Mechanics 2341, PS 6

Problem 2.

A beam of particles moving along the z − axis from z = −∞ is scattered by a perfectly rigid paraboloid $z = a\left(x^2 + y^2\right)$.
Find the differential cross − section for this scattering.

```
$Assumptions = {{m, g, En} ∈ Reals};
```

Functions

```
x[t_]; y[t_]; z[t_]; r[t_]; phi[t_];
```

Paraboloid

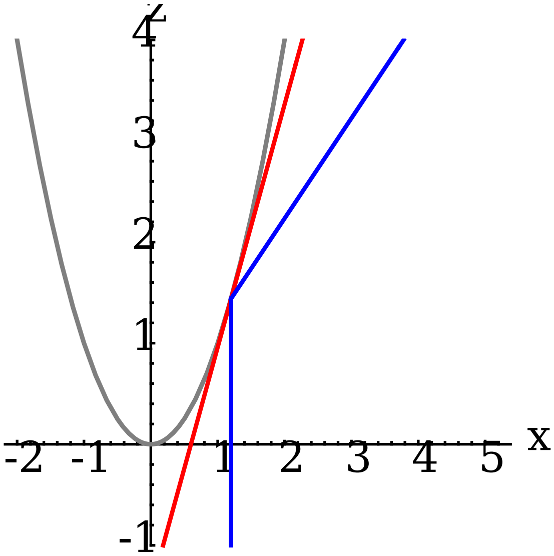
```
z[t] = a (x[t] ^2 + y[t] ^2);
```

```
Plot3D[x^2 + y^2, {x, -2, 2}, {y, -2, 2},
 RegionFunction → Function[{x, y, z}, x^2 + y^2 ≤ 4],
 BoxRatios → Automatic, AxesLabel → {x, y, z}]
```

Consider a particle moving in the xz-plane. Black curve is a parabola z/a = x^2(with a=1), and it is intersection of the paraboloid with the xz-plane. Blue curve is the path of the particle. Before the scattering it is x = x0, and after the scattering it is z/a = (x0-1/4 1/x0)(x - x0)+x0^2 . Red curve is the tangent line to the parabola at x = x0.

```
x0 = 1.2;
```

```
ParametricPlot[{{x, x^2}, {x + 2, 2 x0 (x + 2 - x0) + x0^2},
  {x0, x - 2 + x0^2}, {x + 2 + x0, (x0 - 1 / 4 1 / x0) (x + 2 ) + x0^2}}, {x, -2, 2},
 {AxesLabel → {x, z}}, PlotStyle → {{Thickness[0.009], RGBColor[0.5, 0.5, 0.5]},
   {Thickness[0.0085], RGBColor[1, 0, 0]}, {Thickness[0.0085], RGBColor[0, 0, 1]},
   {Thickness[0.0085], RGBColor[0, 0, 1]}}, PlotRange → {-1, 4}, AspectRatio → 1]
```



- Graphics -

ρ (which is equal to x0) as a function of deflection angle χ

```
ρ = 1 / (2 a) Cot[χ / 2]
```

$$\frac{\mathrm{Cot}\left[\frac{\chi}{2}\right]}{2\,a}$$

Differential cross-section

```
dσ = -FullSimplify[ 2 Pi ρ D[ρ, χ] dχ]
```

$$\frac{\mathrm{d}\chi\,\pi\,\mathrm{Cot}\left[\frac{\chi}{2}\right]\,\mathrm{Csc}\left[\frac{\chi}{2}\right]^2}{4\,a^2}$$

```
dΩ = 2 Pi Sin[χ] dχ;
Simplify[dσ / dΩ]
```

$$\frac{\mathrm{Csc}\left[\frac{\chi}{2}\right]^4}{16\,a^2}$$

So, $d\sigma = \dfrac{\mathrm{Csc}\left[\frac{\chi}{2}\right]^4}{16\,a^2}\,d\Omega$, and the scattering is not isotropic.

Problem 4.

