Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 3
Each set of homework questions is worth 100 marks

Problem 1. Consider the Lagrangian of a particle moving in a potential field

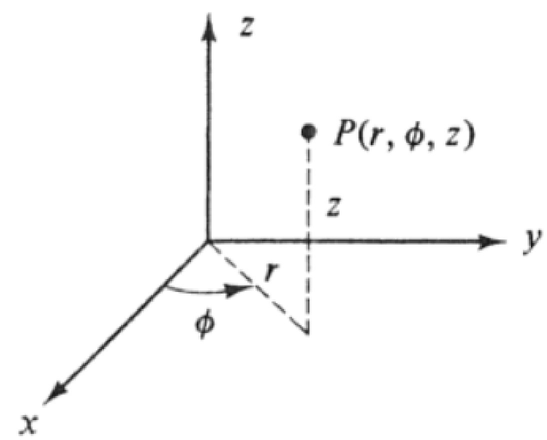
$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(r)\,, \quad r = \sqrt{x^2 + y^2}\,.$$

(a) Introduce the cylindrical coordinates (draw a picture), and derive an expression for the Lagrangian in terms of the coordinates.

Answer: The cylindrical coordinates are

$$x = r\cos\phi\,, \quad y = r\sin\phi\,, \quad z = z\,, \tag{0.1}$$

see the picture



The Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - U(r)\,.$$

(b) Identify the cyclic coordinates, and find the corresponding conserved charges. What is their physical meaning? What symmetries do they correspond to?

Answer: The angle ϕ, and the coordinate z are cyclic, and the conserved charges are their momenta

$$p_\phi = mr^2\dot{\phi}\,, \quad p_z = m\dot{z}\,. \tag{0.2}$$

Here p_ϕ is the component M_z of the angular momentum, while p_z is the linear momentum in the z-direction. They correspond to the rotational symmetry about the z-axis, and the translational symmetry along the z-axis, respectively.

Problem 2. Consider the Lagrangian of a particle moving in a potential field

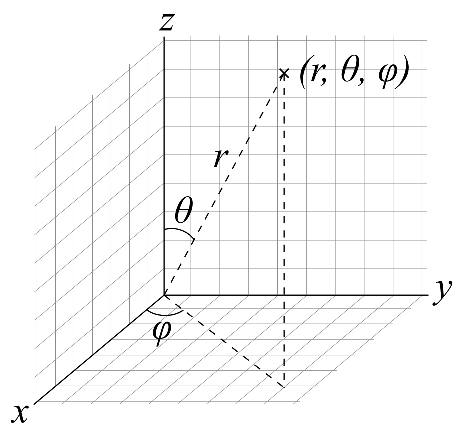
$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(r)\,, \quad r = \sqrt{x^2 + y^2 + z^2}\,.$$

(a) Introduce the spherical coordinates (draw a picture), and derive an expression for the Lagrangian in terms of the coordinates.

Answer: The spherical coordinates are

$$x = r\cos\varphi\sin\theta\,, \quad y = r\sin\varphi\sin\theta\,, \quad z = r\cos\theta\,, \tag{0.3}$$

see the picture



The Lagrangian is

$$L = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\dot\varphi^2) - U(r)\,. \tag{0.4}$$

(b) Identify the cyclic coordinates, and find the corresponding conserved charges. What is their physical meaning? What symmetries do they correspond?

Answer: The angle φ is cyclic, and the conserved charge is its momentum

$$p_\varphi = mr^2\sin^2\theta\dot\varphi\,. \tag{0.5}$$

Here p_φ is the component M_z of the angular momentum. It corresponds to the rotational symmetry about the origin. There are two more conserved charges due to the symmetry.

Problem 3. Consider the following Lagrangian of a relativistic particle moving in a D-dimensional space and interacting with a central potential field ($m\,, c\,, \alpha\,, \beta$ are constants)

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - \frac{\alpha}{r^2}e^{-\beta r^2}\,, \quad v^2 \equiv \vec v^2 = \sum_{i=1}^{D} v_i^2\,, \quad r^2 \equiv \vec r^2 = \sum_{i=1}^{D} x_i^2\,.$$

For some questions below you may use that infinitesimal rotations are parametrised by a skew-symmetric matrix ϵ_{ij}, that is

$$x_i \to x_i' = x_i + \epsilon_{ij}x_j\,, \qquad \epsilon_{ij} + \epsilon_{ji} = 0\,.$$

(a) Show that L is invariant under infinitesimal rotations of the D-dimensional space.

