Lecture 2: Newton’s universal law of gravitation

Read: Ch. 13.1 - 13.4 of “University Physics” (Young and Freedman)

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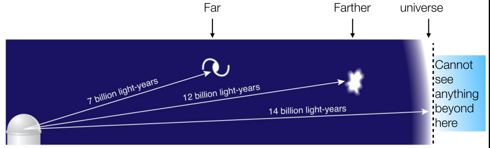
Quick recap of last lecture

Far away means back in time



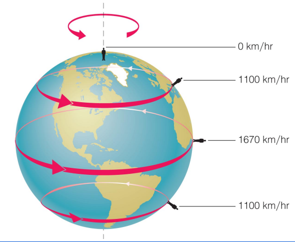
1 light-year = (speed of light) × (1 year)

$$=\left(300{,}000\,\frac{\text{km}}{\cancel{\text{s}}}\right)\times\left(\frac{365\,\cancel{\text{days}}}{1\,\cancel{\text{yr}}}\times\frac{24\,\cancel{\text{hr}}}{1\,\cancel{\text{day}}}\times\frac{60\,\cancel{\text{min}}}{1\,\cancel{\text{hr}}}\times\frac{60\,\cancel{\text{s}}}{1\,\cancel{\text{min}}}\right)$$

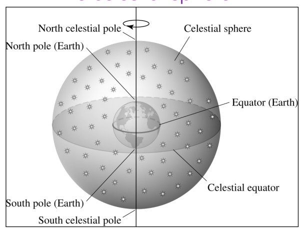
$= 9.46 \times 10^{12}$ km

Our movement through space

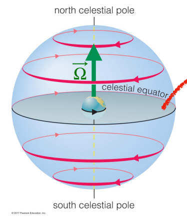
$\vec{v} = \vec{\Omega} \times \vec{r}$ $\qquad v_{\tan} = \Omega r \sin\theta$



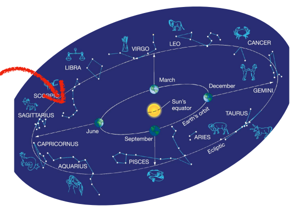
The celestial sphere



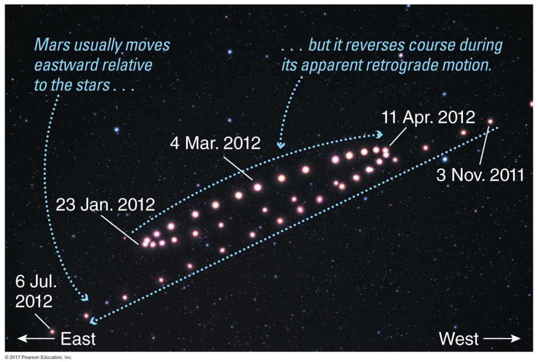
Diurnal motion

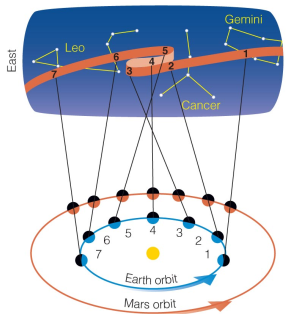


Visibility through year



The apparent motions of planets on the celestial sphere





What we will cover today…

Goal: calculate the gravitational forces that any two bodies exert on each other

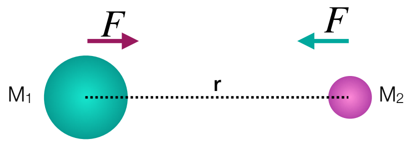
1. Gravitational force
2. The acceleration of gravity
3. Gravitational potential energy
4. Orbits of satellites

Gravitation is the most important force on the scale of planets, stars and galaxies

1. Gravitational force

Any two bodies attract each other through gravitational force that is
- directly proportional to the product of the masses
- inversely proportional to (distance)²

$$F = \frac{GM_1M_2}{r^2}$$



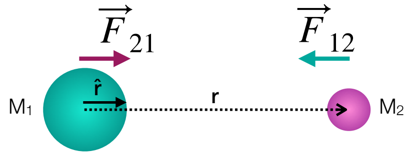
Gravitational constant:
$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.

