6.6 Partial fractions: General case

Definition 6.23 – Proper rational function

A proper rational function is a quotient of two polynomials $P(x)/Q(x)$ such that the degree of the numerator $P(x)$ is smaller than the degree of the denominator $Q(x)$.

Theorem 6.24 – Partial fractions, part 1

Suppose that $f(x)$ is a proper rational function whose denominator is the product of relatively prime polynomials. Then $f(x)$ can be expressed as a sum of proper rational functions whose denominators are these relatively prime polynomials.

- Two polynomials are relatively prime, if they do not have any common divisor other than constant factors. For instance, $(x+1)(x-1)$ and $x^2(x+3)$ are relatively prime, whereas $x(x-1)$ and $x^2(x+3)$ have a non-constant factor in common.

Example 6.25 We use partial fractions to compute the integral

$$\int \frac{x^2+3x-4}{(x^2+1)(x+1)}\,dx.$$

According to the last theorem, the integrand can be expressed in the form

$$\frac{x^2+3x-4}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$x^2+3x-4 = (Ax+B)(x+1) + C(x^2+1)$$

and one may look at some suitable choices of x to find that

$$x = -1, 0, 1 \quad\Longrightarrow\quad -6 = 2C, \qquad -4 = B + C, \qquad 0 = 2A + 2B + 2C.$$

Solving these equations, we now get $C = -3$, $B = -1$ and $A = 4$, which means that

$$\int \frac{x^2+3x-4}{(x^2+1)(x+1)}\,dx = \int \frac{4x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx - \int \frac{3}{x+1}\,dx.$$

The two rightmost integrals are rather easy to compute, and so is the integral

$$\int \frac{4x}{x^2+1}\,dx = \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln(x^2+1) + C,$$

if one substitutes $u = x^2+1$. In view of the last two equations, we must thus have

$$\int \frac{x^2+3x-4}{(x^2+1)(x+1)}\,dx = 2\ln(x^2+1) - \tan^{-1}x - 3\ln|x+1| + C. \qquad \square$$

Example 6.26 We use partial fractions to compute the integral

$$\int \frac{x^3+3x^2+5}{x(x-1)}\,dx.$$

This rational function is not proper because its numerator is cubic and its denominator is only quadratic. Thus, one needs to first use division of polynomials to write

$$\frac{x^3+3x^2+5}{x(x-1)} = \frac{x^3+3x^2+5}{x^2-x} = x+4+\frac{4x+5}{x(x-1)}.$$

Since the rightmost fraction is proper, one may use partial fractions to express it as

$$\frac{4x+5}{x(x-1)} = \frac{A}{x}+\frac{B}{x-1} \qquad \Longrightarrow \qquad 4x+5 = A(x-1)+Bx.$$

Setting $x=0$ gives $5=-A$ and setting $x=1$ gives $9=B$. It easily follows that

$$\frac{x^3+3x^2+5}{x(x-1)} = x+4+\frac{A}{x}+\frac{B}{x-1} = x+4-\frac{5}{x}+\frac{9}{x-1}.$$

Once we now integrate this equation term by term, we may finally conclude that

$$\int \frac{x^3+3x^2+5}{x(x-1)}\,dx = \frac{x^2}{2}+4x-5\ln|x|+9\ln|x-1|+C. \qquad \square$$

Example 6.27 We use a substitution and partial fractions to compute the integral

$$\int \frac{e^{5x}\,dx}{e^{2x}-1}.$$

If we take $u=e^x$, then $du=e^x\,dx$ and the given integral takes the form

$$\int \frac{e^{5x}\,dx}{e^{2x}-1} = \int \frac{e^{4x}\cdot e^x\,dx}{e^{2x}-1} = \int \frac{u^4\,du}{u^2-1}.$$

This is not a proper rational function, so one needs to first use division to write

$$\int \frac{e^{5x}\,dx}{e^{2x}-1} = \int \frac{u^4-1+1}{u^2-1}\,du = \int \left(u^2+1+\frac{1}{u^2-1}\right)du. \tag{6.8}$$

Let us merely focus on the proper rational function. Using partial fractions, we get

$$\frac{1}{u^2-1} = \frac{A}{u-1}+\frac{B}{u+1} \qquad \Longrightarrow \qquad 1 = A(u+1)+B(u-1).$$