How do the coordinates x_i, velocities v_i, and momenta p_i transform under an infinitesimal rotation in the x_2x_5-plane?

Answer: Since L depends only on v^2 and r it is sufficient to show that v^2 and r are invariant. Since under infinitesimal rotations the coordinates are transformed as

$$x_i \to x'_i = x_i + \epsilon_{ij} x_j \,, \qquad \epsilon_{ij} + \epsilon_{ji} = 0 \,, \tag{0.6}$$

and the same expressions for the velocities, we get

$$\delta r^2 = 2x_i \delta x_i = 2x_i \epsilon_{ij} x_j = \epsilon_{ij} x_i x_j + \epsilon_{ji} x_j x_i = 0 \,, \tag{0.7}$$

and similarly for v^2.

Since infinitesimal rotations are parametrised by a skew-symmetric matrix ϵ_{ij}

$$x_i \to x'_i = x_i + \epsilon_{ij} x_j \,, \qquad \epsilon_{ij} + \epsilon_{ji} = 0 \,, \tag{0.8}$$

we get for an infinitesimal rotation in the $x_2 x_5$-plane that

$$\epsilon_{ij} = 0 \;\; \text{unless} \;\; i = 2, j = 5 \;\; \text{or} \;\; i = 5, j = 2 \,, \quad \epsilon_{52} = -\epsilon_{25} \tag{0.9}$$

and therefore

$$x_2 \to x'_2 = x_2 + \epsilon_{25} x_4 \,, \quad x_5 \to x'_5 = x_5 - \epsilon_{25} x_2 \,, \quad x_i \to x'_i = x_i \;\; \text{unless} \;\; i = 2, 5 \,. \tag{0.10}$$

The velocities and momenta transform in the same way.

(b) Find the momentum $\vec{p}$ of the particle as a function of its velocity $\vec{v}$. What is the component of the momentum along the x_4-axis?

Find the velocity $\vec{v}$ of the particle as a function of $\vec{p}$. What is the component of the velocity along the x_2-axis?

Answer:

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \,, \quad p_4 = \frac{m v_4}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{0.11}$$

From the formula above we find

$$p^2 = \frac{m^2 v^2}{1 - \frac{v^2}{c^2}} \quad \Rightarrow \quad \frac{v^2}{c^2} = \frac{p^2}{m^2 c^2 + p^2} \quad \Rightarrow \quad \sqrt{1 - \frac{v^2}{c^2}} = \frac{c\, m}{\sqrt{m^2 c^2 + p^2}} \,. \tag{0.12}$$

Thus,

$$\vec{v} = \frac{c\,\vec{p}}{\sqrt{m^2 c^2 + p^2}} = \frac{1}{\sqrt{1 + \frac{p^2}{m^2 c^2}}} \frac{\vec{p}}{m} \,, \quad v_2 = \frac{1}{\sqrt{1 + \frac{p^2}{m^2 c^2}}} \frac{p_2}{m} \,. \tag{0.13}$$

(c) Use Noether's theorem to find conserved charges, J_{ij}, corresponding to the rotational symmetry of the Lagrangian. How many independent charges are there?

Answer: Since $\delta x_i = \epsilon_{ij} x_j$ we get

$$\frac{1}{2} J_{ij} \epsilon_{ij} = \frac{\partial L}{\partial \dot{x}_i} \delta x_i = p_i \epsilon_{ij} x_j = \frac{1}{2} \epsilon_{ij} (p_i x_j - p_j x_i) \quad \Rightarrow \quad J_{ij} = p_i x_j - p_j x_i \,. \tag{0.14}$$

It is $D(D-1)/2$.

(d) Specialise the formulae for J_{ij} to the $D = 3$ case.

Define the angular momentum $\vec{M}$.

How are J_{ij} for $D = 3$ related to the components M_k of $\vec{M}$?

Express J_{ij} for $D = 3$ in terms of M_i by using ϵ_{ijk}.

Express M_i in terms of J_{ij} by using ϵ_{ijk}.

