Advanced Calculus
MA1132
Exercises 8 Solutions

1. Consider the region R defined by
$$R = \left\{(x, y), \,|\, x^2 + y^2 \leq 9\right\}.$$
Let $f(x, y) = 3 - \sqrt{x^2 + y^2}$. Calculate
$$\iint_R f(x, y)dA.$$
Solution: Representing the region R in polar coordinates, we see that
$$R = \{(r, \theta) \,|\, r \leq 3 \,\text{and}\, 0 \leq \theta \leq 2\pi\}.$$
Thus, the integral becomes
$$\int_0^{2\pi} \int_0^3 (3 - r) r dr d\theta = \int_0^{2\pi} \frac{9}{2} d\theta = 18\pi.$$

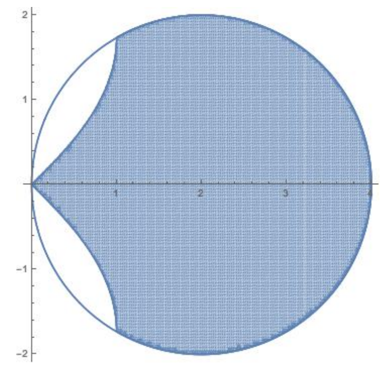
2. Consider the region R that is inside the curve $r = 4\cos\theta$ and outside the curve (lemniscate) $r^2 = -8\cos 2\theta$, where r and θ are the polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$.
   (a) What is the curve $r = 4\cos\theta$?
   (b) Sketch the region R.
   (c) Find the area of R.

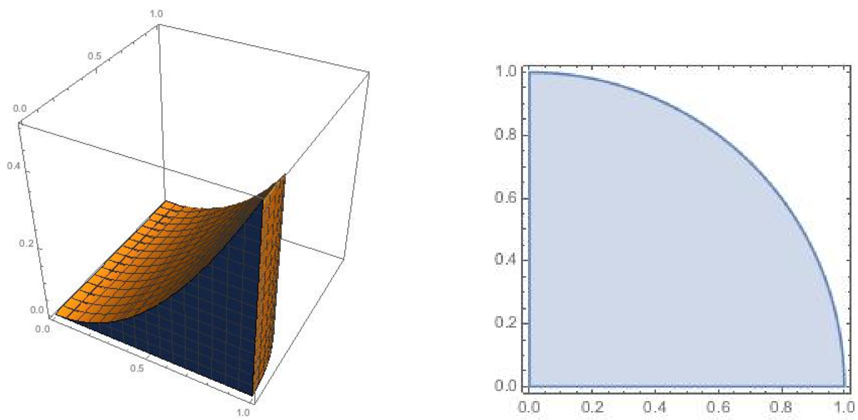
Solution:

(a) $r = 4\cos\theta$ is the circle of radius 2 centred at $(2, 0)$. The curve $r^2 = -8\cos 2\theta$ is the lemniscate $(x^2 + y^2)^2 = 2a^2(y^2 - x^2)$ with the parameter $a = 2$.

(b) The region R is shown below

(c) To find the area of R we need to find the range of θ. It is done by solving the equation
$$4\cos\theta = \sqrt{-8\cos 2\theta} \implies \cos\theta = \frac{1}{2}, \tag{1}$$
which gives $\theta = -\pi/3$ and $\theta = \pi/3$. Thus, the area A of R is
$$A = \iint_R dA = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(16\cos^2\theta + 8\cos 2\theta)d\theta = \frac{8\pi}{3} + 4\sqrt{3}. \tag{2}$$





3. Find the volume V of the solid bounded by the planes $x = 0$, $y = 0$, $z = 0$, the cylinders $az = x^2$, $a > 0$, $x^2 + y^2 = b^2$, and located in the first octant $x \geq 0, y \geq 0, z \geq 0$.