Don't confuse g and G.
g = acceleration due to gravity (9.8m/s² at the Earth's surface)

The universal law of gravitation

Magnitude of the force:

$$F = \frac{GM_1M_2}{r^2}$$



In vector form:

$$\vec{F}_{12} = -\frac{GM_1M_2}{r^2}\hat{r}$$

Pointing against r̂

The unit vector along $\vec{r}$

- Even if the particles have very different masses, the gravitational forces they exert on each other are equal in magnitude

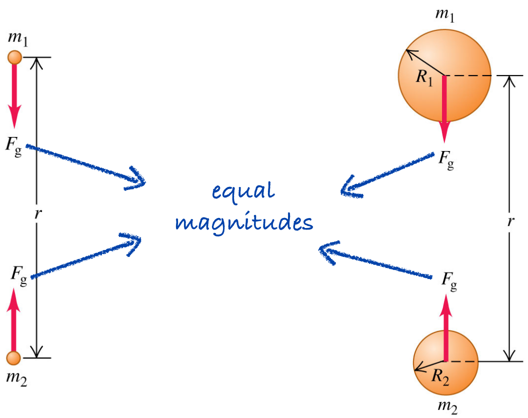
$$F_{12} = F_{21}$$

- Note however that these forces form an action-reaction pair, hence

$$\vec{F}_{12} = -\vec{F}_{21}$$

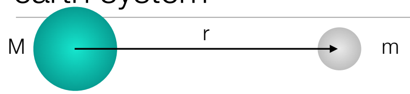
Large body m₁ exerts the same force as particle m₁

- The gravitational effect outside any spherically symmetric mass distribution is the same as though all of the mass were concentrated at its centre.



Useful because moons, planets and stars are usually modelled as spherically symmetric bodies

Example: Gravitational forces on the moon-sun-earth system



$\odot$: Sun symbol
$\oplus$: Earth symbol

$M_\odot = 2 \times 10^{30}$ kg

$M_\oplus = 6 \times 10^{24}$ kg

$m_{\rm moon} = 7 \times 10^{22}$ kg

$r_{\rm SM} = 1\text{au} = 1.5 \times 10^{8}$ km

$r_{\rm EM} = 3.85 \times 10^{5}$ km

A) Earth-moon force:

$$F_{\rm EM} = \frac{Gm_{\rm moon}M_\oplus}{r_{\rm EM}^2}$$

$$= \frac{(6.67 \times 10^{-11})(7 \times 10^{22})(6 \times 10^{24})}{(3.85 \times 10^5 \times 10^3)^2} = 1.9 \times 10^{20}\mathrm{N}$$

B) Sun-moon force:

$$F_{\rm SM} = \frac{Gm_{\rm moon}M_\odot}{r_{\rm SM}^2}$$

$$= \frac{(6.67 \times 10^{-11})(7 \times 10^{22})(2 \times 10^{30})}{(1.5 \times 10^8 \times 10^3)^2} = 4.2 \times 10^{20}\mathrm{N}$$

$$\frac{F_{\rm SM}}{F_{\rm EM}} = 2.2 \quad (!!)$$

The Sun exerts twice as great a force on the moon than the Earth does. Technically, the Moon orbits the Sun… (and not the Earth…)

Conceptual question

The mass of the moon is 1/81 of the mass of the earth. Compared to the gravitational force that the earth exerts on the moon, the gravitational force that the moon exerts on the earth is

A. $81^2 = 6561$ times greater.
B. 81 times greater.
C. equally strong.
D. 1/81 as great.
E. $(1/81)^2 = 1/6561$ as great.

