Local and global minima/maxima

- We say f has a LOCAL maximum at the point x_0, if $f(x) \le f(x_0)$ in some <u>interval</u> around x_0. We say f has a GLOBAL maximum at x_0, if $f(x) \le f(x_0)$ for all x in the domain of f.

1 Single change of signs Suppose f is differentiable and f' changes sign exactly once. If f' changes from $\oplus$ to $\ominus$ at x_0, then ~~*~~ f has a global max at x_0. If it changes from $\ominus$ to $\oplus$, then f has a global min at x_0.

<u>Example 1.</u> We show $f(x) = x^4 - 4x$ attains a min value. In fact, we show $x^4 - 4x \geq -3$ for all x.

Consider $f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x-1)(\underbrace{x^2+x+1})$.

$\hookrightarrow$ a quadratic with no real roots. It is always positive.

The sign of f' is the sign of $x-1$.

Thus $f'(x) > 0$ when $x > 1$
$f'(x) < 0$ when $x < 1$.

		1	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$\therefore$ $f(1)$ is the min value attained and $f(x) \geq f(1) \Rightarrow x^4 - 4x \geq -3$ for all x.

<u>Example 2.</u> We show $xe^{-x} \le e^{-1}$ for all $x \in \mathbb{R}$.
This gives $x \le e^{x-1}$ for all $x \in \mathbb{R}$.

We need to compute the max value of $f(x) = \frac{x}{e^x} = xe^{-x}$.

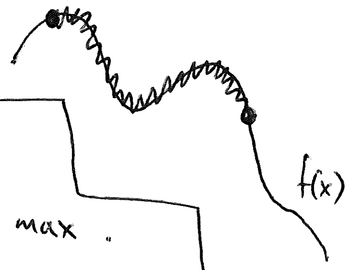
In this case, $f'(x) = \frac{e^x - xe^x}{e^x e^x} = \frac{\cancel{e^x}(1-x)}{\cancel{e^x}\, e^x} = \frac{1-x}{e^x}$.

The denominator is positive, so $f'(x) > 0$ when $x < 1$
$f'(x) < 0$ when $x > 1$.

		1	
f'	$+$		$-$
f	$\nearrow$		$\searrow$

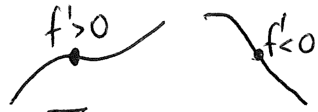
Thus $f(1)$ is the GLOBAL maximum so $f(x) \le f(1)$ for all x, as needed

2 Continuous functions on $[a,b]$



If f is continuous on a finite interval $[a,b]$, then f attains <u>both</u> a global min <u>and</u> a global max.
Moreover, these occur at ... ⓐ the endpoints a, b or
ⓑ the points with $f'(x)=0$ or
ⓒ the points where $f'(x)$ does not exist.

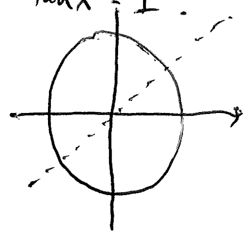
<u>Proof</u>. If we get a min/max at an <u>interior</u> point, then f' can be neither ⊕ nor ⊖ at that point. ▨



<u>Example 1.</u> Consider $f(x) = \sin x + \cos x$ for $0 \le x \le 2\pi$.
We find the min/max values attained. Note that we <u>don't</u> have to worry about the sign of $f'(x)$. In this case,

$$f'(x) = \cos x - \sin x \quad \text{exists at all points}$$

and $f'(x)=0$ when $\cos x = \sin x$, namely when $\tan x = 1$.
We get $x = \pi/4$ and $x = 5\pi/4$ as possible candidates. We check the values at



$x=0$ (endpoint) ... $f(0) = \sin 0 + \cos 0 = 1$

$x = 2\pi$ (endpoint) ... $f(2\pi) = 1$

$x = \pi/4$ $(f'=0)$ ... $f(\pi/4) = \sin \pi/4 + \cos \pi/4 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$.

$x = 5\pi/4$ $(f'=0)$ ... $f(5\pi/4) = \; = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$.

The global min is $-\sqrt{2}$. The global max is $+\sqrt{2}$.

▨ For this problem, one can write

$$\sin x + \cos x = \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$
$$= \sqrt{2}\left(\cos x \cos \pi/4 + \sin x \sin \pi/4\right)$$
$$= \sqrt{2}\cos(x - \pi/4).$$

Example 2. Consider $f(x) = x^4 - 2x^2 - 1$ on $[0, 2]$.

Then $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$. This is defined for all x and $f'(x) = 0$ when $x = 0, 1, -1$.

We exclude $x = -1$ and check the others. We need to check

endpoint .... $f(0) = -1$

endpoint .... $f(2) = 16 - 8 - 1 = 7$

$f(1) = 1 - 2 - 1 = -2$.

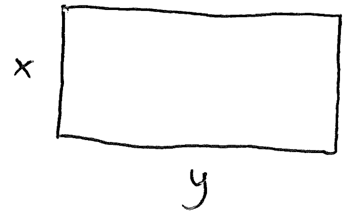
This gives $f(1) = -2$ as the minimum value

and $f(2) = 7$ as the maximum value.

We have proved that $-2 \leq x^4 - 2x^2 - 1 \leq 7$ for all $0 \leq x \leq 2$.

Optimisation problems

① Out of all rectangles with perimeter 40, which one has the largest area?



Let x = width, y = length.

We know $2x + 2y = 40$ and need to maximise Area $= xy$.

- Eliminating y gives $x+y=20 \Rightarrow \underline{y = 20-x}$ and we need to maximise $f(x) = xy = x(20-x) = 20x - x^2$.
- Restrictions on x .... $x \geq 0$ and $y \geq 0$

  $x \geq 0$ and $20 - x \geq 0$ .... so $0 \leq x \leq 20$.

We thus need to max $f(x) = 20x - x^2$ over $[0, 20]$.

Since $f'(x) = 20 - 2x = 2(10-x)$, we get $f'(x) = 0$ when $x=10$, f' exists at all points. Possible min/max values:

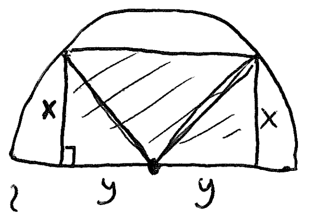
$x = 10$ .... $f(10) = 10(20-10) = 100$

$x = 0$ .... $f(0) = 0$

$x = 20$ .... $f(20) = 0$ as well.

Thus, the largest possible area is 100 (attained when $x=y=10$).

② Consider a rectangle inscribed in a semicircle of radius $r = 4$ with one of its sides along the diameter. How large can the area of the rectangle be?



Let x be the height, let y be half of the other side.

By Pythagoras', $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$.

We need to maximise $A(x) = 2xy = 2x\sqrt{r^2 - x^2}$.

This is the same as maximising $B(x) = 4x^2(r^2 - x^2)$, namely $B(x) = 4x^2r^2 - 4x^4$ .... for a given constant r.

Restrictions on x: we need $x \geq 0$ and also $x \leq r$.
This gives an interval $[0, r]$ that is finite.

We have $B'(x) = 4 \cdot 2x \cdot r^2 - 4 \cdot 4x^3 = 8x(r^2 - 2x^2)$ and this is zero when $x = 0$ and $x^2 = \frac{r^2}{2}$, namely $x = \frac{r}{\sqrt{2}}$.

Possible candidates for a maximum:

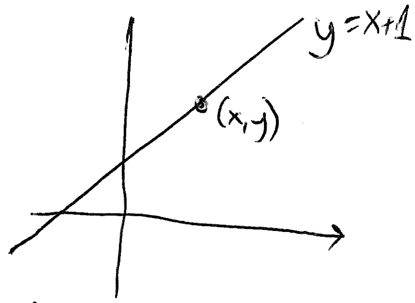
$x = 0$ ----- $B(0) = 0$

$x = \frac{r}{\sqrt{2}}$ ----- $B\left(\frac{r}{\sqrt{2}}\right) = 4 \cdot \frac{r^2}{2} \cdot r^2 - 4 \cdot \frac{r^4}{4} = r^4$

(endpoint) $x = r$ ----- $B(r) = 4r^4 - 4r^4 = 0$.

Thus, the largest possible area is $B(r/\sqrt{2}) = r^4 \Rightarrow A(r/\sqrt{2}) = r^2$.

(3) We find the point on the line $y = x+1$ which is closest to the point $(4, 1)$.



We need to minimise the distance

$$d = \sqrt{(x-4)^2 + (y-1)^2}$$ between (x,y) and $(4,1)$.

This is $d = \sqrt{(x-4)^2 + x^2}$ with $x \in \mathbb{R}$ arbitrary.

We have $$d'(x) = \frac{2(x-4) + 2x}{2\sqrt{(x-4)^2 + x^2}} = \frac{2(x-2)}{\sqrt{(x-4)^2 + x^2}}$$

So $d'(x) > 0$ when $x > 2$
$d'(x) < 0$ when $x < 2$.

		2	
$d'(x)$	$-$		$+$
$d(x)$	$\searrow$		$\nearrow$

This makes $d(2)$ a global minimum (attained when $x=2$, $y=3$).
The shortest distance is then $\sqrt{(2-4)^2 + (3-1)^2} = \sqrt{4+4} = 2\sqrt{2}$.

(4) (Economics) A company produces phones with daily cost $C(x) = 200 + 40x$ with x = number of phones. The price of each phone is $P(x) = 100 - 2x$ with $0 \leq x \leq 50$. How many phones

should be produced to maximise profits?

In this case Profits $\pi(x)$ = Revenues - Cost

$$= x(100-2x) - (200+40x)$$

$$= 100x - 2x^2 - 200 - 40x$$

$$= -2x^2 + 60x - 200.$$

We need to maximise $\pi(x)$ when $0 \le x \le 50$. We have

$$\pi'(x) = -4x + 60 = -4(x-15)$$

So the possible candidates are:

(endpoint) $x=0$ ---- $\pi(0) = -200$

(endpoint) $x=50$ ---- $\pi(50) = 50 \cdot 0 - 200 - 40 \cdot 50 = -2200$

$x=15$ ---- $\pi(15) = 15 \cdot 70 - (200 + 40 \cdot 15) = 250.$

Profits become max when $x=15$.

$\boxed{\text{Related rates}}$ Consider two or more variables that change over time. If those are related, then their rates of change are also related. The differentiation is usually with respect to t.

Example 1. Consider a circular pond whose radius is increasing at 2m/sec. How fast is its area increasing when radius = 4m?

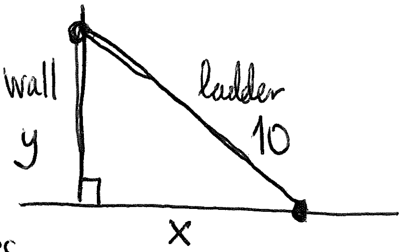
Let $r = r(t)$ be the radius, $A = A(t)$ = the area. Then

$$A = \pi r^2 \quad \text{or} \quad A(t) = \pi \cdot r(t)^2 .$$

We are given $\frac{dr}{dt} = 2m/sec$ and seek $\frac{dA}{dt}$. Since $A(t) = \pi r(t)^2$, we have $\frac{dA}{dt} = \pi \cdot 2r(t)\, r'(t)$ and so

$$\frac{dA}{dt} = \pi \cdot 2 \cdot 4 \cdot 2 = 16\pi \quad m^2/sec.$$

Example 2. A ladder is resting against a wall and it is 10 ft long. If the base of the ladder starts sliding at 1ft/sec along the floor, how fast is the top sliding down the wall when the ladder is 6 ft away from the wall?



We have the relation $x^2 + y^2 = 10^2$, where $x = x(t)$ and $y = y(t)$. Differentiating gives

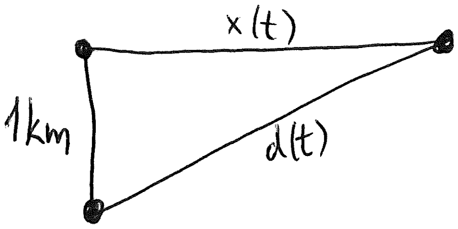
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies x\frac{dx}{dt} + y\frac{dy}{dt} = 0.$$

We know $\frac{dx}{dt} = 1$ and $x = 6$, so $\quad 6 \cdot 1 + y \cdot \frac{dy}{dt} = 0$

$$\text{so} \quad \frac{dy}{dt} = \frac{-6}{y} = -\frac{6}{\sqrt{10^2 - 6^2}}$$

$$\text{and} \quad \frac{dy}{dt} = \frac{-6}{\sqrt{64}} = -3/4 .$$

Example 3. A plane is flying horizontally at 1km altitude and 800 Km/h speed. It passes above a petrol / gas station. How fast is the distance from the station changing when it is 2km further down?



We have $d(t)^2 = x(t)^2 + 1$

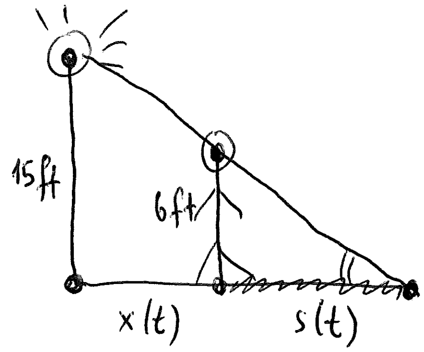
$$\Rightarrow \cancel{2}d(t)\,d'(t) = \cancel{2}x(t)\,x'(t)$$

$$\Rightarrow d'(t) = \frac{x(t)x'(t)}{d(t)}.$$

We know $\frac{dx}{dt} = 800$ and $x = 2$, so $d = \sqrt{2^2+1^2} = \sqrt{5}$.

Thus $d' = \frac{2\cdot 800}{\sqrt{5}} = \frac{1600\sqrt{5}}{5} = 320\sqrt{5}$.

Example 4. A street light is on top of a 15ft pole. A 6ft-tall person is walking away from the pole at 3ft/sec. How fast is the tip of his/her shadow moving?



We know $\frac{dx}{dt} = 3$ and we seek $\frac{ds}{dt}$.

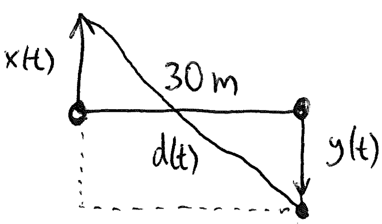
We need to relate s and x. Similar triangles give: $\frac{6}{s} = \frac{15}{x+s}$

So $6x + 6s = 15s$ and $6x = 9s$. Thus

$$6\frac{dx}{dt} = 9\frac{ds}{dt} \Rightarrow \frac{ds}{dt} = \frac{6\frac{dx}{dt}}{9} = \frac{2}{3}\cdot 3 = 2 \text{ ft/sec}.$$

Example 5. A man and a woman start walking at the same time. The man walks north at 3m/sec and the woman walks south at 5m/sec but starting at a point 30 m to the east.

How fast are they separating from one another 5 secs later?



Those are related by: $d(t)^2 = (x(t)+y(t))^2 + 30^2$

Then $\cancel{2}d(t)\,d'(t) = \cancel{2}(\underbrace{x(t)}_{15} + \underbrace{y(t)}_{25})\cdot(\underbrace{x'(t)}_{3} + \underbrace{y'(t)}_{5})$

So $d'(t) = \frac{40\cdot 8}{d(t)} = \frac{320}{\sqrt{40^2+30^2}} = \frac{320}{[illegible]} = 32/5$ m/sec $\square$

$\boxed{\text{Linear approximation}}$ If f is differentiable at the point x_0, then $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$, so $f'(x_0) \simeq \frac{f(x) - f(x_0)}{x - x_0}$ near the point x_0 and $f(x) \simeq f'(x_0)(x - x_0) + f(x_0)$.

