Advanced Calculus
MA1132
Homework Assignment 2

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SOLUTIONS

0. Determine whether the limit exists. If so, find its value

(a)

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{e^{x^2+y^2}-1}$$

Solution: The limit exist as can be seen by switching to the polar coordinates

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{e^{x^2+y^2}-1} = \lim_{r\to 0^+} \frac{e^{-\frac{1}{r}}}{e^{r^2}-1} = \lim_{r\to 0^+} \frac{e^{-\frac{1}{r}}}{r^2} = \lim_{t\to+\infty} t^2 e^{-t} = 0\,. \tag{1}$$

(b)

$$\lim_{(x,y)\to(1,-1)} \frac{1-\cosh(x+y)}{\sin(x^2-y^2)\ln(\frac{2x}{x-y})}$$

Solution: The limit exist as can be seen by switching to the coordinates $x+y=t\,,\ x-y=u$

$$\begin{aligned}\lim_{(x,y)\to(1,-1)} \frac{1-\cosh(x+y)}{\sin(x^2-y^2)\ln(\frac{2x}{x-y})} = \lim_{(t,u)\to(0,2)} \frac{1-\cosh(t)}{\sin(tu)\ln(\frac{t+u}{u})} &= \lim_{t\to 0} \frac{1-\cosh(t)}{\sin(2t)\ln(1+\frac{t}{2})}\\ &= \lim_{t\to 0} \frac{-\frac{1}{2}t^2}{2t\,\frac{t}{2}} = -\frac{1}{2}\,.\end{aligned} \tag{2}$$

(c)

$$\lim_{(x,y)\to(0,0)} \frac{3+\cos(2x)-4\cosh(y)}{1-\sqrt[4]{1+x^2+y^2}}$$

Solution: The limit exist as can be seen by switching to the polar coordinates

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{3+\cos(2x)-4\cosh(y)}{1-\sqrt[4]{1+x^2+y^2}} &= \lim_{r\to 0} \frac{3+\cos(2r\cos\phi)-4\cosh(r\sin\phi)}{1-\sqrt[4]{1+r^2}}\\ = \lim_{r\to 0} \frac{3+1-\frac{1}{2}(2r)^2\cos^2\phi - 4(1+\frac{1}{2}r^2\sin^2\phi)+\mathcal{O}(r^4)}{-\frac{1}{4}r^2+\mathcal{O}(r^4)} &= 8\,.\end{aligned} \tag{3}$$

1. Find all first and second order partial derivatives of the function

$$f(x,y) = x\sin(y\ln(x)),$$

and hence verify that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ for this function.

Solution:

Using the Product and Chain Rules,

$$\frac{\partial f}{\partial x} = \sin(y\ln(x)) + x \cdot \frac{y}{x}\cos(y\ln(x)) = \sin(y\ln(x)) + y\cos(y\ln(x)). \tag{4}$$

Using the Chain Rule,

$$\frac{\partial f}{\partial y} = x\ln(x)\cos(y\ln(x)). \tag{5}$$

Differentiating (4) w.r.t. x, using the Chain Rule,

$$\frac{\partial^2 f}{\partial x^2} = \frac{y}{x}\cos(y\ln(x)) - \frac{y^2}{x}\sin(y\ln(x)) = \frac{y}{x}[\cos(y\ln(x)) - y\sin(y\ln(x))].$$

Differentiating (4) w.r.t. y, using the Product and Chain Rules,

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} &= \ln(x)\cos(y\ln(x)) + \cos(y\ln(x)) - y\ln(x)\sin(y\ln(x))\\ &= (\ln(x)+1)\cos(y\ln(x)) - y\ln(x)\sin(y\ln(x)).\end{aligned}$$

Differentiating (5) w.r.t. x, using the Product and Chain Rules,

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \ln(x)\cos(y\ln(x)) + x\left[\frac{1}{x}\cdot\cos(y\ln(x)) - \ln(x)\cdot\frac{y}{x}\cdot\sin(y\ln(x))\right]\\ &= (\ln(x)+1)\cos(y\ln(x)) - y\ln(x)\sin(y\ln(x))\\ &= \frac{\partial^2 f}{\partial y \partial x}.\end{aligned}$$

Differentiating (5) w.r.t. y, using the Chain Rule,

$$\frac{\partial^2 f}{\partial x \partial y} = -x\ln(x)^2\sin(y\ln(x)).$$

2. Let $f(x,y,z) = x\cos(x+y+z)$.

   (a) Find the directional derivative of f at the point $\left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\right)$ in the direction $(8,-4,-1)$.

