Advanced Calculus
MA1132
Homework 1

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1. Find the derivative of the vector-valued function

$$\mathbf{r}(t) = \left(\frac{t^2-1}{t^2+2}, \sin(t)\ln(t), \cos(\sin(t))\right).$$

What is $\mathbf{r}'(1)$ equal to?

Solution:

To differentiate $\mathbf{r}(t) = (x(t), y(t), z(t))$, we differentiate the three component functions separately.

Using the Quotient Rule,

$$x'(t) = \frac{2t(t^2+2) - 2t(t^2-1)}{(t^2+2)^2} = \frac{2t^3+4t-2t^3+2t}{(t^2+2)^2} = \frac{6t}{(t^2+2)^2}.$$

Using the Product Rule,

$$y'(t) = \cos t \ln(t) + \sin(t)\cdot\frac{1}{t} = \cos(t)\ln(t) + \frac{\sin(t)}{t}.$$

Using the Chain Rule, $z'(t) = \cos(t)(-\sin(\sin(t))) = -\cos(t)\sin(\sin(t))$.

Hence

$$\mathbf{r}'(t) = \left(\frac{6t}{(t^2+2)^2}, \cos(t)\ln(t) + \frac{\sin(t)}{t}, -\cos(t)\sin(\sin(t))\right).$$

Letting $t=1$ in this expression, we obtain

$$\begin{aligned}\mathbf{r}'(1) &= \left(\frac{6(1)}{(1^2+2)^2}, \cos(1)\ln(1) + \frac{\sin(1)}{1}, -\cos(1)\sin(\sin(1)\right)\\ &= \left(\frac{2}{3}, \sin(1), -\cos(1)\sin(\sin(1))\right).\end{aligned}$$

2. Find the derivative of the vector-valued function

$$\mathbf{r}(t) = (2t^3-3t^2+5t+3)\left(\frac{t^2-1}{t^2+2}, \sin(t)\ln(t), \cos(\sin(t))\right).$$

Solution:

Here we use Theorem 1.2.11(c) (Chapter 1 of course notes by Anthony Brown), the solution to Question 1, and the fact that the derivative of $f(t) = 2t^3 - 3t^2 + 5t + 3$ is $f'(t) = 6t^2 - 6t + 5$, to obtain

$$\begin{aligned}\mathbf{r}'(t) =& (2t^3 - 3t^2 + 5t + 3)\left(\frac{6t}{(t^2+2)^2}, \cos(t)\ln(t) + \frac{\sin(t)}{t}, \cos(t)\cos(\sin(t))\right) \\ &+ (6t^2 - 6t + 5)\left(\frac{t^2-1}{t^2+2}, \sin(t)\ln(t), \cos(\sin(t))\right).\end{aligned}$$

3. The parametrisation

$$x = \alpha + a\cosh t\,, \quad y = \beta + b\sinh t\,, \quad -\infty < t < \infty \tag{1}$$

of the hyperbola

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1\,, \quad a > 0\,, \quad b > 0\,, \tag{2}$$

represents only one branch of the hyperbola.

Find a parametrisation which represents both branches of the hyperbola.

Solution: Recall that using the definitions for $\sinh t$ and $\cosh t$, we can rewrite the given parameterization as

$$x(r) = \alpha + a\frac{e^t + e^{-t}}{2}\,, \quad y(r) = \beta + b\frac{e^t - e^{-t}}{2}\,, \quad -\infty < t < \infty. \tag{3}$$

Since $0 < e^t < \infty$, we can reparameterize using $r = e^t$ with $0 < r < \infty$ yielding (with some algebraic manipulation)

$$x = \alpha + \frac{a}{2}\left(r + \frac{1}{r}\right)\,, \quad y = \beta + \frac{b}{2}\left(r - \frac{1}{r}\right)\,, \quad 0 < r < \infty. \tag{4}$$

Lastly, observe that since $-r < 0$, we have that $x(-r) = \alpha - \frac{a}{2}\left(r + \frac{1}{r}\right)$, and it is not hard to show that for all negative values $-r$, $y(-r)$ delivers the correct corresponding y-value, so that the full parameterization of both branches is

$$x = \alpha + \frac{a}{2}(r + \frac{1}{r})\,, \quad y = \beta + \frac{b}{2}(r - \frac{1}{r})\,, \quad -\infty < r < \infty\,, \quad r \neq 0\,. \tag{5}$$

4. Find an arc length parametrization of the curve

$$\mathbf{r}(t) = \cos^3(t)\mathbf{i} + \sin^3(t)\mathbf{j} \quad t \in \left[0, \frac{\pi}{2}\right],$$

in the same direction as the original curve.

