Mechanics 2341, PS 8, 2018

```
FS = FullSimplify;
```

Problem 1.

Find the equations of small oscillations under
friction for a particle moving on the surface of a paraboloid
$z = k\left(x^2 + y^2 + 4x - 6y\right)$ in a uniform gravitational field and being
acted on by a force of friction $F_{fr} = -\lambda\, v$. Solve these equations,
and determine the motion of the particle.

We first rewrite the equation of the paraboloid as
$z = k\left((x+2)^2 + (y-3)^2 - 13\right)$.
It is then clear that the motion is equivalent to the motion on
the paraboloid $z = k\left(x^2 + y^2\right)$.

Lagrangian

```
$Assumptions = {m > 0, k > 0, F > 0, λ > 0};
```

Functions

```
Clear[x, y, z, r, ϕ, t]
x[t_]; y[t_]; z[t_]; r[t_]; ϕ[t_];
```

Lagrangian

```
L = m/2 (D[x[t], t]^2 + D[y[t], t]^2 + D[z[t], t]^2) - m g z[t]
```

$$-g\, m\, z[t] + \frac{1}{2} m \left(x'[t]^2 + y'[t]^2 + z'[t]^2\right)$$

Constraint

```
z[t_] = k (x[t]^2 + y[t]^2);
```

Lagrangian becomes

```
L = L // FS
```

$$\frac{1}{2} m \left(-2\, g\, k\left(x[t]^2 + y[t]^2\right) + x'[t]^2 + y'[t]^2 + 4 k^2 \left(x[t]\, x'[t] + y[t]\, y'[t]\right)^2\right)$$

For small oscillations

```
L2 = 1/2 m (-2 g k (x[t]^2 + y[t]^2) + x'[t]^2 + y'[t]^2)
```

$$\frac{1}{2} m \left(-2\, g\, k\left(x[t]^2 + y[t]^2\right) + x'[t]^2 + y'[t]^2\right)$$

and it is just a system of two decouple harmonic oscillators. Adding the force of friction, we get the eom

```
Eomx = D[D[L2, x′[t]], t] - D[L2, x[t]] + λ x'[t]
```

$2\, g\, k\, m\, x[t] + \lambda\, x'[t] + m\, x''[t]$

```
Eomy = D[D[L2, y′[t]], t] - D[L2, y[t]] + λ y'[t]
```

$2\, g\, k\, m\, y[t] + \lambda\, y'[t] + m\, y''[t]$

and the motion is exactly the same as in § 25 of the LL textbook.

Problem 2.

Determine the motion of an oscillator due to an external force

F(t) = $F_0\, e^{\alpha t}$, t < 0,

F(t) = $F_0\, e^{-\alpha t}$, t > 0,

$\alpha > 0$, $F_0 > 0$.

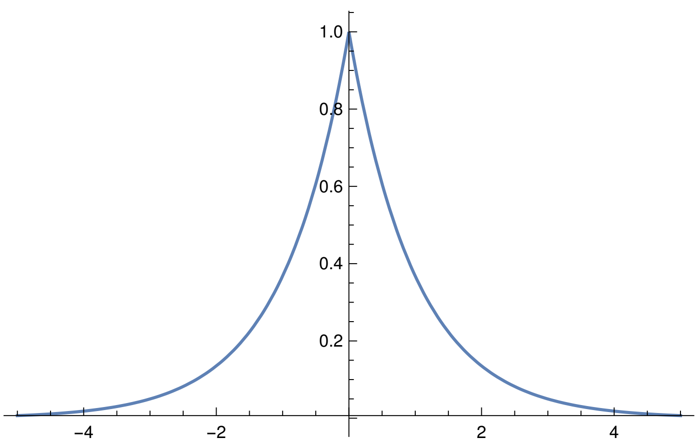
in the presence of friction.

```
Clear[x, y, z, r, ϕ, L, t, ω, F]
x[t_]; y[t_]; z[t_]; r[t_]; ϕ[t_];
$Assumptions = {m > 0, ω > 0, α > 0, β > 0};
```

