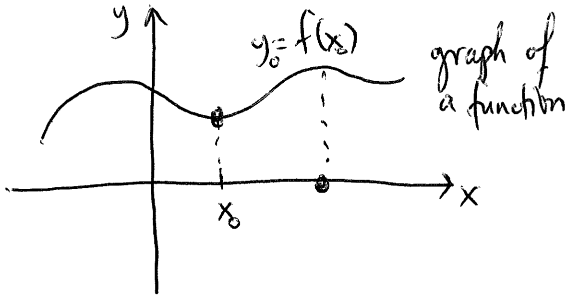
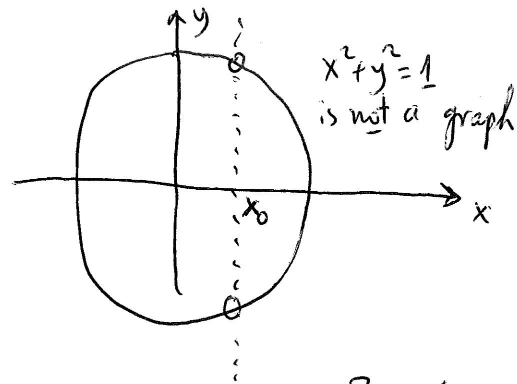
<u>Function</u>. A rule or a formula that assigns a value $f(x)$ to each admissible value of x, say $f(x)=x+1$ or $f(x)=x^2$.

<u>Domain</u>. The admissible values of x.

<u>Range</u>. The possible values of $f(x)$. We usually write $y=f(x)$ and depict the points (x,y) in the xy-plane.





<u>Example</u>. We find the domain and range of $f(x)=\dfrac{3x+4}{2x-1}$.

Domain ---- need $2x-1\neq 0$ or $2x\neq 1$ or $x\neq 1/2$.

Range ---- $y=\dfrac{3x+4}{2x-1}$. What are the limitations for y?

We try to solve for x (in terms of y) to find these limitations:

$$y=\frac{3x+4}{2x-1} \Rightarrow (2x-1)y = 3x+4$$
$$\Rightarrow 2xy-y=3x+4$$
$$\Rightarrow 2xy-3x=y+4$$
$$\Rightarrow x(2y-3)=y+4 \Rightarrow \boxed{x=\frac{y+4}{2y-3}}.$$

Based on this, we get $2y\neq 3$ or $y\neq 3/2$.

<u>Example</u>. Consider $f(x)=\sqrt{\dfrac{1-x}{x}}$.

- For the domain, we need $x\neq 0$ <u>and</u> $\dfrac{1-x}{x}\geq 0$ as well.

We need ① $1-x \geq 0$ and $x \geq 0$, so $\underline{0 \leq x \leq 1}$

$\underline{\underline{\text{or}}}$ ② $1-x \leq 0$ and $x \leq 0$, so $1 \leq x \leq 0$, not possible.

The domain consists of all $0 < x \leq 1$. This is denoted by $[0,1]$. More generally, (a,b) is the set of all points $a < x < b$ with a, b not included and

$[a,b)$ denotes the points $a \leq x < b$

$(a,b]$ stands for $a < x \leq b$

$[a,b]$ stands for $a \leq x \leq b$.

<u>Example</u> (continued) We determine the range of f when

$$f(x) = \sqrt{\frac{1-x}{x}}.$$

Solving the equation $y = \sqrt{\frac{1-x}{x}}$ for x, we get

① $y \geq 0$ and ② $y^2 = \frac{1-x}{x} = \frac{1}{x} - 1 \Rightarrow y^2+1 = \frac{1}{x} \Rightarrow x = \frac{1}{y^2+1}$.

This formula makes sense for any y, so the range consists of all $y \geq 0$. We denote that by $[0, +\infty)$.

$\boxed{\text{Injective}}$ We say $f: X \to Y$ is <u>injective</u> or 1-1 or one-to-one, if f maps distinct x values to distinct y values, namely if

$$\boxed{f(x_1) = f(x_2) \implies x_1 = x_2}$$

For instance, $f(x) = x^2$ is <u>not</u> injective ... $f(-2) = 4 = f(2)$.

<u>Example</u> Consider $f(x) = \dfrac{3x+4}{2x-1}$. We show f is injective.

Suppose $f(x_1) = f(x_2)$. Then $\dfrac{3x_1+4}{2x_1-1} = \dfrac{3x_2+4}{2x_2-1}$

so $(3x_1+4)(2x_2-1) = (2x_1-1)\cdot(3x_2+4)$

so $\cancel{6x_1x_2} - 3x_1 + 8x_2 \cancel{-4} = \cancel{6x_1x_2} - 3x_2 + 8x_1 \cancel{-4}$

so $-11x_1 = -11x_2$, so $x_1 = x_2$.

<u>Example</u>. Consider $f(x) = x^2$. We check f is <u>not</u> injective.

Suppose $f(x_1) = f(x_2)$. Then $x_1^2 = x_2^2$

so $x_1^2 - x_2^2 = 0$

so $(x_1+x_2)(x_1-x_2) = 0$.

This gives $x_1 = x_2$ <u>or</u> $x_1 = -x_2$, so f is not injective.

$\boxed{\text{Surjective}}$ We say $f : X \to Y$ is <u>surjective</u> or onto, if the range of f is all of Y. We say f is <u>bijective</u>, if f is injective and surjective.

<u>Example</u>. Consider $f(x) = \sqrt{1-x^4}$ as a function $f : [-1,1] \to \mathbb{R}$.
The range is ... $y = \sqrt{1-x^4} \overset{\circledast}{\Rightarrow} y^2 = 1-x^4 \Rightarrow x = \sqrt[4]{1-y^2}$.
We need $1-y^2 \geq 0$, namely $-1 \leq y \leq 1$. However $y \geq 0$ so range is $[0,1]$.
Thus $f : [-1,1] \to \mathbb{R}$ is <u>not</u> surjective, but $f : [-1,1] \to [0,1]$ is.

$\boxed{\text{Quadratic functions}}$ Consider $f(x) = ax^2 + bx + c$, where $a \neq 0$

<u>Case 1.</u> If it happens that $\Delta = b^2 - 4ac$ is non-negative, then $x_1 = \dfrac{-b + \sqrt{\Delta}}{2a}$ and $x_2 = \dfrac{-b - \sqrt{\Delta}}{2a}$.
Moreover, one may factor $f(x) = a(x-x_1)(x-x_2)$.

<u>Case 2.</u> If it happens that $\Delta < 0$, then $f(x)$ has no real roots, so $f(x)$ cannot be factored (using real numbers).

<u>Proof.</u>

$$\frac{f(x)}{a} = \underbrace{x^2 + \frac{b}{a}x} + \frac{c}{a} \qquad (x+y)^2 = x^2 + 2xy + y^2$$

$$= x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2$$

$$= \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}.$$

If $4ac - b^2 > 0$, we get a sum of squares and no roots!
This is true when $b^2 - 4ac < 0$. Suppose $4ac - b^2 \leq 0$.

Then
$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2$$

$$= \left(x + \frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\right) = (x-x_2)(x-x_1)$$

Factorisation and range of quadratics

Example 1. Let $f(x) = 2x^2 - 7x + 3$. Then $\Delta = 49 - 4\cdot 2\cdot 3 = 25$ is positive, so we get real roots

$$x = \frac{7 \pm \sqrt{25}}{2\cdot 2} = \frac{7\pm 5}{4} \Rightarrow x_1 = 3 \text{ and } x_2 = 1/2.$$

We can thus factor as $f(x) = a(x-x_1)(x-x_2)$

$$= 2(x-3)(x-1/2) = (x-3)(2x-1).$$

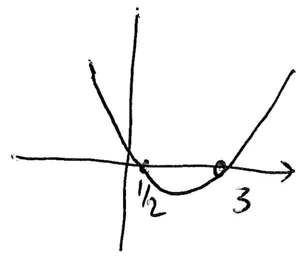
To find the range, we look at $y = 2x^2 - 7x + 3$ and solve for x. We get $\underbrace{2x^2 - 7x + 3 - y}_{ax^2 + bx + c} = 0$ and

we need $\Delta = b^2 - 4ac = 49 - 4\cdot 2\cdot(3-y)$ to be non-negative

so $\Delta = 49 - 8(3-y) = 49 - 24 + 8y \geq 0$

so $25 + 8y \geq 0 \Rightarrow 8y \geq -25 \Rightarrow y \geq -25/8$.

In other words, the range is $[-25/8, +\infty)$.



Example 2. Let $f(x) = -2x^2 + 3x - 4$.

Then $\Delta = b^2 - 4ac = 9 - 4(-2)(-4) = 9 - 32 < 0$

so f has no real roots and no factorisation.

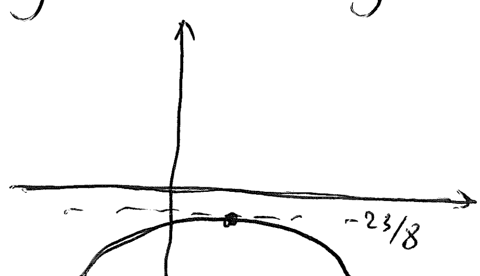
To find the range, $y = -2x^2 + 3x - 4$

$$\Rightarrow -2x^2 + 3x - 4 - y = 0.$$

We need $\Delta = b^2 - 4ac = 9 - 4(-2)(-4-y) = 9 - 8(4+y)$ to be non-negative, so $\Delta = 9 - 32 - 8y \geq 0$

$$\Rightarrow 8y \leq -23 \Rightarrow y \leq -23/8.$$

The graph of f looks roughly like



Polynomial functions. We start with factorisation.

Rational root theorem Consider $f(x) = a_n x^n + \dots + a_2 x^2 + a_1 x + a_0$, a polynomial with integer coefficients. If there is a rational root x_0, then $x_0 = p/q$ with p, q relatively prime and p dividing a_0 and q dividing a_n.

Quick proof. Suppose $x = p/q$ is a root. Then

$$a_n \frac{p^n}{q^n} + \dots + a_2 \frac{p^2}{q^2} + a_1 \frac{p}{q} + a_0 = 0.$$

Thus $a_n p^n + a_{n-1} p^{n-1} q + \dots + a_2 p^2 q^{n-2} + a_1 p q^{n-1} + \underline{a_0 q^n} = 0$.

Now, p divides all terms except for $a_0 q^n \Rightarrow p$ divides $a_0 q^n$.

$\Rightarrow p$ divides a_0. ▨

Example. Take $f(x) = x^3 + x^2 - 3x + 1$.

Possible rational roots .... $x = \pm 1$ are only possibilities.

Check $x = 1$ .... $f(1) = 1 + 1 - 3 + 1 = 0$ so $x=1$ a root

$x = -1$ .... $f(-1) = -1 + 1 + 3 + 1 = 4$ so $x = -1$ not ~~root~~ root.

Factor theorem. Suppose $f(x)$ is a polynomial with x_0 as a root. Then $f(x)$ has $x - x_0$ as a factor: $f(x) = (x - x_0) \cdot g(x)$.

Assuming this, $x = 1$ is a root $\Rightarrow$ $x - 1$ is a factor.

Thus $f(x) = x^3 + x^2 - 3x + 1 = (x-1) \cdot g(x)$ for some polynomial $g(x)$. Needless to say, $g(x) = \dfrac{x^3 + x^2 - 3x + 1}{x - 1}$.

We use division of polynomials to simplify.

$$
\begin{array}{r|l}
 & x^2 + 2x - 1 \\
\hline
x - 1 & x^3 + x^2 - 3x + 1 \\
 & x^3 - x^2 \\
\hline
 & \quad 2x^2 - 3x + 1 \\
 & \quad 2x^2 - 2x \\
\hline
 & \qquad -x + 1 \\
 & \qquad -x + 1 \\
\hline
 & \qquad\quad 0
\end{array}
$$

This gives $x^3 + x^2 - 3x + 1 = (x-1)\underline{(x^2+2x-1)}$.
For the quadratic, $\Delta = 4 + 4 = 8$ is positive, so

$$x = \frac{-2 \pm \sqrt{\Delta}}{2} = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2} \Rightarrow \begin{array}{l} x_1 = -1 + \sqrt{2} \\ x_2 = -1 - \sqrt{2} \end{array}$$

are both roots and $f(x) = (x-1)(x-x_1)(x-x_2)$.

__Factor theorem.__ If $f(x)$ is a polynomial with x_0 as a root, then $f(x) = (x-x_0)\,g(x)$ for some polynomial $g(x)$.

__Proof.__ We divide $f(x)$ by $x-x_0$.

$$\begin{array}{r|l} & x^2 \\ \hline x-1 & x^3 + x - 2 \\ & x^3 - x^2 \\ \hline & x^2 + x - 2 \\ & \vdots \end{array} \qquad\qquad \begin{array}{r|l} & Q(x) \\ \hline x-x_0 & f(x) \\ & \vdots \\ & r \end{array}$$

The division may proceed up until the remainder becomes a constant (no x).

This gives $f(x) = (x-x_0)\cdot Q(x) + r$.

But $0 = f(x_0) = r$, so the remainder is zero and $f(x) = (x-x_0)Q(x)$. $\boxtimes$

__Example.__ We factor $f(x) = \frac{1}{5}x^3 - \frac{1}{2}x^2 + \frac{2}{5}x - \frac{1}{10}$.

Write $10f(x) = 2x^3 - 5x^2 + 4x - 1$.

Rational roots ---- ± 1 and $\pm 1/2$

We check ----

$$10f(1) = 2 - 5 + 4 - 1 = -3 + 3 = 0 \quad \checkmark$$

$$10f(1/2) = \frac{2}{8} - \frac{5}{4} + 2 - 1 = 0 \quad \checkmark$$

and $10f(-1) \neq 0 \neq 10f(-1/2)$.

Thus, only $x = 1$ and $x = 1/2$ are rational roots, so $x - 1$ and $x - 1/2$ are factors.

We use division of polynomials.

$$\begin{array}{r|l} & 2x^2 - 3x + 1 \\ \hline x-1 & 2x^3 - 5x^2 + 4x - 1 \\ & 2x^3 - 2x^2 \\ \hline & -3x^2 + 4x - 1 \\ & -3x^2 + 3x \\ \hline & x - 1 \\ & x - 1 \\ \hline & 0 \end{array}$$

We get

$$\begin{aligned} 10f(x) &= 2x^3 - 5x^2 + 4x - 1 \\ &= (x-1)\underline{(2x^2 - 3x + 1)} \\ &= (x-1)(2x-1)(x-1). \end{aligned}$$

Thus $f(x) = \frac{1}{10}(x-1)^2(2x-1)$.

Example. Let $f(x) = 2x^3 - 9x^2 + 10x - 3$.

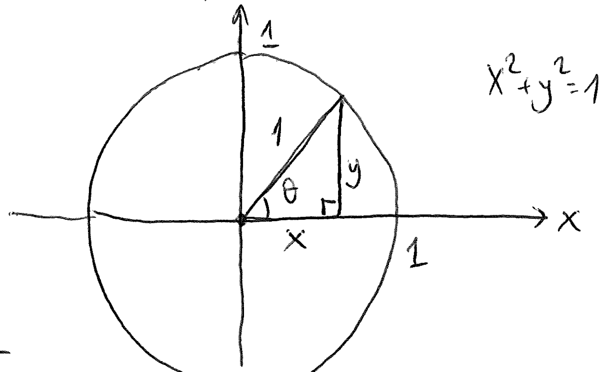
Rational roots .... ± 1, ± 3, $\pm 1/2$, $\pm 3/2$.

We check .... $x=1$, $x=3$, $x=1/2$ are roots.

We get 3 factors $x-1$, $x-3$, $x-1/2$ and f is cubic, so

$$f(x) = 2(x-1)(x-3)(x-1/2)$$
$$= (x-1)(x-3)(2x-1).$$

Trigonometric functions These are defined in terms of a triangle. Consider an angle θ measured in radians:



$$2\pi \text{ radians} = 360°$$
$$\pi/2 \text{ radians} = 90°.$$

We define $\sin\theta = \dfrac{y}{1}$, $\cos\theta = \dfrac{x}{1}$

$\tan\theta = \dfrac{y}{x}$, $\cot\theta = \dfrac{x}{y}$

$\sec\theta = \dfrac{1}{x}$, $\csc\theta = \dfrac{1}{y}$.

- The most important of those are $x = \cos\theta$ & $y = \sin\theta$.

The remaining ones are:

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}.}$$

- The main identity/identities follow(s) from Pythagoras' theorem:

$$x^2 + y^2 = 1$$

$$\cos^2\theta + \sin^2\theta = 1.$$

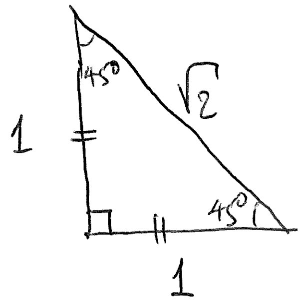
Dividing by $\cos^2\theta$ .... $1 + \tan^2\theta = \sec^2\theta$

Dividing by $\sin^2\theta$ .... $\cot^2\theta + 1 = \csc^2\theta$.

Special values of trigonometric functions

The main/simplest values arise when $\theta = 30^\circ, 45^\circ, 60^\circ$

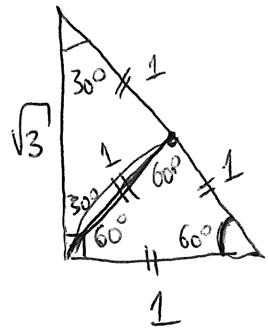
$\theta = \pi/6, \pi/4, \pi/3$ radians.



$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\tan\frac{\pi}{4} = 1$

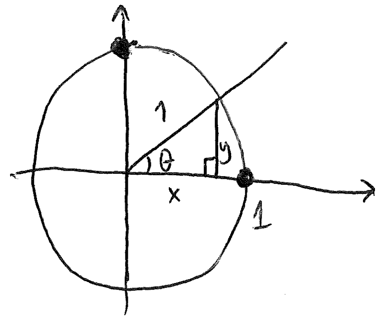


$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{6} = 1/2$

$\cos\frac{\pi}{3} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \sqrt{3}/2$

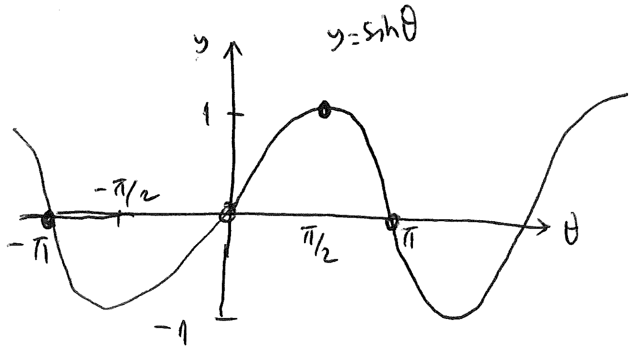
$\tan\frac{\pi}{3} = \sqrt{3}$, $\tan\frac{\pi}{6} = 1/\sqrt{3}$.

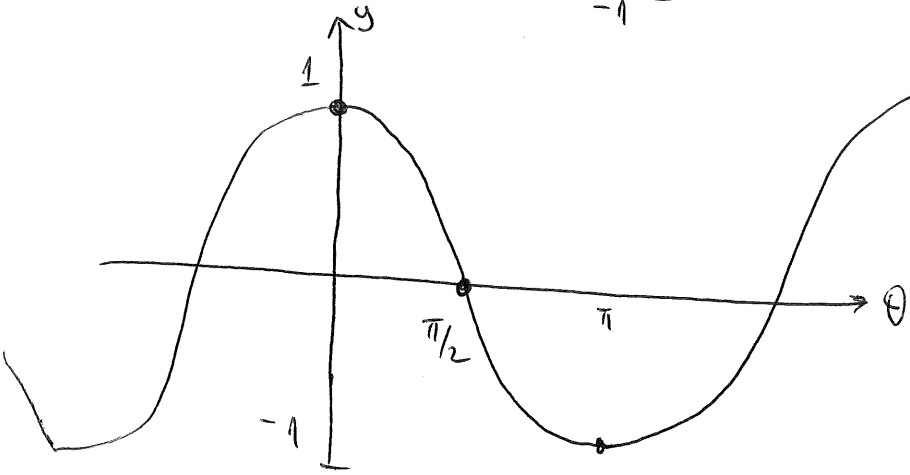
Graphs of sine and cosine



$\sin\theta = \frac{y}{1} = y$

$\cos\theta = \frac{x}{1} = x$





Theorem (Addition formulas for sine and cosine)
Given any angles x, y, one has

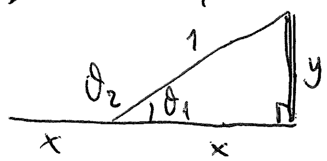
$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

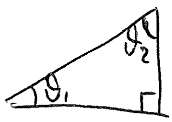
Example. Consider two angles θ_1, θ_2 whose sum is π. Then

$$\sin\theta_2 = \sin(\pi - \theta_1) = \cancelto{0}{\sin\pi}\cdot\cos\theta_1 - \sin\theta_1 \cdot \cancelto{-1}{\cos\pi} = \sin\theta_1,$$

$$\cos\theta_2 = \cos(\pi - \theta_1) = \cancelto{-1}{\cos\pi}\cdot\cos\theta_1 + \cancelto{0}{\sin\pi}\cdot\sin\theta_1 = -\cos\theta_1.$$



Example. Consider two angles θ_1, θ_2 with $\theta_1 + \theta_2 = \frac{\pi}{2}$.