Solution:

Here we are given our reference point $t_0 = 0$, so we don't have to calculate it.

Next, let us calculate the arc length of the curve from $t = 0$ to t.

Using the Chain Rule, we obtain

$$\mathbf{r}'(t) = -3\sin(t)\cos^2(t)\mathbf{i} + 3\cos(t)\sin^2(t)\mathbf{j}.$$

Hence

$$\begin{aligned}
\|\mathbf{r}'(t)\| &= \sqrt{9\sin^2(t)\cos^4(t) + 9\cos^2(t)\sin^4(t)} \\
&= \sqrt{9\sin^2(t)\cos^2(t)(\cos^2(t) + \sin^2(t))} \\
&= \sqrt{9\sin^2(t)\cos^2(t)} \\
&= 3\sin(t)\cos(t).
\end{aligned}$$

Thus the required arc length is

$$s(t) = \int_0^t \|\mathbf{r}'(v)\|\,dv = \int_0^t 3\sin(v)\cos(v)\,dv = \left[\frac{3}{2}\sin^2(v)\right]_0^t = \frac{3}{2}\sin^2(t),$$

where we have integrated 'by inspection' If you can't spot the integral, then you can use the substitution $u = \sin(v)$.

So we have $s = \frac{3}{2}\sin^2(t)$, and solving this for t we obtain

$$s = \frac{3}{2}\sin^2(t) \quad\Longrightarrow\quad \sin^2(t) = \frac{2s}{3} \quad\Longrightarrow\quad \sin t = \sqrt{\frac{2s}{3}} \quad\Longrightarrow\quad t = \sin^{-1}\left(\sqrt{\frac{2s}{3}}\right).$$

Substituting this into $\mathbf{r}(t)$, we obtain

$$\mathbf{r}(s) = \cos^3\left(\sin^{-1}\left(\sqrt{\frac{2s}{3}}\right)\right)\mathbf{i} + \sin^3\left(\sin^{-1}\left(\sqrt{\frac{2s}{3}}\right)\right)\mathbf{j}$$

Now, when $t = 0$, $s = 0$ and when $t = \frac{\pi}{2}$, $s = \frac{3}{2}$, so we obtain the arc length parametrization

$$\mathbf{r}(s) = \cos^3\left(\sin^{-1}\left(\sqrt{\frac{2s}{3}}\right)\right)\mathbf{i} + \sin^3\left(\sin^{-1}\left(\sqrt{\frac{2s}{3}}\right)\right)\mathbf{j} \quad s \in \left[0, \frac{3}{2}\right].$$

Although this is a perfectly good answer, it doesn't look particularly 'nice'. We can get a 'nicer' answer (although of course this is a matter of opinion) by instead of solving $s = \frac{3}{2}\sin^2(t)$ for t, stopping when we got an expression for $\sin(t)$, that is $\sin t = \sqrt{\frac{2s}{3}} = \left(\frac{2s}{3}\right)^{\frac{1}{2}}$. Then $\sin^3(t) = \left(\frac{2s}{3}\right)^{\frac{3}{2}}$ and

$$\cos^3(t) = \left(\sqrt{1 - \sin^2(t)}\right)^3 = \left(1 - \sin^2(t)\right)^{\frac{3}{2}} = \left(1 - \frac{2s}{3}\right)^{\frac{3}{2}}.$$

So we obtain the parametrization

$$\mathbf{r}(s) = \left(1 - \frac{2s}{3}\right)^{\frac{3}{2}}\mathbf{i} + \left(\frac{2s}{3}\right)^{\frac{3}{2}}\mathbf{j} \quad s \in \left[0, \frac{3}{2}\right].$$

5. Prove the Serret-Frenet formulas

   (a) $\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}(s)$

Solution: By definition

$$\mathbf{N}(s) = \frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|} = \frac{\frac{d\mathbf{T}}{ds}}{\kappa(s)}\,. \tag{6}$$

   (b) $\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N}(s)$, where τ is a scalar called the torsion of $\mathbf{r}(s)$.

Solution: By definition

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}\,. \tag{7}$$

Thus $\frac{d\mathbf{B}}{ds}$ is normal to $\mathbf{T}$. Since $\mathbf{B}$ is normal, $\frac{d\mathbf{B}}{ds}$ is also normal to $\mathbf{B}$. Thus, it is proportional to $\mathbf{N}$.