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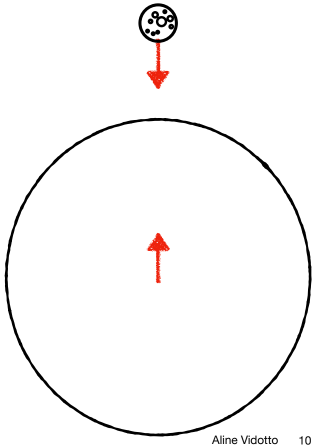


Equal magnitudes:

$$F_{\text{moon, Earth}} = F_{\text{Earth, moon}}$$

but opposite directions:

$$\vec{F}_{\text{moon, Earth}} = -\vec{F}_{\text{Earth, moon}}$$

2. The acceleration of gravity

The acceleration due to gravitational force

- A body's weight is $w = mg$
- The gravitational force exerted by a planet on the body of mass *m* is

$$F_g = \frac{GM_p m}{R_p^2}$$

M_p= mass of the planet
R_p= radius of the planet

- If gravity is the only force acting on the body, then:

$$w = F_g \quad \rightarrow \quad \cancel{m}g = \frac{GM_p\cancel{m}}{R_p^2}$$

- The acceleration due to gravity at the surface of the planet is:

$$\boxed{g = \frac{GM_p}{R_p^2}}$$

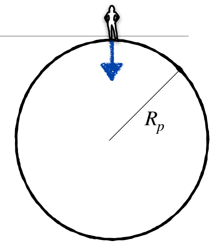
The acceleration is also a vector:

$$\vec{g} = -\frac{GM_p}{R_p^2}\hat{r}$$

Why is it negative?

The acceleration of gravity

- At the Earth's surface:



$$g = \frac{GM_p}{R_p^2} = \frac{(6.67 \times 10^{-11})(6 \times 10^{24})}{(6.4 \times 10^6)^2} = 9.8 \mathrm{m/s^2}$$

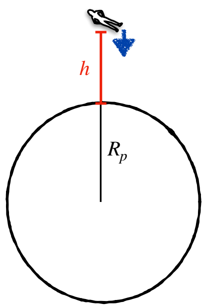
R_p=6400km, M_p=6x10^{24} kg

- What is the acceleration due to gravity for an astronaut in orbit?

$$r = R_p + h$$

$$\rightarrow \quad F_g = \frac{GM_p m}{(R_p + h)^2}$$

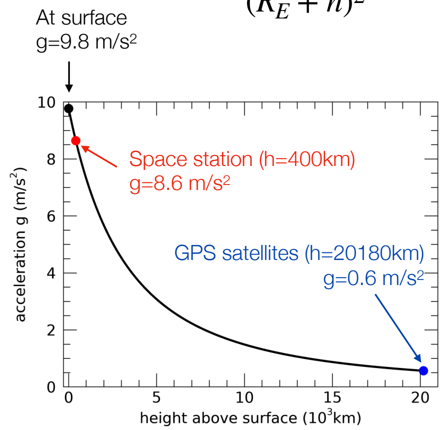
$$\rightarrow \quad g = \frac{F_g}{m} = \frac{GM_p}{(R_p + h)^2}$$



h: height above the surface

The acceleration of gravity at Earth

$$g = \frac{GM_E}{(R_E + h)^2}$$



- The weight of the astronaut decreases with altitude

$$w = mg = m\frac{GM_E}{(R_E + h)^2}$$

$$m\frac{GM_E}{(R_E + h)^2} < m\frac{GM_E}{R_E^2}$$

Weight in orbit | Weight at the surface

Which is farther, the distance from Cork to Belfast, or the distance from you to the international space station if it passes directly overhead?

(a) Cork-Belfast is farther.

421 km (driving)

(b) The space station is farther.

Example: gravity on Mars

- A robotic lander with an Earth weight of 3430N is sent to Mars (radius: R_M = 3.39x10^6m; mass: M_M=6.42x10^{23}kg). Find the weight of the lander on the Martian surface and the acceleration due to gravity.

$$m = \frac{w}{g} = \frac{3430}{9.8} = 350 \text{ kg}$$

$$F_g = \frac{GM_M m}{R_M^2} = \frac{(6.67 \times 10^{-11})(6.42 \times 10^{23})(350)}{(3.39 \times 10^6)^2}$$

$$= 1.3 \times 10^3 \text{ N}$$

$$g_M = \frac{F_g}{m} = \frac{1.3 \times 10^3}{350} = 3.7 \mathrm{m/s^2}$$

Conceptual question

The planet Saturn has about 100 times the mass of the earth and is about 10 times farther from the sun than the Earth is. Compared to the acceleration of the Earth caused by sun's gravitational pull, how great is the acceleration of Saturn due to the Sun's gravitation?

