Advanced Calculus
MA1132
Exercises 1 Solutions

1. Find the domain of the function

$$\mathbf{r}(t) = \left(\sqrt{1-3t}, \tan^{-1}(-t), \ln(9-t^2)\right).$$

Solution: We have to find the intersection of the domains of the functions $x(t) = \sqrt{1-3t}$, $y(t) = \tan^{-1}(-t)$ and $z(t) = \ln(9-t^2)$.

Now, $\sqrt{1-3t}$ is defined whenever

$$1 - 3t \geqslant 0 \quad \Longrightarrow \quad 3t \leqslant 1 \quad \Longrightarrow \quad t \leqslant \frac{1}{3}.$$

Thus the domain of x is $\left(\infty, \frac{1}{3}\right]$.

Next, $\tan^{-1}(-t)$ is always defined, so the domain of y is $\mathbb{R}$.

Finally, $\ln(9-t^2)$ is defined whenever

$$9 - t^2 > 0 \quad \Longrightarrow \quad t^2 < 9 \quad \Longrightarrow \quad -3 < t < 3.$$

Thus the domain of z is $(-3, 3)$.

Hence the domain of $\mathbf{r}$ is $\left(\infty, \frac{1}{3}\right] \cap \mathbb{R} \cap (-3,3) = \left(-3, \frac{1}{3}\right]$.

2. Show the graph of the function

$$\mathbf{r}(t) = t\mathbf{i} + \frac{3-2t}{t}\mathbf{j} + \frac{(t-3)^2}{t}\mathbf{k}$$

lies in the plane $x + 3y - z = 0$.

Solution: To show the graph lies in the plane $x + 3y - z = 0$, all we need to do is to substitute $x = t$, $y = \dfrac{3-2t}{t}$ and $z = \dfrac{(t-3)^2}{t}$ into $x + 3y - z$ and show we get 0.

However

$$x + 3y - z = t + 3 \cdot \frac{3-2t}{t} - \frac{(t-3)^2}{t} = \frac{t^2 + 9 - 6t - t^2 + 6t - 9}{t} = \frac{0}{t} = 0,$$

as required.

3. Describe what the graph of the function given by

$$x(t) = t^2 - t$$
$$y(t) = t - 2$$

looks like.

Solution: From $y = t - 2$ we obtain $t = y + 2$ and on substituting this into $x = t^2 - t$, we obtain

$$x = (y+2)^2 - (y+2) = y^2 + 4y + 4 - y - 2 = y^2 + 3y + 2.$$

So it is a parabola 'laying on its side' (to the right since the coefficient of y^2 is positive), Differentiating x with respect to y, $\dfrac{dx}{dy} = 2y + 3$, so $\dfrac{dx}{dy} = 0$ implies $y = -\frac{3}{2}$. When $y = -\frac{3}{2}$, $x = -\frac{1}{4}$. Thus the turning point lies at $\left(-\frac{1}{4}, -\frac{3}{2}\right)$. Also the y-intercept occurs when

$$y^2 + 3y + 2 = 0 \quad \Longrightarrow \quad (y+2)(y+1) = 0 \quad \Longrightarrow \quad y = -2 \text{ or } y = -1.$$

Finally, letting $y = 0$, we see that the x-intercept is at $x = 2$. Thus the curve is as shown in Figure 1. Note that as t increases, so does y, so the direction of the curve is

