MA1125 – Calculus
Homework #2 solutions

1. Determine the inverse function f^{-1} in each of the following cases.

$$f(x) = 3 - \log_2(2x - 4), \qquad f(x) = \frac{2 \cdot 7^x + 3}{5 \cdot 7^x + 4}.$$

When it comes to the first case, one can easily check that

$$3 - y = \log_2(2x - 4) \iff 2^{3-y} = 2x - 4 \iff 2^{2-y} = x - 2,$$

so the inverse function is defined by $f^{-1}(y) = 2^{2-y} + 2$. When it comes to the second case,

$$y = \frac{2 \cdot 7^x + 3}{5 \cdot 7^x + 4} \iff 5y \cdot 7^x + 4y = 2 \cdot 7^x + 3 \iff 7^x(5y - 2) = 3 - 4y$$

and this gives $7^x = \frac{3-4y}{5y-2}$, so the inverse function is defined by $f^{-1}(y) = \log_7 \frac{3-4y}{5y-2}$.

2. Simplify each of the following expressions.

$$\cos\left(\tan^{-1} x\right), \qquad \sin\left(\cos^{-1} x\right), \qquad \log_2 \frac{4^x + 8^x}{2^x + 4^x}.$$

To simplify the first expression, let $\theta = \tan^{-1} x$ and note that $\tan\theta = x$. When $x \geq 0$, the angle θ arises in a right triangle with an opposite side of length x and an adjacent side of length 1. It follows by Pythagoras' theorem that the hypotenuse has length $\sqrt{1 + x^2}$, so the definition of cosine gives

$$\cos(\tan^{-1} x) = \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{1}{\sqrt{1 + x^2}}.$$

When $x \leq 0$, the last equation holds with $-x$ instead of x. This changes the term $\tan^{-1} x$ by a minus sign, but the cosine remains unchanged, so the equation is still valid.

To simplify the second expression, one may use a similar approach or simply note that

$$\theta = \cos^{-1} x \implies \cos\theta = x \implies \sin^2\theta = 1 - \cos^2\theta = 1 - x^2.$$

Since $\theta = \cos^{-1} x$ lies between 0 and π by definition, $\sin\theta$ is non-negative and

$$\sin^2\theta = 1 - x^2 \implies \sin\theta = \sqrt{1 - x^2}.$$

As for the third expression, one may simplify the given fraction to conclude that

$$\log_2 \frac{4^x + 8^x}{2^x + 4^x} = \log_2 \frac{4^x(1 + 2^x)}{2^x(1 + 2^x)} = \log_2 2^x = x.$$

3. Use the ε-δ definition of limits to compute $\lim_{x\to 2} f(x)$ in the case that

$$f(x) = \left\{ \begin{array}{ll} 2x-5 & \text{if } x \leq 2 \\ 5-3x & \text{if } x > 2 \end{array} \right\}.$$

In this case, x is approaching 2 and $f(x)$ is either $2x-5$ or $5-3x$. We thus expect the limit to be $L = -1$. To prove this formally, we let $\varepsilon > 0$ and estimate the expression

$$|f(x)+1| = \left\{ \begin{array}{ll} |2x-4| & \text{if } x \leq 2 \\ |6-3x| & \text{if } x > 2 \end{array} \right\} = \left\{ \begin{array}{ll} 2|x-2| & \text{if } x \leq 2 \\ 3|x-2| & \text{if } x > 2 \end{array} \right\}.$$

If we assume that $0 \neq |x-2| < \delta$, then we may use the last equation to get

$$|f(x)+1| \leq 3|x-2| < 3\delta.$$

Since our goal is to show that $|f(x)+1| < \varepsilon$, an appropriate choice of δ is thus $\delta = \varepsilon/3$.

4. Compute each of the following limits.

$$L = \lim_{x\to 1} \frac{x^3-4x^2+4x-1}{x-1}, \qquad M = \lim_{x\to 1} \frac{3x^3-7x^2+5x-1}{(x-1)^2}.$$

When it comes to the first limit, division of polynomials gives

$$L = \lim_{x\to 1} \frac{(x-1)(x^2-3x+1)}{x-1} = \lim_{x\to 1} (x^2-3x+1) = 1-3+1 = -1.$$

When it comes to the second limit, division of polynomials gives

$$M = \lim_{x\to 1} \frac{(x^2-2x+1)(3x-1)}{x^2-2x+1} = \lim_{x\to 1} (3x-1) = 3-1 = 2.$$

5. Use the ε-δ definition of limits to compute $\lim_{x\to 3} (5x^2-6x+3)$.

Let $f(x) = 5x^2-6x+3$ for convenience. Then $f(3) = 30$ and one has

$$|f(x)-f(3)| = |5x^2-6x-27| = |x-3| \cdot |5x+9|.$$

The factor $|x-3|$ is related to our usual assumption that $0 \neq |x-3| < \delta$. To estimate the remaining factor $|5x+9|$, we assume that $\delta \leq 1$ for simplicity and we note that

$$\begin{aligned} |x-3| < \delta \leq 1 \quad &\implies \quad -1 < x-3 < 1 \\ &\implies \quad 2 < x < 4 \quad \implies \quad 19 < 5x+9 < 29. \end{aligned}$$

Combining the estimates $|x-3| < \delta$ and $|5x+9| < 29$, one may then conclude that

$$|f(x)-f(3)| = |x-3| \cdot |5x+9| < 29\delta \leq \varepsilon,$$

as long as $\delta \leq \varepsilon/29$ and $\delta \leq 1$. An appropriate choice of δ is thus $\delta = \min(\varepsilon/29, 1)$.