LINEAR ALGEBRA II

Kernels & Images of Linear Maps

Let $f: V \to W$ be a linear map between two vector spaces

Def: The <u>Kernel</u> of f (denoted $\text{Ker} f$) is the set of all $v \in V$ such that $f(v) = 0$ (in W)

Def: The <u>Image</u> of f (denoted $\text{Im} f$) is the set of all $w \in W$ such that $w = f(v)$ for some $v \in V$

Lemma 1 ~~Both cont~~

$\text{Ker} f$ is a subspace of V and $\text{Im} f$ is a subspace of W

Proof: 1 <u>Both contain 0:</u>

$f(0_V) = 0_W \Rightarrow \begin{array}{l} 0_V \in \text{Ker} f \\ 0_W \in \text{Im} f \end{array}$

2 <u>closed under addition</u>

$v_1, v_2 \in \text{Ker} f$

- $f(v_1 + v_2) = f(v_1) + f(v_2) = 0 + 0 = 0$

$\Rightarrow v_1 + v_2 \in \text{Ker} f$

- $w_1, w_2 \in \text{Im} f \Rightarrow w_1 = f(v_1)$ and $w_2 = f(v_2)$

$w_1 + w_2 = f(v_1) + f(v_2) = f(v_1 + v_2)$

$\Rightarrow w_1 + w_2 \in \text{Im} f$

3 <u>closed under rescaling</u>

- $v \in \text{Ker} f \Rightarrow f(c \cdot v) = c \cdot f(v) = c \cdot 0 = 0$ so $c \cdot v \in \text{Ker} f$
- $w \in \text{Im} f$ and $w = f(v)$

$c \cdot w = c \cdot f(v) = f(c \cdot v)$, so $c \cdot w \in \text{Im} f$

Kernels & Images of Linear Maps: Nullity & Rank

Def: The dimension of $\operatorname{Ker} f$ is called the <u>nullity</u> of f (denoted $\operatorname{null} f$)

DEF: The dimension of $\operatorname{Im} f$ is called the <u>rank</u> of f (denoted $\operatorname{rk} f$ or $\operatorname{rank} f$)

Ex 1 suppose $I: V \to V$ and $I(v) = v \ \forall v$

As this is the Identity map
$\operatorname{null} I = \{0\}$ $\qquad$ $\operatorname{rk} I = V$

Ex 2 $O: V \to W$ s.t. $O(v) = 0_W$

then $\operatorname{Ker} O = V$ and $\operatorname{null} O = V$ and $\operatorname{Im} O = \{0\}$

Ex 3 $f: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $f\binom{x}{y} = \binom{y}{0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\binom{x}{y}$ linear map

then $\operatorname{Ker} f = \left\{\binom{x}{0}\right\}$ as y must be 0 $\qquad$ $\operatorname{Im} f = \left\{\binom{y}{0}\right\}$

Rank-Nullity Theorem

suppose $\varphi: V \to W$ linear map.
assume V, W finite dimensional (But V is enough)

$$\operatorname{null}(\varphi) + \operatorname{rk}(\varphi) = \dim V$$

for $\varphi: V \to W$

$$\operatorname{null}(\varphi) + \operatorname{rk}(\varphi) = \dim(V)$$

Proof: choose a basis $e_1, \dots e_n$ of V and $f_1, \dots f_m$ of W

- consider the matrix $A = A_{\varphi, e, f}$

THEN: For every vector $v \in V$

$$\varphi(v)_f = A \cdot v_e$$

collumn of coordinates of $\varphi(v)$ w.r.t $f_1 \dots f_m$ $\qquad$ collumn of coordinates of v w.r.t $e_1, \dots e_n$

$\operatorname{Ker} \varphi$ = solution set to $Ax = 0$.
$\operatorname{Im} \varphi$ = column space of A (span of collumns of A)

Proof that $\operatorname{rk}(A) + \operatorname{null}(A) = \dim(A)$ cont...

Proof Continued....

- $\operatorname{Im} \varphi$ = all vectors of the form $\varphi(v)$, $v \in V$

But: $v = x_1 e_1 + \dots x_n e_n$

$\Rightarrow \varphi(v) = \varphi(x_1 e_1 + \dots + x_n e_n) = x_1\varphi(e_1) + \dots x_n \varphi(e_n)$

$\Rightarrow$ All vectors of the form $\varphi(v)$

$=$ span of $\varphi(e_1), \dots \varphi(e_n)$ = <u>column space of A</u> (by definition of A)

- $\operatorname{null} \varphi = \dim \ker \varphi$ = # of <u>free variables in RREF of A</u>
- Let us show that the <u>collumn space of A</u> and the <u>column space of the RREF of A</u> have the <u>same dimension</u>
- Indeed $\underset{\text{RREF}}{R} = M \cdot A$
where M is an invertible matrix (made up of elementary row operations)
- We note that R and A represent the <u>SAME MAP</u> φ for <u>different choices of coordinates</u>, so the <u>dimension</u> of the column space is <u>EQUAL</u> to $\operatorname{rk}\varphi$ in each case

$$R = \begin{pmatrix} 0 & \boxed{1} & x & x & \boxed{0} & x & x & \boxed{0} & x & \\ 0 & \boxed{0} & 0 & 0 & \boxed{1} & x & x & \boxed{0} & x & x \dots \\ \vdots & \boxed{\vdots} & \vdots & \vdots & \boxed{\vdots} & \vdots & \vdots & \boxed{1} & & \\ 0 & \boxed{0} & 0 & 0 & \boxed{0} & 0 & 0 & \boxed{0} & & \end{pmatrix}$$

1st pivot $\qquad$ Proportional to 1st pivot $\qquad$ Linear Combination of FIRST 2 pivotal entries

- the column space of R has a basis of the pivotal columns

$\operatorname{rk}(\varphi)$ = dim of the column space of R
= # of pivotal variables

- our result purely states that:
Free VARIABLES + # Pivotal VARIABLES = # variables

$\varphi(v)$
$\varphi(x_1 e_1 + \dots + x_n e_n)$
$x_1 \varphi(e_1) + \dots + x_n \varphi(e_n)$
columnspace of A = dim

Note:
$A_{\varphi, e', f'} = M_{f' f} A_{\varphi, e, f} M_{e, e'}$

PROOF OF RANK-NULLITY THEOREM

SECOND PROOF: $\mathcal{U} = \operatorname{Ker}\varphi$ is a Subspace of V

- let $e_1 \cdots e_k$ be a basis of $\mathcal{U}$

It is possible to find $f_1, \cdots f_l \in V$
s.t. $e_1, \cdots e_k, f_1, \cdots, f_l$ Form a basis of V

- IF $e_1 \cdots e_k$ already a basis we ARE DONE
- ELSE, choose $f_1 \notin \mathcal{U}$, s.t. $e_1, \cdots, e_k, f_1$ are linearly independant
- IF they form a basis, we are done
- ELSE, choose $f_2 \notin span(e_1, \cdots e_k, f_1)$ etc....

IN such a situation, $f_1 \cdots f_l$ are said to FORM A BASIS OF V RELATIVE TO THE SUBSPACE $\mathcal{U}$

- NOW consider the subspace S
s.t. $S = span(\varphi(f_1), \cdots \varphi(f_l)) \subset W$

let us show that:
1) $S = \operatorname{Im}\varphi$
2) $\dim S = l$

$\boxed{1)\ v \in V}$ so v
$v = x_1 e_1 + \cdots x_k e_k + y_1 f_1 + \cdots y_l f_l$
(coord for our basis of V)
$\varphi(v) = x_1\varphi(e_1) + \cdots + x_k\varphi(e_k) + y_1\varphi(f_1) + \cdots + y_l\varphi(f_l)$

SINCE $e_1, \cdots e_k \in \operatorname{Ker}\varphi$, $\varphi(e_1) = \cdots = \varphi(e_k) = 0$
and $\varphi(v) = y_1\varphi(f_1) + \cdots y_l\varphi(f_l) \Rightarrow \operatorname{Im}\varphi = S$.

$\boxed{2)}$ $\varphi(f_1) \cdots \varphi(f_l)$ form a spanning set of S.
let us show they are linearly independant

$\mathcal{U} = \operatorname{Ker}\varphi \subset V$
$\uparrow$ BASIS of $e_1, e_2, \ldots e_k$

$\exists W = f_1, \cdots f_l$

s.t. $\boxed{\mathcal{U} \oplus W = V}$

- IF $\mathcal{U} = \operatorname{Ker}\varphi = V$, done
- IF $\mathcal{U} = \operatorname{Ker}\varphi < V$
  - choose any f_1 s.t. $f_1 \notin \mathcal{U}$
  - continue to see if $= V$
  - $f_2 \ldots$

2) $\varphi(f_1) \cdots \varphi(f_l)$ form a spanning set of S
let us show they are linearly independant

- Assume the contrary:

$c_1\varphi(f_1) + \cdots + c_l\varphi(f_l) = 0$ has non-trivial solution for $c_1 \cdots c_l$

$\Rightarrow c_1 f_1 + \cdots + c_l f_l \in \operatorname{Ker}\varphi$
$\Rightarrow c_1 f_1 + \cdots + c_l f_l = d_1 e_1 + \cdots + d_k e_k$
$\lightning$ contradiction since $e_1, \cdots e_k, f_1, \cdots f_l$ linearly indep.

FINALLY $\operatorname{null}\varphi + \operatorname{rk}\varphi = \dim V$
is just $k + l = k + l$

Suppose $\varphi: V \to W$ a LINEAR MAP between two vector SPACES.