   (c) $\frac{d\mathbf{N}}{ds} = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}$

Solution: We know that

$$\mathbf{N} = \mathbf{B} \times \mathbf{T}\,, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}\,, \quad \mathbf{T} = \mathbf{N} \times \mathbf{B}\,. \tag{8}$$

Thus

$$\frac{d\mathbf{N}}{ds} = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} = -\tau(s)\mathbf{N} \times \mathbf{T} + \kappa(s)\mathbf{B} \times \mathbf{N} = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}\,. \tag{9}$$

6. Consider the vector-valued function

$$\mathbf{r}(t) = (4\cos(t), 4\sin(t), 3t), \quad t \in \mathbb{R}.$$

   (a) Find a arc-length parametrization for the function with reference point $t_0 = 0$.
   (b) Calculate the curvature κ of $\mathbf{r}$, using the arc-length parametrization you found in Part (a) and the formula $\kappa(s) = \|\mathbf{T}'(s)\|$.
   (c) Calculate the *torsion* τ of $\mathbf{r}$, using the arc-length parametrization you found in Part (a) and the formula $\tau(s) = -\mathbf{N}(s) \cdot \mathbf{B}'(s)$.
   (d) Plot the path, indicating the direction of increasing s, using Mathematica[1]). You should include the point $\mathbf{r}\left(s = \frac{5\pi}{2}\right)$, the vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ corresponding to this point, and the **TN**-, **TB**- and **NB**- planes corresponding to these vectors.

   Solution:

   (a) Here we are given our reference point $t_0 = 0$, so we don't have to calculate it.
   Next, let us calculate the arc length of the curve from $t = 0$ to t.
   We already did this in lectures, finding that $s = 5t$.
   Thus $t = \frac{s}{5}$ and substituting this into $\mathbf{r}(t)$, we obtain the arc length parametrization

$$\mathbf{r}(s) = \left(4\cos\left(\frac{s}{5}\right), 4\sin\left(\frac{s}{5}\right), \frac{3s}{5}\right) \quad s \in \mathbb{R}.$$

[1] student license available from Trinity for free

(b) Since we now have a unit speed parametrization, using Remark 1.4.2(b),

$$\mathbf{T}(s) = \mathbf{r}'(s) = \left(-\frac{4}{5}\sin\left(\frac{s}{5}\right), \frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right).$$

Hence

$$\mathbf{T}'(s) = \left(-\frac{4}{25}\cos\left(\frac{s}{5}\right), -\frac{4}{25}\sin\left(\frac{s}{5}\right), 0\right).$$

Thus

$$\kappa(s) = \|\mathbf{T}'(s)\| = \sqrt{\frac{4^2}{25^2}\cos^2\left(\frac{s}{5}\right) + \frac{4^2}{25^2}\sin^2\left(\frac{s}{5}\right)} = \sqrt{\frac{4^2}{25^2}} = \frac{4}{25}.$$

Note that the curvature is independent of the parametrization, as we would want for any reasonable measure of 'curvature'.

(c) We need to calculate $\mathbf{N}(s)$, $\mathbf{B}(s)$ and hence $\mathbf{B}'(s)$. Firstly

$$\begin{aligned}\mathbf{N}(s) &= \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} \\ &= \frac{1}{4/25}\left(-\frac{4}{25}\cos\left(\frac{s}{5}\right), -\frac{4}{25}\sin\left(\frac{s}{5}\right)\right) \\ &= \left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right).\end{aligned}$$

Next, using $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$,

$$\begin{aligned}\mathbf{B}(s) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4}{5}\sin\left(\frac{s}{5}\right) & \frac{4}{5}\cos\left(\frac{s}{5}\right) & \frac{3}{5} \\ -\cos\left(\frac{s}{5}\right) & -\sin\left(\frac{s}{5}\right) & 0 \end{vmatrix} \\ &= \mathbf{i}\begin{vmatrix} \frac{4}{5}\cos\left(\frac{s}{5}\right) & \frac{3}{5} \\ -\sin\left(\frac{s}{5}\right) & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -\frac{4}{5}\sin & \frac{3}{5} \\ -\cos\left(\frac{s}{5}\right) & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -\frac{4}{5}\sin\left(\frac{s}{5}\right) & \frac{4}{5}\cos\left(\frac{s}{5}\right) \\ -\cos\left(\frac{s}{5}\right) & -\sin\left(\frac{s}{5}\right) \end{vmatrix} \\ &= \frac{3}{5}\sin\left(\frac{s}{5}\right)\mathbf{i} - \frac{3}{5}\cos\left(\frac{s}{5}\right)\mathbf{j} + \left(\frac{4}{5}\sin^2\left(\frac{s}{5}\right) + \frac{4}{5}\cos^2\left(\frac{s}{5}\right)\right)\mathbf{k} \\ &= \left(\frac{3}{5}\sin\left(\frac{s}{5}\right), -\frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\right).\end{aligned}$$