   Solution: First we need to find a unit vector $\mathbf{u}$. We have

   $$\|(8,-4,-1)\| = \sqrt{8^2+(-4)^2+(-1)^2} = \sqrt{81} = 9,$$

   so $\mathbf{u} = \left(\frac{8}{9}, -\frac{4}{9}, -\frac{1}{9}\right)$.

   If we let $\mathbf{a} = \left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\right)$, then

   $$D_{\mathbf{u}}f(\mathbf{a}) = f_x(\mathbf{a})\left(\frac{8}{9}\right) + f_y(\mathbf{a})\left(-\frac{4}{9}\right) + f_z(\mathbf{a})\left(-\frac{1}{9}\right).$$

Also note that at the point $\left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\right)$, $x+y+z = \frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4} = \frac{\pi}{2}$.

Now

$$\begin{aligned} f_x &= \cos(x+y+z) - x\sin(x+y+z), \quad \text{so} \quad f_x(\mathbf{a}) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{12}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi}{12}, \\ f_y &= -x\sin(x+y+z), \quad \text{so} \quad f_y(\mathbf{a}) = -\frac{\pi}{12}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi}{12} \\ f_z &= -x\sin(x+y+z), \quad \text{so} \quad f_z(\mathbf{a}) = -\frac{\pi}{12}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi}{12}. \end{aligned}$$

Hence

$$D_{\mathbf{u}}f(\mathbf{a}) = -\frac{\pi}{12}\left(\frac{8}{9}\right) - \frac{\pi}{12}\left(-\frac{4}{9}\right) - \frac{\pi}{12}\left(-\frac{1}{9}\right) = -\frac{\pi}{36}.$$

(b) Find the unit vectors in the directions in which f is increasing/decreasing most rapidly at the point $\left(\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}\right)$, and give the rate of increase and decrease, respectively.

Solution:

The directions in which f is increasing/decreasing most rapidly at $\mathbf{a}$ are given by $\nabla f(\mathbf{a})$ and $-\nabla f(\mathbf{a})$.

Using Part (a), $\nabla f(\mathbf{a}) = \left(-\frac{\pi}{12}, -\frac{\pi}{12}, -\frac{\pi}{12}\right)$ and $-\nabla f(\mathbf{a}) = \left(\frac{\pi}{12}, \frac{\pi}{12}, \frac{\pi}{12}\right)$.

Since $\left\|\left(-\frac{\pi}{12}, -\frac{\pi}{12}, -\frac{\pi}{12}\right)\right\| = \sqrt{\left(-\frac{\pi}{12}\right)^2 + \left(-\frac{\pi}{12}\right)^2 + \left(-\frac{\pi}{12}\right)^2} = \sqrt{\frac{3\pi^2}{144}} = \frac{\sqrt{3}\pi}{12}$, unit vectors in the directions in which f is increasing and decreasing most rapidly are $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, respectively.

The maximum rate of increase is $\frac{\sqrt{3}\pi}{12}$ and the maximum rate of decrease is $-\frac{\sqrt{3}\pi}{12}$.

3. Find and classify the critical points of the function $f(x,y) = x^2y - 2xy^2 + 3xy + 4$.

Solution:

Since $\nabla f = (2xy - 2y^2 + 3y, x^2 - 4xy + 3x) = (0,0)$, we must have

$$2xy - 2y^2 + 3y = 0 \tag{6}$$

$$x^2 - 4xy + 3x = 0. \tag{7}$$

Now (6) yields $y(2x - 2y + 3) = 0$, so we must either have $y = 0$ or $2x - 2y + 3 = 0$. We will consider the cases $y = 0$ and $y \neq 0$ separately.

Case 1: $y = 0$.