(a) 100 times greater
(b) 10 times greater
(c) The same
(d) 1/10 as great
(e) 1/100 as great

Conceptual question

The planet Saturn has about 100 times the mass of the earth and is about 10 times farther from the sun than the Earth is. Compared to the acceleration of the Earth caused by sun's gravitational pull, how great is the acceleration of Saturn due to the Sun's gravitation?

$$F_{sun,planet} = \frac{GM_{sun}M_{planet}}{r^2_{sun,planet}}$$

$$g_{sun,planet} = \frac{F_{sun,planet}}{M_{planet}} = \frac{GM_{sun}}{r^2_{sun,planet}}$$

(a) 100 times greater
(b) 10 times greater
(c) The same
(d) 1/10 as great
(e) 1/100 as great

The acceleration of Sun's gravity at Earth's orbit:

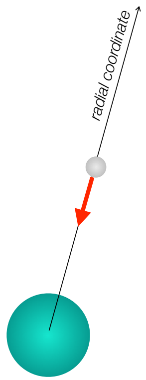
$$g_{sun,E} = \frac{GM_{sun}}{r^2_{sun,E}}$$

The acceleration of Sun's gravity at Saturn's orbit:

$$g_{sun,sat} = \frac{GM_{sun}}{r^2_{sun,sat}}$$

$$\frac{g_{sun,sat}}{g_{sun,E}} = \frac{r^2_{sun,E}}{r^2_{sun,sat}} = \frac{r^2_{sun,E}}{(10\ r_{sun,E})^2} = \frac{1}{100}$$

3. Gravitational potential energy



The gravitational force is directed along a radial line connecting two masses

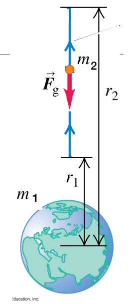
$$\vec{F}_g = -\frac{Gm_1m_2}{r^2}\hat{r}$$

Pointing towards the Earth's centre

The unit vector along $\vec{r}$.

Any force that is directed along a radial line and has magnitude that only depends on the radial coordinate is called a central force

Gravitational potential energy: derivation



- The gravitational potential energy you are familiar with

$$mgh$$

is only valid when the particle is near Earth's surface.
- Goal: derive a more general potential-energy function

$$\vec{F}_g = -\frac{Gm_1m_2}{r^2}\hat{r}$$

Definition: The change in gravitational potential energy U is defined as -1 times the work W done by the gravitational force as the body moves from r_1 to r_2.

$$\Delta U := U_2 - U_1 = -W$$

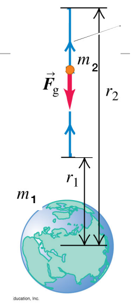
- The work done by a force F is $dW = \vec{F} \cdot d\hat{r}$
  - If force is perpendicular to displacement, dW = 0
  - If force is parallel to displacement, $dW = \vec{F} \cdot d\hat{r} = F\,dr$
- The total work is the sum of the contributions along a segment

$$W = \int_{r_1}^{r_2} dW = \int_{r_1}^{r_2} \vec{F} \cdot d\hat{r} = \int_{r_1}^{r_2} -\frac{Gm_1m_2}{r^2}\hat{r} \cdot d\hat{r} = -Gm_1m_2 \int_{r_1}^{r_2} \frac{dr}{r^2}$$