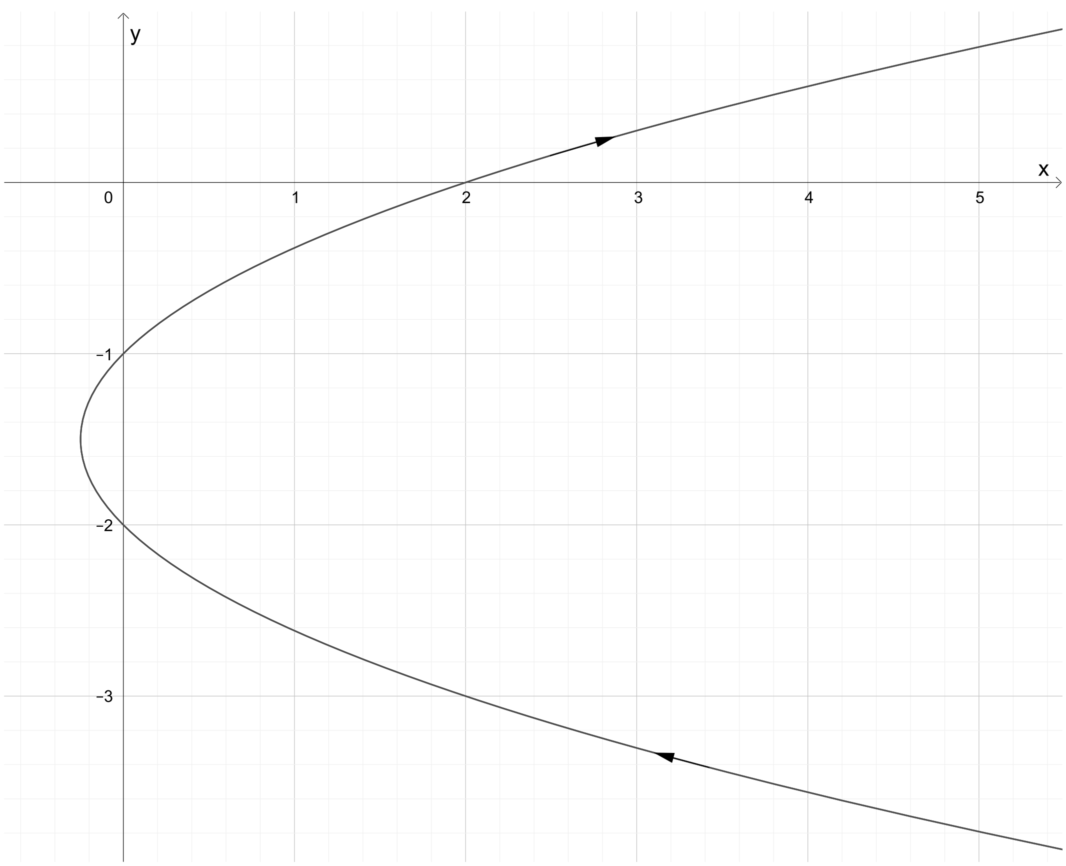

Figure 1: The curve in Question 3.

as I have indicated.

4. Find a vector-valued function which has as its graph, the curve of intersection of the surfaces $x + y = 0$ and $z = \sqrt{2 - x^2 - y^2}$.

   Geometrically, what does this curve look like?

Solution: Here if we let $x = t$, then we obtain $y = -t$ and $z = \sqrt{2 - t^2 - t^2} = \sqrt{2 - 2t^2}$. However we also have to choose our t so that $\sqrt{2 - 2t^2}$ exists. So we need

$$2 - 2t^2 \geqslant 0 \quad \Longrightarrow \quad 2t^2 \leqslant 2 \quad \Longrightarrow \quad t^2 \leqslant 1 \quad \Longrightarrow \quad -1 \leqslant t \leqslant 1.$$

Hence a suitable function is

$$\begin{aligned} \mathbf{r} \colon [-1, 1] &\to \mathbb{R}^3 \\ t &\mapsto (t, -t, \sqrt{2 - 2t^2}). \end{aligned}$$

If you use plotting software (e.g., GeoGebra, Mathematica, Sage, etc) to plot the surfaces then you will see that $x + y = 0$ is a vertical plane and $z = \sqrt{2 - x^2 - y^2}$ is a hemisphere above the xy-plane, and their intersection is a half circle above the xy-plane.

5. Evaluate the limit

$$\lim_{t \to 1} \left(\frac{t^2 + t - 2}{t - 1}, \frac{\ln(t)}{t^3 - t^2 + t - 1}, t^2 - 1 \right)$$

For what values of a, b and c is the function

$$\mathbf{r}(t) = \begin{cases} \left(\dfrac{t^2 + t - 2}{t - 1}, \dfrac{\ln(t)}{t^3 - t^2 + t - 1}, t^2 - 1 \right) & \text{if } t \neq 1 \\ (a, b, c) & \text{if } t = 1 \end{cases}$$

continuous at $t = 1$?