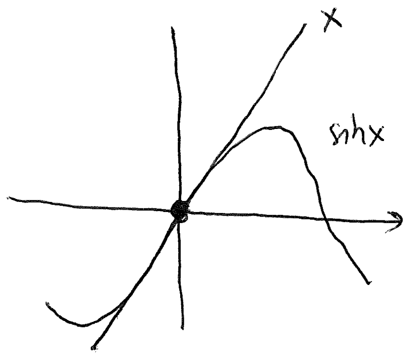
Definition. We call $L(x) = f'(x_0)(x - x_0) + f(x_0)$ the linear approximation (or tangent line approximation) at the point x_0.

Example 1. Consider $f(x) = \sin x$ at the point $x_0 = 0$.

In this case, $f(x_0) = f(0) = \sin 0 = 0$

and $f'(x_0) = f'(0) = \cos 0 = 1$

so the tangent line is $L(x) = 1 \cdot (x - 0) + 0 = x$.

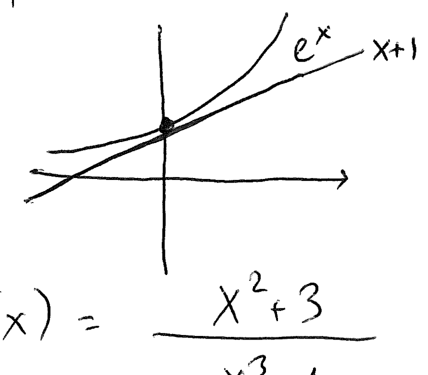


Example 2. Consider $f(x) = e^x$ at the point $x_0 = 0$.

In this case, $f(x_0) = f(0) = e^0 = 1$

and $f'(x_0) = f'(0) = e^0 = 1$, so $L(x) = 1(x - 0) + 1$
$= x + 1$.

Thus, $e^x \simeq x + 1$ for points $x \simeq 0$ ... and $e^{0.1} \simeq 1.1$, for instance.



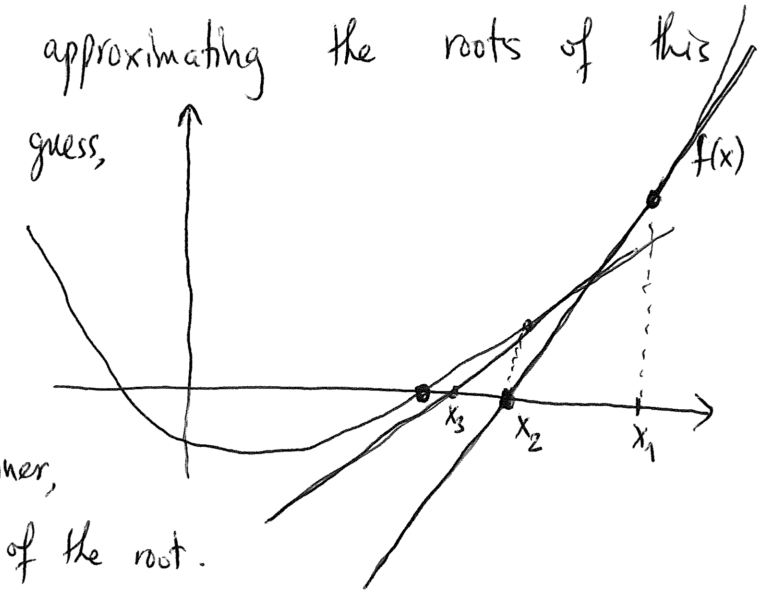
Example 3. Take $f(x) = \frac{x^2 + 3}{x^3 + 1}$ at the point $x_0 = 1$.

In this case, $f'(x) = \frac{2x \cdot (x^3 + 1) - 3x^2 \cdot (x^2 + 3)}{(x^3 + 1)^2} \Rightarrow f'(1) = \frac{4 - 12}{4} = -2$

and $f(1) = \frac{4}{2} = 2$, so the tangent line approximation is

$$L(x) = -2(x - 1) + 2 = -2x + 4.$$

Newton's method Consider an equation of the form $f(x)=0$. There is a standard method for approximating the roots of this equation. We start with an initial guess, say x_1. We approximate $f(x)$ by the tangent line at x_1. We compute the point x_2 at which the line meets the x-axis. Proceeding in this manner, we hope to get an approximation of the root.



The tangent line @ x_1 is $L(x) = f'(x_1)(x-x_1) + f(x_1)$

So we get $L(x)=0$ means $f'(x_1)(x-x_1) = -f(x_1)$

or $x - x_1 = -\dfrac{f(x_1)}{f'(x_1)}$, so $\boxed{x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}}$.

We can introduce this formula and proceed, if $f'(x_1) \neq 0$.

More generally,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}.$$

Example 1. We approximate $\sqrt{2}$ using this method. Obviously, $\sqrt{2}$ is a root of $f(x) = x^2 - 2$. Take $x_1 = 1$ as an initial guess and then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = x_n - \frac{x_n}{2} + \frac{1}{x_n},$$

namely $x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$. We get

$$x_2 = \frac{3}{2} = 1.5 \quad \text{and then} \quad x_3 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} = 1.416666$$

and then $x_4 = 1.4142156$ and $x_5 = 1.414213562$ (with x_6 agreeing to 9 decimal places). If we take $x_1 = 2$ as an initial guess, we get $x_1 = 2$, $x_2 = 3/2$ and the rest stays the same.

Example 2. Consider $f(x) = x^3 - 4x^2 - 3x + 1$ as in the last homework. This has a (unique) root in $(0,2)$. We approximate this root using Newton's method: take $x_1 = 2$ and then define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

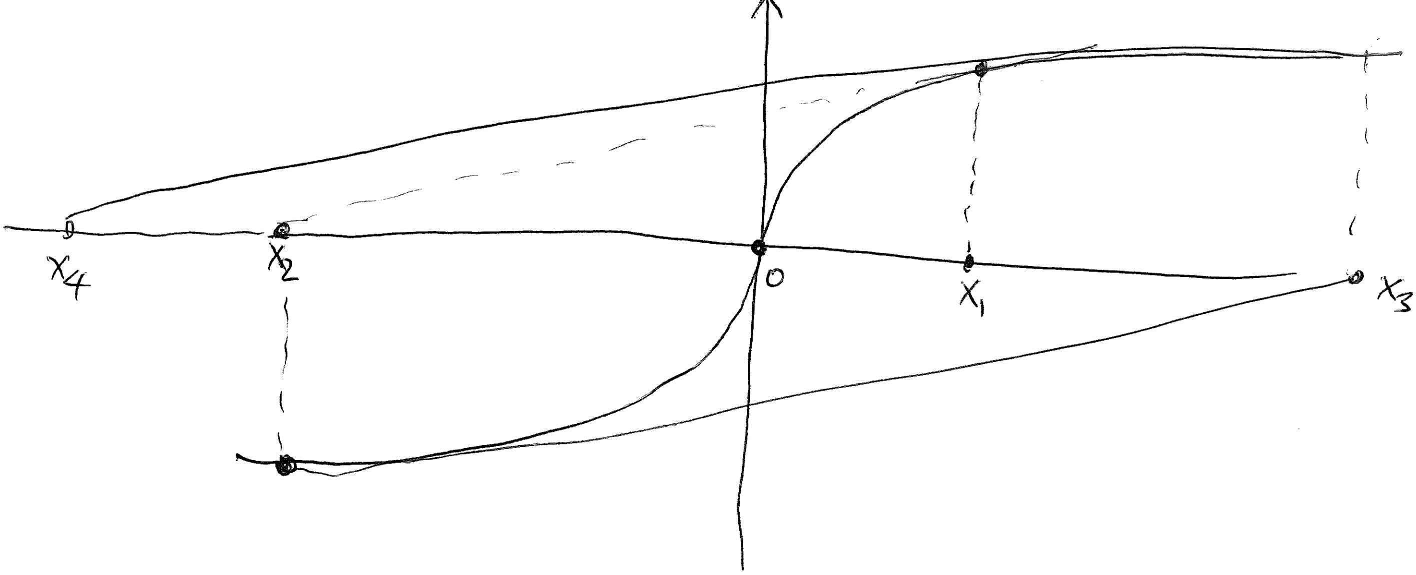
$$= x_n - \frac{\cancel{3}x_n^3 - 4x_n^2 - 3x_n + 1}{3x_n^2 - 8x_n - 3}.$$

In this case,

$$x_1 = 2$$
$$x_2 = 0.1428$$
$$x_3 = 0.2635$$
$$x_4 = 0.253309$$
$$x_5 = 0.253239$$

and one finds the root to be $x = 0.253$ (to 3 decimal digits).

$\boxed{\text{Note:}}$ The method will not work in <u>all</u> cases, but it works quite often (and the approximations approach a limit).



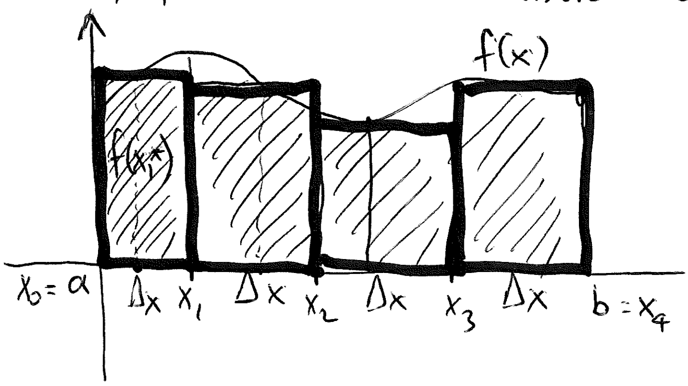
Definition of integral We use Σ to denote sums such as $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5$. We call k the index of summation and $\sum_{k=1}^{5} a_k = \sum_{i=1}^{5} a_i$. One may "shift the index" of summation to get

$$\sum_{k=1}^{6} a_k = \sum_{k=2}^{7} a_{k-1}.$$

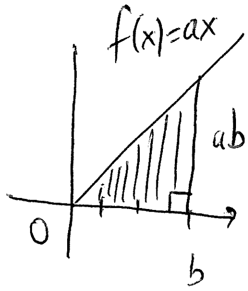
Integral We define $\int_a^b f(x)\,dx$ of a function over a finite interval $[a,b]$ is defined as the limit

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x},$$

where $\Delta x = \frac{b-a}{n}$ and $x_0, x_1, \dots, x_n$ are the points that subdivide the interval into n equal parts $x_0 = a$, $x_1 = a + \Delta x$, $\dots$, $x_n = b$ and the point x_k^* is any point in $[x_k, x_{k+1}]$. We say f is integrable on $[a,b]$, if the limit above exists.



Example. We show $f(x) = ax$ is integrable on $[0,b]$ for any $a, b > 0$ and $\int_0^b ax\,dx = \frac{1}{2}ab^2$.



We have to check $\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x = \frac{1}{2}ab^2}$.

Let $x_0, x_1, \dots, x_n$ be the points that divide $[0,b]$ into n equal parts, $\Delta x = b/n$ is the length of each subinterval and $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x$, $\dots$, $x_n = n\Delta x$.

By assumption, $x_{k-1} \leq x_k^* \leq x_k$

$$\Rightarrow \quad a x_{k-1} \leq a x_k^* \leq a x_k$$

$$\Rightarrow \quad a x_{k-1} \leq f(x_k^*) \leq a x_k$$

$$\Rightarrow \quad a x_{k-1}\,\Delta x \leq f(x_k^*)\,\Delta x \leq a x_k\,\Delta x$$

$$\Rightarrow \quad \boxed{\sum_{k=1}^{n} a x_{k-1}\,\Delta x} \leq \sum_{k=1}^{n} f(x_k^*)\,\Delta x \leq \boxed{\sum_{k=1}^{n} a x_k\,\Delta x}.$$

$$a x_0 \Delta x + a x_1 \Delta x + a x_2 \Delta x + \dots + a x_{n-1} \Delta x$$
$$= a\Delta x\,(x_0 + x_1 + x_2 + \dots + x_{n-1})$$
$$= \frac{ab}{n}\,(0 + 1\Delta x + 2\Delta x + \dots + (n-1)\Delta x)$$
$$= \frac{ab}{n}\cdot\frac{b}{n}\,(0 + 1 + 2 + \dots + (n-1))$$
$$\boxed{\frac{ab^2}{n^2} \cdot \frac{(n-1)n}{2}}$$

Sum on the left

$$a\Delta x\,(x_1 + x_2 + \dots + x_n)$$
$$= a\Delta x\,(1\Delta x + 2\Delta x + \dots + n\Delta x)$$
$$= \boxed{a\left(\frac{b}{n}\right)^2 \frac{n(n+1)}{2}}$$

because
$$S = 1 + 2 + \dots + (n-1)$$
$$S = (n-1) + (n-2) + \dots + 1$$
$$2S = n + n + \dots + n$$
$$= n(n-1)$$

This computation gives

$$\frac{ab^2}{n^2}\,\frac{n(n-1)}{2} \leq \underbrace{\sum_{k=1}^{n} f(x_k^*)\,\Delta x} \leq \frac{ab^2}{n^2}\,\frac{n(n+1)}{2}.$$

We have
$$\lim_{n\to\infty} \frac{ab^2\, n(n-1)}{n^2 \cdot 2} = \frac{ab^2}{2} = \lim_{n\to\infty} \frac{ab^2 \cdot n(n+1)}{n^2 \cdot 2}$$

So it follows by the Squeeze theorem that

$$\int_0^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x = \frac{ab^2}{2}$$

<u>Theorem 1.</u> (Linearity of integral) If f, g are integrable on $[a,b]$ then $f+g$ is integrable and $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. If f is integrable on $[a,b]$ and c is a constant, then cf is integrable and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

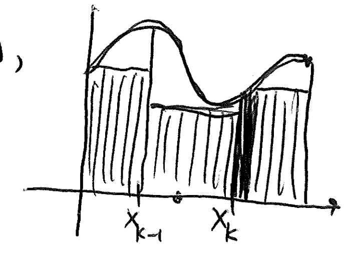
<u>Proof.</u> Let $h(x) = f(x)+g(x)$, for instance. Then

$$
\begin{aligned}
\int_a^b h(x)\,dx &\overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^n h(x_k^*)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\,\Delta x + g(x_k^*)\,\Delta x \\
&= \lim_{n\to\infty} \left[\sum_{k=1}^n f(x_k^*)\,\Delta x + \sum_{k=1}^n g(x_k^*)\,\Delta x \right] \\
&= \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\Delta x + \lim_{n\to\infty} \sum_{k=1}^n g(x_k^*)\,\Delta x_k \\
&= \int f(x)\,dx + \int g(x)\,dx. \quad \blacksquare
\end{aligned}
$$

$\boxed{\text{Integrability}}$ Suppose f is defined on $[a,b]$ and consider the points $x_0, x_1, \ldots, x_n$ that subdivide $[a,b]$ into n subintervals of equal length. We say that f is integrable on $[a,b]$, if

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and x_k^* is arbitrary in $[x_{k-1}, x_k]$.



Theorem 1 (Linearity) If f, g are integrable, then $f+g$ is also and

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

If f is integrable, then so is $c \cdot f$ and $\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$.

