EULER'S ORIGINAL SCRIPTURE

Looking at $\frac{\pi^2}{6} \geq 1 + \frac{1}{4} + \frac{1}{9} + \cdots$

$$\sin x \overset{?}{=} x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)\cdots$$

Because $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$ are roots of $\sin x$

IN FACT:

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots \text{ FOR SMALL } x$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

COEFFICIENT: $\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} \cdots$

EULER'S NOTES CAN ALSO BE FORMALISED

BILINEAR FORMS & QUADRATIC FORMS

suppose $f(x_1, \dots, x_n)$ FUNCTION $\mathbb{R}^n \to \mathbb{R}$

IF f is "smooth enough," then for any point x°

$x^\circ = (x_1^\circ, x_2^\circ, \dots, x_n^\circ)$

WE CAN WRITE THE SECOND-ORDER APPROXIMATION:

$$f\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = f\begin{pmatrix}x_1^\circ\\ \vdots \\ x_n^\circ\end{pmatrix} + \sum_{i=1}^{N} \frac{\delta f}{\delta x_i}(x^\circ)(x_i - x_i^\circ) + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\delta^2 F}{\delta x_i \delta x_j}(x_0)(x_i - x_i^\circ)(x_j - x_j^\circ)$$

+ "SMALL CORRECTION" (relative to $|x - x^\circ|^2$)

IF MAXIMUM/MINIMUM is ATTAINED AT x_0,

THEN for $i \in \{1, \dots, n\}$ we have $\delta x_i f(x^\circ) = 0$

$$\Rightarrow f\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \approx f\begin{pmatrix}x_1^\circ\\ \vdots \\ x_n^\circ\end{pmatrix} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\delta^2 f}{\delta x_i \delta x_j}(x_0)(x_i - x_i^\circ)(x_j - x_j^\circ)$$

$$f\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \approx f\begin{pmatrix}x_1^\circ\\ \vdots \\ x_n^\circ\end{pmatrix} + \frac{1}{2} q(\overrightarrow{x - x^\circ})$$

WHERE $q(y) = \sum_{i,j} a_{ij} y_i y_j \qquad a_{ij} = \frac{\delta^2}{\delta x_i \delta x_j} f(\vec{x}_0)$

DEF: QUADRATIC FORM

Let V be a vector space

A function $q: V \to \mathbb{R}$ is called a quadratic form

IF FOR SOME BASIS $e_1, \dots, e_n$ of V, we have

$$q(y_1 e_1 + \cdots + y_n e_n) = \sum_{i,j=1}^{n} a_{ij} y_i y_j$$

REMARK: This is Enough to check for some coordinate system. It follows by inspection that then the same holds for any coordinate system.

BILINEAR FORMS & QUADRATIC FORMS

EXAMPLE OF QUADRATIC FORM:

$V = \mathbb{R}^n, \quad q(x) = (x, x)$

more generally V is a Euclidean Vector Space, and $q(x)$ is equal to (x, x)

DEF: BILINEAR FORMS

Let V be a vector space.

A function $b: V \times V \to \mathbb{R}$ is called a "BILINEAR FORM" IF:

$b(v_1 + v_2, w) = b(v_1, w) + b(v_2, w) \qquad b(c \cdot v, w) = c \cdot b(v, w)$

$b(v, w_1 + w_2) = b(v, w_1) + b(v, w_2) \qquad b(v, c \cdot w) = c \cdot b(v, w)$

A BILINEAR FORM IS SAID TO BE: SYMMETR

- Symmetric IF $b(v, w) = b(w, v)$
- Positive-Definite IF $b(v, v) \geq 0$, $b(v, v) = 0$ IFF $v = 0$
- Skew-Symmetric IF $b(v, w) = -b(w, v)$

IF b IS A BILINEAR FORM ON V, $e_1, \dots, e_n$ IS A BASIS OF V

THEN: $b(x_1 e_1 + \cdots + x_n e_n, y_1 e_1 + \cdots + y_n e_n) = \sum_{i,j=1}^{n} b_{ij} x_i y_j$

$$= \sum_{i,j=1}^{n} b_{ij} x_i y_j \qquad (b_{ij} = b(e_i, e_j))$$

THIS ALLOWS US TO ASSOCIATE TO A BILINEAR FORM A MATRIX $B = (b_{ij})$

$$\Rightarrow b(x_1 e_1 + \cdots + x_n e_n, \; y_1 e_1 + \cdots + y_n e_n)$$

$$= \underset{1\times n}{x^T} \; \underset{n\times n}{B} \; \underset{n\times 1}{y} \quad \text{gives a } 1\times 1 \text{ matrix (real no)}$$

where $x = (x_1, \dots, x_n) \quad y = (y_1, \dots, y_n)$

EVERY BILINEAR FORM b Gives RISE TO A QUADRATIC FORM q BY THE FORMULA $q(v) = b(v, v)$

BILINEAR FORM / QUADRATIC FORM RELATIONSHIP

EXAMPLE: $V = \mathbb{R}^2$ $b_1\left(\binom{x_1}{x_2},\binom{y_1}{y_2}\right) = x_1y_2 + x_2y_1$