Solution: To evaluate the vector limit, we have to evaluate the three component limits. Firstly

$$\lim_{t \to 1} \frac{t^2 + t - 2}{t - 1} = \lim_{t \to 1} \frac{(t - 1)(t + 2)}{t - 1} = \lim_{t \to 1} (t + 2) = 3.$$

Next, using l'Hôpital's Rule,

$$\lim_{t \to 1} \frac{\ln(t)}{t^3 - t^2 + t - 1} = \lim_{t \to 1} \frac{1/t}{3t^2 - 2t + 1} = \frac{1}{2}.$$

Finally $\lim_{t \to 1} (t^2 - 1) = 1^2 - 1 = 0$.

Hence

$$\lim_{t \to 1} \left(\frac{t^2 + t - 2}{t - 1}, \frac{\ln(t)}{t^3 - t^2 + t - 1}, t^2 - 1 \right) = \left(3, \frac{1}{2}, 0 \right).$$

For the function to be continuous at $t = 1$, we need $\lim_{t \to 1} \mathbf{r}(t) = \mathbf{r}(1)$.

Thus we need $(a, b, c) = \left(3, \frac{1}{2}, 0\right)$.

6. Consider the vector-valued function (with values in $\mathbb{R}^3$)

$$\mathbf{r}(t) = \ln(3 - \sqrt{t})\,\mathbf{i} + (1 + \sqrt{t})\,\mathbf{j} + \frac{(3 - \sqrt{t})^2}{4}\,\mathbf{k} \tag{1}$$

(a) Find the domain $\mathcal{D}(\mathbf{r})$ of the vector-valued function $\mathbf{r}(t)$.

Solution: The domain $\mathcal{D}(\mathbf{r})$ of $\mathbf{r}(t)$ is the intersection of domains of its component functions. Since $\mathcal{D}(\ln(3-\sqrt{t})) = [0\,,9\,)$, $\mathcal{D}(1+\sqrt{t}) = [0,\infty)$ and $\mathcal{D}(\frac{(3-\sqrt{t})^2}{4}) = [0,\infty)$, one gets

$$\mathcal{D}(\mathbf{r}) = [0\,,9\,) \tag{2}$$

that is the vector function $\mathbf{r}(t)$ is defined for $0 \le t < 9$.

(b) Find the derivative $d\mathbf{r}/dt$.

Solution:

$$\frac{d\mathbf{r}}{dt} = -\frac{1}{2\sqrt{t}}\,\frac{1}{3-\sqrt{t}}\,\mathbf{i} + \frac{1}{2\sqrt{t}}\,\mathbf{j} - \frac{1}{2\sqrt{t}}\,\frac{3-\sqrt{t}}{2}\,\mathbf{k}\,. \tag{3}$$

(c) Find the norm $|d\mathbf{r}/dt|$.

Simplify the expressions obtained.

Solution: The magnitude or norm of this vector is

$$\begin{aligned}|\frac{d\mathbf{r}}{dt}| &= \sqrt{\left(-\frac{1}{2\sqrt{t}}\,\frac{1}{3-\sqrt{t}}\right)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2 + \left(-\frac{1}{2\sqrt{t}}\,\frac{3-\sqrt{t}}{2}\right)^2} = \frac{1}{2\sqrt{t}}\sqrt{\left(\frac{1}{3-\sqrt{t}}\right)^2 + 1 + \left(\frac{3-\sqrt{t}}{2}\right)^2} \\ &= \frac{1}{2\sqrt{t}}\sqrt{\left(\frac{1}{3-\sqrt{t}}\right)^2 + 2\frac{1}{3-\sqrt{t}}\,\frac{3-\sqrt{t}}{2} + \left(\frac{3-\sqrt{t}}{2}\right)^2} \\ &= \frac{1}{2\sqrt{t}}\sqrt{\left(\frac{1}{3-\sqrt{t}} + \frac{3-\sqrt{t}}{2}\right)^2} = \frac{1}{2\sqrt{t}}\left(\frac{1}{3-\sqrt{t}} + \frac{3-\sqrt{t}}{2}\right) = -\frac{t-6\sqrt{t}+11}{4\left(t-3\sqrt{t}\right)}\,,\end{aligned} \tag{4}$$

because $\sqrt{t} < 3$.

