Quantum Physics PY1T20/PYU11P20

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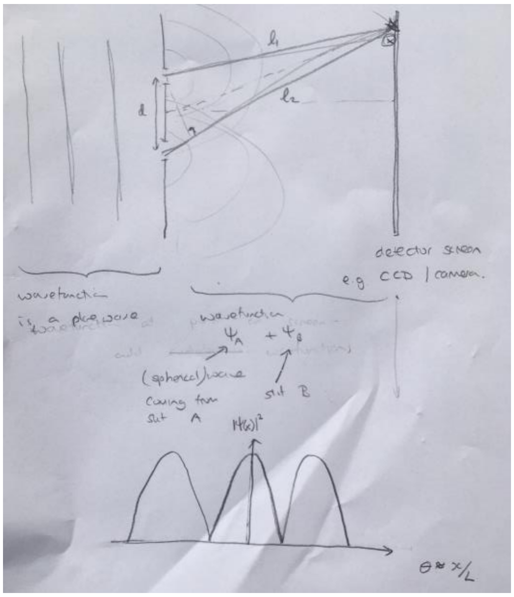
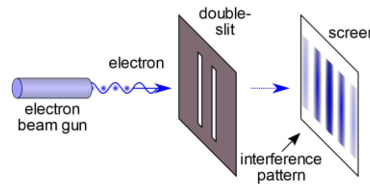
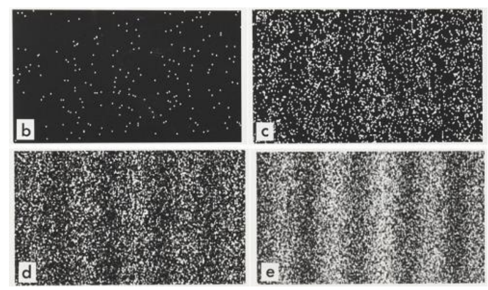
Lecture 5: Wavefunction collapse and motion of wavepackets

- Aims of this lecture:
- See that the wavefunction changes when a measurement is made ('collapse').
- Understand how we observe this.
- Understand how groups of de Broglie waves can be used to make wavepackets, and how these wavepackets move.
- See how a free-particle wave can be formally normalized.

Double slit for single electrons

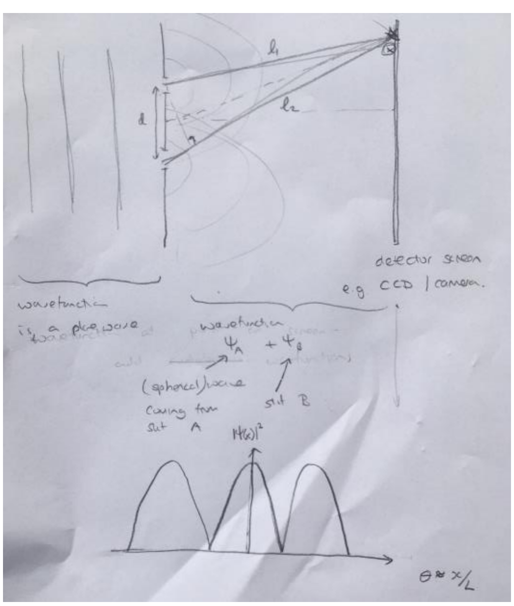






Double slit for single electrons

$$|\psi|^2 = 4|A|^2 \cos^2 [dk\theta/2]$$



Q. Where is the particle before we detect it?

A.: it wasn't really anywhere, we had a wave which told us where the particle was more/less likely to be, but this wave was spread out...

