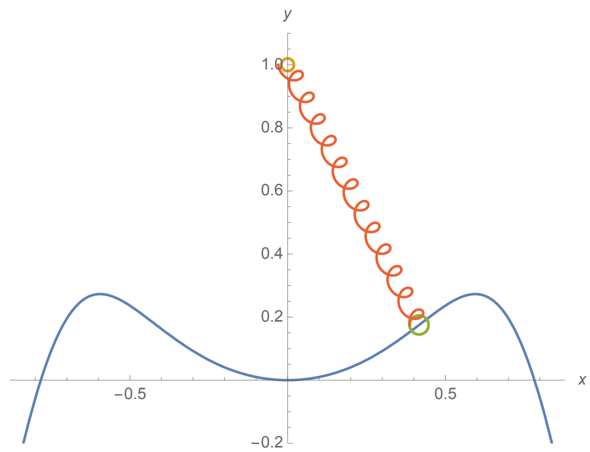
Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 7
Each set of homework questions is worth 100 marks

You may use Mathematica.

Problem 1



Consider a particle of mass m which is free to move along the curve

$$y = l - \sqrt{l^2 - 2x^2 + \frac{x^6}{3g^4}}$$

in the xy-plane, and is attached to an ideal spring whose other end is fixed at a point with coordinates $(0, l)$.
The potential energy of the spring extended to length L is $kL^2/2$.

1. Use Mathematica to plot the curve for

   $l = 1$, and $g = 0.71l\,,\ 0.72l\,,\ 0.73l\,,\ 0.74l$.

   Explain why the curve is discontinuous for $g = 0.73l\,,\ 0.74l$, and find the exact value g_{cr} of g at which the transition occurs.

2. Use x as a generalised coordinate, and find the Lagrangian of the particle.

3. Plot the potential energy for $k = 1$, $l = 1$, and $g = 0.71l\,,\ 0.72l\,,\ 0.73l\,,\ 0.74l$,

   and find stable equilibrium positions.

4. Assume $g < g_{cr}$. For all inequivalent stable equilibrium positions, expand the Lagrangian up to quadratic order in $x - x_0$ and $\dot{x}$, where x_0 is a stable equilibrium position, and find the frequency of small oscillations about it.

Problem 2

Consider the forced oscillations of an oscillator experiencing a force

$$F(t) = \begin{cases} F_0 e^{\alpha t}\cos\beta t & t < 0 \\ F_0 e^{-\alpha t}\cos\beta t & t > 0 \end{cases}\,, \quad F_0 > 0\,,\ \alpha > 0\,,\ \beta > 0\,.$$

The initial energy as $t \to -\infty$ is $E_0 = 0$.

1. Determine the forced oscillations, i.e. find $x(t)$ of the oscillator.

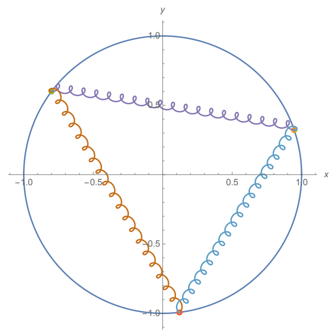
2. Use Mathematica to plot the solution.

3. Find the energy acquired by the oscillator.

4. Analyse the limits (a) $\alpha \to 0$, β fixed, and (b) $\beta \to 0$, α fixed, and explain the results obtained.

Problem 3

Find the normal coordinates and frequencies of small fluctuations of a system of three equal masses connected to each other by identical springs whose equilibrium length is $\sqrt{3}l$ and constrained to move on a circle of raduis l.



Bonus question (each bonus question is worth extra 25 marks)

Find the normal coordinates and frequencies of a system of N equal masses connected to each other by identical ideal springs and constrained to move on a circle.