- It is possible to choose a basis of V (and seperately a basis of W) such that we can get

$$A_{\varphi,e,f} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & & & 0 & 0 \\ 0 & 0 & 1 & 0 & & & 0 & 0 \\ 0 & 0 & 0 & 1 & & & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & & 0 & 0 \end{pmatrix}$$

For $\varphi: V \to W$ we can get

$$A_{\varphi,e,f} = \left(\begin{array}{c|c} I_l & 0 \\ \hline 0 & 0 \end{array}\right)$$

where $l = \operatorname{rk}\varphi$

- IF we CONSIDER LINEAR TRANSFORMATIONS $\varphi: V \to V$, we do not have that much freedom.
(can't choose coordinates "before" and "after" separately)

What we CAN do by changing coordinates is we can replace A by $C^{-1}AC$
and we know various characteristics (e.g.: determinant, trace ....) that do not change

Sums & Direct sums of Subspaces

Let V be a vector space, $V_1, V_2, \dots V_k$ are subspaces of V.

The sum $V_1 + V_2 + \dots + V_k$ is the subset of V consisting of ALL VECTORS:
$v_1 + v_2 + \dots + v_k$ where $v_1 \in V_1, \dots v_k \in V_k$

A sum $V_1 + \dots + V_k$ is said to be DIRECT if $0 + 0 + 0 \dots + 0$ is the only way to find 0 in $V_1 + \dots + V_k$

IF $V_1, \dots V_k$ are one dimensional subspaces ($V_i = \{c u_i\}$), then $V_1 + \dots + V_k = \text{span}(u_1, \dots, u_k)$

AND the sum is direct $\Leftrightarrow$ $u_1 \dots u_k$ are linearly independent

Ex: $\mathbb{R}^4$, $u_1 = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 0 \\ c \\ d \end{pmatrix}$

$\mathcal{U}_1 + \mathcal{U}_2 = \mathbb{R}$

this sum is direct (by inspection)

Notation: If the sum of $V_1, \dots, V_k$ is direct we write $V_1 \oplus V_2 \oplus \dots \oplus V_k$ instead

The sum of 2 subspaces $\mathcal{U}_1, \mathcal{U}_2 \in V$ is direct $\Leftrightarrow$ $\mathcal{U}_1 \cap \mathcal{U}_2 = \{0\}$

$v_1 + v_2 = 0$ NON-TRIVIAL WAY to get zero $\Leftrightarrow$ $v_1 = -v_2$ NONTRIVIAL VECTOR IN $\mathcal{U}_1 \cap \mathcal{U}_2$

LOOKING AT INTERSECTIONS IS NOT ENOUGH WHEN THERE ARE 3+ SUBSPACES

Theorem

Let $\mathcal{U}_1, \mathcal{U}_2$ be subspaces of a vector space V

then
$$\dim(\mathcal{U}_1 + \mathcal{U}_2) = \dim(\mathcal{U}_1) + \dim(\mathcal{U}_2) - \dim(\mathcal{U}_1 \cap \mathcal{U}_2)$$

PROOF

Let $e_1, \dots, e_p$ be a basis of $\mathcal{U}_1 \cap \mathcal{U}_2$

Let $f_1, \dots, f_q$ be s.t. $e_1, \dots e_p, f_1 \dots f_q$ form basis of $\mathcal{U}_1$

Let $g_1, \dots, g_r$ be s.t. $e_1, \dots e_p, g_1 \dots g_r$ form a basis of $\mathcal{U}_2$

Let us show that $e_1, \dots, e_p, f_1, \dots f_q, g_1 \dots g_r$ forms a basis of $\mathcal{U}_1 + \mathcal{U}_2$

1) span: all vectors of $\mathcal{U}_1$ are a combination of $e_1 \dots e_p, f_1 \dots f_q$
all vectors of $\mathcal{U}_2$ are a combination of $e_1 \dots e_p, g_1 \dots g_r$
So every vector from $\mathcal{U}_1 + \mathcal{U}_2$ is a combination of all these vectors

2) LINEAR INDEPENDANCE:

Assume $a_1 e_1 + \dots + a_p e_p + b_1 f_1 + \dots + b_q f_q + c_1 g_1 + \dots + c_r g_r = 0$

$$\underbrace{a_1 e_1 + \dots + a_p e_p + b_1 f_1 + \dots + b_q f_q}_{\mathcal{U}_1} = \underbrace{-c_1 g_1 + \dots + c_r g_r}_{\mathcal{U}_2}$$

Assume is in $\mathcal{U}_1 \cap \mathcal{U}_2$

so this vector can be written as $d_1 e_1 + \dots d_p e_p$

$\therefore -c_1 g_1 - \dots - c_r g_r = d_1 e_1 + \dots + d_p e_p$

$\Rightarrow c_1 = \dots = c_r = d_1 = \dots = d_p = 0$

Since we have a basis of $\mathcal{U}_2$

Now look at LHS, and conclude:

$a_1 = \dots = a_p = b_1 = \dots = b_q = 0$

so these vectors are linearly independant

Finally: $\dim(\mathcal{U}_1 + \mathcal{U}_2) = p + q + r = (p+q) + (q+r) -$
$= \dim(\mathcal{U}_1) + \dim(\mathcal{U}_2) - \dim(\mathcal{U}_1 \cap \mathcal{U}_2)$

In practice, to find a basis of a subspace, one chooses a coordinate system, represents vectors by columns of coordinates, and for the subspace $(u_1, \dots u_k)$ Compute the column echelon form of the matrix $(u_1 \; u_2 \; \dots \; u_k)$; columns of the matrix give a basis of $\mathcal{U}$. Using that, intersections are computed directly

Computing intersections of subspaces & relative bases:

① Let V be a vector space in $\mathbb{R}^n$. $\mathcal{U}_1$ & $\mathcal{U}_2$ are two subspaces of V, given by linear spans $\mathcal{U}_1 = \text{span}(e_1 \dots e_K)$ & $\mathcal{U}_2 = (f_1 \dots f_\ell)$

② The goal is to compute $\mathcal{U}_1 \cap \mathcal{U}_2$

③ Compute convenient bases of $\mathcal{U}_1, \mathcal{U}_2$. Choose some basis of V, and compute co-ordinates of all vectors $e_1 \dots f_\ell$ w.r.t this basis

④ Form a matrix M_1 using columns of co-ordinates of $e_1 \dots e_K$ and another M_2 using columns of coordinates $f_1 \dots f_\ell$.

⑤ Compute the reduced column echlon forms of M_1 & M_2. Non-zero columns of the RCEF of M_1 is a basis of $\mathcal{U}_1$, and the RCEF of M_2 is a basis of $\mathcal{U}_2$

⑥ Form a system of LINEAR Equations w/ UNKNOWNS: $a_1 \dots a_p, b_1 \dots b_q$,

$$\to a_1 g_1 + \dots + a_p g_p = b_1 h_1 + \dots + b_q h_q$$

⑦ Solve this system and for each solution, the vector $a_1 g_1 + \dots + a_p g_p$ is the intersection and all the vectors are obtained in this way!

EX

$V = \mathbb{R}^3$

$$\mathcal{U}_1 = \text{span}\left[\begin{pmatrix}2\\1\\0\\-4\\2\end{pmatrix}, \begin{pmatrix}-4\\1\\3\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\5\\-1\\-1\\14\end{pmatrix}\right]$$

$$\mathcal{U}_2 = \text{span}\left[\begin{pmatrix}2\\1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\-2\\-3\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-2\\-2\\2\end{pmatrix}, \begin{pmatrix}0\\1\\1\\2\\1\end{pmatrix}\right]$$

BASIS OF $\mathcal{U}_1$:

$$M_1^T = \begin{bmatrix}2 & 1 & 0 & -4 & 2\\ -4 & 1 & 3 & -1 & 2\\ 0 & 5 & -1 & -1 & 14\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & -\frac{15}{8} & \frac{1}{2}\\ 0 & 1 & 0 & -\frac{1}{4} & 1\\ 0 & 0 & 1 & -\frac{11}{4} & 1\end{bmatrix}$$

BASIS OF $\mathcal{U}_2$:

$$M_2^T = \begin{bmatrix}2 & 1 & 0 & 1 & 1\\ 2 & -1 & -2 & -3 & -1\\ 1 & 0 & -2 & -2 & 2\\ 0 & 1 & 1 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0 & -\frac{2}{3}\\ 0 & 1 & 0 & 1 & \frac{7}{3}\\ 0 & 0 & 1 & 1 & -\frac{4}{3}\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\mathcal{U}_1 \Rightarrow a_1\begin{pmatrix}1\\0\\0\\-15/8\\1/2\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\\-1/4\\1\end{pmatrix} + a_3\begin{pmatrix}0\\0\\1\\-11/4\\1\end{pmatrix}$$

$$\mathcal{U}_2 \Rightarrow b_1\begin{pmatrix}1\\0\\0\\0\\-2/3\end{pmatrix} + b_2\begin{pmatrix}0\\1\\0\\1\\7/3\end{pmatrix} + b_3\begin{pmatrix}0\\0\\1\\1\\-4/3\end{pmatrix}$$

IN MATRIX FORM: (PART 6)

$$\begin{bmatrix}1 & 0 & 0 & -1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & -1\\ -\frac{15}{8} & -\frac{1}{4} & -\frac{11}{4} & 0 & -1 & -1\\ \frac{1}{2} & 1 & 1 & \frac{2}{3} & -\frac{7}{3} & \frac{4}{3}\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\\b_1\\b_2\\b_3\end{bmatrix}$$

RREF ↓

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 54/13\\ 0 & 1 & 0 & 0 & 0 & -70/13\\ 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & 0 & 54/13\\ 0 & 0 & 0 & 0 & 1 & -70/13\end{bmatrix}$$

$a_1 = -54/13\, t$
$a_2 = 70/13\, t$
$a_3 = t$
$b_1 = -54/13\, t$
$b_2 = 70/13\, t$
$b_3 = t$

INTERSECTION:

$$-\frac{54}{13}t\begin{bmatrix}1\\0\\0\\-\frac{15}{8}\\\frac{1}{2}\end{bmatrix} + \frac{70}{13}t\begin{bmatrix}0\\1\\0\\-\frac{1}{4}\\1\end{bmatrix} + t\begin{bmatrix}0\\0\\1\\-11/4\\1\end{bmatrix} = t\dots$$

COMPUTING A BASIS OF W relative to a subspace W'