Then

$$\mathbf{B}'(s) = \left(\frac{3}{25}\cos\left(\frac{s}{5}\right), \frac{3}{25}\sin\left(\frac{s}{5}\right), 0\right).$$

Finally,

$$
\begin{aligned}
\tau(s) &= -\,\mathbf{N}(s) \cdot \mathbf{B}'(s) \\
&= -\left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right) \cdot \left(\frac{3}{25}\cos\left(\frac{s}{5}\right), \frac{3}{25}\sin\left(\frac{s}{5}\right), 0\right) \\
&= -\left(-\frac{3}{25}\cos^2\left(\frac{s}{5}\right) - \frac{3}{25}\sin^2\left(\frac{s}{5}\right)\right) \\
&= \frac{3}{25}.
\end{aligned}
$$

Again, as we would want, the torsion does not depend on the parametrization.

(d) This time we will use the unit speed parametrization version of the Frenet-Serret equations, as given in Theorem 1.5.7. For each of the equations we will calculate the LHS and show it equals the RHS.

$$
\begin{aligned}
\kappa(s)\mathbf{N}(s) &= \frac{4}{25}\left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right) \\
&= \left(-\frac{4}{25}\cos\left(\frac{s}{5}\right), -\frac{4}{25}\sin\left(\frac{s}{5}\right), 0\right) \\
&= \mathbf{T}'(s),
\end{aligned}
$$

so the first equation holds.

Next

$$
\begin{aligned}
-\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s) &= -\frac{4}{25}\left(-\frac{4}{5}\sin\left(\frac{s}{5}\right), \frac{4}{5}\cos\left(\frac{s}{5}\right), \frac{3}{5}\right) \\
&\quad + \frac{3}{25}\left(\frac{3}{5}\sin\left(\frac{s}{5}\right), -\frac{3}{5}\cos\left(\frac{s}{5}\right), \frac{4}{5}\right) \\
&= \left(\frac{16}{125}\sin\left(\frac{s}{5}\right), -\frac{16}{125}\cos\left(\frac{s}{5}\right), -\frac{12}{125}\right) \\
&\quad + \left(\frac{9}{125}\sin\left(\frac{s}{5}\right), -\frac{9}{125}\cos\left(\frac{s}{5}\right), \frac{12}{125}\right) \\
&= \left(\frac{25}{125}\sin\left(\frac{s}{5}\right), -\frac{25}{125}\cos\left(\frac{s}{5}\right), 0\right) \\
&= \left(\frac{1}{5}\sin\left(\frac{s}{5}\right), -\frac{1}{5}\cos\left(\frac{s}{5}\right), 0\right).
\end{aligned}
$$

However, if we differentiate $\mathbf{N}(s)$, we get

$$
\mathbf{N}'(s) = \left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right)' = \left(\frac{1}{5}\sin\left(\frac{s}{5}\right), -\frac{1}{5}\cos\left(\frac{s}{5}\right), 0\right),
$$

so the second equation also holds.

Finally,

$$\begin{aligned}
-\tau(s)\mathbf{N}(s) &= -\frac{3}{25}\left(-\cos\left(\frac{s}{5}\right), -\sin\left(\frac{s}{5}\right), 0\right) \\
&= \left(\frac{3}{25}\cos\left(\frac{s}{5}\right), \frac{3}{25}\sin\left(\frac{s}{5}\right), 0\right) \\
&= \mathbf{B}'(s),
\end{aligned}$$

so the third equation also holds.

(e) The plot is shown in Figure 1. Note that the **TN**-plane is in red, the **TB**-plane is in green and the **NB**-plane is in yellow.

