Quantum Physics PY1T20/PYU11P20

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Quantum Physics Lecture 15

Time dependence in quantum mechanics

Interference revisited

Course summary

Extra for those that want it: axioms of quantum mechanics

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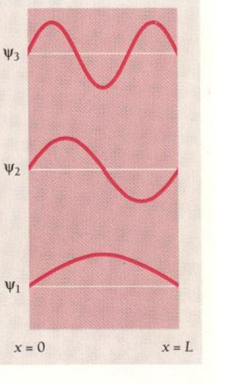
Time-dependent wavefunctions



Results of a measurement at time t can be calculated from the wavefunction
- which obeys the time-dependent Schrodinger equation (TDSE)

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+U(x)\right]\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

- which has special solutions of the form

$$\psi(x,t)=e^{-iEt/\hbar}\psi(x)$$

- where the wavefunction $\psi(x)$ obeys 'steady-state Schrodinger equation'

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+U(x)\right]\psi(x)=E\psi(x)$$

e.g. for the infinite well we have SSSE solutions $\psi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$

with the quantized energies $E_n=\dfrac{n^2\pi^2\hbar^2}{2mL^2}$

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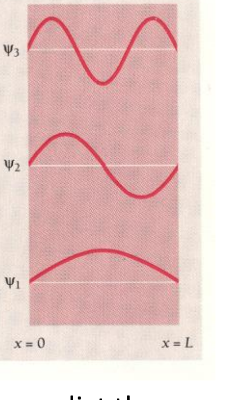
Time-dependent wavefunctions



Key fact: if we know all the solutions of the SSSE for a potential, e.g.

$$\psi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)\qquad E_n=\frac{n^2\pi^2\hbar^2}{2mL^2}$$

and we know the wavefunction at one time (e.g. t=0): $\psi(x,t=0)$

we can work out the wavefunction at any later time $\psi(x,t)$

i.e. we can work out how the 'Schrodinger wave' moves/changes in time, and so predict the results of measurements in situations where the wavefunction is not a stationary state.

Mathematically, we can use the solutions of the SSSE to 'build up' solutions to the TDSE:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+U(x)\right]\psi(x)=E\psi(x)\qquad\qquad\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+U(x)\right]\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}$$

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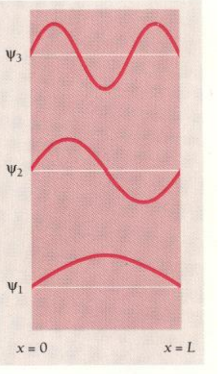
Time-dependence: example



Consider the infinite square well with stationary w/fs $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right)$

We know these are individually solutions to the TDSE, e.g. if

$$\psi(x, t=0) = \psi_1(x) e^{-iE_1 0/\hbar} \quad \text{then} \quad \psi(x,t) = \psi_1(x) e^{-iE_1 t/\hbar}$$

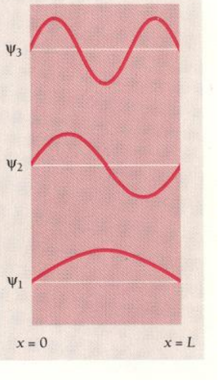
But the TDSE is linear so we can add up solutions of this form, e.g. if

$$\psi(x, t=0) = A\psi_1(x) e^{-iE_1 0/\hbar} + B\psi_2(x) e^{-iE_2 0/\hbar}$$

for some A, B then

$$\psi(x,t) = A\psi_1(x) e^{-iE_1 t/\hbar} + B\psi_2(x) e^{-iE_2 t/\hbar}$$

Time-dependence: example



$$\psi(x, t=0) = A\psi_1(x) e^{-iE_1 0/\hbar} + B\psi_2(x) e^{-iE_2 0/\hbar}$$

$$\Rightarrow \quad \psi(x,t) = A\psi_1(x) e^{-iE_1 t/\hbar} + B\psi_2(x) e^{-iE_2 t/\hbar}$$