Answer: For $D = 3$ there are 3 independent charges

$$J_{12} = p_1 x_2 - p_2 x_1\,, \quad J_{23} = p_2 x_3 - p_3 x_2\,, \quad J_{31} = p_3 x_1 - p_1 x_3\,. \tag{0.15}$$

The angular momentum is

$$\vec{M} = \vec{r} \times \vec{p} \quad \Rightarrow \quad M_1 = x_2 p_3 - x_3 p_2\,, \quad M_2 = x_3 p_1 - x_1 p_3\,, \quad M_3 = x_1 p_2 - x_2 p_1\,. \tag{0.16}$$

Thus

$$J_{12} = -M_3\,, \quad J_{23} = -M_1\,, \quad J_{31} = -M_2\,, \tag{0.17}$$

$$J_{ij} = -\epsilon_{ijk} M_k\,, \qquad M_i = -\frac{1}{2}\epsilon_{ijk} J_{jk}\,. \tag{0.18}$$

(e) Use Noether's theorem to find the energy E of the particle.

Express E in terms of $\vec{v}$ and $\vec{r}$. Express E in terms of $\vec{p}$ and $\vec{r}$.

Answer: We have

$$\begin{aligned} E &= \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L = \vec{p}\cdot\vec{v} - L = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + mc^2\sqrt{1-\frac{v^2}{c^2}} \; + \; \frac{\alpha}{r^2}e^{-\beta\, r^2} \\ &= \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \; + \; \frac{\alpha}{r^2}e^{-\beta\, r^2} = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} \; + \; \frac{\alpha}{r^2}e^{-\beta\, r^2}\,. \end{aligned} \tag{0.19}$$

Problem 4. (To do the problem analyse the solution to Q2b of the 2017 final exam).

Consider the following Lagrangian of a system with two physical degrees of freedom

$$L_\lambda = \frac{m}{2}(-v_0^2 + v_1^2 + v_2^2) \; + \; \lambda(-x_0^2 + x_1^2 + x_2^2 + a^2)\,, \quad x_0 > 0\,,$$

where λ is a Lagrange multiplier. From the previous homework we know that

$$-x_0^2 + x_1^2 + x_2^2 + a^2 = 0\,, \quad x_0 > 0\,,$$

defines a surface which is the upper sheet of a hyperboloid of two sheets obtained by revolving the hyperbola $x_0^2 = x_1^2 + a^2$ in the x_0x_1-plane about the x_0-axis. This system however cannot be interpreted as a constrained system of a particle in three-dimensional Euclidean space $\mathbb{R}^3$ because of the minus sign in front of v_0^2. It describes a particle moving in the two-dimensional Lobachevsky (or hyperbolic) plane H^2 embedded in Minkowski space-time $R^{2,1}$.

(a) Consider a set of all 3×3 matrices A which satisfy the condition

$$A_{ik}\eta_{kn}A_{jn} = \eta_{ij}\,, \quad \textit{summation over } k,n = 0,1,2 \quad \Longleftrightarrow \quad A\eta A^T = \eta \quad \Leftrightarrow \quad A^T\eta A = \eta\,,$$

where $\eta = (\eta_{ij})$, $i,j = 0,1,2$ is the following diagonal matrix

$$\eta = \mathrm{diag}(-1,1,1)\,.$$

This set is denoted as $O(2,1)$, and it is a Lorentz group of pseudo-rotations and reflections of the coordinates x_i in Minkowski space-time $R^{2,1}$. Prove that $O(2,1)$ is a group under the standard matrix multiplication.

Answer: Let us recall the definition of a group.

Definition. A group is a nonempty set G on which there is defined a binary operation $(a,b) \mapsto ab$ satisfying the following properties.

- Closure: If a and b belong to G, then ab is also in G.
- Associativity : $a(bc) = (ab)c$ for all $a,b,c \in G$.
- Identity: There is an element $1 \in G$ such that $a1 = 1a = a$ for all a in G.
- Inverse: If $a \in G$, then there is an element $a^{-1} \in G$: $aa^{-1} = a^{-1}a = 1$.

It is then straightforward to check the properties.

(b) Infinitesimal pseudo-rotations from $O(2,1)$ are parametrised by a matrix ϵ_{ij}

$$x_i \to x_i' = x_i + \epsilon_{ij}x_j\,, \qquad A_{ij} = \delta_{ij} + \epsilon_{ij}\,. \tag{0.20}$$

Find relations between ϵ_{ij}, and write them down explicitly for all values of i , j.