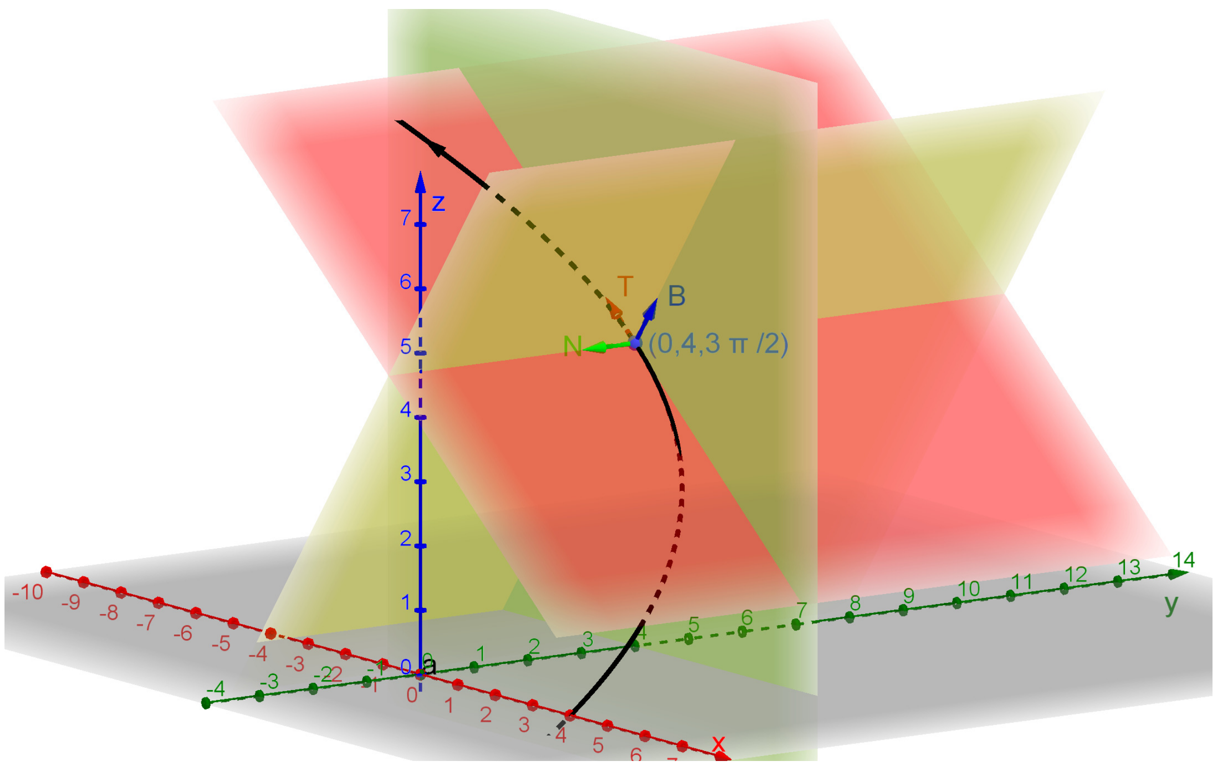


Figure 1:

7. A curve C in the xy-plane is represented by the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\,. \tag{10}$$

In the $x'y'$-plane obtained by rotating the xy-plane through an angle ϕ

$$x' = x\cos\phi + y\sin\phi\,, \quad y' = -x\sin\phi + y\cos\phi\,, \tag{11}$$

the curve C is represented by a similar equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0\,. \tag{12}$$

(a) Express A', B', C', D', E', F' in terms of A, B, C, D, E, F and ϕ.

Solution: Expressing x, y in terms of x', y', one gets

$$x = x' \cos\phi - y' \sin\phi\,, \quad y = x' \sin\phi + y' \cos\phi\,. \tag{13}$$

Substituting these expressions in (10), one finds

$$\begin{aligned}
A' &= A\cos^2(\phi) + B\sin(\phi)\cos(\phi) + C\sin^2(\phi)\,,\\
B' &= -A\sin(2\phi) + B\cos(2\phi) + C\sin(2\phi)\,,\\
C' &= A\sin^2(\phi) - B\sin(\phi)\cos(\phi) + C\cos^2(\phi)\,,\\
D' &= D\cos(\phi) + E\sin(\phi)\,,\\
E' &= E\cos(\phi) - D\sin(\phi)\,,\\
F' &= F\,.
\end{aligned} \tag{14}$$

(b) Prove that if the angle ϕ satisfies

$$\cot 2\phi = \frac{A - C}{B}\,, \tag{15}$$

then the curve C is represented by the equation

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0\,, \tag{16}$$

i.e. $B' = 0$.

Solution: Solving the equation

$$B' = -A\sin(2\phi) + B\cos(2\phi) + C\sin(2\phi) = 0 \tag{17}$$

for ϕ, one obviously finds (15).

8. A curve C is the intersection of the cone

$$z^2 = x^2 + y^2\,, \tag{18}$$

with a plane.

Identify the curve, find a parametric representation and plot the curve in the xyz-space for the planes below. The Mathematica function ParametricPlot3D can be used to plot parametric curves in the xyz-space.