Theorem 2 (Inequalities) If f, g are integrable on $[a,b]$ with $f(x) \le g(x)$ for all $a \le x \le b$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

"Proof." We have $f(x_k^*) \le g(x_k^*)$ for each subinterval $[x_{k-1}, x_k]$

$$\Rightarrow \quad f(x_k^*)\Delta x \le g(x_k^*)\Delta x$$

$$\Rightarrow \quad \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \le \lim_{n\to\infty} \sum_{k=1}^{n} g(x_k^*)\Delta x$$

$$\Rightarrow \quad \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx. \quad \blacksquare$$

More formally, $g(x) \geq f(x)$ means $g(x) - f(x) = h(x) \geq 0$,

so $\int_a^b h(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} h(x_k^*)\,\Delta x \geq 0$ so $\int_a^b g - \int_a^b f \geq 0$.

Example (Non-integrable) Let $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$. Then f is not integrable on $[a,b]$. If this were integrable,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} 0 \cdot \Delta x = 0$$

... for $x_k^* \in \mathbb{Q}$

and $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x = n\Delta x = b-a$$

... for $x_k^* \notin \mathbb{Q}$.

Thus, f is not integrable on $[a,b]$ whenever $b > a$.

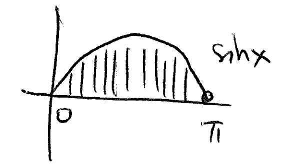
Theorem 3 (Continuous implies integrable) If f is continuous on $[a,b]$, then f is integrable on $[a,b]$, namely $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$ exists.

Proof. Postponed for MA1126.

Theorem 4. Fundamental theorem of calculus, part 1

Suppose f is continuous and $f = F'$ for some function F. Then
$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a).$$

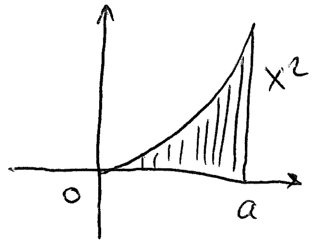
Example 1. We compute $\int_0^{\pi} \sin x \, dx$.



Since $-(\sin x) = (\cos x)'$, we have $(-\cos x)' = \sin x$

so
$$\int_a^b \sin x\,dx = \int_0^{\pi} (\sin x)\,dx = \int_0^{\pi} (-\cos x)'\,dx$$
$$= -\cos\pi + \cos 0 = +1+1 = 2.$$

Example 2. We compute $\int_0^a x^2\,dx$.



We need a function $F(x)$ with $F'(x) = x^2$.

Such a function is $F(x) = \frac{x^3}{3}$. This gives
$$\int_0^a x^2\,dx = \int_0^a \left(\frac{x^3}{3}\right)'dx = \frac{a^3}{3} - \frac{0^3}{3} = \frac{a^3}{3}.$$

Proof of Theorem 4. We need to compute $\int_a^b F'(x)\,dx$, which is $\lim_{n\to\infty} \sum_{k=1}^{n} F'(x_k^*)\Delta x$, with x_k^* arbitrary in $[x_{k-1}, x_k]$.

We pick x_k^* using the MVT .... $\frac{F(b)-F(a)}{b-a} = F'(c)$ for some point $a<c<b$. We use this fact on $[x_{k-1}, x_k]$:
$$\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(x_k^*) \Rightarrow F'(x_k^*)\Delta x = F(x_k) - F(x_{k-1}).$$

We get
$$\int_a^b F'(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left[F(x_k) - F(x_{k-1})\right] = \begin{array}{l} F(x_n) - \cancel{F(x_{n-1})} \\ \cancel{F(x_{n-1})} - \cancel{F(x_{n-2})} \\ \vdots \\ F(x_1) - F(x_0) \end{array} = \begin{array}{l} F(b) \\ -F(a). \end{array}$$

Definition We say that F is an antiderivative of f when F is a function whose derivative is f : $F' = f$. This is denoted by $F(x) = \int f(x)\, dx$.

Definition We defined $\int_a^b f(x)\, dx$ when $a < b$. The remaining cases are defined by $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$ and

$$\int_a^a f(x)\, dx = 0 .$$

Antiderivatives. We'll make a table of known antiderivatives such as

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} , \text{ if } n \neq -1$$

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x|$$

Antiderivatives We say that $F(x)$ is an <u>antiderivative</u> of $f(x)$ when $F(x)$ is a function with $F'(x) = f(x)$. This is denoted by

$$F(x) = \int f(x)\,dx \Rightarrow F'(x) = f(x)$$

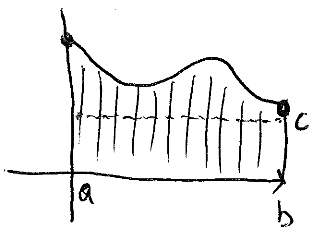
and we call $\int f(x)\,dx$ the <u>indefinite integral</u> of $f(x)$, $\int_a^b f(x)\,dx$ being the <u>definite</u> one.

Table of standard integrals

- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}$, if $n \neq -1$
- $\int \frac{1}{x}\,dx = \ln|x|$
- $\int \cos x\,dx = \sin x$
- $\int \sin x\,dx = -\cos x$
- $\int \sec^2 x\,dx = \tan x$
- $\int \csc^2 x\,dx = -\cot x$
- $\int \sec x \tan x\,dx = \sec x$
- $\int \csc x \cot x\,dx = -\csc x$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x$
- $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$
- $\int e^{ax}\,dx = \frac{1}{a} e^{ax}$, if $a \neq 0$.

- Lots of familiar functions have antiderivatives that are not familiar. Typical examples are $\frac{\sin x}{x}$, $\frac{e^x}{x}$, x^x, e^{-x^2}.

Mean value theorem for integrals If f is continuous on $[a,b]$, its <u>average value</u> is defined by $\frac{1}{b-a}\int_a^b f(x)\,dx$.



This is the constant c that gives the same integral as $f(x)$,

$$\int_a^b c\,dx = c\cdot b - c\cdot a = c(b-a) = \int_a^b f(x)\,dx.$$

$$\boxed{\text{If } f \text{ is continuous on } [a,b], \text{ then } f(x_*) = \frac{1}{b-a}\int_a^b f(x)\,dx \text{ for some } x_*.}$$

<u>Proof.</u> Since f is continuous, it attains a min m and a max M.

Then $m \leq f(x) \leq M$ for all x

So $$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

$$\Rightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

By the intermediate value theorem, since f is continuous, f attains any value between m and M. $\blacksquare$

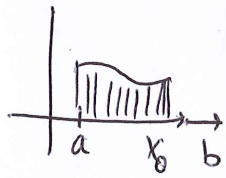
Fundamental theorem of calculus, part 1

If f is continuous and $F' = f$, then $\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$.

We write this as $\int_a^b F'(x)\,dx = \cancel{[F(x)]_a^b}\ \left[\int F'(x)\,dx\right]_a^b = [F(x)]_a^b = F(b) - F(a)$.

Fundamental theorem of calculus, part 2

If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$, namely $\left[\int_a^x f(t)\,dt\right]' = f(x)$.



<u>Proof</u>. We check $F'(x_0) = f(x_0)$ as usual. Then

$$F'(x_0) = \lim_{x\to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0}$$

$$= \lim_{x\to x_0} \underbrace{\frac{1}{x-x_0}\int_{x_0}^x f(t)\,dt}_{f(c)\text{ for some } c \text{ between } x \text{ and } x_0 \text{ by the MVT}} = f(x_0) \text{ by continuity.}$$

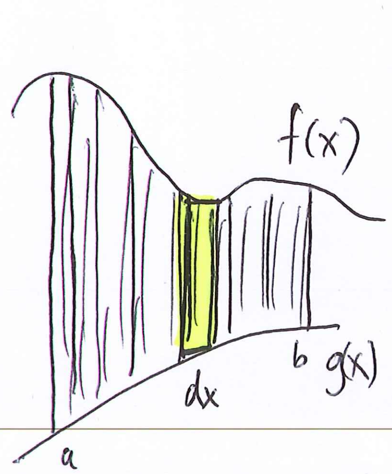
$\blacksquare$

Area between two graphs

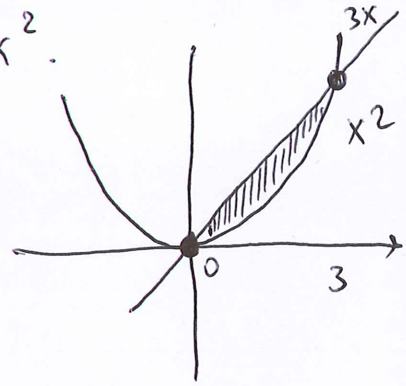
Suppose f, g are continuous on $[a,b]$ and $f(x) \ge g(x)$ for all $a \le x \le b$.

Then the area between the two graphs is

$$\text{Area} = \int_a^b \underbrace{[f(x) - g(x)]}_{\text{height}} \cdot \underbrace{dx}_{\text{base}}.$$



Example. Consider $f(x) = 3x$ and $g(x) = x^2$.

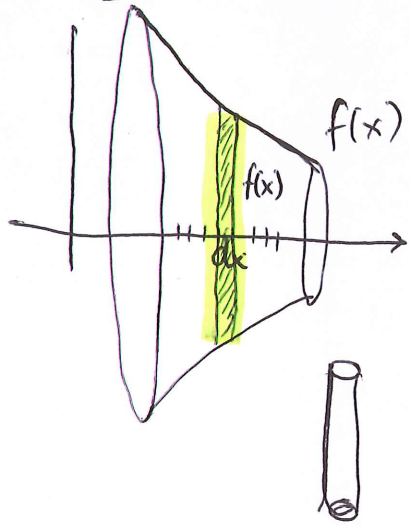


We compute the area of the shaded region. For that region $f(x) \geq g(x)$. The limits of integration are given by

$$f(x) = g(x)$$
$$3x = x^2$$
$$0 = x(x-3) \quad \cdots \quad x = 0, 3.$$

Thus, $\text{Area} = \int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = 9/2.$

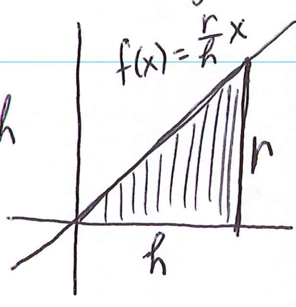
Volume for a solid of revolution



Suppose f is continuous on $[a,b]$ and the graph of f is rotated around the x-axis. Then the resulting solid has

$$\text{Volume} = \int_a^b \underbrace{\pi f(x)^2 \, dx}_{\substack{\text{volume of cylinder} \\ \text{with radius } f(x),\text{ base } dx}}$$

Example. We compute the volume of a cone of radius r and height h. We can obtain such a cone by rotating a triangle around the x-axis. In our case, $f(x) = \frac{r}{h} x$ with $0 \leq x \leq h$ so the volume of the cone is



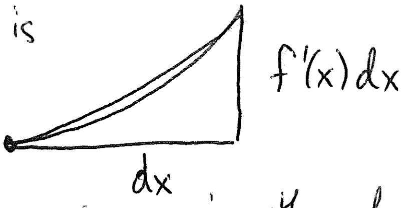
$$V = \int_0^h \pi \left(\frac{rx}{h}\right)^2 dx = \frac{\pi r^2}{h^2} \int_0^h x^2\,dx$$

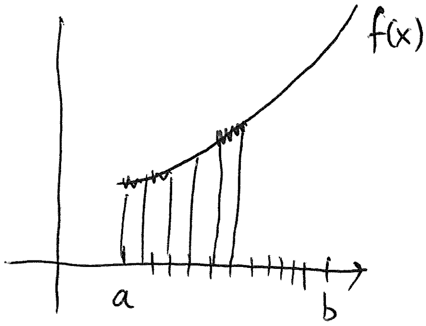
$$= \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2}\left(\frac{h^3}{3} - \frac{0^3}{3}\right) = \frac{1}{3}\pi r^2 h.$$

Arc length of a graph

Suppose f is differentiable and f' is continuous on $[a,b]$. We need to compute the length of its graph over $[a,b]$.

Looking at a small subinterval of length dx, we may assume $f'(x)$ is constant on that interval,



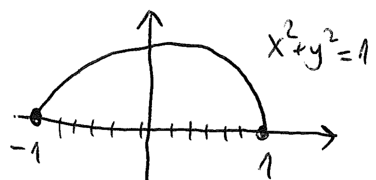


so the length of the curve is the hypotenuse

$$\sqrt{(dx)^2 + f'(x)^2(dx)^2} = \sqrt{1+f'(x)^2}\; dx.$$

Adding those up gives $\boxed{\text{Arc length} = \int_a^b \sqrt{1+f'(x)^2}\, dx.}$

Example. We compute the circumference (2π) of the unit circle. Consider the upper half $y = f(x) = \sqrt{1-x^2}$.



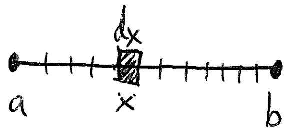
Then $f'(x) = \dfrac{1}{2\sqrt{1-x^2}}(1-x^2)' = \dfrac{-x}{\sqrt{1-x^2}}$

so $f'(x)^2 = \dfrac{x^2}{1-x^2}$ so $1+f'(x)^2 = 1 + \dfrac{x^2}{1-x^2} = \dfrac{1}{1-x^2}$

and Arc length $= \displaystyle\int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx$

$$= \left[\sin^{-1} x\right]_{-1}^{1} = \sin^{-1} 1 - \sin^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

$\boxed{\text{Mass}}$ Consider a thin rod that extends along the points $a \le x \le b$. If the density is constant, say δ, then density $= \dfrac{\text{mass}}{\cancel{\text{volume}}\ \text{length}}$ and mass $= \delta \cdot \text{length} = \delta \cdot (b-a)$.



If the density is arbitrary, given by a function $\delta(x)$, we can argue that a subinterval of length dx contributes mass $= \delta(x) \cdot dx$, so

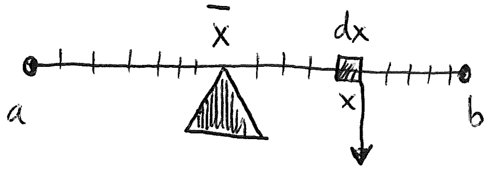
$$\boxed{\text{overall mass} = \int_a^b \delta(x)\, dx.}$$

Example. Consider a thin rod with density $\delta(x) = \sin(\pi x)$, $0 \leq x \leq 1$.

The total mass is $M = \int_0^1 \sin(\pi x)\, dx = \left[-\frac{\cos(\pi x)}{\pi} \right]_0^1$

$$= -\frac{\cos \pi}{\pi} + \frac{\cos 0}{\pi} = \frac{1}{\pi} + \frac{1}{\pi} = 2/\pi .$$

$\boxed{\text{Centre of mass}}$



Consider a thin rod as before. Let $\delta(x)$ be its density. We need to compute its centre of mass, say $\bar{x}$.