Then $\sin\theta_2 = \sin\left(\frac{\pi}{2} - \theta_1\right) = \cancelto{1}{\sin\frac{\pi}{2}}\cos\theta_1 - \sin\theta_1 \cancelto{0}{\cos\frac{\pi}{2}} = \cos\theta_1$

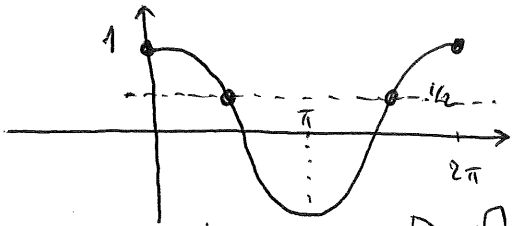
so $\sin\theta_1 = \cos\theta_2$ as well.

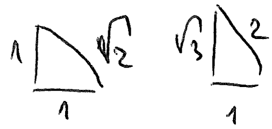
Example. We find angles $0 \le \theta \le 2\pi$ with $2\cos^2\theta + 5\cos\theta = 3$.

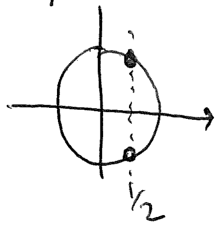
Let $x = \cos\theta$. Then $2x^2 + 5x - 3 = 0$. We have

$$x = \frac{-5 \pm \sqrt{25+24}}{2 \cdot 2} = \frac{-5 \pm 7}{4} \begin{cases} \to x = 1/2 \\ \to x = -3 \end{cases}.$$

Thus $\cos\theta = 1/2$ or $\cos\theta = -3$. We know $-1 \le \cos\theta \le 1$, so $\cos\theta = -3$ is impossible. When $\cos\theta = 1/2$, we get two angles θ: one is $\theta_1 = \pi/3 = 60^\circ$ and the other one is $\theta_2 = 2\pi - \pi/3 = 5\pi/3$.







Exponential functions

- When $a > 0$, we can define the powers a^x for any real number x. We call the function f with $f(x) = a^x$ an exponential function (with base a).
- Integral powers of a are easy to define as
  $$a^2 = a \cdot a, \quad a^3 = a \cdot a \cdot a, \quad a^n = \text{product of } n \text{ copies of } a$$
  when n is positive and $a^{-n} = \frac{1}{a^n}$ by definition.
- The power $a^{1/2}$ stands for the square root of a. This is because $(a^{1/2})^2 = a^{2/2} = a$. Similarly, $a^{1/n}$ stands for the nth root of a.
- This allows us to define $a^{m/n} = (a^{1/n})^m$ for any integers m, n with $n \ne 0$, in particular for all rational numbers: $8^{2/3} = (8^{1/3})^2 = 4$ and $16^{3/4} = 8$.

- Irrational powers like $a^{\sqrt{2}}$ are defined by approximations.

Since $\sqrt{2} = 1.4142135\ldots$ we can look at

$$a^1$$
$$a^{1.4} = a^{14/10} = (a^{1/10})^{14}$$
$$a^{1.41} = a^{141/100} = (a^{1/100})^{141} \quad \text{and so on}$$

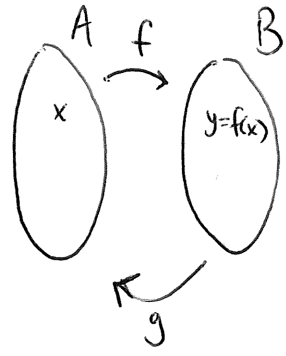
to obtain approximations of the actual power $a^{\sqrt{2}}$.

<u>Basic properties</u>

$$a^x \cdot a^y = a^{x+y}$$
$$a^0 = 1 \quad \text{by definition}$$
$$a^{-x} = \frac{1}{a^x} \quad \text{by above}$$
$$\frac{a^x}{a^y} = a^{x-y} \quad \text{by above}$$
$$(a^x)^y = a^{xy}.$$

Inverse functions

Suppose $f: A \to B$ is bijective.



Then there exists a function $g: B \to A$ such that

$$g(f(x)) = x \quad \text{for all } x \text{ in } A$$

$$f(g(y)) = y \quad \text{for all } y \text{ in } B.$$

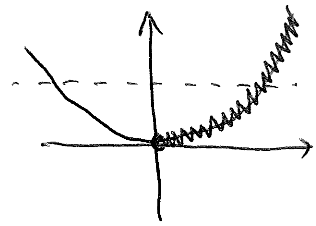
This function is unique, it is called the inverse of f and we write $g = f^{-1}$.

To determine $g = f^{-1}$, we try to solve $y = f(x) \iff x = g(y)$ by solving for x. Then $g(f(x)) = g(y) = x$ and $f(g(y)) = f(x) = y$.

Example 1. (Trivial) Take $f(x) = x^2$ as a function $f: \mathbb{R} \to \mathbb{R}$.

This is not surjective ---- the range is $[0, \infty)$

and not injective ---- since $f(x) = f(-x)$.



Define $f: [0,\infty) \to [0,\infty)$ by $f(x) = x^2$.

Then $y = f(x) \iff y = x^2 \iff x = \pm\sqrt{y} \iff x = \sqrt{y}$ since $x \geq 0$.

The inverse is $g(y) = \sqrt{y}$. For a general function $f: A \to B$ that is bijective $y = f(x) \iff x = g(y)$ has a unique solution (in terms of x).

Example 2. Define $f: [1,\infty) \to [1,\infty)$ by $f(x) = 2x^2 - 4x + 3$.

We claim f is bijective and we find the inverse. One has

$$y = f(x) \iff y = 2x^2 - 4x + 3 \iff 2x^2 - 4x + (3-y) = 0$$

$$\iff x = \frac{4 \pm \sqrt{16 - 8(3-y)}}{4} \iff x = \frac{4 \pm \sqrt{8y - 8}}{4}.$$

This formula makes sense when $y \geq 1$ and we get

$$y = f(x) \iff x = 1 \pm \frac{1}{4}\sqrt{8y-8} \iff x = 1 + \frac{1}{4}\sqrt{8y-8}$$

since $x \geq 1$. This is a unique solution for each $y \geq 1$ and the inverse function is $g(y) = 1 + \frac{1}{4}\sqrt{8y-8}$.

$\boxed{\text{Logarithmic functions}}$ Consider the exponential function $f(x) = a^x$. These are defined when $a > 0$ and they are injective when $a \neq 1$. In fact, $f(x) = a^x$ gives a bijection $f : \mathbb{R} \to (0, \infty)$ when $a \neq 1$. The inverse of this function is $\log_a : (0, \infty) \to \mathbb{R}$. The main properties are: (1) $a^{\log_a x} = x$, (2) $\log_a a^x = x$,

(3) $\log_a (x \cdot y) = \log_a x + \log_a y$ (4) $\log_a x^r = r \cdot \log_a x$

(5) $\log_a \frac{x}{y} = \log_a x - \log_a y$ (6) $\log_a 1 = 0$.

Proof. (1), (2) hold by definition. (3) follows because $a^{x+y} = a^x \cdot a^y$:

$$a^{\log_a x + \log_a y} = a^{\log_a x} \cdot a^{\log_a y} = x \cdot y$$

$$\text{so } \cancel{\log_a a}^{\log_a x + \log_a y} = \cancel{\log} \log_a (x \cdot y).$$

Property (4) holds because

$$x^r = \left(a^{\log_a x}\right)^r = a^{r \cdot \log_a x}$$

$$\text{so } \log_a x^r = \log_a a^{r \log_a x}.$$

(5) holds: $\log_a \frac{x}{y} = \log_a (x \cdot y^{-1}) = \log_a x + \log_a y^{-1}$
$= \log_a x - \log_a y$.

(6) holds since $\log_a 1 = \log_a a^0 = 0$. ∎

Example. $\log_2 18 - 2 \log_2 3 = \log_2 18 - \log_2 3^2 = \log_2 \frac{18}{3^2} = \log_2 2 = 1$.

Example. We find the inverse f^{-1} when $f : (1/3, \infty) \to \mathbb{R}$ and $f(x) = 3 - \log_2 (3x - 1)$.

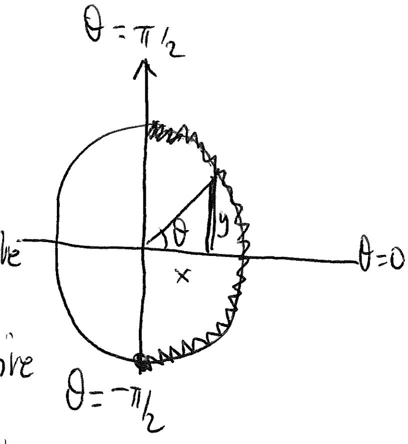
We solve $y = f(x)$ for x. This gives

$$y = f(x) \iff y = 3 - \log_2(3x-1)$$
$$\iff \log_2(3x-1) = 3-y$$
$$\iff \cancel{2}^{\log_2(3x-1)} = 2^{3-y}$$
$$\iff 3x-1 = 2^{3-y}$$
$$\iff 3x = 1 + 2^{3-y} \iff x = \frac{1+2^{3-y}}{3}.$$

$\log_2 x$ is inverse to 2^x

The inverse function is $g(y) = \frac{1+2^{3-y}}{3}$.

Inverse trigonometric functions



None of sine, cosine, tangent is injective.

However sin : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$ is bijective

cos : $[0, \pi] \longrightarrow [-1, 1]$ is bijective

tan : $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$ is bijective.

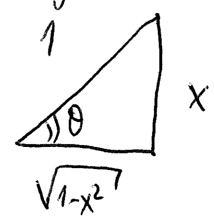
These have an inverse function $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ on the corresponding intervals. Here $\sin^{-1} x$ = angle whose sine is x,

so $\sin^{-1} \frac{\sqrt{3}}{2} = \pi/3$ and $\sin^{-1} 5$ does not exist.

$$\boxed{\begin{array}{c}\sin\theta = x \\ \theta = \sin^{-1} x\end{array}}$$

__Example 1.__ We simplify $\tan(\sin^{-1} x)$.

Here, $\sin^{-1} x$ = angle whose sine is x. We need tangent of this angle. Such an angle appears in the figure and one has $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1-x^2}}$.

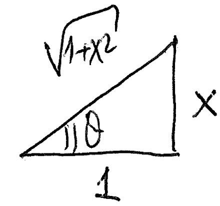


In other words $\tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1-x^2}}$ for any $-1 < \cancel{x} < 1$.

__Example 2.__ We simplify $\sin(\tan^{-1} x)$.

Here, $\theta = \tan^{-1} x$ is an angle with $\tan\theta = x$.

We introduce such an angle as in the figure.



Then $\sin(\tan^{-1} x) = \sin\theta = \dfrac{x}{\sqrt{1+x^2}}$ for any x.

__Algebraic functions__: one can get using polynomials through the standard operations of $+$, $-$, $\times$, $\div$ and roots $\sqrt{\ }$.

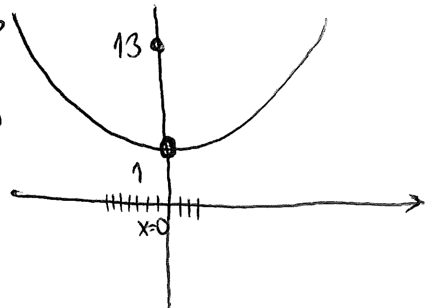
__Transcendental functions__: any function that is not algebraic. Some typical examples are trig functions, inverse trig, exp, logarithms.

Introduction to limits

Consider a function f that is defined near $x = x_0$ but not necessarily at that point. We wish to study the values of $f(x)$ as x approaches x_0. We say that $f(x)$ approaches some value / limit L as x approaches x_0 and write $\boxed{\lim_{x \to x_0} f(x) = L}$ if the values of $f(x)$ get arbitrarily close to L as $x \to x_0$.

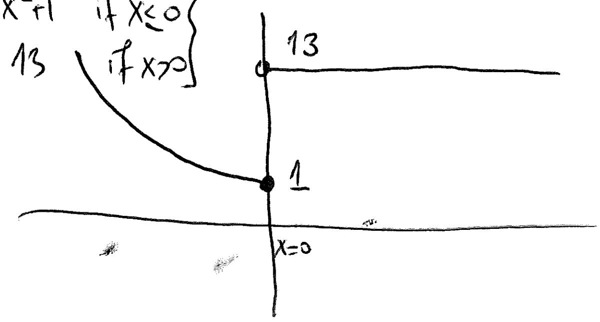
① The behaviour at $x = x_0$ is irrelevant. ② The limit may not exist.

$f(x) = \begin{cases} x^2+1 & \text{if } x \neq 0 \\ 13 & \text{if } x = 0 \end{cases}$



Here $\lim_{x \to 0} f(x) = 1$ but $f(0) = 13$.

$f(x) = \begin{cases} x^2+1 & \text{if } x \leq 0 \\ 13 & \text{if } x > 0 \end{cases}$



Here, $f(x)$ approaches the value 1 if we approach $x=0$ from the left and the value 13 from the right. In this case, $\lim_{x \to 0} f(x)$ does not exist.

③ There are lots of functions for which $\lim_{x \to x_0} f(x) = f(x_0)$. These are called continuous functions and their limits are obtained by letting $x = x_0$. For instance,

$$\lim_{x \to 2} (x^2 + 3x + 1) = 2^2 + 3 \cdot 2 + 1 = 11.$$

There are also cases for which $f(x_0)$ is not even defined, say

$$\lim_{x \to 2} \frac{x^3 - 2x - 4}{x - 2}.$$

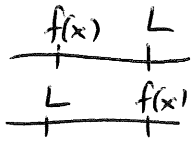
This is defined for $x \neq 2$ but we cannot allow $x = 2$.

For those cases, we still need to study $f(x) \to L$.

Definition ($\varepsilon\delta$ definition) We say $f(x)$ approaches L as x approaches x_0 if given any $\varepsilon > 0$ there exists some $\delta > 0$ such that $0 \neq |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Definition of limits. We need $f(x)$ to approach L as x approaches x_0.

We note that $|f(x)-L| =$ distance between $f(x)$ and L and similarly for $|x-x_0|$ = distance between x_0 and x.



Given <u>any $\varepsilon>0$</u> we need to find $\delta>0$ such that

$$\boxed{0 \neq |x-x_0| < \delta \Rightarrow |f(x)-L| < \varepsilon.}$$

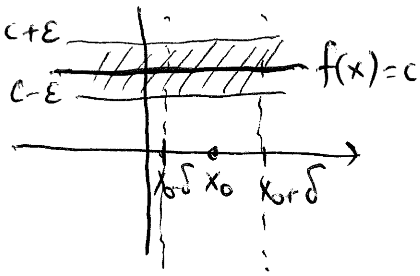
Example 1. (Constant functions) Let $f(x)=c$ for all x.

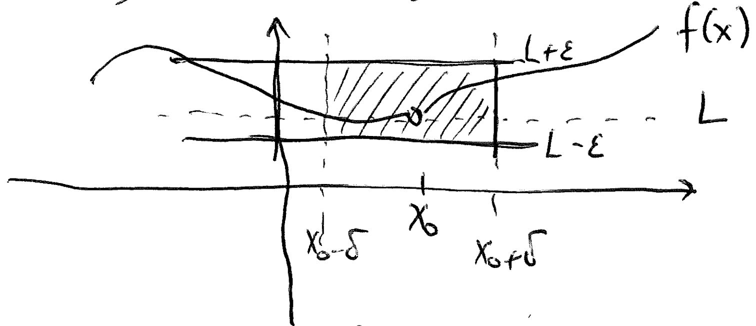
We show that $\boxed{\lim_{x\to x_0} f(x) = c.}$ Let $\varepsilon>0$ be given.

We need $\delta>0$ s.t. $\quad 0 \neq |x-x_0| < \delta \Rightarrow |f(x)-c| < \varepsilon$

namely $0 \neq |x-x_0| < \delta \Rightarrow 0 < \varepsilon$.

This is true for any $\delta>0$, say $\delta=1$.





Example 2. (Linear functions) Let $f(x) = ax+b$ for all x.

We show that $\lim_{x\to x_0} f(x) = ax_0+b$ .... We can take $x=x_0$

We proved this when $a=0$. Suppose $a\neq 0$.

Given $\varepsilon>0$, we need $\delta>0$

$$0 \neq |x-x_0| < \delta \Rightarrow |f(x)-(ax_0+b)| < \varepsilon.$$

In practice, <u>we assume $|x-x_0|<\delta$ and try to estimate $|f(x)-L|$.</u>

The choice of δ is decided later. In our case,

$$|f(x)-L| = |f(x)-ax_0-b| = |ax+\cancel{b}-ax_0-\cancel{b}|$$

$$= |ax-ax_0| = |a|\cdot|x-x_0| < |a|\cdot\delta.$$

If we choose $\delta = \frac{\varepsilon}{|a|}$, then $|f(x)-L| < |a|\cdot\delta = \varepsilon$, as needed.

Example 3. (Piecewise linear) Let $f(x) = \begin{cases} 12x+6 & \text{if } x \leq 1 \\ 7x+11 & \text{if } x > 1 \end{cases}$.

We compute $\lim_{x \to 1} f(x) = 18$. We need to show

given $\varepsilon > 0$ there exists $\delta > 0$: $\boxed{0 \neq |x-1| < \delta \Rightarrow |f(x) - 18| < \varepsilon}$.

Assume $0 \neq |x-1| < \delta$ and estimate

$$|f(x) - 18| = \begin{cases} |12x - 12| & \text{if } x \leq 1 \\ |7x - 7| & \text{if } x > 1 \end{cases} = \begin{cases} 12|x-1| & \text{if } x \leq 1 \\ 7|x-1| & \text{if } x > 1 \end{cases}.$$

$$\leq 12|x-1| < 12\delta.$$

If we choose $\delta = \varepsilon/12$, then $|f(x) - 18| < 12\delta = \varepsilon$, as needed.

Example 4. (No need for $\varepsilon\delta$) We compute the limit

$$\lim_{x \to 2} \frac{3x^2 - 4x - 4}{x - 2}.$$

Note that $f(x)$ is not defined when $x = 2$, still $x \neq 2$ here.

When $x = 2$, the denominator is zero and so is the numerator.

This means $x - 2$ is a factor of both numerator / denominator.

$$\lim_{x \to 2} \frac{3x^2 - 4x - 4}{x - 2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(3x+2)}{\cancel{x-2}}$$

$$\begin{array}{r} 3x + 2 \\ x - 2 \overline{\smash{)}\, 3x^2 - 4x - 4} \\ 3x^2 - 6x \\ \hline 2x - 4 \end{array}$$

$$= \lim_{x \to 2} (3x + 2) = 3 \cdot 2 + 2 \quad \text{(by letting } x = 2\text{)}$$
$$= 8.$$

Useful inequality. ① $|x + y| \leq |x| + |y|$ for all x, y.

② $|x - x_0| < \delta \iff x_0 - \delta < x < x_0 + \delta$.

Here ① follows by squaring both sides

$$|x+y|^2 = (x+y)^2 = x^2 + y^2 + 2xy = |x|^2 + |y|^2 + 2xy \leq |x|^2 + |y|^2 + 2|x| \cdot |y| = (|x| + |y|)^2.$$

Example 5. We use the $\varepsilon\delta$-definition to compute $\lim_{x\to 2} f(x)$ when $f(x) = 3x^2-4x+6$. We expect the limit to be $f(2) = 10$. To prove this, let $\varepsilon>0$ be given and estimate

$$|f(x)-L| = |f(x)-f(2)| = |3x^2-4x-4| = \underline{\underline{|x-2|}} \cdot \underline{\underline{|3x+2|}}.$$

$$\begin{array}{r|l} & 3x+2 \\ \hline x-2 & 3x^2-4x-4 \\ & 3x^2-6x \\ \hline & 2x-4 \end{array}$$

For the first factor, we have $|x-2|<\delta$.
For the second factor, we assume that $\underline{\underline{\delta\le 1}}$ for simplicity.
Then

$$\begin{aligned} |x-2|<\delta\le 1 \quad &\Rightarrow \quad -1<x-2<1 \\ &\Rightarrow \quad 1<x<3 \\ &\Rightarrow \quad 5<3x+2<11 \\ &\Rightarrow \quad |3x+2|<11 \end{aligned}$$

We then have $|f(x)-L| = |x-2|\cdot|3x+2| < \underline{\underline{11\delta \le \varepsilon}}$ provided that $|x-2|<\delta$ and $\delta = \min\{1, \varepsilon/12\}$, for instance.