Eom

```
F[t_] = If[t ≤ 0, F₀ E^(α t), F₀ E^(-α t)];
```

```
Plot[Re[(F[t] /. {F₀ → 1, α → 1})], {t, -5, 5}, PlotRange → All]
```




```
f[t_] = If[t ≤ 0, f₀ E^(α t), f₀ E^(-α t)]; F₀ = m f₀;
```

```
Eom = D[x[t], {t, 2}] + ω₀^2 x[t] - f[t] + 2 λ x'[t]
```

$-\mathrm{If}\left[t \le 0,\ f_0\, e^{\alpha t},\ f_0\, e^{-\alpha t}\right] + \omega_0^2\, x[t] + 2\,\lambda\, x'[t] + x''[t]$

Solution

The general solution is the sum of the general solution of the homogeneous equation and a particular solution of the equation

$x(t) = x_0(t) + x_1(t)$

The general solution of the homogeneous equation is

```
Clear[x]
x0[t_] = x[t] /. DSolve[ω0^2 x[t] + 2 λ x'[t] + x''[t] == 0, x[t], t][[1]]
```

$$e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)} C[1] + e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)} C[2]$$

Note that in general it is complex, and we only need its real part. Then, if $\lambda = \omega_0$ then the general solution is different

```
Clear[x]
xb0[t_] = x[t] /. DSolve[ω0^2 x[t] + 2 ω0 x'[t] + x''[t] == 0, x[t], t][[1]]
```

$$e^{-t\,\omega_0} C[1] + e^{-t\,\omega_0}\, t\, C[2]$$

$t < 0$

A particular solution of the equation is

```
x[t_] = B e^(α t);
```

```
Assuming[t < 0, FS[Solve[Eom == 0, B]]]
```

$$\left\{\left\{B \to \frac{f_0}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2}\right\}\right\}$$

So, for $t \le 0$ the solution is

```
xm[t_] = x0[t] + x[t] /. {B → f0/(α^2 + 2 α λ + ω0^2)}
```

$$e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)} C[1] + e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)} C[2] + \frac{e^{t\,\alpha} f_0}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2}$$

```
xmb[t_] = xb0[t] + x[t] /. {B → f0/(α^2 + 2 α λ + ω0^2)} /. λ -> ω0
```

$$e^{-t\,\omega_0} C[1] + e^{-t\,\omega_0}\, t\, C[2] + \frac{e^{t\,\alpha} f_0}{\alpha^2 + 2\,\alpha\,\omega_0 + \omega_0^2}$$

We see that for both solutions x(t) goes to infinity as t -> - ∞ unless C[1]=C[2]=0. Then for any λ the solution we are interested in is

```
xmg[t_] = e^(t α) f0/(α^2 + 2 α λ + ω0^2);
```

Taking the real part we get the actual solution

```
Xm[t_] = Re[xmg[t]];
```

For real α the solution is real but we would get the same solution for complex α. Then the force would

be the real part of $F_0\, e^{\alpha t}$.
For the solution we find at t = 0

```
x0 = xmg[0]
```

$$\frac{f_0}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2}$$

```
v0 = D[xmg[t], t] /. t → 0
```

$$\frac{\alpha\, f_0}{\alpha^2 + 2\,\alpha\,\lambda + \omega_0^2}$$

These are the initial conditions for t > 0

t > 0

$$x[t\_] = e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)}\,\mathrm{Cp}[1] + e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}\,\mathrm{Cp}[2] + B\, e^{-\alpha t};$$

```
Assuming[t > 0, FS[Eom]]
```

$$e^{-t\,\alpha}\left(-f_0 + B\left(\alpha\left(\alpha - 2\,\lambda\right) + \omega_0^2\right)\right)$$

```
Solve[Assuming[t > 0, FS[Eom]] == 0, B]
```

$$\left\{\left\{B \to \frac{f_0}{\alpha^2 - 2\,\alpha\,\lambda + \omega_0^2}\right\}\right\}$$

$$xp[t\_] = x[t]\ /.\ \left\{B \to \frac{f_0}{\alpha^2 - 2\,\alpha\,\lambda + \omega_0^2}\right\}$$

$$e^{t\left(-\lambda-\sqrt{\lambda^2-\omega_0^2}\right)}\,\mathrm{Cp}[1] + e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}\,\mathrm{Cp}[2] + \frac{e^{-t\,\alpha}\, f_0}{\alpha^2 - 2\,\alpha\,\lambda + \omega_0^2}$$

