Standard deviation of the mean for finite samples

- When *measuring* the mean $\langle x \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i$ of a random variable x governed by $P(x)$, we get an *estimate* of the real mean μ due to the *finite sample size* N.

- $\langle x \rangle \to \mu$ in the limit $N \to \infty$

- How good of an estimate is $\langle x \rangle$ for finite N ?

- Let's calculate this using the error propagation formula we derived in the last lecture.

Standard deviation of the mean

Let μ and σ_x be the true mean and standard deviation of P(x).

We estimate μ by taking N samples and compute the mean

$$\langle x \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i$$

What is the uncertainty in $\langle x \rangle$?

Let's repeat taking N measurements from $P(x)$ many times and compute $\langle x \rangle$ for each trial.

Standard deviation of the mean

- For every set of N measurements we will get a slightly different mean $\langle x \rangle_i$. The distribution of the means $P(\langle x \rangle)$ will be centred around the true mean μ.

- The standard deviation of $P(\langle x \rangle)$ will be **the standard error of the mean $\boldsymbol{\sigma_{\langle x \rangle}}$**.

- We calculate $\sigma_{\langle x \rangle}$ using the error propagation formula for $f(x_1, x_2, .., x_N)$:

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + .. + \left(\frac{\partial f}{\partial x_N}\right)^2 \sigma_{x_N}^2$$

Standard deviation of the mean

$$\langle x \rangle = \frac{1}{N}(x_1 + x_2 + .. + x_N)$$

- Each x_i that appears in the sum is governed by $P(x)$ with a mean μ and standard deviation σ_x.
- $\langle x \rangle$ can be considered a function of N independent variables.
- Therefore,

$$\sigma_{\langle x \rangle} = \sqrt{\left(\frac{\partial \langle x \rangle}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial \langle x \rangle}{\partial x_2}\right)^2 \sigma_{x_2}^2 + .. + \left(\frac{\partial \langle x \rangle}{\partial x_N}\right)^2 \sigma_{x_N}^2}$$

Standard deviation of the mean

- As the x_i are all governed by the same distribution $P(x)$ with standard deviation σ_x :

$$\sigma_{x_1} = .. = \sigma_{x_N} = \sigma_x$$

- The partial derivatives in the sum are also the same

$$\frac{\partial \langle x \rangle}{\partial x_1} = .. = \frac{\partial \langle x \rangle}{\partial x_N} = \frac{1}{N}$$

Standard deviation of the mean

Finally,

$$\sigma_{\langle x\rangle} = \sqrt{\left(\frac{\partial\langle x\rangle}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial\langle x\rangle}{\partial x_2}\right)^2 \sigma_{x_2}^2 + .. + \left(\frac{\partial\langle x\rangle}{\partial x_N}\right)^2 \sigma_{x_N}^2}$$

$$\sigma_{\langle x\rangle} = \sqrt{\left(\frac{1}{N}\right)^2 \sigma_x^2 \ + \left(\frac{1}{N}\right)^2 \sigma_x^2 \ + .. + \left(\frac{1}{N}\right)^2 \sigma_x^2} = \sqrt{N \cdot \left(\frac{1}{N}\right)^2 \sigma_x^2}$$

$$\sigma_{\langle x\rangle} = \frac{\sigma_x}{\sqrt{N}}$$

This is the **error in the mean** and reflects the uncertainty in the estimate of μ given by $\langle x\rangle$ measured from a finite sample size N

Remarks on the standard deviation of the mean

- $\sigma_{\langle x \rangle} = \frac{\sigma_x}{\sqrt{N}}$ reflects the uncertainty in the estimate of the mean.