$b_2\left(\binom{x_1}{x_2},\binom{y_1}{y_2}\right) = 2x_1y_2$

FOR b_1

$q\left(\binom{x_1}{x_2}\right) = b_1\left(\binom{x_1}{x_2},\binom{x_1}{x_2}\right) = 2x_1x_2$

FOR b_2

$q\left(\binom{x_1}{x_2}\right) = b_2\left(\binom{x_1}{x_2},\binom{x_1}{x_2}\right) = 2x_1x_2$

NOT A ONE-TO-ONE MAP

- BILINEAR FORM $\rightarrow$ QUADRATIC FORM (NOT INJECTIVE)
- QUADRATIC FORM $\rightarrow$ BILINEAR FORM ALSO POSSIBLE
  - IT IS POSSIBLE TO UNIQUELY RECONSTRUCT A SYMMETRIC BILINEAR FORM OUT OF A GIVEN QUADRATIC FORM BY THE FORMULA:

$$b(v,w) = \frac{1}{2}\left[q(v+w) - q(v) - q(w)\right]$$

ANOTHER IMPORTANT EXAMPLE OF QUADRATIC FORM

$V = \mathbb{R}^3$ w/ COORDINATES x_1, x_2, x_3, t

$$q(\vec{x}) = x_1^2 + x_2^2 + x_3^2 - t^2$$

THM: NORMAL FORM THEOREM

- IF V IS A vector space, $q: V \rightarrow \mathbb{R}$ a quadratic form, then there exists a basis of V, say $e_1, \dots, e_n$ such that

$$q(x_1, \dots, x_n) = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \dots + \varepsilon_n x_n^2$$

THERE EXISTS BASIS

S.T.: $q\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \varepsilon_1 x_1^2 + \dots + \varepsilon_n x_n^2$

THM: LAW OF INERTIA

- IN theorem 1, the NUMBER of 1's, 0's & (-1)'s do not depend on the choice of basis $e_1, \dots e_n$

THM 3: JACOBI THEOREM

- V - VECTOR SPACE, b - BILINEAR FORM, B - A MATRIX representing b w.r.t. SOME COORD SYSTEMS $e_1, \dots, e_n$

Let $\Delta_0 = 1$ Δ_K = determinant of top left $k \times k$ corner of B

BILINEAR FORM THEOREMS

THM 3: JACOBI THEOREM

V - A VECTOR SPACE b - A BILINEAR FORM

B - A MATRIX representing b w.r.t some basis $e_1, \dots, e_n$

Let $\Delta_0 = 1$ $\Delta_K = \det$(top left $k \times k$ matrix of B)

- Suppose $\Delta_i \neq 0$ FOR ALL i:

THEN THE NUMBER OF $+1$'s

THE NUMBER OF -1's

THE NUMBER OF 0's IS Equal to the number of positive negative And zero Numbers

Among $\frac{A_0}{A_1}, \frac{A_1}{A_2}, \dots \frac{A_{n-1}}{A_n}$

THM 4: SYLVESTER THEOREM

A quadratic form $q(v) = b(v,v)$, a symmetric bilinear form is

POSITIVE DEFINITE IFF $\Delta_K > 0$ FOR ALL $k \in \{1, \dots, n\}$

THM:

- Every quadratic form q on a vectorspace V can be made a signed sum of squares: there exists a basis $e_1, \dots, e_n$ of V s.t.: $q(x_1e_1 + \dots + x_ne_n) = \sum_{i=1}^{n} \varepsilon_i x_i^2$

where all ε_i are $1, 0$ or -1

PROOF OF THEOREMS - NORMAL FORM

$$q(x_1 e_1 + \dots + x_n e_n) = \sum_{i=1}^{n} \varepsilon_i x_i^2, \quad \varepsilon_i \in \{-1, 0, 1\}$$

PROOF: SLOGAN: "IMITATE GRAM-SCHMIDT"

- PROOF BY INDUCTION ON $\dim V$

BASIS OF INDUCTION: $\dim V = 0$
NO BASIS VECTORS
NOTHING TO CHECK (EZPZ)

STEP OF INDUCTION: $\dim V = n$

CASE 1: $q(v) = 0$ for all $v \in V$
THEN TAKE ANY BASIS
ALL $\varepsilon_i = 0$

CASE 2: THERE EXISTS A VECTOR v s.t. $q(v) \neq 0$

- Let us find vectors

$$f_1 = v, f_2, \dots, f_n$$

that form a basis of V

- CONSIDER THE BILINEAR FORM b ASSOCIATED TO THE QUADRATIC FORM q

$$b(v,w) = \tfrac{1}{2}\left(q(v+w) - q(v) - q(w)\right)$$

- USING THIS BILINEAR FORM, WE CAN WRITE:

$f_1' = f_1$ — $b(f_1, f_1) = q(f_1) = q(v) \neq 0$

$f_2' = f_2 - \frac{b(f_2, f_1)}{b(f_1, f_1)} f_1$ — $b(f_1', f_1') = b(v, v) \neq 0$

$\vdots$

$f_n' = f_n - \frac{b(f_n, f_1)}{b(f_1, f_1)} f_1$ — $b(f_k', f_1') =$

THIS GIVES US THAT

$b(f_1, f_1) = q(f_1) = q(v) \neq 0$

$\Rightarrow b(f_1', f_1') = b(v, v) \neq 0$

$b(f_k', f_1') = b\left(f_k - \frac{b(f_k, f_1)}{b(f_1, f_1)} f_1, f_1\right) = b(f_k, f_1) - b(f_k, f_1) = 0$

$b(f_1', f_k') = 0$ by symmetry

- CONSIDER $W = \mathrm{span}(f_2', \dots, f_n')$, and the quadratic form q on it. SINCE $\dim V = n-1$, the induction hypothesis applies, and we can find $e_1, \dots, e_n$ of W s.t.: $q(x_2 e_2, \dots, x_n e_n) = \varepsilon_2 x_2^2 + \dots \varepsilon_n x_n^2$

NORMAL FORM - LAW OF INERTIA PROOFS

- CONSIDER $W = \mathrm{span}(f_2, \dots, f_n)$
q the quadratic form on W

SINCE $\dim V = n-1$, the induction hypothesis applies, and we can find a basis $e_1, \dots, e_n$ of W s.t.: $q(x_2 e_2 + \dots + x_n e_n) = \varepsilon_2 x_2^2 + \dots + x_n e_n^2$

- TAKE $e_1 = \frac{v}{\sqrt{|q(v)|}}$ — $\varepsilon_1 = q(e_1) = \frac{q(v)}{|q(v)|} = \pm 1$

$\Rightarrow$ THE VALUE $q(x_1 e_1 + \dots + x_n e_n)$ is a signed sum of squares

THM: (LAW OF INERTIA) IN THE SIGNED SUM OF SQUARES AS IN THE PREVIOUS THEOREM, THE NUMBERS n_+, n_-, n_0 of ε_i's equal to $1, -1, 0$ respectively does not depend on the choice of basis $e_1, \dots, e_n$

PROOF: Let us 1st prove invariance of n_0

- CONSIDER the associated symmetric bilinear form $b(v,w) = \frac{1}{2}(q(v+w) - q(v) - q(w))$
- THE KERNEL of a symmetric bilinear form b is by definition the set of all vectors v s.t. $b(v,w) = 0$ for all $w \in V$

- Suppose we have a basis

$$e_1, \dots, e_{n_+}, f_1, \dots, f_{n_-}, g_1, \dots, g_{n_0}$$

s.t. $q(x_1 e_1, \dots y f_i \dots z g_{n_0}) = x_1^2 + \dots - y_i^2 \dots y_{n_-}^2$

THEN: $b(x_1 e_1 + \dots + y_{n_-} f_{n_-}, x_1' e_1 + \dots + y_{n_-}' f_{n_-}) =$

$$= x_1 x_1' + \dots + x_{n_+} x_{n_+}' - y_1 y_1' - \dots - y_{n_-} y_{n_-}'$$

- IF $v = x_1 e_1 + \dots z_{n_0} g_{n_0}$ is the kernel, then $b(v, e_i) = b(v, f_j) = b(v, g_k) = 0$
FOR ALL i, j, k

BUT: $b(v, e_1) = x_1 \quad \dots \quad b(v, f_{n_-}) = -y_{n_-}$

SO v IS IN THE KERNEL $\Leftrightarrow v = z_1 g_1 + \dots z_n g_n$

$\Rightarrow n_0$ IS THE KERNEL OF V (DOESN'T DEPEND ON ANY CHOICES)

PROOF: LAW OF INERTIA

- Suppose that

$$e_1, \dots e_{n_+}, f_1, \dots, f_{n_-}, g_1 \dots g_{n_0}$$

and $e_1', \dots e_{n_+}', f_1', \dots, f_{n_-}', g_1' \dots g_{n_0}'$ are two different bases, where our quadratic form is a signed sum of squares

- IF $n_+ \neq n_+'$, then WLOG $n_+ > n_+'$
  - NOTE THAT $n_+ + n_- = n - n_0$

  $\Rightarrow$ $n_- < n_-'$

- CONSIDER THE VECTORS

  $e_1, \dots, e_{n_+}, f_1', \dots f_{n_-}', g_1, \dots, g_{n_0}$

  SINCE WE REPLACED n vectors BY n' VECTORS AND $n_- < n_-'$ THE TOTAL NUMBER OF VECTORS IS GREATER THAN n

- THEREFORE THEY CANNOT BE LINEARLY INDEPENDANT VECTORS:

$$a_1 e_1 + \cdots + a_{n_+} e_{n_+} + b_1 f_1' + \cdots + b_n f_n' + c_1 g_1 + \cdots c_{n_0} g_n^0 = 0$$