Theorem (Limits of sums, products, quotients) Suppose f, g satisfy $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$. Then

$$\lim_{x\to x_0}\left[f(x)+g(x)\right] = L+M,$$

$$\lim_{x\to x_0}\left[f(x)\cdot g(x)\right] = L\cdot M,$$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{(as long as } M\neq 0\text{)}$$

Proof. We check the first two. Let $\varepsilon>0$ be given and estimate

$$\begin{aligned} |f(x)+g(x)-L-M| &= |f(x)-L+g(x)-M| \\ &\le \underbrace{|f(x)-L|}_{\text{as small as needed}} + \underbrace{|g(x)-M|}_{\text{as small as needed}} \end{aligned}$$

More formally, there exists $\delta_1 > 0$ with $0 \neq |x-x_0| < \delta_1 \Rightarrow |f(x)-L| < \varepsilon/2$
and $\delta_2 > 0$ with $0 \neq |x-x_0| < \delta_2 \Rightarrow |g(x)-M| < \varepsilon/2$.

This implies $0 \neq |x-x_0| < \min\{\delta_1, \delta_2\} \Rightarrow |f(x)+g(x)-L-M| < \varepsilon$.

We thus have the statement for sums. For products,

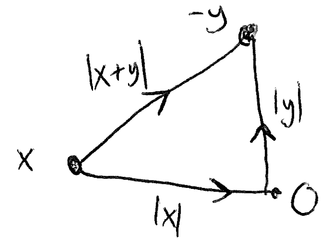
$$|f(x)g(x)-LM| = |\underbrace{f(x)g(x) - Lg(x)} + Lg(x) - LM|$$
$$\leq \underbrace{|f(x)-L|}_{\text{as small as needed}} \cdot |g(x)| + |L| \cdot \underbrace{|g(x)-M|}_{\text{as small as needed, say } \frac{\varepsilon}{2|L|+2}}.$$

For the $|g(x)|$ factor, we argue as before to get

$$|g(x)| = |g(x)-M+M| \leq \underbrace{|g(x)-M|} + |M| \leq 1+|M|$$

so the first term become $|f(x)-L| \cdot |g(x)| < \frac{\varepsilon}{2|M|+2} \cdot (1+|M|) = \frac{\varepsilon}{2}$ ▨

Triangle inequality $|x+y| \leq |x|+|y|$ for all x,y.



> **Theorem** (Limits of polynomials) If f is a polynomial, then
> $$\lim_{x\to x_0} f(x) = f(x_0) \quad \text{---- we can just let } x=x_0 \text{ to compute limits.}$$
> If f, g are polynomials, then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} = \frac{f(x_0)}{g(x_0)}$,
> as long as $g(x_0) \neq 0$.

Example. $\lim_{x\to 1} \frac{x^3+3x+1}{x+2} = \frac{1^3+3+1}{1+2} = \frac{5}{3}$.

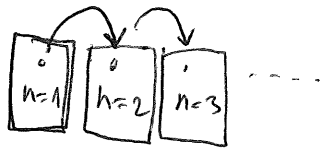
Example. $\lim_{x\to 2} \frac{x^3-5x^2+8x-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x^2-3x+2)}{\cancel{x-2}}$
$= 2^2 - 3\cdot 2 + 2$
$= 0$.

$$\begin{array}{r} x^2-3x+2 \\ x-2 \,\big)\, x^3-5x^2+8x-4 \\ x^3-2x^2 \\ \hline -3x^2+8x-4 \\ -3x^2+6x \\ \hline 2x-4 \end{array}$$

Proof. We need to prove $\lim_{x \to x_0} f(x) = f(x_0)$ when $f(x) = a_0 + a_1 x + \dots + a_n x^n$.

Proofs for polynomials are usually by induction.

① We check the statement when $n=1$.



② We assume the statement for n
and prove/deduce the statement for $n+1$.

If we can check ① - ②, then the statement holds for all n.

In our case, ① refers to $f(x) = a_0 + a_1 x$. We know $\lim_{x \to x_0} f(x) = f(x_0)$ in that case.

② Assume the statement for n. Then

$$\lim_{x \to x_0} \left(a_0 + a_1 x + \dots + a_n x^n + a_{n+1} x^{n+1} \right)$$

$$= \lim_{x \to x_0} \left(a_0 + a_1 x + \dots + a_n x^n \right) + \left(\lim_{x \to x_0} a_{n+1} x \right) \cdot \left(\lim_{x \to x_0} x^n \right)$$

$$= a_0 + a_1 x_0 + \dots + a_n x_0^n + a_{n+1} x_0^{n+1} = f(x_0). \quad \blacksquare$$

$\boxed{\text{One-sided limits}}$ We say that $f(x) \to L$ as $x \to x_0$ from the left and we write $\lim\limits_{x \to x_0^-} f(x) = L$ if, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\underline{x_0 - \delta < x < x_0 \implies |f(x) - L| < \varepsilon}$.

We say that $f(x) \to L$ as $x \to x_0$ from the right and we write $\lim\limits_{x \to x_0^+} f(x) = L$ if, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\underline{x_0 < x < x_0 + \delta \implies |f(x) - L| < \varepsilon}$.

- To say that $\lim\limits_{x \to x_0} f(x) = L$ is to say that

$$\lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x).$$

__Example__. Let $f(x) = \begin{cases} 3x - 1 & \text{if } x \le 2 \\ 4x - 5 & \text{if } x > 2 \end{cases}$. Then

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (3x - 1)$ and we know $\lim\limits_{x \to 2} (3x - 1) = 3 \cdot 2 - 1 = 5$

$\implies \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (3x - 1) = 5$ as well. Similarly,

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4x - 5) = \lim\limits_{x \to 2} (4x - 5) = 4 \cdot 2 - 5 = 3$.

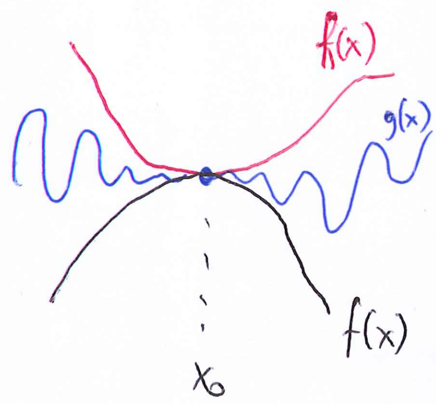
Thus, the one-sided limits exist, but $\lim\limits_{x \to 2} f(x)$ does not exist.

$\boxed{\text{Squeeze law}}$ Suppose $f(x) \le g(x) \le h(x)$ in some interval containing x_0. If we know

$$\lim_{x \to x_0} f(x) = L = \lim_{x \to x_0} h(x),$$

then the middle function g satisfies

$$\lim_{x \to x_0} g(x) = L \text{ as well.}$$



Proof. We are assuming $\underbrace{\lim_{x\to x_0} f(x) = L = \lim_{x\to x_0} h(x)}$.

To show $\lim_{x\to x_0} g(x) = L$, let $\varepsilon>0$ be given. Then there exists δ_1: $0 \neq |x-x_0| < \delta_1 \Rightarrow |f(x)-L| < \varepsilon$

$$\Rightarrow -\varepsilon < f(x) - L < \varepsilon$$

$$\Rightarrow \boxed{L-\varepsilon < f(x) < L+\varepsilon}$$

And similarly, there exists $\delta_2 > 0$:

$$0 \neq |x-x_0| < \delta_2 \Rightarrow \boxed{L-\varepsilon < h(x) < L+\varepsilon}$$

We conclude: $0 \neq |x-x_0| < \min\{\delta_1,\delta_2\} \Rightarrow L-\varepsilon < f(x) \le \underline{g(x)} \le h(x) < L+\varepsilon$

$$\Rightarrow L-\varepsilon < g(x) < L+\varepsilon. \quad \boxtimes$$

Theorem Suppose f is a polynomial or a quotient of polynomials. Then $\lim_{x\to x_0} f(x) = f(x_0)$ as long as $f(x_0)$ is defined.

This is also true for trigonometric or exponential functions.

[We say those functions are continuous.]

Proof. This basically reduces to the case $x_0 = 0$.

Let's look at $\sin x$. Then

$$\lim_{x\to x_0} \sin x = \lim_{x\to x_0} \sin(\underline{x-x_0}+x_0)$$

$$= \lim_{x\to x_0} \sin(x-x_0)\cdot\cos x_0 + \cos(x-x_0)\cdot\sin x_0$$

$$= \boxed{\lim_{x\to x_0} \sin(x-x_0)}\cdot \lim_{x\to x_0}\cos x_0 + \boxed{\lim_{x\to x_0}\cos(x-x_0)}\cdot\lim_{x\to x_0}\sin x_0$$

$y = x - x_0$

$$= \lim_{y\to 0} \sin y \cdot \cos x_0 + \lim_{y\to 0}\cos y \cdot \sin x_0.$$

If we show $\lim_{y\to 0}\sin y = \sin 0 = 0$

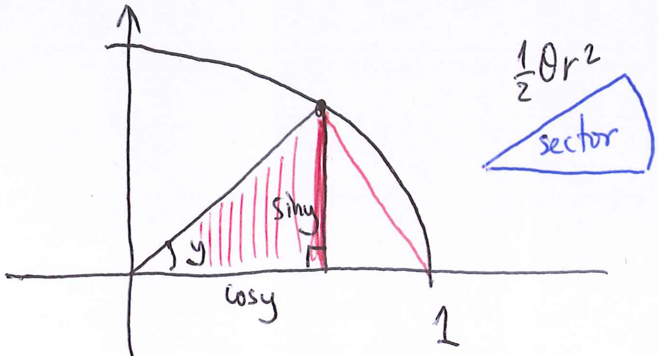
and $\lim_{y\to 0}\cos y = \cos 0 = 1$, then $\lim_{x\to x_0}\sin x = \sin x_0$.

One argues similarly for cosine:

$$\lim_{x\to x_0}\cos x = \lim_{x\to x_0}\cos(x-x_0+x_0) = \boxed{\lim_{x\to x_0}\cos(x-x_0)}\cdot \cos x_0 - \boxed{\lim_{x\to x_0}\sin(x-x_0)}\sin x_0$$

$$= \cos 0 \cdot \cos x_0 = \cos x_0 .$$

Now, let's show $\lim_{y\to 0}\sin y = 0$.



When $y>0$, we compare areas to get

$$\underset{\text{small triangle}}{\frac{1}{2}\sin y\cos y} \le \underset{\text{large triangle}}{\frac{1}{2}\sin y} \le \underset{\text{sector}}{\frac{1}{2}y}.$$

We get $0 \le \sin y \le y$ at least for $y>0$ and then

$\lim_{y\to 0^+} 0 = 0$, $\lim_{y\to 0^+} y = 0$ so $\lim_{y\to 0^+}\sin y = 0$ as well.

When $y<0$, we get $0 \le \sin(-y) \le -y$ by above

$\Rightarrow 0 \le -\sin y \le -y$

$\Rightarrow 0 \ge \sin y \ge y$ and the limit is zero.

For $\lim_{y\to 0}\cos y$, we have

$$\lim_{y\to 0}\cos^2 y = \lim_{y\to 0}(1-\sin^2 y) = 1-0 = 1$$

but $\cos y$ is positive near $y=0$, so $\lim_{y\to 0}\cos y = 1$.

For the exponential functions,

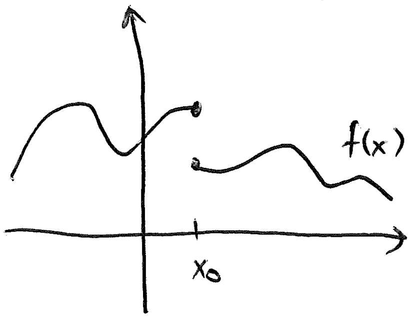
$$\lim_{x\to x_0} a^x = \lim_{x\to x_0} a^{x-x_0+x_0} = \lim_{x\to x_0} a^{x-x_0}\cdot a^{x_0}$$

$$= \boxed{\lim_{y\to 0} a^{y}}\cdot a^{x_0} = a^{x_0}. \quad \square$$

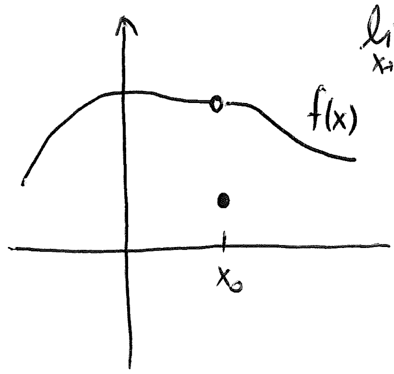
$\boxed{\text{Continuous functions}}$ We say that f is continuous at x_0, if one has ...... ① $\lim_{x\to x_0} f(x) = f(x_0)$. In terms of $\varepsilon\delta$ definition ..... ② given any $\varepsilon>0$ there exists $\delta>0$:

$$0 \neq |x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon .$$

In many cases, we can check continuity using ①, but in some cases we may need to resort to ②.



$\lim_{x\to x_0^-} f(x)$ exists
$\lim_{x\to x_0^+} f(x)$ exists
but not equal
[jump discontinuity]



$\lim_{x\to x_0} f(x)$ exists
but $\neq f(x_0)$
[removable discontinuity]

Examples of continuous functions.

- All polynomials are continuous. ⊙ All rational functions are continuous throughout their domain: $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \frac{P(x)}{Q(x)} = \frac{\lim_{x\to x_0} P(x)}{\lim_{x\to x_0} Q(x)} = \frac{P(x_0)}{Q(x_0)} = f(x_0)$.
- Trigonometric and exponential functions are continuous.
- Sums, products, quotients of cont. functions are continuous
- Square roots are continuous : $f(x)=\sqrt{x}$ is continuous at $x_0 \geq 0$.
- Compositions $f(g(x))$ are continuous as long as f, g are cont.

Proof of last parts. For sums/products/quotients, we can argue that

$$\lim_{x\to x_0} [f(x)\cdot g(x)] = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = f(x_0)\cdot g(x_0) .$$

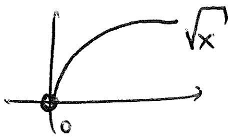
For square roots, we need to check $\lim_{x\to x_0} \sqrt{x} = \sqrt{x_0}$.

We use ε-δ definition .... we need $\boxed{|x-x_0|<\delta \Rightarrow |\sqrt{x}-\sqrt{x_0}|<\varepsilon .}$

Assume $|x-x_0|<\delta$ and estimate $|\sqrt{x}-\sqrt{x_0}| = \frac{|\sqrt{x}-\sqrt{x_0}|\cdot|\sqrt{x}+\sqrt{x_0}|}{\sqrt{x}+\sqrt{x_0}} = \frac{|x-x_0|}{\sqrt{x}+\sqrt{x_0}}$

$$< \frac{\delta}{\sqrt{x}+\sqrt{x_0}} \leq \frac{\delta}{\sqrt{x_0}} .$$

If we choose $\varepsilon = \frac{\delta}{\sqrt{x_0}}$ or $\delta = \varepsilon\sqrt{x_0}$, then $|\sqrt{x}-\sqrt{x_0}| < \varepsilon$ and the continuity follows. This argument works when $x_0 > 0$. The case $x_0 = 0$ is slightly different (but similar).



We'll check compositions tomorrow. $\square$

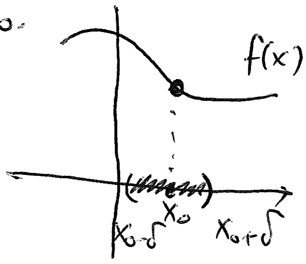
Example. $$\lim_{x\to 2} \frac{x^3 - 2x + 2}{x - 3} = \frac{2^3 - 2\cdot 2 + 2}{2-3} = -6$$

$$\lim_{x\to 2} \sqrt{\frac{x^2-4}{x^3-8}} = \lim_{x\to 2} \sqrt{\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x^2+2x+4)}}$$

$$= \lim_{x\to 2} \sqrt{\frac{x+2}{x^2+2x+4}} = \sqrt{\frac{4}{3\cdot 4}} = \frac{1}{\sqrt{3}}.$$

$$x^2+xy+y^2$$
$$x-y \,\big|\, x^3 - y^3$$
$$x^3 - x^2y$$
$$x^2y - y^3$$
$$x^2y - xy^2$$
$$xy^2 - y^3$$
$$0$$

$\boxed{\text{Continuity \& positivity}}$ Suppose f is continuous at x_0.



① If $f(x_0) > 0$, then there exists $\delta > 0$ s.t.
$$f(x) > 0 \quad \text{on } (x_0-\delta, x_0+\delta).$$

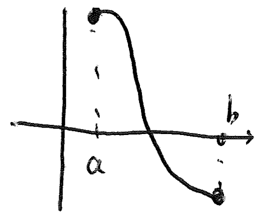
② If $f(x_0) < 0$, then there exists $\delta > 0$ s.t.
$$f(x) < 0 \quad \text{on } (x_0-\delta, x_0+\delta).$$

Quick proof. Given any $\varepsilon > 0$ there exists $\delta > 0$:
$$|x-x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$
namely $$x_0 - \delta < x < x_0+\delta \Rightarrow f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon.$$

~~We pick $\varepsilon =$~~ ① Assume $f(x_0) > 0$. Take $\varepsilon = f(x_0) > 0$.
Then $0 < f(x) < 2f(x_0)$.

② Assume $f(x_0) < 0$. Take $\varepsilon = -f(x_0) > 0$.
Then $+2f(x_0) < f(x) < 0$. $\square$

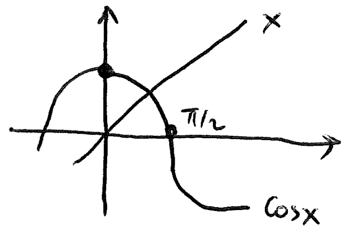
Bolzano's theorem.



Suppose f is <u>continuous</u> on the interval $[a,b]$.
If the values $f(a), f(b)$ have opposite sign,
then there is a point $a<c<b$ such that $f(c)=0$.
This is important and useful for solving equations $f(x)=0$.

<u>Example.</u> Consider $f(x) = x^n + x - 1$ for any positive integer n.
Then f is a polynomial $\Rightarrow$ continuous.
One has $f(0) = -1 < 0$ and $f(1) = 1 > 0$,
so there is a point $0 < c < 1$ such that $f(c) = 0$.
Thus, f has a root in $(0,1)$.

<u>Example.</u> We consider the equation $\cos x = x$.

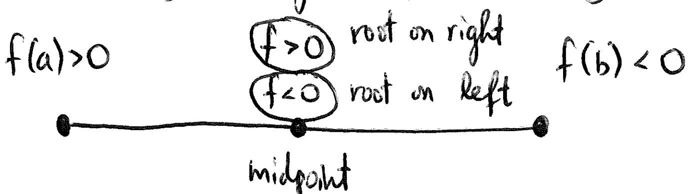


Define $f(x) = \cos x - x$ .... the function to be zero.
We look at the interval $[0, \pi/2]$ and check

$$f(0) = \cos 0 = 1 > 0$$
$$f(\pi/2) = \underbrace{\cos \pi/2}_{0} - \pi/2 < 0 ,$$

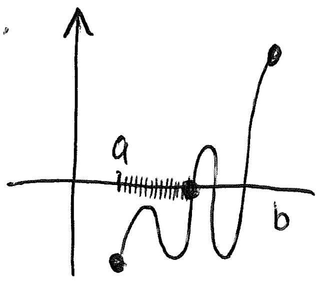
so $f(x) = 0$ for some point $0 < x < \pi/2$.

Digression. One can apply the theorem repeatedly to find the value of the root to any degree of accuracy (decimal points).



If we look at the two subintervals, we can decide which of the two contains the root $\Rightarrow$ split the interval in half and proceed.

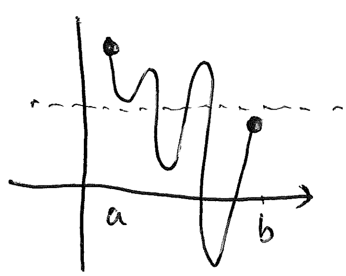
<u>Proof of Bolzano's theorem</u> Suppose $f(a) < 0 < f(b)$.



We look at the <u>largest</u> interval containing a such that $f(x) < 0$ on that interval.
What happens at the end of that interval?
If f is negative there, we contradict maximality.
If f is positive there, it is positive to the left as well, a contradiction. $\square$

Intermediate value theorem (IVT)

Suppose f is continuous on $[a,b]$.
Then f attains any value between $f(a)$ and $f(b)$.



Proof. If $f(a) \le f(b)$ and $f(a) \le c \le f(b)$, then we need to show c is attained, namely $f(x)=c$ has solution.
Define $g(x) = f(x) - c$ ... the difference (to become zero).
Then
$$g(a) = f(a) - c \le 0$$
$$g(b) = f(b) - c \ge 0.$$

If $g(a)=0$, then $f(a)=c$. ✓
If $g(b)=0$, then $f(b)=c$. ✓
If $g(a) < 0 < g(b)$, then there exists a point $a<x<b$ with $g(x)=0$. $\blacksquare$

Composition of continuous functions

The composition $f \circ g$ is defined by ~~f(g(x))~~ $(f\circ g)(x) = f(g(x))$.
If g is continuous at x_0
and f is continuous at $g(x_0)$,
then $f\circ g$ is continuous at x_0 as well.

