Useful formulas

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} \tag{1}$$

$$\left(\vec{A} \times \vec{B}\right)_i = \epsilon_{ijk} A_j B_k \tag{2}$$

PROBLEM 1

1. **Question 3**. Take $l = i$ in equation (1) above. Keep in mind that δ_{ii} stands for $\sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$, so equation (1) becomes

$$\begin{aligned}\epsilon_{ijk}\epsilon_{imn} &= 3\delta_{jm}\delta_{kn} - \delta_{im}\delta_{ji}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{ki} + \delta_{im}\delta_{jn}\delta_{ki} + \delta_{in}\delta_{ji}\delta_{km} - 3\delta_{jn}\delta_{km} \\ &= 3\delta_{jm}\delta_{kn} - \delta_{jm}\delta_{kn} - \delta_{kn}\delta_{jm} + \delta_{km}\delta_{jn} + \delta_{jn}\delta_{km} - 3\delta_{jn}\delta_{km} \\ &= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km},\end{aligned} \tag{3}$$

where in the second equality the relation $\delta_{ij}\delta_{jk} = \delta_{ik}$ was used. So

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{4}$$

2. **Question 4** Looking at the i-th component of the triple product and using (1) yields

$$\begin{aligned}\left[\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right]_i = \epsilon_{ijk} A_j \left(\vec{B} \times \vec{C}\right)_k &= \epsilon_{ijk} A_j \epsilon_{kmn} B_m C_n \\ &= \epsilon_{kij}\epsilon_{kmn} A_j B_m C_n,\end{aligned} \tag{5}$$

where in the second line we have moved the k index of the first LeviCivita symbol to the first position to match the contraction in (4). Using this equation we can rewrite the triple product as

$$\begin{aligned}\left[\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right]_i &= \epsilon_{kij}\epsilon_{kmn} A_j B_m C_n = \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) A_j B_m C_n \\ &= A_n B_i C_n - A_m B_m C_i = (\vec{A} \cdot \vec{C}) B_i - (\vec{A} \cdot \vec{B}) C_i.\end{aligned} \tag{6}$$

And from here we then read that

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}. \tag{7}$$

3. **Question 5** Here we write

$$\begin{aligned}\left(\vec{A} \times \vec{B}\right)_i \left(\vec{C} \times \vec{D}\right)_i &= \epsilon_{ijk} A_j B_k \, \epsilon_{imn} C_m D_n \\ = \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) A_j B_k C_m D_n &= A_m B_n C_m D_n - A_n B_m C_m D_n \\ &= \left(\vec{A} \cdot \vec{C}\right)\left(\vec{B} \cdot \vec{D}\right) - \left(\vec{A} \cdot \vec{D}\right)\left(\vec{B} \cdot \vec{C}\right).\end{aligned} \tag{8}$$

So we find $\left(\vec{A}\times\vec{B}\right)\cdot\left(\vec{C}\times\vec{D}\right)=\left(\vec{A}\cdot\vec{C}\right)\left(\vec{B}\cdot\vec{D}\right)-\left(\vec{A}\cdot\vec{D}\right)\left(\vec{B}\cdot\vec{C}\right)$.

4. **Question 6** Here we start from (4), and make $j=m$. The equation then simplifies to

$$\epsilon_{ijk}\epsilon_{ijn}=\delta_{jj}\delta_{kn}-\delta_{jn}\delta_{kj}=3\delta_{kn}-\delta_{kn}=2\delta_{kn}. \tag{9}$$

5. **Question 7** This problem is easier working it from the end. That is, starting from the transformation property of $\vec{A}=\vec{B}\times\vec{C}$ and showing that it transforms as the product of the transformations of B and C. So let's star from

$$\begin{aligned} A_i=\left(\vec{B}\times\vec{C}\right)_i \quad &\longrightarrow \quad \tilde{A}_i=\left(\vec{\tilde{B}}\times\vec{\tilde{C}}\right)_i \\ =(detR)R_{ij}\epsilon_{jmn}B_mC_n &=(\epsilon_{\alpha\beta\gamma}R_{1\alpha}R_{2\beta}R_{3\gamma})R_{ij}\epsilon_{jmn}B_mC_n \\ &=\epsilon_{\alpha\beta\gamma}\epsilon_{jmn}\ R_{1\alpha}R_{2\beta}R_{3\gamma}R_{ij}B_mC_n. \end{aligned} \tag{10}$$

The levi-civita symbols do not have any index in common, so we need to use (1). Substituting above and summing over α, β and γ (one of which is in the *delta*'s and the other in the R's) yields

$$\begin{aligned} \tilde{A}_i=\Big[&R_{1j}R_{2m}R_{3n}-R_{1m}R_{2j}R_{3n}-R_{1n}R_{2m}R_{3j} \\ &+R_{1m}R_{2n}R_{3j}+R_{1n}R_{2j}R_{3m}-R_{1j}R_{2n}R_{3m}\Big]R_{ij}B_mC_n, \end{aligned} \tag{11}$$

and using $R_{ij}R_{jk}=\delta_{ik}$ this further simplifies too

$$\begin{aligned} \tilde{A}_i=\Big[&\delta_{1i}(R_{2m}R_{3n}-R_{2n}R_{3m}) \\ &+\delta_{2i}(R_{1n}R_{3m}-R_{1m}R_{3n}) \\ &+\delta_{3i}(R_{1m}R_{2n}-R_{2m}R_{1n})\Big]B_mC_n. \end{aligned} \tag{12}$$

Just by inspection, it is clear that the quantity inside the square brackets is $\epsilon_{ijk}R_{jm}R_{kn}$, so this is

$$\begin{aligned} \tilde{A}_1=\epsilon_{ijk}R_{jm}R_{kn}B_mC_n &=\epsilon_{ijk}(R_{jm}B_m)(R_{kn}C_n) \\ &=\epsilon_{ijk}\tilde{B}_j\tilde{C}_k=\left(\vec{\tilde{B}}\times\vec{\tilde{C}}\right)_i, \end{aligned} \tag{13}$$

as we wanted to show.