We see that the solution is singular for $\lambda > \omega_0$ at $\alpha = r_\pm$ where

$$\mathrm{Solve}\left[\alpha^2 - 2\,\alpha\,\lambda + \omega_0^2 == 0,\ \alpha\right]$$

$$\left\{\left\{\alpha \to \lambda - \sqrt{\lambda^2 - \omega_0^2}\right\},\ \left\{\alpha \to \lambda + \sqrt{\lambda^2 - \omega_0^2}\right\}\right\}$$

$$r_+ = \lambda + \sqrt{\lambda^2 - \omega_0^2};\quad r_- = \lambda - \sqrt{\lambda^2 - \omega_0^2};$$

We will see that the full solution is regular at $\alpha = r_\pm$

Cp[i] are found from the initial conditions.
At t = 0 we find

```
xp0 = xp[0]
```

$$\mathrm{Cp}[1] + \mathrm{Cp}[2] + \frac{f_0}{\alpha^2 - 2\,\alpha\,\lambda + \omega_0^2}$$

```
vp0 = D[xp[t], t] /. t → 0
```

$$-\frac{\alpha f_0}{\alpha^2 - 2\alpha\lambda + \omega_0^2} + \mathrm{Cp}[1]\left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right) + \mathrm{Cp}[2]\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)$$

```
Assuming[{α > 0, m > 0, F0 > 0, ω > 0, {Cp[1], Cp[2]} ∈ Complexes},
  Solve[{xp0 == x0, vp0 == v0}, {Cp[1], Cp[2]}]] // FS
```

$$\left\{\left\{\mathrm{Cp}[1] \to -\left(\left(\alpha f_0 \left(\alpha^2 + \omega_0^2 + 2\lambda\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)\right)\right)\Big/\left(\sqrt{\lambda^2 - \omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)\right)\right),\right.\right.$$
$$\left.\left.\mathrm{Cp}[2] \to \left(\alpha f_0\left(\alpha^2 + \omega_0^2 - 2\lambda\left(\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)\right)\right)\Big/\left(\sqrt{\lambda^2 - \omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)\right)\right\}\right\}$$

Thus, the solution which goes to 0 at t -> -∞ is

```
xpg[t_] = xp[t] /. {Cp[1] →
      -((α f0 (α^2 + ω0^2 + 2 λ (-λ + Sqrt[λ^2 - ω0^2])))/(Sqrt[λ^2 - ω0^2] (α (α - 2 λ) + ω0^2) (α (α + 2 λ) + ω0^2))),
     Cp[2] → (α f0 (α^2 + ω0^2 - 2 λ (λ + Sqrt[λ^2 - ω0^2])))/
       (Sqrt[λ^2 - ω0^2] (α (α - 2 λ) + ω0^2) (α (α + 2 λ) + ω0^2))} // FS
```

$$f_0\left(\frac{e^{-t\alpha}}{\alpha^2 - 2\alpha\lambda + \omega_0^2} - \left(e^{-t\left(\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)}\alpha\left(\alpha^2 + \omega_0^2 + 2\lambda\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)\right)\right)\Big/\left(\sqrt{\lambda^2 - \omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)\right) + \left(e^{t\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)}\alpha\left(\alpha^2 + \omega_0^2 - 2\lambda\left(\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)\right)\right)\Big/\left(\sqrt{\lambda^2 - \omega_0^2}\left(\alpha(\alpha - 2\lambda) + \omega_0^2\right)\left(\alpha(\alpha + 2\lambda) + \omega_0^2\right)\right)\right)$$