Key fact: any initial wavefunction $\psi(x, t=0)$ can be written as a superposition of the stationary states (=eigenfunctions of the Hamiltonian defining the problem)

$$\psi(x, t=0) = \sum_n c_n \psi_n(x)$$

for some values of the coefficient $c_1, c_2, c_3, \ldots$

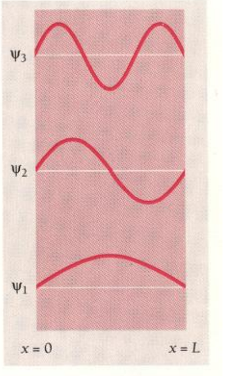
Given a wavefunction in this form we see (?) that the wavefunction at a later time *t* is

$$\psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}$$

What is $|c_n|^2$?

- If $\psi(x,t)$ and $\psi_n(x)$ correctly normalized
- It is : probability that if you measure energy you get E_n
- (See lecture 9 for example of measuring momentum).

Time-dependence: example



Stationary state: no observable property depends on time. But in the general case they do.

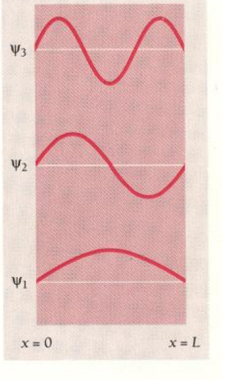
To understand this, let's suppose that at t=0 the particle has 50/50 chance of being in the first/second energy level

More specifically suppose the wavefunction at t=0 is $\psi(x, t=0) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x)$

[i.e. A, B are $1/\sqrt{2}$ - note we've used normalized ψ_1, ψ_2 and chosen A, B so that the initial wavefunction $\psi(x, t=0)$ is also properly normalized. This wavefunction contains *more information* than the statement '50/50 chance' of being in the two energy levels because that only defines the *magnitudes* of the complex numbers A and B.]

Sketch: wavefunction at t=0

Time-dependence: example



If the wavefunction at t=0 is $\psi(x, t=0) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x)$

Wavefunction at a later time is

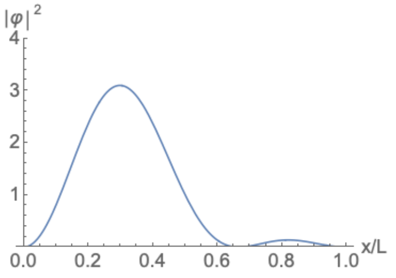
$$\begin{aligned}\psi(x,t) &= \frac{1}{\sqrt{2}}\left(\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}\right) \\ &= \frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}\left(\psi_1(x) + \psi_2(x)e^{-i(E_2-E_1)t/\hbar}\right)\end{aligned}$$

Sketch: wavefunction, ignoring exponential in front, at times for which the plotted quantity is real (and hence sketchable)

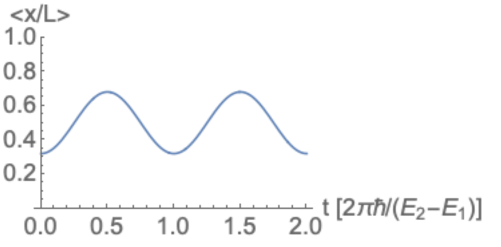
Conclude: the probability density will oscillate, between the functions $|\psi_1(x) + \psi_2(x)|^2/2$ and $|\psi_1(x) - \psi_2(x)|^2/2$ as the different stationary states in the wavefunction go 'in and out of phase' and hence the places in space where we get constructive/destructive interference oscillate.

Time-dependence: example

Probability density in this case oscillates (animation):

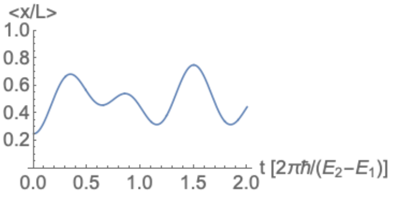


Expectation value <x(t)> :



If there are two stationary states in the wavefunction we get oscillations in observables with a frequency related to the energy difference between the two stationary states. If there are more than two we get more complex motion, because there are many frequencies involved -- cf. adding together oscillations of various frequencies.

