Advanced Calculus
MA1132
Exercises 1 Solutions

1. Find an arc length parametrization of the curve
$$\mathbf{r}(t) = \sin(e^t)\mathbf{i} + \cos(e^t)\mathbf{j} + \sqrt{3}e^t\mathbf{k},$$
starting at the point $(\sin(1), \cos(1), \sqrt{3})$ and going in the same direction as the original curve.

solution: Here we are not given our reference point $t_0 = 0$, so we have to calculate it. Looking at the $\mathbf{k}$ component, we see that $\sqrt{3}e^{t_0} = \sqrt{3}$, so $e^{t_0} = 1$ and $t_0 = 0$.

Next, let us calculate the arc length of the curve from $t = 0$ to t.

Using the Chain Rule, we obtain
$$\mathbf{r}'(t) = e^t\cos(e^t)\mathbf{i} - e^t\sin(e^t)\mathbf{j} + \sqrt{3}e^t\mathbf{k}.$$
Hence
$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{e^{2t}\cos^2(e^t) + e^{2t}\sin^2(e^t) + 3e^{2t}} \\ &= \sqrt{e^{2t}(\cos^2(e^t) + \sin^2(e^t)) + 3e^{2t}} \\ &= \sqrt{e^{2t} + 3e^{2t}} \\ &= \sqrt{4e^{2t}} \\ . &= 2e^t.\end{aligned}$$
Thus the required arc length is
$$s(t) = \int_0^t \|\mathbf{r}'(v)\|\,dv = \int_0^t 2e^v\,dv = [2e^v]_0^t = 2e^t - 2e^0 = 2e^t - 2.$$
So we have $s = 2e^t - 2$, and solving this for t we obtain
$$s = 2e^t - 2 \quad\Longrightarrow\quad 2e^t = s + 2 \quad\Longrightarrow\quad e^t = \frac{s}{2} + 1 \quad\Longrightarrow\quad t = \ln\left(\frac{s}{2} + 1\right).$$
Substituting this into $\mathbf{r}(t)$, we obtain
$$\begin{aligned}\mathbf{r}(s) &= \sin\left[\exp\left(\ln\left(\frac{s}{2}+1\right)\right)\right]\mathbf{i} + \cos\left[\exp\left(\ln\left(\frac{s}{2}+1\right)\right)\right]\mathbf{j} + \sqrt{3}\exp\left(\ln\left(\frac{s}{2}+1\right)\right)\mathbf{k} \\ &= \sin\left(\frac{s}{2}+1\right)\mathbf{i} + \cos\left(\frac{s}{2}+1\right)\mathbf{j} + \sqrt{3}\left(\frac{s}{2}+1\right)\mathbf{k}\end{aligned}$$
Now, when $t = 0$, $s = 2 - 2 = 0$ and as t increases, so does s. Thus we obtain the arc length parametrization
$$\mathbf{r}(s) = \sin\left(\frac{s}{2}+1\right)\mathbf{i} + \cos\left(\frac{s}{2}+1\right)\mathbf{j} + \sqrt{3}\left(\frac{s}{2}+1\right)\mathbf{k} \quad s \in [0, \infty).$$

2. For the curve
$$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}, \quad t \in \mathbb{R},$$

(a) Find $\mathbf{T}(0)$, $\mathbf{N}(0)$ and $\mathbf{B}(0)$, using the definitions.

(b) Find $\mathbf{B}(0)$ using the formula in Remark 1.4.6(e).

solution:

(a) To find $\mathbf{T}(t)$, we will use the formula
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$
Note that we have to find $\mathbf{T}(t)$ rather than just $\mathbf{T}(0)$, since we will need it to find $\mathbf{T}'(0)$, in order to find $\mathbf{N}(0)$.
Now
$$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k},$$
so that
$$\|\mathbf{r}'(t)\| = \sqrt{1 + t^2 + t^4}.$$
Hence
$$\mathbf{T}(t) = \frac{1}{\sqrt{1+t^2+t^4}}\mathbf{i} + \frac{t}{\sqrt{1+t^2+t^4}}\mathbf{j} + \frac{t^2}{\sqrt{1+t^2+t^4}}\mathbf{k},$$
so that
$$\mathbf{T}(0) = \mathbf{i}.$$
Next, to find $\mathbf{N}(0)$, we will use the formula
$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|}.$$
Now, using the Quotient Rule and the Chain Rule,
$$\begin{aligned}\mathbf{T}'(t) = &-\frac{t+2t^3}{(1+t^2+t^4)^{\frac{3}{2}}}\mathbf{i} + \frac{(1+t^2+t^4)^{\frac{1}{2}} - t(-t-2t^3)(1+t^2+t^4)^{-\frac{3}{2}}}{1+t^2+t^4}\mathbf{j} \\ &+ \frac{2t(1+t^2+t^4)^{\frac{1}{2}} - t^2(-t-2t^3)(1+t^2+t^4)^{-\frac{3}{2}}}{1+t^2+t^4}\mathbf{k},\end{aligned}$$
so that
$$\mathbf{T}'(0) = \mathbf{j}.$$
Since this is a unit vector, we also have $\mathbf{N}(0) = \mathbf{j}$.
Finally $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \mathbf{i} \times \mathbf{j} = \mathbf{k}$.

(b) *solution:* We will calculate $\mathbf{B}(0)$ using the formula
$$\mathbf{B}(0) = \frac{\mathbf{r}'(0) \times \mathbf{r}''(0)}{\|\mathbf{r}'(0) \times \mathbf{r}''(0)\|},$$
We have already shown that $\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$. Thus $\mathbf{r}''(t) = \mathbf{j} + 2t\mathbf{k}$.
Hence $\mathbf{r}'(0) = \mathbf{i}$ and $\mathbf{r}''(0) = \mathbf{j}$.
Thus $\mathbf{r}'(0) \times \mathbf{r}''(0) = \mathbf{i} \times \mathbf{j} = \mathbf{k}$.
Since this is a unit vector, we also have $\mathbf{B}(0) = \mathbf{k}$.

3. For the curve
$$\mathbf{r}(t) = e^t \sin(t)\mathbf{i} + e^t \cos(t)\mathbf{j} + 3\mathbf{k}, \quad t \in \mathbb{R},$$

(a) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$, using the definitions.

(b) Find $\mathbf{B}(t)$ using the formula in Remark 1.4.6(e).

solution:

(a) To find $\mathbf{T}(t)$, we will use the formula
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$
Now, using the Product Rule,
$$\begin{aligned}\mathbf{r}'(t) &= (e^t \sin(t) + e^t \cos(t))\mathbf{i} + (e^t \cos(t) - e^t \sin(t))\mathbf{j} \\ &= e^t(\sin(t) + \cos(t))\mathbf{i} + e^t(\cos(t) - \sin(t))\mathbf{j},\end{aligned}$$
so that
$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{e^{2t}(\sin(t) + \cos(t))^2 + e^{2t}(\cos(t) - \sin(t))^2} \\ &= \sqrt{e^{2t}(\sin^2(t) + 2\sin(t)\cos(t) + \cos^2(t)) + e^{2t}(\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t))} \\ &= \sqrt{e^{2t}(2\sin^2(t) + 2\cos^2(t))} \\ &= \sqrt{2e^{2t}} \\ &= \sqrt{2}e^t.\end{aligned}$$
Hence
$$\mathbf{T}(t) = \frac{\sin(t) + \cos(t)}{\sqrt{2}}\mathbf{i} + \frac{\cos(t) - \sin(t)}{\sqrt{2}}\mathbf{j}.$$
Next, to find $\mathbf{N}(t)$, we will use the formula
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$
We have
$$\mathbf{T}'(t) = \frac{\cos(t) - \sin(t)}{\sqrt{2}}\mathbf{i} + \frac{-\sin(t) - \cos(t)}{\sqrt{2}}\mathbf{j},$$
so that
$$\begin{aligned}\|\mathbf{T}'(t)\| &= \sqrt{\frac{(\cos(t) - \sin(t))^2}{2} + \frac{(-\sin(t) - \cos(t))^2}{2}} \\ &= \sqrt{\frac{\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \sin^2(t) + 2\sin(t)\cos(t) + \cos^2(t)}{2}} \\ &= \sqrt{\frac{2\sin^2(t) + 2\cos^2(t)}{2}} \\ &= \sqrt{1} \\ &= 1.\end{aligned}$$

