**MA1125 – Calculus**
**Tutorial solutions #6**

**1.** Let $a_1, a_2, \ldots, a_n$ be some given constants and let $f$ be the function defined by

$$f(x) = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2.$$

Show that $f(x)$ becomes minimum when $x$ is equal to $\overline{x} = (a_1 + a_2 + \ldots + a_n)/n$.

The derivative of the given function can be expressed in the form

$$f'(x) = 2(x - a_1) + 2(x - a_2) + \ldots + 2(x - a_n) = 2(nx - n\overline{x}) = 2n(x - \overline{x}).$$

This means that $f'(x)$ is negative when $x < \overline{x}$ and positive when $x > \overline{x}$. In particular, $f(x)$ is decreasing when $x < \overline{x}$ and increasing when $x > \overline{x}$, so it becomes minimum when $x = \overline{x}$.

**2.** Find the global minimum and the global maximum values that are attained by

$$f(x) = 3x^4 - 16x^3 + 18x^2 - 1, \qquad 0 \leq x \leq 2.$$

The derivative of the given function can be expressed in the form

$$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x - 1)(x - 3).$$

Thus, the only points at which the minimum/maximum value may occur are the points

$$x = 0, \qquad x = 2, \qquad x = 1, \qquad x = 3.$$

We exclude the rightmost point, as it does not lie in the given interval, and we compute

$$f(0) = -1, \qquad f(2) = 48 - 128 + 72 - 1 = -9, \qquad f(1) = 3 - 16 + 18 - 1 = 4.$$

This means that the minimum value is $f(2) = -9$ and the maximum value is $f(1) = 4$.

**3.** Find the linear approximation to the function $f$ at the point $x_0$ in the case that

$$f(x) = \frac{(x^2 + 1)^4 \cdot e^{x^2 - 1}}{\sqrt{3x + 1}}, \qquad x_0 = 1.$$

First, we use logarithmic differentiation to compute the derivative $f'(x)$. Let us write

$$\begin{aligned} \ln f(x) &= \ln(x^2 + 1)^4 + \ln e^{x^2 - 1} - \ln(3x + 1)^{1/2} \\ &= 4\ln(x^2 + 1) + x^2 - 1 - \frac{1}{2}\ln(3x + 1). \end{aligned}$$

Differentiating both sides of this equation, one may use the chain rule to find that

$$\frac{f'(x)}{f(x)} = \frac{4\cdot 2x}{x^2+1} + 2x - \frac{3}{2(3x+1)}.$$

In our case, we have $f(1) = \frac{2^4 e^0}{\sqrt{4}} = 8$, so one may substitute $x = 1$ to conclude that

$$\frac{f'(1)}{f(1)} = \frac{4\cdot 2}{1+1} + 2 - \frac{3}{2\cdot 4} \quad \Longrightarrow \quad f'(1) = 8\left(6 - \frac{3}{8}\right) = 48 - 3 = 45.$$

Since $f(1) = 8$ and $f'(1) = 45$, the linear approximation at the given point is thus

$$L(x) = f'(1)\cdot(x-1) + f(1) = 45(x-1) + 8 = 45x - 37.$$

**4.** The top of a 5m ladder is sliding down a wall at the rate of 0.25 m/sec. How fast is the base sliding away from the wall when the top lies 3 metres above the ground?

Let $x$ be the horizontal distance between the base of the ladder and the wall, and let $y$ be the vertical distance between the top of the ladder and the floor. We must then have

$$x(t)^2 + y(t)^2 = 5^2 \quad \Longrightarrow \quad 2x(t)x'(t) + 2y(t)y'(t) = 0.$$

At the given moment, $y'(t) = -1/4$ and $y(t) = 3$, so it easily follows that

$$x'(t) = -\frac{y(t)y'(t)}{x(t)} = -\frac{y(t)y'(t)}{\sqrt{5^2 - y(t)^2}} = \frac{3/4}{\sqrt{5^2-3^2}} = \frac{3}{16}.$$

**5.** Let $n > 0$ be a given constant. Show that $x^n \ln x \geq -\frac{1}{ne}$ for all $x > 0$.

Setting $f(x) = x^n \ln x$ for convenience, one may use the product rule to find that

$$f'(x) = nx^{n-1}\cdot \ln x + x^n \cdot x^{-1} = x^{n-1}\,(n\ln x + 1).$$

Since $x > 0$ by assumption, the derivative $f'(x)$ is negative if and only if

$$n\ln x + 1 < 0 \quad \Longleftrightarrow \quad \ln x < -1/n \quad \Longleftrightarrow \quad 0 < x < e^{-1/n}.$$

It easily follows that $f(x)$ is decreasing when $0 < x < e^{-1/n}$ and increasing when $x > e^{-1/n}$. In particular, the minimum value of $f(x)$ is attained at the point $x = e^{-1/n}$ and

$$f(x) \geq f(e^{-1/n}) = \left(e^{-1/n}\right)^n \cdot \ln e^{-1/n} \quad \Longrightarrow \quad f(x) \geq -\frac{e^{-1}}{n} = -\frac{1}{ne}.$$

**6.** Find the global minimum and the global maximum values that are attained by

$$f(x) = x^2 \cdot e^{4-2x}, \qquad -1 \le x \le 2.$$

Using both the product rule and the chain rule, one may differentiate $f(x)$ to get

$$f'(x) = 2x \cdot e^{4-2x} + x^2 \cdot e^{4-2x} \cdot (-2) = 2xe^{4-2x} \cdot (1-x).$$

Thus, the only points at which the minimum/maximum value may occur are the points

$$x = -1, \qquad x = 2, \qquad x = 0, \qquad x = 1.$$

The corresponding values that are attained by $f(x)$ are easily found to be

$$f(-1) = e^6, \qquad f(2) = 4e^0 = 4, \qquad f(0) = 0, \qquad f(1) = e^2.$$

In particular, the minimum value is $f(0) = 0$ and the maximum value is $f(-1) = e^6$.

