Notes on Lagrangian Mechanics

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Abstract

This is a part of the Advanced Mechanics course MA2341. These notes are partially based on the textbook "Mechanics" by L.D. Landau and E.M. Lifshitz.

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Chapter 1

THE EQUATIONS OF MOTION

Classical mechanics studies the motion of material bodies which are not too light and not too heavy (then we need quantum mechanics and the theory of general relativity, respectively), e.g. planets, cars, rockets, footballs and so on. We refer to a set of these material bodies as to a mechanical system. Sometimes we can neglect the dimensions of the object we study and regard it as a (point) *particle*. Whether an object may or may not be regarded as a particle depends on the problem concerned. For example, an asteroid moving far enough from Earth may be regarded as a particle but not in considering its collision with Earth.

1.1 Generalised coordinates

To understand the motion of a mechanical system means to know the position of its bodies (and particles they are made of) at any instant of time.

1.1.1 Cartesian coordinates

The position of a particle in a d-dimensional Euclidian space $\mathbb{R}^d$ can be defined by its radius vector

$$\vec{r} = (x_1, \ldots, x_d)\,, \tag{1.1}$$

whose components are its Cartesian coordinates $x_1, \ldots, x_d$. In 3-space we often identify $x \equiv x_1$, $y \equiv x_2$, $z \equiv x_3$. If we know $\vec{r}(t)$ for $t_1 \le t \le t_2$ then the particles trajectory is a parametric curve in the space. The first derivative

$$\vec{v} = \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}} = (\frac{dx_1}{dt}, \ldots, \frac{dx_d}{dt}) = (\dot{x}_1, \ldots, \dot{x}_d) = (v_1, \ldots, v_d) \tag{1.2}$$

is the velocity of the particle while the second derivative $\vec{a} = d^2\vec{r}/dt^2 \equiv \ddot{\vec{r}}$ is its acceleration. The velocity vector is tangent to the trajectory. The norm of the velocity

$$|\vec{v}| = \sqrt{v_1^2 + \cdots + v_d^2} = \sqrt{v_i^2} = v \tag{1.3}$$

is the speed, and the distance travelled by the particle in an infinitesimal time interval dt is

$$ds = vdt = \sqrt{dx_1^2 + \cdots + dx_d^2} = \sqrt{dx_i^2} \implies ds^2 = dx_i^2\,. \tag{1.4}$$

If there are N particles then we need N radius vectors or $d\,N$ coordinates.

1.1.2 Dof and generalised coordinates

Definition. The number of degrees of freedom (dof) of a mechanical system is the number of independent quantities which define uniquely the position of the system.

Thus, a system of N freely moving particles in 3-space has $3N$ dof.

To define the position of a particle one does not have to use the Cartesian coordinates. For example, if a system has axial symmetry, i.e. its properties do not change under rotations about a given line then the cylindrical coordinates are more convenient. Similarly, if a system has spherical symmetry, i.e. its properties do not change under rotations about a given point then one prefers the spherical coordinates.

Definition. Generalised coordinates of a system with s dof are any s quantities $q^1, q^2, \ldots, q^s$ which completely define the position of the system. The first derivatives $\dot{q}^i$ are the generalised velocities.[1]

In what follow we will refer to generalised coordinates and velocities simply as to coordinates and velocities.

1.1.3 Constraints and dof

In some mechanical systems the interaction between different particles restricts their relative position. These restrictions are expressed by means of a set of (holonomic) constraints

$$C_\alpha(q^1, \ldots, q^s, t) = 0\,, \quad \alpha = 1, \ldots, r \leq s\,. \tag{1.5}$$

Each constraint reduces the number of dof by 1, so a constrained system with r independent constraints has $s - r$ dof.

The set of all allowed positions of a mechanical system is called the *configuration space* of the system. In most cases the configuration space of a system is a smooth manifold whose dimension is equal to the number of dof.