In this case (7) yields $x^2 + 3x = 0 \Rightarrow x(x+3) = 0 \Rightarrow x = 0$ or $x = -3$.

Thus $(0,0)$ and $(-3,0)$ are critical points.

Case 2: $y \neq 0$.

Since $2x - 2y + 3 = 0$, it follows that $x = y - \frac{3}{2}$ and if we substitute this in (7), we obtain

$$
\begin{aligned}
\left(y - \frac{3}{2}\right)^2 - 4\left(y - \frac{3}{2}\right) y + 3\left(y - \frac{3}{2}\right) = 0 \\
\Longrightarrow y^2 - 3y + \frac{9}{4} - 4y^2 + 6y + 3y - \frac{9}{2} = 0 \\
\Longrightarrow 3y^2 - 6y + \frac{9}{4} = 0 \\
\Longrightarrow y = \frac{6 \pm \sqrt{36 - 27}}{6} = \frac{6 \pm 3}{6} \\
\Longrightarrow y = \frac{3}{2} \text{ or } y = \frac{1}{2}.
\end{aligned}
$$

Now if $y = \frac{3}{2}$ then $x = 0$ and if $y = \frac{1}{2}$ then $x = -1$.

Thus in this case we obtain the critical points $\left(0, \frac{3}{2}\right)$ and $\left(-1, \frac{1}{2}\right)$.

Now

$$
f_{xx} = 2y, \quad f_{xy} = 2x - 4y + 3 \quad \text{and} \quad f_{yy} = -4x.
$$

Hence $D(0,0) = (0) - (3)^2 = -9$, so there is a saddle point at $(0,0)$.

Next $D(-3,0) = (0)(12) - (-3)^2 = -9 < 0$, so there is also a saddle point at $(-3,0)$.

Next $D\left(0, \frac{3}{2}\right) = (3)(0) - (-3)^2 = -9 < 0$, so there is also a saddle point at $\left(0, \frac{3}{2}\right)$.

Finally, $D\left(-1, \frac{1}{2}\right) = (1)(4) - (-1)^2 = 3 > 0$.

Since $f_{xx} > 0$, there is a local minimum at $\left(-1, \frac{1}{2}\right)$.

4. Show that if $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then

$$
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}.
$$

Solution: We have

$$
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y}. \tag{8}
$$

Then

$$
\begin{aligned}
r = \sqrt{x^2 + y^2} \quad &\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta, \\
\theta = \arctan\frac{y}{x} \quad &\Rightarrow \quad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}.
\end{aligned} \tag{9}
$$

Thus

$$
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{\partial z}{\partial \theta}\frac{\sin\theta}{r}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}. \tag{10}
$$

Differentiating these formulae one gets

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\cos\theta - \frac{\partial z}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial r}\cos\theta - \frac{\partial z}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\partial \theta}{\partial x}\,, \\
&= \left(\frac{\partial^2 z}{\partial r^2}\cos\theta - \frac{\partial^2 z}{\partial r\partial\theta}\frac{\sin\theta}{r} + \frac{\partial z}{\partial \theta}\frac{\sin\theta}{r^2}\right)\cos\theta \\
&- \left(\frac{\partial^2 z}{\partial \theta\partial r}\cos\theta - \frac{\partial z}{\partial r}\sin\theta - \frac{\partial^2 z}{\partial \theta^2}\frac{\sin\theta}{r} - \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\sin\theta}{r}\,, \\
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}\right)\frac{\partial \theta}{\partial y}\,, \\
&= \left(\frac{\partial^2 z}{\partial r^2}\sin\theta + \frac{\partial^2 z}{\partial r\partial\theta}\frac{\cos\theta}{r} - \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r^2}\right)\sin\theta \\
&+ \left(\frac{\partial^2 z}{\partial \theta\partial r}\sin\theta + \frac{\partial z}{\partial r}\cos\theta + \frac{\partial^2 z}{\partial \theta^2}\frac{\cos\theta}{r} - \frac{\partial z}{\partial \theta}\frac{\sin\theta}{r}\right)\frac{\cos\theta}{r}\,.
\end{aligned} \tag{11}
$$

Collecting the terms one gets the formula.