Looking at a small subinterval (of length dx at the point x), we get

torque = propensity to rotate = Force $\times$ displacement

$$= \text{mass} \cdot \text{gravity} \cdot \text{displacement}$$

$$= \underbrace{\delta(x) \cdot dx} \cdot g \cdot (x - \bar{x}).$$

For the centre of mass, the sum of the torques should be zero,

so $$\int_a^b \delta(x) \cancel{g} (x - \bar{x})\, dx = 0$$

so $$\int_a^b x\,\delta(x)\, dx - \bar{x} \int_a^b \delta(x)\, dx = 0$$

and we can solve for the centre of mass

$$\boxed{\bar{x} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx}, \text{ the denominator being mass.}}$$

Example. If the density is constant, say $\delta(x) = \delta$ = constant,

we get $$\bar{x} = \frac{\int_a^b x\,\delta\, dx}{\int_a^b \delta\, dx} = \frac{\cancel{\delta} \int_a^b x\, dx}{\cancel{\delta} \int_a^b 1\, dx} = \frac{\left[\frac{1}{2}x^2\right]_a^b}{\left[x\right]_a^b}$$

$$= \frac{\frac{1}{2}b^2 - \frac{1}{2}a^2}{b-a} = \frac{1}{2} \frac{b^2 - a^2}{b-a} = \frac{b+a}{2} = \text{midpoint}.$$

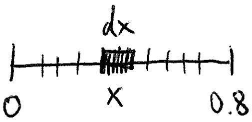
$\boxed{\text{Work}}$ In physics, the <u>work</u> of a force F that moves an object by d units (in the direction of the force) is defined as the product $W = F \cdot d$ in the case that F, d are constant. When either of those is not constant, work is the integral of $\int F(x)\cdot d(x)\,dx$

<u>Example 1.</u> The restoring force of a spring is given by Hooke's law
$F(x) = -kx$ for some (spring) constant k.

If a spring requires a force of 10N to stretch it by 0.5m past equilibrium, then how much work is needed to stretch it by 0.8m?

By assumption, $|F| = 10$ when $x = \frac{1}{2}$ so $k \cdot \frac{1}{2} = 10$ and $\underline{k = 20}$.

We need to find the work of $F(x) = -20x$, a variable force.



Looking at arbitrarily small parts of the spring (x units away of length dx), we get

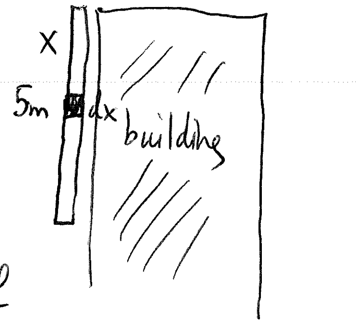
$$\text{work} = \underbrace{\boxed{\text{Force}}}\cdot \underbrace{\text{displacement}} = \underbrace{\boxed{k\cdot x}}\ \underbrace{dx}.$$

Summing up gives

$$\text{work} = \int_0^{0.8} F(x)\,dx = \int_0^{0.8} kx\,dx$$

$$= \int_0^{0.8} 20x\,dx = \left[\frac{\cancel{20}^{10}x^2}{\cancel{2}}\right]_0^{0.8} = 6.4\,J.$$

<u>Example 2.</u> A chain of length 5m is hanging from the top of a building. It <u>weighs 4 kg/m</u> and we need to pull it up to the top. How much work is needed?



Once again, look at the part x metres down and of arbitrarily small length dx. This gives displacement $= x$ and force $=$ mass $\cdot\, g$, where mass $= 4dx$ for that part.

Summing those up gives

$$\text{work} = \int_0^5 \text{force}\cdot\text{displacement} = \int_0^5 4gx\ dx$$

$$= \left[\frac{\cancel{4}^{2}gx^2}{\cancel{2}}\right]_0^5 = 2g\cdot 25 = 50g,$$

where $g = 9.81\ m/sec^2$.

Improper integrals The standard definition of integrability on $[a,b]$ assumes $a \le x \le b$ is finite and $f(x)$ is finite as well. We can slightly extend this definition to handle other cases as well. For instance, $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$, provided that the limit exists. Similarly, one can define

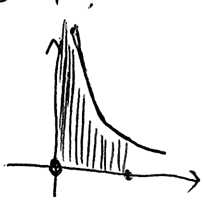
$$\int_a^b f(x)\,dx = \lim_{a_*\to a} \int_{a_*}^b f(x)\,dx$$

when $f(x)$ becomes infinite at a.

<u>Example 1</u>. Consider $\int_0^4 \frac{1}{\sqrt{x}}\,dx$, for instance.

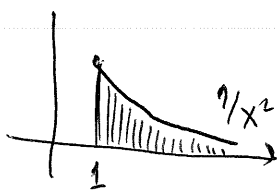
Then $$\int_a^4 \frac{1}{\sqrt{x}}\,dx = \int_a^4 \frac{2}{2\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_a^4$$
$$= 2\sqrt{4} - 2\sqrt{a} = 4 - 2\sqrt{a}.$$

This gives $$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0} \int_a^4 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0}(4-2\sqrt{a}) = 4.$$



<u>Example 2.</u> Consider $\int_1^\infty \frac{1}{x^2}\,dx$.

This is $$\int_1^\infty x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^\infty = \left[-1/x\right]_1^\infty = 0+1 = 1.$$



<u>Example 3.</u> Consider $\int_1^\infty \frac{1}{x}\,dx$.

This is $$\int_1^\infty \frac{1}{x}\,dx = \lim_{L\to\infty} \int_1^L \frac{1}{x}\,dx = \lim_{L\to\infty} \left[\ln x\right]_1^L$$
$$= \lim_{L\to\infty} (\ln L - \ln 1) = \infty.$$

Thus, the integral is <u>not</u> finite.

<u>Example 4.</u> Consider $\int_0^\infty e^{-ax}\,dx$ with $a \neq 0$ fixed. This is

$$\lim_{L\to\infty} \int_0^L e^{-ax}\,dx = \lim_{L\to\infty} \left[\frac{e^{-ax}}{-a}\right]_0^L = \cancel{e^{-at}} \lim_{L\to\infty} \frac{e^{-aL}-1}{-a} = \lim_{L\to\infty} \frac{1-e^{-aL}}{a}.$$

If $a>0$, the limit is $1/a$. If $a<0$, the limit is infinite.

$\boxed{\text{Integration by parts}}$ According to the product rule,

$$[f(x)\cdot g(x)]' = f'(x)\, g(x) + f(x)\cdot g'(x)$$

So $$\int \left[f'(x) g(x) + f(x) g'(x) \right] dx = \int [f(x) g(x)]' \, dx = f(x) g(x)$$

So $$\int f'(x) g(x)\, dx + \int f(x) g'(x)\, dx = f(x)\, g(x).$$

This gives $$\boxed{\int f(x) g'(x)\, dx = f(x) g(x) - \int f'(x) g(x)\, dx}$$

- This formula can also be written as $\underline{\int u\, dv = uv - \int v\, du}$ by taking $u = f(x)$ and $v = g(x)$. In fact,
$$u = f(x) \Rightarrow \frac{du}{dx} = f'(x) \Rightarrow du = f'(x)\, dx.$$

- Some typical examples that involve this formula are:
$\underline{\int p_n(x) \cdot e^{ax}\, dx}$ with p_n = polynomial of degree n

$\underline{\int p_n(x) \cdot \sin(ax)\, dx}$ and $\underline{\int p_n(x) \cos(ax)\, dx}$.

In all those cases, one can take $u = p_n(x)$.

Example 1. $\int x e^x \, dx$ ...... we let $u = x$ and $dv = e^x\, dx$

so $du = dx$ and $v = e^x$.

Then $$\int x e^x\, dx = \int u\, dv = x e^x - \int e^x\, dx = x e^x - e^x + C.$$

Example 2. $\int \underbrace{x^2}\, \sin x\, dx$ ..... this is a double integration by parts.

Take $u = x^2$, $dv = \sin x\, dx$

$du = 2x\, dx$, $v = -\cos x$

Then $$\int x^2 \sin x\, dx = -x^2 \cos x + \int \underbrace{2x \cos x}\, dx.$$

Take $u = 2x$, $dv = \cos x\, dx$

$du = 2\, dx$, $v = \sin x$

Then $$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x - \int 2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

Example 3. Consider $\int x^n \ln x \, dx$ with $n \neq -1$. In this case, we choose $u = \ln x$ (as a function to be differentiated) and $dv = x^n dx$. Then $du = \frac{1}{x} dx$ (contains no logarithms) and $v = \frac{x^{n+1}}{n+1}$, so

$$\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1} \ln x - \int \frac{1}{x} \frac{x^{n+1}}{n+1} dx$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n \, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C.$$

Other reasonable choices of u include
$u = \ln x \; \ldots \; du = \frac{1}{x} dx$
$u = \sin^{-1} x \; \ldots \; du = \frac{1}{\sqrt{1-x^2}} dx$
$u = \tan^{-1} x \; \ldots \; du = \frac{1}{1+x^2} dx$

Example 4. Consider $\int e^{ax} \cos(bx) \, dx$.

We could take $u = e^{ax}$ or $u = \cos(bx)$. Take $u = e^{ax}$, for instance.

Then $u = e^{ax}$, $dv = \cos(bx) \, dx$

$du = a e^{ax}$, $v = \frac{1}{b} \sin(bx)$.

Thus $$\boxed{\int e^{ax} \cos(bx) \, dx = \frac{e^{ax}}{b} \sin(bx) - \int \frac{a}{b} \underline{e^{ax} \sin(bx) \, dx}}$$

We integrate by parts once again. Take

$u = e^{ax}$, $dv = \sin(bx) \, dx$

$du = a e^{ax}$, $v = -\frac{1}{b} \cos(bx)$

We get $$\int e^{ax} \cos(bx) \, dx = \frac{1}{b} e^{ax} \sin(bx) - \frac{a}{b} \left[-\frac{1}{b} e^{ax} \cos(bx) + \frac{a}{b} \int e^{ax} \cos(bx) \right]$$

$$\therefore \int e^{ax} \cos(bx) \, dx = \frac{e^{ax} \sin(bx)}{b} + \frac{a \, e^{ax} \cos(bx)}{b^2} - \frac{a^2}{b^2} \int e^{ax} \cos(bx) \, dx$$

$$\therefore \left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cos(bx) \, dx = \frac{e^{ax} \sin(bx)}{b} + \frac{a e^{ax} \cos(bx)}{b^2} + C.$$

$\boxed{\text{Integration by substitution}}$ Several integrals can be simplified using the formula $\int g(f(x)) \cdot f'(x)\, dx = \int g(u)\, du$ which is dictated by the choice $u = f(x)$ and $du = f'(x)\,dx$.

This is an intermediate step for computing the integral on the left in terms of the variable $u = f(x)$. The choice of u is made to simplify the integral, noting that $du = f'(x)\,dx$ should also be present in the integral.

① $\int \cos(3x+1)\, dx$ ..... $u = 3x+1$, $du = 3dx$ ..... $\int \cos(3x+1)\, dx = \int \cos u \frac{du}{3}$

$\therefore \int \cos(3x+1)\, dx = \frac{1}{3}\int \cos u\, du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3x+1) + C.$

② $\int \frac{2x}{1+x^2}\, dx$ ..... $u = 1+x^2$, $du = 2x\, dx$ ..... $\int \frac{du}{u} = \ln u + C = \ln(1+x^2) + C.$

③ $\int \frac{e^x\, dx}{1+e^{2x}}$ ..... $u = e^x$, $du = e^x dx$ ..... $\int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(e^x) + C.$

④ $\int \frac{x^2}{(x+1)^3}\, dx$ ..... $\boxed{\begin{array}{l} u = x+1 \\ du = dx \end{array}}$ ..... $\int \frac{(u-1)^2}{u^3}\, du$

The integral on the right can be expanded as

$$\int \frac{u^2 - 2u + 1}{u^3}\, du = \int \left(\frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3}\right) du$$

$$= \int \left(\frac{1}{u} - 2u^{-2} + u^{-3}\right) du$$

$$= \ln u + 2u^{-1} - \frac{u^{-2}}{2} + C$$

$$= \ln(x+1) + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C$$

⑤ $\int \frac{\ln x}{x}\,dx$ ----- $\boxed{\begin{array}{l} u=\ln x \\ du=\frac{1}{x}\,dx \end{array}}$ --- $\int u\,du = \frac{u^2}{2}+C$

$= \frac{1}{2}(\ln x)^2 + C.$

⑥ $\int \sin(\sqrt{x})\,\underbrace{dx}$ ---- $\boxed{\begin{array}{l|l} u=\sqrt{x} & x=u^2 \\ du=\frac{1}{2\sqrt{x}}\,dx & dx=2u\,du \end{array}}$

$\int \sin u \cdot 2u\,du = 2\int u\cdot \sin u\,du$

$= 2\left[-u\cos u + \int \cos u\,du\right]$ by integration by parts $u=u$, $dv=\sin u\,du$, $du=du$, $v=-\cos u$

$= -2u\cos u + \sin u + C$

$= -2\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x} + C.$

⑦ $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ ...... $\boxed{\begin{array}{l} \text{let } u=\cos x \\ du=-\sin x\,dx \end{array}}$

$= -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$

$= \ln|\cos x|^{-1} + C = \ln|\sec x| + C.$

⑧ $\underline{\int \tan^{-1}x\,dx}$ or $\underline{\int \sin^{-1}x\,dx}$ or $\underline{\int \ln x\,dx}$.

Those are all similar. They are typical integrations by parts with $u=\tan^{-1}x$, $u=\sin^{-1}x$, $u=\ln x$ respectively. The derivatives are $du=\frac{1}{1+x^2}\,dx$, $du=\frac{1}{\sqrt{1-x^2}}\,dx$, $du=\frac{1}{x}\,dx$ which are much simpler.

For instance

$\int \underbrace{\tan^{-1}x}_{u}\,\underbrace{dx}_{dv} = x\cdot\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$ $\boxed{\begin{array}{ll} u=\tan^{-1}x & dv=dx \\ du=\frac{1}{1+x^2}dx & v=x \end{array}}$

$= x\tan^{-1}x - \int \frac{\frac{1}{2}du}{u}$ ← with $u=1+x^2$, $du=2x\,dx$

$= x\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + C.$

Similarly, $\int \ln x\,dx = x\ln x - \int \frac{1}{\cancel{x}}\cancel{x}\,dx$ int. by parts $\boxed{\begin{array}{l} u=\ln x,\ dv=dx \\ du=\frac{1}{x}dx,\ v=x \end{array}}$

$= x\ln x - x + C.$

On the other hand $\int \frac{\tan^{-1} x}{1+x^2}\,dx$ .... requires a substitution.

Let $u = \tan^{-1} x$, in which case $du = \frac{1}{1+x^2}\,dx$ and we get

$$\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\left(\tan^{-1} x\right)^2 + C.$$

$$\boxed{\text{Integrating } \int \sin^m x \cos^n x \, dx \text{ for any integers } m,n \geq 0}$$

- We use the fact that $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $\sin^2 x + \cos^2 x = 1$.

Ⓐ <u>The substitution $u = \sin x$ helps when n is odd.</u>

This is because powers of $\sin x$ become powers of u

and <u>even</u> powers of $\cos x$ become $(\cos x)^{2k} = (1-\sin^2 x)^k = (1-u^2)^k$

and $du = \cos x \, dx$ requires an additional copy of cosine.

<u>Example 1.</u> $$\int \sin^4 x \cos x \, dx = \int u^4 \, du = \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C.$$

$$\boxed{\begin{array}{l} u = \sin x \\ du = \cos x \, dx \end{array}}$$

<u>Example 2.</u> $$\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cdot \cos^2 x \cdot \cos x \, dx$$
$$= \int \sin^4 x \cdot (1-\sin^2 x) \cdot \cos x \, dx$$
$$= \int u^4 \cdot (1-u^2) \cdot du$$
$$= \int (u^4 - u^6) \, du = \frac{\sin^5 u}{5} - \frac{\sin^7 u}{7} + C.$$

<u>Example 3.</u> $$\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cdot \cos^4 x \cdot \cos x \, dx$$
$$= \int u^4 (1-u^2)^2 \, du$$
$$= \int u^4 (1 - 2u^2 + u^4) \, du = \int (u^4 - 2u^6 + u^8) \, du$$
$$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C.$$

Ⓑ <u>The substitution $u = \cos x$ helps when m is odd</u>

This is because $du = -\sin x \, dx$ takes away one copy of sine

and we can express <u>even</u> powers of sine as $(\sin x)^{2k} = (1-u^2)^k$.

