Quantum Physics PY1T20/PYU11P20

Paul Eastham

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Lecture 4: Meaning of the wavefunction

- In this lecture we will look at:
- How the wavefunction is related to the position of the particle
- How wavefunctions are normalized
- The difference between adding probabilities and adding probability amplitudes.

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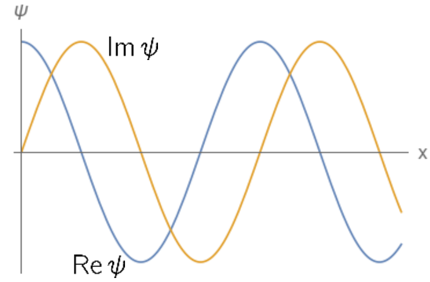
Wavefunction for a free particle

- For a particle with a momentum p, moving freely (V=0), the wavefunction at time t (in 1D):

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

- where $k = \frac{2\pi}{\lambda}$, $p = \frac{h}{\lambda} = \hbar k$

  and $E = \hbar\omega$ is the particle's kinetic energy.



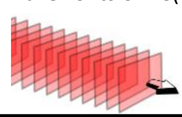
- In three dimensions $\boldsymbol{p}$ and $\boldsymbol{k}$ are vectors and

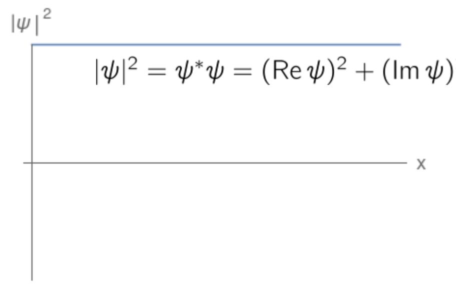
$$\boldsymbol{p} = \hbar\boldsymbol{k} \quad \text{or} \quad \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \hbar \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$$

- and the wavefunction for a particle with momentum $\boldsymbol{p}$ is a plane wave

$$\psi(\boldsymbol{r},t) = Ae^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)} \qquad \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Wavefronts of Re(ψ)





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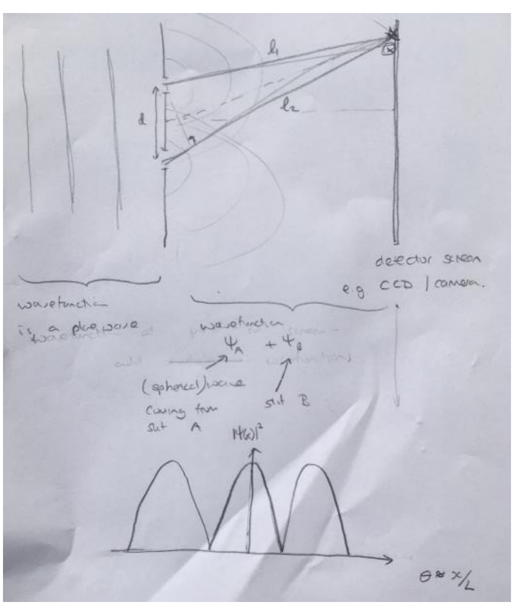
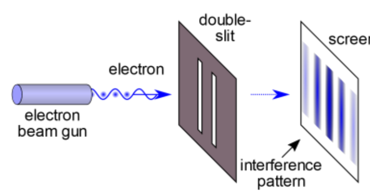
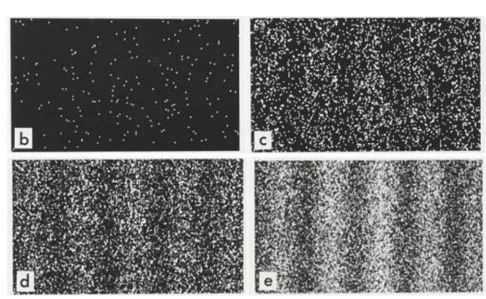
Double slit for single electrons







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What does the wavefunction mean?

- In the double-slit experiment, the electron wavefunction is propagating in the 'usual' wave-like way to the screen.
- For each electron, we get a point somewhere on the screen – when we measure the position of the electron we find it at as a 'true particle' which exists at a point (nearly—see later).
- If we repeat the experiment many times, the number of electrons recorded in different places follows the usual intensity pattern of a double-slit experiment.
- Conclude: the wavefunction tells us the probabilities that, when we measure the position of the particle, it is found in a particular place – more likely to be found where the wavefunction is large, less likely where it is small.

