MA1125 – Calculus
Tutorial solutions #7

1. Find the area of the region enclosed by the graphs of $f(x) = 3x^2$ and $g(x) = x + 2$.

The graph of the parabola $f(x) = 3x^2$ meets the graph of the line $g(x) = x + 2$ when

$$3x^2 = x + 2 \quad \iff \quad 3x^2 - x - 2 = 0 \quad \iff \quad (3x+2)(x-1) = 0.$$

Since the line lies above the parabola at the points $-2/3 \leq x \leq 1$, the area is then

$$\int_{-2/3}^{1} [g(x) - f(x)]\, dx = \int_{-2/3}^{1} [x + 2 - 3x^2]\, dx = \left[\frac{x^2}{2} + 2x - x^3 \right]_{-2/3}^{1} = \frac{125}{54}.$$

2. Compute the volume of the solid that is obtained when the graph of $f(x) = x^2 + 3$ is rotated around the x-axis over the interval $[0, 2]$.

The volume of the resulting solid is the integral of $\pi f(x)^2$ and this is equal to

$$\pi \int_0^2 (x^2 + 3)^2\, dx = \pi \int_0^2 (x^4 + 6x^2 + 9)\, dx = \pi \left[\frac{x^5}{5} + 2x^3 + 9x \right]_0^2 = \frac{202\pi}{5}.$$

3. Compute the length of the graph of $f(x) = \frac{1}{3}(x^2 + 2)^{3/2}$ over the interval $[1, 3]$.

The length of the graph is given by the integral of $\sqrt{1 + f'(x)^2}$. In this case,

$$f'(x) = \frac{1}{3} \cdot \frac{3}{2} \cdot (x^2 + 2)^{1/2} \cdot 2x = x(x^2 + 2)^{1/2},$$

so the expression $1 + f'(x)^2$ can be written in the form

$$1 + f'(x)^2 = 1 + x^2(x^2 + 2) = 1 + x^4 + 2x^2 = (1 + x^2)^2.$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_1^3 \sqrt{1 + f'(x)^2}\, dx = \int_1^3 (1 + x^2)\, dx = \left[x + \frac{x^3}{3} \right]_1^3 = \frac{32}{3}.$$

4. Find both the mass and the centre of mass for a thin rod whose density is given by

$$\delta(x) = x^2 + 4x + 1, \qquad 0 \leq x \leq 2.$$

The mass of the rod is merely the integral of its density function, namely

$$M = \int_0^2 \delta(x)\,dx = \int_0^2 (x^2 + 4x + 1)\,dx = \left[\frac{x^3}{3} + 2x^2 + x\right]_0^2 = \frac{38}{3}.$$

The centre of mass is given by a similar formula and one finds that

$$\overline{x} = \frac{1}{M}\int_0^2 x\delta(x)\,dx = \frac{3}{38}\int_0^2 (x^3 + 4x^2 + x)\,dx = \frac{3}{38}\left[\frac{x^4}{4} + \frac{4x^3}{3} + \frac{x^2}{2}\right]_0^2 = \frac{25}{19}.$$

5. A chain that is 4m long has a uniform density of 3kg/m. If the chain is hanging from the top of a tall building, then how much work is needed to pull it up to the top?

Consider an arbitrarily small part of the chain, say one of length dx, which lies x metres from the top. The work that is needed to pull this part to the top is then

$$\text{Work} = \text{Force} \cdot \text{Displacement} = mg \cdot x = (3\,dx)g \cdot x.$$

Summing up these expressions over all possible values of $0 \leq x \leq 4$, we conclude that

$$\text{Work} = 3g\int_0^4 x\,dx = 3g\left[\frac{x^2}{2}\right]_0^4 = 24g.$$

6. Find the area of the region enclosed by the graphs of $f(x)$ and $g(x)$ in the case that

$$f(x) = \sin x, \qquad g(x) = \cos x, \qquad 0 \leq x \leq \pi/2.$$

The two functions are both non-negative on the interval $[0, \pi/2]$ and one has

$$f(x) \leq g(x) \quad\Longleftrightarrow\quad \sin x \leq \cos x \quad\Longleftrightarrow\quad \tan x \leq 1 \quad\Longleftrightarrow\quad x \in [0, \pi/4].$$

In other words, $f(x) \leq g(x)$ when $0 \leq x \leq \pi/4$ and $g(x) \leq f(x)$ when $\pi/4 \leq x \leq \pi/2$, so

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} [\cos x - \sin x]\,dx + \int_{\pi/4}^{\pi/2} [\sin x - \cos x]\,dx \\
&= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\
&= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 - 0 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 2\sqrt{2} - 2.
\end{aligned}$$

7. The graph of $f(x) = 2e^{6x}$ is rotated around the x-axis over the interval $[0, a]$. If the volume of the resulting solid is equal to π, then what is the value of a?