E.g. Expectation value of x for initial state $(\psi_1 + \psi_2 + \psi_3)/\sqrt{3}$:



Time-dependence: algebra details

Calculation of probability density in two-state case – helps to write w/f as :

$$\psi = \frac{1}{\sqrt{2}} e^{-i(E_1+E_2)t/(2\hbar)} \left(\psi_1(x) e^{-i(E_1-E_2)t/(2\hbar)} + \psi_2(x) e^{i(E_1-E_2)t/(2\hbar)}\right)$$

$$= \frac{1}{\sqrt{2}} e^{-i(E_1+E_2)t/(2\hbar)} \left(\psi_1(x) e^{i\Delta E t/(2\hbar)} + \psi_2(x) e^{-i\Delta E t/(2\hbar)}\right) \qquad \Delta E = E_2 - E_1$$

$$= \frac{1}{\sqrt{2}} e^{-i(E_1+E_2)t/(2\hbar)} \left[(\psi_1 + \psi_2)\cos\frac{\Delta E t}{2\hbar} + i(\psi_1 - \psi_2)\sin\frac{\Delta E t}{2\hbar}\right]$$

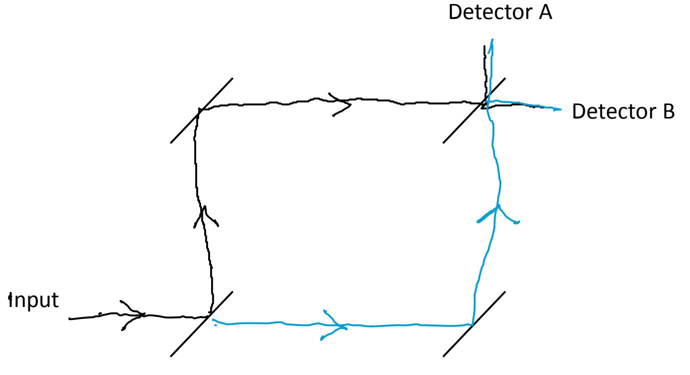
$$\Rightarrow |\psi|^2 = \frac{1}{2}\left[(\psi_1 + \psi_2)^2 \cos^2\frac{\Delta E t}{2\hbar} + (\psi_1 - \psi_2)^2 \sin^2\frac{\Delta E t}{2\hbar}\right]$$

(Since ψ_1, ψ_2 are real).

Interference: the Mach-Zehnder interferometer

Lecture 4 : saw we can understand double-slit interference by adding together probability amplitudes (=wavefunctions) along 'paths' through the experiment + magnitude-squaring to get the probability.

Redo this for a different kind of interferometer : a "Mach-Zehnder" interferometer:



Suppose we send a single photon in

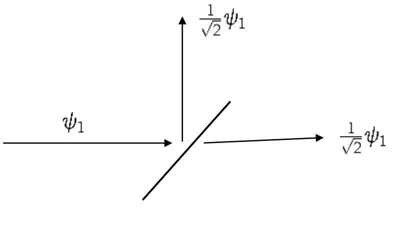
Have single-photon detectors.

What are the possible outcomes?

Detect photon at A
Detect photon at B
Detector photon at A & B??? NO!

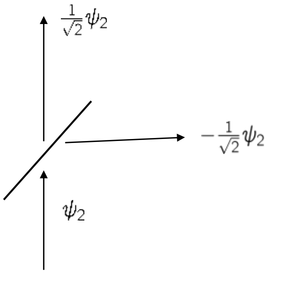
Interference: beam splitters

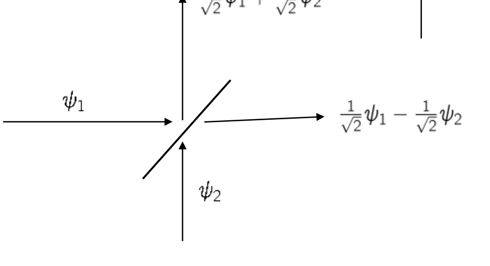
Device which, with probability 0.5, sends an incident photon along one of two different paths :



Could have waves in from both sides?
- Yes – just add the two cases up. Superposition!
- Note that the – sign in one reflection means that at
  - at one output the (in-phase) inputs add constructively
  - at the other they add destructively.