① Let V be a vector space in $\mathbb{R}^n$

② $W \in V$ is a subspace w/ span$(e_1, \dots e_p)$
$W' \in V$ is a subspace w/ span$(f_1 \dots f_q)$

GOAL: compute a basis of W relative to W'

STRATEGY: FORM a MATRIX M' made of coordinates of $f_1 \dots f_q$ and a matrix M from $e_1 \dots e_p$. Compute the RCEF of M' and use it to reduce the matrix M. Then, compute RCEF of that ⇒ Relative BASIS

Ex: $V \in \mathbb{R}^5$, $W = \mathcal{U}_2 = \text{span}\left[\begin{pmatrix}2\\1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\-2\\-3\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-2\\-2\\2\end{pmatrix}, \begin{pmatrix}0\\1\\1\\2\\1\end{pmatrix}\right]$

$$W' = \text{SPAN}\left[\begin{pmatrix}1\\-1\\3\\2\\-7\end{pmatrix}, \begin{pmatrix}3\\1\\1\\2\\-1\end{pmatrix}\right]$$

BASIS OF W': $(M')^T = \begin{pmatrix}1 & -1 & 3 & 2 & -7\\ 3 & 1 & 1 & 2 & -1\end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix}1 & 0 & 1 & 1 & -2\\ 0 & 1 & -2 & -1 & -5\end{pmatrix}$

R.R.E.F. OF M: $(M)^T = \begin{pmatrix}1 & 0 & 0 & 0 & -2/3\\ 0 & 1 & 0 & 1 & 7/3\\ 0 & 0 & 1 & 1 & -4/3\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$ (DONE BEFORE) CAN TRY TO GET m' $\to (m')^T = \begin{pmatrix}r_1 + r_3\\ r_2 - 2r_3\end{pmatrix}$ WORKS, WHICH MEANS $W' \in W$

⑧ Now we reduce basis of W using W':
(i.e: ~~add rows~~ remove pivots of W using pivots of W')

Thus we GET: $\begin{pmatrix}0 & 0 & -1 & -1 & 4/3\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$

CONCLUSION: ~~the matrix~~ $\begin{pmatrix}0\\0\\-1\\-1\\4/3\end{pmatrix}$ FORMS A BASIS OF W relative to W'.

INVARIANT SUBSPACES

Invariant Subspaces

Let V be a vector space.
$\varphi: V \to V$ be a linear transformation.
A subspace $\mathcal{U} \subset V$ is said to be *invariant* under φ (with respect to φ), if $\varphi(\mathcal{U}) \subseteq \mathcal{U}$
THAT IS, $\forall u \in \mathcal{U}$, $\varphi(u) \in \mathcal{U}$

Let v be an eigenvector of $\varphi(v) = c \cdot v$.
Then $\mathrm{span}(v)$ is an invariant subspace,
since $\varphi(a \cdot v) = a\varphi(v) = a \cdot c \cdot v = ac(v)$

Suppose $\mathcal{U} \subset V$ is an invariant subspace, where $e_1 \cdots e_p$ is a basis of $\mathcal{U}$, and $f_1 \cdots f_q$ is a basis of V relative to $\mathcal{U}$
The matrix of φ w.r.t this basis is of the form:

$$\begin{array}{c} p\{ \\ q\{ \end{array} \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right) \quad \begin{array}{c} \\ \underbrace{\quad}_{p} \ \underbrace{\quad}_{q} \end{array}$$

Rectangular corner of zeros

EIGENVALUES ARE ROOTS of $\det(A_{\varphi c} - \lambda I_n) = 0$

Commuting Transformations

Def: The linear transformations $\varphi: V \to V$, $\psi: V \to V$ <u>commute</u> if $\varphi \circ \psi = \psi \circ \varphi$
or, in other words, $\varphi(\psi(v)) = \psi(\varphi(v))$ for all $v \in V$

Theorem: let $\varphi_i: V \to V$ $(i = 1, \dots m)$ be pairwise commuting linear transformations of a vector space V
THEN: these transformations have a <u>common eigenvector</u>
Recall: we assume that scalars are complex numbers, so that every linear transformation has an eigenvalue

Proof: Induction on $\dim(V)$

- If $\dim(V) = 1$, then the only basis vector of V is a common eigenvector of φ
- Assume $\dim V > 1$
  - If each φ_i is a ~~scalad~~ scalar multiple of the Identity transformation, there is nothing to prove, any vector is a common eigenvector
  - W.L.O.G: φ_1 is <u>not</u> a multiple of the identity transformation. Let c be an eigenvalue of φ_1, and let $\mathcal{U}$ be the solution space to $\varphi_1(v) = c \cdot v$.
    We have $0 \subsetneq \mathcal{U} \subsetneq V$ (since c is an eigenvector, and φ is <u>not</u> a multiple of Identity)
  - <u>We claim that $\mathcal{U}$ is invariant under all φ_i.</u>
    Indeed if $\varphi_1(v) = c \cdot v$, then
    $\varphi_1(\varphi_i(v)) = \varphi_i(\varphi_1(v)) = \varphi_i(c \cdot v) = c \cdot \varphi_i(v) \Rightarrow \varphi_i(v) \in \mathcal{U}$

<u>By the induction hypothesis</u>:
$\varphi_1, \dots \varphi_k$ have a common eigenvector in $\mathcal{U}$ ∎

Case: $\varphi^2=\varphi$ (like orthogonal projections in 3D)

- All eigenvectors are of 0's and 1's.
- INDEED, IF v is an eigenvector for an eigenvalue c THEN $\varphi(\varphi(v)) = c^2 v$, $\varphi(v)=c\cdot v \Rightarrow c^2 \cdot v = c\cdot v \Rightarrow c^2=c$

CONSIDER THE SUBSPACE $\operatorname{Ker}\varphi$

LEMMA: we have $\operatorname{Ker}(\varphi)\cap \operatorname{Im}(\varphi) = \{0\}$

PROOF: Let $v \in \operatorname{Ker}(\varphi)\cap\operatorname{Im}(\varphi)$, s.t. $\varphi(v)=0$ and $v=\varphi(w)$

THEN: $0=\varphi(v)=\varphi(\varphi(w))=\varphi(w)=v$

Let us show that $V=\operatorname{Ker}\varphi\oplus\operatorname{Im}\varphi$

- FIRST, WE HAVE JUST SHOWN THAT THE SUM OF $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ is direct.
- ALSO, $\dim(\operatorname{Ker}\varphi\oplus\operatorname{Im}\varphi) = \dim\operatorname{Ker}\varphi + \dim\operatorname{Im}\varphi = \operatorname{null}\varphi + \operatorname{rk}\varphi = \dim V$
- SO THE SUBSPACE $\operatorname{Ker}\varphi\oplus\operatorname{Im}\varphi$ coincides with V

MOREOVER: $\operatorname{Ker}\varphi$ & $\operatorname{Im}\varphi$ ARE invariant under φ:

- If $v\in\operatorname{Ker}\varphi$, then $\varphi(v)=0\in\operatorname{Ker}\varphi$
- If $v\in\operatorname{Im}\varphi$ s.t. $v=\varphi(w)$, then $\varphi(v)=\varphi(\varphi(w))=\varphi(w)=v\in\operatorname{Im}$

CONCLUSION: IF $\varphi^2=\varphi$, then V has a BASIS OF EIGENVECTORS of φ s.t. the matrix of φ becomes the matrix on the right, with $m=\operatorname{rk}\varphi$

$$\begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}$$

CASE: $\varphi^2=0$

NOTE: if $\varphi^2=0$, then $\operatorname{Im}\varphi\subseteq\operatorname{Ker}\varphi$: IF $v=\varphi(w)$, $\varphi(v)=\varphi(\varphi(w))=0$

$\therefore$ The approach that worked for $\varphi^2=\varphi$ no longer works

Let $e_1,\dots e_k$ be A BASIS OF V relative to $\operatorname{Ker}\varphi$

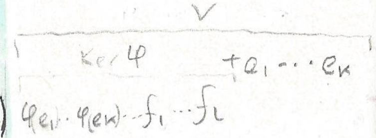
$\Rightarrow \varphi(e_1),\dots\varphi(e_k)\in\operatorname{Im}\varphi\subseteq\operatorname{Ker}\varphi$

CASE: $\varphi^2=0$

NOTE IF $\varphi^2=0$, then $\operatorname{Im}\varphi\subseteq\operatorname{Ker}\varphi$

(IF $v=\varphi(w)$ then $\varphi(v)=\varphi(\varphi(w))=0$

therefore, our approach that worked for $\varphi^2=\varphi$ is no longer valid

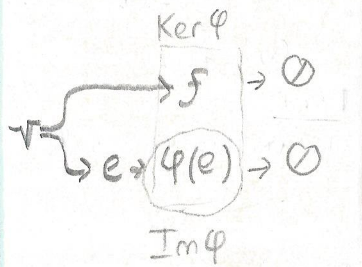


- Let $e_1,\dots e_k$ be a basis of V relative to $\operatorname{Ker}\varphi$

$(\varphi[e_1],\dots\varphi[e_k]\in\operatorname{Im}\varphi\subseteq\operatorname{Ker}\varphi)$

- TAKE $f_1\dots f_l$ that form a basis of $\operatorname{Ker}(\varphi)$ relative to $(\varphi(e_1)\dots\varphi(e_k))$

we shall show $e_1\dots e_k, \varphi(e_1)\dots\varphi(e_k), f_1\dots f_l$ form a basis of V



- FIRST: $\varphi(e_1)\dots\varphi(e_k), f_1,\dots f_l$ form a basis of $\operatorname{Ker}\varphi$
- IF $c_1\varphi(e_1)+\dots+c_k\varphi(e_k)+d_1f_1+\dots+d_lf_l=0$

THEN $d_1=\dots=d_l=0$ once $f_1,\dots f_l$ form a relative basis

$$0=c_1\varphi(e_1)+\dots+c_k\varphi(e_k)=\varphi(c_1e_1+\dots+c_ke_k)$$

$\Rightarrow c_1e_1+\dots+c_ke_k\in\operatorname{Ker}\varphi$

$\Rightarrow c_1=\dots=c_k=0$

- This shows us they are linearly independent
- Adjoining $e_1\dots e_k$, we get a basis of V