Hence

$$\mathbf{N}(t) = \frac{\cos(t) - \sin(t)}{\sqrt{2}}\mathbf{i} + \frac{-\sin(t) - \cos(t)}{\sqrt{2}}\mathbf{j}$$

Finally we will calculate $\mathbf{B}(t)$ using the formula $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.
We have

$$\begin{aligned}
\mathbf{B}(t) = &\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sin(t)+\cos(t)}{\sqrt{2}} & \frac{\cos(t)-\sin(t)}{\sqrt{2}} & 0 \\ \frac{\cos(t)-\sin(t)}{\sqrt{2}} & \frac{-\sin(t)-\cos(t)}{\sqrt{2}} & 0 \end{vmatrix} \\
=&\mathbf{i}\begin{vmatrix} \frac{\cos(t)-\sin(t)}{\sqrt{2}} & 0 \\ \frac{-\sin(t)-\cos(t)}{\sqrt{2}} & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\sin(t)+\cos(t)}{\sqrt{2}} & 0 \\ \frac{\cos(t)-\sin(t)}{\sqrt{2}} & 0 \end{vmatrix} \\
&+ \mathbf{k}\begin{vmatrix} \frac{\sin(t)+\cos(t)}{\sqrt{2}} & \frac{\cos(t)-\sin(t)}{\sqrt{2}} \\ \frac{\cos(t)-\sin(t)}{\sqrt{2}} & \frac{-\sin(t)-\cos(t)}{\sqrt{2}} \end{vmatrix} \\
=&\left[\left(\frac{\sin(t)+\cos(t)}{\sqrt{2}}\right)\left(\frac{-\sin(t)-\cos(t)}{\sqrt{2}}\right) - \left(\frac{\cos(t)-\sin(t)}{\sqrt{2}}\right)\left(\frac{\cos(t)-\sin(t)}{\sqrt{2}}\right)\right]\mathbf{k} \\
=&\frac{-\sin^2(t) - 2\sin(t)\cos(t) - \cos^2(t) - (\cos^2(t) - 2\sin(t)\cos(t) + \sin^2(t))}{2}\mathbf{k} \\
=&\frac{-2\sin^2(t) - 2\cos^2(t))}{2}\mathbf{k} \\
=&-\mathbf{k}.
\end{aligned}$$

(b) *solution:* We will calculate $\mathbf{B}(t)$ using the formula

$$\mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|},$$

We have already shown that

$$\mathbf{r}'(t) = e^t(\sin(t) + \cos(t))\mathbf{i} + e^t(\cos(t) - \sin(t))\mathbf{j}.$$

Thus, using the Product Rule,

$$\begin{aligned}
\mathbf{r}''(t) =&[e^t(\sin(t) + \cos(t)) + e^t(\cos(t) - \sin(t))]\mathbf{i} \\
&+ [e^t(\cos(t) - \sin(t)) + e^t(-\sin(t) - \cos(t))]\mathbf{j} \\
=&2e^t\cos(t)\mathbf{i} - 2e^t\sin(t)\mathbf{j}
\end{aligned}$$