Could also come in from the other side?
- Yes but effect is not quite identical. One of the reflections must have a – sign compared with the other.





Info: which output has constructive/destructive interference depends on which reflection has the – sign, which depends on how the device is constructed. But it must be there somewhere otherwise the device would violate energy/probability conservation. One photon in = one photon out!

Interference: the Mach-Zehnder interferometer

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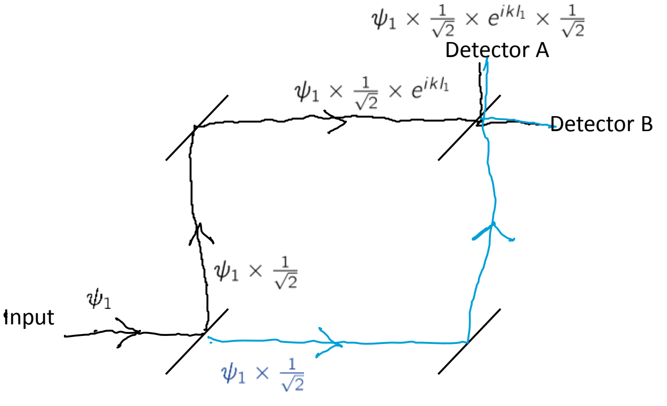
Redo this for a different kind of interferometer : a "Mach-Zehnder" interferometer:



Wavefunction at A :

Wavefunction at B :

e.g. l_1=l_2,

Probability (detect A) :

Probability (detect B) :

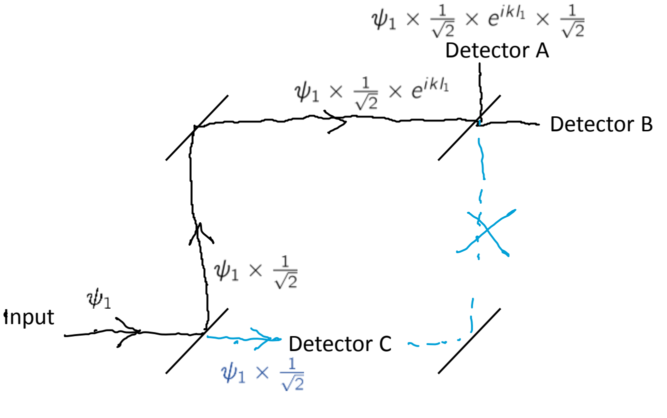
cf classically:

Probability (detect A) :
Probability (detect B) :

Which way did the photon go?

In the double-slit experiment if we close one slit (so we know for sure which way the photon went) we lose the interference pattern. Here see same effect: what happens if we add a detector C in one arm?

Outcomes now: detect A, detect B, or detect C.



Wavefunction at A :

Wavefunction at B :

Wavefunction at C :

Probability (detect A) :

Probability (detect B) :

Probability (detect C) :

which are just the classical probability results! We lost the interference effect because we measured which way the photon went!

Elitzur-Vaidman rocket

This device shows us that quantum mechanics lets us do things that are impossible classically, like measurement.

Thought experiment: firework rockets are triggered by detecting a single photon.

You have a supply of these, but some of the triggers are dud

– they will not launch the rocket, and instead just let the photon through.

You need a sure-fire working rocket that hasn't been triggered. How can you possibly get one?

