# MA1125 – Calculus
## Tutorial solutions #2

**1.** Determine the inverse function $f^{-1}$ in each of the following cases.

$$f(x) = \log_3(2x-5) - 1, \qquad f(x) = \frac{2 \cdot 5^x + 7}{3 \cdot 5^x - 4}.$$

When it comes to the first case, one can easily check that

$$y + 1 = \log_3(2x-5) \quad \Longleftrightarrow \quad 3^{y+1} = 2x - 5 \quad \Longleftrightarrow \quad x = \frac{5 + 3^{y+1}}{2},$$

so the inverse function is defined by $f^{-1}(y) = \frac{5+3^{y+1}}{2}$. When it comes to the second case,

$$y = \frac{2 \cdot 5^x + 7}{3 \cdot 5^x - 4} \quad \Longleftrightarrow \quad 3y \cdot 5^x - 4y = 2 \cdot 5^x + 7 \quad \Longleftrightarrow \quad 4y + 7 = 5^x(3y-2)$$

and this gives $5^x = \frac{4y+7}{3y-2}$, so the inverse function is defined by $f^{-1}(y) = \log_5 \frac{4y+7}{3y-2}$.

**2.** Simplify each of the following expressions.

$$\sec\left(\tan^{-1} x\right), \qquad \cos\left(\sin^{-1} x\right), \qquad \log_2 18 - 2\log_2 3.$$

To simplify the first expression, let $\theta = \tan^{-1} x$ and note that $\tan\theta = x$. When $x \geq 0$, the angle $\theta$ arises in a right triangle with an opposite side of length $x$ and an adjacent side of length 1. It follows by Pythagoras' theorem that the hypotenuse has length $\sqrt{1+x^2}$, so the definition of secant gives

$$\sec(\tan^{-1} x) = \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \sqrt{1+x^2}.$$

When $x \leq 0$, the last equation holds with $-x$ instead of $x$. This changes the term $\tan^{-1} x$ by a minus sign, but the secant remains unchanged, so the equation is still valid.

To simplify the second expression, one may use a similar approach or simply note that

$$\theta = \sin^{-1} x \quad \Longrightarrow \quad \sin\theta = x \quad \Longrightarrow \quad \cos^2\theta = 1 - \sin^2\theta = 1 - x^2.$$

Since $\theta = \sin^{-1} x$ lies between $-\pi/2$ and $\pi/2$ by definition, $\cos\theta$ is non-negative and

$$\cos^2\theta = 1 - x^2 \quad \Longrightarrow \quad \cos\theta = \sqrt{1-x^2}.$$

As for the third expression, the standard properties of the logarithmic function give

$$\log_2 18 - 2\log_2 3 = \log_2 18 - \log_2 3^2 = \log_2 \frac{18}{3^2} = \log_2 2^1 = 1.$$

**3.** Use the $\varepsilon$-$\delta$ definition of limits to compute $\lim_{x\to 3} f(x)$ in the case that

$$f(x) = \left\{ \begin{array}{ll} 3x-7 & \text{if } x \leq 3 \\ 8-2x & \text{if } x > 3 \end{array} \right\}.$$

Note that $x$ is approaching 3 and that $f(x)$ is either $3x-7$ or $8-2x$. We thus expect the limit to be $L = 2$. To prove this formally, we let $\varepsilon > 0$ and estimate the expression

$$|f(x) - 2| = \left\{ \begin{array}{ll} |3x-9| & \text{if } x \leq 3 \\ |6-2x| & \text{if } x > 3 \end{array} \right\} = \left\{ \begin{array}{ll} 3|x-3| & \text{if } x \leq 3 \\ 2|x-3| & \text{if } x > 3 \end{array} \right\}.$$

If we assume that $0 \neq |x-3| < \delta$, then we may use the last equation to get

$$|f(x) - 2| \leq 3|x-3| < 3\delta.$$

Since our goal is to show that $|f(x) - 2| < \varepsilon$, an appropriate choice of $\delta$ is thus $\delta = \varepsilon/3$.

**4.** Compute each of the following limits.

$$L = \lim_{x\to 2} \frac{x^3 - 2x^2 + 5x - 1}{x-3}, \qquad M = \lim_{x\to 2} \frac{x^3 - 3x^2 + 4x - 4}{x-2}.$$

The first limit is the limit of a rational function which is defined at $x = 2$, so

$$L = \frac{2^3 - 2\cdot 2^2 + 5\cdot 2 - 1}{2-3} = -9.$$

The second limit involves a rational function which can be simplified. In fact, one has

$$M = \lim_{x\to 2} \frac{(x-2)(x^2 - x + 2)}{x-2} = \lim_{x\to 2} (x^2 - x + 2) = 2^2 - 2 + 2 = 4.$$

**5.** Use the $\varepsilon$-$\delta$ definition of limits to compute $\lim_{x\to 3} (3x^2 - 7x + 2)$.

Let $f(x) = 3x^2 - 7x + 2$ for convenience. Then $f(3) = 8$ and one has

$$|f(x) - f(3)| = |3x^2 - 7x - 6| = |x-3| \cdot |3x+2|.$$

The factor $|x-3|$ is related to our usual assumption that $0 \neq |x-3| < \delta$. To estimate the remaining factor $|3x+2|$, we assume that $\delta \leq 1$ for simplicity and note that

$$\begin{aligned} |x-3| < \delta \leq 1 \quad &\Longrightarrow \quad -1 < x-3 < 1 \\ &\Longrightarrow \quad 2 < x < 4 \quad \Longrightarrow \quad 8 < 3x+2 < 14. \end{aligned}$$

Combining the estimates $|x-3| < \delta$ and $|3x+2| < 14$, one may then conclude that

$$|f(x) - f(3)| = |x-3| \cdot |3x+2| < 14\delta \leq \varepsilon,$$

as long as $\delta \leq \varepsilon/14$ and $\delta \leq 1$. An appropriate choice of $\delta$ is thus $\delta = \min(\varepsilon/14, 1)$.