- NOT ALL COFFICIENTS ARE EQUAL TO ZERO

$$V_i = a_1 e_1 + \cdots + a_{n_+} e_{n_+} + c_1 g_1 + \cdots + c_{n_0} g_{n_0} = -(b_1 f_1' + \cdots + b_{n_-} f_{n_-}')$$

$$q(v) = a_1^2 + \cdots + a_n^2 = -b_1^2 - \cdots - b_{n_-'}^2$$

$\Rightarrow$ $a_1 = \cdots = a_{n_+} = b_1 = \cdots = b_{n_-} = 0$

$\Rightarrow$ ALL COEFFICEENTS ARE EQUAL TO ZERO

SUBSTITUTING BACK WE GET $c_1 = \dots c_n = 0$

CONTRADICTION

$\rightarrow$ comes from assuming that $n_+' \neq n_+$

PROOF: JACOBI THEOREM

- SUPPOSE $e_1, \dots, e_n$ IS A BASIS OF V AND THAT $\Delta_i \neq 0$ for all $i = 1, \dots, n$

  Then there exists a basis of $f_1, \dots, f_n$ s.t.:

- $q(x_1 f_1 + \cdots + x_n f_n) = \frac{1}{\Delta_1} x_1^2 + \frac{\Delta_1}{\Delta_2} x_2^2 + \cdots + \frac{\Delta_{n-1}}{\Delta_n} x_n^2$

- EQUIVALENTLY: $b(f_i, f_j) = \begin{cases} 0, & i \neq j \\ \frac{\Delta_{i-1}}{\Delta_i}, & i = j \end{cases}$ (Let $\Delta_0 = 1$)

PROOF: WE LOOK FOR A BASIS OF THE FORM

$$f_1 = \alpha_{11} e_1$$

$$f_2 = \alpha_{12} e_1 + \alpha_{22} e_2$$

$$\vdots$$

$$f_n = \alpha_{1n} e_1 + \cdots + \alpha_{nn} e_n$$

AND SOLVE EQUATIONS:

$$b(f_i, e_i) = \begin{cases} 0, & \boxed{j = 1, \dots, i-1} \text{ ENSURE DIAGONAL MATRIX} \\ 1, & \boxed{j = i} \text{ ENSURE THAT \# EQTNS IS SAME AS \# UNKNOWNS} \end{cases}$$

$i > j$ $b(f_i, f_j) = b(f_i, \alpha_{1j} e_1 + \cdots + \alpha_{jj} e_j)$

$$= \alpha_{1j} b(f_i, e_1) + \cdots + \alpha_{2j} b(f_i, e_j) = 0$$

$i < j$ $b(f_i, f_j) = b(f_j, f_i) = 0$ by SYMMETRY

- FOR THE GIVEN i, WE HAVE:

$b(f_i, e_1) = 0$

$b(f_i, e_{i-1}) = 0$

$b(f_i, e_i) = 1$

$\Rightarrow$ $b(\alpha_{1i} e_1 + \cdots + \alpha_{ii} e_i, e_1) = b(e_1, e_1)\alpha_{1i} + \cdots + b(e_i, e_1)\alpha_{ii} = 0$

$b(f_i, e_{i-1}) = b(e_1, e_{i-1})\alpha_{1i} + \cdots + b(e_i, e_{i-1})\alpha_{ii} = 0$

$b(f_i, e_i) = b(e_1, e_i)\alpha_{1i} + \cdots + b(e_i, e_i)\alpha_{ii} = 0$

- $\alpha_{1i}, \alpha_{2i}, \dots \alpha_{ii}$ are unknowns. This system of coefficients is the $i \times i$ matrix left top corner submatrix of B

SINCE $\Delta_i \neq 0$, this system of equations has EXACTLY ONE SOLUTION

THIS SHOWS THAT FOR OUR CONSTRANTS, $f_1, \dots f_n$ IS UNIQUE.

AND $b(f_i, f_j) = 0$ FOR $i \neq j$

LET US COMPUTE $b(f_i, f_i)$

PROOF: JACOBI, SYLVESTER

LET US COMPUTE $b(f_i, f_i)$

$$b(f_i, \alpha_{1i}e_i + \cdots + \alpha_{ii}e_i) = \alpha_{ii} b(f_i, e_i) = \alpha_{ii}$$