Proof. To check $f\circ g$ is continuous, let $\varepsilon>0$ be given.
We need $\delta>0$: $\quad |x-x_0|<\delta \Rightarrow |f(g(x)) - f(g(x_0))| < \varepsilon$.

(1) We are assuming f continuous at $g(x_0)$.
Then there exists $\delta_1>0$: $|z-g(x_0)| < \delta_1 \Rightarrow |f(z) - f(g(x_0))| < \varepsilon$

(2) We are assuming g continuous at x_0.
Then there exists $\delta_2 > 0$: $|x-x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \delta_1$

Thus $|x-x_0| < \delta_2 \Rightarrow |g(x)-g(x_0)| < \delta_1$
$\Rightarrow |f(g(x)) - f(g(x_0))| < \varepsilon$
with $z=g(x)$. $\blacksquare$

$\boxed{\text{Infinite limits}}$ We write $\lim_{x \to x_0^-} f(x) = +\infty$, if the values $f(x)$ become arbitrarily large as $x \to x_0$ from the left.

One similarly defines $\lim_{x \to x_0^+} f(x) = +\infty$, $\lim_{x \to x_0^-} f(x) = -\infty$, $\lim_{x \to x_0^+} f(x) = -\infty$.

Note #1. We can formally define those by requiring: given any number $N > 0$ there exists $\delta > 0$: $x_0 - \delta < x < x_0 \Rightarrow f(x) > N$.

Note #2. Zero denominators will be allowed in the context of limits. In those cases, one needs to know if 0 stands for small positive/negative.

Example 1. $\lim_{x \to 3^-} \frac{1}{x-3} = -\infty$, $\lim_{x \to 3^+} \frac{1}{x-3} = +\infty$.

Example 2. $\lim_{x \to 3^-} \frac{3x+11}{x-3} = \lim_{x \to 3^-} \frac{20}{x-3} = -\infty$.

Example 3. $\lim_{x \to 2^+} \frac{4+3x}{2-x} = \lim_{x \to 2^+} \frac{10}{2-x} = -\infty$.

Example 4. $\lim_{x \to 1^-} \frac{6x^2+4}{2x^3-5x^2+2x+1} = \lim_{x \to 1^-} \frac{10}{2x^3-5x^2+2x+1}$

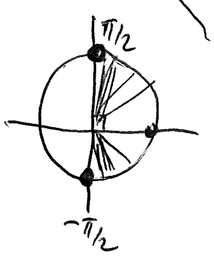
We use division of polynomials to get

$$\lim_{x \to 1^-} \frac{10}{(x-1)(2x^2-3x-1)} = \lim_{x \to 1^-} \frac{10}{(x-1)(-2)} = +\infty.$$

$$\begin{array}{r|l} & 2x^2-3x-1 \\ \hline x-1 & 2x^3-5x^2+2x+1 \\ & \underline{2x^3-2x^2} \\ & -3x^2+2x+1 \\ & \underline{-3x^2+3x} \\ & -x+1 \\ & 0 \end{array}$$

Example 5. $\lim_{x \to 1^\pm} \frac{4x^2-4}{2x^3-5x^2+2x+1} = \lim_{x \to 1^\pm} \frac{4\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(2x^2-3x-1)} = \frac{4 \cdot 2}{-2} = -4.$

Example 6. $\lim_{x \to \pi/2^-} \tan x = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \lim_{x \to \frac{\pi}{2}^-} \frac{1}{\cos x} = +\infty$



$\lim_{x \to -\frac{\pi}{2}^+} \tan x = \lim_{x \to -\pi/2^+} \frac{\sin x}{\cos x} = \lim_{x \to -\frac{\pi}{2}^+} \frac{-1}{\cos x} = -\infty.$

This implies that $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is surjective (by the IVT).

$\boxed{\text{Warning}}$. The usual rules for limits are still applicable. However, it is $\boxed{\text{not}}$ true that $\frac{\infty}{\infty}$ is equal to 1 (in general),

$0 \cdot \infty$ is equal to 0 (in general)

$\infty - \infty$ is equal to 0

1^{∞} is equal to 1.

$\boxed{\text{Limits at infinity}}$ We write $\lim_{x \to +\infty} f(x) = L$, if the values $f(x)$ approach L as becomes arbitrarily large, and we similarly define $\lim_{x \to -\infty} f(x) = L$, $\lim_{x \to \pm\infty} f(x) = +\infty$, $\lim_{x \to \pm\infty} f(x) = -\infty$.

Theorem. 1. $\lim_{x \to +\infty} x^p = +\infty$, if $p > 0$.

2. $$\lim_{x \to -\infty} x^n = \begin{cases} +\infty, & \text{if } n > 0 \text{ is even} \\ -\infty, & \text{if } n > 0 \text{ is odd} \end{cases}$$

3. $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$ for any integer $n > 0$.

4. If $f(x) = a_n x^n + \dots + a_1 x + a_0$ is a polynomial, then

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} a_n x^n \quad \text{(just highest term)}.$$

Example.
$$\lim_{x \to \pm\infty} \frac{3x^2 + 2x + 1}{4x^2 - 3x + 5} = \lim_{x \to \pm\infty} \frac{x^2(3 + 2/x + 1/x^2)}{x^2(4 - 3/x + 5/x^2)} = \lim_{x \to \pm\infty} \frac{3\cancel{x^2}}{4\cancel{x^2}} = 3/4.$$

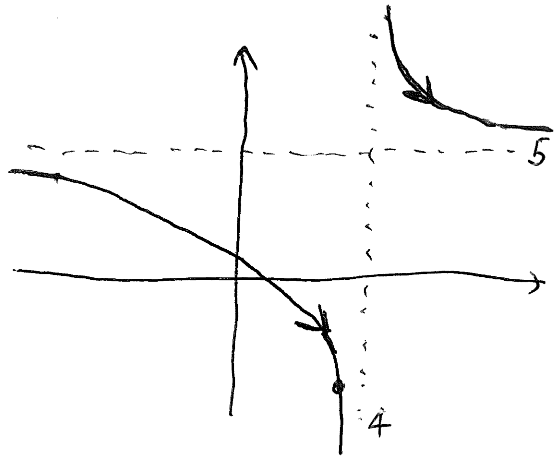
$$\lim_{x \to \pm\infty} \frac{3x^2 + 2x + 1}{4x^3 - 3x + 2} = \lim_{x \to \pm\infty} \frac{3x^2}{4x^3} = \lim_{x \to \pm\infty} \frac{3}{4x} = 0.$$

Example. We analyse the function $f(x) = \dfrac{5x+2}{x-4}$.

The domain consists of points $x \neq 4$. To find the range,

$$y = \frac{5x+2}{x-4} \iff xy - 4y = 5x+2 \iff xy - 5x = 4y+2 \iff x = \frac{4y+2}{y-5}.$$

The range consists of all $y \neq 5$.



Behaviour near $x=4$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{5x+2}{x-4} = \lim_{x\to 4^-} \frac{22}{x-4} = -\infty$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{5x+2}{x-4} = \lim_{x\to 4} \frac{22}{x-4} = +\infty$$

Limits at infinity

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{5x+2}{x-4} = \lim_{x\to -\infty} \frac{5\cancel{x}}{\cancel{x}} = 5$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{5x+2}{x-4} = \lim_{x\to +\infty} \frac{5x}{x} = 5.$$

By the intermediate value theorem, any interval that contains a value smaller than 5 contains all values smaller than 5. The same is true for intervals with values bigger than 5. Finally,

$$f(x) = \frac{5x+2}{x-4} = \frac{5(x-4)+22}{x-4} = 5 + \frac{22}{x-4}$$

and larger values of x yield smaller values of $f(x)$.

Theorem (Limits of exponentials and logarithms)

① Suppose $a > 1$. Then

$$\lim_{x\to +\infty} a^x = +\infty, \quad \lim_{x\to -\infty} a^x = 0, \quad \lim_{x\to +\infty} \log_a x = +\infty, \quad \lim_{x\to 0^+} \log_a x = -\infty$$

② Suppose $0 < a < 1$. Then

$\lim\limits_{x\to+\infty} a^x = 0$, $\lim\limits_{x\to-\infty} a^x = +\infty$, $\lim\limits_{x\to 0^+} \log_a x = +\infty$, $\lim\limits_{x\to+\infty} \log_a x = -\infty$.

$\boxed{\text{Derivatives}}$ We wish to study the behaviour of $f(x)$ and the way it changes with respect to x.

<u>Average rate of change</u>: between the points x_0 and x_1 is defined as

$$\frac{f(x_1)-f(x_0)}{x_1-x_0}.$$

<u>Instantaneous rate of change</u>: the rate at which $f(x)$ changes at $x=x_0$ is defined as $\lim\limits_{x_1\to x_0} \dfrac{f(x_1)-f(x_0)}{x_1-x_0}$, if this limit exists.

If it does exist, we call it the derivative at x_0, we say that f is differentiable at that point and we write

$$\boxed{f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}.}$$

<u>Example 1.</u> Consider a linear function $f(x) = ax+b$. Then

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{ax+\cancel{b}-ax_0-\cancel{b}}{x-x_0}$$

$$= \lim_{x\to x_0} \frac{ax-ax_0}{x-x_0} = \lim_{x\to x_0} \frac{a\cancel{(x-x_0)}}{\cancel{x-x_0}} = a.$$

Thus f is diff. at all points and $f'(x_0)=a$ for all x_0.

<u>Example 2.</u> Consider a quadratic, say $f(x)=x^2$. Then

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{x^2-x_0^2}{x-x_0}$$

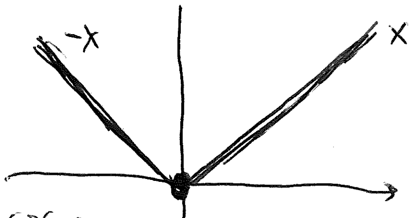
$$= \lim_{x\to x_0} \frac{\cancel{(x-x_0)}(x+x_0)}{\cancel{x-x_0}} = \lim_{x\to x_0} (x+x_0) = 2x_0.$$

This gives $f'(x_0)=2x_0$ or simply $f'(x)=2x$.

Absolute value function. Define $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$.

This is denoted by $|x| \geq 0$. Note that
$$|a \cdot b| = |a| \cdot |b| \quad \text{and} \quad |x| = |-x| \quad \text{by definition.}$$

This function satisfies $f'(x) = 1$ if $x > 0$
$f'(x) = -1$ if $x < 0$



but $f'(0)$ is not defined because of the corner:
$$f'(0) = \lim_{x \to 0} \frac{f(x) - \cancel{f(0)}}{x - \cancel{0}} = \lim_{x \to 0} \frac{|x|}{x} \text{ does not exist as } \frac{|x|}{x} = \pm 1.$$

Theorem (Differentiable $\Rightarrow$ Continuous)
If f is differentiable at x_0, then f is continuous at x_0.

Proof. We know $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists.

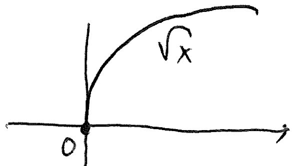
Then
$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$$
$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) = f'(x_0) \cdot 0 = 0. \quad \blacksquare$$

Example 3. We compute the derivative of $f(x) = 1/x$, $x \neq 0$.

We have
$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0} = \frac{\frac{x_0 - x}{x x_0}}{x - x_0} = \frac{\cancel{x_0 - x}^{\,-1}}{\cancel{(x - x_0)} x x_0}$$
$$\Rightarrow f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \left(-\frac{1}{x x_0} \right) = -\frac{1}{x_0^2}.$$

We'll write $f'(x) = -1/x^2$.

Example 4. We compute $f'(x)$ when $f(x) = \sqrt{x}$. We get
$$\lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{\sqrt{x} - \sqrt{x_0}}}{\cancel{(\sqrt{x} - \sqrt{x_0})}(\sqrt{x} + \sqrt{x_0})} = \frac{1}{2\sqrt{x_0}} \quad \text{for all } x_0 > 0.$$



When $x_0 = 0$, we get $\lim_{x \to 0+} \frac{1}{\sqrt{x}} = +\infty$.

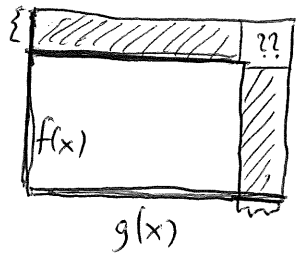
__Theorem__ (Derivatives of sums & constant multiples)

- If f, g are differentiable at x, so is $f+g$ and
$$[f(x)+g(x)]' = f'(x) + g'(x).$$
- If f is differentiable at x_0, and c is a fixed constant,
$$[c \cdot f(x)]' = c \cdot f'(x).$$

__Proof.__ By definition, the derivative is a limit and $h(x) = f(x)+g(x)$ satisfies
$$h'(x_0) = \lim_{x\to x_0} \frac{h(x)-h(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x)+g(x)-f(x_0)-g(x_0)}{x-x_0}$$
$$= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} = f'(x_0)+g'(x_0).$$
This proves the first part. For the second part, let $h(x) = c\cdot f(x)$ to get
$$h'(x_0) = \lim_{x\to x_0} \frac{h(x)-h(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{c\cdot f(x) - c f(x_0)}{x-x_0}$$
$$= \boxed{\lim_{x\to x_0} c} \cdot \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$
$$= c f'(x_0). \quad \square$$

__Examples__
$$(3x^2+4x+2)' = (3x^2)' + (4x+2)' = 3(x^2)' + 4 = 3\cdot 2x + 4$$
$$= 6x+4$$
and
$$\left(\sqrt{x} + \frac{5}{x}\right)' = \frac{1}{2\sqrt{x}} + 5\left(\frac{1}{x}\right)' = \frac{1}{2\sqrt{x}} - \frac{5}{x^2}, \quad \text{if } x>0.$$

$\boxed{\text{Product rule}}$ Suppose f, g are differentiable at x_0. Then $h = f\cdot g$ is also and one has
$$h'(x_0) = f'(x_0)\cdot g(x_0) + f(x_0)\cdot g'(x_0)$$



__Proof.__ We have
$$h'(x_0) = \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}$$
$$= \lim_{x\to x_0} \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x-x_0}$$

$$= \lim_{x \to x_0} \left[f(x) \cdot \frac{g(x)-g(x_0)}{x-x_0} + g(x_0) \cdot \frac{f(x)-f(x_0)}{x-x_0} \right]$$

$$= \underbrace{f(x_0)}_{\text{by continuity}} \cdot g'(x_0) + g(x_0) \cdot f'(x_0). \quad \blacksquare$$

<u>Derivatives of powers.</u> If n is a positive integer, then $\underline{(x^n)' = nx^{n-1}}$.

Proof. (by induction) When $n=1$, we get $(x^1)' = x' = 1 = 1x^0$.

Assume for n. To prove it for $n+1$, we have

$$(x^{n+1})' = (x \cdot x^n)' = x' \cdot x^n + (x^n)' \cdot x$$

$$= x^n + n x^{n} = (1+n) x^n. \quad \blacksquare$$

Product rule. One has $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

Quotient rule. One has $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$.

Proof. Let $H(x) = \frac{f(x)}{g(x)}$ for simplicity. Then

$$H'(x_0) = \lim_{x\to x_0} \frac{H(x)-H(x_0)}{x-x_0} \quad \text{with} \quad H(x)-H(x_0) = \frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)},$$

so $$H(x)-H(x_0) = \frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}$$

so $$\frac{H(x)-H(x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{g(x)\cdot(x-x_0)} + \frac{f(x_0)\cdot[g(x_0)-g(x)]}{g(x)g(x_0)\cdot(x-x_0)}.$$

Taking the limit as $x \to x_0$ gives
$$H'(x_0) = \frac{f'(x_0)}{g(x_0)} - \frac{f(x_0)g'(x_0)}{g(x_0)^2} = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$
because $\lim_{x\to x_0} g(x) = g(x_0)$ by continuity. ▨

Example. Let $f(x) = \frac{x}{x^2+1}$. Then $f'(x) = \frac{1\cdot(x^2+1) - 2x\cdot x}{(x^2+1)^2}$

So $$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2}.$$

Derivatives of trigonometric functions All six trigonometric functions are differentiable and their derivatives are:

$(\sin x)' = \cos x$ .... $(\tan x)' = \sec^2 x$ .... $(\sec x)' = \sec x \tan x$

$(\cos x)' = -\sin x$ .... $(\cot x)' = -\csc^2 x$ .... $(\csc x)' = -\csc x \cot x$

▨ The main formulas are those for sine and cosine.

For instance $$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - (\cos x)'\sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

and $$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\cancel{0\cdot\cos x} - (\cos x)'\cdot 1}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan x.$$

Example. $(x^2 \cdot \sin x)' = (x^2)' \sin x + x^2 (\sin x)' = 2x \sin x + x^2 \cos x$.

$\boxed{\text{Proof of } (\sin x)' = \cos x.}$ The proof is a bit tricky.

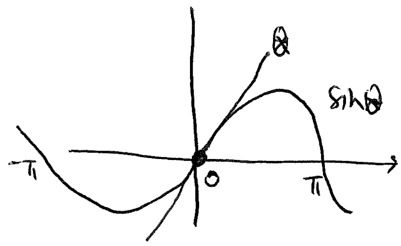
Let $f(x) = \sin x$ and write $f'(x_0) = \lim_{x \to x_0} \frac{\sin x - \sin x_0}{x - x_0}$.

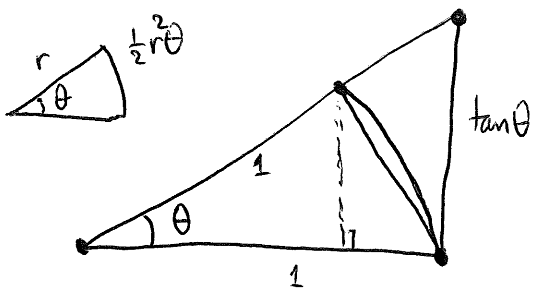
We introduce the difference $z = x - x_0$ (which goes to zero).

Then
$$f'(x_0) = \lim_{z \to 0} \frac{\sin(z + x_0) - \sin x_0}{z}$$
$$= \lim_{z \to 0} \frac{\sin z \cos x_0 + \cos z \sin x_0 - \sin x_0}{z}$$
$$= \cos x_0 \cdot \boxed{\lim_{z \to 0} \frac{\sin z}{z}} \to 1 + \sin x_0 \cdot \boxed{\lim_{z \to 0} \frac{\cos z - 1}{z}} \to 0$$

We claim these limits are 1 and 0, respectively. The proof of the first is geometric and the second follows since
$$\lim_{z \to 0} \frac{\cos z - 1}{z} \cdot \frac{\cos z + 1}{\cos z + 1} = \lim_{z \to 0} \frac{\cos^2 z - 1}{z \cdot 2} = \lim_{z \to 0} \frac{-\sin^2 z}{2z} =$$
$$= \lim_{z \to 0} -\frac{1}{2} \cdot \frac{\sin z}{z} \cdot \sin z = 0$$
by continuity of sine & cosine. Assuming this, we get $f'(x_0) = \cos x_0$. ▨

$\boxed{\text{Proof of } \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.}$





Area of small triangle $\le$ Area of sector $\le$ Area of big triangle

$$\Rightarrow \frac{1}{2} \cdot 1 \cdot \sin\theta \le \frac{1}{2} \cdot 1^2 \cdot \theta \le \frac{1}{2} \cdot 1 \cdot \tan\theta$$
$$\Rightarrow \sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta}$$
$$\Rightarrow \cos\theta \le \frac{\sin\theta}{\theta} \le 1 \quad \text{when } 0 < \theta < \pi/2$$
$$\Rightarrow \cos\theta \le \frac{\sin\theta}{\theta} \le 1 \quad \text{when } -\pi/2 < \theta < \pi/2.$$

As $\theta \to 0$, we get $\cos\theta \to \cos 0 = 1$ and $\sin\theta/\theta \to 1$ by Squeeze Law. ▨

$\boxed{\text{Derivative of exponential functions}}$

Take $f(x) = a^x$ with $a > 0$. To compute $f'(x_0)$, we write

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{a^x - a^{x_0}}{x - x_0} \quad \text{and let } \boxed{z = x - x_0} \; \boxed{x = z + x_0}$$

Then $f'(x_0) = \lim\limits_{z \to 0} \dfrac{a^{z+x_0} - a^{x_0}}{z} = a^{x_0} \cdot \lim\limits_{z \to 0} \dfrac{a^z - 1}{z}$.

This gives $\boxed{f'(x_0) = a^{x_0} \cdot \text{Constant}}$ with $\text{Constant} = \lim\limits_{z \to 0} \dfrac{a^z - 1}{z}$.

The formula becomes simple if $\text{Constant} = 1$, in which case

$$(a^{x_0})' = a^{x_0} \quad \text{or} \quad (a^x)' = a^x.$$

We need $\dfrac{a^z - 1}{z} \simeq 1$ or $a^z \approx z + 1$ or $a \simeq (1+z)^{1/z}$.