6. **Question 8** Taking $J_{ij}=\epsilon_{ijk}M_k$ we inmediately see that every element in the diagonal will yield 0 since we will have a repeated index in the levi-civita. The rest of the components are easily computed, for example $J_{21}=\epsilon_{21k}M_k=-\epsilon_{12k}M_k$,

and since we're forced to choose $k = 3$ to avoid repetition then $J_{21} = -M_3$. The end result is

$$J = \begin{bmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{bmatrix} \tag{14}$$

7. **Question 9** Using the definition of J in the previous exercise and (9) then

$$\epsilon_{ijk} J_{ij} N_k = \epsilon_{ijk}\epsilon_{ijl} N_k M_l = 2\delta_{kl} N_k M_l = 2(M \cdot N). \tag{15}$$

The constant of proportionality is then $\Lambda = 2$.

PROBLEM 2

1. **Question 1** Recalling that $A = MN$ and $w = M \cdot v$ are expressed in coordinates as $A_{ij} = M_{ik}N_{kj}$ and $w_i = M_{ij}v_j$ then

$$\begin{aligned} M_{jk}M_{ij} = M_{ij}M_{jk} \quad &\longrightarrow \quad (M^2)_{ik} \\ M_{jk}M_{ji} \quad &\longrightarrow \quad (M^T M)_{ki} \\ M_{ij}M_{ij} \quad &\longrightarrow \quad tr(MM^T) \\ M_{ij}v_j \quad &\longrightarrow \quad M \cdot v \\ M_{ij}v_i \quad &\longrightarrow \quad v^T \cdot M \end{aligned} \tag{16}$$

2. **Question 2**

$$\begin{aligned} (MM^T)_{ki} = M_{km}(M^T)_{mi} &= M_{km}M_{im} \\ (MM^T)_{ik} = M_{im}(M^T)_{mk} &= M_{km}M_{im} \\ v^T \cdot v &= v_i v_i \\ v^T \cdot M \cdot v &= M_{ij}v_i v_j \end{aligned} \tag{17}$$

PROBLEM 3

1. **Question 1** Recalling that $\frac{\partial q^i}{\partial q^j} = \delta_{ij}$ then

$$\begin{aligned} \frac{\delta q^j}{\delta q^i} &= \delta_{ij}, \\ \frac{\delta q^j}{\delta q^i} &= \delta_{ii} = d \text{ (where } d \text{ is the dimension of space)}, \\ \frac{\partial}{\partial q^i}\left(M_{jk}q^j q^k\right) &= M_{jk}\delta_{ij}q^k + M_{jk}q^j\delta_{ik} = 2M_{ik}q^k \\ \frac{\partial}{\partial q^i}\left(M_{jk}q^i q^k\right) &= dM_{jk}q^k + M_{jk}q^i\delta_{ik} = (d+1)M_{jk}q^k \end{aligned} \tag{18}$$

2. **Question 2** Same computations as above yield

$$\begin{aligned}\frac{\partial}{\partial q^i}\left(M_{jkl}q^jq^kq^l\right) &= M_{ikl}q^kq^l + M_{jil}q^jq^l + M_{jki}q^jq^k \\ &= (M_{ikl} + M_{kil} + M_{lki})q^kq^l \\ \frac{\partial}{\partial q^i}\left(M_{jkl}q^jq^iq^l\right) &= M_{ikl}q^iq^l + dM_{jkl}q^jq^l + M_{jki}q^jq^i \\ (M_{jkl} + dM_{jkl} + M_{jkl})q^jq^l &= (d+2)M_{jkl}q^jq^l.\end{aligned} \tag{19}$$

Note that the first computation can't be simplified further unless we know the symmetry properties of M.

3. **Question 3** Here we just need to differentiate twice

$$\begin{aligned}\frac{\partial^2}{\partial q^l \partial q^i}(M_{jk}q^jq^k) = \frac{\partial}{\partial q^l}(M_{ik}q^k + M_{ki}q^k) = M_{il} + M_{li} \\ \frac{\partial^2}{\partial q^i \partial q^i}(M_{jk}q^jq^k) = \frac{\partial}{\partial q^i}(M_{ik}q^k + M_{ki}q^k) = M_{ii} + M_{ii} = 2Tr(M)\end{aligned} \tag{20}$$

And for the last one

$$\begin{aligned}\frac{\partial^2}{\partial q^i \partial q^l}(M_{jkm}q^jq^kq^l) = \frac{\partial}{\partial q^i}(dM_{jkm}q^jq^k + M_{jlm}q^jq^l + M_{lkm}q^kq^l) \\ = \frac{\partial}{\partial q^i}\left[\left(\left(d+1\right)M_{lkm} + M_{klm}\right)q^kq^l\right] = \left(\left(d+1\right)M_{lkm} + M_{klm}\right)\left(\delta_{ik}q^l + \delta_{il}q^k\right) \\ = \left(d+2\right)\left[M_{kim} + M_{ikm}\right]q^k.\end{aligned} \tag{21}$$