Gravitational potential energy: derivation

$$W = -Gm_1m_2\int_{r_1}^{r_2}\frac{dr}{r^2} = -Gm_1m_2\frac{(-1)}{r}\Big|_{r_1}^{r_2}$$

$$= \frac{Gm_1m_2}{r_2} - \frac{Gm_1m_2}{r_1}$$



- From our definition

$$\Delta U := U_2 - U_1 = -W$$

we now have:

$$\Delta U := U_2 - U_1 = -\left(\frac{Gm_1m_2}{r_2} - \frac{Gm_1m_2}{r_1}\right)$$

$$U_2 = -\frac{Gm_1m_2}{r_2} \qquad U_1 = -\frac{Gm_1m_2}{r_1}$$

In general:

$$U = -\frac{Gm_1m_2}{r}$$

Ex: Gravitational potential energy of objects in orbit

$$U = -\frac{Gm_{obj}M_E}{r}$$

$$U = -\frac{Gm_{obj}M_E}{(R_E + h)}$$

- When the object moves away from the Earth, r increases and U becomes less negative (i.e., U increases)
- When the object "falls" towards Earth, r decreases and U becomes more negative (i.e., U decreases)

- A 70-kg astronaut in orbit has more potential energy than when they are at the surface of the Earth

$$U_{surface} = -\frac{Gm_{obj}M_E}{(R_E)} = -4.38\times10^9\text{J}$$

$$U_{space\ station} = -\frac{Gm_{obj}M_E}{(R_E + 400\text{km})} = -4.12\times10^9\text{J}$$

- An astronaut who goes (from the surface) to the space station gains ΔU potential energy:

$$\Delta U = (U_{space\ station} - U_{surface}) = [-4.12 - (-4.38)]\times10^9 = 0.26\times10^9\text{J}$$

Homework: show that

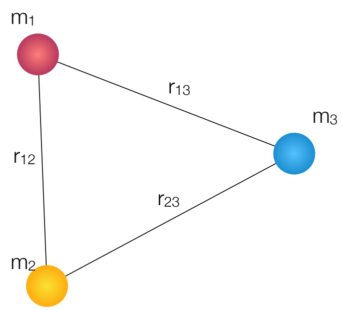
$$mgh = 0.27\times10^9\text{J}$$

Why are these ~ the same?

Total potential energy

Example: stars in multiple systems, clusters of stars, stars in one galaxy, clusters of galaxies.

- The total potential energy in a system with more than two particles is the sum of the potential energies of each pair of particles
- Interpretation
  - U_{tot}: work needed to separate the particles by an infinite distance



$$U_{tot} = U_{12} + U_{23} + U_{13} = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_1m_3}{r_{13}}$$

Conceptual question

A satellite is moving around the earth in a circular orbit. Over the course of one complete orbit, the Earth's gravitational force does

(a) positive work on the satellite.

(b) negative work on the satellite.

(c) positive work on the satellite during part of the orbit and negative work on the satellite during the other part.

(d) positive, negative, or zero work on the satellite during various parts of the orbit.

(e) zero work on the satellite at all points in the orbit.

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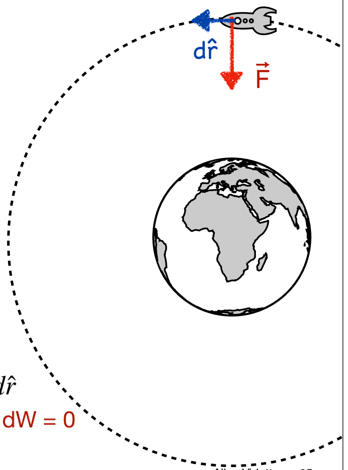
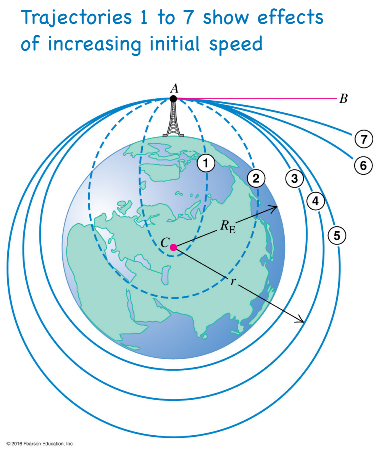


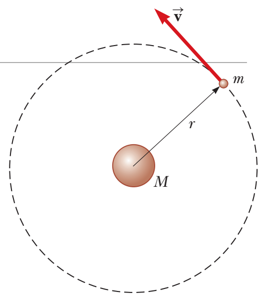
- The work done by a force F is $dW = \vec{F} \cdot d\hat{r}$
  - If force is perpendicular to displacement, dW = 0

4. Orbits of satellites



- The trajectory of a projectile fired from a great height (ignoring air resistance) depends on its initial speed.
- Trajectories 1 to 5: closed orbits
  - They are ellipses (or segments of ellipses with a focus on C)
  - trajectory 4 is a circle (special case of an ellipse)
- Trajectories 6 to 7: open orbits

Energy considerations



- Consider an object of mass m moving with a speed v in the vicinity of a massive object of mass M, where M >> m.