When $u=1$, this gives $1=2A$. When $u=-1$, it gives $1=-2B$. In particular, one has

$$u^2+1+\frac{1}{u^2-1} = u^2+1+\frac{A}{u-1}+\frac{B}{u+1} = u^2+1+\frac{1/2}{u-1}-\frac{1/2}{u+1}$$

and each of these terms can be easily integrated. Returning to (6.8), we conclude that

$$\begin{aligned}\int \frac{e^{5x}\,dx}{e^{2x}-1} &= \frac{1}{3}\,u^3+u+\frac{1}{2}\,\ln|u-1|-\frac{1}{2}\,\ln|u+1|+C \\ &= \frac{1}{3}\,e^{3x}+e^x+\frac{1}{2}\,\ln|e^x-1|-\frac{1}{2}\,\ln(e^x+1)+C. \qquad \square\end{aligned}$$

6.7 Partial fractions: Special case

Theorem 6.28 – Partial fractions, part 2

Every proper rational function whose denominator $Q(x)$ is the product of linear and quadratic factors can be expressed as the sum of fractions that have the form

$$\frac{A_j}{(ax+b)^j}, \qquad \frac{A_j x + B_j}{(ax^2+bx+c)^j}. \tag{6.9}$$

The former fraction arises for each $1 \leq j \leq m$ whenever $(ax+b)^m$ divides $Q(x)$, while the latter fraction arises for each $1 \leq j \leq m$ whenever $(ax^2+bx+c)^m$ divides $Q(x)$.

- When a proper rational function is decomposed into fractions of the form (6.9), the denominators are known in advance, but the numerators need to be determined.

Example 6.29 We use partial fractions to compute the integral

$$\int \frac{3x^2-4}{x(x-1)^2}\,dx.$$

According to the last theorem, the integrand can be expressed in the form

$$\frac{3x^2-4}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$3x^2 - 4 = A(x-1)^2 + Bx(x-1) + Cx$$

and one may look at some suitable choices of x to find that

$$x = -1, 0, 1 \quad \Longrightarrow \quad -1 = 4A + 2B - C, \qquad -4 = A, \qquad -1 = C.$$

Since $2B = C - 4A - 1 = 14$, we get $B = 7$ and the partial fractions decomposition reads

$$\int \frac{3x^2-4}{x(x-1)^2}\,dx = -\int \frac{4}{x}\,dx + \int \frac{7}{x-1}\,dx - \int \frac{1}{(x-1)^2}\,dx.$$

The integrals on the right hand side are all easy to compute and one finds that

$$\int \frac{3x^2-4}{x(x-1)^2}\,dx = -4\ln|x| + 7\ln|x-1| + (x-1)^{-1} + C. \qquad \square$$

Example 6.30 We use partial fractions to compute the integral

$$\int \frac{2x-5}{x^2(x^2+2x+2)}\,dx.$$

According to the last theorem, the integrand can be expressed in the form

$$\frac{2x-5}{x^2(x^2+2x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2x+2} \tag{6.10}$$

for some constants A, B, C, D that need to be determined. Clearing denominators gives

$$2x-5 = Ax(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)x^2$$

and this identity must hold for all x. One may look at some suitable choices of x, as usual, or else compare coefficients of x. In this case, the latter approach is easier and one has

$$A+C=0, \qquad 2A+B+D=0, \qquad 2A+2B=2, \qquad 2B=-5.$$

Solving the last system of equations, it is now easy to show that

$$B=-5/2, \qquad A=1-B=7/2, \qquad D=-B-2A=-9/2, \qquad C=-A=-7/2.$$

In particular, the partial fractions decomposition (6.10) has the form

$$\frac{2x-5}{x^2(x^2+2x+2)} = \frac{7/2}{x} - \frac{5/2}{x^2} + \frac{-7x/2-9/2}{x^2+2x+2}.$$

The first two terms on the right hand side can be easily integrated and one finds that

$$\int \frac{2x-5}{x^2(x^2+2x+2)}\,dx = \frac{7}{2}\ln|x| + \frac{5}{2x} - \frac{1}{2}\int \frac{7x+9}{x^2+2x+2}\,dx. \tag{6.11}$$

