Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 5
Each set of homework questions is worth 100 marks

Problem 1

Consider a particle moving on the surface of a paraboloid $z = k(x^2 - 4x + y^2 + 2y)$, $k > 0$ in a uniform gravitational field $\vec{F} = \{0, 0, -mg\}$.

(a) Find conserved quantities and use them to reduce the system to a one-dimensional one.

(b) Find and sketch the effective potential.

(c) Find the conditions for the trajectory of the particle to be a circle, and find the period of revolution. How does it depend on the energy?

(d) Find the conditions for the trajectory of the particle to be a segment of a parabola, and use Mathematica to find the period of oscillations, and expand it at small and large values of the energy, and also at small and large values of k. Explain the results obtained.

(e) Use Mathematica to plot $t = t(r)$, $r = r(t)$, $\phi = \phi(r)$, $r = r(\phi)$ for some values of the parameters.

Problem 2

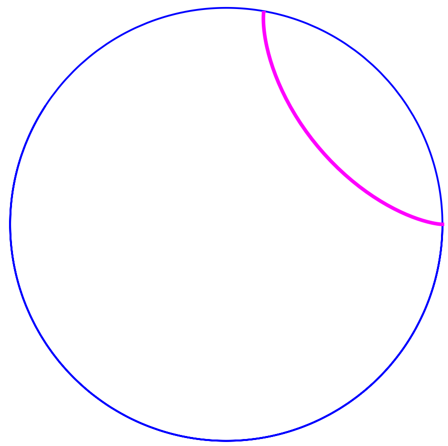
(a) Derive the formula for the deflection angle of a particle scattered by a central field which asymptotes to a constant at infinity.

(b) Rewrite the formula obtained in terms of the impact parameter and the velocity of the particle at infinity.

(c) Consider a particle moving in the following central field

$$U = \frac{\alpha - \beta\, r - \gamma\, r^2}{r^2}\,, \quad \alpha > 0\,,\ \beta > 0\,, \gamma > 0\,.$$

  (i) Sketch the potential.
  (ii) Where is it a repulsive field? Where is it an attractive field?
  (iii) Sketch the effective potential.
  (iv) What types of motion are possible? Find the turning points.
  (v) Find the deflection angle of the particle if it is scattered by this central field, and express it as a function of the impact parameter and the velocity of the particle at infinity.



Bonus question (each bonus question is worth extra 25 marks)

Find the shape of a tunnel drilled through the Moon such that the travel time between two points on the surface of the Moon under the force of gravity is minimized. Assume the Moon is spherical and homogeneous.

Hint: Prove that the shape is the hypocycloid

$$x(\theta) = (R-r)\cos\frac{r}{R}\theta + r\cos\frac{R-r}{R}\theta$$
$$y(\theta) = (R-r)\sin\frac{r}{R}\theta - r\sin\frac{R-r}{R}\theta$$

See also

http://mathworld.wolfram.com/Hypocycloid.html