Answer: We get

$$\epsilon_{ij}\eta_{jj} + \eta_{ii}\epsilon_{ji} = 0 \quad \Longleftrightarrow \quad \epsilon_{ji} = -\eta_{ii}\eta_{jj}\epsilon_{ij} \quad \textit{no summation over } i \textit{ or } j\ ! \tag{0.21}$$

Thus,

$$\begin{gathered}\epsilon_{ii} = 0\,, \quad i = 0,1,2\,,\\ \epsilon_{12} = -\epsilon_{21}\,, \quad \epsilon_{01} = +\epsilon_{10}\,, \quad \epsilon_{02} = +\epsilon_{20}\,.\end{gathered} \tag{0.22}$$

(c) Prove that L_λ is invariant under the $O(2,1)$ group of pseudo-rotations and reflections of the coordinates x_i:

$$x_i \to A_{ij}x_j\,, \quad \textit{summation over } j\ !\,,$$

where $A \in O(2,1)$ is a 3×3 matrix.

Answer: To prove that L_λ is invariant under the $SO(2,1)$ group of rotations we first notice that it can be written as

$$L_\lambda = \frac{m}{2}v^T\eta v\, +\, \lambda(x^T\eta x + a^2)\,, \quad x_0 > 0\,, \tag{0.23}$$

where x^T is a row: $x^T = (x_0, x_1, x_2)$, and similarly for v. Thus, it is sufficient to prove that $x^T\eta x$ and $v^T\eta v$ are invariant. We have

$$x^T\eta x \to (Ax)^T\eta Ax = x^TA^T\eta Ax = x^T\eta x\,, \tag{0.24}$$

and similarly for $v^T\eta v$.

(d) Specify any continuous symmetries and use Noether's theorem to construct the corresponding conserved quantities.

Answer: The system is invariant under the pseudo-rotations which lead to the conservation of the following quantities

$$\frac{1}{2}J_{ij}\epsilon_{ij} = p_i\epsilon_{ij}x_j = \frac{1}{2}(p_ix_j - \eta_{ii}\eta_{jj}p_jx_i)\epsilon_{ij}\,, \quad p_i = \frac{\partial L_\lambda}{\partial v_i} = m\eta_{ii}v_i \tag{0.25}$$

and therefore

$$J_{ij} = p_ix_j - \eta_{ii}\eta_{jj}p_jx_i = m\eta_{ii}(v_ix_j - v_jx_i)\,. \tag{0.26}$$

Since L_λ has no explicit time dependence the energy is also conserved

$$E = \frac{m}{2}(v_1^2 + v_2^2 - v_0^2)\,, \tag{0.27}$$

where the coordinates and velocities are subject to the constraints

$$x_1^2 + x_2^2 - x_0^2 + a^2 = 0\,, \quad v_1x_1 + v_2x_2 - v_0x_0 = 0\,. \tag{0.28}$$

(e) A convenient choice of generalised coordinates in $\mathbb{R}^{2,1}$ is

$$x_1 = r\cos\phi\sinh\zeta\,, \quad x_2 = r\sin\phi\sinh\zeta\,, \quad x_0 = r\cosh\zeta\,, \tag{0.29}$$

which is an analog of spherical coordinates in $\mathbb{R}^3$.

By using these coordinates solve the constraint.

Derive an expression for the reduced Lagrangian L, and express the conserved quantities in terms of ϕ and ζ, and their time derivatives (velocities).

Answer: The constraint is $r = a$. Thus the reduced Lagrangian is

$$L = \frac{ma^2}{2}(\dot\zeta^2 + \dot\phi^2\sinh^2\zeta)\,. \tag{0.30}$$

The energy is obviously equal to

$$E = \frac{ma^2}{2}(\dot\zeta^2 + \dot\phi^2\sinh^2\zeta)\,. \tag{0.31}$$

To express J_{ij} in terms of ϕ and ζ, we first compute v_i

$$\begin{aligned} v_1 &= -a\dot\phi\sin\phi\sinh\zeta + a\dot\zeta\cos\phi\cosh\zeta\,, \\ v_2 &= a\dot\phi\cos\phi\sinh\zeta + a\dot\zeta\sin\phi\cosh\zeta\,, \\ v_0 &= a\dot\zeta\sinh\zeta\,, \end{aligned} \tag{0.32}$$