<u>Example 4.</u> $$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \cdot \sin x \, dx$$

$$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array}}$$

$$= -\int (1-u^2) u^2 \, du = \int (u^4 - u^2) \, dx = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$$

(C) When m,n are both even, one may use the half-angle formulas

More precisely, $\cos(2x) = \cos(x+x) = \cos^2 x - \sin^2 x$

gives $\cos(2x) = 1-\sin^2 x - \sin^2 x \Rightarrow \boxed{\sin^2 x = \frac{1-\cos(2x)}{2}}$

and also $\cos(2x) = \cos^2 x - (1-\cos^2 x)$ $\quad \boxed{\cos^2 x = \frac{1+\cos(2x)}{2}}$

Example 5. $\displaystyle\int \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} dx = \frac{1}{2}\int (1-\cos(2x))\, dx$

$$= \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C = \frac{x}{2} - \frac{\sin(2x)}{4} + C.$$

Example 6. $\displaystyle\int \sin^2 x \cos^2 x \, dx = \int \frac{1-\cos(2x)}{2}\,\frac{1+\cos(2x)}{2}\, dx$

$$= \frac{1}{4}\int (1-\cos^2(2x))\, dx = \frac{1}{4}\int \left(1 - \frac{1+\cos(4x)}{2}\right) dx$$

$$= \frac{1}{4}\left[\frac{x}{2} - \frac{\sin(4x)}{8}\right] + C.$$

Example 7.(*) $\displaystyle\int \sin^3 x \, dx = \int \sin^2 x \cdot \sin x \, dx$ $\quad \boxed{u=\cos x}$ $\quad du = -\sin x \, dx$

$$= \int (1-\cos^2 x)\sin x \, dx$$

$$= -\int (1-u^2)\, du = -\cos x + \frac{\cos^3 x}{3} + C.$$

Integrating $\int \sec^m x \cdot \tan^n x \, dx$ for integers $m,n \geq 0$.

- In this case, $(\sec x)' = \sec x \tan x$, $(\tan x)' = \sec^2 x$, $\quad$ ~~$\sin^2 + \cos^2 = 1$~~ $\quad \tan^2 x + 1 = \sec^2 x$.

(A) The substitution $u = \sec x$ helps when n is odd

This is because $du = \sec x \tan x dx$ introduces one copy of $\tan x$ and we can deal with even powers of $\tan x$.

Example 1. $\displaystyle\int \tan^3 x \sec^2 x \, dx = \int \tan^2 x \sec x \tan x \sec x \, dx$ $\quad$ $u = \sec x$, $du = \sec x \tan dx$

$$= \int (u^2-1)\, u \, du$$

$$= \int (u^3 - u)\, du = \frac{\sec^4 x}{4} - \frac{\sec^2 x}{2} + C.$$

(B) The substitution $u = \tan x$ helps when m is even

This is because $du = \sec^2 x\, dx$ contains two copies of secant.

Example 2. $\int \tan^3 x \sec^4 x\, dx = \int \tan^3 x \cdot \sec^2 x\, \underbrace{\sec^2 x\, dx}$

$$\boxed{\begin{array}{c} u = \tan x \\ du = \sec^2 x\, dx \end{array}} \quad \boxed{\sec^2 x = 1 + \tan^2 x}$$

$$= \int u^3 (1 + u^2)\, du$$

$$= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C.$$

(C) When m is odd and n is even, we can express the integral in terms of secant

$$\boxed{\text{Integrating } \int \sec^m x \cdot \tan^n x \, dx \text{ for all integers } m, n \geq 0}$$

- The substitution $u = \sec x$ works whenever n is odd since $du = \sec x \tan x$ requires a copy of tangent. The substitution $u = \tan x$ works whenever m is even since $du = \sec^2 x \, dx$ requires $\sec^2 x$.
- For the remaining case of m odd and n even, one can express the integrand in terms of secant alone:

$$\int \sec^m x \cdot \tan^n x \, dx = \int (\sec x)^{2k+1} (\tan x)^{2\ell} \, dx$$
$$= \int (\sec x)^{2k+1} (\sec^2 x - 1)^{\ell} \, dx.$$

$\boxed{\text{Recursive formulas}}$ There are several formulas that relate an integral I_n depending on an integer n to integrals I_m for smaller values of m. The most standard/useful ones are:

① $$\int \underline{\underline{x^n}} \, \underline{e^{ax} \, dx} = \frac{x^n}{a} e^{ax} - \int \frac{n}{a} x^{n-1} e^{ax} \, dx$$

$\boxed{\begin{array}{l} u = x^n, \ dv = e^{ax} dx \\ du = n x^{n-1} dx \end{array}}$ $\boxed{v = \frac{e^{ax}}{a}}$

$$\int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx.$$

$$I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}.$$

For instance, $I_0 = \int x^0 e^{ax} \, dx = \frac{1}{a} e^{ax} + C$

and then $I_1 = \frac{1}{a} x e^{ax} - \frac{1}{a} I_0 = \frac{1}{a} x e^{ax} - \frac{1}{a^2} e^{ax} + C,$

so $I_2 = \frac{1}{a} x^2 e^{ax} - \frac{2}{a} I_1 = \frac{1}{a} x^2 e^{ax} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^3} e^{ax} + C.$

② Powers of sine $\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cdot \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

③ Powers of cosine $\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \cdot \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

Proof of ②: $\int \sin^n x \, dx = \int \sin^{n-1} x \; \sin x \, dx$

$u = \sin^{n-1} x$, $dv = \sin x \, dx$
$du = (n-1) \sin^{n-2} x \cos x$, $v = -\cos x$

$$= -\sin^{n-1} x \cdot \cos x + \int \cos x \, (n-1) \sin^{n-2} x \cdot \cos x \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$

This gives $\int \sin^n x \, dx = -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \, dx + (1-n) \int \sin^n x \, dx.$

so $n \int \sin^n x \, dx = -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \, dx.$ ∎

For instance, $I_0 = \int (\sin x)^0 \, dx = x + C$

$I_1 = \int \sin x \, dx = -\cos x + C$

$I_2 = -\frac{1}{2} \sin x \cos x + \frac{1}{2} I_0 = -\frac{1}{2} \sin x \cos x + \frac{x}{2} + C.$

④ Powers of secant ... There is a similar formula for $\int \sec^n x \, dx$.

In this case,

$$\int \sec^n x \, dx = \int \sec^{n-2} x \cdot \sec^2 x \, dx \quad \ldots$$

$u = \sec^{n-2} x$, $dv = \sec^2 x \, dx$
$du = (n-2) \sec^{n-2} x \tan x$
$v = \tan x$

$$= \sec^{n-2} x \cdot \tan x - \int (n-2) \sec^{n-2} x \cdot \tan^2 x \, dx$$

$$= \sec^{n-2} x \cdot \tan x + (2-n) \int \sec^{n-2} x \, (\sec^2 x - 1) \, dx$$

and $\underline{\int \sec^n x \, dx} = \sec^{n-2} x \tan x + (2-n) \underline{\int \sec^n x \, dx} + (n-2) \int \sec^{n-2} x \, dx$

$\therefore (n-1) \int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx.$

● The integral of secant is $\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C.}$

This is because $[\ln|\sec x + \tan x|]' = \frac{1}{\sec x + \tan x} (\sec x + \tan x)'$

$$= \frac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x)$$

$$= \frac{1}{\cancel{\sec x + \tan x}} \cdot \sec x \, (\cancel{\tan x + \sec x}).$$

$\boxed{\text{Trigonometric substitutions}}$ Those are substitutions that help simplify square roots like $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ with $a>0$.

- In the first case, $\sqrt{a^2-x^2} = \sqrt{a^2\left(1-\frac{x^2}{a^2}\right)} = a\sqrt{1-\left(\frac{x}{a}\right)^2}$.

Taking $\frac{x}{a} = \sin\theta$ gives $1-\left(\frac{x}{a}\right)^2 = 1-\sin^2\theta = \cos^2\theta$ and we can simplify the square root. Note that $-a \le x \le a$ so $-1 \le \frac{x}{a} \le 1$ and we can take $-\pi/2 \le \theta \le \pi/2$. Then

$$\sqrt{a^2-x^2} = a\sqrt{\cos^2\theta} = a\cos\theta \quad \text{because } \cos\theta \geq 0 \text{ here.}$$

$$\boxed{\sqrt{a^2-x^2} = a\cos\theta, \quad \text{where } x = a\sin\theta \text{ and } -\pi/2 \le \theta \le \pi/2.}$$

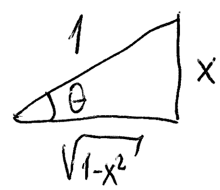
- For instance, $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\sqrt{1-\sin^2\theta}}} = \theta + C = \sin^{-1} x + C$. $(x = \sin\theta)$

Example 1. We compute $\displaystyle\int \sqrt{1-x^2}\, dx$.

Take $x = \sin\theta$ with $-\pi/2 \le \theta \le \pi/2$. Then $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$ and $dx = \cos\theta\, d\theta$. We get $\displaystyle\int \sqrt{1-x^2}\, dx = \int \cos^2\theta\, d\theta$

and $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ so
$$\int \sqrt{1-x^2}\, dx = \frac{1}{2}\int (1+\cos(2\theta))\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right] + C.$$

We need to express this in terms of x, where $x = \sin\theta$.
First, $\theta = \sin^{-1} x$. Also, $\sin(2\theta) = 2\sin\theta\cos\theta = 2x\sqrt{1-x^2}$ by above.
The integral is $\displaystyle\int \sqrt{1-x^2}\, dx = \frac{1}{2}\left[\sin^{-1} x + x\sqrt{1-x^2}\right] + C$.



Example 2. We compute $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$.

In this case, $\sqrt{4-x^2} = \sqrt{4\left(1-\frac{x^2}{4}\right)} = 2\sqrt{1-\left(\frac{x}{2}\right)^2}$ so we take $x = 2\sin\theta$.

Once again, $-\pi/2 \le \theta \le \pi/2$ and $\sqrt{4-x^2} = 2\sqrt{1-\sin^2\theta} = 2\cos\theta$

and $dx = 2\cos\theta \, d\theta$.

We get $$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{\cancel{2\cos\theta}\, d\theta}{4\sin^2\theta \cdot \cancel{2\cos\theta}}$$

$$= \frac{1}{4}\int \csc^2\theta \, d\theta.$$

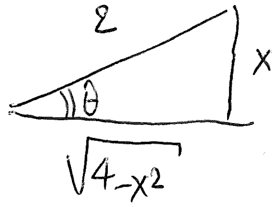
Note that $\int \sec^2\theta \, d\theta = \tan\theta + C$

and $\int \csc^2\theta \, d\theta = -\cot\theta + C$. We get $\frac{1}{4}(-\cot\theta) + C$

and we need to express this in terms of x.

Since $x = 2\sin\theta$, $\sin\theta = \frac{x}{2}$ and

$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{4-x^2}}{x}$



$$\therefore \int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{\sqrt{4-x^2}}{4x} + C.$$

$\boxed{\text{Consider integrals involving } \sqrt{a^2+x^2},\ a>0.}$

To simplify those, write $\sqrt{a^2\left(1+\frac{x^2}{a^2}\right)} = a\sqrt{1+\left(\frac{x}{a}\right)^2}$ so we can take $\frac{x}{a} = \tan\theta$ to get $a\sqrt{1+\tan^2\theta} = a\sqrt{\sec^2\theta} = a\sec\theta$ for any $-\pi/2 < \theta < \pi/2$. We thus have:

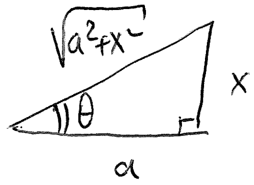
$$\boxed{x = a\tan\theta \quad \text{------} \quad \sqrt{a^2+x^2} = a\sec\theta \quad \text{-----} \quad dx = a\sec^2\theta\, d\theta}$$

Example 3. We compute $\int \frac{dx}{\sqrt{x^2+25}} = \int \frac{dx}{\sqrt{x^2+a^2}}$.

Let $x = a\tan\theta$ as above. We get $\int \frac{\cancel{a}\sec^{\cancel{2}}\theta\, d\theta}{\cancel{a\sec\theta}}$

which gives $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$. In our case,

$\frac{x}{a} = \tan\theta$ and so $\sec\theta = \frac{1}{\cos\theta} = \frac{\sqrt{a^2+x^2}}{a}$.



$$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|\frac{1}{a}\sqrt{a^2+x^2} + \frac{x}{a}\right| + C.$$

$\boxed{\text{Integrals involving } \sqrt{x^2-a^2},\ a>0.}$

$1+\tan^2 = \sec^2$

● Since $\sqrt{x^2-a^2} = a\sqrt{\left(\frac{x}{a}\right)^2-1}$, we can take $\frac{x}{a} = \sec\theta$.
Then $\sqrt{x^2-a^2} = a\sqrt{\tan^2\theta} = a|\tan\theta|$ depending on the values of θ.
We get $\tan\theta$ for $0<\theta<\pi/2$ and $-\tan\theta$ for $-\pi/2<\theta<0$.
The rest of the approach applies verbatim.

$\boxed{\text{Quadratic factors with no real roots.}}$

● The typical example is $\int \frac{dx}{x^2+1} = \tan^{-1} x + C$.

Here, one could substitute $x = \tan\theta$ to get $\int \frac{\cancel{\sec^2\theta}\, d\theta}{\cancel{1+\tan^2\theta}} = \theta + C = \tan^{-1}x + C$.

More generally, $\int \frac{dx}{ax^2+bx+c}$ with ax^2+bx+c having no real roots (and thus cannot be factored) is related to $\tan^{-1} x$.

<u>Example</u>. Consider $\int \frac{dx}{x^2-4x+5}$. If we complete the square, we can write $x^2-4x+5 = \underbrace{x^2-4x+4}+1 = (x-2)^2+1$. Thus

$$\int \frac{dx}{x^2-4x+5} = \int \frac{dx}{(x-2)^2+1} = \int \frac{du}{u^2+1} = \tan^{-1}u \ldots \quad \underline{\text{with } u = x-2}$$
$$= \tan^{-1}(x-2) + C.$$

We can similarly deal with

with $u = \frac{x-c}{a}$

$$\int \frac{dx}{(x-c)^2+a^2} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x-c}{a}\right)^2+1} = \frac{1}{a^2}\int \frac{a\,du}{u^2+1} = \frac{1}{a}\tan^{-1}\left(\frac{x-c}{a}\right).$$

Integration of rational functions $\frac{P(x)}{Q(x)}$

- There is a standard method for integrating PROPER rational functions $\frac{P(x)}{Q(x)}$, ones with degree $P(x)$ < degree $Q(x)$. If that is not the case, the fraction $\frac{P(x)}{Q(x)}$ can be simplified using division.

Example (Improper case) Consider $\int \frac{3x^3+x-1}{x+2}\,dx$.