REARRANGING, WE GET A BASIS:

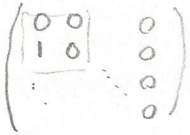
$e_1, \varphi(e_1), e_2, \varphi(e_2), \dots e_k, \varphi(e_k), f_1, \dots f_l$

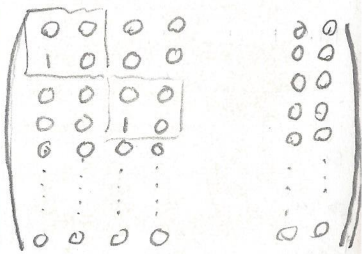
Let us write a matrix of φ w.r.t this BASIS

$\varphi(e_1)$ is a basis vector

$\varphi(\varphi(e_1))=0$

$\varphi(f_1)=\dots\varphi(f_l)=0$





CASE: $\varphi^k=0$

WE ASSUME that k is smallest exponent w/ $\varphi^k=0$

consider the sequence of subspaces $\underset{N_1}{\operatorname{Ker}\varphi}, \underset{N_2}{\operatorname{Ker}\varphi^2}, \dots \underset{N_k}{\operatorname{Ker}\varphi^k}=V$

Let $e_1,\dots e_s$ be a basis of N_k relative to N_{k-1}

CASE OF $\varphi^K = 0$

CASE: $\varphi^K = 0$

- We assume that K is the smallest k s.t. $\varphi^K = 0$
- Consider a sequece of subspaces

$$\underset{N_1}{\operatorname{Ker}\varphi},\ \underset{N_2}{\operatorname{Ker}\varphi^2},\ \dots\ \underset{N_{K-1}}{\operatorname{Ker}\varphi^{K-1}},\ \underset{N_K}{\operatorname{Ker}\varphi^{K}} = V$$

- Let $e_1 \dots e_s$ be a basis of N_K relative to N_{K-1}
- consider $\varphi(e_1), \dots \varphi(e_s)$. These vectors belong to N_{K-1}
- Let $f_1, \dots f_t$ be a basis of N_{K-1} relative to $\operatorname{span}(\varphi(e_1), \dots \varphi(e_s) + N_{K-2})$

Lemma:

The vectors $e_1 \dots e_s, \varphi(e_1) \dots \varphi(e_s), \varphi^2(e_1) \dots \varphi^2(e_s)$
$f_1, \dots f_t, \varphi(f_1) \dots \varphi(f_t)$
are linearly independant relative to N_{K-3}

PROOF. SUPPOSE THAT:

$$a_1^{(1)}e_1 + \dots + a_s^{(1)}e_s + \overbrace{a_{(1)}^{(2)}\varphi(e_1) + \dots a_s^{(2)}\varphi(e_s) + a_1^{(3)}\varphi^2(e_1) \dots a_s^{(3)}\varphi^2(e_s) + b_1^{(1)}f_1 + \dots b_t^{(1)}f_t + b_1^{(2)}\varphi(f_1) + \dots + b_t^{(2)}\varphi(f_t)}^{\text{IN } N_{K-1}}$$

is AN ELEMENT of N_{K-3}

THIS IMPLIES $a_1^{(1)}e_1 + \dots a_s^{(1)}e_s \in N_{K-1}$

$\Rightarrow a_1^{(1)} = \dots = a_s^{(1)} = 0$ since these vectors form a relative basis

so, $b_1^{(1)} = \dots = b_t^{(1)} = 0$ since $f_1 \dots f_t$ form a relative basis

so $a_1^{(2)} = \dots = a_s^{(2)} = 0$ since $e_1 \dots e_s$ form a relative basis

~~LEMMA~~: so $b_1^{(2)} = \dots b_t^{(2)} = 0$ since $f_1, \dots f_t$ are a relative basis

$\underbrace{e_1 \dots e_s}_{\text{BASIS relative to } N_{K-1}},\ \varphi(e_1) \dots \varphi(e_s),\ \varphi^2(e_1), \dots, \varphi^2(e_s)$

$f_1 \dots f_t,\ \varphi(f_1) \dots \varphi(f_t)$

$g_1 \dots g_t$

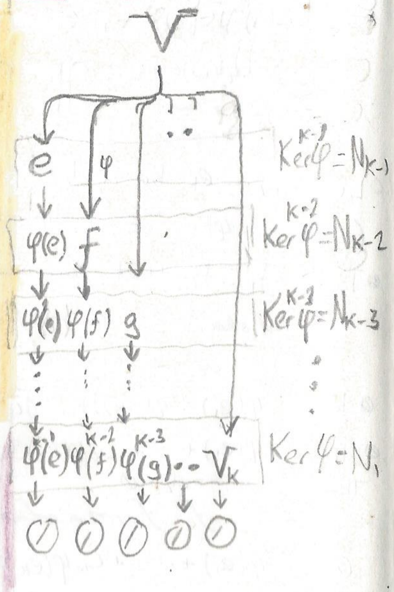
BASIS Relative to N_{K-2}

BASIS RELATIVE TO N_{K-3}

- CONTINUE PROCEDURE of completing vectors linearly independant relative to N_{K-i} to a relative basis
- AFTER THIS PROCEDURE, WE OBTAIN A BASIS OF V OF THE FORM:

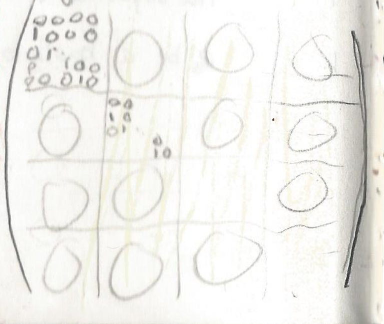
$e_1, \varphi(e_1) \dots \varphi^{K-1}(e_1), e_2, \varphi(e_2) \dots \varphi^{K-1}(e_2) \dots f_1, \varphi(f_1) \dots \varphi^{K-2}(f_1) \dots$

GIVING A MATRIX AS ON THE RIGHT



$$a^{(1)}e + a^{(2)}\varphi(e) + a^{(3)}\varphi^2(e) + b^{(1)}f + b^{(2)}\varphi(f)\Big|_{N_{K-3}}$$

For each "thread", $e, \varphi(e) \dots \varphi^{s-1}(e)$, we get an $s \times s$ block: $\begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}$



$\varphi^K = 0$ EXAMPLES

Ex $V = \mathbb{R}^3$, φ is multiplication by $A = \begin{pmatrix} -3 & 1 & -1 \\ -12 & 4 & -4 \\ -3 & 1 & -1 \end{pmatrix}$

$A^2 = 0$, so $\varphi^2 = 0$

$$\begin{pmatrix} -3 & 1 & -1 \\ -12 & 4 & -4 \\ -3 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -3 & 1 & -1 \\ -12 & 4 & -4 \\ -3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{V = \operatorname{Ker}\varphi^2}$ $\boxed{\operatorname{Ker}\varphi \subset \operatorname{Ker}\varphi^2}$

$\dim V = 3$
$\operatorname{null}\varphi = 3 - \operatorname{rk}\varphi = 2$
$\operatorname{Ker}\varphi = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} ; -3x + y - z = 0 \right\}$

y, z ARE free variables $\Rightarrow$ Basis of $\operatorname{Ker}\varphi$: $\begin{pmatrix} 1/3 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}$

WE NOW FIND the BASIS OF V relative to $\operatorname{Ker}\varphi$

- RCEF OF the basis is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 1 \end{pmatrix}$, which we now use to reduce the basis of V $\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}$ giving us the MATRIX $\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}$,

SO A RELATIVE BASIS OF V is $f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$A \cdot f = \varphi(f) = \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}$

TO PROCEED, we look for a basis of $\operatorname{Ker}\varphi$ relative to $\operatorname{span}(\varphi(f))$

$\varphi(f) = \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}$, BASIS OF $\operatorname{Ker}\varphi$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 1 \end{pmatrix}$, Reducing, we get $\begin{pmatrix} 0 & 0 \\ -4 & 1 \\ -4 & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$

So we have $g = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ forms a relative BASIS.

$V \to f \to \varphi(f) \to 0$ THREAD OF LENGTH 2 $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -1 \end{pmatrix}$

$V \to g \to 0$ THREAD OF Length 1 $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Ex: $V = \mathbb{R}^3$, φ IS MULTIPLICATION BY $A = \begin{pmatrix} 21 & -7 & 8 \\ 60 & -20 & 23 \\ -3 & 1 & -1 \end{pmatrix}$

$A = \begin{pmatrix} 21 & 7 & 8 \\ 60 & -20 & 23 \\ -3 & 1 & -1 \end{pmatrix}$ $A^2 = \begin{pmatrix} -3 & 1 & -1 \\ -9 & +3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $A^3 = 0$ so $\varphi^3 = 0$

- We KNOW $\underset{\text{Null}=1}{\operatorname{Ker}\varphi} \subset \underset{\text{Null}=2}{\operatorname{Ker}\varphi^2} \subset \underset{\dim=3}{V}$
- FIRST STEP: basis of V relative to $\operatorname{Ker}(\varphi^2)$ AS BEFORE:

$\operatorname{Ker}(\varphi^2) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : -3x + y - z = 0 \right\}$ has a basis $\begin{pmatrix} +1/3 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \to f = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

f, $\varphi(f) = \begin{pmatrix} 8 \\ 23 \\ -1 \end{pmatrix}$, $\varphi^2(f) = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$

$V \supset \operatorname{Ker}\varphi \supset \operatorname{Ker}^2\varphi$ SO ~~SO~~/ BASIS CAN BECOME $\begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$

EXAMPLES

Ex 3: $V = \mathbb{R}^4$, φ is multiplication by A

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix} \to A^2 = \emptyset \qquad \underset{\dim = 2}{\operatorname{Ker}\varphi} \subset \underset{\dim = 4}{V}$$