5. Consider the surface

$$z = f(x,y) = \ln\left(\frac{1}{2}e^{2/3}\sqrt[3]{8x^2 - 6xy^2 - y^3 + 32 - 12\sin(2x-y)}\right)\,.$$

(a) Find an equation for the tangent plane to the surface at the point $P = (1, 2, z_0)$ where $z_0 = f(1, 2)$.

(b) Find points of intersection of the tangent plane with the x-, y- and z-axes.

(c) Sketch the tangent plane, and show the point $P = (1, 2, z_0)$ on it.

(d) Find parametric equations for the normal line to the surface at the point $P = (1, 2, z_0)$.

(e) Sketch the normal line to the surface at the point $P = (1, 2, z_0)$.

Solution:

(a) We first find

$$8x^2 - 6xy^2 - y^3 + 32 - 12\sin(2x-y)|_P = 8\,, \quad z_0 = \frac{2}{3}\,,$$

and then simplify

$$
\begin{aligned}
z &= \ln\left(\frac{1}{2}e^{2/3}\sqrt[3]{8x^2 - 6xy^2 - y^3 + 32 - 12\sin(2x-y)}\right) \\
&= \frac{1}{3}\ln(8x^2 - 6xy^2 - y^3 + 32 - 12\sin(2x-y)) + \frac{2}{3} - \ln 2\,.
\end{aligned} \tag{12}
$$

Then, we compute the partial derivatives at P

$$\frac{\partial}{\partial x}z|_P = -\frac{4}{3}\,.$$

$$\frac{\partial}{\partial y} z|_P = -1\,.$$

The tangent plane equation is given by

$$z = \frac{2}{3} - \frac{4}{3}(x-1) - 1(y-2) = 4 - \frac{4}{3}x - y\,.$$

(b) $(3,0,0)$, $(0,4,0)$, $(0,0,4)$

(c) The tangent plane is the one through the points in (b).

(d) The normal line to the surface (and the tangent plane) is given by

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \frac{2}{3}\mathbf{k} + t\big(\frac{4}{3}\mathbf{i} + \mathbf{j} + \mathbf{k}\big)\,.$$

(e) The normal line is perpendicular to the plane.

6. Show that the equation of the plane that is tangent to the cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

at (x_0, y_0, z_0) can be written in the form

$$\frac{x_0}{a^2}x + \frac{y_0}{b^2}y - \frac{z_0}{c^2}z = 0\,.$$

Solution: Consider the function $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}$. The gradient ∇F

$$\nabla F = 2\frac{x}{a^2}\mathbf{e}_1 + 2\frac{y}{b^2}\mathbf{e}_2 - 2\frac{z}{c^2}\mathbf{e}_3 \tag{13}$$

is normal to the level surfaces of F, and therefore to tangent planes to the level surfaces of F. Thus, the equation of the plane tangent to the cone at (x_0, y_0, z_0) can be written in the form

$$\frac{1}{2}(\mathbf{r} - \mathbf{r}_0) \cdot \nabla F(x_0, y_0, z_0) = 0\,. \tag{14}$$

Explicitly one gets

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} - \frac{zz_0}{c^2} - (\frac{x_0x_0}{a^2} + \frac{y_0y_0}{b^2} - \frac{z_0z_0}{c^2}) = 0\,, \tag{15}$$

which proves the formula.

7. Prove: If the surfaces $z = f(x_1, \ldots, x_n)$ and $z = g(x_1, \ldots, x_n)$ intersect at $P = (x_1^o, \ldots, x_n^o, z^o)$, and if f and g are differentiable at $(x_1^o, \ldots, x_n^o)$, then the normal lines at P are perpendicular if and only if

$$\sum_{i=1}^{n} \frac{\partial f(x_1^o, \ldots, x_n^o)}{\partial x_i} \frac{\partial g(x_1^o, \ldots, x_n^o)}{\partial x_i} = -1\,.$$

Solution: Consider the functions $F(x_1, \ldots, x_n, z) = f(x_1, \ldots, x_n) - z$ and $G(x_1, \ldots, x_n, z) = g(x_1, \ldots, x_n) - z$. The normal lines to the level surfaces of F and G are parallel to ∇F and ∇G. Since

$$\nabla F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathbf{e}_i - \mathbf{e}_z \,, \quad \nabla G = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \mathbf{e}_i - \mathbf{e}_z \,, \tag{16}$$

one gets that ∇F and ∇G are perpendicular if and only if

$$\nabla F \cdot \nabla G = 1 + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} = 0 \,. \tag{17}$$

8. Consider the function

$$f(x, y) = x^2 - xy^2 - 3x + y^4 + 5$$

Locate all relative maxima, relative minima, and saddle points, if any. Use Mathematica to plot its graph.