Take $\boxed{a = \lim\limits_{z \to 0} (1+z)^{1/z}}$ and then we get $(a^x)' = a^x$.

We usually denote this constant by $\boxed{e = \lim\limits_{z \to 0} (1+z)^{1/z}.}$

Roughly speaking $z = -1/2$ ---- gives $(1+z)^{1/z} = (1 - 1/2)^{-2} = 4$

$z = 1/2$ ---- gives $(1+z)^{1/z} = (1 + 1/2)^{2} = 2.25$.

Actually, $e \simeq 2.71828\ldots$ approximately.

<u>Definition</u> The exponential function $f(x) = e^x$ satisfies $(e^x)' = e^x$ and the inverse of this function is the logarithmic function $g(x) = \ln x = \log_e x$. Every other $\log_a$ function satisfies

$$\log_a x = \frac{\ln x}{\ln a}.$$

The formula follows by noting that

$$\ln x = \ln a^{\log_a x} = \log_a x \cdot \ln a.$$

$\boxed{\text{Derivatives of inverse functions}}$ Suppose $f : A \to B$ is bijective and differentiable. Let $g : B \to A$ be the inverse function. Then

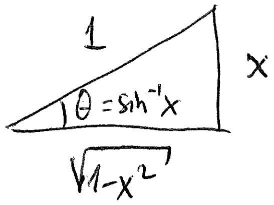
$$g'(x) = \frac{1}{f'(g(x))} \quad \text{as long as } f'(g(x)) \neq 0.$$

Example 1. Consider $f(x) = e^x$ and $g(x) = \ln x$. Then

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cancel{f}(\cancel{g}(x))} = \frac{1}{x} \quad \text{and} \quad (\ln x)' = 1/x.$$

Example 2. Consider $f(x) = \sin x$ and $g(x) = \sin^{-1} x$, where $-1 \le x \le 1$.

Then $$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\sin^{-1} x)}.$$



When $x \geq 0$, we get $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

When $x \leq 0$, we get $\cos(\sin^{-1}(-x)) = \sqrt{1-(-x)^2}$

$\cos(\sin^{-1} x) = \sqrt{1-x^2}$, so the formula still holds.

Thus $$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \quad \text{for all } -1 < x < 1.$$

Example 3. Consider $f(x) = \tan x$ with $x \in \mathbb{R}$ and $g(x) = \tan^{-1} x$ with $x \in (-\pi/2, \pi/2)$. Then $$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sec^2(\tan^{-1} x)}.$$ We know

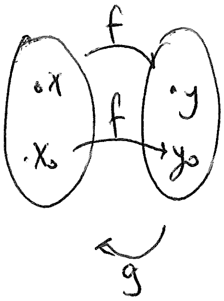
$$\sin^2 x + \cos^2 x = 1 \implies \tan^2 x + 1 = \sec^2 x$$

$$\implies \sec^2(\tan^{-1} x) = 1 + \tan^2(\tan^{-1} x) = 1 + x^2.$$

$(x^n)' = nx^{n-1}$	$(1/x)' = -1/x^2$	$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	$(\sin x)' = \cos x$
$(\cos x)' = -\sin x$	$(\sec x)' = \sec x \tan x$	$(\tan x)' = \sec^2 x$	$(e^x)' = e^x$
$(\ln x)' = 1/x$	$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$	$(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$	$(\tan^{-1} x)' = \frac{1}{1+x^2}$

Proof of main formula $\quad g'(x) = \dfrac{1}{f'(g(x))}$ whenever $g = f^{-1}$.

Let $y_0 = f(x_0)$ and $y = f(x)$ so that $x = g(y)$.
We define $z = x - x_0$ for simplicity and $w = y - y_0$. We compute



$$g'(y_0) = \lim_{y\to y_0} \frac{g(y)-g(y_0)}{y-y_0}$$

$$= \lim_{y\to y_0} \frac{\cancel{g(f}(x_0+z)) - \cancel{g(f}(x_0))}{f(x) - f(x_0)}$$

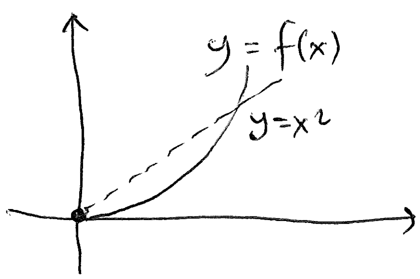
$$= \lim_{y\to y_0} \frac{z}{f(x)-f(x_0)} = \lim_{y\to y_0} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}} .$$

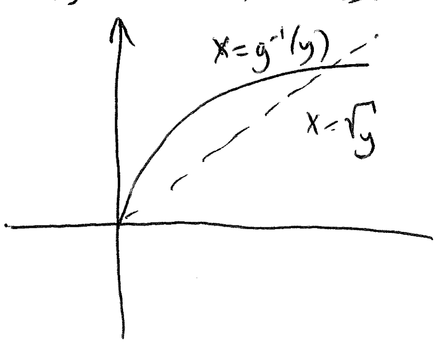
If we had the limit $\lim\limits_{x\to x_0}$, this would be

$$\frac{1}{\lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}$$

It thus remains to check $y \to y_0$ implies $x \to x_0$,
namely $y \to y_0$ implies $g(y) \to g(y_0)$.

This holds because $g = f^{-1}$ is continuous. ▨





$\boxed{\text{Chain rule.}}$ This rule applies to compositions of functions.

Suppose f is diff. at x_0 and g is diff. at $f(x_0)$. Then the composition $f \circ g$ is diff. at x_0 and

$$[f(g(x))]' = f'(g(x)) \cdot g'(x).$$

For instance

$$[\sin(g(x))]' = \cos(g(x)) \cdot g'(x),$$

$$[\ln g(x)]' = \frac{1}{g(x)} \cdot g'(x),$$

and

$$\left(e^{g(x)}\right)' = e^{g(x)} \cdot g'(x).$$

Example 1. Take $f(x) = \tan(\ln x)$. Then $f'(x) = \sec^2(\ln x) \cdot \frac{1}{x}$.

Example 2. Take $f(x) = \sin(3x^2+x+1)$. Then

$$f'(x) = \cos(3x^2+x+1) \cdot (3x^2+x+1)' = (6x+1) \cdot \cos(3x^2+x+1)$$

Example 3. Take $f(x) = \sin^{-1}(\sqrt{x})$. Since $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$, we get

$$f'(x) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot (\sqrt{x})' = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}.$$

Example 4. Take $f(x) = \sec(\tan(3x))$. By the chain rule

$(\sec x)' = \sec x \tan x$

$(\tan x)' = \sec^2 x$

$$\begin{aligned} f'(x) &= \underbrace{\sec(\tan(3x)) \cdot \tan(\tan(3x))} \cdot (\tan(3x))' \\ &= \sec(\tan(3x)) \cdot \tan(\tan(3x)) \cdot \underbrace{\sec^2(3x) \cdot (3x)'} \\ &= 3\sec(\tan(3x)) \cdot \tan(\tan(3x)) \cdot \sec^2(3x). \end{aligned}$$

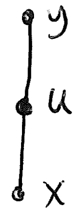
Example 5. Take $f(x) = e^{\tan^{-1}(\cos(x^2+1))}$. Then

$$\begin{aligned} f'(x) &= e^{\tan^{-1}(\cos(x^2+1))} \cdot \left[\tan^{-1}(\cos(x^2+1))\right]' \\ &= e^{\tan^{-1}(\cos(x^2+1))} \cdot \frac{1}{1+\cos^2(x^2+1)} \cdot \left[\cos(x^2+1)\right]' \end{aligned}$$

$(\tan^{-1} x)' = \frac{1}{1+x^2}$

$$= e^{\tan^{-1}(\cos(x^2+1))} \cdot \frac{1}{1+\cos^2(x^2+1)} \left[-\sin(x^2+1)\right] \cdot 2x .$$

<u>Leibniz notation</u> When dealing with several functions at the same time, one may use the notation $y' = \frac{dy}{dx}$ instead of $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$. It is used when $y = f(u)$ and $u = g(x)$, in which case $y = f(g(x))$ and one needs to introduce $\frac{dy}{du} = f'(u)$ and also $\frac{dy}{dx} = f'(g(x)) \cdot g'(x) = f'(u) \cdot g'(x)$



so $$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

<u>Proof of chain rule.</u> We have $y = f(g(x)) = f(u)$ with $u = g(x)$.

The derivative at x_0 is

$$\lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} = \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \cdot \frac{g(x) - g(x_0)}{x - x_0}$$

$$= \underbrace{\lim_{x\to x_0} \frac{f(u) - f(u_0)}{u - u_0}}_{\text{since } x\to x_0,\ g(x)\to g(x_0) \text{ by cont.}} \cdot \underbrace{\lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0}}_{\text{clearly } g'(x_0)}.$$

So $u \to u_0$ and $\lim_{u\to u_0} \frac{f(u)-f(u_0)}{u-u_0} = f'(u_0) = f'(g(x_0))$. ∎

<u>Example 6.</u> Suppose $y = 3u^2$, $u = \sin(\cos x)$. Then

$$\frac{dy}{du} = 6u,$$

$$\frac{du}{dx} = \cos(\cos x) \cdot (\cos x)' = -\sin x \cdot \cos(\cos x)$$

and $$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -6u \cdot \sin x \cdot \cos(\cos x)$$

Example 7. Suppose $y = \sqrt{u}$, $u = \frac{z+1}{z-1}$, $z = \tan(\sec x)$.

Strictly speaking $y = \sqrt{\frac{\tan(\sec x)+1}{\tan(\sec x)-1}}$ but this is difficult to diff.

Let's compute

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

$$\frac{du}{dz} = \frac{z-1-(z+1)}{(z-1)^2} = \frac{-2}{(z-1)^2}$$

$$\frac{dz}{dx} = \sec^2(\sec x)\cdot(\sec x)' = \sec^2(\sec x)\cdot\sec x\cdot\tan x.$$

Then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dz}\cdot\frac{dz}{dx}$$

$$= \frac{-1}{\sqrt{u}\,(z-1)^2}\cdot\sec^2(\sec x)\cdot\sec x\cdot\tan x.$$

Derivative of inverse functions: $f(x)$ and $g(x)$ = inverse of f.

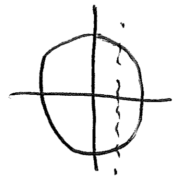
Then $f(g(x)) = x$

$$f'(g(x))\cdot g'(x) = 1$$

So $$g'(x) = \frac{1}{f'(g(x))}.$$

Implicit differentiation. Suppose that we are given a formula that relates x and y such as $x^3 + y^3 + x\sin y = 2$. This formula expresses y as a function of x, but it may be difficult to solve. We can still compute $\frac{dy}{dx} = y'$ easily.

Example 1. Suppose $x^2 + y^2 = 1$. We compute y' by differentiating both sides. We get $2x + 2y \cdot y' = 0$ and so $x + yy' = 0 \Rightarrow y' = -x/y$.



Example 2. Suppose $x^3 + \cos y = y^4 - \sec x$.

Then
$$3x^2 - \sin y \cdot y' = 4y^3 \cdot y' - \sec x \tan x$$
$$\Rightarrow y'(4y^3 + \sin y) = 3x^2 + \sec x \tan x$$
$$\Rightarrow y' = \frac{3x^2 + \sec x \tan x}{4y^3 + \sin y}.$$

Example 3. Suppose $\sin(xy) + \ln(xy^2) = x \cdot y$.

Then
$$\cos(xy) \cdot (xy)' + \frac{1}{xy^2} \cdot (xy^2)' = y + xy'$$
$$\Rightarrow \cos(xy) \cdot (y + xy') + \frac{1}{xy^2}(y^2 + 2yy'x) = y + xy'$$
$$\Rightarrow y\cos(xy) + xy'\cos(xy) + \frac{1}{x} + \frac{2y'}{y} = y + xy'$$
$$\Rightarrow y'\left[x\cos(xy) + \frac{2}{y} - x\right] = y - 1/x - y\cos(xy)$$
$$\Rightarrow y' = \frac{y - 1/x - y\cos(xy)}{x\cos(xy) + 2/y - x}.$$

Example 4. Suppose $\tan\frac{x}{y} + \sqrt{\cos(xy)} = x$. Then
$$\sec^2\frac{x}{y}\left(\frac{x}{y}\right)' + \frac{1}{2\sqrt{\cos(xy)}} \cdot [\cos(xy)]' = 1$$
$$\Rightarrow \sec^2\frac{x}{y} \cdot \frac{y - xy'}{y^2} - \frac{\sin(xy)}{2\sqrt{\cos(xy)}} \cdot (xy)' = 1$$

$$\Rightarrow \quad \sec^2\frac{x}{y}\cdot\frac{1}{y} - \frac{xy'}{y^2}\cdot\sec^2\frac{x}{y} - \frac{y\sin(xy)}{2\sqrt{\cos(xy)}} - \frac{y'x\sin(xy)}{2\sqrt{\cos(xy)}} = 1$$

$$\Rightarrow \quad y'\left[\frac{x}{y^2}\sec^2\frac{x}{y} + \frac{x\sin(xy)}{2\sqrt{\cos(xy)}}\right] = \frac{1}{y}\sec^2\frac{x}{y} - \frac{y\sin(xy)}{2\sqrt{\cos(xy)}} - 1$$

and we can solve for y'.

$\boxed{\text{Logarithmic differentiation}}$ We can use $\ln$ to simplify

$$\ln(xy) = \ln x + \ln y, \quad \ln\frac{x}{y} = \ln x - \ln y, \quad \ln x^r = r\ln x.$$

This allows us to deal with products, quotients, powers more easily.

Example 1. We compute the derivative of x^n for any n.

Let $y = x^n \Rightarrow \ln y = \ln x^n = n\ln x$

$$\Rightarrow \quad (\ln y)' = (n\ln x)'$$

$$\Rightarrow \quad \frac{1}{y}\cdot y' = \frac{n}{x}$$

$$\Rightarrow \quad y' = \frac{n}{x}\cdot y = \frac{n}{x}\cdot x^n = n x^{n-1}.$$

This proves the formula $(x^n)' = nx^{n-1}$ for any number n.

For instance $(1/x)' = (x^{-1})' = -1x^{-2} = -1/x^2$

and $(1/x^2)' = (x^{-2})' = -2x^{-3} = -2/x^3$.

$\boxed{\text{Note}}$ The formula $(\ln x)' = \frac{1}{x}$ only applies when $x > 0$.

However, one also has $\boxed{[\ln|x|]' = 1/x}$ for all $x \neq 0$.

This is because $[\ln(-x)]' = \frac{1}{-x}(-x)' = 1/x$ when $x < 0$.

One could thus argue that

$$y = x^n \Rightarrow \ln|y| = \ln|x|^n = n\ln|x| \Rightarrow \frac{1}{y}y' = \frac{n}{x} \quad \text{as before.}$$

Example 2. Consider $y = \dfrac{(x^2+1)^3 \cdot \sqrt{x^4+2x} \cdot \sec^2 x}{(3\cos x - 4\tan x)^4}$

Then $\ln y = \ln (x^2+1)^3 + \ln \sqrt{x^4+2x} + \ln \sec^2 x - \ln (3\cos x - 4\tan x)^4$

so $\ln y = 3 \ln (x^2+1) + \frac{1}{2} \ln (x^4+2x) + 2 \ln(\sec x) - 4 \ln (3\cos x - 4\tan x)$

So $\frac{1}{y} y' = \dfrac{3 \cdot 2x}{x^2+1} + \frac{1}{2} \dfrac{4x^3+2}{x^4+2x} + \dfrac{2 \cdot \cancel{\sec x} \tan x}{\cancel{\sec x}} - \dfrac{4(-3\sin x - 4\sec^2 x)}{3\cos x - 4\tan}$

so $y' = y \left(\dfrac{6x}{x^2+1} + \dfrac{2x^3+1}{x^4+2x} + 2\tan x + \dfrac{4(3\sin x + 4\sec^2 x)}{3\cos x - 4\tan x} \right).$

Example 3. Consider $y = a^x$, the exponential function with base a.
We checked $(a^x)' = Ca^x$, where $C = \lim_{z \to 0} \dfrac{a^z - 1}{z}$ is a constant.
In fact, $y = a^x \Rightarrow \ln y = \ln a^x = x \ln a$

$\Rightarrow \frac{1}{y} \cdot y' = \ln a \Rightarrow y' = y \ln a$
$= a^x \cdot \ln a.$

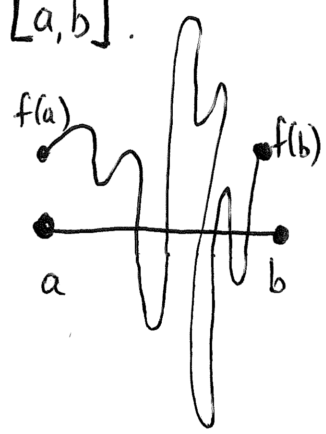
Example 4. Consider $y = x^x$. Note that $(x^n)' = nx^{n-1}$ only applies when n is constant. In this case,

$$\ln y = \ln x^x = x \cdot \ln x$$

$$\Rightarrow \frac{1}{y} \cdot y' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

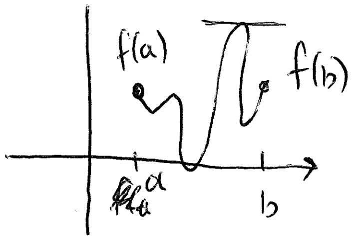
$$\Rightarrow y' = y(\ln x + 1) = x^x (\ln x + 1)$$

Extreme value theorem. Suppose f is continuous on $[a,b]$.
Then f must attain a minimum value $f(x_1)$ and a maximum value $f(x_2)$ on $[a,b]$.



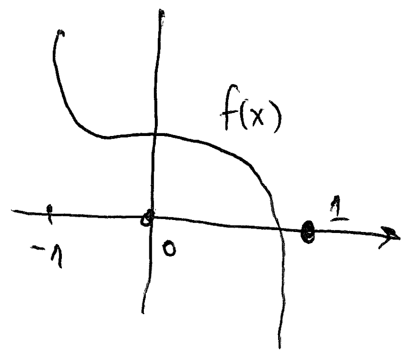
Proof. (Postponed for MA1126).

Rolle's theorem. Suppose f is differentiable on $[a,b]$ and continuous on (a,b) with $f(a)=f(b)$.
Then there exists a point $a<c<b$ with $f'(c)=0$.



Example 1. We show $f(x)=x^5-5x^3+1$ has a root in $(0,1)$.
In fact, we show $f(x)$ has exactly one such root.

Existence of roots: We use Bolzano's theorem.
We know f is continuous on $[0,1]$, while $f(0)=1$ is positive and $f(1)=-3$ is negative.
Thus f has a root $0<c<1$.



Uniqueness of roots: Suppose we have two roots in $(0,1)$, say $0<a<b<1$.
Then $f(a)=0=f(b)$ and f is diff/continuous on $[a,b]$.
By Rolle's theorem, $f'(x)=5x^4-15x^2$ must have a root in (a,b). But $f'(x)=5x^2(x^2-3)$ has roots $x=0,\pm\sqrt{3}$ and none of those lies in $(0,1)$. This is a contradiction that implies f has only one root in $(0,1)$.

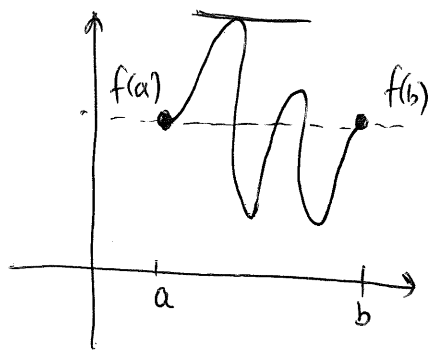
Remark. Between any two roots of f lies a root of f'.
Between any n roots of f lie $n-1$ roots of f'.

Example 2. Consider $f(x) = x^3 + (a+1)x + a^2 - 1$ with $a>0$.
One can only graph this function for given values of a.
We show there is a <u>unique</u> real root in $(-a, a+1)$.

{Existence} Once again, f is continuous on $[-a, a+1]$
with $f(-a) = -a^3 - a(\cancel{a}+1) + \cancel{a^2} - 1 = -a^3 - a - 1 < 0$
and
$$f(a+1) = (a+1)^3 + (a+1)^2 + (a+1)(a-1)$$
$$= (a+1)\left((a+1)^2 + a + \cancel{1} + a - \cancel{1}\right) > 0.$$
We thus get a root in $(-a, a+1)$ by Bolzano's.

{Uniqueness} Assume we have two roots $x_1 < x_2$.
Then f' has a root $x_1 < x_3 < x_2$ by Rolle's theorem.
In our case, $f'(x) = 3x^2 + a + 1 > 0$ for all x,
so f' does not have roots, a contradiction.

