Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 10
Each set of homework questions is worth 100 marks

Problem 1. Any rotation matrix G belonging to the Lie group $SO(3)$ can be written as the following product

$$G = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

where ϕ, θ and ψ are the Euler angles.

Introduce the following matrix

$$J = \frac{dG}{dt} G^{-1} ,$$

where G^{-1} is the inverse matrix.

(a) Show that the matrix J belongs to the Lie algebra $so(3)$, that is it is skew-symmetric: $J^T = -J$, where J^T is the transposed matrix.

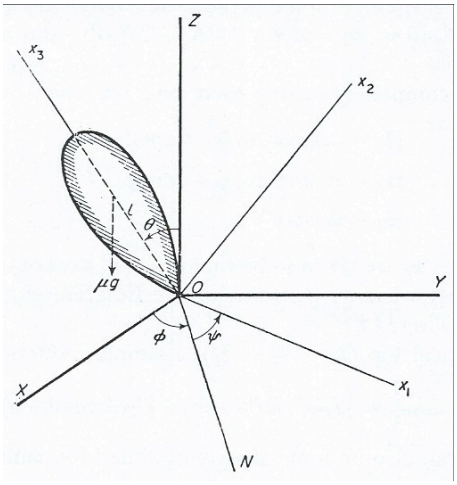
(b) Show that the components Ω_i of the angular velocity vector $\vec{\Omega}$ along the moving axes x_1, x_2, x_3 are expressed through the components J_{ij} as follows

$$\Omega_i = \frac{1}{2} \epsilon_{ijk} J_{jk} ,$$

where ϵ_{ijk} is the skew-symmetric tensor.

Problem 2. Consider the Lagrangian of a heavy symmetric top whose lowest point is fixed (Lagrange's top)

$$L = \frac{1}{2}(I_1 + ml^2)(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta\,.$$



(a) Which of the coordinates of the top are cyclic? Find the integrals of motion corresponding to the cyclic angles, and relate them to the angular momentum of the top.

(b) Use the integrals of the motion to reduce the problem of the motion of the top to a one-dimensional one.

(c) Find the effective potential and the effective Lagrangian.

Use Mathematica to plot the effective potential for $m = g = l = \tilde{I}_1 = 1$, and

i) $M_z = 12$, $M_3 = 18$; ii) $M_z = 1$, $M_3 = 1/6$. Explain the plots.

(d) Let the top rotate about the vertical axis. Explain why such a rotation is possible only for $M_z = M_3$.

(e) Simplify the effective potential for $M_z = M_3$, and use Mathematica to plot the effective potential for $m = g = l = \tilde{I}_1 = 1$, and i) $M_z = M_3 = 3$; ii) $M_z = M_3 = 1/2$.

Explain the plots.

(f) Find the exact condition for the rotation about the vertical axis to be stable.

(g) Find the frequency of small oscillations in the θ-direction if the top was shifted from the equilibrium position. What type of motion does the top undergo?

Problem 3. A symmetric top with a fixed centre of mass experiences a constant torque $\vec{K}$. Use the Euler angles to find the angular velocity of the top as a function of time. The initial angular momentum is proportional to $\vec{K}$: $\vec{M}(0) = \alpha\vec{K}$, where α is a constant.