(a) $z = 2\,.$

Solution: It is a circle of radius 2: $x = 2\cos t\,,\ y = 2\sin t\,,\ z = 2$

(b) $y = 0\,.$

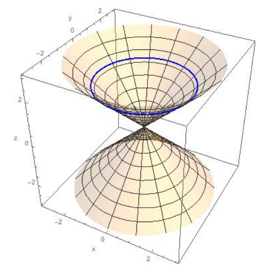
Solution: It is two intersecting lines: $y = 0\,,\ x = t\,,\ z = \pm t$

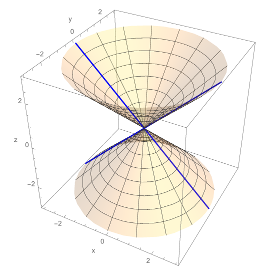
(c) $x = 1\,.$

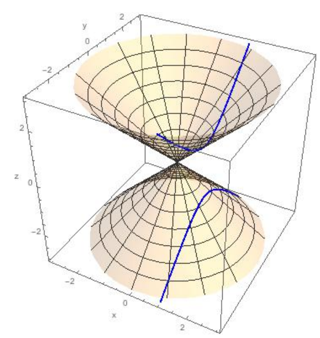
Solution: It is a hyperbola: $x = 1\,,\ y = \frac{1}{2}(t - \frac{1}{t})\,,\ z = \frac{1}{2}(t + \frac{1}{t})$

(d) $x + y = 1\,.$

Solution: It is a hyperbola: $x = \frac{1}{2} + \frac{1}{4}(t - \frac{1}{t})\,,\ y = \frac{1}{2} - \frac{1}{4}(t - \frac{1}{t})\,,\ z = \frac{1}{2^{3/2}}(t + \frac{1}{t})$







(e) $x + z = 1$.

Solution: It is a parabola: $x = \frac{1}{2} - \frac{t^2}{2}\,,\ y = t\,,\ z = 1 - t$

(f) $x + y + z = 1$.

Solution: It is a hyperbola: $x = 1 + t\,,\ y = 1 + \frac{1}{2t}\,,\ z = -1 - t - \frac{1}{2t}$

9. A curve C is the intersection of the cone

$$x^2 + y^2 - z^2 = 0\,, \tag{19}$$

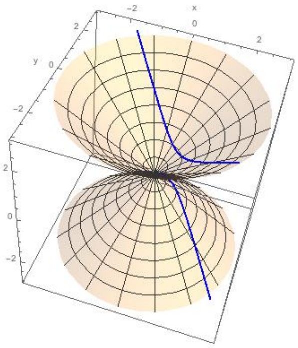
with the plane

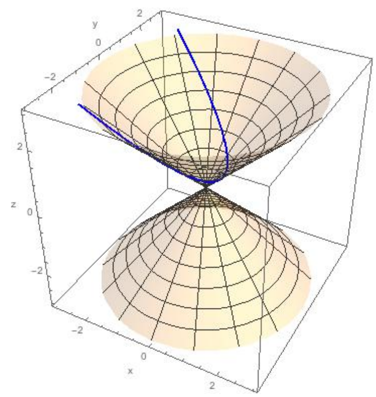
$$ax + by + cz + d = 0\,. \tag{20}$$

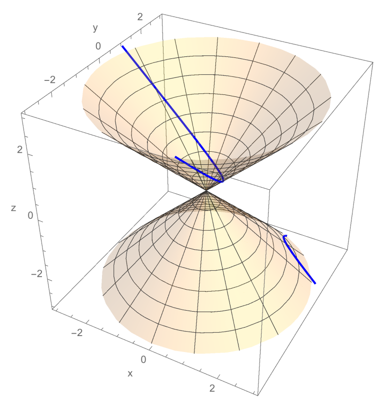
Prove that the curve C is one of these 1) a circle; 2) an ellipse; 3) a parabola; 4) a hyperbola; 5) a pair of intersecting lines; 6) a single line; 7) a point.

Solution: By means of a rotation and a shift of coordinates the plane $ax + by + cz + d = 0$ can become the $x'y'$-plane in the transformed coordinates, i.e. it is given by the equation $z' = 0$. Since rotations and shifts are linear transformations of coordinates the equation $x^2 + y^2 - z^2 = 0$ will take the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0\,, \tag{21}$$







where we took into account that $z' = 0$. The statement then follows from the result of Question 7.

10. Consider elliptic coordinates (u, v)

$$x = l\,u\,v\,, \quad y^2 = l^2(u^2-1)(1-v^2)\,, \quad u \geq 1\,,\ -1 \leq v \leq 1\,,\ l \text{ is a dimensionfull constant}$$

(a) Show that curves of constant u are ellipses, while curves of constant v are hyperbolae.