Example 1. A simple planar pendulum is a particle moving in 2-space and attached to a rod of length l whose other end is fixed. The Cartesian coordinates x and y of the particle are subject to the constraint[2]

$$C(x, y) = x^2 + y^2 - l^2 = 0\,, \tag{1.6}$$

which defines a circle of radius l centred at the origin. This circle, S^1, is the configuration space of the pendulum. In polar coordinates

$$x = r\cos\phi\,, \quad y = r\sin\phi\,, \tag{1.7}$$

the constraint takes the form $r = l$ which makes obvious that the angle ϕ is the only generalised coordinate of the system.

Example 2. A simple pendulum is a particle moving in 3-space and attached to a rod of length l whose other end is fixed. The Cartesian coordinates x, y and z of the particle are subject to the constraint

$$C(x, y, z) = x^2 + y^2 + z^2 - l^2 = 0\,. \tag{1.8}$$

[1] Throughout the course we mainly use upper indices for generalised coordinates q^i unless they are Cartesian coordinates for which we prefer the lower indices. Cartesian coordinates with lower and upper indices are identified $x^i \equiv x_i$.

[2] Here and in what follows the origin coincides with the fixed end of the rod.

which defines a sphere of radius l centred at the origin. This sphere, S^2, is the configuration space of the pendulum. In spherical coordinates

$$x = r\cos\varphi\sin\theta\,, \quad y = r\sin\varphi\sin\theta\,, \quad z = r\cos\theta\,, \tag{1.9}$$

the constraint takes the form $r = l$ which makes obvious that the angles φ and θ are the two generalised coordinates of the system.

Example 3. A coplanar double pendulum. The Cartesian coordinates x_1, y_1 and x_2, y_2 of the two particles are subject to the constraints

$$\begin{aligned} C_1(x_1, y_1, x_2, y_2) &= x_1^2 + y_1^2 - l_1^2 = 0\,, \\ C_2(x_1, y_1, x_2, y_2) &= (x_2 - x_1)^2 + (y_2 - y_1)^2 - l_2^2 = 0\,, \end{aligned} \tag{1.10}$$

and the system has two dof. Since each of the constraints defines a circle, the configuration space of the double pendulum is a torus $T^2 = S^1 \times S^1$.

Introducing polar coordinates for x_1, y_1 and $x_2 - x_1$, $y_2 - y_1$

$$x_1 = r_1\cos\phi_1\,, \quad y_1 = r_1\sin\phi_1\,, \quad x_2 - x_1 = r_2\cos\phi_2\,, \quad y_2 - y_1 = r_2\sin\phi_2\,, \tag{1.11}$$

one gets

$$\begin{aligned} C_1(r_1, \phi_1, r_2, \phi_2) &= r_1^2 - l_1^2 = 0\,, \\ C_2(r_1, \phi_1, r_2, \phi_2) &= r_2^2 - l_2^2 = 0\,, \end{aligned} \tag{1.12}$$

and the angles ϕ_1 and ϕ_2 are the two generalised coordinates of the system.

Example 4. Find the number of dof of a system of N particles in d-dimensional space moving such that the distance between the particles do not vary.

1. $N = 2$, $d = 2$.
2. $N = 3$, $d = 2$.
3. $N = n \geq 4$, $d = 2$.
4. $N = 2$, $d = 3$.
5. $N = 3$, $d = 3$.
6. $N = 4$, $d = 3$.
7. $N = n \geq 5$, $d = 3$.

1.2 Hamilton's principle of least action

1.2.1 Equations of motion

If we know the position of a system at a given instant of time we cannot say anything about its position at any other instant because the particles of the system can have any velocities, accelerations, and so on. However, for many systems if all the coordinates and velocities of a system are known at a given instant of time then the position of the system is completely determined, and the coordinates can be found at any instant. This implies in particular that the accelerations $\ddot{q}^i$ at that instant are uniquely defined by q_i and $\dot{q}^i$, and therefore $\ddot{q}^i$ are functions of q^i and $\dot{q}^i$. Since the same must hold for any instant of time we conclude that

$$\ddot{q}^i = f^i(q^1, \ldots, q^s, \dot{q}^1, \ldots, \dot{q}^s, t)\,, \quad i = 1, \ldots, s\,. \tag{1.13}$$

Definition. The relations between the coordinates, velocities and accelerations are called the equations of motion (eom).