Determine the differential cross - section for small - angle scattering in a field

U[r, a, κ] = κ/(r^2 + a^2)^(1/2)
where κ and a are constants.

```
$Assumptions = {{m1, a, ρ} ∈ Reals, ρ > 0, a > 0, κ > 0};
```

Potential

```
U[r_, a_, κ_] = κ / (r^2 + a^2)^(1/2)
```

$$\frac{\kappa}{\sqrt{a^2+r^2}}$$

The angle of scattering θ_1 as a function of ρ

```
θ1 = -2 ρ / (m1 v∞^2) Integrate[D[U[r, a, κ], r] / (r^2 - ρ^2)^(1/2), {r, ρ, Infinity}]
```

$$\frac{2\,\kappa\,\rho}{\left(a^2+\rho^2\right)\,m_1\,v_\infty^2}$$

ρ as a function of θ_1-> θ

$$\mathbf{Solve}\left[\frac{2\,\kappa\,\rho}{\left(a^2+\rho^2\right)\,m_1\,v_\infty^2}==\theta,\ \rho\right]$$

$$\left\{\left\{\rho\to\frac{\kappa-\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}}{\theta\,m_1\,v_\infty^2}\right\},\ \left\{\rho\to\frac{\kappa+\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}}{\theta\,m_1\,v_\infty^2}\right\}\right\}$$

We have to choose the second solution because at a=0 the potential is the Coulomb one, and in tht case $\rho \sim 1/\theta$

$$\rho=\frac{\kappa+\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}}{\theta\,m_1\,v_\infty^2};$$

Differential cross-section as a function of θ

```
dσ = -FullSimplify[2 Pi ρ D[ρ, θ] dθ]
```

```
General::spell : Possible spelling error: new symbol name "dθ" is similar to existing symbols
    {dσ, dχ}. More…
```

$$\frac{2\,d\theta\,\pi\,\kappa\left(\kappa+\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}\right)^2}{\theta^3\,m_1^2\,v_\infty^4\,\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}}$$

```
dΩ = 2 Pi θ dθ;
```

```
General::spell : Possible spelling error: new symbol name "dΩ" is similar to existing symbols
    {dθ, dσ, dχ}. More…
```

```
Simplify[dσ / dΩ]
```

$$\frac{\kappa\left(\kappa+\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}\right)^2}{\theta^4\,m_1^2\,v_\infty^4\,\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}}$$

So, $d\sigma = \dfrac{\kappa\left(\kappa+\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}\right)^2}{\theta^4\,m_1^2\,v_\infty^4\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}}\,d\Omega$, and the scattering is not isotropic.

$$\text{Simplify}\left[\text{Series}\left[\frac{\kappa\left(\kappa+\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}\right)^2}{\theta^4\,m_1^2\,v_\infty^4\sqrt{\kappa^2-a^2\,\theta^2\,m_1^2\,v_\infty^4}},\ \{a,0,4\}\right]\right]$$

$$\frac{4\,\kappa^2}{\theta^4\,m_1^2\,v_\infty^4}+\frac{m_1^2\,v_\infty^4\,a^4}{4\,\kappa^2}+O[a]^5$$

The first term agrees with the small scattering angle expansion of Rutherford's formula

U = α/r^2

```
Solve[1 - ρ^2/r^2 - 2α/(m v^2 r^2) == 0, r]
```

$$\left\{\left\{r\to-\frac{\sqrt{2\,\alpha+m\,v^2\,\rho^2}}{\sqrt{m}\,v}\right\},\left\{r\to\frac{\sqrt{2\,\alpha+m\,v^2\,\rho^2}}{\sqrt{m}\,v}\right\}\right\}$$

```
Assuming[{m > 0, ρ > 0, α > 0, v > 0}, Integrate[
  ρ/r^2/(1 - ρ^2/r^2 - 2α/(m v^2 r^2))^(1/2), {r, Sqrt[2 α + m v^2 ρ^2]/(Sqrt[m] v), Infinity}]]
```

$$\frac{\pi\,\rho}{2\sqrt{\frac{2\,\alpha}{m\,v^2}+\rho^2}}$$

```
ρ/r^2/(1 - ρ^2/r^2 - 2α/(m v^2 r^2))^(1/2)
```

$$\frac{\rho}{r^2\sqrt{1-\frac{2\,\alpha}{m\,r^2\,v^2}-\frac{\rho^2}{r^2}}}$$