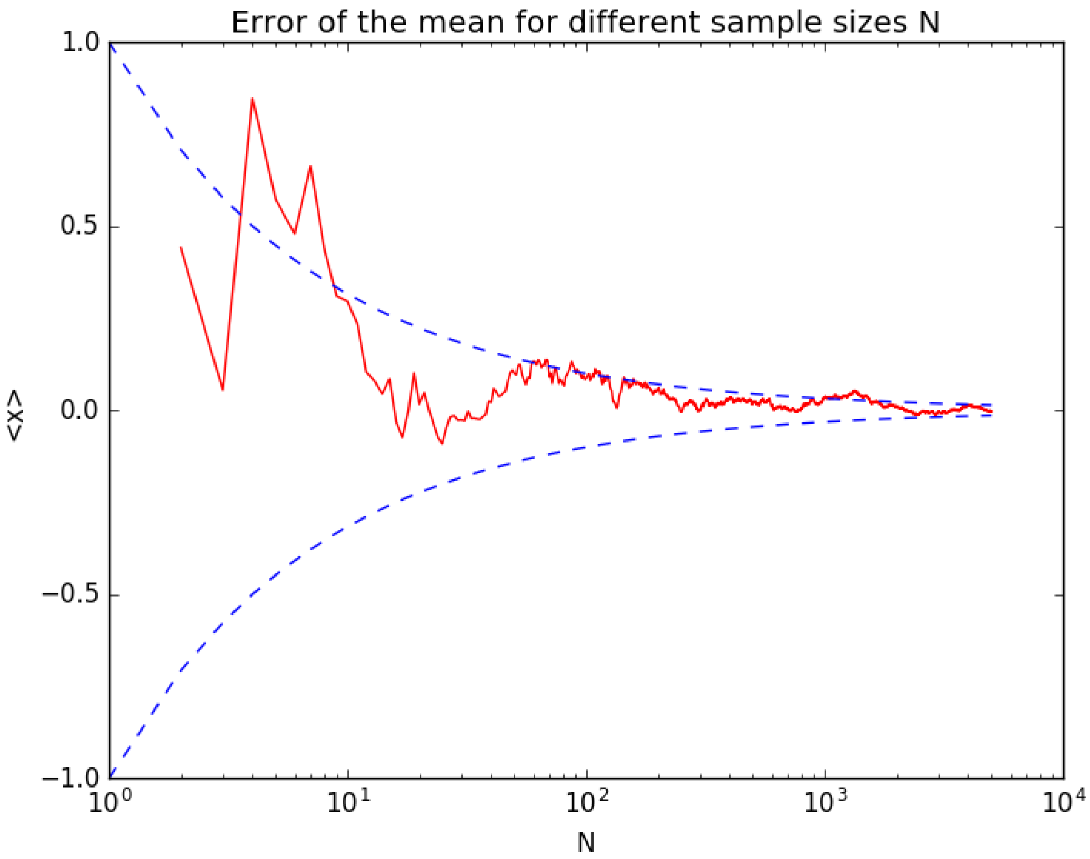
- Error of the mean goes to zero as $N \to \infty$ as it should since $\langle x \rangle \to \mu$ in that limit.

- Don't confuse error of the mean $\sigma_{\langle x \rangle}$ with the standard deviation σ_x. The latter reflects the uncertainty in the measurements (i.e. error bars) which will not decrease as N increases, although larger N will give a better estimate of σ_x.

- Derivation does not assume any particular distribution $P(x)$.

Standard deviation of the mean

Plot shows $\langle x \rangle$ calculated from N x_i's drawn from a Gaussian with $\mu = 0$ and $\sigma_x = 1$. Blue dotted line is $\pm\sigma_x/\sqrt{N}$.