Solution: We first find all critical points

$$f_x(x, y) = 2x - y^2 - 3 = 0 \,, \quad f_y(x, y) = -2xy + 4y^3 = 0 \,.$$

From the first equation we find x in terms of y

$$x = \frac{y^2}{2} + \frac{3}{2} \,,$$

and substituting it to the second equation, we derive the following equation for y

$$3y^3 - 3y = 0 \,.$$

There are three solutions to this equation

$$y = 0 \,, \; y = -1 \,, \; y = 1 \,,$$

and, therefore, three critical points

$$(x = \frac{3}{2} \,, \; y = 0) \,, \quad (x = 2 \,, \; y = -1) \,, \quad (x = 2 \,, \; y = 1) \,.$$

Computing the values of f at critical points, we get

$$f(\frac{3}{2}, 0) = \frac{11}{4} \,, \quad f(2, -1) = 2 \,, \quad f(2, 1) = 2 \,.$$

To find out if they are maximum, minimum or saddle points we use the second derivative test. To this end we compute

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 2 \,, \; \frac{\partial^2 f}{\partial y^2}(x, y) = -2x + 12y^2 \,, \; \frac{\partial^2 f}{\partial x \partial y}(x, y) = -2y \,,$$

and

$$D(x,y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 20y^2 - 4x\,,$$

Computing D and $\frac{\partial^2 f}{\partial x^2}$ for the three critical points, we get

$$D(\frac{3}{2},0) = -6\,, \quad \frac{\partial^2 f}{\partial x^2}(\frac{3}{2},0) = 2\,,$$

and therefore $(0\,,\,\frac{3}{2})$ is a saddle point.

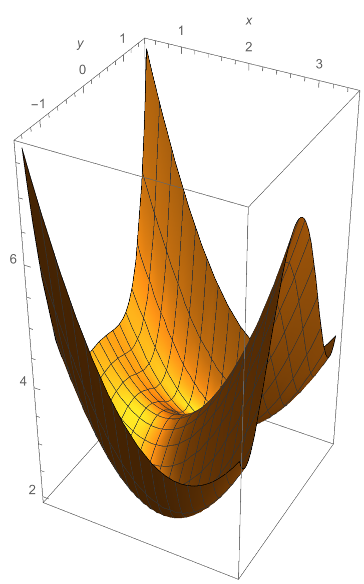
$$D(2,-1) = 12\,, \quad \frac{\partial^2 f}{\partial x^2}(2,-1) = 2\,,$$

and therefore $(2,-1)$ is a relative minimum.

$$D(2,1) = 12\,, \quad \frac{\partial^2 f}{\partial x^2}(2,1) = 2\,,$$

and therefore $(2,1)$ is a relative minimum too.

The graph of the function is shown below



9. Consider the function

$$z = 3e^{y-\frac{\pi}{4}}\cos x - 2e^{\frac{\pi}{2}-x}\sin y$$

(a) Find

$$iii)\ \frac{\partial^2 z}{\partial x \partial y}(\frac{\pi}{2},\frac{\pi}{4})\,, \qquad iv)\ \frac{\partial^2 z}{\partial y \partial x}(\frac{\pi}{2},\frac{\pi}{4})\,.$$