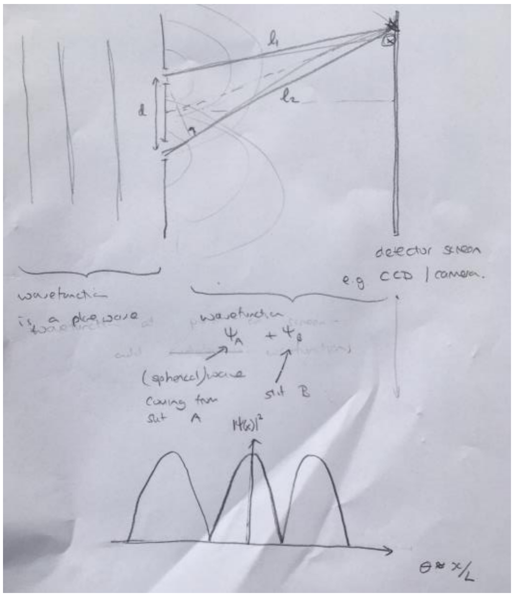
When we detect the particle though, we find it as a whole thing, somewhere specific on the screen. You never get a single particle being detected at two points on the screen.

Q. Where is the particle after we detect it?

Answer: wherever we found it to be! If we measure the position of the particle, and find it somewhere, then do the same measurement again, we still find it there.

Collapse of the wavefunction

$$|\psi|^2 = 4|A|^2 \cos^2 [dk\theta/2]$$



Q. Where is the particle after we detect it?

Answer: wherever we found it to be! If we measure the position of the particle, and find it somewhere, then do the same measurement again, we still find it there.

Conclusion: the wavefunction must change when we measure the particle's position, so that if we measure again we find the particle (near) where it was before.

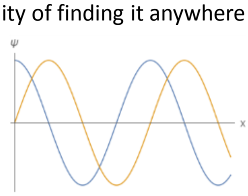
This is a general feature in quantum mechanics: when we measure some physical quantity, and get a definite value, the wavefunction changes so that this value is reproducible.

'Collapse of the wavefunction'

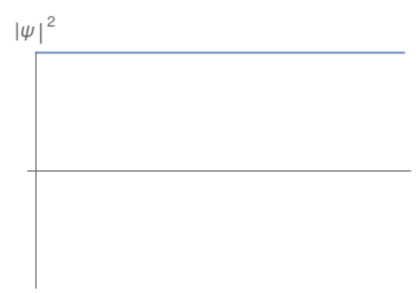
Wavepackets and wavefunction collapse

Suppose we have a free particle of momentum p in 1D: constant probability of finding it anywhere in space.

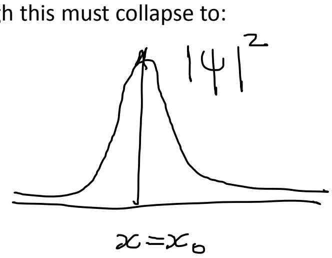
$$\psi(x,t) = Ae^{i(kx-\omega t)}$$



$$\Rightarrow |\psi|^2 = |A|^2$$



After measurement where the particle is found at x_0 though this must collapse to:



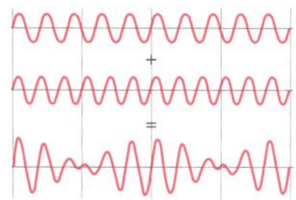
= A wavegroup of many de Broglie waves, with many different momenta

Wave groups

In a classical/quantum harmonic wave, $y = A\cos(\omega t - kx)$ $\psi = Ae^{i(kx-\omega t)}$ energy/probability is spread uniformly throughout in space. They cannot be associated with energy/particle known to be somewhere in space because of their infinite extent.

If we want to make a wave which concentrates the energy/particle in space we need to consider a "wave group".

Simplest example is beats:
2 waves of slightly different frequencies+wavelengths



$$y_1 = A\cos(\omega t - kx)$$

$$y_2 = A\cos\left[(\omega + \Delta\omega)t - (k + \Delta k)x\right]$$

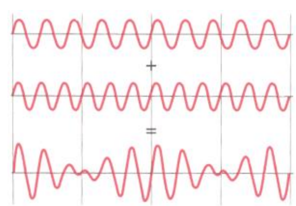
+ (using cos a + cos b= 2 cos((a+b)/2) cos ((a-b)/2):

$$y = 2A\cos \tfrac{1}{2}\left[(2\omega + \Delta\omega)t - (2k + \Delta k)x\right]\cos \tfrac{1}{2}\left[\Delta\omega t - \Delta kx\right]$$