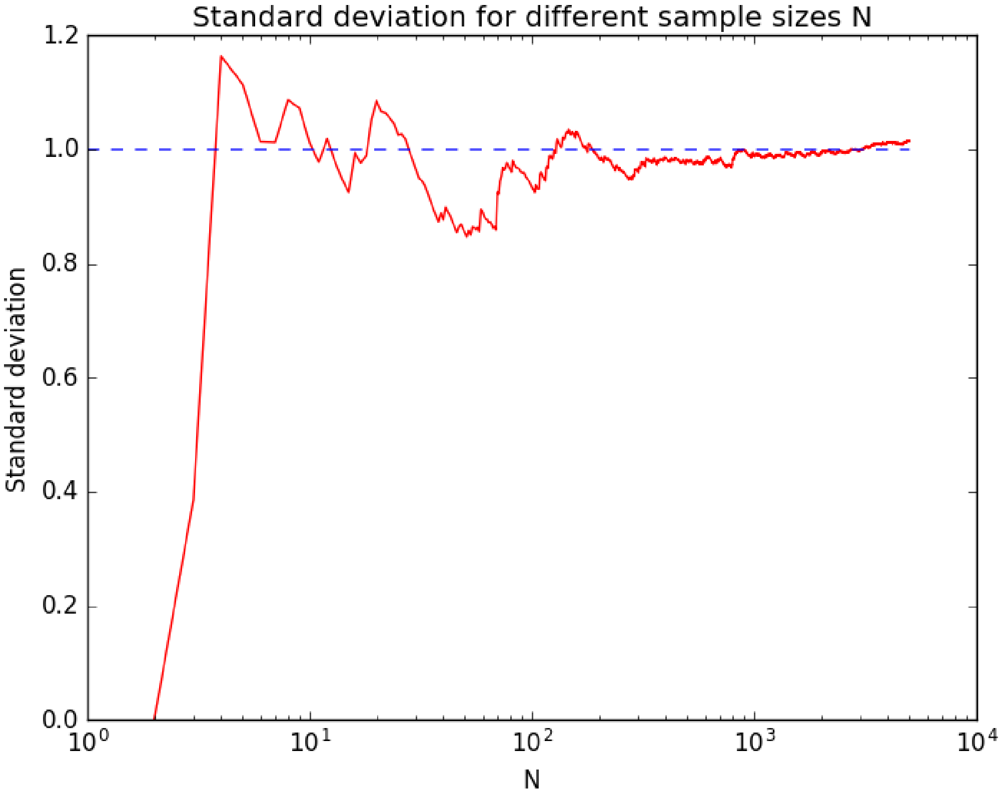


As N increases $\langle x \rangle$ gets closer to the true mean $\mu = 0$. $\langle x \rangle$ bounded by the error of the mean

Error of the standard deviation

Similar to the error of the mean, there is an error of the standard deviation ($= \sigma_x/\sqrt{2(N-1)}$).

$\sigma_{x,N}$ measured over a finite sample size N approaches true standard deviation (blue dotted line) in the limit $N \to \infty$



The Poisson process

Many counting experiments are governed by a Poisson process:

- the number of car accidents at a site or in an area
- the requests for individual documents on a web server
- Number of earthquakes in a certain area
- the number of deaths from horse kicks in the Prussian army.
- Incidence rate of rare diseases.
- Nuclear decay

What do all these processes have in common?

Conditions for a Poisson process

- Counting measurements.
- Events have low probability of occurring.
- Non-overlapping events.
- Constant long-time average rate.

Can be thought of as a binomial process in the limit of small success probability p and large N.

Recall binomial distribution: $B_{N,p}(x) = \binom{N}{x} p^x q^{N-x}$

With mean $\mu = Np$, and $\sigma = \sqrt{Np(1-p)}$.
Note $q = 1 - p$.

Poisson distribution - derivation

- Rewrite binomial distribution:

$$B_{N,p}(x) = \binom{N}{x} p^x (1-p)^{N-x}$$
$$= \frac{1}{x!} \frac{N!}{(N-x)!} p^x (1-p)^{-x} (1-p)^N$$

Now take the limit of low success probability $p \ll 1$. This implies that the success probability $B_{N,p}(x)$ is close to zero for large x, the number of successful events. Therefore, $x \ll N$ in the region of interest which is the limit of low success rate.

x terms

$$\frac{N!}{(N-x)!} = N \cdot (N-1) \cdot .. \cdot (N-x+1) \approx N^x$$

Poisson distribution - derivation

So the binomial distribution simplifies to

$$B_{N,p}(x) = \frac{1}{x!} N^x p^x (1-p)^{-x} (1-p)^N$$

$N^x p^x = (Np)^x = \mu^x$. Binomial expansion of fourth term yields

$(1-p)^{-x} \approx 1 + px + .. \approx 1$ since $p \ll 1$

Last term can be recast using $\mu = Np$:

$$(1-p)^N = \left[(1-p)^{\frac{1}{p}}\right]^{\mu}$$

Since $\lim_{p \to 0} (1+p)^{1/p} = \frac{1}{e}$, we finally obtain

$$P(x) = \frac{1}{x!} \mu^x e^{-\mu}$$

The Poisson distribution

$$P_\mu(x) = e^{-\mu}\,\frac{\mu^x}{x!}$$

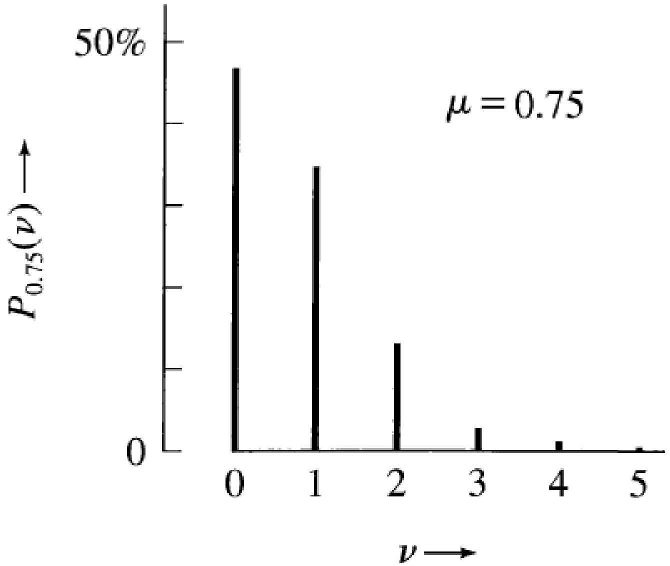
- Poisson distribution is the limit of a binomial distribution for small success probability p and large N.
- It is a **discrete** distribution.
- Poisson distribution only has *one* parameter – the mean μ.
- Standard deviation directly related to mean: $\sigma_x = \sqrt{\mu}$
- Similar to the binomial distribution, the Poisson distribution can be approximated by a Gaussian for large N. For the Poisson distribution this means large μ. (Recall binomial mean: $\mu = Np$)

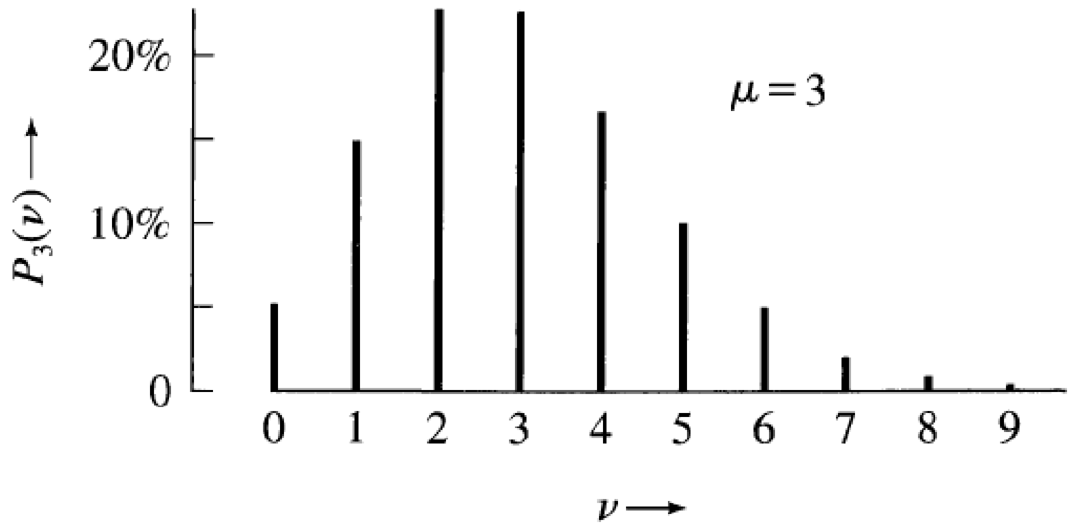
Gaussian: $\mathrm{G}_{\mu,\sigma}(\mathrm{x}) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Binomial: $B_{N,p}(x) = \binom{N}{x}p^x q^{N-x} \approx \frac{1}{\sqrt{2\pi Npq}}e^{-\frac{(x-Np)^2}{2Npq}}$ for large N

Poisson: $P_\mu(x) = e^{-\mu}\,\frac{\mu^x}{x!} \approx \frac{1}{\sqrt{2\pi\mu}}e^{-\frac{(x-\mu)^2}{2\mu}}$ for large μ (good even for $\mu \gtrsim 10$)