Course Summary

- Quantum mechanics reconciles the wave-like and particle-like behaviours seen in nature.
- It does this by describing the 'state of a system' using a wavefunction, which obeys a wave-equation.
- And by adding the statement about what the wavefunction means: if you measure the position of the particle, you find it in a certain place with probability (density).
- An additional effect: if you measure the position of the particle, the wavefunction 'collapses' to a spike around where you found it.
- Key background for these discoveries :
  - The photoelectric effect (shows that measuring 'energy of an EM wave' gives you a discrete answer)
  - Davisson-Germer experiment (shows that 'particles' diffract)
- Leading to de-Broglie proposal relating the wavelength of a particle to its momentum
- Leading to Schrodinger's equation, the uncertainty principle, quantization of energy....
- For more details : see the learning outcomes on Blackboard (in the schedule document) + past exam papers.

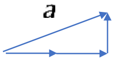
Axioms of quantum mechanics: vectors in Hilbert space

The formalism we have seen is not quite general enough – it is a special case of a more general formalism.
For interest & potentially future use this is summed up here:

The general case is expressed in terms of a ‘set of vectors’, which form a ‘Hilbert space’.

- The state of a system is a vector Hilbert space $\boldsymbol{\psi}$ (usually chosen to have unit length, i.e. ‘normalized’).

- Like the vector spaces you are used to, one can introduce an orthonormal basis, and write any vector ‘in Hilbert space’ in terms of its components (coordinates). e.g. unit vectors i, j in Cartesian coordinates so e.g.



$$\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} \longleftrightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

- You’re used to ‘real vector spaces’, where the components of a vector are real numbers.
- ‘Hilbert spaces’ are those where they are complex numbers.

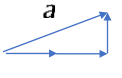
Axioms of quantum mechanics: operators in Hilbert space

Another important object in Hilbert space is an operator.

An operator ‘acts on’ a vector (any vector in the space) to give you another vector (some other vector in the space):

$$\boldsymbol{O}\boldsymbol{a} = \boldsymbol{b}$$

If you introduce a basis, so that vectors are represented by columns of numbers, then (linear) operators are represented by matrices of numbers.



$$\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} \longleftrightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Axioms of quantum mechanics: operators and observables

In this formalism, the matrices corresponding to observables have real eigenvalues and complete, orthogonal eigenvectors: e.g. for the 'momentum' operator $\boldsymbol{P}$ there are (many) eigenvectors $\boldsymbol{p}$, with eigenvalues p.

$$\boldsymbol{P}\boldsymbol{p} = p\boldsymbol{p}$$

These eigenvectors form a basis: any vector, including the state vector, can be written as

$$\boldsymbol{\psi} = \sum_{\text{eigenvectors of } \boldsymbol{P}} c_p \boldsymbol{p}$$

Measurement rule: if you measure the quantity $\boldsymbol{P}$ (say), you get the eigenvalue p with probability $|c_p|^2$ (if $\boldsymbol{\psi}$ and $\boldsymbol{p}$ are chosen to have length 1, i.e. are normalized vectors).

After the measurement, the state vector collapses (to the eigenvector corresponding to the eigenvalue you got).

Schrodinger's equation : the state-vector is time-dependent, and evolves in time according to the differential equation:

$$\boldsymbol{H}\boldsymbol{\psi}(t) = i\hbar\frac{d\boldsymbol{\psi}(t)}{dt}$$

Axioms of quantum mechanics: connection to wavefunctions

The connection to wavefunctions? Well, the position operator is an observable, so we can use its eigenvectors as the basis. These correspond to situations in which the particle is definitely at some position x, so let us label them $\boldsymbol{x}$.

The wavefunction is the components of the state vector in the position basis. $\boldsymbol{\psi} \longleftrightarrow c(x)$

If the position x were discrete (say, a particle which can be found in one of three places, labelled '1', '2', '3'), this would just be a set of numbers :

$$\boldsymbol{\psi} \longleftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Usually position x is continuous, so that rather than having a set of numbers labelled by a discrete index '1, 2, 3 (which we'd write as a column vector), we have a set of numbers labelled by a continuous index = a function.