Hence

$$\mathbf{r}'(t)\times\mathbf{r}''(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ e^t(\sin(t)+\cos(t)) & e^t(\cos(t)-\sin(t)) & 0\\ 2e^t\cos(t) & -2e^t\sin(t) & 0\end{vmatrix}$$
$$\begin{aligned}
=&\mathbf{i}\begin{vmatrix} e^t(\cos(t)-\sin(t)) & 0\\ -2e^t\sin(t) & 0\end{vmatrix}-\mathbf{j}\begin{vmatrix} e^t(\sin(t)+\cos(t)) & 0\\ 2e^t\cos(t) & 0\end{vmatrix}\\
&+\mathbf{k}\begin{vmatrix} e^t(\sin(t)+\cos(t)) & e^t(\cos(t)-\sin(t))\\ 2e^t\cos(t) & -2e^t\sin(t)\end{vmatrix}\\
=&\left[\left(e^t(\sin(t)+\cos(t))\right)\left(-2e^t\sin(t)\right)-\left(e^t(\cos(t)-\sin(t))\right)\left(2e^t\cos(t)\right)\right]\mathbf{k}\\
=&\left[2e^{2t}(-\sin^2(t)-\cos(t)\sin(t)-\cos^2(t)+\sin(t)\cos(t))\right]\mathbf{k}\\
=&\left[2e^{2t}(-\sin^2(t)-\cos^2(t))\right]\mathbf{k}\\
=&-2e^{2t}\mathbf{k}.
\end{aligned}$$

Then $\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|=2e^{2t}$, and hence

$$\mathbf{B}(t)=\frac{\mathbf{r}'(t)\times\mathbf{r}''(t)}{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}=\frac{-2e^{2t}\mathbf{k}}{2e^{2t}}=-\mathbf{k}.$$

4. Consider the vector function (with values in $\mathbb{R}^3$)

$$\mathbf{r}(t)=\ln(3-\sqrt{t})\,\mathbf{i}+(1+\sqrt{t})\,\mathbf{j}+\frac{(3-\sqrt{t})^2}{4}\,\mathbf{k} \tag{1}$$

(a) Find the arc length of the graph of $\mathbf{r}(t)$ if $1\le t\le 4$.

Solution: The arc length of the graph of $\mathbf{r}(t)$ is given by the definite integral

$$\begin{aligned}
L=\int_1^4 |\frac{d\mathbf{r}}{dt}|\,dt \;&=\; \int_1^4 \frac{1}{2\sqrt{t}}\left(\frac{1}{3-\sqrt{t}}+\frac{3-\sqrt{t}}{2}\right)dt=\int_1^2\left(\frac{1}{3-v}+\frac{3-v}{2}\right)dv\\
&=\;\left(-\ln(3-v)-\frac{(3-v)^2}{4}\right)\Big|_1^2=\frac{3}{4}+\ln 2\,,
\end{aligned}$$

where we have made the substitution $v=\sqrt{t}$.

(b) Find a negative change of parameter from t to s where s is an arc length parameter of the curve having $\mathbf{r}(4)$ as its reference point.

Solution: The arc length parameter s can be found as follows

$$\begin{aligned}
s=-\int_4^t |\frac{d\mathbf{r}}{du}|\,du= \;&=\; \int_t^4 \frac{1}{2\sqrt{u}}\left(\frac{1}{3-\sqrt{u}}+\frac{3-\sqrt{u}}{2}\right)du=\int_{\sqrt{t}}^2\left(\frac{1}{3-v}+\frac{3-v}{2}\right)dv\\
&=\;\left(-\ln(3-v)-\frac{(3-v)^2}{4}\right)\Big|_{\sqrt{t}}^2=-\frac{1}{4}+\frac{(3-\sqrt{t})^2}{4}+\ln(3-\sqrt{t})\,,
\end{aligned}$$

where we have made the substitution $v=\sqrt{u}$.

5. Consider parabolic coordinates (μ, ν)

$$x = \mu\, \nu\,, \quad y = \frac{1}{2}(\mu^2 - \nu^2)\,. \tag{2}$$

(a) Recall that if a parabola is described by the equation $y = a(x-h)^2 + k$, then its vertex is at (h, k) and its focus is at $(h, k + 1/(4a))$.
Show that if you hold ν constant, the resulting curves form confocal parabolae that open upwards (i.e., towards $+y$), while holding μ constant resultsin curves which are confocal parabolae that open downwards (i.e., towards $-y$). Where are the foci of all these parabolae located?