Poisson distributions for different μ's

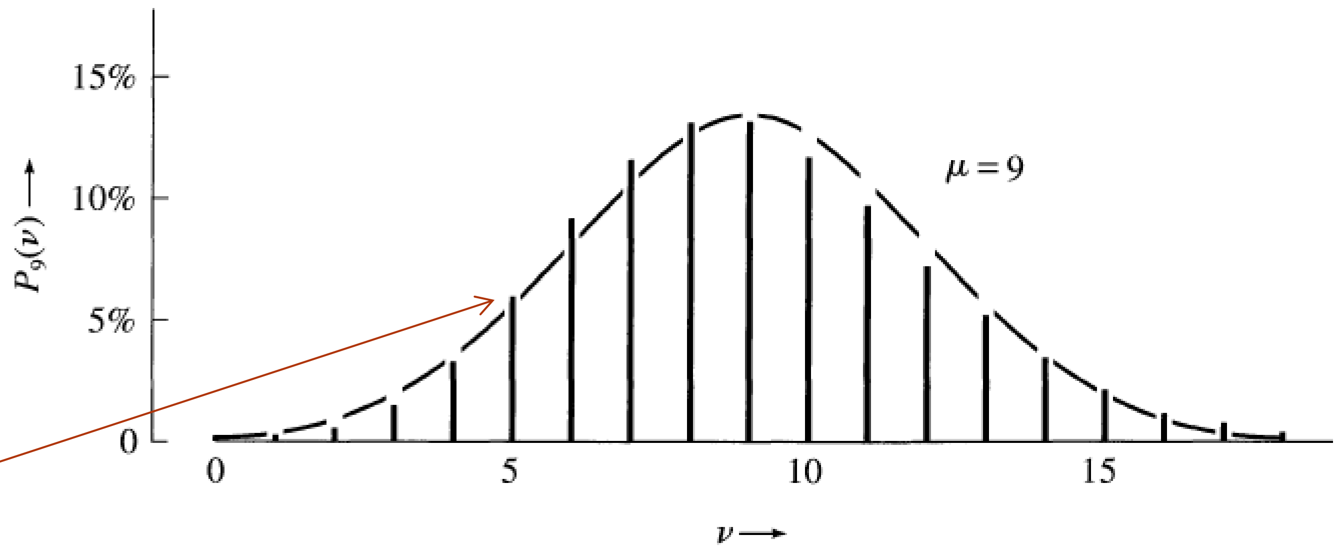




Asymmetric at low μ.

Becomes more bell shaped at larger μ

Gaussian approximation not bad even for $\mu = 9$



Poisson distribution - Normalisation

Check normalisation condition:

$$\sum_{x=0}^{\infty} P_\mu(x) = \sum_{x=0}^{\infty} e^{-\mu}\,\frac{\mu^x}{x!}$$

$$= e^{-\mu}\sum_{i=0}^{\infty}\frac{\mu^x}{x!} = e^{-\mu}\left(\frac{\mu^0}{0!}+\frac{\mu^1}{1!}+\frac{\mu^2}{2!}+..\right)$$

$$= e^{-\mu}\left(\frac{\mu^0}{0!}+\frac{\mu^1}{1!}+\frac{\mu^2}{2!}+..\right) = e^{-\mu}\left(1+\mu+\frac{\mu^2}{2!}+..\right)$$

Note $x! = 0$. The term in the brackets is just a Taylor expansion of the exponential e^μ, so

$$\sum_{i=0}^{\infty} P_\mu(x) = e^{-\mu}e^{\mu} = 1$$

The mean of the Poisson distribution

Start from

$$\sum_{x=0}^{\infty} P_\mu(x) = \sum_{x=0}^{\infty} e^{-\mu}\,\frac{\mu^x}{x!} = 1$$

Differentiate w.r.t μ (use product rule):

$$\sum_{x=0}^{\infty}\left(-e^{-\mu}\,\frac{\mu^x}{x!} + x e^{-\mu}\,\frac{\mu^{x-1}}{x!}\right) = 0$$

$$-\sum_{x=0}^{\infty} e^{-\mu}\,\frac{\mu^x}{x!} + \sum_{x=0}^{\infty} x e^{-\mu}\,\frac{\mu^{x-1}}{x!} = 0$$

$$-1 + \sum_{x=0}^{\infty} x e^{-\mu}\,\frac{\mu^{x-1}}{x!} = 0$$

In the last step we used the normalisation condition.