Solution: It follows from (Derive!)

$$\frac{x^2}{l^2u^2} + \frac{y^2}{l^2(u^2-1)} = 1\,, \quad \frac{x^2}{l^2v^2} - \frac{y^2}{l^2(1-v^2)} = 1\,. \tag{22}$$

(b) Show that in the elliptic coordinates a curve given by the parametric equations $u = u(t)$, $v = v(t)$ for $a \le t \le b$ has arc length

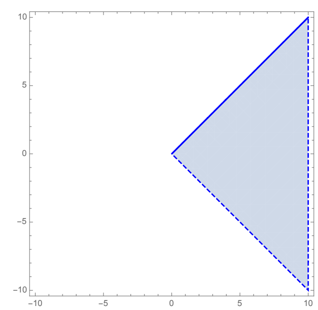
$$L = l \int_a^b \sqrt{\frac{u^2 - v^2}{u^2 - 1}\left(\frac{du}{dt}\right)^2 + \frac{u^2 - v^2}{1 - v^2}\left(\frac{dv}{dt}\right)^2}\, dt\,. \tag{23}$$

Solution: Straightforward computation

11. Sketch the domain of f (you may use Mathematica). Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included. Determine whether the domain is an open set, closed set, or neither.

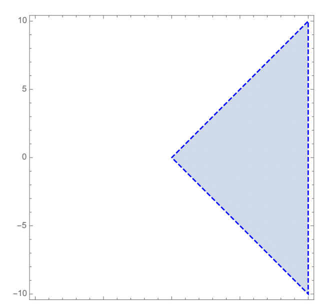
(a) $f(x,y) = \frac{\sqrt{x-y}}{\sqrt{x+y}}$

Solution: The domain consists of the points satisfying $x - y \ge 0\,,\ x + y > 0$, and it is neither an open set nor a closed set



(b) $f(x,y) = \frac{\ln(x+y)}{\sqrt{x-y}}$

Solution: The domain consists of the points satisfying $x - y > 0\,,\ x + y > 0$, and it is an open set

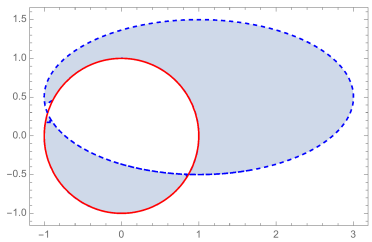


(c) $f(x,y) = \sqrt{\frac{x^2+y^2-1}{2-x^2+2x-4y^2+4y}}$

Solution: The domain consists of the points satisfying $x^2 + y^2 - 1 \ge 0\,,\ 2 - x^2 + 2x - 4y^2 + 4y > 0$ and $x^2 + y^2 - 1 \le 0\,,\ 2 - x^2 + 2x - 4y^2 + 4y < 0$, and it is neither an open set nor a closed set

12. Sketch the level curve $z = k$ for the specified values of k

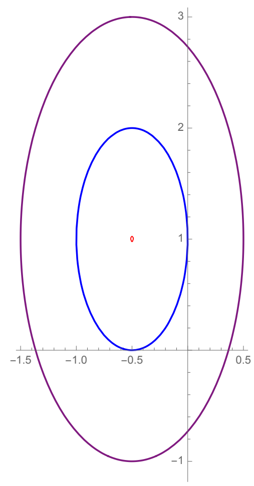
$$z = 4x^2 + 4x + y^2 - 2y\,, \quad k = -2, -1, 2\,.$$



Solution: The level curve equation $4x^2 + 4x + y^2 - 2y = k$ can be written as

$$4(x + \frac{1}{2})^2 + (y - 1)^2 = k + 2\,,$$

and, therefore, the level curves are ellipses with the centre located at $(-1/2, 1)$. For $k = -2$ the level curve is just the point $(-1/2, 1)$.



13. [Problem shifted to second homework. Solution therefore not shown] Determine whether the limit exists. If so, find its value

    (a)
    $$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{e^{x^2+y^2} - 1}$$

    (b)
    $$\lim_{(x,y)\to(1,-1)} \frac{1 - \cosh(x+y)}{\sin(x^2 - y^2)\ln(\frac{2x}{x-y})}$$

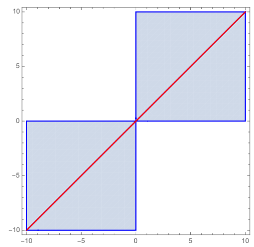
    (c)
    $$\lim_{(x,y)\to(0,0)} \frac{3 + \cos(2x) - 4\cosh(y)}{1 - \sqrt[4]{1 + x^2 + y^2}}$$

14. Consider the function

$$f(x,y) = \int_x^y \frac{dt}{\sqrt{\left(t^2 - x^2\right)\left(y^2 - t^2\right)}}$$

(a) Sketch the domain of f. Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included. Determine whether the domain is an open set, closed set, or neither.

