TRINITY COLLEGE DUBLIN
THE UNIVERSITY OF DUBLIN

School of Mathematics

JF Mathematics
JF Theoretical Physics
JF Two Subject Mod

Trinity Term 2015

MA1123 — Analysis I

Wednesday, April 29 Sports Centre 14.00 – 17.00

Prof. D. O'Donovan

Instructions to Candidates:

ANSWER ALL QUESTIONS:

All questions carry equal marks.

Materials Permitted for this Examination:

Formulae and Tables tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Give the $\epsilon - \delta$ definition of f(x) is continuous at x = a.

   (b) Use the definition of limit to evaluate $\lim_{x \to 1} x^3 + 2x - 1$.

   (c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ and the left hand and right hand limit of f(x) at x = a both exist and are equal, then the limit at a exists.

   (d) Prove that if the derivative of f(x) exists at x = a, then f(x) is continuous at x = a. Prove or disprove the converse.

2. (a) Find the quadratic approximation to $\sqrt{37}$, and explain exactly what this approximation is giving you.

   (b) Find $\frac{dy}{dx}$ if

      i. $y = \ln(\sqrt{cosx^3})$

      ii. $xy^2 + x^3y = \cos(xy)$

      iii. $y = e^{x^4} \sin x \ln x \tan x$

      iv. $y = \ln 3t, x = t^3 + t^2$

   (c) Given that the derivative of $y = x^r$ is rx^{r-1}, for r any integer prove the same formula for r any rational. Then defining $\ln x$ as an integral show how one defines x^r for any real number r and derive the same formula.

3. (a) State and prove Rolle's Theorem. State the Mean Value Theorem.

   (b) Solve the following

      i. $(x^2 + 1)\frac{dy}{dx} = y$

      ii. $\frac{dy}{dx} + 3y = \exp 3x$

   (c) A rectangle is to be inscribed inside a semicircle of radius 2 cm. What are the dimensions of the rectangle of largest possible area?

4. (a) Let f(x) be defined on [0,1] as 1 on all rational numbers and 0 on all irrationals. Explain why this function is not Riemann integrable.

   (b) Integrate the following.

      i. $\int x^2 \ln x dx$

ii. $\int x \ln x^2 dx$

iii. $\int \sin^4 x \cos^2 x dx$

iv. $\displaystyle\int \frac{x+1}{x^2+x+1} dx$

v. $\displaystyle\int \frac{x^2+x+1}{(x-1)^2(x-2)} dx$

5. (a) Find the area of the region bounded by $y = x^2$ and $y = x + 1$.

(b) How is $\int_0^1 \frac{1}{x^2} dx$ defined?

(c) Find the volume of the solid of revolution gotten by revolving the region bounded by $y = x^2, x = 0$, and $y = 1$ about the y-axis, first by the method of disks, and then by the method of cylindrical shells.

6. (a) Define $\lim_{n\to\infty} a_n = L$, and $\sum_{n=1}^{\infty} a_n = S$.

(b) Do the following series converge or diverge? Give reasons.

i. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-2}$

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(c) Prove that if $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$ then $\sum_{n=1}^{\infty}(a_n + b_n) = a + b$.

(d) Find for what values of x the following power series converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n!}$$