MA1125 – Calculus
Tutorial solutions #3

1. Show that there exists a real number $0 < x < \pi/2$ that satisfies the equation

$$x^3 \cos x + x^2 \sin x = 2.$$

Consider the function f which is defined by $f(x) = x^3 \cos x + x^2 \sin x - 2$. Being the sum of continuous functions, f is then continuous and one can easily check that

$$f(0) = -2 < 0, \qquad f(\pi/2) = \frac{\pi^2}{4} - 2 = \frac{\pi^2 - 8}{4} > 0.$$

In view of Bolzano's theorem, this already implies that f has a root $0 < x < \pi/2$.

2. For which values of a, b is the function f continuous at the point $x = 3$? Explain.

$$f(x) = \left\{ \begin{array}{cl} 2x^2 + ax + b & \text{if } x < 3 \\ 2a + b + 1 & \text{if } x = 3 \\ 5x^2 - bx + 2a & \text{if } x > 3 \end{array} \right\}.$$

Since f is a polynomial on the intervals $(-\infty, 3)$ and $(3, +\infty)$, it should be clear that

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (2x^2 + ax + b) = 3a + b + 18,$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (5x^2 - bx + 2a) = 2a - 3b + 45.$$

In particular, the function f is continuous at the given point if and only if

$$3a + b + 18 = 2a - 3b + 45 = 2a + b + 1.$$

Solving this system of equations, one obtains a unique solution which is given by

$$45 - 3b = b + 1 \quad\Longrightarrow\quad 4b = 44 \quad\Longrightarrow\quad b = 11 \quad\Longrightarrow\quad a = 27 - 4b = -17.$$

In other words, f is continuous at the given point if and only if $a = -17$ and $b = 11$.

3. Show that $f(x) = x^3 - 3x^2 + 1$ has three roots in the interval $(-1, 3)$. Hint: you need only consider the values that are attained by f at the integers $-1 \leq x \leq 3$.

Being a polynomial, the given function is continuous and one can easily check that

$$f(-1) = -3, \qquad f(0) = 1, \qquad f(1) = -1, \qquad f(2) = -3, \qquad f(3) = 1.$$

Since the values $f(-1)$ and $f(0)$ have opposite signs, f has a root that lies in $(-1, 0)$. The same argument yields a second root in $(0, 1)$ and also a third root in $(2, 3)$.

4. Compute each of the following limits.

$$L = \lim_{x\to+\infty} \frac{2x^4 - 7x + 3}{3x^4 - 5x^2 + 1}, \qquad M = \lim_{x\to 2^-} \frac{2x^2 + 3x - 4}{3x^3 - 7x^2 + 4x - 4}.$$

Since the first limit involves infinite values of x, it should be clear that

$$L = \lim_{x\to+\infty} \frac{2x^4 - 7x + 3}{3x^4 - 5x^2 + 1} = \lim_{x\to+\infty} \frac{2x^4}{3x^4} = \frac{2}{3}.$$

For the second limit, the denominator becomes zero when $x = 2$, while the numerator is nonzero at that point. Thus, one needs to factor the denominator and this gives

$$M = \lim_{x\to 2^-} \frac{2x^2 + 3x - 4}{(x-2)(3x^2 - x + 2)} = \lim_{x\to 2^-} \frac{10}{12(x-2)} = -\infty.$$

5. Use the definition of the derivative to compute $f'(x_0)$ in each of the following cases.

$$f(x) = 3x^2, \qquad f(x) = 2/x, \qquad f(x) = (2x+3)^2.$$

The derivative of the first function is given by the limit

$$f'(x_0) = \lim_{x\to x_0} \frac{3x^2 - 3x_0^2}{x - x_0} = \lim_{x\to x_0} \frac{3(x - x_0)(x + x_0)}{x - x_0} = \lim_{x\to x_0} 3(x + x_0) = 6x_0.$$