**7.** Find the point on the graph of $y = 2\sqrt{x}$ which lies closest to the point $(2, 1)$.

The distance between the point $(x, y)$ and the point $(2, 1)$ is given by the formula

$$d(x) = \sqrt{(x-2)^2 + (y-1)^2} = \sqrt{(x-2)^2 + (2\sqrt{x}-1)^2}.$$

The value of $x$ that minimises this expression is the value of $x$ that minimises its square

$$f(x) = d(x)^2 = (x-2)^2 + (2\sqrt{x}-1)^2.$$

Let us then worry about $f(x)$, instead. Using the chain rule, one finds that

$$f'(x) = 2(x-2) + 2(2\sqrt{x}-1) \cdot \frac{2}{2\sqrt{x}} = 2\left(x - 2 + 2 - \frac{1}{\sqrt{x}}\right) = \frac{2(x^{3/2}-1)}{\sqrt{x}}.$$

This means that $f'(x)$ is negative when $0 < x < 1$ and positive when $x > 1$, so $f(x)$ attains its minimum value when $x = 1$. Thus, the closest point is the point $(x, y) = (1, 2)$.

**8.** Find the largest possible area for a rectangle that is inscribed inside a semicircle of radius $r > 0$, if one side of the rectangle lies along the diameter of the semicircle.

We may assume that the semicircle is the upper half of the circle $x^2 + y^2 = r^2$. If the vertices of the rectangle are $(\pm x, 0)$ and $(\pm x, y)$, then the area of the rectangle is

$$A(x) = 2x \cdot y = 2x \cdot \sqrt{r^2 - x^2}, \qquad 0 \le x \le r.$$

The value of $x$ that maximises this expression is the value of $x$ that maximises its square

$$f(x) = 4x^2(r^2 - x^2) = 4r^2x^2 - 4x^4, \qquad 0 \leq x \leq r.$$

Let us then worry about $f(x)$, instead. The derivative of this function is given by

$$f'(x) = 8r^2x - 16x^3 = 8x(r^2 - 2x^2) = 8x(r - x\sqrt{2})(r + x\sqrt{2}).$$

Thus, the only points at which the maximum value may occur are the points

$$x = 0, \qquad x = r, \qquad x = \frac{r}{\sqrt{2}}.$$

Since $f(0) = f(r) = 0$, the maximum value is $f(r/\sqrt{2})$ and the largest possible area is

$$A(r/\sqrt{2}) = \frac{2r}{\sqrt{2}} \cdot \sqrt{r^2 - \frac{r^2}{2}} = \frac{2r}{\sqrt{2}} \cdot \frac{r}{\sqrt{2}} = r^2.$$

**9.** Two cars are driving in opposite directions along two parallel roads which are 300m apart. If one is driving at 50 m/sec and the other is driving at 30 m/sec, how fast is the distance between them changing 5 seconds after they pass one another?

Let us denote by $x$ and $y$ the displacements of the two cars after they pass one another. Then $x + y$ and 300 are the sides of a right triangle whose hypotenuse is the distance $z$ between the two cars. In view of Pythagoras' theorem, we must then have

$$z(t)^2 = (x(t) + y(t))^2 + 300^2 \quad \Longrightarrow \quad 2z(t)z'(t) = 2(x(t) + y(t)) \cdot (x'(t) + y'(t)).$$

At the given moment, $x'(t) = 50$, $y'(t) = 30$ and $x(t) + y(t) = 5 \cdot 50 + 5 \cdot 30 = 400$, so

$$z'(t) = \frac{400 \cdot 80}{\sqrt{400^2 + 300^2}} = \frac{400 \cdot 80}{500} = \frac{320}{5} = 64.$$

**10.** Show that $f(x) = x^4 + 5x - 1$ has a unique root in $(0, 1)$ and use Newton's method with initial guess $x_1 = 0$ to approximate this root within two decimal places.

The existence of a root in $(0, 1)$ follows by Bolzano's theorem, as $f$ is continuous with

$$f(0) = -1, \qquad f(1) = 1 + 5 - 1 = 5.$$

Moreover, the root is unique because $f'(x) = 4x^3 + 5$ is positive on $(0, 1)$, so $f$ is increasing on this interval. To use Newton's method, we repeatedly apply the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 + 5x_n - 1}{4x_n^3 + 5}.$$

Starting with the initial guess $x_1 = 0$, one obtains the approximations

$$x_1 = 0, \qquad x_2 = 0.2, \qquad x_3 = 0.1996820350, \qquad x_4 = 0.1996820302.$$

This suggests that the unique root in $(0, 1)$ is roughly 0.1996820 to seven decimal places.