  Example: a planet moving around the Sun, a satellite in orbit around the Earth, or a comet making a one-time flyby of the Sun (open orbit)

- The total mechanical energy is

$$E = K + U$$
$$= \frac{1}{2}mv^2 - \frac{GMm}{r}$$

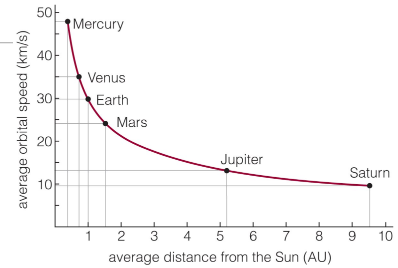
- Depending on the value of v, E can be negative, positive or zero
  - E<0: system is gravitationally bound (e.g., Earth-Sun)
  - if E≥0: object is gravitationally unbound (e.g., comets in parabolic orbits)

Velocities in circular orbits

- Assuming circular orbits with radius r:

$$F_{cent} = F_g$$

$$\left.\begin{aligned} F_g &= \frac{GmM}{r^2} \\ F_{cent} &= \frac{m}{r}v_{orb}^2 \end{aligned}\right\} \quad \boxed{v_{orb} = \sqrt{\frac{GM}{r}}}$$

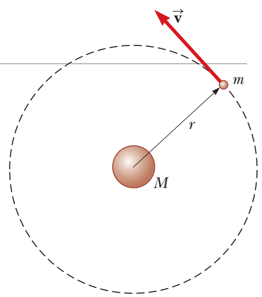


- The energy of a body in circular orbit is:

$$E = \frac{mv_{orb}^2}{2} - \frac{GmM}{r} = \frac{m}{2}\left(\frac{GM}{r}\right) - \frac{GmM}{r} = -\frac{GmM}{2r}$$

- Circular orbits has negative energies (E<0)
  - we say that such an orbit is gravitationally bound

Orbital periods



$$v_{orb} = \sqrt{\frac{GM}{r}}$$

- The velocity of the orbit can also be written as

$$v_{orb} = \frac{\text{circumference}}{\text{time for one orbit}}$$

$$v_{orb} = \frac{2\pi r}{P}$$ period

- Thus the period of the orbit can be written as:

$$P = \frac{2\pi r}{v_{orb}} = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

Larger orbits (larger r) have smaller orbital velocities and longer periods.
Example: Jupiter: r=5au
- smaller orbital velocity than Earth's
- one Jupiter year is longer than one terrestrial year

Example: a satellite orbit

- You wish to put a 1000kg satellite into a circular orbit 300km above the Earth's surface. What speed, period and radial acceleration will it have?

$$r = R_E + h = (6400 + 300)\ \text{km} = 6.7\times10^6\ \text{m}$$

$$v_{orb} = \sqrt{\frac{GM_E}{r}} = \sqrt{\frac{(6.67\times10^{-11})(6\times10^{24})}{6.7\times10^6}} = 7730\ \text{m/s}$$

$$P = \frac{2\pi r}{v_{orb}} = \frac{2\pi(6.7\times10^6)}{7730} = 5445\text{s} \simeq 91\ \text{min}$$

$$a = \frac{v_{orb}^2}{r} = \frac{7700^2}{(6.7\times10^6)} = 8.92\ \text{m/s}^2$$

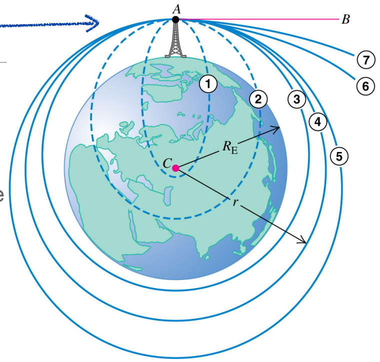
Note: This is the value of g at 300km!

