t]^2\right) + 3 m\, \phi_{cm}'[t]^2\right)$$

Excluding $\zeta_3[t]$, one finds that the oscillatory modes are then described by the Lagrangian

In[∘]:= `ξ3[t_] = -Sum[ξi[t], {i, 1, 2}]`

Out[∘]= $-\zeta_1[t] - \zeta_2[t]$

In[∘]:= `L = L /. ϕcm'[t] → 0 // FS`

Out[∘]=
$$\frac{1}{2} l^2 \left(k \left(-15 - 2 \mathrm{Sin}\left[\frac{\pi}{6} - \zeta_1[t] - 2\zeta_2[t]\right] + 2\sqrt{6}\sqrt{1 + \mathrm{Sin}\left[\frac{\pi}{6} - \zeta_1[t] - 2\zeta_2[t]\right]} - 2 \mathrm{Sin}\left[\frac{\pi}{6} - \zeta_1[t] + \zeta_2[t]\right] + 2\sqrt{6}\sqrt{1 + \mathrm{Sin}\left[\frac{\pi}{6} - \zeta_1[t] + \zeta_2[t]\right]} - 2 \mathrm{Sin}\left[\frac{\pi}{6} + 2\zeta_1[t] + \zeta_2[t]\right] + 2\sqrt{6}\sqrt{1 + \mathrm{Sin}\left[\frac{\pi}{6} + 2\zeta_1[t] + \zeta_2[t]\right]}\right) + 2 m \left(\zeta_1'[t]^2 + \zeta_1'[t]\,\zeta_2'[t] + \zeta_2'[t]^2\right)\right)$$

Note that the Lagrangian is invariant under the group Z_3 generated by the shifts of all ζ_i by $\frac{2\pi}{3}$: ζ_i -> ζ_i + $\frac{2\pi}{3}$. This is just the symmetry of a triangle which obviously is the equilibrium configuration of the system.

Expanding L up to quadratic order in the fields one gets

In[∘]:= `L2 = Normal[Series[(L /. {ξi_[t] → eps ξi[t]}), {eps, 0, 2}]] /. eps → 1`

Out[∘]=
$$-\frac{3}{4}\left(k\, l^2 \zeta_1[t]^2 + k\, l^2 \zeta_1[t]\,\zeta_2[t] + k\, l^2 \zeta_2[t]^2\right) + l^2 m\, \zeta_1'[t]^2 + l^2 m\, \zeta_1'[t]\,\zeta_2'[t] + l^2 m\, \zeta_2'[t]^2$$

The kinetic energy and the mass constants matrix are

In[∘]:= `T = L2 /. ξi_[t] → 0 // FS`

Out[∘]= $l^2 m \left(\zeta_1'[t]^2 + \zeta_1'[t]\,\zeta_2'[t] + \zeta_2'[t]^2\right)$

In[∘]:= `M = Table[D[D[T, ξi'[t]], ξj'[t]], {i, 1, 2}, {j, 1, 2}]; M // MF`

Out[∘]//MatrixForm=
$$\begin{pmatrix} 2 l^2 m & l^2 m \\ l^2 m & 2 l^2 m \end{pmatrix}$$

In[•]:= `T - 1/2 Sum[M[[i, j]] ξi'[t] ξj'[t], {i, 1, 2}, {j, 1, 2}] // FS`

Out[•]= 0

The eigenvalues and eigenvectors of the mass matrix are

In[•]:= `Eigensystem[M]`

Out[•]= $\{\{3\, l^2 m, l^2 m\}, \{\{1, 1\}, \{-1, 1\}\}\}$

The eigenvectors are orthogonal, and we get

In[•]:= `DM = DiagonalMatrix[Eigensystem[M][[1]]]`

Out[•]= $\{\{3\, l^2 m, 0\}, \{0, l^2 m\}\}$

In[•]:= `V = Transpose[{1/2^(1/2) {1, 1}, 1/2^(1/2) {-1, 1}}]; V // MF`

Out[•]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

In[•]:= `V.Transpose[V] // FS // MF`

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In[•]:= `M - V.DM.Transpose[V] // FS // MF`

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Introduce new coordinates which reduce T to the canonical form

In[•]:= `Do[ξi[t_] = Sum[ V[[i, j]] / DM[[j, j]]^(1/2) ξj[t], {j, 1, 2}], {i, 1, 2}]`

In[•]:= `T = 1/2 Sum[M[[i, j]] ξi'[t] ξj'[t], {i, 1, 2}, {j, 1, 2}] // FS`

Out[•]= $\frac{1}{2}\left(\xi_1'[t]^2 + \xi_2'[t]^2\right)$

In terms of these new coordinates the potential and the spring constants matrix are