We now focus on the remaining part. If we let $u=x^2+2x+2$, then $du=2(x+1)\,dx$ and this substitution is useful for numerators that are multiples of $x+1$ such as

$$\int \frac{7(x+1)}{x^2+2x+2}\,dx = \frac{7}{2}\int \frac{du}{u} = \frac{7}{2}\ln|u| = \frac{7}{2}\ln|x^2+2x+2|.$$

Comparing the last two equations, one may thus establish the identity

$$\begin{aligned}\int \frac{2x-5}{x^2(x^2+2x+2)}\,dx &= \frac{7}{2}\ln|x| + \frac{5}{2x} - \frac{1}{2}\int \frac{7x+7}{x^2+2x+2}\,dx - \int \frac{dx}{x^2+2x+2} \\ &= \frac{7}{2}\ln|x| + \frac{5}{2x} - \frac{7}{4}\ln|x^2+2x+2| - \int \frac{dx}{x^2+2x+2}.\end{aligned}$$

To compute the rightmost integral, one needs to complete the square and write

$$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{(x+1)^2+1} = \int \frac{du}{u^2+1},$$

where $u=x+1$. This integral is then $\tan^{-1} u = \tan^{-1}(x+1)$, so we finally get

$$\int \frac{2x-5}{x^2(x^2+2x+2)}\,dx = \frac{7}{2}\ln|x| + \frac{5}{2x} - \frac{7}{4}\ln|x^2+2x+2| - \tan^{-1}(x+1) + C. \qquad \square$$

Chapter 7

Sequences and series

7.1 Convergence of sequences

Definition 7.1 – Convergence of sequences

A sequence is a function that is defined on the set $\mathbb{N}$ of natural numbers. Its values are usually denoted by writing a_n for each $n \in \mathbb{N}$. We say that the sequence $\{a_n\}$ converges, if a_n approaches a finite limit as $n \to \infty$. Otherwise, we say that $\{a_n\}$ diverges.

Definition 7.2 – Monotonicity

A sequence $\{a_n\}$ is called monotonic, if it is either increasing, in which case $a_n \leq a_{n+1}$ for each $n \in \mathbb{N}$, or else decreasing, in which case $a_n \geq a_{n+1}$ for each $n \in \mathbb{N}$.

Theorem 7.3 – Monotonic and bounded

If a sequence is monotonic and bounded, then the sequence is also convergent.

- When a precise formula for a_n is known, one may use that formula to compute the limit of a_n and prove convergence. However, a precise formula is not always available.

Example 7.4 We show that each of the following sequences converges.

$$a_n = \sqrt{\frac{8n^2+3}{2n^2+5}}, \qquad b_n = \frac{3+\sin n}{n^2}, \qquad c_n = \left(1+\frac{1}{n}\right)^n.$$

Since the limit of a square root is the square root of the limit, it should be clear that

$$\lim_{n\to\infty} \frac{8n^2+3}{2n^2+5} = \lim_{n\to\infty} \frac{8n^2}{2n^2} = 4 \quad\Longrightarrow\quad \lim_{n\to\infty} a_n = \sqrt{4} = 2.$$

The limit of the second sequence is zero because $2/n^2 \leq b_n \leq 4/n^2$ for each $n \geq 1$. This means that b_n is squeezed between two sequences that converge to zero. Finally, one has

$$c_n = \left(1+\frac{1}{n}\right)^n \quad\Longrightarrow\quad \ln c_n = n \cdot \ln\left(1+\frac{1}{n}\right) = \frac{\ln(1+1/n)}{1/n}.$$

This is a limit of the form 0/0, so one may use L'Hôpital's rule to conclude that

$$\lim_{n\to\infty} \ln c_n = \lim_{n\to\infty} \frac{(1+1/n)^{-1} \cdot (1/n)'}{(1/n)'} = 1 \quad \Longrightarrow \quad \lim_{n\to\infty} c_n = e^1 = e. \qquad \square$$

Example 7.5 There are two different ways of checking that $a_n = n/(n+1)$ is increasing. First of all, one may use derivatives. If we define $f(x) = x/(x+1)$ for each $x \geq 1$, then

$$f'(x) = \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0.$$

This makes $f(x)$ increasing for all $x \geq 1$ and thus a_n is increasing for all $n \geq 1$. It is also possible to check this directly. To show that a_n is increasing, one needs to show that

$$a_n \leq a_{n+1} \quad \Longleftrightarrow \quad \frac{n}{n+1} \leq \frac{n+1}{n+2} \quad \Longleftrightarrow \quad n^2 + 2n \leq n^2 + 2n + 1.$$