$$g = \frac{GM_E}{r^2} = \frac{(6.67\times10^{-11})(6\times10^{24})}{(6.7\times10^6)^2} = 8.92\text{m/s}^2$$

Gravity is the centripetal acceleration of a satellite in orbit

Escape Velocity

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$



- Depending on the launch velocity, E can be negative, positive or zero
  - E<0: system is gravitationally bound (e.g., Satellite-Earth, Earth-Sun)
  - E=0: system has just enough energy to "escape" and becomes unbound
- What is the minimum velocity required for a projectile to escape the gravitational potential of Earth (at its surface)?

$$E = 0 \quad \rightarrow \quad \frac{mv_{esc}^2}{2} = \frac{GM_E m}{R_E} \quad \rightarrow \quad v_{esc} = \sqrt{\frac{2GM_E}{R_E}}$$

Example: escape speed of a rocket

- Calculate the escape speed from the Earth for a 5,000-kg spacecraft and determine the kinetic energy it must have at the Earth's surface to move infinitely far away from the Earth.

$$v_{esc} = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{\frac{2(6.67\times10^{-11})(6\times10^{24})}{6.4\times10^6}}$$

$$= 1.1\times10^4\ \text{m/s} = 11\ \text{km/s}$$

- Velocity to escape Earth from sea level is ≈ 11 km/s (or about 40,000 km/h).
- It does **not** depend on the mass of the escaping object: a spacecraft has the same minimum speed as a H_2 molecule escaping Earth.

$$K = \frac{1}{2}mv_{esc}^2 = \frac{1}{2}(5000)(1.1\times10^4)^2 = 3\times10^{11}\ \text{J}$$

Conceptual question

Superman stands on top of a very tall mountain and throws a baseball horizontally with a speed such that the baseball goes into a circular orbit around the Earth. While the baseball is in orbit, what is the magnitude of the acceleration of the ball?

(a) It depends on how fast the baseball is thrown.
(b) It is zero because the ball does not fall to the ground.
(c) It is slightly less than 9.80 m/s^2.
(d) It is equal to 9.80 m/s^2.
(e) I don't know

Conceptual question

Superman stands on top of a very tall mountain and throws a baseball horizontally with a speed such that the baseball goes into a circular orbit around the Earth. While the baseball is in orbit, what is the magnitude of the acceleration of the ball?

(a) It depends on how fast the baseball is thrown.
(b) It is zero because the ball does not fall to the ground.
(c) It is slightly less than 9.80 m/s^2.
(d) It is equal to 9.80 m/s^2.
(e) I don't know

Extra slides - read at home

Why do all objects fall at the same rate?

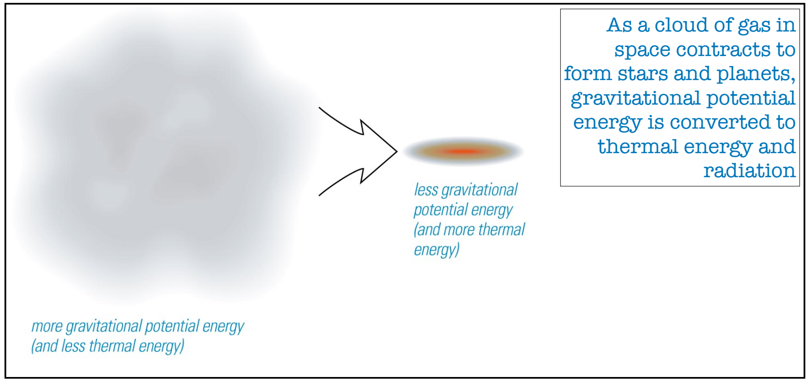
- The gravitational acceleration of an object like a rock does not depend on its mass because M_{rock} in the equation for acceleration cancels M_{rock} in the equation for gravitational force.