In[•]:= `U = -L2 /. ξi_'[t] → 0 // FS`

Out[•]= $\frac{3\, k\left(\xi_1[t]^2 + \xi_2[t]^2\right)}{8\, m}$

In[•]:= `K = Table[D[D[U, ξi[t]], ξj[t]], {i, 1, 2}, {j, 1, 2}]; K // MF`

Out[•]//MatrixForm=

$$\begin{pmatrix} \frac{3k}{4m} & 0 \\ 0 & \frac{3k}{4m} \end{pmatrix}$$

In[•]:= `U - 1/2 Sum[K[[i, j]] ξi[t] ξj[t], {i, 1, 2}, {j, 1, 2}] // FS`

Out[•]= 0

In[•]:=
```
Eigensystem[K]
```

Out[•]= $\left\{\left\{\frac{3\,k}{4\,m}, \frac{3\,k}{4\,m}\right\}, \{\{0, 1\}, \{1, 0\}\}\right\}$

In[•]:=
```
DK = DiagonalMatrix[Eigensystem[K][[1]]]
```

Out[•]= $\left\{\left\{\frac{3\,k}{4\,m}, 0\right\}, \left\{0, \frac{3\,k}{4\,m}\right\}\right\}$

In[•]:=
```
W = Transpose[{{0, 1}, {1, 0}}]; W // MF
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In[•]:=
```
W.Transpose[W] // MF
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In[•]:=
```
K - W.DK.Transpose[W]
```

Out[•]= {{0, 0}, {0, 0}}

Finally the normal coordinates and the Lagrangian are

In[•]:=
```
Do[ξi[t_] = Sum[ W[[i, j]] αj[t], {j, 1, 2}], {i, 1, 2}]
```

In[•]:=
```
1/2 Sum[K[[i, j]] ξi[t] ξj[t], {i, 1, 2}, {j, 1, 2}] // FS
```

Out[•]= $\frac{3\,k\,\left(\alpha_1[t]^2 + \alpha_2[t]^2\right)}{8\,m}$

In[•]:=
```
LL = Collect[L2, {α1'[t]^2, α2'[t]^2, α3'[t]^2, α1[t]^2, α2[t]^2, α3[t]^2}, FS]
```

Out[•]= $-\frac{3\,k\,\alpha_1[t]^2}{8\,m} - \frac{3\,k\,\alpha_2[t]^2}{8\,m} + \frac{1}{2}\,\alpha_1'[t]^2 + \frac{1}{2}\,\alpha_2'[t]^2$

Thus the frequencies are

In[•]:=
```
Do[ωi = (-Coefficient[LL, αi[t]^2] / Coefficient[LL, αi'[t]^2])^(1/2), {i, 1, 2}]
ω1
ω2
```

Out[•]= $\frac{1}{2}\sqrt{3}\,\sqrt{\frac{k}{m}}$

Out[•]= $\frac{1}{2}\sqrt{3}\,\sqrt{\frac{k}{m}}$

The original coordinates in terms of normal ones are

In[•]:= **ϕ_1[t] // FS // Expand**

Out[•]= $$-\frac{\alpha_1[t]}{\sqrt{2}\, l \sqrt{m}} + \frac{\alpha_2[t]}{\sqrt{6}\, l \sqrt{m}} + \phi_{cm}[t]$$

In[•]:= **ϕ_2[t] // FS // Expand**

Out[•]= $$\frac{\alpha_1[t]}{\sqrt{2}\, l \sqrt{m}} + \frac{\alpha_2[t]}{\sqrt{6}\, l \sqrt{m}} + \phi_{cm}[t]$$

In[•]:= **ϕ_3[t] // FS // Expand**

Out[•]= $$-\frac{\sqrt{\frac{2}{3}}\ \alpha_2[t]}{l \sqrt{m}} + \phi_{cm}[t]$$

In[•]:= **L // FS**

Out[•]= $$\frac{1}{2}\left(k\, l^2 \left(-15 + 2\,\mathrm{Cos}\left[\frac{\sqrt{\frac{3}{2}}\ \alpha_2[t]}{l\sqrt{m}}\right]\left(-\mathrm{Cos}\left[\frac{\alpha_1[t]}{\sqrt{2}\, l\sqrt{m}}\right] + \sqrt{3}\ \mathrm{Sin}\left[\frac{\alpha_1[t]}{\sqrt{2}\, l\sqrt{m}}\right]\right) - 2\,\mathrm{Sin}\left[\frac{\pi}{6} + \frac{\sqrt{2}\ \alpha_1[t]}{l\sqrt{m}}\right] + 2\sqrt{6}\ \sqrt{1 + \mathrm{Sin}\left[\frac{\pi}{6} + \frac{\sqrt{2}\ \alpha_1[t]}{l\sqrt{m}}\right]} + 2\sqrt{6}\ \sqrt{\left(1 + \mathrm{Sin}\left[\frac{1}{6}\left(\pi - \frac{3\sqrt{2}\ \left(\alpha_1[t] - \sqrt{3}\ \alpha_2[t]\right)}{l\sqrt{m}}\right)\right]\right)} + 2\sqrt{6}\ \sqrt{\left(1 + \mathrm{Sin}\left[\frac{1}{6}\left(\pi - \frac{3\sqrt{2}\ \left(\alpha_1[t] + \sqrt{3}\ \alpha_2[t]\right)}{l\sqrt{m}}\right)\right]\right)}\right) + \alpha_1'[t]^2 + \alpha_2'[t]^2\right)$$