MA1125 – Calculus
Homework #7 solutions

1. Find the area of the region enclosed by the graphs of $f(x) = 2x^2$ and $g(x) = x + 6$.

The graph of the parabola $f(x) = 2x^2$ meets the graph of the line $g(x) = x + 6$ when

$$2x^2 = x + 6 \quad \Longleftrightarrow \quad 2x^2 - x - 6 = 0 \quad \Longleftrightarrow \quad (2x + 3)(x - 2) = 0.$$

Since the line lies above the parabola at the points $-3/2 \leq x \leq 2$, the area is then

$$\int_{-3/2}^{2} [g(x) - f(x)]\,dx = \int_{-3/2}^{2} \left[x + 6 - 2x^2\right]\,dx = \left[\frac{x^2}{2} + 6x - \frac{2x^3}{3}\right]_{-3/2}^{2} = \frac{343}{24}.$$

2. Compute the volume of a sphere of radius $r > 0$. Hint: one may obtain such a sphere by rotating the upper semicircle $f(x) = \sqrt{r^2 - x^2}$ around the x-axis.

The volume of the sphere is the integral of $\pi f(x)^2$ and this is given by

$$\int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \pi\left(\frac{2r^3}{3} + \frac{2r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

3. Compute the length of the graph of $f(x) = \frac{1}{3}x^{3/2}$ over the interval $[0, 5]$.

The length of the graph is given by the integral of $\sqrt{1 + f'(x)^2}$ and one has

$$f'(x) = \frac{1}{3} \cdot \frac{3}{2} \cdot x^{1/2} = \frac{\sqrt{x}}{2} \quad \Longrightarrow \quad 1 + f'(x)^2 = 1 + \frac{x}{4} = \frac{x+4}{4}.$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_0^5 \sqrt{1 + f'(x)^2}\,dx = \int_0^5 \frac{(x+4)^{1/2}}{2}\,dx = \left[\frac{(x+4)^{3/2}}{3}\right]_0^5 = \frac{3^3 - 2^3}{3} = \frac{19}{3}.$$

4. Find both the mass and the centre of mass for a thin rod whose density is given by

$$\delta(x) = x^3 + 2x^2 + 5x, \qquad 0 \le x \le 2.$$

The mass of the rod is merely the integral of its density function, namely

$$M = \int_0^2 \delta(x)\,dx = \int_0^2 (x^3 + 2x^2 + 5x)\,dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{5x^2}{2}\right]_0^2 = \frac{58}{3}.$$

The centre of mass is given by a similar formula and one finds that

$$\overline{x} = \frac{1}{M}\int_0^2 x\delta(x)\,dx = \frac{3}{58}\int_0^2 (x^4 + 2x^3 + 5x^2)\,dx = \frac{3}{58}\left[\frac{x^5}{5} + \frac{x^4}{2} + \frac{5x^3}{3}\right]_0^2 = \frac{208}{145}.$$

5. A cylindrical tank of radius 2m and height 3m is full with water of density 1000kg/m^3. How much work is needed to pump out the water through a hole in the top?

Consider a cross section of the tank which has arbitrarily small height dx and lies x metres from the top. The volume of this cylindrical cross section is

$$V = \pi \cdot \text{radius}^2 \cdot \text{height} = 4\pi\,dx.$$

Its mass is volume times density, namely $m = 4000\pi\,dx$, and the force needed to pump out this part is mass times acceleration, namely mg. The overall amount of work is thus

$$\text{Work} = \int mg \cdot x = \int_0^3 4{,}000\pi g \cdot x\,dx = 4{,}000\pi g\left[\frac{x^2}{2}\right]_0^3 = 18{,}000\pi g.$$