$$a_{\text{rock}} = \frac{F_g}{M_{\text{rock}}} \qquad F_g = \frac{GM_E M_{\text{rock}}}{R_E^2}$$

$$a_{\text{rock}} = \frac{GM_E \cancel{M_{\text{rock}}}}{R_E^2 \cancel{M_{\text{rock}}}} = \frac{GM_E}{R_E^2}$$

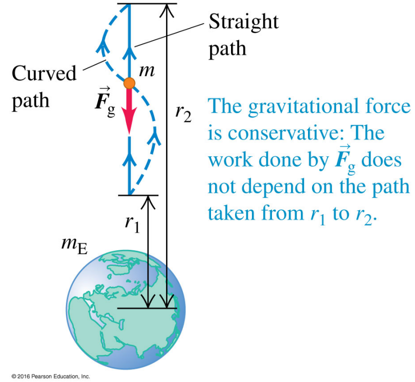
Aside: Conserving energy (more generally)

- We only talked about mechanical energy in orbits, in which kinetic energy is transformed in gravitational energy (or vice-versa). More types of energy exist!
- In space, an object or gas cloud has more gravitational energy when it is spread out than when it contracts.



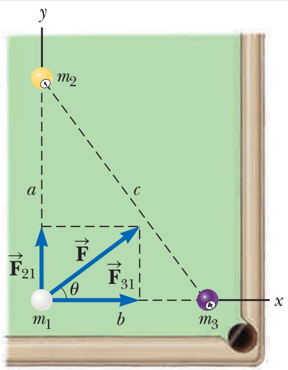
The gravitational force is conservative

Conservative force



Example

Three 0.300-kg billiard balls are placed on a table at the corners of a right triangle as shown in Figure 13.3. The sides of the triangle are of lengths a = 0.400 m, b = 0.300 m, and c = 0.500 m. Calculate the gravitational force vector on the cue ball (designated m_1) resulting from the other two balls as well as the magnitude and direction of this force.



- The force exerted on m1 by m2

$$\vec{F}_{21} = \frac{Gm_1m_2}{a^2}\hat{j}$$

$$= \frac{6.67\times10^{-11}(0.3)(0.3)}{0.4^2}\hat{j}$$

$$= 3.7\times10^{-11}\hat{j}\,\mathrm{N}$$

- The force exerted on m1 by m3

$$\vec{F}_{31} = \frac{Gm_1m_3}{b^2}\hat{i} \quad = 6.67\times10^{-11}\hat{i}\,\mathrm{N}$$

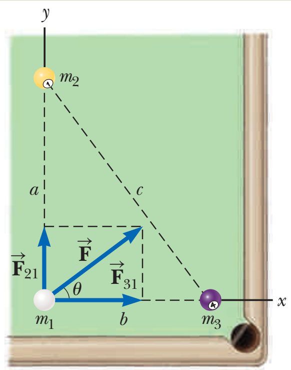
- The net gravitational force on m1 is found by adding force vectors

$$\vec{F} = \vec{F}_{31} + \vec{F}_{21} = (6.67\hat{i} + 3.75\hat{j})\times10^{-11}\mathrm{N}$$

Example (cont.)

$$\vec{F} = \vec{F}_{31} + \vec{F}_{21} = (6.67\hat{i} + 3.75\hat{j})\times10^{-11}\mathrm{N}$$

Three 0.300-kg billiard balls are placed on a table at the corners of a right triangle as shown in Figure 13.3. The sides of the triangle are of lengths a = 0.400 m, b = 0.300 m, and c = 0.500 m. Calculate the gravitational force vector on the cue ball (designated m_1) resulting from the other two balls as well as the magnitude and direction of this force.



- The magnitude of the force is

$$F = \sqrt{F_{31}^2 + F_{21}^2} = 7.66\times10^{-11}\mathrm{N}$$

- The direction of the force is found by evaluating the angle θ

$$\tan\theta = \frac{F_y}{F_x} = \frac{F_{21}}{F_{31}} = \frac{3.75\times10^{-11}}{6.67\times10^{-11}} = 0.562$$

$$\theta = 29.4^o$$

- Gravitational forces between everyday objects have extremely small magnitudes!