BY CRAMER'S FORMULA:

$$\alpha_{ii} = \frac{\det\begin{pmatrix} b(e_1,e_1) & b(e_2,e_1) & \cdots & b(e_{i-1},e_1) & 0 \\ b(e_1,e_2) & & & & 0 \\ \vdots & & & & 0 \\ b(e_i,e_{i-1}) & \cdots & \cdots & b(e_i,e_i) & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}}{\Delta_i}$$

$$\Rightarrow \boxed{\alpha_{ii} = \frac{\Delta_{i-1}}{\Delta_i}}$$

QED

THM: SYLVESTER'S THEOREM

- THE GIVEN SYMMERTRIC BILINEAR FORM IS POSITIVE DEFINITE IFF all Δ_i are positive (ENOUGH TO CHECK FOR SOME BASIS) THEN THE JACCOBI THEOREM APPLIES

PROOF: Assume that all $\Delta_i > 0$

THEN THE JACOBI THEOREM APPLIES

$$q(x_i f_1 + \cdots + x_n f_n) = \frac{1}{\Delta_1} x_1^2 + \cdots + \frac{\Delta_{n-1}}{\Delta_n} x_n^2 > 0$$

UNLESS $x_1 = \cdots = x_n = 0$

- Assume that the BILINEAR FORM IS POSITIVE DEFINITE. Let us show $\Delta_i \neq 0$

→ ASSUME $\Delta_i = 0$

we shal demonstrate that there exists $v = c_1 e_1 + \cdots + c_i e_i$

s.t. $b(v, e_1) = \cdots = b(v, e_i) = 0$

$$b(v, e_1) = b(e_1, e_1)c_1 + \cdots + b(e_i, e_1)c_i = 0$$

$$\vdots$$

$$b(v, e_i) = b(e_1, e_i)c_1 + \cdots + b(e_i, e_i)c_i = 0$$

THIS IS A HOMOGENEOUS SYSTEM OF i UNKOWNS, zero determinant of the matrix.

$\Rightarrow$ HAS NON-TRIVIAL SOLUTIONS $c_1, \cdots, c_i$

SYLVESTER, SYMMETRIC BILINEAR FORM

$$b(v,v) = b(v, c_1 e_1 + \cdots + c_i e_i) = 0$$

WITH $c_1 \cdots c_i$ NON-ZERO TRIVIAL SOLUTION

$\Rightarrow$ BY POSITIVITY, $v = 0$, A CONTRADICTION

THEREFORE $\Delta_i \neq 0$ for all i, so the JACOBI THEOREM APPLIES

$$q(x_1 f_1 + \cdots + x_n f_n) = \frac{1}{\Delta_1} x_1^2 + \cdots + \frac{\Delta_{n-1}}{\Delta_n} x_n^2$$

- We HAVE $\Delta_0 = 1, \Delta_1, \ldots, \Delta_n$

IF SOME OF THE Δ_i's are negative, THEN THE FIST negative one of these is

$$q(f_i) = \frac{\Delta_{i-1}}{\Delta_i} < 0 \quad \text{CONTRADICTION}$$

- A BILINEAR FORM IS SAID TO BE *NEGATIVE DEFINITE* IF $b(v,v) < 0$ for all non-zero vectors.

Sylvester's THEOREM IMPLIES THAT ~~THE~~ A FORM IS *NEGATIVE DEFINITE* IFF all numbers $(-1)^i \Delta_i$ are positive (CONSIDER NEW FORM $b' = -b$)

- THE TRIPLE (n_+, n_-, n_0) IS OFTEN CALLED THE *SIGNATURE OF A BILINEAR FORM*

SIGNATURE: (n_+, n_-, n_0)

THM: LET B be the MATRIX OF A SYMMETRIC BILINEAR FORM b w.r.t some BASIS.

THEN: (1) all eigenvalues of b are real numbers

(2) THE NUMBERS n_+, n_-, n_0 are equal to the number of positive, negative and zero eigenvalues of B respectively

SYMMETRIC BILINEAR FORMS:

① $\lambda_i \in \mathbb{R}$

② $\lambda_0^{\pm} \to (n_+, n_-, n_0)$

PROOF: CONSIDER $\mathbb{R}^n$ and define a linear transformation $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ where $\varphi(x) = B \cdot x$

Let us equip $\mathbb{R}^n$ WITH THE STANDARD POSITIVE DEFINITE SCALAR PRODUCT

- WE HAVE THE FOLLOWING EQUATION:

$(Bx, y) = (x, By)$ for any vectors x, y

$(x, y) = y^T \cdot x$

$(Bx, y) = y^T \cdot Bx$

$(x, By) = (By)^T \cdot x = y^T B^T \cdot x = y^T B \cdot x$

BECAUSE $B^T = B$ AS IT IS SYMMETRIC

- IN OTHER WORDS, φ is an SYMMETRIC LINEAR TRANSFORMATION

DEF: LET V BE A EUCLIDEAN VECTOR SPACE. A LINEAR TRANSFORMATION φ IS SAID TO BE SYMMETRIC IF $(\varphi(x), y) = (x, \varphi(y))$ $\forall x, y \in V$

- LET US CONSIDER THE FUNCTION $q(x) = (\varphi(x), x)$

$q(x) = (\varphi(x), x)$ ON THE "UNIT SPHERE" OF ALL VECTORS x s.t. $|x| = 1$

- THE UNIT SPHERE IN $\mathbb{R}^n$ IS compact.