Since the rightmost inequality is obviously true, the leftmost inequality holds as well. $\square$

Example 7.6 We show that $a_n = \frac{2^n}{n!}$ is decreasing for all $n \geq 1$. In this case, we have

$$a_n \geq a_{n+1} \quad \Longleftrightarrow \quad \frac{2^n}{n!} \geq \frac{2^{n+1}}{(n+1)!} \quad \Longleftrightarrow \quad n+1 \geq 2.$$

Since the rightmost inequality is obviously true, the leftmost inequality holds as well. $\square$

Example 7.7 We find the limit of the sequence $\{a_n\}$ which is defined by $a_1 = 1$ and

$$a_{n+1} = \sqrt{2a_n} \quad \text{for each } n \geq 1.$$

To show that this sequence converges, we shall first show that

$$1 \leq a_n \leq a_{n+1} \leq 2 \quad \text{for each } n \geq 1. \tag{7.1}$$

When $n = 1$, this statement asserts that $1 \leq 1 \leq \sqrt{2} \leq 2$, so it is certainly true. Suppose that it is true for some n. Multiplying by 2 and taking square roots, we then find that

$$\begin{aligned} 2 \leq 2a_n \leq 2a_{n+1} \leq 4 \quad &\Longrightarrow \quad \sqrt{2} \leq \sqrt{2a_n} \leq \sqrt{2a_{n+1}} \leq 2 \\ &\Longrightarrow \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 2. \end{aligned}$$

In particular, the statement holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, one may then argue that

$$a_{n+1} = \sqrt{2a_n} \quad \Longrightarrow \quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} \quad \Longrightarrow \quad L = \sqrt{2L}.$$

This gives $L^2 = 2L$, so either $L = 0$ or else $L = 2$. On the other hand, we must also have

$$1 \leq a_n \leq 2 \quad \Longrightarrow \quad 1 \leq \lim_{n\to\infty} a_n \leq 2 \quad \Longrightarrow \quad 1 \leq L \leq 2$$

because of equation (7.1). We conclude that the limit of the sequence is $L = 2$. $\square$

7.2 Convergence of series

Definition 7.8 – Partial sums

Given a sequence $\{a_n\}$, we define the sequence of its partial sums $\{s_n\}$ by

$$s_n = a_1 + a_2 + \ldots + a_n \quad \text{for each } n \geq 1.$$

Definition 7.9 – Infinite series

Let $\{a_n\}$ be a given sequence and let $\{s_n\}$ be the sequence of its partial sums. In the case that $\{s_n\}$ happens to converge, one may introduce the series

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} s_N$$

and we say that the series converges. Otherwise, we say that the series diverges.

Theorem 7.10 – nth term test

If the series $\sum_{n=1}^{\infty} a_n$ converges, then its nth term must satisfy $\lim_{n\to\infty} a_n = 0$.

Theorem 7.11 – Geometric series

The geometric series $\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$, in which case

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}.$$

- There are very few series whose exact value can be determined explicitly. Thus, we shall mainly worry about the convergence of a series and not its exact value.
- The nth term test implies the divergence of the series $\sum_{n=1}^{\infty} a_n$ whenever a_n fails to approach zero, but it does not provide any conclusions when a_n approaches zero.

Example 7.12 In view of the nth term test, each of the following series is divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n+1}, \qquad \sum_{n=1}^{\infty} \frac{2n+1}{3n+2}, \qquad \sum_{n=1}^{\infty} (-1)^n, \qquad \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n. \qquad \square$$

Example 7.13 We use the formula for a geometric series to compute the sum

$$S = \sum_{n=1}^{\infty} \frac{2^{n+2}}{3^{2n+1}}.$$

If we first isolate the part of the exponent that depends on n, then we can simplify to get

$$S = \frac{4}{3} \cdot \sum_{n=1}^{\infty} \frac{2^n}{3^{2n}} = \frac{4}{3} \cdot \sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^n = \frac{4}{3} \cdot \frac{2/9}{1-2/9} = \frac{8}{21}. \qquad \square$$