The volume of the resulting solid is the integral of $\pi f(x)^2$ and this is given by

$$\text{Volume} = \pi \int_0^a 4e^{12x}\,dx = 4\pi \left[\frac{e^{12x}}{12}\right]_0^a = \frac{\pi}{3}(e^{12a} - 1).$$

Since the volume must be equal to π by assumption, it easily follows that

$$e^{12a} - 1 = 3 \quad\Longrightarrow\quad e^{12a} = 4 \quad\Longrightarrow\quad 12a = \ln 4 \quad\Longrightarrow\quad a = \frac{\ln 2^2}{12} = \frac{\ln 2}{6}.$$

8. Compute the length of the graph of $f(x) = x^{3/2} - \frac{1}{3}x^{1/2}$ over the interval $[0, 2]$.

The length of the graph is given by the integral of $\sqrt{1 + f'(x)^2}$. In this case,

$$f'(x) = \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-1/2} \quad\Longrightarrow\quad 1 + f'(x)^2 = 1 + \frac{9}{4}x + \frac{1}{36x} - \frac{1}{2}$$

and one may use a common denominator to write this expression in the form

$$1 + f'(x)^2 = \frac{18x + (9x)^2 + 1}{36x} = \frac{(9x+1)^2}{36x}.$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_0^2 \frac{9x+1}{6\sqrt{x}}\,dx = \frac{1}{6}\int_0^2 \left(9x^{1/2} + x^{-1/2}\right)\,dx = \frac{1}{6}\left[\frac{9x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2}\right]_0^2 = \frac{7}{3}\sqrt{2}.$$

9. Show that the function f is integrable on $[0, 1]$ for any given constants a, b when

$$f(x) = \left\{ \begin{array}{ll} a & \text{if } x \neq 0 \\ b & \text{if } x = 0 \end{array} \right\}.$$

Let $x_0, x_1, \ldots, x_n$ be the points that divide the interval $[0, 1]$ into n subintervals of equal length. To show that f is integrable on $[0, 1]$, we need to compute the limit

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\,\Delta x$$

for any choice of points $x_k^* \in [x_{k-1}, x_k]$. When $x_1^* > 0$, we have $x_k^* > 0$ for all $k \geq 1$ and so

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^n \frac{a}{n} = \lim_{n\to\infty} n \cdot \frac{a}{n} = a.$$

When $x_1^* = 0$, on the other hand, we have $x_k^* > 0$ for all $k \geq 2$ and the limit is still

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \left[\frac{b}{n} + \sum_{k=2}^n \frac{a}{n}\right] = \lim_{n\to\infty} \left[\frac{b}{n} + \frac{(n-1)a}{n}\right] = a.$$

10. Compute each of the following improper integrals.

$$I_1 = \int_2^\infty \frac{dx}{(x-1)^5}, \qquad I_2 = \int_2^3 \frac{dx}{\sqrt[4]{x-2}}, \qquad I_3 = \int_0^\infty \frac{dx}{x^2+1}.$$

When it comes to the first integral, one easily finds that

$$I_1 = \lim_{L\to\infty} \int_2^L (x-1)^{-5}\, dx = \lim_{L\to\infty} \left[-\frac{1}{4}(x-1)^{-4} \right]_2^L = \frac{1}{4}.$$

When it comes to the second integral, one similarly finds that

$$I_2 = \lim_{a\to 2^+} \int_a^3 (x-2)^{-1/4}\, dx = \lim_{a\to 2^+} \left[\frac{4}{3}(x-2)^{3/4} \right]_a^3 = \frac{4}{3}.$$

Finally, the third integral is related to the inverse tangent function and one has

$$I_3 = \lim_{L\to\infty} \int_0^L \frac{dx}{x^2+1} = \lim_{L\to\infty} (\tan^{-1} L - \tan^{-1} 0) = \frac{\pi}{2}.$$