Note that it is not singular at $\lambda = \omega_0$

```
xpgb[t_] = Series[xpg[t], {λ, ω0, 0}] // Normal
```

$$f_0\left(\frac{e^{-t\alpha}}{(\alpha - \omega_0)^2} + \frac{2 e^{-t\omega_0}\alpha\left(t\alpha^2 - 2\omega_0 - t\omega_0^2\right)}{(\alpha - \omega_0)^2(\alpha + \omega_0)^2}\right)$$

and then at $\alpha = \omega_0$

```
xpgb2[t_] = Series[xpgb[t], {α, ω0, 0}] // Normal
```

$$\frac{e^{-t\omega_0} f_0\left(1 + 2t\omega_0 + 2t^2\omega_0^2\right)}{4\omega_0^2}$$

Note that it is not singular at $\alpha = r_+$ and $\alpha = r_-$

```
xpg2[t_] = Series[xpg[t], {α, r₊, 0}] // Normal
```

$$f_0\left(-\frac{e^{-t\left(\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}}{4\sqrt{\lambda^2-\omega_0^2}\left(\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}+\frac{e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}\left(\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}{4\lambda\left(\lambda^2-\omega_0^2\right)}-\frac{e^{-t\lambda-t\sqrt{\lambda^2-\omega_0^2}}\left(2t\lambda^2-2t\omega_0^2+\sqrt{\lambda^2-\omega_0^2}\right)}{4\left(\lambda^2-\omega_0^2\right)^{3/2}}\right)$$

```
xpg3[t_] = Series[xpg[t], {α, r₋, 0}] // Normal
```

$$f_0\left(\frac{e^{-t\left(\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}\left(\lambda-\sqrt{\lambda^2-\omega_0^2}\right)}{4\lambda\left(\lambda^2-\omega_0^2\right)}-\frac{e^{t\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}}{4\sqrt{\lambda^2-\omega_0^2}\left(-\lambda+\sqrt{\lambda^2-\omega_0^2}\right)}-\frac{e^{-t\lambda+t\sqrt{\lambda^2-\omega_0^2}}\left(-2t\lambda^2+2t\omega_0^2+\sqrt{\lambda^2-\omega_0^2}\right)}{4\left(\lambda^2-\omega_0^2\right)^{3/2}}\right)$$

```
xpg0[t_] = Series[xpg[t], {α, 0, 0}] // Normal
```

$$\frac{f_0}{\omega_0^2}$$

```
xpginfty[t_] = Series[xpg[t], {α, Infinity, 0}] // Normal // FS
```

$$\frac{2\,e^{-t\lambda}\,\mathrm{Sinh}\left[t\sqrt{\lambda^2-\omega_0^2}\right]f_0}{\alpha\sqrt{\lambda^2-\omega_0^2}}$$