(d) Find the unit tangent vector $\mathbf{T}$ for all values of t in $\mathcal{D}(\mathbf{r})$.

Solution: The unit tangent vector is

$$\mathbf{T} = \frac{\frac{d\mathbf{r}}{dt}}{|\frac{d\mathbf{r}}{dt}|} = -\frac{2}{t-6\sqrt{t}+11}\,\mathbf{i} - \frac{2\left(\sqrt{t}-3\right)}{t-6\sqrt{t}+11}\,\mathbf{j} + \frac{-t+6\sqrt{t}-9}{t-6\sqrt{t}+11}\,\mathbf{k}\,. \tag{5}$$

(e) Find the vector equation of the line tangent to the graph of $\mathbf{r}(t)$ at the point $P_0(0,3,\frac{1}{4})$ on the curve.

Solution: The point $P_0(0,3,\frac{1}{4})$ on the curve corresponds to $t=4$. We find

$$\mathbf{r}_0 = \mathbf{r}(4) = 3\mathbf{j} + \frac{1}{4}\mathbf{k}\,, \quad \mathbf{v}_0 = \frac{d\mathbf{r}}{dt}(4) = -\frac{1}{4}\mathbf{i} + \frac{1}{4}\mathbf{j} - \frac{1}{8}\,\mathbf{k}\,. \tag{6}$$

Thus the tangent line equation is

$$\mathbf{r} = \mathbf{r}_0 + (t-4)\,\mathbf{v}_0 = (1-\frac{t}{4})\,\mathbf{i} + (2+\frac{t}{4})\,\mathbf{j} + (\frac{3}{4}-\frac{t}{8})\,\mathbf{k}\,. \tag{7}$$

Note that the same line is also described by the following equation which is obtained from the one above by rescaling and shifting the parameter t: $t \to -4t+4$

$$\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v}_0 = t\,\mathbf{i} + (3-t)\,\mathbf{j} + \frac{1}{4}(2t+1)\,\mathbf{k}\,. \tag{8}$$

7. Find the indefinite integral

$$\int \ln(t)\mathbf{i} + \sin(t)e^{\cos(t)}\mathbf{j} + \frac{3t+2}{t^2-4}\mathbf{k}\,dt.$$

Solution: To integrate $\mathbf{r}(t) = (x(t), y(t), z(t))$, we integrate the three component functions separately.

To integrate $x(t) = \ln(t)$, we integrate by parts:

$$\int \ln(t)\,dt = t\ln(|t|) - \int t \cdot \frac{1}{t}\,dt = t\ln(|t|) - \int 1\,dt = t\ln(|t|) - t + c.$$

To integrate $y(t) = \sin(t)e^{\cos(t)}$, we integrate by inspection (or we can use the substitution $u = \cos(t)$ if we don't spot the integral):

$$\int \sin(t)e^{\cos(t)}\,dt = -e^{\cos(t)} + c.$$

To integrate $z(t) = \dfrac{3t+2}{t^2-4}$, we integrate using partial fractions:

$$\int \frac{3t+2}{t^2-4}\,dt = \int \frac{1}{t+2} + \frac{2}{t-2}\,dt = \ln(|t+2|) + 2\ln(|t-2|) + c = \ln[|(t+2)(t-2)^2|] + c.$$

Hence

$$\int \ln(t)\mathbf{i} + \sin(t)e^{\cos(t)}\mathbf{j} + \frac{3t+2}{t^2-4}\mathbf{k}\,dt = (t\ln(|t|) - t)\mathbf{i} - e^{\cos(t)}\mathbf{j} + \ln[|(t+2)(t-2)^2|]\mathbf{k} + \mathbf{c}.$$

Note the constant is a constant vector rather than a constant number.