Computing J_{ij} one gets

$$\begin{aligned} J_{12} &= m(v_1x_2 - v_2x_1) = -ma^2\dot\phi\sinh^2\zeta\,, \\ J_{01} &= m(v_1x_0 - v_0x_1) = ma^2(\dot\zeta\cos\phi - \dot\phi\sinh\zeta\cosh\zeta\sin\phi)\,, \\ J_{02} &= m(v_1x_0 - v_0x_1) = ma^2(\dot\zeta\sin\phi + \dot\phi\sinh\zeta\cosh\zeta\cos\phi)\,. \end{aligned} \tag{0.33}$$

(f) Check explicitly that the conserved quantities you've found are indeed conserved.

Answer: The eom are

$$\begin{aligned}
&\ddot{\zeta} = \dot{\phi}^2 \sinh\zeta \cosh\zeta \,, \\
&\frac{d}{dt}(\dot{\phi}\sinh^2\zeta) = 0 \,.
\end{aligned} \tag{0.34}$$

Thus,

$$\begin{aligned}
\frac{d}{dt}E &= \frac{ma^2}{2}(2\ddot{\zeta}\dot{\zeta} + \dot{\phi}^2\sinh^4\zeta\frac{d}{dt}\frac{1}{\sinh^2\zeta}) = ma^2\dot{\zeta}(\ddot{\zeta} - \dot{\phi}^2\sinh\zeta\cosh\zeta) = 0 \,, \\
\frac{d}{dt}J_{12} &= \frac{d}{dt}(-ma^2\dot{\phi}\sinh^2\zeta) = 0 \,, \\
\frac{d}{dt}J_{01} &= ma^2\frac{d}{dt}(\dot{\zeta}\cos\phi - \dot{\phi}\sinh\zeta\cosh\zeta\sin\phi) = ma^2(\ddot{\zeta}\cos\phi - \dot{\zeta}\dot{\phi}\sin\phi - \dot{\phi}\sinh^2\zeta\frac{d}{dt}\frac{\cosh\zeta\sin\phi}{\sinh\zeta}) \\
&= ma^2(\ddot{\zeta}\cos\phi - \dot{\zeta}\dot{\phi}\sin\phi - \dot{\phi}\sinh^2\zeta(-\frac{\dot{\zeta}\sin\phi}{\sinh^2\zeta} + \frac{\cosh\zeta\cos\phi\,\dot{\phi}}{\sinh\zeta})) = 0 \,, \\
\frac{d}{dt}J_{02} &= ma^2\frac{d}{dt}(\dot{\zeta}\sin\phi + \dot{\phi}\sinh\zeta\cosh\zeta\cos\phi) = ma^2(\ddot{\zeta}\sin\phi + \dot{\zeta}\dot{\phi}\cos\phi + \dot{\phi}\sinh^2\zeta\frac{d}{dt}\frac{\cosh\zeta\cos\phi}{\sinh\zeta}) \\
&= ma^2(\ddot{\zeta}\sin\phi + \dot{\zeta}\dot{\phi}\cos\phi + \dot{\phi}\sinh^2\zeta(-\frac{\dot{\zeta}\cos\phi}{\sinh^2\zeta} - \frac{\cosh\zeta\sin\phi\,\dot{\phi}}{\sinh\zeta})) = 0 \,.
\end{aligned} \tag{0.35}$$

(g) Express the conserved quantities in terms of ϕ and ζ, and their conjugated momenta p_ϕ and p_ζ.

Answer: We have

$$p_\phi = ma^2\dot{\phi}\sinh^2\zeta \,, \quad p_\zeta = ma^2\dot{\zeta} \,, \tag{0.36}$$

Thus,

$$\begin{aligned}
E &= \frac{p_\zeta^2}{2ma^2} + \frac{p_\phi^2}{2ma^2\sinh^2\zeta} \,, \\
J_{12} &= -p_\phi \,, \\
J_{01} &= p_\zeta\cos\phi - p_\phi\coth\zeta\sin\phi \,, \\
J_{02} &= p_\zeta\sin\phi + p_\phi\coth\zeta\cos\phi \,.
\end{aligned} \tag{0.37}$$