FIRST STEP: BASIS OF V relative to $\ker(\varphi)$

RREF of A: $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

CASE OF AN ARBITRARY TRANSFORMATION ($\varphi^K \stackrel{?}{=} \emptyset$ NOT ASSURED)

- we cannot assume $\varphi^K = \emptyset$
- STILL, we will consider $N_i = \operatorname{Ker}\varphi^i$, $i \geq 1$

so, $N_1 \subset N_2 \subset \dots \subset N_i \subset \dots$

because if $\varphi^i(v) = \emptyset$, then $\varphi^{i+1}(v) = \varphi(\overset{0}{\varphi^i(v)}) = \emptyset$

- SINCE V is finite-dimensional, these inclusions of subspaces cannot all be strict. (stict inclusion strictly increases dimension)

- Let us assume $N_K = N_{K+1}$, and choose the smallest possible K for this

→ Let us show that we have:

$N_K = N_{K+1} = N_{K+2} = \dots$

Percisely, $N_{K+L} = N_{K+L+1}$ by induction on L

- INITIAL CASE: $\ell = \emptyset$
- STEP OF INDUCTION: Suppose $N_{K+L} = N_{K+\ell+1}$ for some L
- WANT: $N_{K+\ell+1} \stackrel{?}{=} N_{K+\ell+2}$

we let $v \in N_{K+L+2}$ so that $\varphi^{K+L+2}(v) = \emptyset$

$\varphi^{K+L+2}(v) = \varphi^{K+L+1}(\varphi(v)) \rightsquigarrow \varphi(v) \in N_{K+L+1} = N_{K+L}$

$\Rightarrow \varphi^{K+L}(\varphi(v)) = \emptyset = \varphi^{K+\ell+1}(v)$

$\Rightarrow v \in N_{K+L+1}$

- SINCE $N_{K+L+1} \subset N_{K+L+2}$ AND WE JUST PROVED THAT $N_{K+L+2} \subset N_{K+L+1}$ WE CONCLUDE $N_{K+L+1} = N_{K+L+2}$

- LET US NOW SHOW:

$N_K = \operatorname{Ker}\varphi^K \cap \operatorname{Im}\varphi^K = \{\emptyset\}$

(REMARK: IF $\varphi^2 = \emptyset$, then $\operatorname{Im}\varphi \subset \operatorname{Ker}\varphi$, so our choice of K is important here)

- Let $v \in \operatorname{Ker}\varphi^K \cap \operatorname{Im}\varphi^K$ s.t $\varphi^K(v) = \emptyset$, $v = \varphi^K(w)$

$\Rightarrow \varphi^K(\varphi^K(w)) = \varphi^{2K}(w) = \emptyset$

$\Rightarrow w \in N_{2K} = N_K \Rightarrow \varphi^K(w) = \emptyset$

AS A CONSEQUENCE: $V = \operatorname{Ker}\varphi^K \oplus \operatorname{Im}\varphi^K$

$\operatorname{Ker}\varphi^K \cap \operatorname{Im}\varphi^K = \{\emptyset\}$

$V = \operatorname{Ker}\varphi^K \oplus \operatorname{Im}\varphi^K$

$\operatorname{Ker}\varphi^K \cap \operatorname{Im}\varphi^K = \{\emptyset\} \Rightarrow$ SUM IS DIRECT

$$\dim(\operatorname{Ker}\varphi^K \oplus \operatorname{Im}\varphi^K) = \dim \operatorname{Ker}\varphi^K + \dim \operatorname{Im}\varphi^K = \operatorname{null}\varphi^K + \operatorname{rk}\varphi^K = \dim V$$

$\Rightarrow \operatorname{Ker}\varphi^K \oplus \operatorname{Im}\varphi^K = V$

NOW LET US PROVE THE FOLLOWING:

1. $\operatorname{Ker}\varphi^K$, $\operatorname{Im}\varphi^K$ are invariant subspaces of φ
2. ON $\operatorname{Ker}\varphi^K$, we have $\varphi^K = \emptyset$, and and we have that φ has only the zero eigenvalue
3. ON $\operatorname{Im}\varphi^K$, φ has NO zero eigenvalues

PROOF OF 1-3:

1) IF $v \in \operatorname{Ker}\varphi^K$ s.t $\varphi^K(v) = \emptyset$, then $\varphi^{K+1}(v) = \varphi(\varphi^K(v)) = \varphi^K(\varphi(v)) = \emptyset$

so $\varphi(v) \in \operatorname{Ker}\varphi^K$

IF $v \in \operatorname{Im}\varphi^K$, so that $v = \varphi^K(w)$,

THEN $\varphi(v) = \varphi(\varphi^K(w)) = \varphi^{K+1}(w) = \varphi^K(\varphi(w))$

SO $\varphi(v) \in \operatorname{Im}\varphi^K$

2) IF $v \in \operatorname{Ker}\varphi^K$, then $\varphi^K(v) = \emptyset$, and if

IF $\varphi(v) = \lambda \cdot v$ then $\varphi^K(v) = \emptyset = \lambda^K v \Rightarrow \lambda^K = 0 = \lambda$

3) SUPPOSE $v \in \operatorname{Im}\varphi^K$ s.t. $v = \varphi^K(w)$

SUPPOSE $\lambda = \emptyset$ is AN EIGENVECTOR of φ

that is, $\varphi(v) = 0 \cdot v \Rightarrow \varphi^K(v) = \emptyset \Rightarrow v \in \operatorname{Ker}\varphi^K$

BUT $\operatorname{Ker}\varphi^K \cap \operatorname{Im}\varphi^K = \{\emptyset\} \Rightarrow v = \emptyset$ NOT EIGENVECTOR

REMARK:

INVARIANCE OF $\operatorname{Ker}\varphi^K$, $\operatorname{Im}\varphi^K$ assumes that statements 2, 3 make sense

Now we can prove (by induction on a number of distinct eigen values of φ) the following:

"THERE EXISTS A DIRECT SUM DECOMPOSITION

$$V = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \dots \oplus \mathcal{U}_m \text{ s.t.}$$

1) Each of the subspaces is invariant under φ

2) For each $\mathcal{U}_i$, φ has just one eigenvalue λ_i on $\mathcal{U}_i$

3) For each $\mathcal{U}_i$, there exists an exponent n_i s.t. $(\varphi - \lambda_i I)^{n_i} = \emptyset$ on $\mathcal{U}_i$"

PROOF:

- BASIS OF INDUCTION: φ has just one eigenvalue λ on V
- CONSIDER $\varphi_\lambda = \varphi - \lambda I$

BY OUR PREVIOUS RESULTS: there exists k_λ s.t $V = \underbrace{\operatorname{Ker}\varphi_\lambda^{k_\lambda}}_{(*)\ \neq\{0\}} \oplus \underbrace{\operatorname{Im}\varphi_\lambda^{k_\lambda}}_{=\emptyset\ (**)}$

(*) Follows from the fact that $\operatorname{Ker}\varphi_\lambda \neq \{0\}$

(**) Follows from the fact that on $\operatorname{Im}\varphi_\lambda^{k_\lambda}$, φ_λ has NO zero ~~eigen~~ eigenvalues

REMARK: For $V = V_1 \oplus V_2$, with V_1, V_2 invariant under φ then $\chi_{\varphi,V}(t) = \chi_{\varphi,V_1}(t) \cdot \chi_{\varphi,V_2}(t)$

$$A_{\varphi,V} = \left(\begin{array}{c|c} A_{\varphi,V_1} & \emptyset \\ \hline \emptyset & A_{\varphi,V_2} \end{array}\right) \qquad A_{\varphi,V} - tI = \left(\begin{array}{c|c} A_{\varphi,V_1} - tI & \emptyset \\ \hline \emptyset & A_{\varphi,V_2} - tI \end{array}\right)$$

NOTE: χ denotes characteristic polynomials

- For step of induction, do the same for λ being one of the eigenvalues. Then we can put $\mathcal{U}_1 = \operatorname{Ker}\varphi_\lambda^{k_\lambda}$, and on the invariant subspace $\operatorname{Im}\varphi_\lambda^{k_\lambda}$ φ has eigenvalues different from λ, so we replace V with $\operatorname{Im}\varphi_\lambda^{k_\lambda}$ and apply the induction hypothesis

THM: Let V be a finite-dimensional vector space ($\mathbb{C}$) and let $\varphi: V \to V$ be a lineer transformation

Then: there exists a basis of V of the form:

$$e_1^{(1)}, \dots, e_{m_1}^{(1)}$$
$$e_1^{(2)}, \dots, e_{m_2}^{(2)}$$
$$\vdots$$
$$e_1^{(s)}, \dots, e_{m_s}^{(s)}$$

and scalars $\lambda_1, \dots \lambda_s$ s.t:

$$(\varphi-\lambda_1 I)e_1^{(1)} = e_2^{(1)}, \; (\varphi-\lambda_1 I)\cdot e_2^{(1)} = e_3^{(1)}, \dots, (\varphi-\lambda I)e_{m_1}^{(1)} = 0$$
$$\vdots$$
$$(\varphi-\lambda_s I)e_1^{(s)} = e_2^{(s)}, \qquad (\varphi-\lambda I)e_{m_s}^{(s)} = 0$$

- WITH RESPECT to this BASIS, the matrix of φ is:

$$A = \begin{pmatrix} J_{m_1}(\lambda_1) & 0 & & \\ 0 & J_{m_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & 0 & & \end{pmatrix} \quad \text{where } J_m(\lambda) = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 1 & \lambda & 0 \cdot & 0 \\ 0 & 1 & \lambda \cdot & 0 \\ 0 & \cdot & \cdot & \vdots \\ 0 & \cdots & 1 & \lambda \end{pmatrix}$$

called the *JORDAN NORMAL FORM*

SUMMARY OF THE STRATEGEY:
(OF A JORDAN BASIS)