<u>Proof of Rolle's theorem.</u>

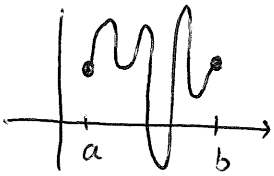


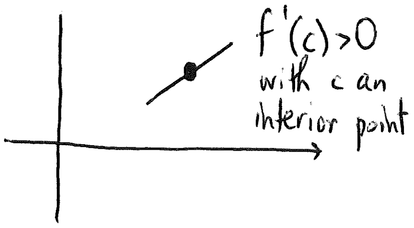
It could happen that f is constant.
In that case, $f' = 0$ at all points.
Otherwise, there is a minimum/maximum value attained. We claim that

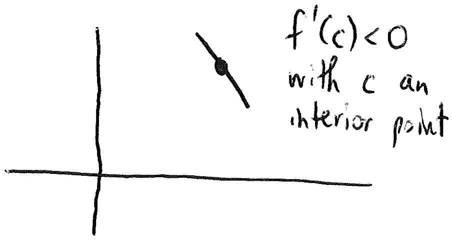
$$\boxed{\begin{array}{l} f(x_1) \text{ maximum} \Rightarrow f'(x_1) = 0. \\ \text{and } x_1 \neq a, b \end{array}}$$

Similarly for $f(x_2)$ minimum. Indeed, suppose
$$f'(x_1) = \lim_{x \to x_1} \frac{f(x) - f(x_1)}{x - x_1} > 0.$$
Then $\dfrac{f(x)-f(x_1)}{x-x_1} > 0$ when x close to x_1.
If $x < x_1$, then $f(x) < f(x_1)$.
If $x > x_1$, then $f(x) > f(x_1)$. This contradicts $f(x_1)$ being maximum.

Rolle's theorem. Suppose f differentiable on $[a,b]$ and continuous on (a,b) with $f(a)=f(b)$. Then $f'(c)=0$ for some point $a<c<b$.



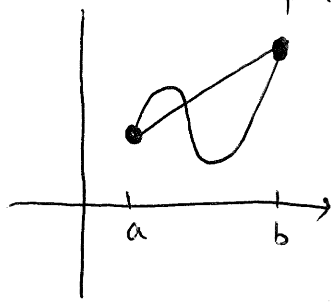




Mean value theorem. Suppose f differentiable on $[a,b]$ and continuous on (a,b).

Then $\boxed{f'(c) = \dfrac{f(b)-f(a)}{b-a}}$ for some point $a<c<b$.

In other words, instantaneous rate of change = average rate of change (at some point).



Proof. We look at the line through the points $(a,f(a))$ and $(b,f(b))$. Consider

$$\cancel{L(x) = \frac{f(b)-f(a)}{b-a}\cdot(x-a)+f(a)}$$

$$\cancel{f(b)\cdot(x-a)}$$

$$\cancel{L(x) = [f(b)-f(a)]\cdot(x-a) + (b-a)\cdot(x-}$$

$$\cancel{L(x) = \frac{f(b)-f(a)}{b-a}\cdot(x-a) + }$$

~~Then~~ $\cancel{L(a) = f(a)} \quad \cancel{L(a) = f}$

and $\cancel{L(b) = f}$

$$H(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$$

Then $H(a) = f(a) - \dfrac{f(b)-f(a)}{b-a}\cdot(a-a) - f(a) = 0$

$H(b) = f(b) - \dfrac{f(b)-f(a)}{\cancel{b-a}}\cancel{(b-a)} - f(a) = 0$

$\therefore$ $H'(c)=0$ and $f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0$ by Rolle's theorem. $\boxtimes$

Example 1. We can use the MVT to estimate $|f(x) - f(y)|$.
For instance $\underline{\underline{|\sin x - \sin y| \leq |x-y|}}$ for all x, y.
When $x=y$, this is clear. When $x \neq y$, we need $\dfrac{|\sin x - \sin y|}{|x-y|} \leq 1$.
By the MVT, $\dfrac{\sin x - \sin y}{x-y} = \dfrac{f(x)-f(y)}{x-y} = f'(c)$ for some c,
Since $f(x) = \sin x$, $f'(x) = \cos x$ and $\left|\dfrac{\sin x - \sin y}{x-y}\right| = |\cos c| \leq 1$.
We thus get $|\sin x - \sin y| \leq |x-y|$ and $|\sin x| \leq |x|$ as well.

Example 2. A similar argument gives $|\tan^{-1} x - \tan^{-1} y| \leq |x-y|$.
In this case, $f(x) = \tan^{-1} x$ and $f'(x) = \dfrac{1}{1+x^2} \leq 1$ so we can argue as before to get $\dfrac{|\tan^{-1} x - \tan^{-1} y|}{|x-y|} = |f'(c)| \leq 1$.

L'HÔPITAL'S RULE Suppose f, g are differentiable with

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty.$$

Then

$$\underline{\underline{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}}.$$

This only works for $0/0$ and ∞/∞.

Example 1. $\displaystyle \lim_{x\to 2} \frac{x^4-16}{3x-6} \overset{LHR}{=} \lim_{x\to 2} \frac{4x^3}{3} = \frac{32}{3}$.

Example 2. $\displaystyle \lim_{x\to 0} \frac{e^x-1}{\sin(2x)} \overset{LHR}{=} \lim_{x\to 0} \frac{e^x}{2\cdot\cos(2x)} = \frac{1}{2}$.

Example 3. $\displaystyle \lim_{x\to \infty} \frac{\ln x}{x^2+1} \overset{LHR}{=} \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$.

Example 4. $\displaystyle \lim_{x\to \infty} \frac{4x^2+3x+1}{e^x} \overset{LHR}{=} \lim_{x\to\infty} \frac{8x+3}{e^x} \overset{LHR}{=} \lim_{x\to\infty} \frac{8}{e^x} = 0.$

Other types of limits

- $0/0$ and ∞/∞ $\rightsquigarrow$ L'Hôpital's rule
- $0 \cdot \infty$ $\rightsquigarrow$ rearrange terms to get either $0/0$ or ∞/∞.
- Limits involving exponents like 1^∞ and ∞^0 and 0^0 can be treated using logarithms.

Example 5.

$$\lim_{x\to 0^+} \underset{\substack{\downarrow \\ 0}}{x} \cdot \underset{\substack{\downarrow \\ -\infty}}{\ln x} = \lim_{x\to 0^+} \frac{\ln x}{1/x} \quad \cdots\cdots \quad \frac{-\infty}{+\infty}$$

$$\overset{LHR}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

L'Hôpital's rule for $0/0$ and ∞/∞ limits

Suppose f, g are differentiable and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$ is either 0 or $\pm\infty$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. This is also true for one-sided limits and limits at infinity.

<u>Proof.</u> We shall prove the $x\to 0^+$ case and deduce the other cases.

Namely,
$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{z\to 0^+} \frac{f(z+a)}{g(z+a)} = \lim_{z\to 0^+} \frac{f'(z+a)}{g'(z+a)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \quad \cdots \quad z = x-a$$

$$\lim_{x\to +\infty} \frac{f(x)}{g(x)} = \lim_{z\to 0^+} \frac{f(1/z)}{g(1/z)} = \lim_{z\to 0^+} \frac{f'(1/z)\,\cancel{(1/z)'}}{g'(1/z)\,\cancel{(1/z)'}} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \quad \cdots \quad z = 1/x.$$

The other cases are similar. Suppose now that $x\to 0^+$ and $L=0$.

We apply <u>Rolle's theorem</u> to $H(x) = f(x)g(x_0) - f(x_0)g(x)$, where x_0 is fixed. This is differentiable on $[0, x_0]$ with

$$H(0) = \overset{0}{\cancel{f(0)}}\,g(x_0) - f(x_0)\,\overset{0}{\cancel{g(0)}} = 0$$
$$H(x_0) = f(x_0)g(x_0) - f(x_0)g(x_0) = 0.$$

We can apply Rolle's theorem to get $H'(x)=0$ for some $0<x<x_0$.

Thus
$$H'(x) = f'(x)g(x_0) - f(x_0)g'(x) = 0$$
and
$$f'(x)g(x_0) = f(x_0)g'(x)$$
and
$$\frac{f'(x)}{g'(x)} = \frac{f(x_0)}{g(x_0)} \quad \cdots \quad \text{for some } 0<x<x_0.$$

We take $x_0 \to 0$ and $x\to 0$ to get $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x_0\to 0^+} \frac{f(x_0)}{g(x_0)}$.

- This proves the case $L=0$. For the case $L=\infty$, $\lim_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{\infty}{\infty}$ $\Rightarrow$ write it as $\lim_{x\to a} \frac{1/g(x)}{1/f(x)}$ (which is $0/0$)

Thus
$$\lim_{x\to a} \frac{f}{g} = \lim_{x\to a} \frac{1/g}{1/f} = \lim_{x\to a} \frac{-g(x)^{-2} g'(x)}{-f(x)^{-2} f'(x)} = \lim \frac{f(x)^2 g'(x)}{g(x)^2 f'(x)}$$

and so $\lim_{x\to a} \frac{g(x)}{f(x)} = \lim \frac{g'(x)}{f'(x)}$ by rearranging terms. ∎

Computations of limits

① $0/0$ and ∞/∞ $\longrightarrow$ use L'Hôpital's rule.

② $0 \cdot \infty$ limits $\longrightarrow$ express as $0/0$ and ∞/∞ by rearranging.

③ 1^∞, 0^0, ∞^0 limits $\longrightarrow$ introduce $\ln$ to eliminate exponents.

Example 1. Consider $L = \lim\limits_{z \to 0} (1+az)^{1/z}$, where a is fixed.

This is 1^∞. Write

$$\ln L = \ln \lim_{z\to 0} (1+az)^{1/z}$$

$$= \lim_{z \to 0} \ln (1+az)^{1/z} \quad \text{since } \ln \text{ is continuous}$$

$$= \lim_{z\to 0} \frac{1}{z} \cdot \ln(1+az).$$

This is $\infty . \ln 1 = \infty . 0$. We rearrange terms to get

$$\ln L = \lim_{z \to 0} \frac{\ln(1+az)}{z} \quad \left(\text{which is } \tfrac{0}{0} \text{ now}\right)$$

$$= \lim_{z\to 0} \frac{\frac{1}{1+az} \cdot a}{1} = a.$$

This proves $\ln L = a \Rightarrow e^{\ln L} = e^a \Rightarrow L = e^a$.

Example 2. Consider $L = \lim\limits_{x \to \infty} \dfrac{\ln x}{x^p}$, where $p > 0$ is fixed.

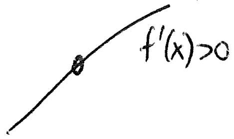
This is $\dfrac{\infty}{\infty}$ and so

$$L = \lim_{x \to \infty} \frac{(\ln x)'}{(x^p)'} = \lim_{x\to\infty} \frac{1/x}{p x^{p-1}}$$

$$= \lim_{x \to \infty} \frac{1}{p x^p} = 0.$$

In particular, $\ln x$ grows much more slowly than x^p for any $p > 0$.

Monotonicity We say that f is increasing on some interval I when $f(a) < f(b)$ for all $a<b$ within the interval. We say that f is decreasing on some interval I when $f(a) > f(b)$ for all $a<b$ within I.



Theorem. If f is differentiable on I, then

① $f'(x) > 0$ for all $x \in I \Rightarrow f(x)$ is increasing on I

② $f'(x) < 0$ for all $x \in I \Rightarrow f(x)$ is decreasing on I.

Example. ⊙ We know $a<b \Rightarrow 5a < 5b$. This is because $f(x) = 5x$ is increasing (since $f'(x) = 5$ is positive).

⊙ On the other hand $a<b \Rightarrow -2a > -2b$ since $f(x) = -2x$ has derivative -2 and $f(x)$ is decreasing.

⊙ $a<b \Rightarrow e^a < e^b$ since e^x is increasing

⊙ $0<a<b \Rightarrow \ln a < \ln b$ since $(\ln x)' = 1/x > 0$ and $\ln x \nearrow$.

On the other hand $a<b \Rightarrow a^2 < b^2$ is NOT TRUE, in general. We need $f(x) = x^2$ to be increasing, namely $f'(x) = 2x$ to be positive.

This gives $0<a<b \Rightarrow a^2 < b^2$

and $a<b<0 \Rightarrow a^2 > b^2$.

Proof of theorem. We need to study the monotonicity of f.

Suppose $f'(x) > 0$ within I. Suppose $a<b$. We need $f(a) < f(b)$, namely $f(b) - f(a) > 0$. By the MVT,

$$\frac{f(b)-f(a)}{b-a} = f'(c) \quad \text{for some } a<c<b.$$

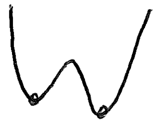
Thus $f(b) - f(a) = f'(c) \cdot (b-a) > 0$.

Monotonicity

1. If $f'(x)>0$ on some interval, then f is increasing on the interval.
2. If $f'(x)<0$ on some interval, then f is decreasing on the interval.

<u>Example</u>. Consider $f(x) = x^4 - 8x^2 + 6$, for instance.
To check for monotonicity, we compute $f'(x) = 4x^3 - 16x$ and determine its sign. We factor $f'(x) = 4x(x^2-4) = 4x(x+2)(x-2)$.
To determine the sign, we worry about each factor separately.

		-2		0		2	
$4x$	$-$		$-$		$+$		$+$
$x+2$	$-$		$+$		$+$		$+$
$x-2$	$-$		$-$		$-$		$+$
$f'(x)$	$-$		$+$		$-$		$+$

This gives $f'(x)>0$ when $x \in (-2,0) \cup (2,\infty)$,
while $f'(x)<0$ when $x \in (-\infty,-2) \cup (0,2)$.
The graph of the function looks roughly like .

<u>Example.</u> Consider $f(x) = x^3 \ln x$, for instance.
In this case,

$$f'(x) = 3x^2 \cdot \ln x + x^3 (\ln x)'$$
$$= 3x^2 \ln x + x^2 = x^2(3\ln x + 1).$$

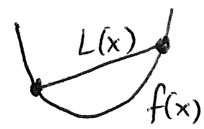
Note that $x>0$ because of $\ln x$. We have

$$f'(x)>0 \iff 3\ln x + 1 > 0$$
$$\iff 3\ln x > -1 \iff \ln x > -1/3$$
$$\iff e^{\ln x} > e^{-1/3}.$$

Thus, f is increasing when $x > e^{-1/3}$
and f is decreasing when $0<x<e^{-1/3}$. The graph looks like 

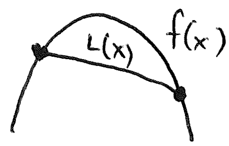
$\boxed{\text{Concavity}}$ We say that f is <u>concave up</u> on some interval I

if $$f(x) < \frac{f(b)-f(a)}{b-a}\cdot(x-a) + f(a)$$ ---- for all points $a<x<b$ in I.



We say that f is <u>concave down</u> on some interval I, if

$$f(x) > \frac{f(b)-f(a)}{b-a}\cdot(x-a) + f(a)$$ for all points $a<x<b$ in I.



<u>Theorem</u> (Concavity test) Suppose f is twice differentiable on I.

① If $f''(x)>0$ on I, then f is concave up on I.

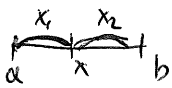
② If $f''(x)<0$ on I, then f is concave down on I.

<u>Proof.</u> If $f''(x)>0$, then $f'(x)$ is increasing. We need to prove

$$\frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(a)}{b-a} \quad \text{for all } a<x<b.$$

By the MVT, the LHS equals $f'(x_1)$ for some $a<x_1<x$.

By the MVT applied to $[x,b]$,



$$\frac{f(x)-f(b)}{x-b} = f'(x_2) \quad \text{for some } x<x_2<b.$$

Since f' is increasing,

$$f'(x_1) < f'(x_2)$$

$$\Rightarrow \frac{f(x)-f(a)}{x-a} < \frac{f(x)-f(b)}{x-b} = \frac{f(b)-f(x)}{b-x}$$

$$\Rightarrow \underline{bf(x)} - \cancel{xf(x)} - bf(a) + xf(a) < xf(b) - \cancel{xf(x)} - af(b) + \underline{\underline{af(x)}}$$

$$\Rightarrow (b-a)\ f(x) < x\,[f(b)-f(a)] + bf(a) - af(b)$$

$$\Rightarrow (b-a)\,[f(x)-f(a)] < x\,(f(b)-f(a)) + \cancel{bf(a)} - af(b) - \cancel{bf(a)} + af(a)$$

$$\Rightarrow (b-a)\,(f(x)-f(a)) < (x-a)\,(f(b)-f(a)).$$ ▨

Example. Consider $f(x) = ax^2 + bx + c$. To check concavity, we compute $f'(x) = 2ax + b$, $f''(x) = 2a$. We get f to be concave up when $a > 0$ and concave down when $a < 0$.

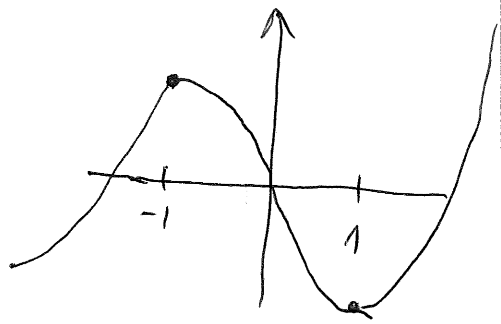
Example. Consider $f(x) = x^3 - 3x + 1$.

Then $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

and $f''(x) = 6x$.

Thus f is concave up if $x > 0$

concave down if $x < 0$.



For monotonicity,

f is increasing on $(-\infty, -1)$

$(1, \infty)$

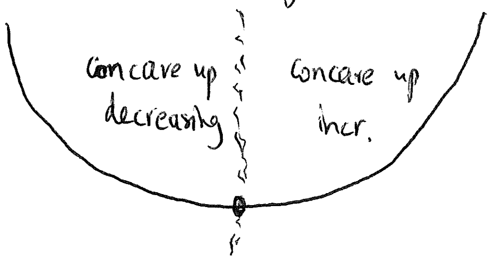
decreasing on $(-1, 1)$.

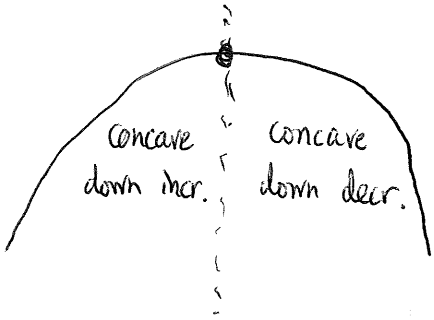
		-1		1	
$3(x-1)$	$-$		$-$		$+$
$x+1$	$-$		$+$		$+$
$f'(x)$	$+$		$-$		$+$

Monotonicity, concavity & graphing

- $f' > 0$ means f is increasing
- $f' < 0$ means f is decreasing
- $f'' > 0$ means f is concave up $\cup$
- $f'' < 0$ means f is concave down $\cap$

This gives <u>four</u> types of behaviour.





<u>Example 1.</u> Consider $f(x) = x^3 - 3x^2 - 9x$.

Then $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x+1)(x-3)$

and $f''(x) = 6x - 6 = 6(x-1)$.

The points $x = -1, 3$ are important because of f'

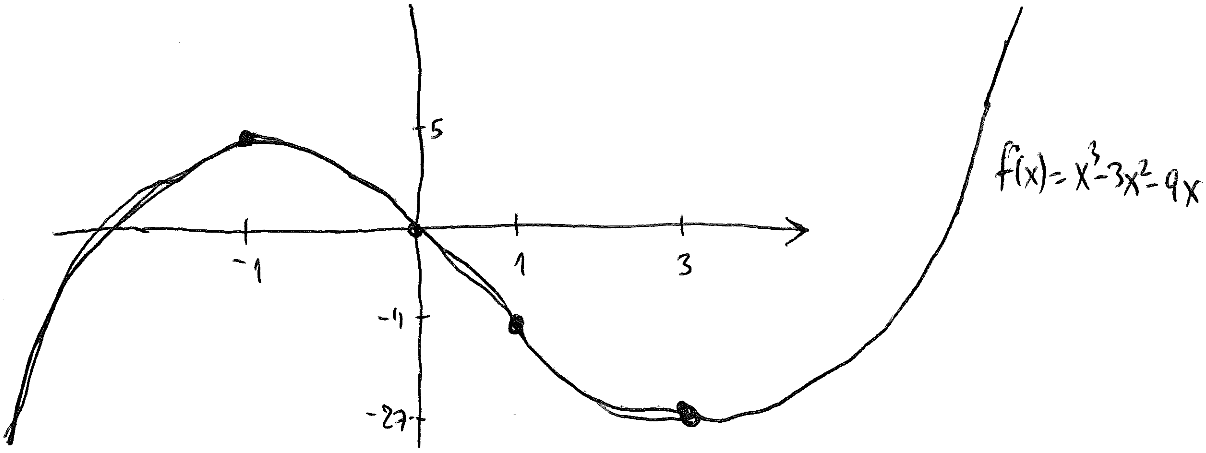
and $x = 1$ is important because of f''.

	-1	1	3	
$3(x+1)$	$-$	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$-$	$+$
$f''(x)$	$-$	$-$	$+$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

We can use this table to make a sketch of the graph.

