MA1125 – Calculus
Tutorial solutions #1

1. Find the domain and the range of the function f which is defined by

$$f(x) = \frac{4-3x}{6-5x}.$$

The domain consists of all points $x \neq 6/5$. To find the range, we note that

$$y = \frac{4-3x}{6-5x} \iff 6y - 5xy = 4 - 3x \iff 6y - 4 = 5xy - 3x$$
$$\iff x(5y-3) = 6y - 4 \iff x = \frac{6y-4}{5y-3}.$$

The rightmost formula determines the value of x that satisfies $y = f(x)$. Since the formula makes sense for any number $y \neq 3/5$, the range consists of all numbers $y \neq 3/5$.

2. Find the domain and the range of the function f which is defined by

$$f(x) = \sqrt{x - x^2}.$$

When it comes to the domain, one needs $x - x^2 = x(1-x)$ to be non-negative, so the factors $x, 1-x$ must have the same sign. If $x \geq 0$, then $1 - x \geq 0$ and this gives $0 \leq x \leq 1$. If $x \leq 0$, then $1 - x \leq 0$ and this gives $1 \leq x \leq 0$, which is absurd. Thus, only the former case may arise and the domain is $[0, 1]$. To find the range, we note that $y = f(x) \geq 0$ and

$$y^2 = x - x^2 \iff x^2 - x + y^2 = 0 \iff x = \frac{1 \pm \sqrt{1-4y^2}}{2}.$$

Since the rightmost formula is only defined when $4y^2 \leq 1$, the range is then $[0, 1/2]$.

3. Show that the function $f\colon (0,1) \to (0,\infty)$ is bijective in the case that

$$f(x) = \frac{1}{x} - 1.$$

To show that the given function is injective, we note that

$$f(x_1) = f(x_2) \implies \frac{1}{x_1} - 1 = \frac{1}{x_2} - 1 \implies \frac{1}{x_1} = \frac{1}{x_2} \implies x_1 = x_2.$$

To show that the given function is surjective, we note that

$$y = f(x) \iff y = \frac{1}{x} - 1 \iff \frac{1}{x} = y + 1 \iff x = \frac{1}{y+1}.$$

The rightmost formula determines the value of x such that $y = f(x)$ and we need to check that $0 < x < 1$ if and only if $y > 0$. When $y > 0$, we have $y + 1 > 1 > 0$, so $0 < x < 1$. When $0 < x < 1$, we have $0 < 1 < \frac{1}{x}$ and this gives $y > 0$, as needed.

4. Express the following polynomials as the product of linear factors.

$$f(x) = 2x^3 - 7x^2 + 9, \qquad g(x) = x^3 - \frac{3x}{4} - \frac{1}{4}.$$

The possible rational roots for the first polynomial are $\pm 1, \pm 3, \pm 9, \pm 1/2, \pm 3/2, \pm 9/2$. Checking the first few, one finds that $x = -1$ and $x = 3$ are both roots. This implies that both $x + 1$ and $x - 3$ must be factors, so it easily follows by division that

$$f(x) = (x + 1)(2x^2 - 9x + 9) = (x + 1)(x - 3)(2x - 3).$$

Let us now turn to the second polynomial and clear denominators to write

$$4g(x) = 4x^3 - 3x - 1.$$

The possible rational roots are $\pm 1, \pm 1/2, \pm 1/4$. Checking these possibilities, one finds that only $x = 1$ and $x = -1/2$ are actual roots. It easily follows by division that

$$4g(x) = (x - 1)(4x^2 + 4x + 1) = (x - 1)(2x + 1)^2 \quad \Longrightarrow \quad g(x) = \frac{1}{4}(x - 1)(2x + 1)^2.$$

5. Use the addition formulas for sine and cosine to prove the identity

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \cdot \tan\beta}.$$

By definition, the tangent of an angle is the quotient of its sine and cosine, so

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta}{\cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta}.$$

Once we now divide both the numerator and the denominator by $\cos\alpha \cdot \cos\beta$, we get