**6.** For which value of $a$ does the limit $\lim_{x\to 2} f(x)$ exist? Explain.

$$f(x) = \left\{ \begin{array}{ll} 2x^2 - ax + 3 & \text{if } x \le 2 \\ 4x^3 + 3x - a & \text{if } x > 2 \end{array} \right\}.$$

Since the given function is a polynomial on the interval $(-\infty, 2)$, its limit from the left is

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x^2 - ax + 3) = 8 - 2a + 3 = 11 - 2a.$$

The same argument applies for the interval $(2, +\infty)$, so the limit from the right is

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x^3 + 3x - a) = 32 + 6 - a = 38 - a.$$

To ensure that the given function has a limit as $x$ approaches 2, one must then have

$$11 - 2a = 38 - a \quad\iff\quad a = -27.$$

**7.** Determine the inverse function $f^{-1}$ in the case that $f\colon [2,\infty) \to [1,\infty)$ is defined by

$$f(x) = 2x^2 - 8x + 9.$$

Using the quadratic formula to solve the equation $y = f(x)$ for $x$, one finds that

$$2x^2 - 8x + (9 - y) = 0 \quad\implies\quad x = \frac{8 \pm \sqrt{64 - 8(9-y)}}{4} = \frac{8 \pm \sqrt{8y - 8}}{4}.$$

Since $y \ge 1$, the square root is obviously defined. Since $x \ge 2$, however, one needs to have

$$x = \frac{8 + \sqrt{8y-8}}{4} = 2 + \frac{\sqrt{2y-2}}{2} \quad\implies\quad f^{-1}(y) = 2 + \frac{\sqrt{2y-2}}{2}.$$

**8.** Compute each of the following limits.

$$L = \lim_{x\to 3} \frac{x^3 - 5x^2 + 7x - 3}{x - 3}, \qquad M = \lim_{x\to 3} \frac{2x^3 - 9x^2 + 27}{(x-3)^2}.$$

When it comes to the first limit, division of polynomials gives

$$L = \lim_{x\to 3} \frac{(x-3)(x^2 - 2x + 1)}{x - 3} = \lim_{x\to 3} (x^2 - 2x + 1) = 9 - 6 + 1 = 4.$$

When it comes to the second limit, division of polynomials gives

$$M = \lim_{x\to 3} \frac{(x^2 - 6x + 9)(2x + 3)}{x^2 - 6x + 9} = \lim_{x\to 3} (2x + 3) = 6 + 3 = 9.$$

**9.** Use the $\varepsilon$-$\delta$ definition of limits to compute $\lim_{x\to 2} \frac{1}{x}$.

To show that the limit is $L = \frac{1}{2}$, we let $\varepsilon > 0$ be given and we estimate the expression

$$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{2|x|}.$$

Assume that $0 \neq |x-2| < \delta$ and that $\delta \leq 1$ for simplicity. We must then have

$$\begin{aligned} |x-2| < \delta \leq 1 \quad &\implies \quad -1 < x - 2 < 1 \\ &\implies \quad 1 < x < 3 \quad \implies \quad \frac{1}{2|x|} = \frac{1}{2x} < \frac{1}{2}. \end{aligned}$$

Once we now combine these estimates, we may actually conclude that

$$|f(x) - L| = \frac{|x-2|}{2|x|} < \frac{\delta}{2|x|} < \frac{\delta}{2} \leq \varepsilon,$$

as long as $\delta \leq 2\varepsilon$ and $\delta \leq 1$. An appropriate choice of $\delta$ is thus $\delta = \min(2\varepsilon, 1)$.

**10.** Use the $\varepsilon$-$\delta$ definition of limits to compute $\lim_{x\to 2} (4x^2 - 5x + 1)$.

Let $f(x) = 4x^2 - 5x + 1$ for convenience. Then $f(2) = 7$ and one has

$$|f(x) - f(2)| = |4x^2 - 5x - 6| = |x-2| \cdot |4x+3|.$$

The factor $|x-2|$ is related to our usual assumption that $0 \neq |x-2| < \delta$. To estimate the remaining factor $|4x+3|$, we assume that $\delta \leq 1$ for simplicity and we find that

$$\begin{aligned} |x-2| < \delta \leq 1 \quad &\implies \quad -1 < x - 2 < 1 \\ &\implies \quad 1 < x < 3 \quad \implies \quad 7 < 4x + 3 < 15. \end{aligned}$$

Combining the estimates $|x-2| < \delta$ and $|4x+3| < 15$, one may now conclude that

$$|f(x) - f(2)| = |x-2| \cdot |4x+3| < 15\delta \leq \varepsilon,$$

as long as $\delta \leq \varepsilon/15$ and $\delta \leq 1$. An appropriate choice of $\delta$ is thus $\delta = \min(\varepsilon/15, 1)$.