Solution: The solid, and its projection R onto the xy-plane are shown below $(a = 2, b = 1)$ Thus, the volume is

$$\begin{aligned} V &= \iint_R \frac{1}{a} x^2 dxdy = \frac{1}{a} \int_0^{\pi/2} \int_0^b r^2 \cos^2\theta \, r \, drd\theta = \frac{b^4}{4a} \int_0^{\pi/2} \cos^2\theta d\theta \\ &= \frac{b^4}{4a} \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) d\theta = \frac{\pi b^4}{16a} \,. \end{aligned} \tag{3}$$

4. Find the surface area of the portion of the surface $z = 2x + y^2$ that is above the triangular region with vertices $(0, 0)$, $(0, 1)$ and $(1, 1)$.

   Solution:

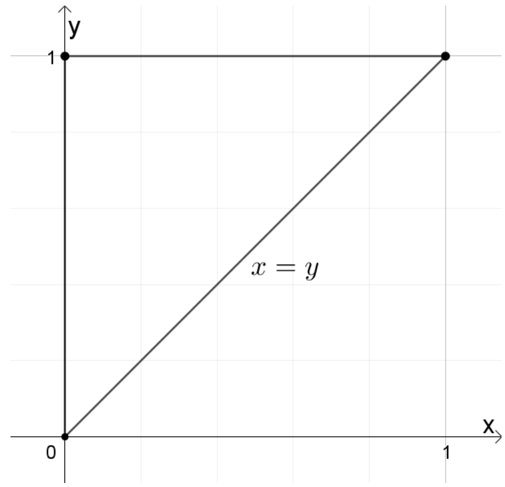
   We have $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 2y$.

   Since

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{2^2 + (2y)^2 + 1} = \sqrt{5 + 4y^2},$$

   it is easier to integrate with respect to x first.

Using the sketch,
we see that the area is



$$
\begin{aligned}
\int_0^1 \int_0^y \sqrt{5+4y^2}\,dx\,dy
&= \int_0^1 \left[x\sqrt{5+4y^2} \right]_0^y dy \\
&= \int_0^1 y\sqrt{5+4y^2}\,dy \\
&= \left[\frac{1}{12}\left(5+4y^2\right)^{\frac{3}{2}} \right]_0^1 \\
&= \frac{9^{\frac{3}{2}} - 5^{\frac{3}{2}}}{12} \\
&= \frac{27-5\sqrt{5}}{12}.
\end{aligned}
$$

5. Find the surface area of the portion of the paraboloid $2z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 8$.

   Solution:

   We have $z = \dfrac{x^2+y^2}{2}$, so that $\dfrac{\partial z}{\partial x} = x$ and $\dfrac{\partial z}{\partial y} = y$.

   Hence
$$
\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{x^2+y^2+1} = \sqrt{r^2+1}.
$$

   Thus the surface area is
$$
\begin{aligned}
\int_0^{2\pi} \int_0^{\sqrt{8}} \sqrt{r^2+1}\, r\,dr\,d\theta &= \int_0^{2\pi} \left[\frac{1}{3}\left(r^2+1\right)^{\frac{3}{2}} \right]_0^{\sqrt{8}} d\theta \\
&= \int_0^{2\pi} \frac{9^{\frac{3}{2}} - 1^{\frac{3}{2}}}{3}\,d\theta \\
&= \int_0^{2\pi} \frac{26}{3}\,d\theta \\
&= \left[\frac{26}{3}\theta \right]_{)}^{2\pi} \\
&= \frac{52\pi}{3}.
\end{aligned}
$$

6. Find a parametric representation of

   (a) the elliptic cone
$$
z = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}.
$$

Solution: It is a generalisation of the parametrisation of a circular cone

$$x = au\cos\theta\,, \quad y = bu\sin\theta\,, \quad z = u\,, \quad 0 \le \theta \le 2\pi\,,\ u \ge 0\,.$$

(b) the hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\,.$$

Solution: It is a generalisation of the parametrisation of a surface of revolution

$$x = a\sqrt{u^2+1}\cos\theta\,, \quad y = b\sqrt{u^2+1}\sin\theta\,, \quad z = cu\,, \quad 0 \le \theta \le 2\pi\,,\ -\infty \le u \le \infty\,.$$