To compute the derivative of the second function, we begin by writing

$$f(x) - f(x_0) = \frac{2}{x} - \frac{2}{x_0} = \frac{2(x_0 - x)}{xx_0}.$$

Once we now divide this expression by $x - x_0$, we may also conclude that

$$f'(x_0) = \lim_{x\to x_0} \frac{2(x_0 - x)}{(x - x_0)xx_0} = \lim_{x\to x_0} \frac{-2}{xx_0} = -\frac{2}{x_0^2}.$$

Finally, the derivative of the third function is given by the limit

$$f'(x_0) = \lim_{x\to x_0} \frac{(2x+3)^2 - (2x_0+3)^2}{x - x_0} = \lim_{x\to x_0} \frac{(2x + 2x_0 + 6)(2x - 2x_0)}{x - x_0} = 4(2x_0 + 3).$$

6. Show that there exists a real number $0 < x < \pi/2$ that satisfies the equation

$$x^2 + x - 1 = \sin x.$$

Consider the function f which is defined by $f(x) = x^2 + x - 1 - \sin x$. Being the sum of continuous functions, f is then continuous and one can easily check that

$$f(0) = -1 < 0, \qquad f(\pi/2) = \frac{\pi^2}{4} + \frac{\pi}{2} - 2 > \frac{\pi^2 - 8}{4} > 0.$$

In view of Bolzano's theorem, this already implies that f has a root $0 < x < \pi/2$.

7. Show that $f(x) = 3x^3 - 5x + 1$ has three roots in the interval $(-2, 2)$. Hint: you need only consider the values that are attained by f at the integers $-2 \leq x \leq 2$.

Being a polynomial, the given function is continuous and one can easily check that

$$f(-2) = -13, \qquad f(-1) = 3, \qquad f(0) = 1, \qquad f(1) = -1, \qquad f(2) = 15.$$

Since the values $f(-2)$ and $f(-1)$ have opposite signs, f has a root that lies in $(-2, -1)$. The same argument yields a second root in $(0, 1)$ and also a third root in $(1, 2)$.

8. Compute each of the following limits.

$$L = \lim_{x\to-\infty} \frac{6x^3 - 5x^2 + 7}{5x^4 - 3x + 1}, \qquad M = \lim_{x\to2^+} \frac{x^3 + x^2 - 5x - 2}{x^3 - 5x^2 + 8x - 4}.$$

Since the first limit involves infinite values of x, it should be clear that

$$L = \lim_{x\to-\infty} \frac{6x^3 - 5x^2 + 7}{5x^4 - 3x + 1} = \lim_{x\to-\infty} \frac{6x^3}{5x^4} = \lim_{x\to-\infty} \frac{6}{5x} = 0.$$

For the second limit, both the numerator and the denominator become zero when $x = 2$, so one needs to factor each of these expressions. Using division of polynomials, we get

$$M = \lim_{x\to2^+} \frac{(x-2)(x^2 + 3x + 1)}{(x-2)^2(x-1)} = \lim_{x\to2^+} \frac{11}{x-2} = +\infty.$$

9. Use the Squeeze Theorem to show that $\lim_{x\to0} x^2 \sin(1/x) = 0$.

Since $-1 \leq \sin x \leq 1$ for all x, one has $-1 \leq \sin(1/x) \leq 1$ for all $x \neq 0$ and

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2.$$

On the other hand, both $-x^2$ and x^2 approach zero as $x \to 0$, so this also implies

$$\lim_{x\to0} x^2 \sin\frac{1}{x} = 0.$$

10. Suppose that f is continuous with $f(0) < 1$. Show that there exists some $\delta > 0$ such that $f(x) < 1$ for all $-\delta < x < \delta$. Hint: use the ε-δ definition for some suitable ε.

Since $\varepsilon = 1 - f(0)$ is positive by assumption, there exists some $\delta > 0$ such that

$$|x - 0| < \delta \quad\Longrightarrow\quad |f(x) - f(0)| < \varepsilon \quad\Longrightarrow\quad |f(x) - f(0)| < 1 - f(0).$$

Rearranging terms to simplify this equation, one may thus conclude that

$$-\delta < x < \delta \quad\Longrightarrow\quad f(0) - 1 < f(x) - f(0) < 1 - f(0) \quad\Longrightarrow\quad f(x) < 1.$$