Solution:

$$iii)\ \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\frac{\partial z}{\partial y}=\frac{\partial}{\partial x}\left(3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\cos y\right)=-3e^{y-\frac{\pi}{4}}\sin x+2e^{\frac{\pi}{2}-x}\cos y\ \Rightarrow\ \frac{\partial^2 z}{\partial x\partial y}(\frac{\pi}{2},\frac{\pi}{4})=-3+\sqrt{2}\,.$$

$$iv)\ \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\frac{\partial z}{\partial x}=\frac{\partial}{\partial y}\left(-3e^{y-\frac{\pi}{4}}\sin x+2e^{\frac{\pi}{2}-x}\sin y\right)=-3e^{y-\frac{\pi}{4}}\sin x+2e^{\frac{\pi}{2}-x}\cos y\ \Rightarrow\ \frac{\partial^2 z}{\partial x\partial y}(\frac{\pi}{2},\frac{\pi}{4})=-3+\sqrt{2}\,.$$

(b) Find the slope of the surface $z=3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\sin y$ in the y-direction at the point $(\frac{\pi}{3},\frac{\pi}{6})$.

Solution: The slope k_y is equal to

$$k_y=\frac{\partial z}{\partial y}(\frac{\pi}{3},\frac{\pi}{6})=3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\cos y|_{x=\frac{\pi}{3},y=\frac{\pi}{6}}=\frac{3}{2}e^{-\frac{\pi}{12}}-\sqrt{3}e^{\frac{\pi}{6}}\approx-1.76936\,.$$

(c) Show that the function $z=3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\sin y$ satisfies Laplace's equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0\,.$$

Solution: To this end we compute the following derivatives

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(-3e^{y-\frac{\pi}{4}}\sin x+2e^{\frac{\pi}{2}-x}\sin y\right)=-3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\sin y\,,$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(3e^{y-\frac{\pi}{4}}\cos x-2e^{\frac{\pi}{2}-x}\cos y\right)=3e^{y-\frac{\pi}{4}}\cos x+2e^{\frac{\pi}{2}-x}\sin y\,.$$

The sum of these two expressions is obviously 0.

10. The equations of motion of a system of n particles are given by

$$m_i\ddot{x}_i=-\frac{\partial U(x_1,\dots,x_n)}{\partial x_i}\,,\quad \ddot{x}_i=\frac{d^2x_i}{dt^2}\,,\quad i=1,2,\dots,n\,,$$

where m_i is the mass and x_i is the coordinate of the i-th particle, and $U(x_1,\dots,x_n)$ is the potential energy of the system.

(a) Consider a system of n particles moving in a central field

$$U(x_1,\dots,x_n)=V(r)\,,\quad r=\Big|\sum_{i=1}^{n}x_i\mathbf{e}_i\Big|\,,$$

where V is a smooth function of a single variable.

i. Find the equations of motion of the first particle (x_1).
ii. Find the equations of motion of the second particle (x_2).
iii. Find the equations of motion of the last particle (x_n).
iv. Find the equations of motion of the i-th particle (x_i) for $1<i<n$.

v. Write the equations of motion of the i-th particle (x_i) for $1 \leq i \leq n$ by using the Kronecker delta δ_{ij}.

Solution: We have for all i'th

$$m_i \ddot{x}_i = -\frac{\partial U(x_1, \ldots, x_n)}{\partial x_i} = -\frac{\partial}{\partial x_i} V(r) = -V'(r)\frac{x_i}{r}\,, \quad i = 1, \ldots, n\,.$$

(b) Find the equations of motion of a system of n particles with the rational Calogero-Moser potential

$$U(x_1, \ldots, x_n) = \sum_{i,j=1, i\neq j}^{n} \frac{\alpha}{(x_i - x_j)^2}\,.$$

Solution: We have

$$m_i \ddot{x}_i = -\frac{\partial U(x_1, \ldots, x_n)}{\partial x_i} = -\frac{\partial}{\partial x_i} \sum_{j,k=1, j\neq k}^{n} \frac{\alpha}{(x_j - x_k)^2} = \sum_{j,k=1, j\neq k}^{n} \frac{2\alpha}{(x_j - x_k)^3}(\delta_{ij} - \delta_{ik})$$
$$= \sum_{k=1, k\neq i}^{n} \frac{2\alpha}{(x_i - x_k)^3} - \sum_{j=1, j\neq i}^{n} \frac{2\alpha}{(x_j - x_i)^3} = \sum_{j=1, j\neq i}^{n} \frac{4\alpha}{(x_i - x_j)^3}\,. \tag{18}$$

11. Compute the differential df of

$$f(x_1, x_2, \ldots, x_n) = (\frac{1}{2} + x_1)^{\alpha_1}(\frac{1}{2} + x_2)^{\alpha_2} \cdots (\frac{1}{2} + x_n)^{\alpha_n}\,,$$

and find its local linear approximation at $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.