The mean of the Poisson distribution

$$\sum_{x=0}^{\infty} x e^{-\mu} \frac{\mu^{x-1}}{x!} = 1$$

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Multiplying both sides with μ, we get

$$\sum_{x=0}^{\infty} x e^{-\mu} \frac{\mu^{x}}{x!} = \sum_{x=0}^{\infty} x P_{\mu}(x) = \langle x \rangle = \mu$$

So the mean of the Poisson distribution is just

$$\boxed{\langle x \rangle = \mu}$$

The standard deviation of the Poisson distribution

Use $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$. Already derived $\langle x \rangle = \mu$.

To calculate second moment $\langle x^2 \rangle$, start with

$$\langle x \rangle = \sum_{x=0}^{\infty} x P_\mu(x) = \sum_{x=0}^{\infty} x e^{-\mu} \frac{\mu^x}{x!} = \mu$$

Differentiate w.r.t μ (product rule):

$$\sum_{x=0}^{\infty} x e^{-\mu} \frac{\mu^x}{x!} = \mu$$

$$\sum_{x=0}^{\infty} x^2 e^{-\mu} \frac{\mu^{x-1}}{x!} - \sum_{x=0}^{\infty} x e^{-\mu} \frac{\mu^x}{x!} = 1$$

Multiply both sides with μ

The standard deviation of the Poisson distribution

$$\sum_{x=0}^{\infty} x^2 e^{-\mu} \frac{\mu^x}{x!} - \mu \sum_{x=0}^{\infty} x e^{-\mu} \frac{\mu^x}{x!} = \mu$$

$$\sum_{x=0}^{\infty} x^2 P_\mu(x) - \mu \sum_{x=0}^{\infty} x P_\mu(x) = \mu$$

$$\langle x^2 \rangle - \mu \langle x \rangle = \mu$$

$$\langle x^2 \rangle = \mu^2 + \mu$$

Therefore,

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\mu^2 + \mu - \mu^2} = \sqrt{\mu}$$

Note: Binomial $\sigma = \sqrt{Np(1-p)}$. For small p this reduces to $\sigma = \sqrt{Np} = \sqrt{\mu}$

$$\boxed{\sigma_x = \sqrt{\mu}}$$

Example: Nuclear decay

- The standard example in Physics for a Poisson process is nuclear decay. A radioactive sample may have 10^{20} nuclei (large N), each having a decay probability of down to $\sim 10^{-20}$ in one second (small p)

- Number of atoms dN decaying in time dt proportional to total number of atoms N, where the proportionality constant is the decay constant λ (units 1/s), which tells us the probability of an atom to decay per unit time.

$$\frac{dN}{dt} = -\lambda \cdot N(t)$$

Example: Nuclear decay

- Solution to this differential equation is
$$N(t) = N_0 e^{-\lambda t}$$

- At $t = \tau$, half of the atoms have decayed. τ is the half-life:
$$N(t = \tau) = \frac{1}{2} N_0 = N_0 e^{-\lambda \tau}$$
$$\tau = \frac{\ln 2}{\lambda}$$

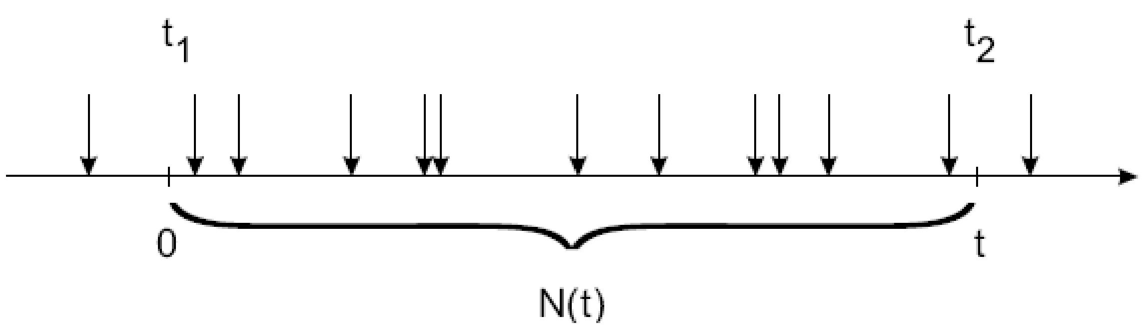
- E.g. Cs^{137} has an half life of 30 years, which corresponds to $\lambda = 7.3 \cdot 10^{-10}\ s^{-1}$. This is the probability for a Caesium atom to decay in 1 second.