Solution: It follows from (Derive!)

$$y = \frac{1}{2}(\frac{x^2}{\nu^2} - \nu^2)\,, \quad y = \frac{1}{2}(\mu^2 - \frac{x^2}{\mu^2})\,. \tag{3}$$

The foci are located at $(0, 0)$.

(b) Show that in parabolic coordinates a curve given by the parametric equations $\mu = \mu(t)$, $\nu = \nu(t)$ for $a \le t \le b$ has arc length

$$L = \int_a^b \sqrt{(\mu^2 + \nu^2)\left(\left(\frac{d\mu}{dt}\right)^2 + \left(\frac{d\nu}{dt}\right)^2\right)}\, dt\,. \tag{4}$$

Solution: Straightforward computation

6. Consider the vector function

$$\mathbf{r}(t) = e^{-t}\,\mathbf{i} + e^{-t}\cos t\,\mathbf{j} - e^{-t}\sin t\,\mathbf{k}\,. \tag{5}$$

(a) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$, at $t = 0$.

Solution: The unit tangent vector is

$$\mathbf{T}(t) = \frac{\frac{d\mathbf{r}}{dt}}{|\frac{d\mathbf{r}}{dt}|} = \frac{-e^{-t}\,\mathbf{i} - e^{-t}(\cos t + \sin t)\,\mathbf{j} + e^{-t}(\sin t - \cos t)\,\mathbf{k}}{e^{-t}\sqrt{1 + (\cos t + \sin t)^2 + (\sin t - \cos t)^2}} = -\frac{\mathbf{i} + (\cos t + \sin t)\,\mathbf{j} + (\cos t - \sin t)\,\mathbf{k}}{\sqrt{3}} \tag{6}$$

The unit tangent vector at $t = 0$ is

$$\mathbf{T} \equiv \mathbf{T}(0) = -\frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}\,. \tag{7}$$

The unit normal vector is

$$\mathbf{N}(t) = \frac{\frac{d\mathbf{T}}{dt}}{|\frac{d\mathbf{T}}{dt}|} = -\frac{(\cos t - \sin t)\,\mathbf{j} + (-\sin t - \cos t)\,\mathbf{k}}{\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}} = \frac{(-\cos t + \sin t)\,\mathbf{j} + (\sin t + \cos t)\,\mathbf{k}}{\sqrt{2}} \tag{8}$$

The unit normal vector at $t = 0$ is

$$\mathbf{N} \equiv \mathbf{N}(0) = \frac{-\mathbf{j} + \mathbf{k}}{\sqrt{2}}\,. \tag{9}$$

The unit binormal vector at $t = 0$ is

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{-2\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{6}}\,. \tag{10}$$

(b) Find equations for the **TN**-plane at $t = 0$.

Solution: The **TN**-plane is normal to **B**. Thus, its equation at $t = 0$ is

$$(\mathbf{r} - \mathbf{r}(0)) \cdot \mathbf{B} = 0\,, \quad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,, \quad \mathbf{r}(0) = \mathbf{i} + \mathbf{j}\,, \tag{11}$$

or, explicitly

$$2(x-1) - (y-1) - z = 0\,. \tag{12}$$

(c) Find equations for the **NB**-plane at $t = 0$.

Solution: The **NB**-plane is normal to **T**. Thus, its equation at $t = 0$ is

$$(\mathbf{r} - \mathbf{r}(0)) \cdot \mathbf{T} = 0\,, \quad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,, \quad \mathbf{r}(0) = \mathbf{i} + \mathbf{j}\,, \tag{13}$$

or, explicitly

$$(x-1) + (y-1) + z = 0\,. \tag{14}$$

(d) Find equations for the **TB**-plane at $t = 0$.

Solution: The **TB**-plane is normal to **N**. Thus, its equation at $t = 0$ is

$$(\mathbf{r} - \mathbf{r}(0)) \cdot \mathbf{N} = 0\,, \quad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,, \quad \mathbf{r}(0) = \mathbf{i} + \mathbf{j}\,, \tag{15}$$

or, explicitly

$$-(y-1) + z = 0\,. \tag{16}$$