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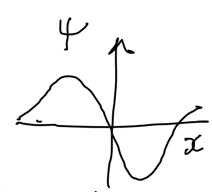
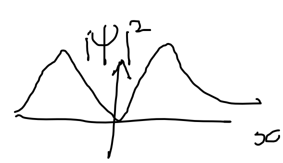


Relation of wavefunction to experiment

- The wavefunction is directly related to a measurement of the position of the particle, e.g. what happens when it hit a screen or detector at some particular position x.
- It tells us about the probability that the particle is found at that particular position – more likely to be found where the wavefunction is large, less likely where it is small.
- Specifically:

$$\int_{x_1}^{x_2} |\psi(x,t)|^2 dx = \int_{x_1}^{x_2} \psi^* \psi dx$$

is the probability that the particle is found between x_1 and x_2



- '$|\psi|^2$ is the probability density of the position of the particle'
- Recall: probability density of x is a thing you integrate over x to get probability.
- Wavefunction is complex but $|\psi|^2$ is real and positive, as probabilities have to be.

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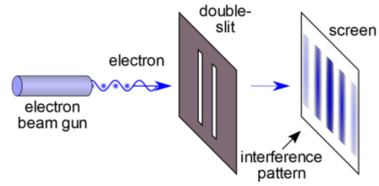
Double slit for single electrons: maths

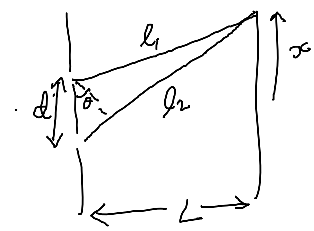
At a point distance x from centre of screen wavefunction is



$$\psi = Ae^{i(kl_1-\omega t)} + Ae^{i(kl_2-\omega t)}$$

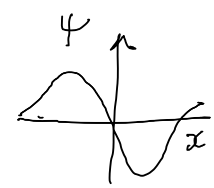
So the probability density of the electron's position across the screen is

$$\begin{aligned}|\psi|^2 &= |A|^2 \left|e^{ikl_1} + e^{ikl_2}\right|^2 \\ &= |A|^2 \left|e^{ik(l_1+l_2)/2}\left(e^{ik(l_1-l_2)/2} + e^{-ik(l_1-l_2)/2}\right)\right|^2 \\ &= |A|^2 \times \left|e^{ik(l_1+l_2)/2}\right|^2 \times \left|2\cos\left[\frac{k(l_1-l_2)}{2}\right]\right|^2 \\ &= 4|A|^2\cos^2[dk\theta/2]\end{aligned}$$

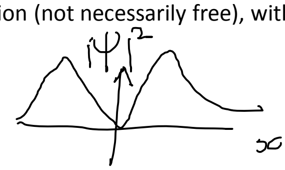


$$(l_2 - l_1) \approx d\sin\theta \approx d\theta \approx d(x/L)$$

Normalization of wavefunctions



- Suppose we have a particle in one-dimension (not necessarily free), with wavefunction $\psi(x)$
- Probability density of position = $|\psi|^2$



- Now we measure position. We must find it somewhere so $\int_{-\infty}^{+\infty} |\psi|^2 dx = 1$. Probabilities must add to 1!
- How can we ensure this? What if this integral is not 1 for our 'wavefunction'?
- Answer: multiply the wavefunction by a constant, call it *N*, and choose *N* so that the integral is 1
- N is a 'normalization factor' and doing this gives a 'normalized wavefunction'.
- To interpret $|\psi|^2$ as a probability (density), ψ <u>must be a normalized wavefunction</u>.
- If you don't do this you can get relative probabilities, which is sometimes all you need, but not absolute ones.

Normalization of wavefunctions: double slit

- We can get some insight into this process from our result for the double slit $|\psi|^2 = 4|A|^2 \cos^2 [dk\theta(x)/2]$
- Where the constant A came from the overall amplitude of the wave(s).
- This overall amplitude factor should, in principle, be determined by the normalization condition – probabilities add to 1!
- But in this case we would probably not bother (our electrons won't all end up on the screen anyway), and instead be content with relative probabilities, i.e. the number of electrons hitting a detector at one point in the pattern compared with the number hitting a detector at another.

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Probability and probability amplitude

- The quantity $|\psi|^2$ is 'the probability distribution of the position of the particle'
- Sometimes just 'the probability of the position'.
- The wavefunction itself ψ is the corresponding 'probability amplitude'
  - Probability amplitudes are things we take the magnitude-squared of to get probabilities
- In classical probability you add up probabilities: prob (a or b)=prob (a) + prob (b)
  - (e.g. the probability of getting a 1 or a 2 on a throw of a dice is 1/6+1/6=1/3).
- In quantum mechanics you add up probability amplitudes – you superpose wavefunctions.
- And do the magnitude-squared at the end to turn them into probabilities.
- So you get interference effects, which cannot be captured in terms of the probabilities of the outcomes alone...

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Quantum interference vs classical probability

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Lecture 4: Meaning of the wavefunction

- When we measure the position of a particle, we find it at a particular place
    - Not as a smeared out wavefunction!
- The wavefunction tells us the probability of finding the particle at a particular position.
- But only if the wavefunction is correctly normalized.
- Quantum mechanics differs from classical physics because we add up wavefunctions and then take the magnitude-squared to get the probabilities of outcomes.

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