- Since a typical sample contains many atoms (1 gram has $\sim 10^{21}$ atoms), get many decays per second.

Example: Nuclear decay

- For radioactive samples with long half lifes, $\frac{dN}{dt} = -\lambda N_0 e^{-\lambda t} \approx constant$ over experimental time scales ($\lambda \ll t \rightarrow e^{-\lambda t} \approx 1$).

- In this case, nuclear decay is a Poisson process with a constant rate of events λN_0, even though decays occur very irregularly over any time interval.



Example: Nuclear decay

- Decays can be measured with a Geiger counter very precisely (assume no uncertainty in recording the number of counts).

- Let's say for a particular radioactive source we measure 1249 counts in 7 minutes. What is the statistical uncertainty in the number of counts?

$$N = 1249 \pm \sqrt{1249} = 1249 \pm 35$$

- What is the rate of decay (in 1/min) including the uncertainty?

$$R_{total} = \frac{1249 \pm 35}{7min} = 178 \pm 5$$

Example: Nuclear decay

- Since there are also counts from the natural background radiation, need to measure counts without the source and subtract it from the previously measured rate R_{total}.

- Without the source we measure 50 counts in 3 minutes. What is the rate and its uncertainty?

$$R_{bkgrnd} = \frac{50 \pm \sqrt{50}}{3min} = 17 \pm 2$$

- Therefore, the rate of the source is

$$R_{source} = R_{total} - R_{bkgrnd} = 161 \pm \sqrt{5^2 + 2^2} = 161 \pm 5$$

Remarks

- Important: In most applications of the Poisson distribution one is interested in the rate of events (number of events per unit time).

- **The square root error $\sqrt{N}$ applies to the raw count of the events , not the rate!**
- Previous example: error in the rate is $\frac{\sqrt{1249}}{7min}$ not $\sqrt{\frac{1249}{7min}}$

Gaussian approximation

- Consider the Poisson distribution with $\mu = 64$. The probability of $x = 72$ is

$$P_{64}(72) = e^{-64}\frac{64^{72}}{72!} = 2.9\%$$

- This is tedious to calculate and can be problematic on a pocket calculator : $64^{72} \approx 10^{130}$

- Instead use Gaussian approximation

$$P_{64}(72) \approx \int_{71.5}^{72.5} G_{64,\sqrt{64}}(72)dx = 3.0\%$$

$$\approx G_{64,\sqrt{64}}(72)dx = G_{64,\sqrt{64}}(72)\cdot 1$$

Note: $dx = 72.5 - 71.5 = 1$

Gaussian approximation

- Calculating $Prob_{Poisson}(x \geq 72) = P_{64}(72) + P_{64}(73) + ..$ is even more tedious.

- Use Gaussian approximation $Prob_{Poisson}(x \geq 72) \approx Prob_{Gaussian}(x \geq 71.5)$. ***Since x is continuous variable in Gaussian distribution use lowest number that rounds up to 72 (so-called continuity correction). Likewise, for*** $Prob_{Poisson}(x \leq 72) \approx Prob_{Gaussian}(x \leq 72.5)$

- 71.5 is 7.5 above the mean ($\mu = 64$), which is $\frac{7.5}{\sqrt{64}} = 0.94$ standard deviations above the mean.

- Use error function to calculate probability $Prob_{Gaussian}(x \geq 71.5, \mu = 64, \sigma = 8) = \frac{[1-\mathrm{erf}(\mathrm{t}=0.94)]}{2} =$ 17.4%

Mastering physics issues

- Rounding tolerance for numerical answers is 2% by default.

- Default precision: <u>Always state the result with **3 significant digits**</u>, e.g. 15.15789=15.2 .

- If units are required, a separate answer box for it will be there. Single box means numerical answer only. **In my questions I only ask you for the numerical answer (in units that are stated in the question)**