Advanced Calculus
MA1132
Homework Assignment 1

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To be completed and handed in AT THE BEGINNING of tutorial on Friday, 22. February

NO LATE ASSIGNMENTS WILL BE ACCEPTED. IF YOU CANNOT ATTEND TUTORIALS, PLEASE MAKE ARRANGEMENTS TO EMAIL YOUR SOLUTIONS TO YOUR TUTOR

1. Find the derivative of the vector-valued function

$$\mathbf{r}(t) = \left(\frac{t^2 - 1}{t^2 + 2}, \sin(t)\ln(t), \cos(\sin(t)) \right).$$

   What is $\mathbf{r}'(1)$ equal to?

2. Find the derivative of the vector-valued function

$$\mathbf{r}(t) = (2t^3 - 3t^2 + 5t + 3) \left(\frac{t^2 - 1}{t^2 + 2}, \sin(t)\ln(t), \cos(\sin(t)) \right).$$

3. The parametrisation

$$x = \alpha + a\cosh t\,, \quad y = \beta + b\sinh t\,, \quad -\infty < t < \infty \tag{1}$$

   of the hyperbola

$$\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1\,, \quad a > 0\,, \quad b > 0\,, \tag{2}$$

   represents only one branch of the hyperbola.

   Find a parametrisation which represents both branches of the hyperbola.

4. Find an arc length parametrization of the curve

$$\mathbf{r}(t) = \cos^3(t)\mathbf{i} + \sin^3(t)\mathbf{j} \quad t \in \left[0, \frac{\pi}{2}\right],$$

   in the same direction as the original curve.

5. Prove the Serret-Frenet formulae

   (a) $\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}(s)$

   (b) $\frac{d\mathbf{B}}{ds} = -\tau(s)\mathbf{N}(s)$, where τ is a scalar called the *torsion* of $\mathbf{r}(s)$.

   (c) $\frac{d\mathbf{N}}{ds} = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}$

6. Consider the vector-valued function

$$\mathbf{r}(t) = (4\cos(t), 4\sin(t), 3t), \quad t \in \mathbb{R}.$$

(a) Find a arc-length parametrization for the function with reference point $t_0 = 0$.

(b) Calculate the curvature κ of $\mathbf{r}$, using the arc-length parametrization you found in Part (a) and the formula $\kappa(s) = \|\mathbf{T}'(s)\|$.

(c) Calculate the *torsion* τ of $\mathbf{r}$, using the arc-length parametrization you found in Part (a) and the formula $\tau(s) = -\mathbf{N}(s) \cdot \mathbf{B}'(s)$.

(d) Plot the path, indicating the direction of increasing s, using Mathematica[1]). You should include the point $\mathbf{r}\left(s = \frac{5\pi}{2}\right)$, the vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ corresponding to this point, and the **TN**-, **TB**- and **NB**- planes corresponding to these vectors.

7. A curve C in the xy-plane is represented by the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\,. \tag{3}$$

In the $x'y'$-plane obtained by rotating the xy-plane through an angle ϕ

$$x' = x\cos\phi + y\sin\phi\,, \quad y' = -x\sin\phi + y\cos\phi\,, \tag{4}$$

the curve C is represented by a similar equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0\,. \tag{5}$$

(a) Express A', B', C', D', E', F' in terms of A, B, C, D, E, F and ϕ.

(b) Prove that if the angle ϕ satisfies

$$\cot 2\phi = \frac{A - C}{B}\,, \tag{6}$$

then the curve C is represented by the equation

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0\,, \tag{7}$$

i.e. $B' = 0$.

8. A curve C is the intersection of the cone

$$z^2 = x^2 + y^2\,, \tag{8}$$

with a plane.

Identify the curve, find a parametric representation and plot the curve in the xyz-space for the planes below. The Mathematica function `ParametricPlot3D[]` can be used to plot parametric curves in the xyz-space.

(a) $z = 2\,.$

[1] student license available from Trinity for free

(b) $y = 0$.

(c) $x = 1$.

(d) $x + y = 1$.

(e) $x + z = 1$.

(f) $x + y + z = 1$.

9. A curve C is the intersection of the cone

$$x^2 + y^2 - z^2 = 0\,, \tag{9}$$

with the plane

$$ax + by + cz + d = 0\,. \tag{10}$$

Which type figure (show with proof) is C: a circle; an ellipse; a parabola; a hyperbola; a pair of intersecting lines; a single line; a point?

10. Consider elliptic coordinates (u, v)

$$x = l\,u\,v\,, \quad y^2 = l^2(u^2-1)(1-v^2)\,, \quad u \geq 1\,,\ -1 \leq v \leq 1\,,\ l \text{ is a constant}$$

(a) Show that curves of constant u (i.e., for which we hold u at a constant value) are ellipses, while curves of constant v are hyperbolae.

(b) Show that in the elliptic coordinates a curve given by the parametric equations $u = u(t)$, $v = v(t)$ for $a \leq t \leq b$ has arc length

$$L = l \int_a^b \sqrt{\frac{u^2-v^2}{u^2-1}\left(\frac{du}{dt}\right)^2 + \frac{u^2-v^2}{1-v^2}\left(\frac{dv}{dt}\right)^2}\, dt\,. \tag{11}$$

11. Sketch the domain of f (you may use Mathematica). Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included. Determine whether the domain is an open set, closed set, or neither.

(a) $f(x,y) = \frac{\sqrt{x-y}}{\sqrt{x+y}}$

(b) $f(x,y) = \frac{\ln(x+y)}{\sqrt{x-y}}$

(c) $f(x,y) = \sqrt{\frac{x^2+y^2-1}{2-x^2+2x-4y^2+4y}}$

12. Sketch the level curve $z = k$ for the specified values of k

$$z = 4x^2 + 4x + y^2 - 2y\,, \quad k = -2, -1, 2\,.$$

13. Determine whether the limit exists. If so, find its value

(a)

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{e^{x^2+y^2}-1}$$

(b)

$$\lim_{(x,y)\to(1,-1)} \frac{1-\cosh(x+y)}{\sin(x^2-y^2)\ln(\frac{2x}{x-y})}$$

(c)

$$\lim_{(x,y)\to(0,0)} \frac{3+\cos(2x)-4\cosh(y)}{1-\sqrt[4]{1+x^2+y^2}}$$

14. Consider the function

$$f(x,y) = \int_x^y \frac{dt}{\sqrt{\left(t^2-x^2\right)\left(y^2-t^2\right)}}$$

(a) Sketch the domain of f. Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included. Determine whether the domain is an open set, closed set, or neither.

(b) Assume that $0 < x < y$, and determine if the following limits exist. If so, find its value. If the limit does not exist find the leading behaviour of the function in a neighbourhood of the limiting point, e.g. $\lim_{(x,y)\to(0,0)} \ln(\sqrt{1+(x+y)^2}-1)$ does not exist and the leading behaviour of this function in a neighbourhood of $(0,0)$ is $2\ln(x+y)$.

i.

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

ii.

$$\lim_{(x,y)\to(1,1)} f(x,y)$$

iii.

$$\lim_{(x,y)\to(0,1)} f(x,y)$$