MA1125 – Calculus
Tutorial solutions #8

1. Compute each of the following indefinite integrals.

$$\int \frac{x^2}{x^3+1}\,dx, \qquad \int \frac{x^2}{x+1}\,dx.$$

For the first integral, we use the substitution $u = x^3 + 1$. Since $du = 3x^2\,dx$, we get

$$\int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int \frac{du}{u} = \frac{\ln|u|}{3} + C = \frac{\ln|x^3+1|}{3} + C.$$

For the second integral, we let $u = x + 1$. This gives $du = dx$, so it easily follows that

$$\begin{aligned}\int \frac{x^2}{x+1}\,dx &= \int \frac{(u-1)^2}{u}\,du = \int \frac{u^2-2u+1}{u}\,du = \int \left(u - 2 + \frac{1}{u}\right) du \\ &= \frac{u^2}{2} - 2u + \ln|u| + C = \frac{(x+1)^2}{2} - 2(x+1) + \ln|x+1| + C.\end{aligned}$$

2. Compute each of the following indefinite integrals.

$$\int \sin^2 x \cdot \cos^3 x\,dx, \qquad \int \sec^5 x \cdot \tan x\,dx.$$

For the first integral, we use the substitution $u = \sin x$. Since $du = \cos x\,dx$, we get

$$\begin{aligned}\int \sin^2 x \cdot \cos^3 x\,dx &= \int \sin^2 x \cdot \cos^2 x \cdot \cos x\,dx = \int u^2(1-u^2)\,du \\ &= \int (u^2 - u^4)\,du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.\end{aligned}$$

For the second integral, we let $u = \sec x$. This gives $du = \sec x \tan x\,dx$ and so

$$\int \sec^5 x \cdot \tan x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{\sec^5 x}{5} + C.$$

3. Compute each of the following indefinite integrals.

$$\int \sin^{-1} x\, dx, \qquad \int e^{\sqrt{x}}\, dx.$$

For the first integral, let $u = \sin^{-1} x$ and $dv = dx$. Then $du = \frac{dx}{\sqrt{1-x^2}}$ and $v = x$, so

$$\int \sin^{-1} x\, dx = uv - \int v\, du = x \sin^{-1} x - \int \frac{x\, dx}{\sqrt{1-x^2}}.$$

To compute the rightmost integral, we let $w = 1 - x^2$. This gives $dw = -2x\, dx$ and

$$\begin{aligned} \int \sin^{-1} x\, dx &= x \sin^{-1} x + \frac{1}{2} \int \frac{dw}{\sqrt{w}} = x \sin^{-1} x + \frac{1}{2} \int w^{-1/2}\, dw \\ &= x \sin^{-1} x + w^{1/2} + C = x \sin^{-1} x + \sqrt{1-x^2} + C. \end{aligned}$$

Finally, we integrate $e^{\sqrt{x}}$. If we let $u = \sqrt{x}$, then $x = u^2$ and $dx = 2u\, du$, so

$$\int e^{\sqrt{x}}\, dx = 2 \int u e^u\, du.$$

Once we now integrate by parts with $dv = e^u\, du$, we get $v = e^u$ and also

$$\int e^{\sqrt{x}}\, dx = 2ue^u - 2 \int e^u\, du = 2ue^u - 2e^u + C = 2\sqrt{x} e^{\sqrt{x}} - 2\sqrt{x} + C.$$

4. Find the area of the region enclosed by the graphs of $f(x) = e^{2x}$ and $g(x) = 4e^x - 3$.

Letting $z = e^x$ for simplicity, we get $f(x) = z^2$ and $g(x) = 4z - 3$. It easily follows that

$$f(x) \leq g(x) \iff z^2 \leq 4z - 3 \iff (z-3)(z-1) \leq 0 \iff 1 \leq z \leq 3.$$

In other words, $f(x) \leq g(x)$ if and only if $0 \leq x \leq \ln 3$, so the area of the region is

$$\begin{aligned} \text{Area} &= \int_0^{\ln 3} [g(x) - f(x)]\, dx = \int_0^{\ln 3} (4e^x - 3 - e^{2x})\, dx \\ &= \left[4e^x - 3x - \frac{1}{2} e^{2x} \right]_0^{\ln 3} = 4 - 3\ln 3. \end{aligned}$$

5. Find the volume of the solid that is obtained by rotating the graph of $f(x) = \tan x$ around the x-axis over the interval $[0, \pi/4]$.