This can be simplified. Using division, we get

$$\begin{array}{r} 3x^2-6x+13 \\ x+2\ \overline{)\ 3x^3+x-1} \\ \underline{3x^3+6x^2} \\ -6x^2+x-1 \\ \underline{-6x^2-12x} \\ 13x-1 \\ \underline{13x+26} \\ -27 \end{array}$$

$$\frac{3x^3+x-1}{x+2} = \cancel{(x+2)}(3x^2-6x+13) - \frac{27}{x+2}$$

so $$\int \frac{3x^3+x-1}{x+2}\,dx = \int \left(3x^2-6x+13-\frac{27}{x+2}\right)dx$$

$$= x^3 - 3x^2 + 13x - 27\ln|x+2| + C.$$

- Consider the ~~case~~ main (proper) case $\frac{P(x)}{Q(x)}$ with $\deg P < \deg Q$.

In that case, it is useful to express $Q(x) = Q_1(x)Q_2(x)$ as a product and express $\frac{P(x)}{Q(x)}$ as a sum of two fractions. The main idea is the following:

$$\frac{3}{x-1} + \frac{2}{x+4} = \frac{3(x+4)+2(x-1)}{(x-1)(x+4)} = \frac{5x+10}{(x-1)(x+4)}$$

so that the sum $\frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)} = \frac{P_3(x)}{Q_1(x)Q_2(x)}$. We need to run this computation backwards: $\frac{P_3(x)}{Q_1(x)Q_2(x)}$ needs to written as $\frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)}$.

Theorem (Partial fraction decomposition) Consider a PROPER rational function $\frac{P(x)}{Q(x)}$ whose denominator factors as $Q(x) = Q_1(x)Q_2(x)\cdots Q_m(x)$ with the polynomials Q_i being "relatively prime" (no common divisors other than constants). Then one can write $P(x)/Q(x)$ as the sum of partial fractions

$$\frac{P(x)}{Q(x)} = \frac{P_1(x)}{Q_1(x)} + \dots + \frac{P_m(x)}{Q_m(x)}$$ with $\frac{P_i(x)}{Q_i(x)}$ proper for each i.

The denominators are given. The numerators need to be found.

Example 1. Consider $\frac{3x-5}{x^2-4x+3} = \frac{3x-5}{(x-1)(x-3)}$. By the theorem,

$$\frac{3x-5}{x^2-4x+3} = \frac{A}{x-1} + \frac{B}{x-3}$$ for some constants A, B

$$\therefore \frac{3x-5}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3} = \frac{A(x-3)+B(x-1)}{(x-1)(x-3)}$$

$$\therefore \boxed{3x-5 = A(x-3)+B(x-1)}$$ ...... this holds for all x!!!

Let $x=1$ to get $3-5 = -2A \Rightarrow A=1$

Let $x=3$ to get $9-5 = 2B \Rightarrow B=2$

Going back to the original function $\frac{3x-5}{x^2-4x+3} = \frac{1}{x-1} + \frac{2}{x-3}$

so $$\int \frac{3x-5}{x^2-4x+3}\,dx = \ln|x-1| + 2\ln|x-3| + C.$$

One could also find A, B by comparing coefficients:

$$3x-5 = A(x-3)+B(x-1) \quad \ldots\ldots \quad \begin{cases} A+B=3 \\ -3A-B=-5 \end{cases}$$

Example 2. Consider $\frac{x^2+4}{x^3-x} = \frac{x^2+4}{x(x^2-1)} = \frac{x^2+4}{x(x-1)(x+1)}$.

We can write $$\frac{x^2+4}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

for some constants A, B, C. Clear denominators to get

$$\boxed{x^2+4 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \quad \ldots \text{ for all } x.}$$

When $x=1$, we get $1+4 = 2B$ and $B = 5/2$

$x=0$, we get $4 = -A$ and $A=-4$

$x=-1$, we get $1+4 = C(-1)(-2)$ and $C = 5/2$.

Thus $$\int \frac{x^2+4}{x(x-1)(x+1)}\,dx = \int \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right) dx$$

$$= -4\ln|x| + \frac{5}{2}\ln|x-1| + \frac{5}{2}\ln|x+1| + C.$$

$\boxed{\text{Repeated factors}}$ Consider $\frac{P(x)}{Q(x)}$ with $Q(x) = x^2(x+1)$, for instance. In this case, the factors x, x cannot be separated (because they have a common factor). The partial fractions decomposition is

$$\frac{3x-5}{x^2(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1}$$

$$= \underbrace{\frac{A}{x} + \frac{B}{x^2}}_{\text{two terms for } x^2} + \underbrace{\frac{C}{x+1}}_{\text{one term for } x+1}.$$

Partial fractions decomposition - General case

Consider a PROPER rational function $\frac{P(x)}{Q(x)}$ in the case that Q can be factored. If Q is the product of m polynomials, then the quotient $\frac{P(x)}{Q(x)}$ is the sum of m proper rational functions.

- for each linear factor $ax+b$, we get a fraction $\frac{A}{ax+b}$.
- for each repeated linear factor $(ax+b)^m$, we get the fractions $\frac{A_i}{(ax+b)^i}$ for each $1 \le i \le m$.
- for each repeated quadratic factor $(ax^2+bx+c)^m$ that cannot be factored, we get the fractions $\frac{A_i x + B_i}{(ax^2+bx+c)^i}$ for each i.

Examples.

$$\frac{x^3-x+2}{x^2(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3} + \frac{D}{x-1}$$

$$\frac{3x^2+5x-2}{(x^2+1)(x-1)(x+3)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+3} + \frac{E}{(x+3)^2}$$

$$\frac{x^4-3x+5}{x^3(x^2+4)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}.$$

- Repeated factors contribute several terms because x^3 contributes $\frac{Ax^2+Bx+C}{x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}$. Similarly $(x-1)^2$ contributes $\frac{Au+B}{u^2}$ with $u = x-1$ and that is $\frac{A}{x-1} + \frac{B}{(x-1)^2}$.

The same is true for quadratic factors.

<u>Example</u>. Consider $\int \frac{x+2}{x^2(x-1)}\,dx$, for instance.

In this case, $\frac{x+2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

$$\Longrightarrow \quad \boxed{x+2 = Ax(x-1) + B(x-1) + Cx^2 \;\text{---- for all } x.}$$

When $x=0$, we get $2 = -B$ and $B=-2$
When $x=1$, we get $3 = C$
When $x=2$, we get $4 = 2A+B+4C = 2A-2+12$ and $A=-3$.

Thus
$$\int \frac{x+2}{x^2(x-1)}\,dx = \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}\right) dx$$
$$= -3\ln|x| + \frac{2}{x} + 3\ln|x-1| + C_0.$$

- Once we use partial fractions, we only end up with terms

$$\int \frac{dx}{(ax+b)^i} = \frac{1}{a}\int \frac{du}{u^i} = \begin{cases} \frac{1}{a}\,\frac{u^{1-i}}{1-i} & \text{if } i\neq 1 \\ \frac{1}{a}\ln|u| & \text{if } i=1 \end{cases} \qquad \begin{aligned} u &= ax+b \\ du &= a\,dx. \end{aligned}$$

Those arise through linear factors. For quadratic factors, we get

$$\int \frac{x\,dx}{(ax^2+bx+c)^i} \quad \text{and} \quad \int \frac{dx}{(ax^2+bx+c)^i}.$$ The quadratic

can be reduced to $ax^2+bx+c = a\left(x^2 + 2\frac{b}{2a}x + \frac{c}{a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right)$

$$= a\left(x+\frac{b}{2a}\right)^2 + a\,\frac{4ac-b^2}{4a^2}.$$

This becomes $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + ak^2$

and then $a u^2 + ak^2 = a(u^2+k^2)$.

We can substitute $u = x+\frac{b}{2a}$ to end up with $\int \frac{du}{(u^2+k^2)^i}$.

Euclidean algorithm (division algorithm)

Consider two integers x, y. Then $\gcd(x,y)$ can be expressed as a linear combination of x, y: $\gcd(x,y) = ax + by$ for some a, b.

$$8 = 5 \cdot 1 + 3$$
$$5 = 3 \cdot 1 + 2$$
$$3 = 2 \cdot 1 + 1 \leftarrow \text{gcd}$$
$$2 = 1 \cdot 2 + \underline{\underline{0}}$$

We run the algorithm backwards

$$1 = 3 - 2$$
$$= 3 - (5 - 3) = 2\cdot 3 - 5$$
$$= 2\cdot(8-5) - 5 = 2\cdot 8 - 3\cdot 5.$$

This shows that $\gcd(x,y) = ax + b\cdot y$.

Suppose now that $\frac{P(x)}{Q(x)} = \frac{P(x)}{Q_1(x)Q_2(x)}$ with Q_1, Q_2 relatively prime.

In this case, the gcd will be equal to 1 (a constant).

Consider x^3 and $x+1$, for instance.

$$x^3 = (x+1)\cdot(x^2-x+1) - 1$$

$x^3+1 = (x+1)(x^2-x+1)$

Then $1 = \underline{\underline{(x+1)}}(x^2-x+1) - \underline{\underline{x}}^3$. In the case of Q_1, Q_2 we get $P_1(x)\underline{\underline{Q_1(x)}} + P_2(x)\underline{\underline{Q_2(x)}} = 1$ for some polynomials $P_1(x), P_2(x)$ and then

$$\frac{P_1(x)\cancel{Q_1(x)}P(x)}{\cancel{Q_1(x)}Q_2(x)} + \frac{P_2(x)\cancel{Q_2(x)}P(x)}{Q_1(x)\cancel{Q_2(x)}} = \frac{P(x)}{Q_1(x)Q_2(x)}.$$

This gives the partial fractions decomposition $\frac{P_1 P}{Q_2} + \frac{P_2 P}{Q_1}$.

Example (Partial fractions) We compute $\int \frac{3x+1}{x^4-1}\,dx$.

The denominator factors as $x^4-1 = (x^2-1)(x^2+1) = (x+1)(x-1)(x^2+1)$ and we get

$$\frac{3x+1}{x^4-1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}.$$

Thus

$$3x+1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x^2+1)^{(x-1)} \text{ for all } x.$$

When
$x=1$, $4 = 4B$ and $B=1$
$x=-1$, $-2=-4A$ and $A = 1/2$
$x=0$, $1 = -A+B-D$ and $D = \cancel{B}-A-\cancel{1} = -1/2$
$x=2$, $7 = 5A+15B+3(2C+D)$ and $C=-3/2$.

Thus

$$\int \frac{3x+1}{x^4-1}\,dx = \int \frac{1/2}{x+1}\,dx + \int \frac{dx}{x-1} \cancel{+} -\frac{3}{2}\int \frac{x}{x^2+1}\,dx - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\ln|x+1| + \ln|x-1| - \frac{3}{4}\ln(x^2+1) - \frac{1}{2}\tan^{-1}x + C.$$

$u = x^2+1$

Integration examples

① $\int \frac{e^x\,dx}{1+e^x} = \int \frac{du}{u} = \ln(1+e^x)+C$ $\quad\boxed{u = 1+e^x}$

② $\int \frac{\cancel{e^x}\,dx}{\cancel{e^x}(1+e^x)} = \int \frac{du}{(u-1)u} = \int \frac{A}{u} + \frac{B}{u-1}\,du$

$= A\ln u + B\ln(u-1) = A\ln(1+e^x) + Bx.$

③ $\int e^{x^2}\,dx = \int \frac{1}{2\sqrt{u}}e^u\,du$ = not elementary = not a familiar function $\quad\boxed{u=x^2}$ $\boxed{\sqrt{u}=x}$

④ $\int 2xe^{x^2}\,dx = \int e^u\,du = e^{x^2}+C.$

(5) $\int 2x^3 e^{x^2}\,dx = \int u \underbrace{e^u}_{dv}\,du = ue^u - \int e^u\,du$

$= x^2 e^{x^2} - e^{x^2} + C.$

(6) $\int \frac{2x\,dx}{(1+x^2)^4} = \text{easy} = -\frac{1}{3}(1+x^2)^{-3} + C.$ $\boxed{u=1+x^2}$

(7) $\int \frac{2\,dx}{(1+x^2)^4} =$ hard ... reduction formula (tutorial problem)

(8) $\int \cos\sqrt{x}\,dx = \int 2u \underbrace{\cos u\,du}_{dv}$ $\boxed{\begin{array}{c} u=\sqrt{x} \\ \hline x=u^2 \end{array}}$

$= 2u\sin u - \int 2\sin u\,du$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$

(9) $\int \frac{x}{x^3-1}\,dx = \int \frac{x}{(x-1)(x^2+x+1)}\,dx$

$= \int \frac{A}{x-1} + \int \frac{Bx+C}{x^2+x+1}\,dx.$

We need to integrate $\frac{x}{x^2+x+1}$ and $\frac{1}{x^2+x+1}$.

The discriminant is negative, so we cannot factor.

Completing the square gives:

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\underbrace{x^2+x+\frac{1}{4}}+\frac{3}{4}} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \int \frac{\frac{4}{3}\,dx}{\frac{4}{3}(x+\frac{1}{2})^2+1} = \int \frac{\frac{4}{3}\,\frac{\sqrt{3}}{2}\,du}{u^2+1} \qquad u=\frac{2}{\sqrt{3}}(x+\tfrac{1}{2})$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x+\tfrac{1}{2})\right) + C.$$

⊙ To integrate $\int \frac{x \, dx}{x^2+x+1}$, we try $u = x^2+x+1$.

This gives $du = (2x+1)\,dx = 2(x+1/2)\,dx$ so

$$\int \frac{x+1/2-1/2}{x^2+x+1}\,dx = \int \frac{x+1/2}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{dx}{x^2+x+1}$$

$$= \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x+1/2)\right) + C.$$

$\boxed{\text{Sequences}}$ A sequence is a function defined on the set $\mathbb{N}$ of positive integers. This is determined by $f(1), f(2), \dots$ and it is usually denoted by $a_1, a_2, \dots$

$\boxed{\text{Convergent}}$ A sequence is convergent, if $\lim_{n\to\infty} a_n$ exists. For instance, $a_n = 1/n$ is convergent with $\lim_{n\to\infty} a_n = 0$, $a_n = (1+1/n)^n$ is convergent with $\lim_{n\to\infty} (1+1/n)^n = e$, while $a_n = (-1)^n$ is divergent with $a_n = \pm 1$.

- We also define factorials $n! = 1 \cdot 2 \cdot \dots \cdot n$.

$\boxed{\text{Monotonicity}}$ The sequence $\{a_n\}$ is monotonic, if it is increasing with $a_n \leq a_{n+1}$ for all n, or decreasing with $a_n \geq a_{n+1}$ for all n.

<u>Example</u>. Consider $a_n = \frac{n}{n+1}$. Then $a_1 = 1/2$, $a_2 = 2/3$, $a_3 = 3/4, \dots$ seems increasing.

We can check ① $f(x) = \frac{x}{x+1} \Rightarrow f'(x) = \frac{\cancel{x}+1-\cancel{x}}{(x+1)^2} > 0 \Rightarrow f(x)$ increases.

② $a_n \leq a_{n+1} \Leftrightarrow \frac{n}{n+1} \leq \frac{n+1}{n+2} \Leftrightarrow n^2+2n \leq n^2+2n+1$ ✓

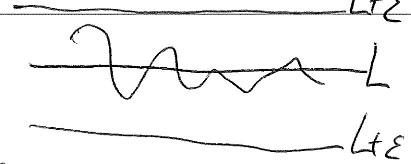
<u>Example</u>. Consider $a_n = \frac{2^n}{n!}$. Then $a_1 = 2$, $a_2 = \frac{2^2}{2!} = 2$, $a_3 = \frac{8}{6} = 4/3$ seems decreasing. In fact, $a_n \geq a_{n+1} \Leftrightarrow \frac{2^n}{n!} \geq \frac{2^{n+1}}{(n+1)!} \Leftrightarrow \frac{2^n}{2^{n+1}} \geq \frac{n!}{(n+1)!}$

$$\Leftrightarrow \frac{1}{2} \geq \frac{1}{n+1} \Leftrightarrow n+1 \geq 2 \Leftrightarrow n \geq 1.$$

Thus, the given sequence is monotonic.