1) Compute $\chi_\varphi(t)$ & determine eigenvalues
2) For each eigenvalue λ, take $\varphi_\lambda = \varphi - \lambda I$ and look at the square subspaces
$\operatorname{Ker}\varphi_\lambda \subset \operatorname{Ker}\varphi_\lambda^2 \subset \dots$
FIND where this sequence stablises, and compute all threads for λ
3) JOIN Together the results

Ex 1 $V = \mathbb{R}^3$, φ is multiplication by $A = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}$

$\chi_A(t) = \det(A - tI) = -t + 2t^2 - t^3 = -t(1-t)^2$

we get eigen values 0, 1

$$A - 1I = \begin{pmatrix} -3 & 2 & 1 \\ -7 & 3 & 2 \\ 5 & 0 & -1 \end{pmatrix}, \quad (A-I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 10 & -5 & -3 \\ -20 & 10 & 6 \end{pmatrix}$$

rk = 2 null = 1   rk = 1 null = 2

- IN this case, we get AT LEAST one thread of length at least 2.
- For $\lambda = 0$, we get at least one thread of at least length 1
- SINCE $\dim V = 3$ this means we get:
  EXACTLY one thread of length 2 for $\lambda = 1$
  one thread of length 1 for $\lambda = 0$
- As A CONSEQUENCE, $\operatorname{null}(A-I)^3 = 2$
  (as otherwise would have a thread of length 3)

So $\ker(A-I) \subset \operatorname{Ker}(A-I)^2 = \operatorname{Ker}(A-I)^3 = \dots$

BASIS OF $\operatorname{Ker}(A-I)^2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 10x - 5y - 3z = 0 \right\} = \left[\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3/10 \\ 0 \\ 1 \end{pmatrix} \right]$

y, z free variables

BASIS OF $\operatorname{KeR}(A-I) = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$

- REDUCING THE TWO BASIS VECTORS, USING THE EIGENVECTOR, WE OBTAIN $e = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}$

$(A-I)\cdot e = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$

FOR $\lambda = 0$, BASIS OF $\operatorname{Ker} A$ is $\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

AND WITH THE BASIS:

$$\begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \text{ we get } A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Ex 2: $V = \mathbb{R}^4$, $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 11 & 6 & -4 & -4 \\ 22 & 15 & -8 & -9 \\ -3 & -2 & 1 & 2 \end{pmatrix}$

$$\chi_\varphi(t) = \det(A - tI) = 1 - 2t^2 + t^4 = (1-t)^2(1+t)^2 \Rightarrow \lambda = \{1,-1\}$$

$$(A+I) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 11 & 7 & -4 & -4 \\ 22 & 15 & -7 & -9 \\ -3 & -2 & 1 & 3 \end{pmatrix} \begin{matrix} rk=3 \\ null=1 \end{matrix} \qquad (A+I)^2 = \begin{pmatrix} 12 & 8 & -4 & -4 \\ 12 & 8 & -4 & -4 \\ 60 & 40 & -20 & -24 \\ -12 & -8 & 4 & 8 \end{pmatrix} \begin{matrix} rk=2 \\ null=2 \end{matrix}$$

$$(A-I) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 11 & 5 & -4 & 4 \\ 22 & 15 & -9 & -9 \\ -2 & -2 & 1 & 1 \end{pmatrix} \begin{matrix} rk=3 \\ null=1 \end{matrix} \qquad (A-I)^2 = \begin{pmatrix} 12 & 4 & -4 & -4 \\ -32 & -16 & 12 & 12 \\ -28 & -20 & 12 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} rk=2 \\ null=2 \end{matrix}$$

- SINCE $\dim V = 4$, AND WE HAVE AT LEAST ONE THREAD OF LENGTH 2 for both $\lambda = \pm 1$

$\Rightarrow$ we HAVE EXACTLY one thread of length 2 for each eigenvalue

APPLICATIONS OF JORDAN DECOMPOSITION THEOREM

① CLASSIFICATION OF LINEAR TRANSFORMATIONS $\varphi: V \to V$

a) Jordan normal form of φ is *invariant*

- Numbers on diagonals + Sizes of Blocks is determined uniquely
- Numbers on the diagonal = eigenvalues
- DO NOT DEPEND ON ANY CHOICES

THEN: suppose $\varphi^K = 0$

$$N_1 = \operatorname{Ker}\varphi \subset N_2 = \operatorname{Ker}\varphi^2 \subset \cdots \subset N_K = \operatorname{Ker}\varphi^K = V$$

m_K = number of threads of length k

m_{K-1} = number of threads of length $k-1$

$\vdots$

m_1 = number of threads of length 1

- $m_K = \dim N_K - \dim N_{K-1}$

$m_{K-1} + m_K = \dim N_{K-1} - \dim N_{K-2}$

$m_{K-2} + m_{K-1} + m_K = \dim N_{K-2} - \dim N_{K-1}$

$\vdots$

$m_1 + m_2 + \cdots + m_K = \dim N_1$

FOR ANY eigenvalue λ, we consider $N_{\lambda,i} := \operatorname{Ker}(\varphi - \lambda I)^i$. Look where these stablize, and use the previous result

CONCLUSION: Let A, B be two $n \times n$ matrices. They are called SIMILAR IF $A = C^{-1}BC$ for some invertible matrix C.

GEOMETRICALLY: A, B represent some linear transformation w.r.t two different bases (and C is the transition matrix)

Two matrices are SIMILAR IF AND ONLY IF THEY HAVE THE SAME JORDAN NORMAL FORM.

Cayley-Hamilton Theorem

② Cayley-Hamilton Theorem

Recall: for an $n \times n$ matrix A, we have that the characteristic polynomial $\chi_A(t)$

$$\chi_A(t) = \det(A - tI) = a_0 + a_1 t + \dots + a_n t^n$$

THEOREM:

$$\chi_A(A) = a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^k = \emptyset$$

<u>WRONG PROOF:</u> $\chi_A(t) = \det(A - tI) \Rightarrow \chi_A(A) = \det(A - A \cdot I) = \emptyset$

COUNTER EXAMPLE FOR TRACE:

$tr(A - tI)|_{t=A} = \emptyset$ but $tr(A - tI) = trA - n \cdot t \neq \emptyset$

so $tr(A \cdot I) - nA = tr(A - tI)|_{t=0} \neq \emptyset$

PROOF: Roots of $\chi_A(t)$ are percisely the eigen values, so:

BUT CORRECT

$$\chi_A(t) = (\lambda_1 - t)^{m} \dots (\lambda_m - t)^{n_m}$$

where $\lambda_1, \dots \lambda_m$ are all destinct eigenvalues of A

$$\Rightarrow \chi_A(A) = (\lambda_1 I - A)^{m} \dots (\lambda_m I - A)^{n_k}$$

AS WE KNOW THAT

$$\det(A - tI) = \det(C^{-1}(A - tI)C) = \det(C^{-1}AC - tI)$$

AND THUS $\chi_A(t) = \chi_{C^{-1}AC}(t)$

MEANING WE CAN THEREFORE ASSUME THAT A IS ALREADY IN JORDAN NORMAL FORM.

It is enough to show that $\chi_A(J_p(\lambda)) = \emptyset$, (where $J_p(\lambda)$ is one JORDAN BLOCK OF A.)

In $\chi_A(t)$, we have factors $(\lambda - t)^s$, and substituting $J_p(\lambda)$ we get that $(\lambda \cdot I - J_p(\lambda))^s$. This is

This is equal to $\emptyset$ by inspection as long as $s \geq p$.

But s, the multiplicity of λ, is sum of sizes of all JORDAN BLOCKS ▨

Cayleigh-Hamilton Proof & Jordan Matrices Powers

ALTERNATE PROOF:

<u>OBSERVATION:</u> C-H theorem is obvious if A has a basis of eigenvectors.

$$\chi_A(A) = (\lambda_1 I - A)^{n_1} \dots (\lambda_m I - A)^{n_m}$$

- let $v_1, \dots v_n$ be a basis of eigenvectors

IF $A \cdot v = \lambda v$,

Then $(\lambda \cdot I - A) \cdot v = \emptyset$

SO EACH BASIS VECTOR IS ANNIHILATED BY $\chi_A(A)$, so $\chi_A(A) = \emptyset$

<u>ANALYTIC TWIST:</u>

IF A IS ANY MATRIX, then we can change entries of A by arbitrarily small numbers, getting a matrix w/ distinct eigenvalues

$\Rightarrow$ WITH A BASIS OF EIGENVECTORS

TAKING THE LIMIT OF THOSE MATRICES AS THEY APPROACH A, AND NOTICE THAT $\chi_A(A)$ IS A CONTINUOUS FUNCTION, we get $\chi_A(A) = \emptyset$

▨ Computing powers of Jordan Matrices.