We compute
$$f(-1) = -1 - 3 + 9 = 5$$
$$f(1) = 1 - 3 - 9 = -11$$
$$f(3) = \cancel{27} - \cancel{3\cdot 9} - 27 = -27$$

along with $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} x^3 = -\infty$ and $\lim\limits_{x\to+\infty} f(x) = +\infty$.



Example 2. Consider $f(x) = \dfrac{x^2+3}{1-x}$, where $x \neq 1$.

In this case, the quotient rule gives

$$f'(x) = \frac{2x(1-x) + x^2+3}{(1-x)^2} = \frac{-x^2+2x+3}{(1-x)^2} = \frac{-(x^2-2x-3)}{(1-x)^2} = \frac{-(x+1)(x-3)}{(1-x)^2}$$

and also

$$f''(x) = \frac{(-2x+2)(1-x)^{\cancel{2}} + 2\cancel{(1-x)}(-x^2+2x+3)}{(1-x)^{\cancel{4}\,3}}$$

$$= \frac{-\cancel{2x}+\cancel{2x^2}+2-\cancel{2x}-\cancel{2x^2}+\cancel{4x}+6}{(1-x)^3} = \frac{8}{(1-x)^3}.$$

The points of interest are $x=-1, 3$ and $x=1$ (which is not in the domain).

		-1		1		3	
$-(x+1)$	$+$		$-$		$-$		$-$
$x-3$	$-$		$-$		$-$		$+$
$f'(x)$	$-$		$+$		$+$		$-$
$f''(x)$	$+$		$+$		$-$		$-$
$f(x)$							

It remains to compute $f(-1) = \dfrac{1+3}{1+1} = 2$, $f(3) = \dfrac{3^2+3}{1-3} = -6$

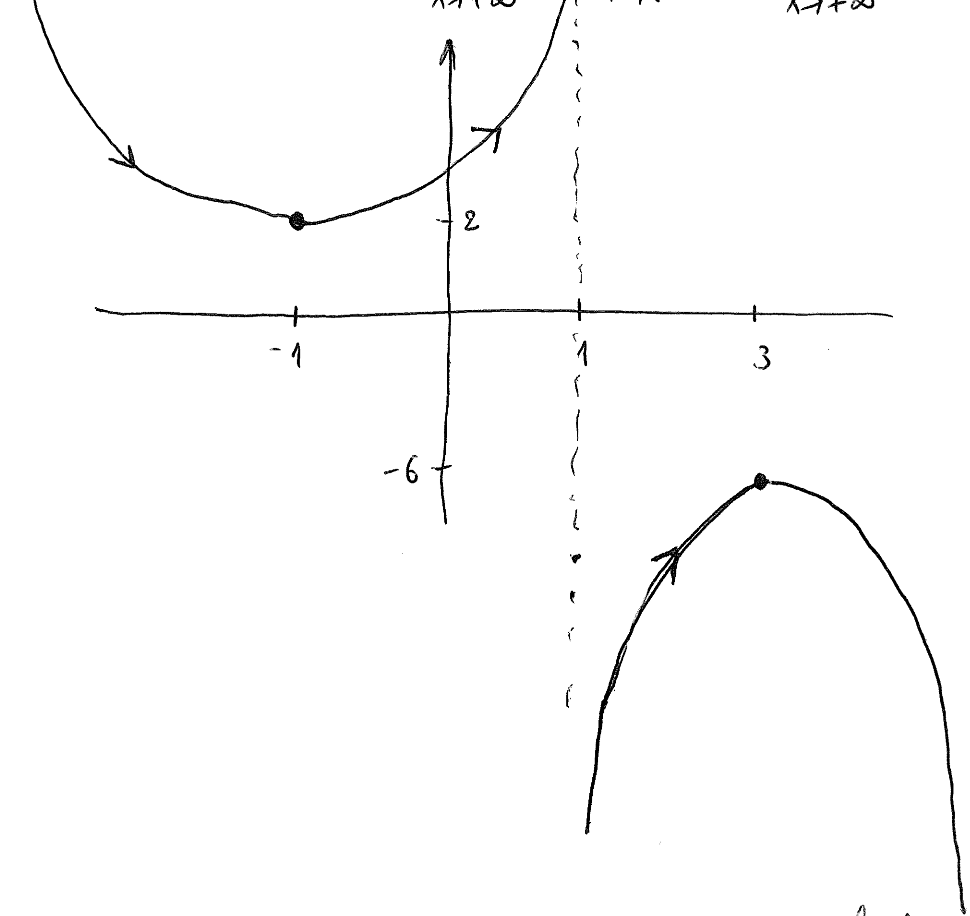
limits near $x=1$ .... $\displaystyle\lim_{x\to 1^-} \frac{x^2+3}{1-x} = \lim_{x\to 1^-} \frac{4}{1-x} = +\infty$

$$\lim_{x\to 1^+} \frac{x^2+3}{1-x} = \lim_{x\to 1^+} \frac{4}{1-x} = -\infty$$

Limits at infinity

$$\lim_{x\to-\infty} \frac{x^2+3}{1-x} = \lim_{x\to-\infty} \frac{x^2}{-x} = +\infty$$

$$\lim_{x\to+\infty} \frac{x^2+3}{1-x} = \lim_{x\to+\infty} \frac{x^2}{-x} = -\infty .$$



$$f(x) = \frac{x^2+3}{1-x}$$

This approaches $+\infty$ as $x\to-\infty$

as $x\to1^-$

and $-\infty$ as $x\to+\infty$

as $x\to1^+$.

Local minimum: is a point x_0 at which $f(x)$ attains a minimum value on some interval around x_0, namely

$$f(x) \geq f(x_0) \quad \text{for all } x \text{ in that interval.}$$

Local maximum: is defined similarly with $f(x) \leq f(x_0)$ in an interval around x_0.

Vertical asymptote: is a vertical line $x = x_0$ (with x_0 finite) such that $\lim_{x\to x_0^{\pm}} f(x)$ is infinite $\pm\infty$.

Horizontal asymptote: is a horizontal line $y = L$ (with L finite) such that $\lim_{x\to\pm\infty} f(x) = L$.

Example 3. Consider $f(x) = x^2 e^x$.

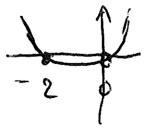
In this case, $f'(x) = \underset{\sim}{2xe^x} + x^2e^x = xe^x(2+x)$

and $f''(x) = \underset{\sim\sim\sim}{2e^x + 2xe^x} + 2xe^x + x^2e^x$

$= e^x(2+4x+x^2)$.

- To say $f'(x) > 0$ is to say $xe^x(x+2) > 0$, namely $x(x+2) > 0$.

We get roots at $x=0$, $x=-2$ and $x(x+2) > 0 \iff x \in (-\infty,-2) \cup (0,\infty)$.



		-2		0	
xe^x	$-$		$-$		$+$
$x+2$	$-$		$+$		$+$
$f'(x)$	$+$		$-$		$+$

- To say $f''(x) > 0$ is to say

$x^2+4x+2 > 0$. Since $\Delta = b^2-4ac = 4^2-4.2 = 8$,

we get two real roots $x = \dfrac{-4 \pm \sqrt{8}}{2} = \dfrac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$.

This gives $x^2+4x+2 > 0 \iff x \in (-\infty, -2-\sqrt{2}) \cup (-2+\sqrt{2}, \infty)$.

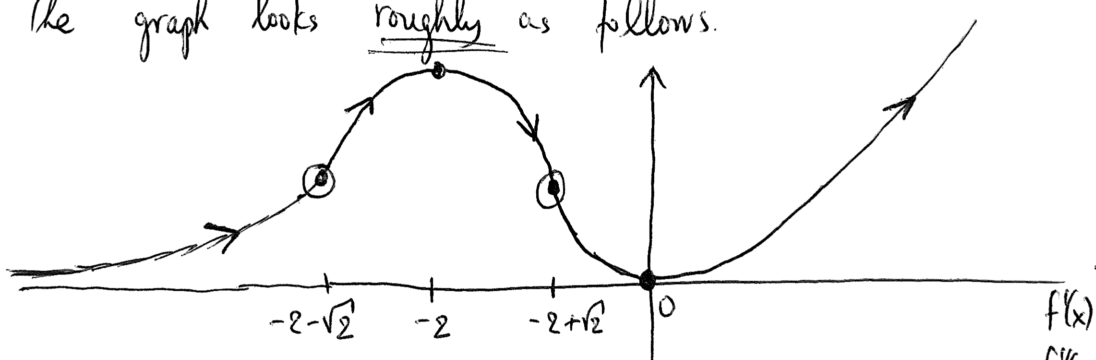
- **Behaviour as $x \to \pm\infty$.**

$\lim\limits_{x\to+\infty} x^2e^x = +\infty$ is clear.

$\lim\limits_{x\to-\infty} x^2e^x = \lim\limits_{x\to-\infty} \dfrac{x^2}{e^{-x}}$ $\left(\dfrac{\infty}{\infty} \text{ now}\right)$ $= \lim\limits_{x\to-\infty} \dfrac{2x}{-e^{-x}}$ $\left(\dfrac{-\infty}{\infty}\right)$

$= \lim\limits_{x\to-\infty} \dfrac{2}{e^{-x}} = 0$.

The graph looks roughly as follows.



$f(x) = x^2e^x$

		$-2-\sqrt{2}$		-2		$-2+\sqrt{2}$		0	
$f'(x)$	$+$		$+$		$-$		$-$		$+$
$f''(x)$	$+$		$-$		$-$		$+$		$+$
$f(x)$	$\nearrow$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

The concavity changes at two points. Those are called inflection points.

Local minimum and maximum We say x_0 is a point of local minimum, if $f(x_0) \le f(x)$ for all x in some interval around x_0.

First derivative test. Suppose that f' changes sign at x_0.

① If f' changes from being negative $\ominus$ to positive $\oplus$, then x_0 is a point of local minimum.

② If f' changes from $\oplus$ to $\ominus$ at x_0, then we have a local max.

Second derivative test. Suppose that $f'(x_0) = 0$.

① If $f''(x_0) > 0$, then x_0 is a point of local min.

② If $f''(x_0) < 0$, then x_0 is a point of local max.

Proofs. For the first derivative test, suppose $f' < 0$ to the left of x_0 and $f' > 0$ to the right of x_0.

Then f decreases on the left, so $f(x) \geq f(x_0)$ there and f increases on the right, so $f(x) \geq f(x_0)$ there as well.

For the second derivative test, $f''(x_0) > 0$ implies f' is increasing at x_0. Thus f' is negative on the left, positive on the right $\Rightarrow$ minimum $\square$

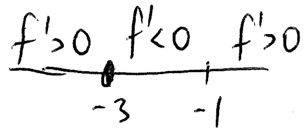
Example 1. Consider $f(x) = x(x+3)^2$.

Then $f'(x) = (x+3)^2 + 2x(x+3) = (x+3)(x+3+2x)$

$\Rightarrow$ $f'(x) = 3(x+3)(x+1)$.

This gives $f'(x) < 0$ when $-3 < x < -1$

and $f'(x) > 0$ when $x < -3$ or $x > -1$.



The derivative changes from $\oplus$ to $\ominus$ at $x = -3$ ---- a local max

from $\ominus$ to $\oplus$ at $x = -1$ ---- a local min

⊙ One could also use the second derivative test.

Since $f'(x) = 3(x^2 + 4x + 3)$, $f''(x) = 3(2x+4) = 6(x+2)$.

At $x = -3$ ---- $f''(x) = -6 < 0$ ---- a local max

At $x = -1$ ---- $f''(x) = 6 > 0$ ---- a local min

Example 2. Consider $f(x) = \dfrac{x^2}{x^4+4}$.

Then $f'(x) = \dfrac{2x(x^4+4) - 4x^3 \cdot x^2}{(x^4+4)^2} = \dfrac{2x^5 + 8x - 4x^5}{(x^4+4)^2}$

$\Rightarrow \quad f'(x) = \dfrac{8x - 2x^5}{(x^4+4)^2} = \dfrac{2x(4-x^4)}{(x^4+4)^2}.$

$\Rightarrow \quad f'(x) = \dfrac{2x(2-x^2)(2+x^2)}{(x^4+4)^2}.$

This is zero when $x=0$ or $x = \pm\sqrt{2}$. To determine the sign of f', as usual, we can make a table

		$-\sqrt{2}$		0		$\sqrt{2}$	
$2x$	$-$		$-$		$+$		$+$
$\sqrt{2}-x$	$+$		$+$		$+$		$-$
$\sqrt{2}+x$	$-$		$+$		$+$		$+$
$f'(x)$	$+$		$-$		$+$		$-$
		local max		local min		local max	

Local and global minima/maxima

- We say f has a LOCAL maximum at the point x_0, 
if $f(x) \le f(x_0)$ in some interval around x_0. We say f has a GLOBAL maximum at x_0, if $f(x) \le f(x_0)$ for all x in the domain of f.

1 Single change of signs Suppose f is differentiable and f' changes sign exactly once. If f' changes from $+$ to $-$ at x_0, then f has a global max at x_0. If it changes from $-$ to $+$, then f has a global min at x_0.

Example 1. We show $f(x) = x^4 - 4x$ attains a min value.
In fact, we show $x^4 - 4x \geq -3$ for all x.

Consider $f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x-1)(\underbrace{x^2+x+1})$.

a quadratic with no real roots. It is always positive.

The sign of f' is the sign of $x-1$.

Thus $f'(x) > 0$ when $x > 1$
$f'(x) < 0$ when $x < 1$.

		1	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

$\therefore$ $f(1)$ is the min value attained
and $f(x) \geq f(1) \Rightarrow x^4 - 4x \geq -3$ for all x.

Example 2. We show $xe^{-x} \le e^{-1}$ for all $x \in \mathbb{R}$.
This gives $x \le e^{x-1}$ for all $x \in \mathbb{R}$.

We need to compute the max value of $f(x) = \dfrac{x}{e^x} = xe^{-x}$.

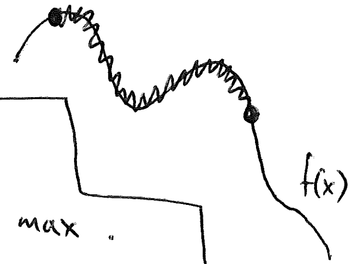
In this case, $f'(x) = \dfrac{e^x - xe^x}{e^x e^x} = \dfrac{\cancel{e^x}(1-x)}{\cancel{e^x}e^x} = \dfrac{1-x}{e^x}$.

The denominator is positive, so $f'(x) > 0$ when $x < 1$
$f'(x) < 0$ when $x > 1$.

		1	
f'	$+$		$-$
f	$\nearrow$		$\searrow$

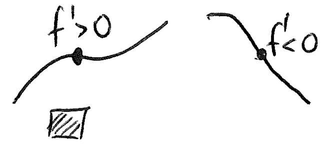
Thus $f(1)$ is the GLOBAL maximum
so $f(x) \le f(1)$ for all x, as needed

2 Continuous functions on $[a,b]$



If f is continuous on a finite interval $[a,b]$, then f attains <u>both</u> a global min <u>and</u> a global max. Moreover, these occur at ... (a) the endpoints a, b or
(b) the points with $f'(x)=0$ or
(c) the points where $f'(x)$ does not exist.

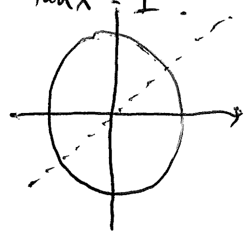
<u>Proof.</u> If we get a min/max at an <u>interior</u> point, then f' can be neither ⊕ nor ⊖ at that point. ▨



<u>Example 1.</u> Consider $f(x) = \sin x + \cos x$ for $0 \le x \le 2\pi$.
We find the min/max values attained. Note that we <u>don't</u> have to worry about the sign of $f'(x)$. In this case,

$$f'(x) = \cos x - \sin x \quad \text{exists at all points}$$

and $f'(x)=0$ when $\cos x = \sin x$, namely when $\tan x = 1$.



We get $x = \pi/4$ and $x = 5\pi/4$ as possible candidates. We check the values at

$x=0$ (endpoint) ... $f(0) = \sin 0 + \cos 0 = 1$

$x=2\pi$ (endpoint) ... $f(2\pi) = 1$

$x=\pi/4$ $(f'=0)$ ... $f(\pi/4) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$.

$x=5\pi/4$ $(f'=0)$ ... $f(5\pi/4) = \qquad = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$.

The global min is $-\sqrt{2}$. The global max is $+\sqrt{2}$.

◍ For this problem, one can write

$$\begin{aligned}\sin x + \cos x &= \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)\\ &= \sqrt{2}\left(\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}\right)\\ &= \sqrt{2}\cos\left(x - \frac{\pi}{4}\right).\end{aligned}$$

Example 2. Consider $f(x) = x^4 - 2x^2 - 1$ on $[0, 2]$.

Then $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$. This is defined for all x and $f'(x) = 0$ when $x = 0, 1, -1$. We exclude $x = -1$ and check the others. We need to check

endpoint ... $f(0) = -1$

endpoint .... $f(2) = 16 - 8 - 1 = 7$

$f(1) = 1 - 2 - 1 = -2$.

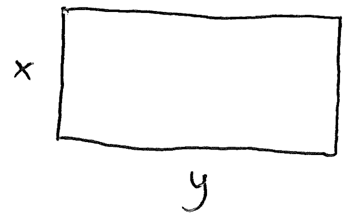
This gives $f(1) = -2$ as the minimum value

and $f(2) = 7$ as the maximum value.

We have proved that $-2 \le x^4 - 2x^2 - 1 \le 7$ for all $0 \le x \le 2$.

Optimisation problems

(1) Out of all rectangles with perimeter 40, which one has the largest area?



Let x = width, y = length.

We know $2x + 2y = 40$ and need to maximise Area $= xy$.

- Eliminating y gives $x + y = 20 \Rightarrow \underline{y = 20 - x}$ and we need to maximise $f(x) = xy = x(20-x) = 20x - x^2$.
- Restrictions on x .... $x \geq 0$ and $y \geq 0$

  $x \geq 0$ and $20 - x \geq 0$ .... so $0 \leq x \leq 20$.

We thus need to max $f(x) = 20x - x^2$ over $[0, 20]$.

Since $f'(x) = 20 - 2x = 2(10 - x)$, we get $f'(x) = 0$ when $x = 10$, f' exists at all points. Possible min/max values:

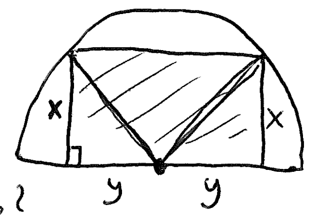
$x = 10$ .... $f(10) = 10(20-10) = 100$

$x = 0$ .... $f(0) = 0$

$x = 20$ .... $f(20) = 0$ as well.

Thus, the largest possible area is 100 (attained when $x = y = 10$).

(2) <u>Consider a rectangle inscribed in a semicircle of radius $r = 4$ with one of its sides along the diameter. How large can the area of the rectangle be?</u>



Let x be the height, let y be half of the other side.

By Pythagoras', $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2 \Rightarrow y = \sqrt{r^2 - x^2}$.

We need to maximise $A(x) = 2xy = 2x\sqrt{r^2 - x^2}$.

This is the same as maximising $B(x) = 4x^2(r^2 - x^2)$, namely $B(x) = 4x^2r^2 - 4x^4$ .... for a given constant r.

Restrictions on x: we need $x \geq 0$ and also $x \leq r$.
This gives an interval $[0,r]$ that is finite.

We have $B'(x) = 4 \cdot 2x \cdot r^2 - 4 \cdot 4x^3 = 8x(r^2 - 2x^2)$ and this is zero when $x=0$ and $x^2 = \frac{r^2}{2}$, namely $x = \frac{r}{\sqrt{2}}$.

Possible candidates for a maximum:

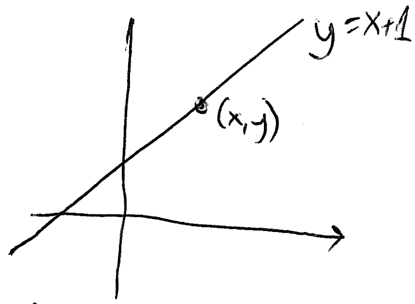
$x=0$ ---- $B(0) = 0$

$x = \frac{r}{\sqrt{2}}$ ---- $B\left(\frac{r}{\sqrt{2}}\right) = 4 \cdot \frac{r^2}{2} \cdot r^2 - 4 \cdot \frac{r^4}{4} = r^4$

(endpoint) $x = r$ ---- $B(r) = 4r^4 - 4r^4 = 0$.

Thus, the largest possible area is $B(r/\sqrt{2}) = r^4 \Rightarrow A(r/\sqrt{2}) = r^2$.

(3) We find the point on the line $y=x+1$ which is closest to the point $(4,1)$.



We need to minimise the distance

$$d = \sqrt{(x-4)^2 + (y-1)^2}$$ between (x,y) and $(4,1)$.

This is $d = \sqrt{(x-4)^2 + x^2}$ with $x \in \mathbb{R}$ arbitrary.

We have $d'(x) = \dfrac{2(x-4) + 2x}{2\sqrt{(x-4)^2 + x^2}} = \dfrac{2(x-2)}{\sqrt{(x-4)^2+x^2}}$

So $d'(x) > 0$ when $x > 2$
$d'(x) < 0$ when $x < 2$.