(which implies that every continuous function on it is bounded & attains it's maximum & minimum value)

- Let $m = \min_{|x|=1} q(x)$ $\quad M = \max_{|x|=1} q(x)$

$\Rightarrow m \leq q(x) \leq M$ for $|x| = 1$, and for some x, these inequalities will be equalities

THEN: for all vectors in $\mathbb{R}^n$, we have:

- $m|x|^2 \leq q(x) \leq M(x)$

$\Rightarrow q(x) - M(x, x) \leq 0$

$(B(x), x) - M(x, x) \leq 0$

$(Bx, x) - (Mx, x) \leq 0$

$((B - MI_n)x, x) \leq 0$

EQUALITY for $x \neq 0$ s.t. $\frac{x}{|x|}$ is one of the points where M is attained on the unit sphere

- WE CALL $B - MI = A = (a_{ij})$

$\sum_{i,j} A_{ij} x_i x_j \to MAX$

$\frac{\delta}{\delta x_k} \sum_{i,j} A_{ij} x_i x_j = \sum_i A_{ik} x_i + \sum_j A_{kj} x_j$

BUT IT IS SYMMETRIC, SO

$= 2 \sum_i A_{ik} x_i$

- IF AT A POINT x^0 WE HAVE MAXIMUM/MINIMUM

THEN: $Ax^0 = 0$

$\Rightarrow (B - MI)x^0 = 0$

$\Rightarrow Bx^0 = Mx^0$

- NOW CONSIDER $U = \text{span}(x_0)^\perp$

Let us check that U is invariant under φ

Let $u \in U$, $\varphi(u, x^0) = 0$

$\varphi(u, x_0) = (u, \varphi(x_0)) = (u, Mx^0) = M(u, x^0) = 0$

- CONSIDER THE VECTORSPACE U, and the SYMMETRIC TRANSFORMATION φ.

So we can argue BY INDUCTION ON DIMENSION [illegible] ORTHONORMAL BASIS OF EIGENVECTORS

NOW THAT WE HAVE CONSTRUCTED AN ORTHONORMAL BASIS OF EIGENVECTORS, let us take an orthonormal basis of eigenvectors $e_1, \dots, e_n$ and compute the matrix of the quadratic form $q(x)$ with respect to this basis.

$$q(e_i, e_j) = (Be_i, e_j) = (\lambda_i e_i, e_j) = \begin{cases} 0, & i \neq j \\ \lambda_i, & i = j \end{cases}$$

THIS MEANS THAT

$$q(x_1 e_1 + \dots + x_n e_n) = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2$$

WHICH IMPLIES OUR STATEMENT ON THE SIGNATURE OF q ▨

DON'T EVEN NEED THE ACCURATE EIGENVALUES, AS AN ESTIMATE IS GOOD ENOUGH FOR GETTING JUST THE SIGN

REMARK:

Let B be a symmetric matrix, and x_1, x_2 be eigenvectors s.t. $\lambda_1 \neq \lambda_2$ (THE EIGENVALUES ARE DIFFERENT)

THEN: $(B(x_1), x_2) = (x_1, B(x_2))$

$$\Rightarrow (\lambda_1 x_1, x_2) = (x_1, \lambda_2 x_2)$$

$$\Rightarrow \lambda_1 (x_1, x_2) = \lambda_2 (x_1, x_2)$$

$$\Rightarrow (\lambda_1 - \lambda_2)(x_1, x_2) = 0$$

$$\lambda_1 \neq \lambda_2 \quad \Rightarrow (x_1, x_2) = 0$$

$\lambda_1 \to x_1$

$\lambda_2 \to x_2$

$\lambda_1 \neq \lambda_2$

$\Rightarrow (x_1, x_2) = 0$

THIS REMARK MEANS THAT IN PRACTICE, when searching for an orthonormal basis of eigenvectors, MOST OF THE TIME, we just need find eigenvectors, and they are automatically orthogonal.

WHEN we have ~~orthogon~~ repeated eigenvalues, we'd need to GRAHM-SCHMIDT ORTHOGONALISE vectors with the same eigenvalue

ORTHOGONAL MATRICES

DEF: AN $n \times n$ matrix A is ORTHOGONAL IF $A^T \cdot A = I_n$

(IN OTHER WORDS, $A^T = A^{-1}$)

(ALSO, COLUMNS OF A ARE ORTHONORMAL)