- A $n \times n$ matrix, J, the jorden form of A
  C is transition matrix to a JORDAN BASIS

$J = C^{-1} A C$

$A = C \cdot J \cdot C^{-1}$

$A^m = C \cdot J^m C^{-1}$

SO IT IS ENOUGH TO COMPUTE $J_k(\lambda)^m$

$$J_k(\lambda)^m = \begin{pmatrix} \lambda^m & \emptyset & \emptyset & \cdots & & \emptyset \\ m\lambda^{m-1} & \lambda^m & \emptyset & & & \emptyset \\ \binom{m}{2}\lambda^{m-2} & m\lambda^{m-1} & \lambda^m & & & \vdots \\ \binom{m}{3}\lambda^{m-3} & \binom{m}{2}\lambda^{m-2} & m\lambda^{m-1} & & & \\ & & & & \lambda^m & \emptyset \\ \cdots \binom{m}{i}\lambda^{m-i} \cdots & \binom{m}{3}\lambda^{m-3} & \binom{m}{2}\lambda^{m-2} & m\lambda^{m-1} & & \lambda^m \end{pmatrix} k$$

IF $k > m$, then for $i > m$ we have zeros on the i^{th} diagonal

$J_k(\lambda) = \lambda I_k + B$, $B = J_k(0)$, $\boxed{J_k(\lambda)^m = (\lambda I_k + B)^m}$

JORDAN MATRICES POWERS

- we get $J_k(\lambda)^m = (\lambda I_k + B)^m$

USING THE BINOMIAL THEOREM THAT

$(a+b)^m = a^m + \binom{m}{1} a^{m-1} b + \dots$

which when proven only requires that we have $ab = ba$, which is OK with Identity MATRIX

- $J_k(\lambda)^m = \lambda^m I + \binom{m}{1}\lambda^{m-1} B + \dots + \binom{m}{i}\lambda^{m-i} B^i + \dots$

EXAMPLE

CONSIDER THE SEQUENCE $\{x_n\}$, where:

$x_0 = 7 \qquad x_1 = 3 \qquad x_{n+2} = -10x_{n+1} + 25x_n$

- $V_n = \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}$, $V_{n+1} = \begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ -10x_{n+1} - 25x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -25 & -10 \end{pmatrix}\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}$

THUS WE GET A MATRIX $A = \begin{pmatrix} 0 & 1 \\ 25 & -10 \end{pmatrix}$

$V_n = A^n V_0 = A^n \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

$\chi(t) = \det\begin{pmatrix} -t & 1 \\ -25 & -10-t \end{pmatrix} = t^2 + 10t + 25 = (t+5)^2$
$\Rightarrow t = -5$

$(A + 5I) = \begin{pmatrix} 5 & 1 \\ -25 & -5 \end{pmatrix}$

BASIS:

$\mathrm{Ker}(A+5I) = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : 5x + y = 0\right\} = \begin{pmatrix} -1/5 \\ 1 \end{pmatrix} = \begin{pmatrix} +1 \\ -5 \end{pmatrix}$

Relative basis gives us $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as A RELATIVE BASIS

THREAD: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $(A+5I)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

So $C = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}$ is the transition matrix to our JORDAN BASIS

$J = \begin{pmatrix} -5 & 0 \\ 1 & -5 \end{pmatrix}$

$J^n = \begin{pmatrix} -5 & 0 \\ 1 & -5 \end{pmatrix}^n = \begin{pmatrix} (-5)^n & 0 \\ n(-5)^{n-1} & (-5)^n \end{pmatrix}$ $\boxed{\text{NOT } (-5^n)}$

$C^{-1} = \left(\begin{array}{cc|cc} 1 & -5 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow A^n = CJ^nC^{-1} = \begin{pmatrix} (-5)^n - n(-5)^n & n(-5)^{n-1} \\ -n(-5)^{n+1} & (-5)^n + n(-5)^n \end{pmatrix}$

$\Rightarrow A^n\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} (-5)^{n-1}(38n - 35) \\ (-5)^n \cdot (38n + 3) \end{pmatrix} \Rightarrow \boxed{x_n = (-5)^{n-1}(38n - 35)}$

FURTHER APPLICATIONS

PROBLEM 1: given an $n \times n$ matrix A of (Real) numbers & an unknown vector function $x(t)$ satisfying:

$$\frac{dx(t)}{dt} = A\,x(t), \text{ find } x(t)$$

$\boxed{\text{ANSWER: } x(t) = e^{tA} \cdot x_0}$

$$e^{tA} = I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \dots + \frac{t^nA^n}{n!} + \dots$$

PUTTING $A = CJC^{-1}$, we get:

$$e^{tA} = C e^{tJ} C^{-1}$$

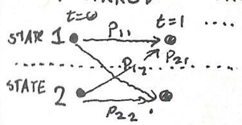
PROBLEM 2: (PARTICULAR) $f(t)$ an unknown function

$a f''(t) + b f'(t) + c = 0$, $a, b, c \in \mathbb{R}$

$x(t) = \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix} \qquad \frac{dx(t)}{dt} = \begin{pmatrix} f'(t) \\ f''(t) \end{pmatrix} = \begin{pmatrix} f'(t) \\ -\frac{b}{a}f'(t) - \frac{c}{a}f(t) \end{pmatrix}$

$\Rightarrow \frac{dx(t)}{dt} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} x(t)$

- MARKOV CHAINS: (STATISTICS/ PROBABILITY)



.... $t=2$

P_{22}, P_{11}

$P_{21} = 1 - P_{11}$

$P_{12} = 1 - P_{22}$

- IN THE BEGINNING, probability p for state 1, q for state 2

- PROBABILITY TO BE IN STATE 1 after 1 second:

$P_{11}p + P_{21}q$ AFTER 1 SECOND IN STATE 1

$P_{12}p + P_{22}q$ AFTER 2 SECONDS IN STATE 2

$\begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} P_{11}p + P_{21}q \\ P_{12}p + P_{22}q \end{pmatrix} = \begin{pmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}$

EUCLIDEAN VECTOR SPACES

Def Let V be a vector space (over $\mathbb{R}$)

V IS SAID TO HAVE A STRUCTURE OF A EUCLIDEAN VECTOR SPACE IF IT IS EQUIPPED WITH A FUNCTION

$$V \times V \to \mathbb{R}$$
$$v_1, v_2 \mapsto (v_1, v_2)$$

CALLED THE "SCALAR PRODUCT" WITH THE FOLLOWING PROPERTIES:

(I) **BILINEAR:** $(v_1+v_2, w) = (v_1, w) + (v_2, w)$
$(c \cdot v, w) = c \cdot (v, w)$
$(v, w_1 + w_2) = (v, w_1) + (v, w_2)$
$(v, c \cdot w) = c \cdot (v, w)$

(II) **SYMMETRIC:** $(v, w) = (w, v)$

(III) **POSITIVE:** $(v, v) \geq 0$ AND $(v, v) = 0$ ONLY for $v = 0$

EXAMPLES:

1) Geometric Scalar Product (2D & 3D)

☆ 2) $\mathbb{R}^n$, with the product: $\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) = x_1 y_1 + \dots + x_n y_n$

3) V = all continuous functions on the interval $[0,1]$

$$(f(t), g(t)) := \int_0^1 f(t) g(t) \, dt$$

EUCLIDEAN VECTOR SPACES - ORTHOGONALITY

OBSERVATION: V vector space, $e_1, \dots e_n$ basis of V

- IF V is a Euclidean Vector Space

Then: $(x_1 e_1 + \dots + x_n e_n, y_1 e_1 + \dots + y_n e_n)$
$= x_1 y_1 (e_1, e_1) + x_1 y_2 (e_1, e_2) + x_2 y_1 (e_1, e_2) + \dots$
$= \sum_{i,j=1}^{n} x_i y_j (e_i, e_j)$

A system of vectors $v_1, \dots, v_k$ in a Euclidean Vector Space V is said to be

orthogonal if $(v_i, v_j) = 0$ for $i \neq j$

orthonormal if $\begin{cases} (v_i, v_j) = 0 \text{ for } i \neq j \\ (v_i, v_i) = 1 \text{ for all } i \end{cases}$

EXAMPLE: $\mathbb{R}^n$ standard unit vectors

NOTE: AN ORTHONORMAL BASIS $e_1, \dots, e_n$ is orthogonal IF & ONLY IF

$(x_1 e_1 + \dots + x_n e_n, y_1 e_1 + \dots + y_n e_n) = x_1 y_1 + \dots + x_n y_n$

FOR ALL $x_1, \dots x_n, y_1, \dots y_n$

Lemma: Orthogonal system of vectors is Linearly independent

PROOF: Suppose $x_1 v_1 + \dots + x_n v_n = 0$ for some $x_1, \dots, x_K$

$0 = (x_1 v_1 + x_2 v_2 + \dots + x_n v_n, x_1 v_1 + \dots + x_n v_n)$

$0 = x_1^2 + x_2^2 + \dots + x_n^2$ as v_i are orthonormal

$\Rightarrow x_1 = 0 = x_2 = x_3 = \dots = x_n$ ▨

Gram-Schmidt Orthogonalisation

Thm: Every Finite-Dimensional Euclidean Vector Space has an Orthonormal Basis

Proof: Let us give a procedure:

Gram-Schmidt Orthogonalisation

Turning a basis $f_1, \dots, f_n$ of V into an orthogonal one

- This is an inductive procedure, where by the kth step we get a system of vectors:

$$e_1, e_2, \dots, e_{k}, f_k, \dots, f_n$$

s.t.: $\begin{cases} e_1, e_2, \dots, e_k \text{ form an orthogonal system of vectors} \\ \operatorname{span}(e_1, \dots, e_{k-1}) = \operatorname{span}(f_1, \dots, f_{k-1}) \end{cases}$

- THE K^{th} STEP OF THE PROCEDURE

replace f_k by a vector of the form:

$$e_k = f_k - a_1 e_1 - a_2 e_2 - \dots - a_{k-1} e_{k-1}$$

To find $a_1, \dots, a_{k-1}$ we look at orthogonality conditions:

$$0 = (e_k, e_1) = (f_k, e_1) - a_1 (e_1, e_1)$$
$$0 = (e_k, e_2) = (f_k, e_2) - a_1 (e_2, e_2)$$
$$\vdots$$
$$0 = (e_k, e_{k-1}) = (f_k, e_{k-1}) - a_{k-1} (e_{k-1}, e_{k-1})$$

Note that we cannot have $(e_i, e_i) = 0$ since that would imply $e_i = 0$ in which case $\operatorname{span}(f_1, \dots, f_{k-1})$ can be spanned by less than $k-1$ vectors which is not possible for a basis

so: $a_1 = \dfrac{(f_k, e_1)}{(e_1, e_1)} \quad \dots \quad a_{k-1} = \dfrac{(f_k, e_{k-1})}{(e_{k-1}, e_{k-1})}$

- After n steps we get an orthogonal basis $e_1, \dots, e_n$ and we only need to make it orthonormal: $\frac{1}{\sqrt{(e_1, e_1)}} e_1, \frac{1}{\sqrt{(e_2, e_2)}} e_2, \dots, \frac{1}{\sqrt{(e_n, e_n)}} e_n$

Orthonormal System of Vectors

Ex: $V = \mathbb{R}^3 \quad f_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, f_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, f_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}$

- The Gram-Schmidt Orthogonalisation gives:

1) there are no vectors to modify f_1 so $e_1 := f_1$
2) then we modify f_2 using e_1:

$$e_2 = f_2 - \frac{(f_2, e_1)}{(e_1, e_1)} e_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} = \boxed{\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix}}$$

3) then we modify f_3 using e_1 and e_2

$$e_3 = f_3 - \frac{(f_3, e_1)}{(e_1, e_1)} e_1 - \frac{(f_3, e_2)}{(e_2, e_2)} = \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{\frac{1}{2}}{\frac{3}{2}}\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix} = \begin{pmatrix}-2/3\\2/3\\2/3\end{pmatrix}$$

So: we normalise to get a basis:

$$\frac{1}{\sqrt{(e_1, e_1)}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{(e_2, e_2)}}\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix}, \frac{1}{\sqrt{(e_3, e_3)}}\begin{pmatrix}-2/3\\2/3\\2/3\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \sqrt{\frac{2}{3}}\begin{pmatrix}1/2\\-1/2\\1\end{pmatrix}, \sqrt{\frac{3}{4}}\begin{pmatrix}-2/3\\2/3\\2/3\end{pmatrix}$$

Remark: It follows that every orthonormal system of vectors $e_1, \dots, e_k$ can be included in an orthonormal basis

Indeed, it is linearly independant, so can be included in a basis $e_1, \dots, e_k, e_{k+1}, \dots, e_n$

Applying Gram-Schmidt to this basis, we get an orthonormal basis $e_1, \dots, e_k, h_{k+1}, \dots, h_n$

(Gram-Schmidt doesn't change already orthogonal vectors)

Length & Angles in Euclidean Space

- Let V be a Euclidean Space
- For $v \in V$ we define length as $\sqrt{(v,v)}$ (notation: $|v|$)
- For $v, w \in V$ s.t. $v \neq 0$ and $w \neq 0$ we define the angle between v and w as the only solution for φ between 0 and $180°$ to the equation: $\cos\varphi = \frac{(v,w)}{|v||w|}$

$$|v| = \sqrt{(v,v)}$$

$$\cos\varphi = \frac{(v,w)}{|v||w|}$$

CAUCHY-SCHWARTZ INEQUALITY

THM: FOR ANY TWO VECTORS v, w IN A euclidean vector space V, we have:

$$(v, w)^2 \leq (v,v)(w,w)$$

WITH EQUALITY ONLY IF v, w are proportional

$$(V,W)^2 \leq (V,V)(W,W)$$

PROOF: IF $v = 0$, then trivial, so let $v \neq 0$

- CONSIDER THE quantity:

$(tv - w, tv - w)$, where t arbitrary scalar

$= (tv, tv - w) - (w, tv - w)$

$= t(v, tv) - t(v, w) - t(w, v) + (w, w)$

$= t^2(v,v) - 2t(v,w) + (w,w)$

IS A QUADRATIC POLYNOMIAL WITH POSITIVE LEADING COEFICIENTS

- THIS QUADRATIC POLYNOMIAL ONLY ASSUMES NON-NEGATIVE VALUES

$\Rightarrow$ POLYNOMIAL HAS A MAXIMUM of 1 root

- CANNOT HAVE 2 DISTINCT ROOTS
- DISCRIMINANT CANNOT BE POSITIVE

$$D = (2(v,w))^2 - 4(v,v)(w,w) \leq 0$$

$$\Rightarrow (v,w)^2 \leq (v,v)(w,w)$$

ALSO: $D = 0$ iff the quadratic polynomial $(tv - w, tv - w)$ has a root t_0, so $t_0 v = w$ and Vectors are Proportional

CAUCHY PROOF 2, ORTHOGONAL COMPLIMENTS

PROOF 2: consider span(v, w) subspace of V

- It is at most two-dimensional
- WLOG, IT IS TWO-DIMENSIONAL
- IF v and w are proportional, the statement is trivial
- By choosing A SYSTEM of COORDINATES WISELY, WE IDENTIFY IT WITH $\mathbb{R}^2$ WITH STANDARD SCALAR PRODUCT
- IN $\mathbb{R}^2$, $(v,w) = |v||w|\cos\varphi$
- THUS THE IDENTITY HOLDS ∎
- Using angles between vectors, we see that $(v,w) = 0$ IF AND ONLY IF THE ANGE BETWEEN IS $90°$

DEF: ORTHOGONAL COMPLIMENT

- Let V be a Euclidean Vector Space and $\mathcal{U}$ a subspace of V. the set of all vectors v s.t $(u,v) = 0$ for all $u \in \mathcal{U}$ is called the ORTHOGONAL COMPLIMENT OF $\mathcal{U}$ and is denoted as $\mathcal{U}^\perp$

$v \in \mathcal{U}^\perp$ IF FOR $u \in \mathcal{U}$ $(v, u) = 0$

LEMMA: FOR EVERY SUBSPACE $\mathcal{U}$, $\mathcal{U}^\perp$ is also a subspace

1) $0 \in \mathcal{U}^\perp$
2) $(u, v_1 + v_2) = (u, v_1) + (u, v_2)$, so so IF $v_1, v_2 \in \mathcal{U}^\perp$ then $v_1 + v_2 \in \mathcal{U}^\perp$
3) For $(u, c \cdot v) = c \cdot (u, v)$ so IF $v \in \mathcal{U}^\perp$ then $c \cdot v \in \mathcal{U}^\perp$

Orthogonal Compliments

① For every subspace $\mathcal{U}$, we have that:

$$\mathcal{U} \cap \mathcal{U}^{\perp} = \{0\}$$

- IF $v \in \mathcal{U} \cap \mathcal{U}^{\perp}$

  THEN $(\underset{\text{vector in } \mathcal{U}}{v}, \underset{\text{vector in } \mathcal{U}^{\perp}}{v}) = 0 \Rightarrow v = 0$

- THIS SHOWS US THAT THE SUM IS DIRECT

- IF $\mathcal{U}$ IS FINITE-DIMENSIONAL, THEN $\mathcal{U} \oplus \mathcal{U}^{\perp} = V$

PROOF: Let $e_1, \dots, e_k$ be an orthonormal BASIS of ~~the~~ $\mathcal{U}$

- We should show $\mathcal{U} \oplus \mathcal{U}^{\perp} = V$, so we would like to represent each vector v as $v = u_1 + u_2$ where $u_1 \in \mathcal{U}$, $u_2 \in \mathcal{U}^{\perp}$

- $u_1 = c_1 e_1 + \dots + c_k e_k$
- suppose we found such a decomposition

$$(v_1, e_1) = (c_1 e_1 + \dots + c_k e_k + u_2, e_1) = c_1$$

$$\vdots$$

$$(v, e_k) = c_k$$

SINCE $(u_2, e_i) = 0$, $(e_i, e_i) = 1$, $(e_i, e_j) = 0$ FOR $i \neq j$

- NOW DEFINE $\vec{u}_2 = \vec{v} - (v, e_1)\hat{e}_1 - \dots - (v, e_k)\hat{e}_k$

  NOTE THAT $u_2 \in \mathcal{U}^{\perp}$

  AS $(u_2, e_i) = (v, e_i) - (v, e_i) = 0$

  so u_2 is orthogonal to all basis vectors of $\mathcal{U}$, and so, all vectors of $\mathcal{U}$

Bassel's Inequality

- Let V be a Euclidean Vector Space. $v \in V$

  Let $e_1, \dots, e_k$ be an orthonormal system of vectors

THEN: $(v, v) \geq (v, e_1)^2 + \dots + (v, e_k)^2$

PROOF: TAKE $\mathcal{U} = \text{span}(e_1, \dots, e_k)$

$$\text{so } \vec{v} = \overbrace{(v, e_1)\hat{e}_1 + \dots + (v, e_k)\hat{e}_k}^{u_1} + \vec{u}_2$$

$$\text{so } (v, v) = (u_1 + u_2, u_1 + u_2)$$

$$= (u_1, u_1) + \underbrace{\cancel{(u_1, u_2)}}_{0} + \underbrace{\cancel{(u_2, u_1)}}_{0} + (u_2, u_2)$$

$$(v, v) = (u_1, u_1) + (u_2, u_2)$$

$$(v, v) = (u_1, u_1) + (u_2, u_2) \geq (c_1 e_1 + \dots + c_k e_k, c_1 e_1 + \dots + c_k e_k)$$

$$(v, v) \geq c_1^2 + c_2^2 + \dots + c_k^2 = (v, e_1)^2 + \dots + (v, e_k)^2$$

APPLICATION

- CONSIDER A SPACE of continuous functions ON THE INTERVAL $[-1, 1]$ w/ SCALAR PRODUCTS

$$(f(t), g(t)) = \int_{-1}^{1} f(t) g(t)\, dt$$

- CONSIDER THE FUNCTIONS $\sin(\pi t), \sin(2\pi t), \dots, \sin(n\pi t)$

They form an orthonormal system:

$$(\sin(k\pi t), \sin(L\pi t)) = \int_{-1}^{1} \sin(k\pi t)\sin(L\pi t)\, dt$$

$$= \int_{-1}^{1} \left(\frac{1}{2}\left[\cos((k-L)\pi t) - \cos((k+L)\pi t)\right]\right) dt$$

$$= \begin{cases} 0 & \text{IF } k \neq L \\ \int_{-1}^{1} \frac{1}{2}\, dt = 1 & \text{IF } k = L \end{cases}$$

- TAKE $h(t) = t$.

  THEN $(h(t), \sin(k\pi t)) = \int_{-1}^{1} t \sin(k\pi t) = \frac{2(-1)^{k-1}}{k\pi}$

  $(h(t), h(t)) = \int_{-1}^{1} t^2\, dt = \frac{2}{3}$

- BY BELL'S INEQUALITY: $\frac{2}{3} \geq \left(\frac{2}{\pi}\right)^2 + \left(\frac{2}{2\pi}\right)^2 + \dots + \left(\frac{2}{n\pi}\right)^2$

$$\frac{\pi^2}{6} \geq 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2}$$