The volume of the solid is the integral of $\pi f(x)^2$ and this is given by

$$\text{Volume} = \pi \int_0^{\pi/4} \tan^2 x\, dx = \pi \int_0^{\pi/4} (\sec^2 x - 1)\, dx = \pi \Big[\tan x - x \Big]_0^{\pi/4} = \pi - \frac{\pi^2}{4}.$$

6. Compute each of the following indefinite integrals.

$$\int \frac{dx}{(1+x)\sqrt{x}}, \qquad \int x(\ln x)^2\,dx.$$

For the first integral, we let $u = \sqrt{x}$. This gives $x = u^2$ and $dx = 2u\,du$, so

$$\int \frac{dx}{(1+x)\sqrt{x}} = \int \frac{2u\,du}{(1+u^2)u} = \int \frac{2\,du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C.$$

For the second integral, we let $u = (\ln x)^2$ and $dv = x\,dx$. Then $du = \frac{2\ln x}{x}\,dx$ and $v = \frac{x^2}{2}$, so

$$\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \int \frac{2\ln x}{x}\cdot\frac{x^2}{2}\,dx = \frac{x^2}{2}(\ln x)^2 - \int x(\ln x)\,dx.$$

Next, we take $u = \ln x$ and $dv = x\,dx$. Since $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$, we conclude that

$$\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \int \frac{x}{2}\,dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C.$$

7. Compute each of the following indefinite integrals.

$$\int \frac{dx}{(x^2+4)^2}, \qquad \int x^2\sqrt{1-x^2}\,dx.$$

For the first integral, let $x = 2\tan\theta$ for some angle $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and note that

$$x^2 + 4 = 4\tan^2\theta + 4 = 4\sec^2\theta, \qquad dx = 2\sec^2\theta\,d\theta.$$

The given integral can thus be expressed in the form

$$\int \frac{dx}{(x^2+4)^2} = \int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta} = \frac{1}{8}\int \cos^2\theta\,d\theta.$$

Using the half-angle formula for cosine, one may now simplify to arrive at

$$\int \frac{dx}{(x^2+4)^2} = \frac{1}{16}\int (1+\cos(2\theta))\,d\theta = \frac{1}{16}\left(\theta + \frac{1}{2}\sin(2\theta)\right) = \frac{1}{16}\left(\theta + \sin\theta\cos\theta\right).$$

We need to express this equation in terms of $x = 2\tan\theta$. When $x \geq 0$, the angle θ appears in a right triangle with an opposite side of length x and an adjacent side of length 2. This makes the hypotenuse of length $\sqrt{x^2+4}$, so one finds that

$$\begin{aligned} \int \frac{dx}{(x^2+4)^2} &= \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{1}{16}\cdot\frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}} \\ &= \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{x}{8(x^2+4)}. \end{aligned}$$

When $x = 2\tan\theta \leq 0$, the expression $\theta + \sin\theta\cos\theta$ changes by a minus sign and the same is true for the right hand side of the last equation. Thus, the equation remains valid.

Finally, we look at the integral of $x^2\sqrt{1-x^2}$. Taking $x = \sin\theta$, we get

$$\begin{aligned}\int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta\cos^2\theta\,d\theta &= \int \frac{1-\cos(2\theta)}{2}\cdot\frac{1+\cos(2\theta)}{2}\,d\theta \\ &= \frac{1}{4}\int (1-\cos^2(2\theta))\,d\theta = \frac{1}{4}\int \left(1 - \frac{1+\cos(4\theta)}{2}\right)d\theta \\ &= \frac{1}{8}\int (1-\cos(4\theta))\,d\theta = \frac{1}{8}\left(\theta - \frac{1}{4}\sin(4\theta)\right).\end{aligned}$$