Solution: The differential of the function is defined by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i\,.$$

Computing partial derivatives, one gets

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i}(\frac{1}{2}+x_1)^{\alpha_1}(\frac{1}{2}+x_2)^{\alpha_2} \cdots (\frac{1}{2}+x_n)^{\alpha_n} = \frac{\alpha_i}{\frac{1}{2} + x_i}(\frac{1}{2}+x_1)^{\alpha_1}(\frac{1}{2}+x_2)^{\alpha_2} \cdots (\frac{1}{2}+x_n)^{\alpha_n} = \frac{\alpha_i}{\frac{1}{2} + x_i} f\,.$$

The differential of the function is given by the formula

$$df = f \sum_{i=1}^{n} \frac{\alpha_i}{\frac{1}{2} + x_i} dx_i = f \sum_{i=1}^{n} \alpha_i \, d \ln(\frac{1}{2} + x_i)\,.$$

The local linear approximation of the function at $(1, \ldots, 1)$ is given by the formula

$$L(x_1, x_2, \ldots, x_n) = f(\frac{1}{2}, \ldots, \frac{1}{2}) + \sum_{i=1}^{n} \frac{\partial f(1, \ldots, 1)}{\partial x_i}(x_i - \frac{1}{2})\,.$$

We obviously have $f(\frac{1}{2}, \ldots, \frac{1}{2}) = 1$, and $\frac{\partial f(\frac{1}{2}, \ldots, \frac{1}{2})}{\partial x_i} = \alpha_i$. Thus

$$L(x_1, x_2, \ldots, x_n) = 1 + \sum_{i=1}^{n} \alpha_i (x_i - \frac{1}{2})\,.$$

12. Consider the function $f(x,y,z) = \cos(x^2+y^2+z^2)$. Find the Taylor series expansion of $f(x,y,z)$ about the point $\mathbf{x}_0 = (0,0,0)$ up to the third order.

Solution: One begins by computing the derivatives of f and noticing that they are all zero at $(0,0,0)$. This means that the Taylor polynomial at the origin is

$$T(x,y,z) = f(0,0,0) + \mathcal{O}\left(\max\left(x^4,y^4,z^4\right)\right) = 1 + \mathcal{O}\left(\max\left(x^4,y^4,z^4\right)\right).$$

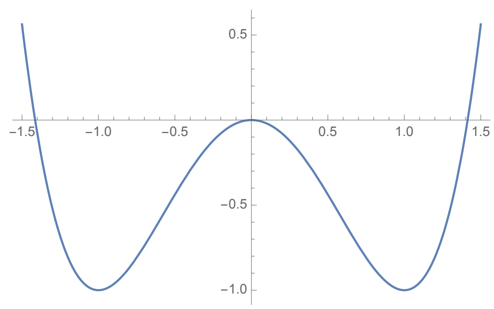
This indicates that near the origin, the function behaves almost like a constant function.

13. Consider the "Higgs" potential

$$U(x_1,\ldots,x_n) = -\frac{\kappa^2}{2}r^2 + \frac{\lambda^2}{4}r^4\,, \quad r = \Big|\sum_{i=1}^{n} x_i\mathbf{e}_i\Big|\,, \quad \kappa>0, \lambda>0\,.$$

(a) Plot the potential for $n=1$, and for $\kappa=\lambda=2$.

Solution: The curve is shown below



(b) Plot the potential for $n=2$, and for $\kappa=\lambda=2$.