$\boxed{\text{Bounded}}$ We say that a sequence $\{a_n\}$ is bounded, if there is a number $M > 0$ such that $|a_n| \leq M$ for all n.

<u>Theorem 1. If a sequence is convergent (with a finite limit), then it is bounded.</u>



<u>Proof</u>. One has $L = \lim_{n\to\infty} a_n$. Thus, given any $\varepsilon > 0$ we get $L - \varepsilon \leq a_n \leq L + \varepsilon$ for large n,

Say for $n \geq N$. Pick $\varepsilon = 1$ for simplicity. Then

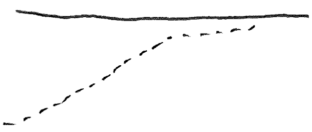
$$-|L|-1 \leq L-1 \leq a_n \leq L+1 \leq |L|+1 \quad \text{for any } n \geq N$$

$$|a_n| \leq |L|+1 \quad \text{for any } n \geq N.$$

We also have $\quad |a_n| \leq \max\{|a_1|, |a_2|, \dots, |a_N|\}$ for any $n \leq N$.

Then $\quad |a_n| \leq |L|+1 + \max\{\dots\}$ in any case. $\boxed{}$

Theorem 2. If a sequence is monotonic and bounded, then it attains a limit as $n \to \infty$, so it converges.



Example. Define $a_1 = 2$, $a_2 = \sqrt{2}$, $a_3 = \sqrt{\sqrt{2}}$ and so on.
In other words, let $a_1 = 2$ and $a_{n+1} = \sqrt{a_n}$ for all $n \geq 1$.
We show that a_n is monotonic (decreasing) and bounded. In fact, we show that $0 \leq a_{n+1} \leq a_n \leq 2$ for all n.

- When $n=1$ .... we get $0 \leq a_2 \leq a_1 \leq 2$, namely $0 \leq \sqrt{2} \leq 2 \leq 2$ and that is fine.
- Assume that $0 \leq a_{n+1} \leq a_n \leq 2$ for some n. Taking square roots, we get $0 \leq \sqrt{a_{n+1}} \leq \sqrt{a_n} \leq \sqrt{2}$
  so $0 \leq a_{n+2} \leq a_{n+1} \leq \sqrt{2} \leq 2$
  and the statement holds for $n+1$ as well.
- This proves $\{a_n\}$ is monotonic & bounded $\Longrightarrow$ convergent.

Thus, $L = \lim_{n\to\infty} a_n$ exists. To find the limit,
we recall that $\quad a_{n+1} = \sqrt{a_n}$

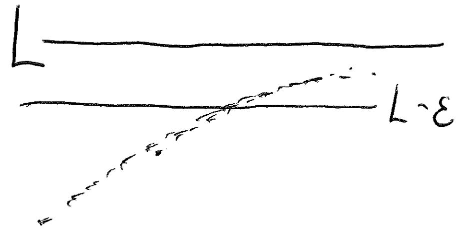
Then $\quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n}$

so $\quad L = \sqrt{L} \quad$ so $\quad L^2 = L$

and we get $L = 0$ or $L = 1$.

In fact $a_n = 2^{1/2^n}$ and so $\lim_{n\to\infty} a_n = 2^0 = 1$.

__Proof of Theorem 2.__ Suppose $\{a_n\}$ is increasing and bounded. We define L to be the "smallest possible" bound, the supremum. Then the limit is L for the following reason.



Consider $L-\varepsilon$ with $\varepsilon > 0$ arbitrarily small.

Then $L - \varepsilon \leq a_n \leq L$ for large enough n

This means that the terms a_n belong to $[L-\varepsilon, L]$ for large enough n. Since $\varepsilon > 0$ is arbitrary, we get $\lim a_n = L$. ▨

$\boxed{\text{Series}}$ Given a sequence $\{a_n\}$ of terms $a_1, a_2, a_3, \ldots$ we define S_n = the nth partial sum by $S_n = a_1 + a_2 + \cdots + a_n$.

If this partial sum converges (if S_n has a limit), then we may introduce the infinite sum (the infinite series) by

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N .$$

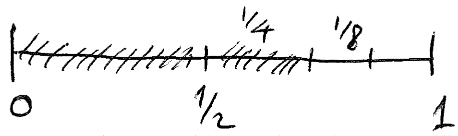
● We say the __series__ $\sum_{n=1}^{\infty} a_n$ converges, if the __sequence__ S_n converges (if S_n has a limit as $n\to\infty$).

__Example.__ Let $a_n = (-1)^n$ so that $a_n = -1$ if n odd, $a_n = +1$ if n even.

Then
$$S_1 = a_1 = -1$$
$$S_2 = a_1 + a_2 = -1 + 1 = 0$$
$$S_3 = -1 + 1 - 1 = -1$$
and so on. These terms do not have a limit as $n\to\infty$, so S_n diverges and $\sum a_n = \sum (-1)^n$ diverges.

__Example.__ Let $a_n = \frac{1}{2^n}$ so that $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{8}$ and so on.

Then $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$



$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ and so on.

These converge and we get $S_n = \frac{2^n - 1}{2^n} \longrightarrow 1$, so $\sum a_n$ converges.

Theorem (Nth term test) If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.
In other words, if $\lim_{n\to\infty} a_n$ is not zero, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. $\sum a_n$ converges means that $\lim_{n\to\infty} s_n$ exists. Let $L = \lim_{n\to\infty} s_n$.
Since $a_n = s_n - s_{n-1}$, we get $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0$. ☐

Examples $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges because $\lim_{n\to\infty} \frac{n}{n+1} = 1$

$\sum_{n=1}^{\infty} (1+1/n)^n$ diverges because $\lim_{n\to\infty} (1+1/n)^n = e$

If $\lim a_n = 0$, however, then $\sum a_n$ may converge or diverge.

Geometric series

Consider the series $\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \dots$ for some given x.
To check convergence, consider $s_n = x + x^2 + \dots + x^n$. We need s_n to have a limit. In this case $xs_n = x^2 + x^3 + \dots + x^{n+1}$ is a shifted version of the sum, so

$$x s_n - s_n = x^{n+1} - x$$
$$(x-1) s_n = x^{n+1} - x$$

and $s_n = \frac{x^{n+1} - x}{x-1}$, if $x \neq 1$.

Theorem (Convergence of $\sum x^n$). If $|x| < 1$, then $\sum_{n=1}^{\infty} x^n$ converges.
If $|x| \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges.

Proof. If $|x| < 1$, then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{x^{n+1} - x}{x-1} = \frac{0-x}{x-1} = \frac{x}{1-x}$

This gives $\sum_{n=1}^{\infty} x^n = \lim_{n\to\infty} s_n = \frac{x}{1-x}$, if $|x| < 1$.

If $x = 1$, then $a_n = x^n = 1$, so $\sum a_n$ diverges by above.
If $x = -1$, then $a_n = (-1)^n$, so a_n does not go to 0 and we get divergence. If $x > 1$ or $x < -1$, we get a similar issue. ☐

Example. $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ is divergent and $\sum_{n=1}^{\infty} \frac{2^{n+2}}{3^{2n+1}}$ is convergent.

In fact,
$$\sum_{n=1}^{\infty} \frac{2^{n+2}}{3^{2n+1}} = \sum_{n=1}^{\infty} \frac{2^n \cdot 4}{9^n \cdot 3} = \frac{4}{3} \sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^n$$
$$= \frac{4}{3} \frac{x}{1-x} = \frac{4}{3} \frac{2/9}{7/9} = \frac{8}{21}.$$

$$\boxed{\textbf{Example.}\ \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \quad \text{and} \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \ \text{if } |x| < 1.}$$

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = x^{-1} \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}.$$

$\boxed{\text{Non-negative sequences}}$ If $\{a_n\}$ is non-negative, then $S_n = a_1 + \cdots + a_n$ is increasing. If S_n is bounded, then S_n converges.

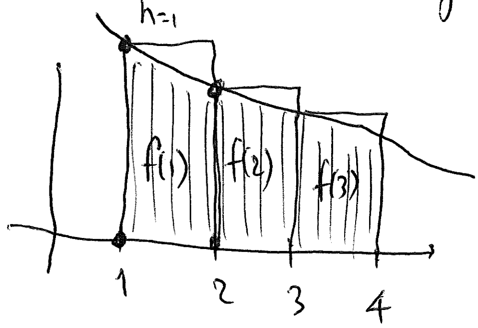
If S_n is convergent, then S_n is bounded by Theorem 1.

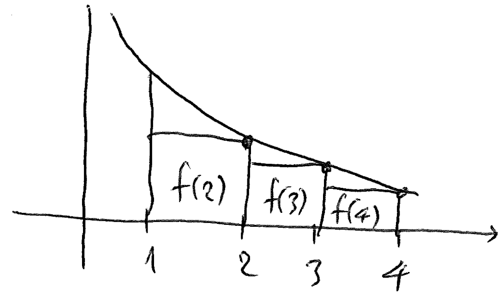
Convergence of non-negative series

- Recall that $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$ converges, if the partial sums $S_N = \sum_{n=1}^{N} a_n$ have a limit as $N \to \infty$.
- For non-negative series, S_N is increasing, hence

$$S_N \text{ is bounded} \iff S_N \text{ is convergent}$$

Integral test Suppose $f(x)$ is continuous, decreasing and non-negative on $[1, \infty)$.
Then $\sum_{n=1}^{\infty} f(n)$ converges $\iff$ $\int_1^{\infty} f(x)\,dx$ is bounded / finite.





This gives $f(2) + f(3) + f(4) \le \int_1^4 f(x)\,dx \le f(1) + f(2) + f(3)$

Proof. Since f is decreasing, we get

$$f(n) \geq f(x) \geq f(n+1) \qquad \text{for all} \qquad n \le x \le n+1.$$

~~Adding~~ Integrating these inequalities gives

$$f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1) \qquad \text{for all } n.$$

We can add these up to conclude that

$$\sum_{n=1}^{N-1} f(n) \geq \int_1^N f(x)\,dx \geq \sum_{n=1}^{N-1} f(n+1).$$

If the series converges, the leftmost sum is finite $\Rightarrow \int_1^N f(x)\,dx$ is finite as well. If the integral is finite, then the rightmost sum is finite and so the series converges. $\blacksquare$

Geometric series $\sum_{n=1}^{\infty} x^n$ converges $\iff$ $|x| < 1$.

$\boxed{\text{p-series}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff$ $p>1$.

We use the integral test to check convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Take $f(n) = \frac{1}{n^p}$ or $f(x) = \frac{1}{x^p} = x^{-p}$ on $[1,\infty)$.

This is continuous, positive, while $f'(x) = -p x^{-p-1} \leq 0$ when $p \geq 0$.

By the integral test, we need to ensure $\int_1^{\infty} f(x)\,dx$ is finite.

Write $\displaystyle\int_1^{\infty} f(x)\,dx = \lim_{L\to\infty} \int_1^L x^{-p}\,dx = \lim_{L\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^L$ if $p \neq 1$

$$= \lim_{L\to\infty} \frac{L^{1-p}-1}{1-p}.$$

<u>If $p>1$</u>, the exponent $1-p$ is negative, so $L^{1-p}\to 0$ and the integral is finite. <u>If $p<1$</u>, then $L^{1-p}\to\infty$ and the series diverges.

<u>If $p=1$</u>, then $\displaystyle\int_1^{\infty} f(x)\,dx = \lim_{L\to\infty}\int_1^L \frac{1}{x}\,dx = \ln L - \ln 1 \to \infty$.

Finally, when $p<0$, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges since $\frac{1}{n^p}\to\infty$.

$\boxed{\text{Conclusion: } \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^p} \text{ converges} \iff p>1.}$

Comparison test Suppose $\{a_n\}, \{b_n\}$ are <u>non-negative</u> sequences.

1. If $\sum b_n$ converges and $a_n \leq b_n$ for large n, then $\sum a_n$ converges. Smaller than convergent is convergent.
2. If $\sum a_n$ diverges and $a_n \leq b_n$ for large n, then $\sum b_n$ diverges. Larger than divergent is divergent.

<u>Example 1.</u> ~~Let~~ Consider $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+2^n}$.

One can argue that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+2^n} \leq \sum_{n=1}^{\infty}\frac{1}{2^n} = \underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n}_{\text{geometric series that converges}}$

Thus, $\sum \frac{1}{n+2^n}$ is smaller than convergent $\Rightarrow$ convergent.

Example 2. Consider $\sum_{n=1}^{\infty} \frac{1}{n^3+2n}$.

If we say $\sum_{n=1}^{\infty} \frac{1}{n^3+2n} \leq \sum_{n=1}^{\infty} \frac{1}{2n} = \underbrace{\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}}_{\text{divergent}}$, then we get no conclusions.

If we say $\sum_{n=1}^{\infty} \frac{1}{n^3+2n} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^3}}_{\text{convergent } (p=3)}$, then we get convergence.

Example 3. Consider $\sum_{n=1}^{\infty} \frac{1}{n+2}$.

We compare this with $\sum 1/n$, a divergent p-series.

Unfortunately $\sum \frac{1}{n+2} \geq \sum \frac{1}{n}$ is not true.

However $\sum_{n=1}^{\infty} \frac{1}{n+2} \geq \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$ is true because $3n \geq n+2$
$2n \geq 2$.

This implies $\sum_{n=1}^{\infty} \frac{1}{n+2}$ diverges.

Tests of convergence for non-negative series

① **Integral test** .... $\sum_{n=1}^{\infty} f(n)$ converges $\Leftrightarrow$ $\int_1^{\infty} f(x)\,dx$ is finite.

This implies that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Leftrightarrow$ $p>1$.

② **Comparison test**: Smaller than convergent is convergent.
Bigger than divergent is divergent.

③ **Limit comparison test**: Suppose $\{a_n\}, \{b_n\}$ are non-negative with $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Then $\sum_{n=1}^{\infty} a_n$ converges $\Longleftrightarrow$ $\sum_{n=1}^{\infty} b_n$ converges.

Example 1. Consider $\sum_{n=1}^{\infty} \frac{3n^2+5n+2}{\sqrt{n^6+2n+1}}$. Call this $\sum a_n$.

We compare this with $\sum b_n$, $b_n = \frac{3n^2}{\sqrt{n^6}} = \frac{3n^2}{n^3} = \frac{3}{n}$.

We check $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^2+5n+2}{3n^2} \frac{\sqrt{n^6}}{\sqrt{n^6+2n+1}} = 1$.

Then $\sum a_n$ converges $\Leftrightarrow$ $\sum b_n$ converges. But $\sum b_n = \sum \frac{3}{n} = 3\sum_{n=1} \frac{1}{n}$ diverges (p-series with $p=1$). So $\sum a_n$ diverges as well.

Example 2. Consider $\sum_{n=1}^{\infty} \frac{5n^4+5n^2+6}{n^6+3n+4} = \sum a_n$.

We compare with $\sum \frac{5n^4}{n^6} = \sum \frac{5}{n^2} = \sum b_n$, convergent.