ORTHOGONAL

$A^T \cdot A = I_n$

THM: A matrix A is orthogonal IF AND ONLY IF the associative linear transformation on φ of $\mathbb{R}^n$ preserves the scalar product

$$(\varphi(x), \varphi(y)) = (x, y) \quad \forall x, y \in \mathbb{R}$$

PROOF: $(\varphi(x), \varphi(y)) = (Ax, Ay) = (Ay)^T (Ax)$

$$= y^T A^T A x$$

SO, $(\varphi(x), \varphi(y)) = (x, y)$

$\Leftrightarrow (A^T A x, y) = (x, y) \Leftrightarrow (A^T A x - x, y) = 0$

$\Leftrightarrow A^T A x - x = 0$ FOR ALL x

$\Leftrightarrow (A^T A - I_n) x = 0$

$\Leftrightarrow A^T A = I_n$ ☐

Orthogonal Matrices

Examples: $n = 2$

$\Rightarrow A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^T A = I_N$

$\Rightarrow \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ orthonormal

$\Rightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$

Remark: IF A IS ORTHOGONAL, THEN $\det(A) = \pm 1$

- suppose $\det(A) = 1$,
  THEN $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$ (OTHERWISE, $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \end{pmatrix}$)
  so: $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

Example: $n = 3$

- IN THIS CASE, A HAS a real eigenvalue, (SINCE $\chi(t)$ is of degree 3, so has different signs as $t \to \pm\infty$)
- Let c be an eigenvalue, v an eigenvector

$\Rightarrow (Av, Av) = (v, v)$

$= (cv, cv) = c^2(v, v) = (v, v)$

$\Rightarrow c^2 = 1 \Rightarrow c = \pm 1$

- Assume $\det A = 1$
  → THE ORTHOGONAL COMPLIMENT OF v IS AN ~~ORTHOG~~ INVARIANT SUBSPACE OF A.
- suppose $(v, w) = 0$
  THEN: $(Aw, v) = (Aw, \frac{1}{c}Av) = (Aw, A(\frac{1}{c} \cdot v))$
  $\Rightarrow (Aw, A(\frac{1}{c}v)) = (w, \frac{1}{c}v) = 0$
- IF $c = -1$
  THEN ON $\mathrm{span}(v)^\perp$, THE ASSOCIATED LINER

3D Orthogonal Matrix Example

- IF $c = -1$ $(\det A = 1)$
  THEN on $\mathrm{span}(v)^\perp$, the associated linear transformation has determinant -1

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ 0 & & \end{pmatrix}$ — has $\mathrm{span}(v^\perp)$ transforms

- THE CHARACTARISTIC polynomial χ of B is $t^2 - kt - 1$ for some k

⇒ THE POLYNOMIAL HAS A REAL ROOT

~~THE~~ ~~ROOTS~~ OF $t^2 - kt - 1$ are also eigenvalues of A, so equal to ± 1, and cannot both be equal to -1

- SO EVERY ORTHOGONAL MATRIX A WITH DET(A) has an eigenvalue equal to 1
  → Let $Av = v$
- TAKE THE ORTHOGONAL COMPLIMENT $\mathrm{span}(v)^\perp$
  - IT IS INVARIANT
- FOR THE DIRECT SUM DECOMPOSITION
  $\mathbb{R}^3 = \mathrm{span}(v) \oplus \mathrm{span}(v)^\perp$, A is represented by

$A \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{C}} \\ 0 & & \end{pmatrix}$

where C is orthogonal, and $\det(C) = 1$

- SO $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$
- IS THE ROTATION ABOUT v THROUGH THE ANGLE OF φ

DEF: Let V be a vector space over $\mathbb{C}$

V has a structure of a HERMITIAN VECTOR SPACE IF IT IS EQUIPPED W/ A FUNCTION: $V \times V \to \mathbb{C}$

$$v, w \longmapsto \langle v, w \rangle$$

SATISFYING THESE CONDITIONS:

① SESQUILINEAR

$$\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$$
$$\langle v_1, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v_2, w \rangle$$
$$\langle cv, w \rangle = c \langle v, w \rangle$$
$$\langle v, cw \rangle = \bar{c} \langle v, w \rangle$$

② "SYMMETRY"

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

- THIS IMPLIES THAT $\langle v, v \rangle = \overline{\langle v, v \rangle}, \Rightarrow \langle v, v \rangle$ Real

③ "POSITIVITY"

$$\langle v, v \rangle \geq 0$$

AND EQUAL TO ZERO IFF v EQUALS ZERO

Ex $V = \mathbb{C}^n$

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2 + \cdots + x_n \bar{y}_n$$

$$\Rightarrow \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$$

- FOR A HERMITIAN VECTORSPACE, CAN DEFINE AN orthonormal basis and prove that every Hermitian space has an orthonormal basis. (ANALAGOUS TO GRAHM-SCHMIDT)

→ THEREFORE, AFTER

- FOR A HERMITIAN VECTOR SPACE, WE can define an orthogonal basis, and that every Hermitian space has an orthonormal basis (ANALAGOUS to GRAHM-SCHMIDT)

  (Therefore, after a change of basis, every space becomes $\mathbb{C}^n$ with the "standard" structure.)

- Also, can define orthogonal compliments and prove their properties.

