Advanced Calculus
MA1132
Tutorial Exercises 5

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To be completed before and during tutorials of Friday, 1. March

1. (a) Find the total differential of the function $f(x, y, z) = \dfrac{xyz}{x + y + z}$ at a point (a, b, c).

   (b) Use your answer to Part (a) to estimate $f(-1.04, -1.98, 3.97)$ using the total differential at $f(-1, -2, 4)$ and compare it with the actual value.

2. Let $f(x, y, z) = \tan^{-1}\left(\dfrac{x}{y + z}\right)$.

   (a) Find the directional derivative of f at the point $(4, 2, 2)$ in the direction $(2, 3, 4)$.

   (b) Find the unit vectors in the directions in which f is increasing/decreasing most rapidly at the point $(4, 2, 2)$, and give the rate of increase and decrease, respectively.

3. Given that $z = 3x^2 - y^2$, find all the points at which $\|\nabla z\| = 6$.

4. Given that $z = 3x + y^2$, find $\nabla\|\nabla z\|$ at the point $(5, 2)$.

5. Find an equation for the tangent plane and a parametric equation for the normal line to the graph of the function $f(x, y) = \sqrt{x} + \sqrt{y}$ at the point $(4, 9)$.

6. Find the parametric equation of the tangent line to the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $x + 2y + 2z = 20$ at the point $(4, 3, 5)$

7. Let $\mathbf{u}_r$ be a unit vector whose counterclockwise angle from the positive x-axis is θ, and let $\mathbf{u}_\theta$ be a unit vector 90% counterclockwise from $\mathbf{u}_r$. Show that if $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, then
$$\nabla z = \frac{\partial z}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial z}{\partial \theta}\mathbf{u}_\theta\,.$$

8. Consider the surface
$$z = f(x, y) = \ln\frac{\sqrt[3]{2x^2 - 3xy^2 + 3\cos(2x + 3y) - 3y^3 + 18}}{2}$$

   (a) Find an equation for the tangent plane to the surface at the point $P(3, -2, z_0)$ where $z_0 = f(3, -2)$.

   (b) Sketch the tangent plane.

   (c) Find parametric equations for the normal line to the surface at the point $P(3, -2, z_0)$.

(d) Sketch the normal line to the surface at the point $P(3, -2, z_0)$.

9. Show that the equation of the plane that is tangent to the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

at (x_0, y_0, z_0) can be written in the form

$$\frac{x_0 x}{a^2} + \frac{y y_0}{b^2} - \frac{z z_0}{c^2} = 1\,.$$