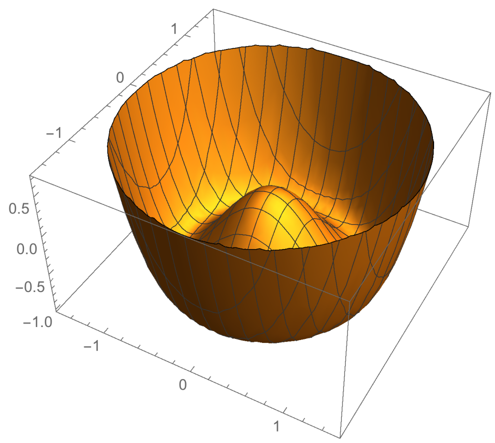
Solution: The surface is shown below

It is a surface of revolution obtained by revolving the graph of the curve above about the vertical axis.

(c) Find the Taylor series expansion of the "Higgs" potential about the point $x_1^o = \frac{\kappa}{\lambda}$, $x_i^o = 0$, $i=2,\ldots,n$ up to the fourth order in $y_i \equiv x_i - x_i^o$. Use Mathematica to check your answer.

Solution: We have

$$\begin{aligned}
U(\frac{\kappa}{\lambda},0,\ldots,0) &= -\frac{\kappa^4}{4\lambda^2}\,,\\
\frac{\partial U(x_1,\ldots,x_n)}{\partial x_i} &= -\kappa^2 x_i + \lambda^2 r^2 x_i \quad \Rightarrow \quad \frac{\partial U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_i} = 0\,,\\
\frac{\partial^2 U(x_1,\ldots,x_n)}{\partial x_i \partial x_j} &= (-\kappa^2+\lambda^2 r^2)\delta_{ij} + 2\lambda^2 x_i x_j \quad \Rightarrow\\
\frac{\partial^2 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_1^2} &= 2\kappa^2\,, \quad \frac{\partial^2 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_i \partial x_j} = 0 \text{ if } i\neq 1, j\neq 1\,,
\end{aligned} \tag{19}$$



$$
\begin{aligned}
&\frac{\partial^3 U(x_1,\ldots,x_n)}{\partial x_i \partial x_j \partial x_k} = 2\lambda^2 x_k \delta_{ij} + 2\lambda^2 x_i \delta_{jk} + 2\lambda^2 x_j \delta_{ik} \quad \Rightarrow \\
&\frac{\partial^3 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_1^3} = 6\kappa\lambda\,, \quad \frac{\partial^3 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_1^2 \partial x_j} = 0\,, \quad \frac{\partial^3 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_1 \partial x_j \partial x_j} = 2\kappa\lambda\,, \quad j = 2,\ldots,n \\
&\frac{\partial^3 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_i \partial x_j \partial x_k} = 0\,, \quad i,j,k = 2,\ldots,n\,,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\frac{\partial^4 U(x_1,\ldots,x_n)}{\partial x_i \partial x_j \partial x_k \partial x_l} = 2\lambda^2 \delta_{kl}\delta_{ij} + 2\lambda^2 \delta_{il}\delta_{jk} + 2\lambda^2 \delta_{jl}\delta_{ik} \quad \Rightarrow \\
&\frac{\partial^4 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_i \partial x_j \partial x_k \partial x_l} = 2\lambda^2(\delta_{kl}\delta_{ij} + \delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik})\,, \\
&\frac{\partial^4 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_i^4} = 6\lambda^2\,, \\
&\frac{\partial^4 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_i^3 \partial x_l} = 0\,, \quad l \neq i\,, \\
&\frac{\partial^4 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_i^2 \partial x_k^2} = 2\lambda^2 \quad i \neq k\,, \\
&\frac{\partial^4 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_i^2 \partial x_k \partial x_l} = 0 \quad i \neq k, k \neq l\,, \\
&\frac{\partial^4 U(\frac{\kappa}{\lambda},0,\ldots,0)}{\partial x_i \partial x_j \partial x_k \partial x_l} = 0 \quad i \neq j,k,l\,.
\end{aligned}
\tag{21}
$$

Thus, one gets

$$U(x_1,\ldots,x_n) = -\frac{\kappa^4}{4\lambda^2} + \kappa^2 y_1^2 + \kappa\lambda y_1^3 + \kappa\lambda y_1 \sum_{k=2}^{n} y_k^2 + \frac{1}{4}\lambda^2 \left(\sum_{k=1}^{n} y_k^2\right)^2 \ . \tag{22}$$