DEF (Adjoint Linear Map)

Let V be a hermitian vector space, $\varphi: V \to V$ a linear map

- THE ADJOINT LINEAR MAP $\varphi^\dagger$ is defined by the formula: $\langle \varphi(v), w \rangle = \langle v, \varphi^\dagger(w) \rangle$

  (THIS DEFINES $\varphi^\dagger(w)$ UNIQUELY, AND ONE CAN TAKE VECTORS OF AN ORTHOGONAL BASIS IN PLACE OF v.)

DEF: Symmetric, Unitary, Normal,

- A LINEAR TRANSFORMATION IS SAID TO BE

SYMMETRIC IF $\varphi^\dagger = \varphi$ OR $\langle \varphi(v), w \rangle = \langle v, \varphi(w) \rangle$

UNITARY IF $\varphi^\dagger \varphi = I$ OR $\langle \varphi(v), \varphi(w) \rangle = \langle v, w \rangle$

NORMAL IF $\varphi^\dagger \varphi = \varphi \varphi^\dagger$ (INCLUDES SYMMETRIC & UNITARY)

NORMAL TRANSFORMATIONS

THEOREM:

A NORMAL TRANSFORMATION OF A HERMITIAN VECTOR SPACE has an orthonormal basis of eigenvectors

PROOF: Since $\varphi\varphi^{\dagger} = \varphi^{\dagger}\varphi$, they have a common eigenvector:

$$\varphi(v) = c(v), \quad \varphi^{\dagger}(v) = c'v$$

$$\langle \varphi(v), v \rangle = \langle v, \varphi^{\dagger}(v) \rangle$$

$$\Rightarrow c\langle v, v\rangle = \overline{c'}\langle v, v\rangle \Rightarrow c = \overline{c'}$$

- Let us show that $\mathrm{span}(v)^{\perp}$ is invariant under both φ and $\varphi^{\dagger}$
- Then $\langle \varphi(w), v\rangle = \langle w, \varphi(v)\rangle = \langle w, \bar{c}v\rangle = c\langle w, v\rangle$

$$\langle \varphi^{\dagger}(w), v\rangle = \overline{\langle v, \varphi^{\dagger}(w)\rangle} = \overline{\langle \varphi(v), w\rangle} = \overline{\langle cv, w\rangle} = \overline{c\langle v, w\rangle} = 0$$

- Since $\mathrm{Span}(v)^{\perp}$ is invariant under $\varphi, \varphi^{\dagger}$, can proceed by induction □

COROLLARY
① Every symmetric linear transformation of a hermitian vector space has an orthogonal basis of vectors

② All eigen values of a symmetric linear trasformation are real

REMARK: A symmetric real matrix can be regarded as a symmetric transformation of $\mathbb{C}^n$, and our result gives another proof of the fact that a symmetric real matrix is diagonalisable

APPLICATIONS: IMAGE RECOGNITION

- Suppose we have a database of grey scale photos in a standardised format of, lets say, 100×100 pixels.

→ Each photo is a vector with 10000 entries

→ Each individual pixel can be regarded as a random variable, and we have some measurements of that variable

COVARIANCE MATRIX $\left(E(X_i - EX_i)(X_j - EX_j)\right)$

- Eigenvectors of this matrix correspond to independant (probabalistic) d ections.
- $\mathcal{U}$ = matrix whose columns are vectors of intensities. $(v_1 | \dots | v_k)$
  k is the number of faces in the database

$\bar{v} = \frac{v_1 + \dots + v_k}{k}$, THE AVERAGE

$\overline{v_i} = v_i - \bar{v}$

- $\overline{\mathcal{U}} = (\overline{v_1} / \dots / \overline{v_k})$
- Thus we get that
  $\mathcal{U}\mathcal{U}^T$ = covariance matrix (up to scalar multiple)
- Eigenvalues corresponding to zero eigenvalues are not useful ones.

GENERAL RESULT:

For any $m \times n$ matrix A, nonzero eigenvalues of AA^T are the same as non-zero eigenvectors of A

JUSTIFICATION $A^TAv = \lambda v$

$(AA^T)Av = \lambda Av$

So Av is an eigenvector with eigenvalue for AA^T

∘ NOW LET US CONSIDER

$\bar{\mathcal{U}}^T \bar{\mathcal{U}}$, AND FIND IT'S EIGENVALUES FOR

NON-ZERO EIGENVALUES BY $\bar{\mathcal{U}}$.

THIS RECOVERS EIGENVECTORS FOR THE COVARIANCE MATRIX

EXPECTATIONS FOR THE EXAM

- **Know all standard methods:**
  - Compute basis relative to a subspace
  - FIND JORDAN FORMS/BASES & Powers of a MATRIX
  - Compute signature of quadratic forms
  - etc...

- **TIME MANAGEMENT IS IMPORTANT:**
  - Number of marks $\approx$ Time expectation
  - IF "small problem" takes too long $\rightarrow$ SKIP

- **ALL ANSWERS NEED CLEAR & COMPLETE JUSTIFICATION**
  - Guessing right answer is not enough