Advanced Calculus
MA1132
Tutorial Exercises 9

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To be completed before and during tutorials of Friday, 5. April

Moment of inertia: The tendency of a solid to resist a change in rotational motion about an axis is measured by its *moment of inertia* about that axis. If the solid occupies a region G in an xyz-coordinate system, and if its density function $\delta(x, y, z)$ is continuous on G, then the moments of inertia about the x-axis, the y-axis, and the z-axis are denoted by I_x, I_y, and I_z, respectively, and are defined by

$$\begin{aligned} I_x &= \iiint_G (y^2 + z^2)\delta(x, y, z)dV \,, \\ I_y &= \iiint_G (x^2 + z^2)\delta(x, y, z)dV \,, \\ I_z &= \iiint_G (x^2 + y^2)\delta(x, y, z)dV \,. \end{aligned} \tag{1}$$

Newton's law of gravitation: Let a solid occupy a region G in an xyz-coordinate system, and let its density function $\delta(x, y, z)$ be continuous on G. Then the gravitational force $\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$ exerted by the solid on a point particle of mass m located at (ξ, η, ζ) is given by

$$\begin{aligned} F_x(\xi, \eta, \zeta) &= gm \iiint_G \frac{x - \xi}{r^3}\delta(x, y, z)dV \,, \\ F_y(\xi, \eta, \zeta) &= gm \iiint_G \frac{y - \eta}{r^3}\delta(x, y, z)dV \,, \\ F_z(\xi, \eta, \zeta) &= gm \iiint_G \frac{z - \zeta}{r^3}\delta(x, y, z)dV \,, \\ r &= \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2} \,, \end{aligned} \tag{2}$$

where G is the gravitational constant.

The force can be obtained from the gravitational potential field $U(\xi, \eta, \zeta)$ as follows

$$\begin{aligned} &U(\xi, \eta, \zeta) = -g \iiint_G \frac{1}{r}\delta(x, y, z)dV \,, \\ &F_x(\xi, \eta, \zeta) = -m\frac{\partial U(\xi, \eta, \zeta)}{\partial \xi} \,, \quad F_y(\xi, \eta, \zeta) = -m\frac{\partial U(\xi, \eta, \zeta)}{\partial \eta} \,, \quad F_z(\xi, \eta, \zeta) = -m\frac{\partial U(\xi, \eta, \zeta)}{\partial \zeta} \,. \end{aligned} \tag{3}$$

In what follows we set $m = 1$, $g = 1$, and consider homogeneous solids with $\delta(x, y, z) = 1$.

1. Consider the solid G bounded by the surface $x^2 + y^2 + z^2 = a^2$.

   (a) What is the surface $x^2 + y^2 + z^2 = a^2$?

   (b) Find the volume V of the solid G.

   (c) Find the moments of inertia of the solid G.

   (d) Find the gravitational force $\mathbf{F}(\xi, \eta, \zeta)$ exerted on a point particle located at (ξ, η, ζ) by the solid G.

   (e) Find the gravitational potential field $U(\xi, \eta, \zeta)$ of the solid G.

2. Use spherical coordinates $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$. Consider the solid G bounded above by the surface $r = a$ and below by the surface $\phi = \gamma$.

   (a) What is the surface $r = a$?

   (b) What is the surface $\phi = \gamma$?

   (c) Sketch the solid G.

   (d) Sketch the projection of the solid G onto the xy-plane.

   (e) Find the volume V of the solid G. Specify your answer for $\gamma = \pi/2$ and $\gamma = \pi$ and explain the result.

   (f) Find the centroid of the solid G. Specify your answer for $\gamma = \pi/2$ and $\gamma = \pi$.

   (g) Find the moments of inertia of the solid G. Specify your answer for $\gamma = \pi/2$ and $\gamma = \pi$.

   (h) Set $\gamma = \pi/2$, and find the gravitational force exerted on a point particle by the solid G if the point particle is located at $(0, 0, \zeta)$.

   (i) Set $\gamma = \pi/2$, and find the gravitational potential field $U(0, 0, \zeta)$ of the solid G.