Taking the real part we get the actual solution

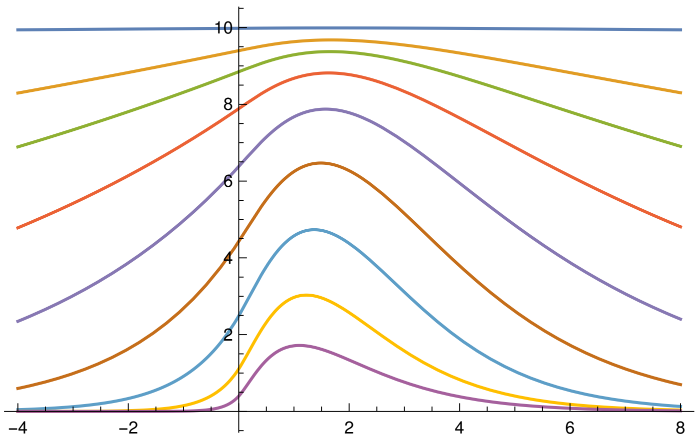
```
Xp[t_] = Re[xpg[t]];
```

The solution

```
X[t_] = If[t ≤ 0, Xm[t], Xp[t]];
```

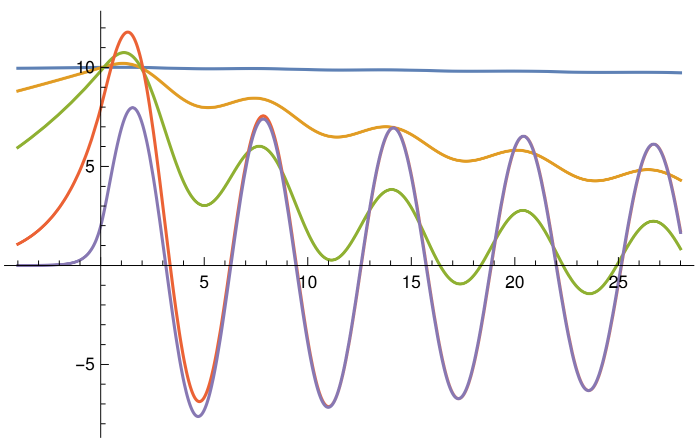
The real part of the solution depends on whether λ is less or greater than ω_0.
If $\lambda > \omega_0$ then the soution is real. Its plot looks like that

```
Plot[{X[t] /. {α → 1/1000, ω₀ -> 1, λ → 1.01, f₀ -> 10},
  X[t] /. {α → 1/32, ω₀ -> 1, λ → 1.01, f₀ -> 10},
  X[t] /. {α → 1/16, ω₀ -> 1, λ → 1.01, f₀ -> 10},
  X[t] /. {α → 1/8, ω₀ -> 1, λ → 1.01, f₀ -> 10},
  X[t] /. {α → 1 / 4, ω₀ -> 1, λ → 1.01, f₀ -> 10},
  X[t] /. {α → 1/2, ω₀ -> 1, λ → 1.01, f₀ -> 10},
  X[t] /. {α → 1, ω₀ -> 1, λ → 1.01, f₀ -> 10}, X[t] /. {α → 2, ω₀ -> 1, λ → 1.01, f₀ -> 10},
  X[t] /. {α → 4, ω₀ -> 1, λ → 1.01, f₀ -> 10}}, {t, -4, 8}]
```

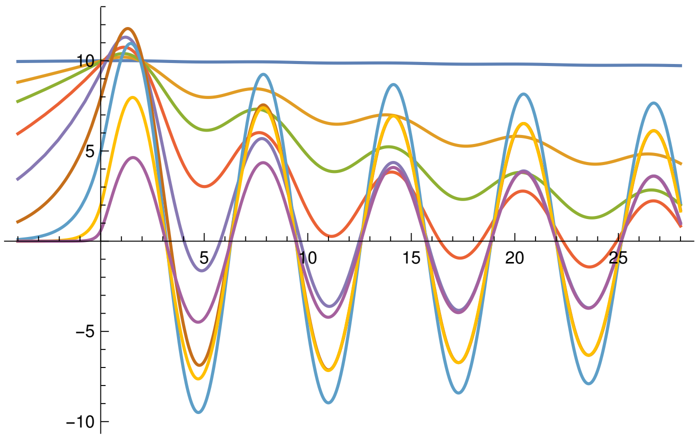


If $\lambda < \omega_0$ then the soution is real. Its plot looks like that

```
Plot[{X[t] /. {α → 1/1000, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 1/32, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 1/8, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 1/2, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 2, ω₀ -> 1, λ → .01, f₀ -> 10}}, {t, -4, 28}]
```



```
Plot[{X[t] /. {α → 1/1000, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 1/32, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 1/16, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 1/8, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 1/4, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 1/2, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 1, ω₀ -> 1, λ → .01, f₀ -> 10}, X[t] /. {α → 2, ω₀ -> 1, λ → .01, f₀ -> 10},
  X[t] /. {α → 4, ω₀ -> 1, λ → .01, f₀ -> 10}}, {t, -4, 28}]
```



Problem 3.