As before, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n^4+5n^2+6}{5n^4} \frac{n^6}{n^6+3n+4} = 1$.

Thus $\sum a_n$ converges as well.

Proof of limit comparison. By assumption $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Then $\frac{1}{2} \le \frac{a_n}{b_n} \le \frac{3}{2}$ for large enough n, say for $n \geq N$.

This gives $\frac{1}{2} b_n \le a_n \le \frac{3}{2} b_n$ for $n \geq N$. By comparison,

$\sum_{n=N}^{\infty} b_n$ convergent $\Rightarrow$ $\sum_{n=N}^{\infty} a_n \le \frac{3}{2}\sum_{n=N}^{\infty} b_n$ convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent. ▨

④ **Ratio test** Suppose $\{a_n\}$ is non-negative and let

$$\boxed{L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}}.$$

If $L<1$, $\sum a_n$ converges.
If $L>1$, $\sum a_n$ diverges.
If $L=1$, we get <u>no</u> conclusions.

Example. Consider $\sum \frac{n \cdot 2^n}{n^2+1}$ so that $a_n = \frac{n \cdot 2^n}{n^2+1}$.

Then $$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} \frac{2^{n+1}}{2^n} \frac{n^2+1}{(n+1)^2+1}$$

$$= \underbrace{\lim_{n\to\infty} \frac{n+1}{n}}_{=1} \cdot \underbrace{\lim \frac{2^{n+1}}{2^n}}_{\text{equal to } 2} \cdot \underbrace{\lim \frac{n^2+1}{(n+1)^2+1}}_{=1}$$

So $L=2$ and $\sum a_n$ diverges.

Example. Consider $\sum_{n=1}^{\infty} \frac{(3n+2)\cdot 5^n}{n!} = \sum_{n=1}^{\infty} a_n$.

We use the ratio test. In this case,

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3(n+1)+2}{3n+2} \cdot \frac{5^{n+1}}{5^n} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{3n\cancel{+5}}{3n\cancel{+2}} \cdot \lim_{n\to\infty} \cancel{\frac{5^{n+1}}{5^n}}\ 5 \cdot \lim_{n\to\infty} \frac{1}{n+1}$$

$$= 1 \cdot 5 \cdot 0 = 0.$$

Since $L<1$, we get convergence.

Remark: The ratio test helps when a_n involves factorials / exponents.
The limit comparison test helps when a_n involves polynomials.

<u>Proof of ratio test.</u> We look at $\sum a_n$ and let $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Roughly speaking, $a_{n+1} \simeq L \cdot a_n$ for large enough n, say $n \geq N$.

Then

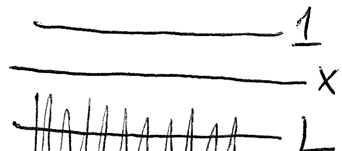
$$a_{N+1} \simeq L a_N$$
$$a_{N+2} \simeq L a_{N+1} \simeq L^2 a_N$$
$$a_{N+3} \simeq L a_{N+2} \simeq L^3 a_N$$

so

$$a_{N+1} + a_{N+2} + a_{N+3} + \dots \simeq L a_N + L^2 a_N + L^3 a_N + \dots$$
$$= (L + L^2 + L^3 + \dots)\, a_N,$$

a geometric series.

For a formal proof, suppose $L < 1$ and let $X = \frac{1+L}{2}$ so that $L < X < 1$.



Since $\frac{a_{n+1}}{a_n} \longrightarrow L$, we get $\frac{a_{n+1}}{a_n} < X$ for large n, say $n \geq N$. Then

$$a_{N+1} < X a_N$$
$$a_{N+2} < X a_{N+1} < X^2 a_N$$
$$a_{N+3} < X a_{N+2} < X^3 a_N$$

and so

$$a_{N+1} + a_{N+2} + a_{N+3} + \dots < (X + X^2 + X^3 + \dots)\, a_N$$
$$= \frac{X}{1-X} a_N, \text{ convergent.}$$

By comparison $\sum_{n=N+1}^{\infty} a_n$ converges, so $\sum_{n=1}^{\infty} a_n$ converges. $\blacksquare$

Arbitrary series and absolute convergence

We say that the series $\sum a_n$ converges <u>absolutely</u>, if ~~the~~ it converges when absolute values are introduced, namely if $\sum |a_n|$ converges.

<u>Theorem</u> (Absolute convergence implies convergence).

If $\sum |a_n|$ converges, then $\sum a_n$ converges.

<u>Proof</u>. Every number is the difference of two non-negative numbers. More precisely, let

$$b_n = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0 \end{cases} \quad \text{and} \quad c_n = \begin{cases} 0, & \text{if } a_n \geq 0 \\ -a_n, & \text{if } a_n < 0 \end{cases}.$$

Then
1. b_n, c_n are non-negative
2. $b_n - c_n = a_n$
3. $b_n + c_n = \begin{cases} a_n & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0 \end{cases} = |a_n|$.

We can thus concentrate on the series $\sum b_n, \sum c_n, \sum |a_n|$ .... non-neg.

Since $\sum b_n \leq \sum b_n + \sum c_n = \sum |a_n|$, we get $\sum b_n$ conv. by comparison.

Since $\sum c_n \leq \sum b_n + \sum c_n = \sum |a_n|$, we get $\sum c_n$ convergent.

Then $\sum a_n = \sum b_n - \sum c_n$ is convergent as well. $\square$

<u>Example 1</u>. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin n}{n^3+1}$.

We look at $\displaystyle\sum_{n=1}^{\infty} \left| \frac{n \sin n}{n^3+1} \right| \leq \sum_{n=1}^{\infty} \frac{n}{n^3+1} \leq \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

The series on the right is a p-series with $p=2$, so it converges, so $\displaystyle\sum \left| \frac{n \sin n}{n^3+1} \right|$ converges by comparison

and $\displaystyle\sum \frac{n \sin n}{n^3+1}$ converges by the theorem.

Example 2. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{2^n}$.

We look at $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n (n+1)}{2^n} \right| = \sum_{n=1}^{\infty} \frac{n+1}{2^n}$.

We use the ratio test and compute

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+2}{n+1} \, \frac{2^n}{2^{n+1}}$$

$$= \lim_{n\to\infty} \overset{\to 1}{\frac{n+2}{n+1}} \cdot \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}.$$

Since $L<1$, the series $\sum \frac{n+1}{2^n}$ converges and $\sum \frac{(-1)^n (n+1)}{2^n}$ converges.

Ratio test for arbitrary series. Suppose $\{a_n\}$ is arbitrary and let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

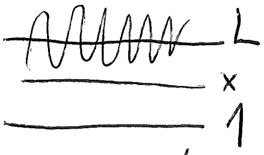
If $L<1$, then $\sum a_n$ conv.
If $L>1$, then $\sum a_n$ diverges.

Proof. We look at $\sum |a_n|$ and apply the old ratio test for

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

If $L<1$, then $\sum |a_n|$ converges by the old ratio test $\Rightarrow \sum a_n$ converges as well.

When $L>1$, we let $x = \frac{1+L}{2}$ be the average.



Then $\left|\frac{a_{n+1}}{a_n}\right| \to L$, so $\left|\frac{a_{n+1}}{a_n}\right| > x$ for large n, say $n \geq N$.

Then $|a_{N+1}| > x\,|a_N|$, $|a_{N+2}| > x^2 |a_N|$ and so on.

This gives $|a_{N+i}| > |a_N|$ for all $i \geq 1$, so the nth term does not approach zero and the series diverges. $\square$

Example 3. Consider $\displaystyle\sum \frac{(-1)^n \cdot 8^n \, n!}{n^n}$.

We use the ratio test. Let us compute

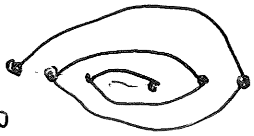
$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \, \frac{8^{n+1}}{8^n} \, \frac{(n+1)!}{n!} \, \frac{n^n}{(n+1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{8\cancel{(n+1)}\, n^n}{(n+1)^{n\cancel{+1}}} = \lim_{n\to\infty} \frac{8 n^n}{(n+1)^n}.$$

Note that $\lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^n\right]^{-1} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = 1/e.$

We end up with $L = 8/e > 1$, so the series diverges.

$\boxed{\text{Alternating series test}}$ Suppose $\{a_n\}$ is non-neg.



and decreasing to zero. Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges. [This only applies for special alternating series.]

Example. Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$.

We need to check $a_n = \frac{n}{n^2+1}$ is decreasing to zero.

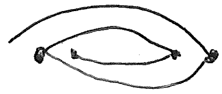
In fact, $f(x) = \frac{x}{x^2+1} \Longrightarrow f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$

for all $x \geq 1$. And $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2} = 0.$

$\boxed{\text{Alternating series test}}$ Suppose $\{a_n\}$ is non-negative and decreasing to zero. Then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent.

For instance, $\sum \frac{(-1)^{n-1}}{n}$ and $\sum \frac{(-1)^{n-1}}{n^2}$ are convergent.

Proof. We look at the partial sums S_n.



The even partial sums $S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2n-1} - a_{2n})$ are increasing because $S_{2n+2} = S_{2n} + a_{2n+1} - a_{2n+2} \geq S_{2n}$.

The odd partial sums S_{2n+1} are similarly decreasing. Thus, there exists a limit for the even sums $L_1 = \lim_{n\to\infty} S_{2n}$ and a limit for the odd sums $L_2 = \lim_{n\to\infty} S_{2n+1}$. However,

$$L_2 - L_1 = \lim_{n\to\infty} S_{2n+1} - S_{2n} = \lim_{n\to\infty} a_{2n+1} = 0 \text{ by assumption,}$$

so the two limits coincide and $S_n \to L_1 = L_2$ for all n.

$\boxed{\text{Power series}}$ One can define functions $f(x)$ as infinite sums. A power series has the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$, namely $f(x) = a_0 + a_1 x + a_2 x^2 + \dots$ for some coefficients a_n. This expression is defined as long as the series converges.

Example 1. Consider $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$.

The ratio test gives $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x^{n+1}|}{|x|^n} = |x|$

Thus, $f(x)$ converges when $|x| < 1$ and diverges when $|x| > 1$.

Example 2. Consider $f(x) = \sum_{n=0}^{\infty} \frac{n x^n}{3^n}$.

The ratio test gives $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \frac{x^{n+1}}{x^n} \frac{3^n}{3^{n+1}} \right|$

$= \lim_{n\to\infty} \underbrace{\frac{n+1}{n}}_{\to 1} |x| \cdot \frac{1}{3} = \frac{|x|}{3}$. We get conv. if $L < 1$, if $|x| < 3$;

div. if $L > 1$, if $|x| > 3$.

Example 3. Consider $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. In this case,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for any given x. Thus $L<1$ for any x
and the series converges for all x.

Definition (Radius of convergence) We say that $\sum a_n x^n$ has radius of convergence R when the series converges if $|x|<R$
and diverges if $|x|>R$.

This can be determined using the ratio test since

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| .$$

One gets convergence when $L<1$ or $|x| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$
and divergence when $L>1$ or $|x| > \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$.

The radius of convergence might be infinite, so $R=\infty$ is allowed.

Theorem. (Differentiation of power series) Consider a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R.
Then $g(x) = \sum_{n=0}^{\infty} a_n n x^{n-1}$ has the same radius of convergence
and $f'(x) = g(x)$ for all $|x|<R$

It is not true that $\left[\sum_{n=0}^{\infty} f_n(x) \right]' = \sum_{n=0}^{\infty} f_n'(x)$, in general.
This is true for (a) finite sums and (b) power series $\sum a_n x^n$.

Example 4. Consider $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
as in Example 3 above. This converges for all x. Its radius of convergence is $R=\infty$. We can thus differentiate term by term.

We get $f'(x) = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = f(x).$$

This is the exponential function $f(x) = e^x$ defined for all x.

One could also write

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f'(x) = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$\Rightarrow f'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ by shifting the index.

Taylor polynomials and Taylor series

- Suppose $f(x)$ is a function that is differentiable an infinite number of times at $x=0$. The linear approximation at that point is $L(x) = f(0) + f'(0)(x-0) = f(0) + f'(0)\,x$. This satisfies:

$$L(0) = f(0)$$

and $L'(x) = f'(0)$, so $L'(0) = f'(0)$. Thus, L and f have the same value and the same derivative at $x=0$.

- Consider a polynomial $\underline{T_N(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_N x^N}$ with unknown coefficients $a_0, a_1, \dots, a_N$. We want to make sure that T_N and f agree at $x=0$ up until the N<u>th</u> derivative.

This gives ---- ⊙ $f(0) = T_N(0)$ so $\boxed{f(0) = a_0}$

⊙ $T_N'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + N a_N x^{N-1}$, so $\boxed{a_1 = T_N'(0) = f'(0)}$

⊙ $T_N''(x) = 2a_2 + 6a_3 x + \dots + N(N-1)a_N x^{N-2}$, so $\boxed{2a_2 = T_N''(0) = f''(0)}$

⊙ $T_N'''(x) = 3 \cdot 2 \cdot 1 a_3 +$ higher order terms, so $\boxed{3!\,a_3 = T_N'''(0) = f'''(0)}$

More generally, we get $n!\,a_n = f^{(n)}(0) =$ nth derivative at 0

with $f^{(0)} = f$, $f^{(1)} = f'$ etc.

Definition (Taylor polynomial) We define the Nth Taylor polynomial of a function f at the point $x=0$ by the formula

$$T_N(x) = \sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} \cdot x^n .$$

The corresponding formula at the point $x = x_0$ is

$$T_N(x) = \sum_{n=0}^{N} a_n (x-x_0)^n = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n .$$

Example 1 (Exponential) Consider $f(x) = e^x$. In this case $f(x) = f'(x) = f''(x) = \dots = f^{(n)}(x)$ for all x so $f^{(n)}(0) = f(0) = 1$. The Taylor polynomial is $T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^N}{N!}$.

Example 2 (Sine function) Consider $f(x) = \sin x$. We need $f^{(n)}(0)$. In this case, $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$ and $f^{(4)}(x) = \sin x = f^{(0)}(x)$. Thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$, $f^{(4)}(0) = 0$ and the remaining ones arepeat this pattern. The Taylor polynomials have the form

$$T_N(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots + \frac{f^{(N)}(0)}{N!} x^N$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$$

So they only contain odd powers of x.

Example 3 (Cosine function) Consider $f(x) = \cos x$. Using the same approach, we get $T_N(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$.

$\boxed{\text{Taylor series}}$ We define the Taylor series of f to be

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n ,$$ provided that the series converges.

This series can be differentiated term by term when $|x| < R$.

One can show that $T(x) = f(x)$ for the standard functions that we had above. Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Assuming these formulas for <u>definitions</u> of $e^x, \sin x, \cos x$:

$$(e^x)' = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \qquad (*)$$

So $(e^x e^{-x})' = (e^x)e^{-x} - e^x(e^{-x}) = 0$ by $(*)$

So $e^x e^{-x} = \text{constant} = e^0 e^{-0} = \underline{1}$ and $e^{-x} = 1/e^x$.

Similarly, $$(\sin x)' = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x$$

and $$(\cos x)' = \sum_{n=1}^{\infty} \frac{(-1)^n 2n\, x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = -\sin x.$$

Then $(\sin^2 x + \cos^2 x)' = 2\sin x \cos x + 2\cos x(-\sin x) = 0$

so $\sin^2 x + \cos^2 x = \text{constant} = \sin^2 0 + \cos^2 0 = 0 + \underline{1}$.