It remains to simplify the right hand side. The addition formulas for sine and cosine give

$$\sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = 4\sin\theta\cos\theta\cdot(\cos^2\theta - \sin^2\theta).$$

Since $\sin\theta = x$, one has $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$ and so

$$\sin(4\theta) = 4x\sqrt{1-x^2}\cdot(1-x^2-x^2) = 4x(1-x^2)^{3/2} - 4x^3\sqrt{1-x^2}.$$

Once we now combine the above computations, we may finally conclude that

$$\int x^2\sqrt{1-x^2}\,dx = \frac{1}{8}\sin^{-1}x - \frac{x}{8}(1-x^2)^{3/2} + \frac{x^3}{8}\sqrt{1-x^2} + C.$$

8. Compute the length of the graph of $f(x) = \frac{1}{2}x^2$ over the interval $[0,1]$.

The length of the graph is the integral of $\sqrt{1+f'(x)^2}$ and this is given by

$$\text{Arc length} = \int_0^1 \sqrt{1+x^2}\,dx.$$

To compute this integral, we let $x = \tan\theta$ for some angle $0 \leq \theta \leq \pi/4$ and we note that

$$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$$

because $\cos\theta \geq 0$. Since we also have $dx = \sec^2\theta\,d\theta$, we can then write

$$\text{Arc length} = \int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\pi/4} \sec^3\theta\,d\theta.$$

Next, we take $u = \sec\theta$ and $dv = \sec^2\theta$. Since $du = \sec\theta\tan\theta\,d\theta$ and $v = \tan\theta$, we get

$$\begin{aligned}\text{Arc length} &= \Big[\sec\theta\tan\theta\Big]_0^{\pi/4} - \int_0^{\pi/4} \sec\theta\tan^2\theta\,d\theta \\ &= \Big[\sec\theta\tan\theta\Big]_0^{\pi/4} - \int_0^{\pi/4} \sec\theta(\sec^2\theta - 1)\,d\theta.\end{aligned}$$

The integral of $\sec^3\theta$ on the right hand side is equal to the original integral, so we may move it to the left hand side and then divide by 2. This means that

$$\begin{aligned}\text{Arc length} &= \frac{1}{2}\Big[\sec\theta\tan\theta\Big]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4}\sec\theta\,d\theta \\ &= \frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4} = \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln(\sqrt{2}-1).\end{aligned}$$

9. Let $a > 0$ be given. Use integration by parts to find a reduction formula for

$$I_n = \int \frac{dx}{(x^2+a^2)^n}.$$

If we let $u = (x^2+a^2)^{-n}$ and $dv = dx$, then $du = -2nx(x^2+a^2)^{-n-1}\,dx$ and $v = x$, so

$$\begin{aligned}I_n &= x(x^2+a^2)^{-n} + 2n\int x^2(x^2+a^2)^{-n-1}\,dx \\ &= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}\,dx \\ &= \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1}.\end{aligned}$$

Rearranging terms, one may thus express the integral I_{n+1} in terms of I_n to find that

$$I_{n+1} = \frac{2n-1}{2na^2}\cdot I_n - \frac{x}{2na^2(x^2+a^2)^n}.$$

10. Use integration by parts to compute the indefinite integral

$$\int \sin(\ln x)\,dx.$$

Letting $u = \sin(\ln x)$ and $dv = dx$, we get $du = \cos(\ln x)\cdot\frac{dx}{x}$ and $v = x$, so

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

Letting $u = \cos(\ln x)$ and $dv = dx$, we similarly get $du = -\sin(\ln x)\cdot\frac{dx}{x}$ and $v = x$, so

$$\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx.$$

Once we now combine the last two equations, we get an identity of the form

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx.$$

Moving the rightmost integral to the left hand side, we may thus conclude that

$$\int \sin(\ln x)\,dx = \frac{x}{2}\sin(\ln x) - \frac{x}{2}\cos(\ln x) + C.$$