MA1125 – Calculus
Homework #3 solutions

1. Show that there exists a real number $0 < x < \pi$ that satisfies the equation
$$x^2 = \frac{x^2+1}{2+\sin x} + 4.$$

Consider the function f which is defined as the difference of the two sides, namely
$$f(x) = \frac{x^2+1}{2+\sin x} + 4 - x^2.$$
Being a composition of continuous functions, f is then continuous and we also have
$$f(0) = \frac{1}{2} + 4 > 0, \qquad f(\pi) = \frac{\pi^2+1}{2} + 4 - \pi^2 = \frac{9-\pi^2}{2} < 0.$$
In view of Bolzano's theorem, this already implies that f has a root $0 < x < \pi$.

2. For which values of a, b is the function f continuous at the point $x = 2$? Explain.
$$f(x) = \left\{ \begin{array}{cc} 2x^3 - ax^2 + bx & \text{if } x < 2 \\ a^2 + b & \text{if } x = 2 \\ 2x^2 + bx - a & \text{if } x > 2 \end{array} \right\}.$$

Since f is a polynomial on the intervals $(-\infty, 2)$ and $(2, +\infty)$, it should be clear that
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x^3 - ax^2 + bx) = 16 - 4a + 2b,$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (2x^2 + bx - a) = 8 + 2b - a.$$
In particular, the function f is continuous at the given point if and only if
$$16 - 4a + 2b = 8 + 2b - a = a^2 + b.$$
Solving this system of equations, one obtains a unique solution which is given by
$$16 - 4a = 8 - a \quad\Longrightarrow\quad 3a = 8 \quad\Longrightarrow\quad a = \frac{8}{3} \quad\Longrightarrow\quad b = a^2 + a - 8 = \frac{16}{9}.$$
In other words, f is continuous at the given point if and only if $a = 8/3$ and $b = 16/9$.

3. Show that $f(x) = x^5 - x^2 - 3x + 1$ has three roots in the interval $(-2, 2)$. Hint: you need only consider the values that are attained by f at the points ± 2, ± 1 and 0.

Being a polynomial, the given function is continuous and one can easily check that

$$f(-2) = -29, \qquad f(-1) = 2, \qquad f(0) = 1, \qquad f(1) = -2, \qquad f(2) = 23.$$

Since the values $f(-2)$ and $f(-1)$ have opposite signs, f has a root that lies in $(-2, -1)$. The same argument yields a second root in $(0, 1)$ and also a third root in $(1, 2)$.

4. Compute each of the following limits.

$$L = \lim_{x \to +\infty} \frac{3x^3 - 2x + 4}{5x^3 - x^2 + 7}, \qquad M = \lim_{x \to 2^-} \frac{x^3 + 5x^2 - 4}{3x^3 - 16x + 8}.$$

Since the first limit involves infinite values of x, it should be clear that

$$L = \lim_{x \to +\infty} \frac{3x^3 - 2x + 4}{5x^3 - x^2 + 7} = \lim_{x \to +\infty} \frac{3x^3}{5x^3} = \frac{3}{5}.$$

For the second limit, the denominator becomes zero when $x = 2$, while the numerator is nonzero at that point. Thus, one needs to factor the denominator and this gives

$$M = \lim_{x \to 2^-} \frac{x^3 + 5x^2 - 4}{(x-2)(3x^2 + 6x - 4)} = \lim_{x \to 2^-} \frac{24}{20(x-2)} = -\infty.$$

5. Use the definition of the derivative to compute $f'(x_0)$ in each of the following cases.

$$f(x) = x^3, \qquad f(x) = 1/x^2, \qquad f(x) = (3x+4)^2.$$

The derivative of the first function is given by the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{x^3 - x_0^3}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x^2 + xx_0 + x_0^2)}{x - x_0} = x_0^2 + x_0^2 + x_0^2 = 3x_0^2.$$

To compute the derivative of the second function, we begin by writing

$$f(x) - f(x_0) = \frac{1}{x^2} - \frac{1}{x_0^2} = \frac{(x_0 - x)(x_0 + x)}{x^2 x_0^2}.$$

Once we now divide this expression by $x - x_0$, we may also conclude that

$$f'(x_0) = \lim_{x \to x_0} \frac{-(x_0 + x)}{x^2 x_0^2} = -\frac{2x_0}{x_0^4} = -\frac{2}{x_0^3}.$$

Finally, the derivative of the third function is given by the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{(3x+4)^2 - (3x_0+4)^2}{x - x_0} = \lim_{x \to x_0} \frac{(3x + 3x_0 + 8)(3x - 3x_0)}{x - x_0} = 6(3x_0 + 4).$$