For $\Delta\omega$ and Δk small: $y = 2A\cos\left[\omega t - kx\right]\cos\left[\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right]$

i.e. the basic wave 'modulated' by 'beat' frequency $\Delta\omega/2$ and wavenumber $\Delta k/2$

Propagation of wave groups



$$y = 2A\cos\left[\omega t - kx\right]\cos\left[\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right]$$

The fast oscillations move at velocity $v = \omega/k$

The slow oscillations move at velocity of the group (beat) : $v_g = \dfrac{\Delta\omega/2}{\Delta k/2} = \dfrac{\Delta\omega}{\Delta k}$

Generalisation for a group of many waves: $v_g = \dfrac{d\omega}{dk}$

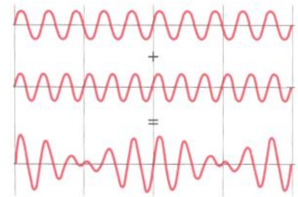
Q: can we associate these with the velocity of the particle v_{cl} in classical mechanics?

v (='phase velocity')? $E = hf$ $p = mv = h/\lambda$

$$\therefore v = f\lambda = hf\frac{\lambda}{h} = \frac{E}{p} = \frac{mc^2}{mv_{\mathrm{cl}}} = \frac{c^2}{v_{\mathrm{cl}}}$$

No good! (Unless v_{cl}=c, i.e. photons)

Propagation of wave groups



Q: can we associate these with the velocity of the particle v_{cl} in classical mechanics?

v (='phase velocity')? No good! (Unless v_{cl}=c, i.e. photons)

v_g (= 'group velocity')? $v_g = \frac{d\omega}{dk} = \frac{dE}{dp}$ because $E = \hbar\omega, p = \hbar k$

Non-relativistic particle: $E = \frac{1}{2}mv_{\text{cl}}^2 = \frac{p^2}{2m} \Rightarrow \frac{dE}{dp} = \frac{p}{m} = v_{\text{cl}}$

Holds relativistically too: derivation:

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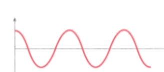
Group velocity and dispersion of wavepackets

$$v_g = \frac{d\omega}{dk} = v$$



$$\left(v_p = \frac{\omega}{k} \right)$$

In a wave with definite wavelength and frequency, the particle is uniformly distributed in space. Wavefronts move at v_p. We can make a 'wave group' built from many such waves, which concentrates the probability into a particular region of space.

Each component moves at v_p, and the packet moves at v_g.

The velocity of the packet, v_g, recovers the result we expect from classical mechanics.

Good! If we make something which is a particle in the sense that it is 'a lump of energy in some bit of space', then it moves the way we expect.

Note also: for a wavepacket to move at < c, we must have $v_g = \frac{d\omega}{dk} \neq v_p = \frac{\omega}{k}$.

The de Broglie waves for massive particles must be dispersive, i.e. the phase velocity depends on wavelength. This in turn means that the wavepacket doesn't just move, but spreads in time.

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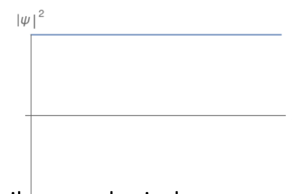
Normalization of wavefunctions: free particles

- Consider again our wavefunction for a free particle in one-dimension $\psi(x,t) = Ae^{i(kx-\omega t)}$
- A is a normalization factor which we should fix so that $\int_{-\infty}^{+\infty} |\psi|^2 = 1$



- Unfortunately this integral is infinite, so we can't do this.
- This is just because this wavefunction is an idealization which does not describe any physical system (no infinite strings either!).
- We can fix this, and calculate the normalization factor, if we acknowledge that the particle is not anywhere in the universe, but somewhere in a finite but large region. For 1D, say it is definitely somewhere in an interval of length L. Then the wavefunction is zero outside this region so we get

$$\int_0^L |\psi|^2 dx = |A|^2 L = 1 \Rightarrow A = \frac{1}{\sqrt{L}}$$

What does this mean?

- This 'box normalization' is a mathematical trick needed to calculate the normalization of the harmonic wave. It's not needed for physical, non-idealised wavefunctions.

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