Advanced Calculus
MA1132
Tutorial Exercises 4

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To be completed before and during tutorials of Friday, 22. February

1. (a) Use the chain rule to find $\frac{df}{dt}$ if

$$f(x,y) = \cosh^2(xy), \quad x(t) = \frac{t}{2}, \quad y(t) = e^t.$$

   (b) Use the chain rule to find $\frac{df}{dt}$ if

$$f(x,y,z) = \ln(3x^2 - 2y + 4z^3), \quad x(t) = t^{\frac{1}{2}}, \quad y(t) = t^{\frac{2}{3}}, \quad z(t) = t^{-2}.$$

2. Use appropriate forms of the chain rule to find $\frac{\partial z}{\partial u}$ where

$$z = \sin\frac{x}{2}\cos 2y\,; \quad x = 2u + 3v\,, \quad y = u^3 - 2v^2\,.$$

3. A function $f(x_1,\ldots,x_n)$ is said to be homogeneous of degree k if $f(tx_1,\ldots tx_n) = t^k f(x_1,\ldots,x_n)$ for $t > 0$. Show that it satisfies

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf\,.$$

4. Consider the function

$$z = 3e^{y-\frac{\pi}{4}}\cos x - 2e^{\frac{\pi}{2}-x}\sin y$$

   (a) Find

$$iii)\ \frac{\partial^2 z}{\partial x \partial y}\left(\frac{\pi}{2}, \frac{\pi}{4}\right), \qquad iv)\ \frac{\partial^2 z}{\partial y \partial x}\left(\frac{\pi}{2}, \frac{\pi}{4}\right).$$

   (b) Find the slope of the surface $z = 3e^{y-\frac{\pi}{4}}\cos x - 2e^{\frac{\pi}{2}-x}\sin y$ in the y-direction at the point $(\frac{\pi}{3}, \frac{\pi}{6})$.

   (c) Show that the function $z = 3e^{y-\frac{\pi}{4}}\cos x - 2e^{\frac{\pi}{2}-x}\sin y$ satisfies Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0\,.$$

5. The equations of motion of a system of n particles are given by

$$m_i \ddot{x}_i = -\frac{\partial U(x_1, \dots, x_n)}{\partial x_i}\,, \quad \ddot{x}_i = \frac{d^2 x_i}{dt^2}\,, \quad i = 1, 2, \dots, n\,,$$

where m_i is the mass and x_i is the coordinate of the i-th particle, and $U(x_1, \dots, x_n)$ is the potential energy of the system.

(a) Find the equations of motion of a system of n particles moving in a Coulomb field

$$U(x_1, \dots, x_n) = \frac{\alpha}{r}\,, \quad r = \Big|\sum_{i=1}^{n} x_i \mathbf{e}_i\Big|\,.$$

(b) Find the equations of motion of a system of n coupled harmonic oscillators

$$U(x_1, \dots, x_n) = \sum_{i=1}^{n-1} \frac{\kappa}{2}(x_{i+1} - x_i)^2\,,$$

(c) Find the equations of motion of a system of n particles with pairwise interaction

$$U(x_1, \dots, x_n) = \sum_{i,j=1, i\neq j}^{n} V(x_i - x_j)\,.$$

Here V is an even function of a single variable, and we use the notation

$$\sum_{i,j=1, i\neq j}^{n} a_{ij} \equiv \sum_{j=1}^{n} \sum_{i=1, i\neq j}^{n} a_{ij} = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_{ij}\,.$$

6. The Taylor series is given by

$$f(\vec{x}) = \sum_{k_1, \dots, k_n = 0}^{\infty} \frac{\partial_1^{k_1} \cdots \partial_n^{k_n} f(\vec{x^o})}{k_1! \cdots k_n!} \Delta x_1^{k_1} \cdots \Delta x_n^{k_n}\,, \tag{1}$$

where we denote

$$f(x_1, \dots, x_n) \equiv f(\vec{x})\,, \quad f(x_1^o, \dots, x_n^o) \equiv f(\vec{x^o})\,, \quad x_i - x_i^o \equiv \Delta x_i \tag{2}$$

and $\partial_i^0 f \equiv f$; $\partial_i^k f \equiv \frac{\partial^k f}{\partial x_i^k}$ is the k-th partial derivative of f with respect to x_i.

The Taylor series can be equivalently written as

$$f(\vec{x}) = \sum_{q=0}^{\infty} \frac{1}{q!} \sum_{i_1, \dots, i_q = 1}^{n} \frac{\partial^q f(\vec{x^o})}{\partial x_{i_1} \cdots \partial x_{i_q}} \Delta x_{i_1} \cdots \Delta x_{i_q}\,. \tag{3}$$

(a) Check the equality for functions of three variables by computing the Taylor series expansion up to the third order.

(b) Check the equality by computing the Taylor series expansion up to the third order.

(c) Show that the Taylor series can be equivalently written as in (3) for functions of n variables