Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 3
Each set of homework questions is worth 100 marks

Problem 1. Consider the Lagrangian of a particle moving in a potential field

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(r)\,, \quad r = \sqrt{x^2 + y^2}\,.$$

(a) Introduce the cylindrical coordinates (draw a picture), and derive an expression for the Lagrangian in terms of the coordinates.

(b) Identify the cyclic coordinates, and find the corresponding conserved charges. What is their physical meaning? What symmetries do they correspond to?

Problem 2. Consider the Lagrangian of a particle moving in a potential field

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(r)\,, \quad r = \sqrt{x^2 + y^2 + z^2}\,.$$

(a) Introduce the spherical coordinates (draw a picture), and derive an expression for the Lagrangian in terms of the coordinates.

(b) Identify the cyclic coordinates, and find the corresponding conserved charges. What is their physical meaning? What symmetries do they correspond?

Problem 3. Consider the following Lagrangian of a relativistic particle moving in a D-dimensional space and interacting with a central potential field ($m\,, c\,, \alpha\,, \beta$ are constants)

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - \frac{\alpha}{r^2}e^{-\beta\, r^2}\,, \quad v^2 \equiv \vec{v}^2 = \sum_{i=1}^{D} v_i^2\,, \quad r^2 \equiv \vec{r}^2 = \sum_{i=1}^{D} x_i^2\,.$$

For some questions below you may use that infinitesimal rotations are parametrised by a skew-symmetric matrix ϵ_{ij}, that is

$$x_i \to x_i' = x_i + \epsilon_{ij}x_j\,, \qquad \epsilon_{ij} + \epsilon_{ji} = 0\,.$$

(a) Show that L is invariant under infinitesimal rotations of the D-dimensional space.

How do the coordinates x_i, velocities v_i, and momenta p_i transform under an infinitesimal rotation in the x_2x_5-plane?

(b) Find the momentum $\vec{p}$ of the particle as a function of its velocity $\vec{v}$. What is the component of the momentum along the x_4-axis?

Find the velocity $\vec{v}$ of the particle as a function of $\vec{p}$. What is the component of the velocity along the x_2-axis?

(c) Use Noether's theorem to find conserved charges, J_{ij}, corresponding to the rotational symmetry of the Lagrangian. How many independent charges are there?

(d) Specialise the formulae for J_{ij} to the $D = 3$ case.

Define the angular momentum $\vec{M}$.

How are J_{ij} for $D = 3$ related to the components M_k of $\vec{M}$?

Express J_{ij} for $D = 3$ in terms of M_i by using ϵ_{ijk}.

Express M_i in terms of J_{ij} by using ϵ_{ijk}.

(e) Use Noether's theorem to find the energy E of the particle.

Express E in terms of $\vec{v}$ and $\vec{r}$. Express E in terms of $\vec{p}$ and $\vec{r}$.

Problem 4. (To do the problem analyse the solution to Q2b of the 2017 final exam).

Consider the following Lagrangian of a system with two physical degrees of freedom

$$L_\lambda = \frac{m}{2}(-v_0^2 + v_1^2 + v_2^2) + \lambda(-x_0^2 + x_1^2 + x_2^2 + a^2)\,, \quad x_0 > 0\,,$$

where λ is a Lagrange multiplier. From the previous homework we know that

$$-x_0^2 + x_1^2 + x_2^2 + a^2 = 0\,, \quad x_0 > 0\,,$$

defines a surface which is the upper sheet of a hyperboloid of two sheets obtained by revolving the hyperbola $x_0^2 = x_1^2 + a^2$ in the x_0x_1-plane about the x_0-axis. This system however cannot be interpreted as a constrained system of a particle in three-dimensional Euclidean space $\mathbb{R}^3$ because of the minus sign in front of v_0^2. It describes a particle moving in the two-dimensional Lobachevsky (or hyperbolic) plane H^2 embedded in Minkowski space-time $R^{2,1}$.

(a) Consider a set of all 3×3 matrices A which satisfy the condition

$$A_{ik}\eta_{kn}A_{jn} = \eta_{ij}\,, \quad \textit{summation over } k, n = 0, 1, 2 \quad \Longleftrightarrow \quad A\eta A^T = \eta \quad \Leftrightarrow \quad A^T\eta A = \eta\,,$$

where $\eta = (\eta_{ij})$, $i, j = 0, 1, 2$ is the following diagonal matrix

$$\eta = \mathrm{diag}(-1, 1, 1)\,.$$

This set is denoted as $O(2,1)$, and it is a Lorentz group of pseudo-rotations and reflections of the coordinates x_i in Minkowski space-time $R^{2,1}$. Prove that $O(2,1)$ is a group under the standard matrix multiplication.

(b) Infinitesimal pseudo-rotations from $O(2,1)$ are parametrised by a matrix ϵ_{ij}

$$x_i \to x_i' = x_i + \epsilon_{ij}x_j\,, \qquad A_{ij} = \delta_{ij} + \epsilon_{ij}\,. \tag{0.1}$$

Find relations between ϵ_{ij}, and write them down explicitly for all values of i, j.

(c) Prove that L_λ is invariant under the $O(2,1)$ group of pseudo-rotations and reflections of the coordinates x_i:

$$x_i \to A_{ij} x_j \,, \quad \textit{summation over } j \,!\,,$$

where $A \in O(2,1)$ is a 3×3 matrix.

(d) Specify any continuous symmetries and use Noether's theorem to construct the corresponding conserved quantities.

(e) A convenient choice of generalised coordinates in $\mathbb{R}^{2,1}$ is

$$x_1 = r \cos\phi \sinh\zeta \,, \quad x_2 = r \sin\phi \sinh\zeta \,, \quad x_0 = r \cosh\zeta \,, \tag{0.2}$$

which is an analog of spherical coordinates in $\mathbb{R}^3$.

By using these coordinates solve the constraint.

Derive an expression for the reduced Lagrangian L, and express the conserved quantities in terms of ϕ and ζ, and their time derivatives (velocities).

(f) Check explicitly that the conserved quantities you've found are indeed conserved.

(g) Express the conserved quantities in terms of ϕ and ζ, and their conjugated momenta p_ϕ and p_ζ.