MA1125 – Calculus
Homework #6 solutions

1. Find the global minimum and the global maximum values that are attained by

$$f(x) = x^3 - 6x^2 + 9x - 5, \qquad 0 \leq x \leq 2.$$

The derivative of the given function can be expressed in the form

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3).$$

Thus, the only points at which the minimum/maximum value may occur are the points

$$x = 0, \qquad x = 2, \qquad x = 1, \qquad x = 3.$$

We exclude the rightmost point, as it does not lie in the given interval, and we compute

$$f(0) = -5, \qquad f(2) = 8 - 24 + 18 - 5 = -3, \qquad f(1) = 1 - 6 + 9 - 5 = -1.$$

This means that the minimum value is $f(0) = -5$ and the maximum value is $f(1) = -1$.

2. If a right triangle has a hypotenuse of length $a > 0$, how large can its area be?

Let us denote by x, y the other sides of the triangle. Then $x^2 + y^2 = a^2$ and the area is

$$A(x) = \frac{1}{2}\,xy = \frac{1}{2}\,x\sqrt{a^2 - x^2}, \qquad 0 \leq x \leq a.$$

The value of x that maximises this expression is the value of x that maximises its square

$$f(x) = A(x)^2 = \frac{1}{4}\,x^2(a^2 - x^2) = \frac{1}{4}\,(a^2x^2 - x^4).$$

Let us then worry about $f(x)$, instead. The derivative of this function is given by

$$f'(x) = \frac{1}{4}\,(2a^2x - 4x^3) = \frac{x}{2}\,(a^2 - 2x^2).$$

Thus, the only points at which the maximum value may occur are the points

$$x = 0, \qquad x = a, \qquad x = \frac{a}{\sqrt{2}}.$$

Since $f(0) = f(a) = 0$, the maximum value is $f(a/\sqrt{2})$ and the largest possible area is

$$A(a/\sqrt{2}) = \frac{a}{2\sqrt{2}} \cdot \sqrt{a^2 - \frac{a^2}{2}} = \frac{a}{2\sqrt{2}} \cdot \frac{a}{\sqrt{2}} = \frac{a^2}{4}.$$

3. A balloon is rising vertically at the rate of 1 m/sec. When it reaches 48m above the ground, a bicycle passes under it moving at 3 m/sec along a flat, straight road. How fast is the distance between the bicycle and the balloon increasing 16 seconds later?

Let x be the horizontal distance between the balloon and the bicycle, and let y be the height of the balloon. Then x, y are the sides of a right triangle whose hypotenuse is the distance z between the balloon and the bicycle. It follows by Pythagoras' theorem that

$$x(t)^2 + y(t)^2 = z(t)^2 \quad \Longrightarrow \quad 2x(t)x'(t) + 2y(t)y'(t) = 2z(t)z'(t).$$

At the given moment, $x'(t) = 3$ and $y'(t) = 1$, while $x(t) = 16 \cdot 3$ and $y(t) = 48 + 16$, so

$$z'(t) = \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}} = \frac{48 \cdot 3 + 64}{\sqrt{48^2 + 64^2}} = \frac{208}{80} = \frac{13}{5}.$$

4. Find the linear approximation to the function f at the point x_0 in the case that

$$f(x) = \frac{x^3 - 2x + 4}{x^2 + 2}, \qquad x_0 = 0.$$

To find the derivative of $f(x)$ at the given point, we use the quotient rule to get

$$f'(x) = \frac{(3x^2 - 2) \cdot (x^2 + 2) - 2x \cdot (x^3 - 2x + 4)}{(x^2 + 2)^2} \quad \Longrightarrow \quad f'(0) = -\frac{4}{2^2} = -1.$$

Since $f(0) = 4/2 = 2$, the linear approximation is thus $L(x) = -(x - 0) + 2 = 2 - x$.

5. Show that $f(x) = x^3 - 4x + 1$ has two roots in $(0, 2)$ and use Newton's method with initial guesses $x_1 = 0, 2$ to approximate these roots within two decimal places.

To prove existence using Bolzano's theorem, we note that f is continuous with

$$f(0) = 1, \qquad f(1) = 1 - 4 + 1 = -2, \qquad f(2) = 8 - 8 + 1 = 1.$$

In view of Bolzano's theorem, f must then have a root in $(0, 1)$ and another root in $(1, 2)$, so it has two roots in $(0, 2)$. Suppose that it has three roots in $(0, 2)$. Then f' must have two roots in this interval by Rolle's theorem. On the other hand, $f'(x) = 3x^2 - 4$ has only one root in $(0, 2)$. This implies that f can only have two roots in $(0, 2)$.

To use Newton's method to approximate the roots, we repeatedly apply the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 4x_n + 1}{3x_n^2 - 4}.$$

Starting with the initial guess $x_1 = 0$, one obtains the approximations

$$x_1 = 0, \qquad x_2 = 0.25, \qquad x_3 = 0.2540983607, \qquad x_4 = 0.2541016884.$$

Starting with the initial guess $x_1 = 2$, one obtains the approximations

$$x_1 = 2, \qquad x_2 = 1.875, \qquad x_3 = 1.860978520, \qquad x_4 = 1.860805879.$$

This suggests that the two roots are roughly 0.25 and 1.86 within two decimal places.