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \cdot \tan\beta}.$$

6. Show that the function $f\colon (0, \infty) \to \mathbb{R}$ is injective in the case that

$$f(x) = \frac{2x - 1}{3x + 2}.$$

We assume that $f(x_1) = f(x_2)$ and we clear denominators to get

$$\begin{aligned} \frac{2x_1 - 1}{3x_1 + 2} = \frac{2x_2 - 1}{3x_2 + 2} \quad &\Longrightarrow \quad (2x_1 - 1)(3x_2 + 2) = (2x_2 - 1)(3x_1 + 2) \\ &\Longrightarrow \quad 6x_1x_2 - 3x_2 + 4x_1 - 2 = 6x_1x_2 - 3x_1 + 4x_2 - 2. \end{aligned}$$

Once we now cancel the common terms, we may easily conclude that

$$-3x_2 + 4x_1 = -3x_1 + 4x_2 \quad \Longrightarrow \quad 7x_1 = 7x_2 \quad \Longrightarrow \quad x_1 = x_2.$$

7. Find the roots of the polynomial $f(x) = x^3 + x^2 - 5x - 2$.

The only possible rational roots are $\pm 1, \pm 2$ and one may check each of those to see that only $x = 2$ is a root. This implies that $x - 2$ is a factor and division of polynomials gives

$$f(x) = (x-2)(x^2 + 3x + 1).$$

To find the roots of the quadratic factor, one may use the quadratic formula to get

$$x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}.$$

8. Determine the range of the quadratic $f(x) = ax^2 + bx + c$ in the case that $a > 0$.

We use the standard approach and solve $y = f(x)$ in terms of x. This gives

$$y = ax^2 + bx + c \quad \Longrightarrow \quad ax^2 + bx + (c - y) = 0 \quad \Longrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4a(c-y)}}{2a}$$

and we need the discriminant to be non-negative, so we need to have

$$b^2 - 4ac + 4ay \geq 0 \quad \Longrightarrow \quad 4ay \geq 4ac - b^2 \quad \Longrightarrow \quad y \geq \frac{4ac - b^2}{4a}.$$

In other words, the range of the quadratic has the form $[y_*, +\infty)$, where $y_* = \frac{4ac-b^2}{4a}$.

9. Relate the sines and the cosines of two angles θ_1, θ_2 whose sum is equal to 2π.

Since $\theta_1 + \theta_2 = 2\pi$ by assumption, the addition formulas for sine and cosine give

$$\sin\theta_2 = \sin(2\pi - \theta_1) = \sin(2\pi) \cdot \cos\theta_1 - \cos(2\pi) \cdot \sin\theta_1,$$
$$\cos\theta_2 = \cos(2\pi - \theta_1) = \cos(2\pi) \cdot \cos\theta_1 + \sin(2\pi) \cdot \sin\theta_1.$$

On the other hand, $\sin(2\pi) = 0$ and $\cos(2\pi) = 1$ by definition, so it easily follows that

$$\sin\theta_2 = -\sin\theta_1, \qquad \cos\theta_2 = \cos\theta_1.$$

10. Determine all angles $0 \leq \theta \leq 2\pi$ such that $2\cos^2\theta + 7\cos\theta = 4$.

Letting $x = \cos\theta$ for convenience, we get $2x^2 + 7x - 4 = 0$ and thus

$$x = \frac{-7 \pm \sqrt{49 + 4 \cdot 8}}{2 \cdot 2} = \frac{-7 \pm \sqrt{81}}{4} = \frac{-7 \pm 9}{4} \quad \Longrightarrow \quad x = \frac{1}{2}, -4.$$

Since $x = \cos\theta$ must lie between -1 and 1, the only relevant solution is $x = \cos\theta = \frac{1}{2}$. In view of the graph of the cosine function, there should be two angles $0 \leq \theta \leq 2\pi$ that satisfy this condition. The first one is $\theta_1 = \frac{\pi}{3}$ and the second one is $\theta_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.