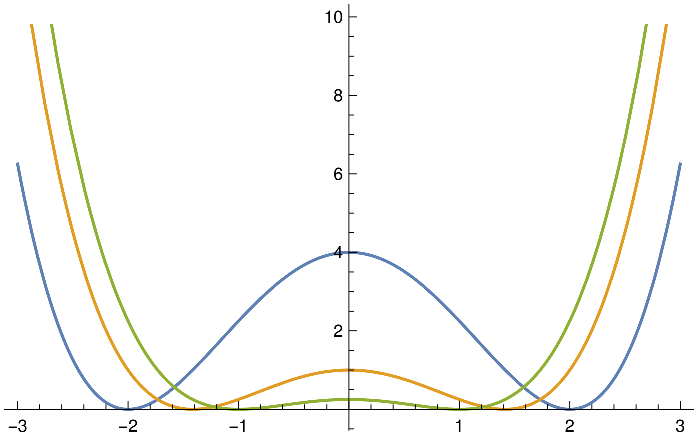
The motion of a particle of mass m in the ``Mexican hat'' potential is described by the potential
$U = \frac{k^2}{4 g} - \frac{k x^2}{2} + \frac{g x^4}{4}$, k>0, g>0.
Plot the potential.
Find positions of stable equilibrium, the equations of anharmonic oscillations in the vicinity of the equilibrium positions, and the fundamental frequencies up to the second order in the amplitude of the oscillations.

```
Clear[x, y, z, r, ϕ, t, ω, F, k]
x[t_]; y[t_]; ϕ[t_]; ξ[t_]; ξ[t_];
$Assumptions = {m > 0, g > 0, k > 0};
```

```
Clear[x, g, k, m]
U = k^2/4/g - k/2 x^2  + g/4 x^4
```

$$\frac{k^2}{4 g} - \frac{k x^2}{2} + \frac{g x^4}{4}$$

```
Plot[{U /. {g → 1, k → 4}, U /. {g → 1, k → 2}, U /. {g → 1, k → 1}}, {x, -3, 3}]
```



It is clear that there are two equivalent stable equilibria.

```
Solve[D[U, x] == 0, x]
```

$$\left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{k}}{\sqrt{g}}\right\}, \left\{x \to \frac{\sqrt{k}}{\sqrt{g}}\right\}\right\}$$

$$D[U, \{x, 2\}] \;/.\; \left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{k}}{\sqrt{g}}\right\}, \left\{x \to \frac{\sqrt{k}}{\sqrt{g}}\right\}\right\}$$

$$\{-k, 2\,k, 2\,k\}$$

$$U \;/.\; \left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{k}}{\sqrt{g}}\right\}, \left\{x \to \frac{\sqrt{k}}{\sqrt{g}}\right\}\right\}$$

$$\left\{\frac{k^2}{4\,g}, 0, 0\right\}$$

We choose

$$x0 = \frac{\sqrt{k}}{\sqrt{g}};$$

Eom are

$$Eom = m\,D[x[t], \{t, 2\}] + \left(D[U, x] \;/.\; x \to x[t]\right)$$

$$-k\,x[t] + g\,x[t]^3 + m\,x''[t]$$

Introducing

```
x[t_] = x0 + y[t];
```

Introducing we find the equations of anharmonic oscillations in the vicinity of the equilibrium position

$$Eom = Eom \big/ m \;//\; FS \;//\; Expand$$

$$\frac{2\,k\,y[t]}{m} + \frac{3\,\sqrt{g\,k}\;y[t]^2}{m} + \frac{g\,y[t]^3}{m} + y''[t]$$

Thus, $\omega_0{}^2 = 2\,k/m$.

```
ω0 = (2 k / m) ^ (1/2)
```

$\sqrt{2}\,\sqrt{\frac{k}{m}}$

To find the fundamental frequencies up to the second order in the amplitude of the oscillations, we introduce the parameters α and β

```
α = Coefficient[Eom, y[t]^2]
```

$\frac{3\sqrt{g\,k}}{m}$

```
β = Coefficient[Eom, y[t]^3]
```

$\frac{g}{m}$

Then, the second order frequency is

```
ω(2) = a^2 (3 β/8/ω0 - 5/12 α^2/ω0^3) // FS
```

$-\frac{3\,a^2\,g}{2\sqrt{2}\,\sqrt{k\,m}}$

```
ω(2) / ω0 // FS
```

$-\frac{3\,a^2\,g}{4\,k}$