Solution: The integral gives a real number if $\left(t^2 - x^2\right)\left(y^2 - t^2\right) > 0$ for all $x \le t \le y$ if $x < y$ or for all $y \le t \le x$ if $y < x$. It is easy to see that the solutions are $x > 0, y > 0, x \neq y$ and $x < 0, y < 0, x \neq y$, i.e. these are points in the first and third quadrants excluding the points on the line $y = x$, see the picture



(b) Assume that $0 < x < y$, and determine whether the limit exists. If so, find its value.

[Treated as bonus]If the limit does not exist find the leading behaviour of the function in a neighbourhood of the limiting point, e.g. $\lim_{(x,y)\to(0,0)} \ln(\sqrt{1+(x+y)^2} - 1)$ does not exist and the leading behaviour of this function in a neighbourhood of $(0,0)$ is $2\ln(x+y)$.

i.
$$\lim_{(x,y)\to(0,0)} f(x,y)$$

Solution: The limit does not exist as can be seen by making the substitution $t = yz$. Then
$$f(x,y) = \frac{1}{y}\int_{x/y}^{1} \frac{dz}{\sqrt{\left(z^2 - \frac{x^2}{y^2}\right)\left(1 - z^2\right)}} . \tag{24}$$

Assuming now that we approach $(0,0)$ along the line $x/y = \alpha$, $\alpha < 1$ we see that the integral
$$\int_{\alpha}^{1} \frac{dz}{\sqrt{\left(z^2 - \alpha^2\right)\left(1 - z^2\right)}}$$
exist and therefore $f(x,y)$ diverges as $1/y$ (or $1/x$) as $(x,y) \to (0,0)$. In fact,
$$\int_{\alpha}^{1} \frac{dz}{\sqrt{\left(z^2 - \alpha^2\right)\left(1 - z^2\right)}} = K(1-\alpha^2)\,,$$
where $K(x)$ is the complete elliptic integral of the first kind.

ii.
$$\lim_{(x,y)\to(1,1)} f(x,y)$$

Solution: We need to understand if there is a limit of $\int_\alpha^1 \frac{dz}{\sqrt{\left(z^2-\alpha^2\right)\left(1-z^2\right)}}$ as $\alpha \to 1$.

To this end we make the substitution

$$z = \epsilon\, w + 1 - \epsilon\,, \quad \epsilon = 1 - \alpha\,. \tag{25}$$

Then the integral takes the form

$$\int_\alpha^1 \frac{dz}{\sqrt{\left(z^2-\alpha^2\right)\left(1-z^2\right)}} = \int_0^1 \frac{dw}{\sqrt{w\left(1-w\right)\left(2+\epsilon w-2)\right)\left(2+\epsilon(w-1)\right)}}\,. \tag{26}$$

Now the limit $\epsilon \to 0$ is straightforward and one gets

$$\begin{aligned} \lim_{(x,y)\to(1,1)} f(x,y) &= \lim_{\epsilon\to 0}\int_0^1 \frac{dw}{\sqrt{w\left(1-w\right)\left(2+\epsilon(w-2)\right)\left(2+\epsilon(w-1)\right)}} \\ &= \int_0^1 \frac{dw}{2\sqrt{w\left(1-w\right)}} = \frac{\pi}{2}\,. \end{aligned} \tag{27}$$

iii.

$$\lim_{(x,y)\to(0,1)} f(x,y)$$

Solution: Setting $\alpha = 0$ one gets the integral

$$\int_0^1 \frac{dz}{z\sqrt{1-z^2}}$$

which is obviously divergent. To find the leading behaviour of $f(x,y)$ we expand $1/\sqrt{1-z^2}$ in the Taylor series in powers of z

$$\int_\alpha^1 \frac{dz}{\sqrt{\left(z^2-\alpha^2\right)\left(1-z^2\right)}} = \int_\alpha^1 \frac{dz}{\sqrt{z^2-\alpha^2}} + \int_\alpha^1 \frac{z^2\,dz}{2\sqrt{z^2-\alpha^2}} + \cdots \tag{28}$$

It is now clear that only the first integral produces a divergent result as $\alpha \to 0$. Computing it one gets

$$\int_\alpha^1 \frac{dz}{\sqrt{z^2-\alpha^2}} = \log\left(\frac{\sqrt{1-\alpha^2}+1}{\alpha}\right) \to -\log\alpha \quad \text{as} \quad \alpha \to 0\,. \tag{29}$$

Thus the leading behaviour of $f(x,y)$ is

$$f(x,y) \to -\log x + \text{subleading} \tag{30}$$