		2	
$d'(x)$	$-$		$+$
$d(x)$	$\searrow$		$\nearrow$

This makes $d(2)$ a global minimum (attained when $x=2$, $y=3$).
The shortest distance is then $\sqrt{(2-4)^2 + (3-1)^2} = \sqrt{4+4} = 2\sqrt{2}$.

(4) (Economics) A company produces phones with daily cost $C(x) = 200 + 40x$ with x = number of phones. The price of each phone is $P(x) = 100 - 2x$ with $0 \leq x \leq 50$. How many phones

should be produced to maximise profits?

In this case Profits $\Pi(x)$ = Revenues − Cost

$$= x(100-2x) - (200+40x)$$
$$= 100x - 2x^2 - 200 - 40x$$
$$= -2x^2 + 60x - 200.$$

We need to maximise $\Pi(x)$ when $0 \leq x \leq 50$. We have

$$\Pi'(x) = -4x + 60 = -4(x-15)$$

So the possible candidates are:

(endpoint) $x=0$ ---- $\Pi(0) = -200$

(endpoint) $x=50$ ---- $\Pi(50) = 50 \cdot 0 - 200 - 40 \cdot 50 = -2200$

$x=15$ ---- $\Pi(15) = 15 \cdot 70 - (200 + 40 \cdot 15) = 250.$

Profits become max when $x=15$.

Related rates

Consider two or more variables that change over time. If those are related, then their rates of change are also related. The differentiation is usually with respect to t.

Example 1. Consider a circular pond whose radius is increasing at 2m/sec. How fast is its area increasing when radius = 4m?

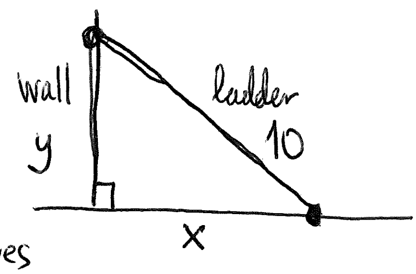
Let $r = r(t)$ be the radius, $A = A(t)$ = the area. Then

$$A = \pi r^2 \quad \text{or} \quad A(t) = \pi \cdot r(t)^2.$$

We are given $\frac{dr}{dt} = 2\text{m/sec}$ and seek $\frac{dA}{dt}$. Since $A(t) = \pi r(t)^2$, we have $\frac{dA}{dt} = \pi \cdot 2 r(t) r'(t)$ and so

$$\frac{dA}{dt} = \pi \cdot 2 \cdot 4 \cdot 2 = 16\pi \ \text{m}^2/\text{sec}.$$

Example 2. A ladder is resting against a wall and it is 10 ft long. If the base of the ladder starts sliding at 1ft/sec along the floor, how fast is the top sliding down the wall when the ladder is 6 ft away from the wall?



We have the relation $x^2 + y^2 = 10^2$, where $x = x(t)$ and $y = y(t)$. Differentiating gives

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \implies x \frac{dx}{dt} + y \frac{dy}{dt} = 0.$$

We know $\frac{dx}{dt} = 1$ and $x = 6$, so

$$6 \cdot 1 + y \cdot \frac{dy}{dt} = 0$$

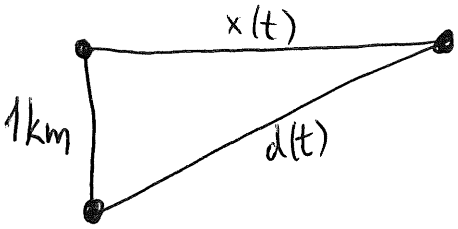
So

$$\frac{dy}{dt} = \frac{-6}{y} = -\frac{6}{\sqrt{10^2 - 6^2}}$$

and

$$\frac{dy}{dt} = \frac{-6}{\sqrt{64}} = -3/4.$$

Example 3. A plane is flying horizontally at 1km altitude and 800 Km/h speed. It passes above a petrol/gas station. How fast is the distance from the station changing when it is 2km ~~[illegible]~~ further down?



We have $d(t)^2 = x(t)^2 + 1$

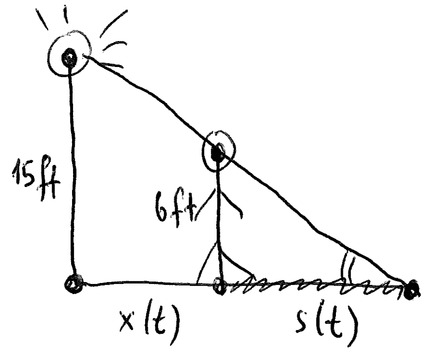
$$\Rightarrow \cancel{2}\, d(t)\, d'(t) = \cancel{2}\, x(t)\, x'(t)$$

$$\Rightarrow d'(t) = \frac{x(t)x'(t)}{d(t)}.$$

We know $\frac{dx}{dt} = 800$ and $x = 2$, so $d = \sqrt{2^2+1^2} = \sqrt{5}$.

Thus $d' = \frac{2\cdot 800}{\sqrt{5}} = \frac{1600\sqrt{5}}{5} = 320\sqrt{5}$.

Example 4. A street light is on top of a 15ft pole. A 6ft-tall person is walking away from the pole at 3ft/sec. How fast is the tip of his/her shadow moving?



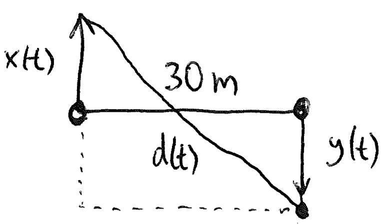
We know $\frac{dx}{dt} = 3$ and we seek $\frac{ds}{dt}$.

We need to relate s and x. Similar triangles give: $\frac{6}{s} = \frac{15}{x+s}$

So $6x + 6s = 15s$ and $6x = 9s$. Thus

$$6\frac{dx}{dt} = 9\frac{ds}{dt} \Rightarrow \frac{ds}{dt} = \frac{6\frac{dx}{dt}}{9} = \frac{2}{3}\cdot 3 = 2 \text{ ft/sec}.$$

Example 5. A man and a woman start walking at the same time. The man walks north at 3m/sec and the woman walks south at 5m/sec but starting at a point 30 m to the east. How fast are they separating from one another 5 secs later?



Those are related by: $d(t)^2 = (x(t)+y(t))^2 + 30^2$

Then $\cancel{2}\, d(t)\, d'(t) = \cancel{2}\,\underbrace{(x(t)}_{15} + \underbrace{y(t))}_{25} \cdot (\underbrace{x'(t)}_{3} + \underbrace{y'(t)}_{5})$

So $d'(t) = \frac{40\cdot 8}{d(t)} = \frac{320}{\sqrt{40^2+30^2}} = \frac{320}{[illegible]} = 32/5$ m/sec

Linear approximation If f is differentiable at the point x_0, then $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$, so $f'(x_0) \approx \frac{f(x)-f(x_0)}{x-x_0}$ near the point x_0 and $f(x) \approx f'(x_0)(x-x_0) + f(x_0)$.

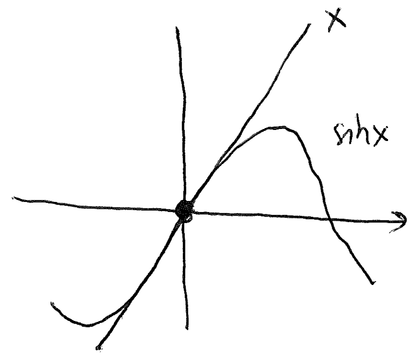
Definition. We call $L(x) = f'(x_0)(x-x_0) + f(x_0)$ the linear approximation (or tangent line approximation) at the point x_0.

Example 1. Consider $f(x) = \sin x$ at the point $x_0 = 0$.

In this case, $f(x_0) = f(0) = \sin 0 = 0$

and $f'(x_0) = f'(0) = \cos 0 = 1$

so the tangent line is $L(x) = 1 \cdot (x-0) + 0 = x$.

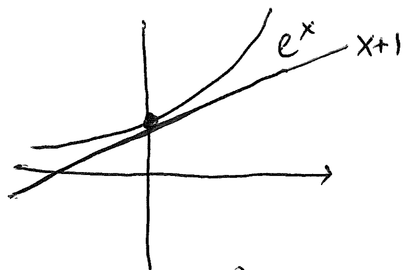


Example 2. Consider $f(x) = e^x$ at the point $x_0 = 0$.

In this case, $f(x_0) = f(0) = e^0 = 1$

and $f'(x_0) = f'(0) = e^0 = 1$, so $L(x) = 1(x-0) + 1$
$= x + 1$.

Thus, $e^x \approx x+1$ for points $x \approx 0$ ... and $e^{0.1} \approx 1.1$, for instance.



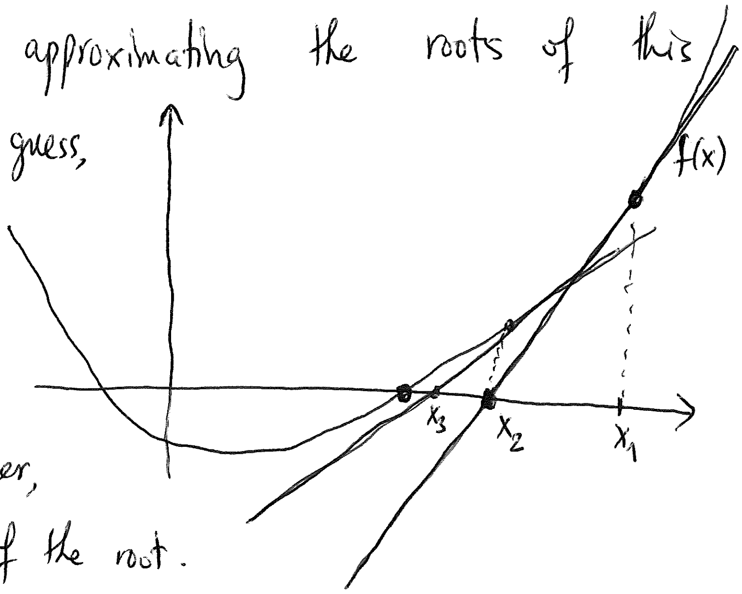
Example 3. Take $f(x) = \frac{x^2+3}{x^3+1}$ at the point $x_0 = 1$.

In this case, $f'(x) = \frac{2x \cdot (x^3+1) - 3x^2 \cdot (x^2+3)}{(x^3+1)^2} \Rightarrow f'(1) = \frac{4-12}{4} = -2$

and $f(1) = \frac{4}{2} = 2$, so the tangent line approximation is

$$L(x) = -2(x-1) + 2 = -2x + 4.$$

Newton's method Consider an equation of the form $f(x)=0$. There is a standard method for approximating the roots of this equation. We start with an initial guess, say x_1. We approximate $f(x)$ by the tangent line at x_1. We compute the point x_2 at which the line meets the x-axis. Proceeding in this manner, we hope to get an approximation of the root.



The tangent line @ x_1 is $L(x) = f'(x_1)(x-x_1) + f(x_1)$

So we get $L(x)=0$ means $f'(x_1)(x-x_1) = -f(x_1)$

or $x - x_1 = -\dfrac{f(x_1)}{f'(x_1)}$, so $\boxed{x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}}$.

We can introduce this formula and proceed, if $f'(x_1) \neq 0$.

More generally, $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$.

__Example 1.__ We approximate $\sqrt{2}$ using this method. Obviously, $\sqrt{2}$ is a root of $\underline{f(x) = x^2 - 2}$. Take $\underline{x_1 = 1}$ as an initial guess and then

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = x_n - \frac{x_n}{2} + \frac{1}{x_n},$$

namely $x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$. We get

$x_2 = \frac{3}{2} = 1.5$ and then $x_3 = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} = 1.416666$

and then $x_4 = 1.4142156$ and $x_5 = 1.414213562$ (with x_6 agreeing to 9 decimal places). If we take $\underline{x_1 = 2}$ as an initial guess, we get $x_1 = 2$, $x_2 = 3/2$ and the rest stays the same.

Example 2. Consider $f(x) = x^3 - 4x^2 - 3x + 1$ as in the last homework. This has a (unique) root in $(0,2)$. We approximate this root using Newton's method: take $x_1 = 2$ and then define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

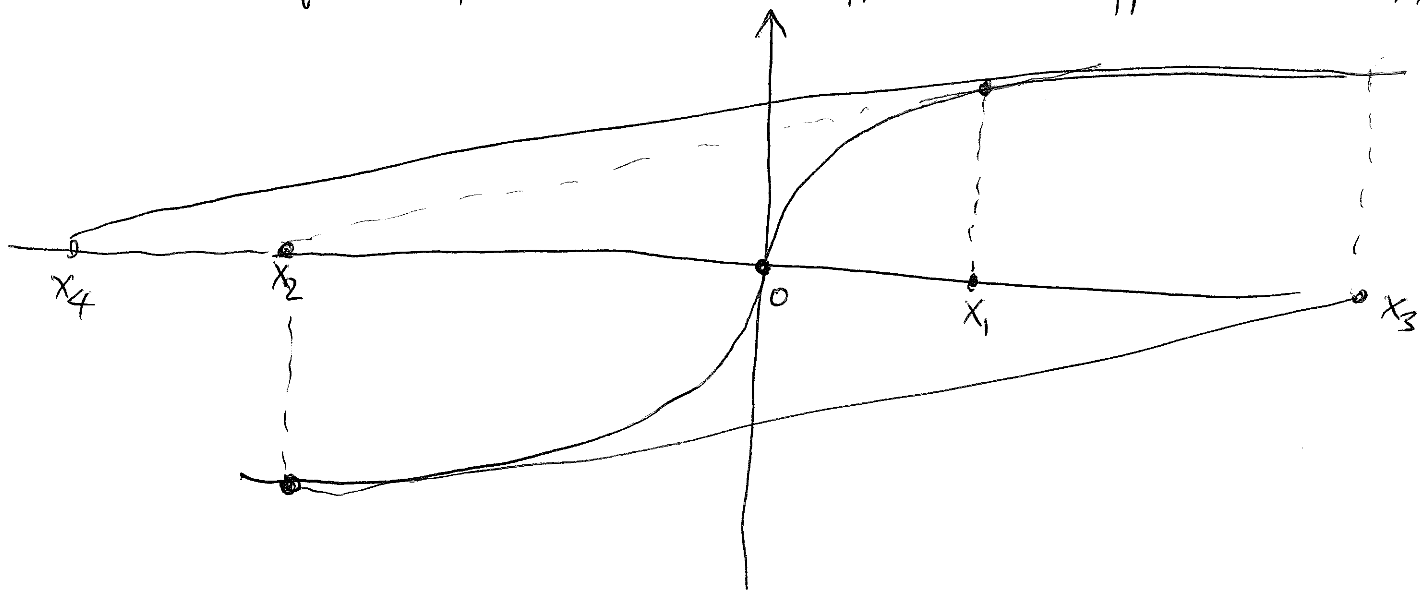
$$= x_n - \frac{\cancel{3}x_n^3 - 4x_n^2 - 3x_n + 1}{3x_n^2 - 8x_n - 3}.$$

In this case,

$$x_1 = 2$$
$$x_2 = 0.1428$$
$$x_3 = 0.2635$$
$$x_4 = 0.253309$$
$$x_5 = 0.253239$$

and one finds the root to be $x = 0.253$ (to 3 decimal digits).

$\boxed{\text{Note:}}$ The method will not work in all cases, but it works quite often (and the approximations approach a limit).

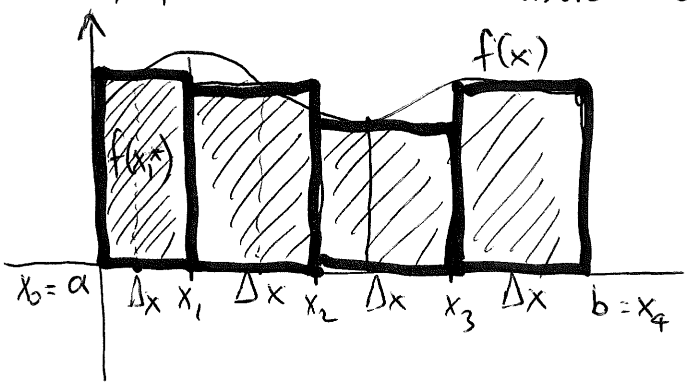


$\boxed{\text{Definition of integral}}$ We use Σ to denote sums such as $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5$. We call k the index of summation and $\sum_{k=1}^{5} a_k = \sum_{i=1}^{5} a_i$. One may "shift the index" of summation to get $\sum_{k=1}^{6} a_k = \sum_{k=2}^{7} a_{k-1}$.

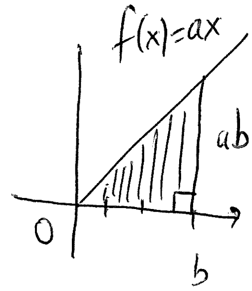
Integral We define $\int_a^b f(x)\,dx$ of a function over a finite interval $[a,b]$ is defined as the limit

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x},$$

where $\Delta x = \frac{b-a}{n}$ and $x_0, x_1, \ldots, x_n$ are the points that subdivide the interval into n equal parts $x_0 = a$, $x_1 = a + \Delta x$, $\ldots$, $x_n = b$ and the point x_k^* is any point in $[x_k, x_{k+1}]$. We say f is integrable on $[a,b]$, if the limit above exists.



Example. We show $f(x) = ax$ is integrable on $[0,b]$ for any $a, b > 0$ and $\int_0^b ax\,dx = \frac{1}{2}ab^2$.



We have to check $\boxed{\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x = \frac{1}{2}ab^2}$.

Let $x_0, x_1, \ldots, x_n$ be the points that divide $[0,b]$ into n equal parts, $\Delta x = b/n$ is the length of each subinterval and $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x$, $\ldots$, $x_n = n\Delta x$.

By assumption, $x_{k-1} \leq x_k^* \leq x_k$

$$\Rightarrow \quad a x_{k-1} \leq a x_k^* \leq a x_k$$

$$\Rightarrow \quad a x_{k-1} \leq f(x_k^*) \leq a x_k$$

$$\Rightarrow \quad a x_{k-1} \Delta x \leq f(x_k^*) \Delta x \leq a x_k \Delta x$$

$$\Rightarrow \quad \boxed{\sum_{k=1}^{n} a x_{k-1} \Delta x} \leq \sum_{k=1}^{n} f(x_k^*) \Delta x \leq \boxed{\sum_{k=1}^{n} a x_k \Delta x}.$$

$$a x_0 \Delta x + a x_1 \Delta x + a x_2 \Delta x + \dots + a x_{n-1} \Delta x$$

$$= a \Delta x \left(x_0 + x_1 + x_2 + \dots + x_{n-1}\right)$$

$$= \frac{ab}{n} \left(0 + 1\Delta x + 2\Delta x + \dots + (n-1)\Delta x\right)$$

$$= \frac{ab}{n} \cdot \frac{b}{n} \left(0 + 1 + 2 + \dots + (n-1)\right)$$

$$\boxed{\frac{ab^2}{n^2} \cdot \frac{(n-1)n}{2}}$$

Sum on the left

$$a \Delta x \left(x_1 + x_2 + \dots + x_n\right)$$

$$= a \Delta x \left(1\Delta x + 2\Delta x + \dots + n\Delta x\right)$$

$$= \boxed{a \left(\frac{b}{n}\right)^2 \frac{n(n+1)}{2}}$$

because $S = 1 + 2 + \dots + (n-1)$

$$S = (n-1) + (n-2) + \dots + 1$$

$$2S = n + n + \dots + n$$

$$= n(n-1)$$

This computation gives

$$\frac{ab^2}{n^2} \frac{n(n-1)}{2} \leq \underbrace{\sum_{k=1}^{n} f(x_k^*) \Delta x} \leq \frac{ab^2}{n^2} \frac{n(n+1)}{2}.$$

We have $\displaystyle \lim_{n \to \infty} \frac{ab^2\, n(n-1)}{n^2 \cdot 2} = \frac{ab^2}{2} = \lim_{n \to \infty} \frac{ab^2 \cdot n(n+1)}{n^2 \cdot 2}$

So it follows by the Squeeze theorem that

$$\int_0^b f(x)\, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x = \frac{ab^2}{2}$$

__Theorem 1.__ (Linearity of integral) If f, g are integrable on $[a,b]$ then $f+g$ is integrable and $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
If f is integrable on $[a,b]$ and c is a constant, then cf is integrable and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

__Proof.__ Let $h(x) = f(x)+g(x)$, for instance. Then

$$\int_a^b h(x)\,dx \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{k=1}^{n} h(x_k^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x + g(x_k^*)\,\Delta x$$

$$= \lim_{n\to\infty} \left[\sum_{k=1}^{n} f(x_k^*)\,\Delta x + \sum_{k=1}^{n} g(x_k^*)\,\Delta x \right]$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x + \lim_{n\to\infty} \sum_{k=1}^{n} g(x_k^*)\,\Delta x_k$$

$$= \int f(x)\,dx + \int g(x)\,dx. \qquad \blacksquare$$