MA1125 – Calculus
Tutorial solutions #4

1. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = \ln(\sec x) + e^{\tan x}, \qquad y = \sin(\sec^2(4x)).$$

When it comes to the first function, one may use the chain rule to get

$$y' = \frac{1}{\sec x} \cdot \sec x \tan x + e^{\tan x} \sec^2 x = \tan x + e^{\tan x} \sec^2 x.$$

When it comes to the second function, one similarly finds that

$$\begin{aligned} y' &= \cos(\sec^2(4x)) \cdot [\sec^2(4x)]' \\ &= \cos(\sec^2(4x)) \cdot 2\sec(4x) \cdot [\sec(4x)]' \\ &= \cos(\sec^2(4x)) \cdot 2\sec(4x) \cdot 4\sec(4x)\tan(4x) \\ &= 8\cos(\sec^2(4x)) \cdot \sec^2(4x) \cdot \tan(4x). \end{aligned}$$

2. Compute the derivative $y' = \frac{dy}{dx}$ in the case that $x^2 \sin y = y^2 e^x$.

We differentiate both sides of the equation and then rearrange terms. This gives

$$\begin{aligned} 2x\sin y + x^2 y' \cos y = 2yy'e^x + y^2 e^x \quad &\Longrightarrow \quad (x^2\cos y - 2ye^x)y' = y^2e^x - 2x\sin y \\ &\Longrightarrow \quad y' = \frac{y^2e^x - 2x\sin y}{x^2\cos y - 2ye^x}. \end{aligned}$$

3. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = x^2 \cdot \tan^{-1}(2x), \qquad y = (x \cdot \sin x)^x.$$

When it comes to the first function, we use the product rule and the chain rule to get

$$y' = 2x \cdot \tan^{-1}(2x) + x^2 \cdot \frac{2}{(2x)^2 + 1} = 2x \cdot \tan^{-1}(2x) + \frac{2x^2}{4x^2 + 1}.$$

When it comes to the second function, logarithmic differentiation gives

$$\begin{aligned} \ln y = x\ln(x \cdot \sin x) \quad &\Longrightarrow \quad \frac{y'}{y} = \ln(x\sin x) + x \cdot \frac{1}{x\sin x} \cdot (\sin x + x\cos x) \\ &\Longrightarrow \quad y' = y \cdot (\ln(x\sin x) + 1 + x\cot x) \\ &\Longrightarrow \quad y' = (x \cdot \sin x)^x \cdot (\ln(x\sin x) + 1 + x\cot x). \end{aligned}$$

4. Compute the derivative $f'(x_0)$ in the case that

$$f(x) = \frac{(x^3+5x^2+2)^3 \cdot e^{\sin x}}{\sqrt{x^2+4x+1}}, \qquad x_0 = 0.$$

First, we use logarithmic differentiation to determine $f'(x)$. In this case, we have

$$\begin{aligned} \ln|f(x)| &= \ln|x^3+5x^2+2|^3 + \ln e^{\sin x} - \ln|x^2+4x+1|^{1/2} \\ &= 3\ln|x^3+5x^2+2| + \sin x - \frac{1}{2}\ln|x^2+4x+1|. \end{aligned}$$

Differentiating both sides of this equation, one easily finds that

$$\frac{f'(x)}{f(x)} = \frac{3(3x^2+10x)}{x^3+5x^2+2} + \cos x - \frac{2x+4}{2(x^2+4x+1)}.$$

To compute the derivative $f'(0)$, one may then substitute $x = 0$ to conclude that

$$\frac{f'(0)}{f(0)} = 0 + \cos 0 - \frac{4}{2} = -1 \quad \Longrightarrow \quad f'(0) = -f(0) = -8.$$

5. Compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$y = \sin^{-1} u, \qquad u = \ln(2z^2+3z+1), \qquad z = \frac{3x-1}{2x+5}.$$

Differentiating the given equations, one easily finds that

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}, \qquad \frac{du}{dz} = \frac{4z+3}{2z^2+3z+1}, \qquad \frac{dz}{dx} = \frac{3(2x+5)-2(3x-1)}{(2x+5)^2} = \frac{17}{(2x+5)^2}.$$

According to the chain rule, the derivative $\frac{dy}{dx}$ is the product of these factors, namely

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dz}\frac{dz}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{4z+3}{2z^2+3z+1} \cdot \frac{17}{(2x+5)^2}.$$

6. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = (e^{2x}+x^3)^4, \qquad y = \tan(x\sin x).$$

When it comes to the first function, one may use the chain rule to get

$$y' = 4(e^{2x}+x^3)^3 \cdot (e^{2x}+x^3)' = 4(e^{2x}+x^3)^3 \cdot (2e^{2x}+3x^2).$$

When it comes to the second function, one similarly finds that

$$y' = \sec^2(x\sin x) \cdot (x\sin x)' = \sec^2(x\sin x) \cdot (\sin x + x\cos x).$$

7. Compute the derivative $y' = \frac{dy}{dx}$ in the case that $x^2 + y^2 = \sin(xy)$.

Differentiating both sides of the given equation, one finds that

$$2x + 2yy' = \cos(xy) \cdot (y + xy') = y\cos(xy) + xy'\cos(xy).$$

Once we now rearrange terms and solve for y', we may conclude that

$$(2y - x\cos(xy)) \cdot y' = y\cos(xy) - 2x \quad \Longrightarrow \quad y' = \frac{y\cos(xy) - 2x}{2y - x\cos(xy)}.$$

8. Compute the derivative $f'(x_0)$ in the case that

$$f(x) = \frac{(x^2+3x+1)^4 \cdot \sqrt{2x + \cos x}}{(e^x + x)^3}, \qquad x_0 = 0.$$

First, we use logarithmic differentiation to determine $f'(x)$. In this case, we have

$$\begin{aligned} \ln|f(x)| &= \ln|x^2+3x+1|^4 + \ln|2x+\cos x|^{1/2} - \ln|e^x + x|^3 \\ &= 4\ln|x^2+3x+1| + \frac{1}{2}\ln|2x+\cos x| - 3\ln|e^x+x|. \end{aligned}$$

Differentiating both sides of this equation, one may use the chain rule to get

$$\frac{f'(x)}{f(x)} = \frac{4(2x+3)}{x^2+3x+1} + \frac{2-\sin x}{2(2x+\cos x)} - \frac{3(e^x+1)}{e^x+x}.$$

To compute the derivative $f'(0)$, one may then substitute $x = 0$ to conclude that

$$\frac{f'(0)}{f(0)} = 4 \cdot 3 + \frac{2}{2} - 3 \cdot 2 = 7 \quad \Longrightarrow \quad f'(0) = 7f(0) = 7.$$

9. Compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$y = \frac{2u-1}{3u+1}, \qquad u = \sin(e^z), \qquad z = \tan^{-1}(x^2).$$

Differentiating the given equations, one easily finds that

$$\frac{dy}{du} = \frac{2(3u+1) - 3(2u-1)}{(3u+1)^2} = \frac{5}{(3u+1)^2}, \qquad \frac{du}{dz} = e^z\cos(e^z), \qquad \frac{dz}{dx} = \frac{2x}{x^4+1}.$$

According to the chain rule, the derivative $\frac{dy}{dx}$ is the product of these factors, namely

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dz}\frac{dz}{dx} = \frac{5}{(3u+1)^2} \cdot e^z\cos(e^z) \cdot \frac{2x}{x^4+1}.$$

10. Compute the derivative $f'(1)$ in the case that $x^2 f(x) + x f(x)^3 = 2$ for all x.

Letting $y = f(x)$ for convenience, we get $x^2 y + xy^3 = 2$ and this implies that

$$\begin{aligned} 2xy + x^2 y' + y^3 + 3xy^2 y' = 0 \quad &\implies \quad (x^2 + 3xy^2) y' = -2xy - y^3 \\ &\implies \quad y' = -\frac{y(2x + y^2)}{x(x + 3y^2)}. \end{aligned}$$

We need to evaluate this expression at the point $x = 1$. At that point, one has

$$x^2 y + xy^3 = 2 \quad \implies \quad y + y^3 = 2 \quad \implies \quad y^3 + y - 2 = 0.$$

It is easy to see that $y = 1$ is a solution. In fact, it is the only real solution because

$$y^3 + y - 2 = (y - 1)(y^2 + y + 2)$$

and the quadratic factor has no real roots. This gives $y = 1$ at the point $x = 1$, so

$$f'(x) = -\frac{y(2x + y^2)}{x(x + 3y^2)} \quad \implies \quad f'(1) = -\frac{3}{4}.$$