Problem 1

Let ϵ_{ijk}, $i, j, k = 1, 2, 3$ be the antisymmetric (Levi-Civita) tensor with $\epsilon_{123} = 1$, and let

$$\vec{A} = \{A_1, A_2, A_3\}\,, \quad \vec{B} = \{B_1, B_2, B_3\}\,, \quad \vec{C} = \{C_1, C_2, C_3\}$$

be three arbitrary vectors.

1. Check that
$$\big(\vec{A} \times \vec{B}\big)_i = \epsilon_{ijk} A_j B_k\,, \quad i = 1, 2, 3\,. \tag{1}$$
Here and in what follows the repeated indices are summed over.

2. Check that
$$\epsilon_{ijk}\epsilon_{\ell mn} = \delta_{i\ell}\delta_{jm}\delta_{kn} - \delta_{im}\delta_{j\ell}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{k\ell} + \delta_{im}\delta_{jn}\delta_{k\ell} + \delta_{in}\delta_{j\ell}\delta_{km} - \delta_{i\ell}\delta_{jn}\delta_{km} \tag{2}$$

3. Use (2) to show that
$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{3}$$

4. Use (1) and (3) to show that
$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \tag{4}$$

5. Use (1) and (3) to simplify
$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) \tag{5}$$

6. Use (3) to show that
$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn} \tag{6}$$

7. Let $\vec{B}$ and $\vec{C}$ be polar vectors, i.e transforming as
$$\tilde{B}_i = R_{ij} B_j\ , \tag{7}$$
under orthogonal transformations. Prove that the vector $\vec{A} = \vec{B} \times \vec{C}$ transforms as
$$\tilde{A}_i = \det R\; R_{ij} A_j\ , \tag{8}$$
i.e. it is a pseudo or axial vector.

8. Define $J_{ij} = \epsilon_{ijk} M_k$ for some vector $\vec{M} = \{M_1, M_2, M_3\}$. Write down J explicitly as a 3×3 matrix. Compute $\epsilon_{ijk} J_{jk}$.

   This question establishes a one-to-one correspondence between vectors $\vec{M}$ and rank-2 antisymmetric tensors J in 3-dimensional space

9. Show that $\epsilon_{ijk} J_{ij} N_k = \Lambda\, \vec{M} \cdot \vec{N}$, find the coefficient of proportionality Λ.

Problem 2

Let M be an $n \times n$ matrix with entries M_{ij}, and v be an n-dimensional column with entries v_i.

1. You have the following combinations
$$M_{jk} M_{ij}\,, \qquad M_{jk} M_{ji}\,, \qquad M_{ij} M_{ij}\,, \qquad M_{ij} v_j\,, \qquad M_{ij} v_i \tag{9}$$
Identify them as components of the following objects
$$M \cdot v\,, \qquad v^{\mathrm{T}} \cdot M\,, \qquad \mathrm{tr} M^2\,, \qquad (M^2)_{ik}\,, \qquad (M^{\mathrm{T}} M)_{ki}\,, \qquad \mathrm{tr} M M^{\mathrm{T}}\,. \tag{10}$$

2. How would you write
$$(M M^{\mathrm{T}})_{ki}\,, \quad (M M^{\mathrm{T}})_{ik}\,, \quad v^{\mathrm{T}} \cdot v\,, \quad v^{\mathrm{T}} \cdot M \cdot v \tag{11}$$
using indices and Einstein's summation convention?

Problem 3

Let q^i, $i = 1, \ldots, s$ be generalised coordinates. Compute the quantities below

1. $$\frac{\partial q^j}{\partial q^i} = \quad , \quad \frac{\partial q^i}{\partial q^i} = \quad , \quad \frac{\partial}{\partial q^i}(M_{jk} q^j q^k) = \quad , \quad \frac{\partial}{\partial q^i}(M_{jk} q^i q^k) = \quad , \tag{12}$$

   Do not assume that M_{ij} is symmetric with respect to index permutations.

2. $$\frac{\partial}{\partial q^i}(M_{jkl} q^j q^k q^l) = \quad , \quad \frac{\partial}{\partial q^i}(M_{jkl} q^j q^i q^l) = \quad , \tag{13}$$

   Do not assume that M_{ijk} is symmetric with respect to index permutations

3. $$\frac{\partial^2}{\partial q^l \partial q^i}(M_{jk} q^j q^k) = \quad , \quad \frac{\partial^2}{\partial q^i \partial q^i}(M_{jk} q^j q^k) = \quad , \quad \frac{\partial^2}{\partial q^i \partial q^l}(M_{jkm} q^j q^k q^l) = \quad , \tag{14}$$

   Do not assume that M_{ij} is symmetric with respect to index permutations.