They are second-order differential equations for $q^i(t)$.

For brevity, we will often denote by q the set of all the coordinates $q^1, \ldots, q^s$, and by $\dot{q}$ and $\ddot{q}$ the sets of all the velocities and accelerations, respectively. Then, the eom are written as

$$\ddot{q} = f(q, \dot{q}, t)\,. \tag{1.14}$$

It is clear that the eom also define all higher-derivatives of q at any instant of time.

1.2.2 Hamilton's principle

Hamilton's principle or the principle of least action:

A mechanical system with s dof is characterised by a definite function

$$L(q^1, \ldots, q^s, \dot{q}^1, \ldots, \dot{q}^s, t) \quad \text{or} \quad L(q, \dot{q}, t)\,,$$

called the Lagrangian of the system. Then, the motion of the system between the position with coordinates $q_{(1)} \equiv q(t_1)$ at the instant t_1 and the position with coordinates $q_{(2)} \equiv q(t_2)$ at the instant t_2 is such that the integral

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt\,, \tag{1.15}$$

takes the least possible value (for t_2 sufficiently close to t_1).

Definition. The integral is called the action of the mechanical system.

L depends on $q, \dot{q}$ but not on $\ddot{q}\,, \dddot{q}, \ldots$ because we assume that the motion of the system is completely determined by coordinates and velocities.

1.2.3 Equations of motion from Hamilton's principle

Hamilton's principle can be used to derive the eom. Let $q = q(t)$ be the function (a set of s functions for a system with s dof) for which S is a minimum. Then S is increased when $q(t)$ is replaced by any function of the form

$$q(t) \to q(t) + \delta q(t)\,, \quad \delta q(t_1) = \delta q(t_2) = 0\,, \tag{1.16}$$

where $\delta q(t)$ is a function which is small everywhere in the interval $[t_1, t_2]$, and it vanishes at t_1 and t_2 because the initial and final positions of the system are fixed.

Definition. $\delta q(t)$ is called a variation of the function $q(t)$.

Thus, according to Hamilton's principle the following condition must hold

$$\begin{aligned}\Delta S = \Delta \int_{t_1}^{t_2} L(q,\dot q,t)dt &= \int_{t_1}^{t_2} L(q+\delta q,\dot q+\delta\dot q,t)dt - \int_{t_1}^{t_2} L(q,\dot q,t)dt \\ &= \int_{t_1}^{t_2} \Big(L(q+\delta q,\dot q+\delta\dot q,t) - L(q,\dot q,t)\Big)dt = \int_{t_1}^{t_2} \Delta L(q,\dot q,t)\,dt \ge 0\,,\end{aligned} \tag{1.17}$$

for any variation δq.

1.2.4 Examples

Example 1. Consider a particle with the action

$$S = \frac{m}{2}\int_{t_1}^{t_2} \dot{\vec r}^{\,2}\,dt\,, \quad \dot{\vec r} = (\dot x,\dot y,\dot z)\,, \quad \dot{\vec r}^{\,2} = \dot{\vec r}\cdot\dot{\vec r} = \dot x^2+\dot y^2+\dot z^2\,. \tag{1.18}$$

The Lagrangian is

$$L = \frac{m}{2}\,\dot{\vec r}^{\,2}\,. \tag{1.19}$$

Let $\vec r = \vec r(t)$ be the vector-valued function for which S is a minimum, and $\delta\vec r(t)$ is its variation. Then ΔS is

$$\begin{aligned}\Delta S &= \frac{m}{2}\int_{t_1}^{t_2}\big((\dot{\vec r}(t)+\delta\dot{\vec r}(t))^2 - \dot{\vec r}^{\,2}\big)\,dt = m\int_{t_1}^{t_2}\dot{\vec r}(t)\cdot\delta\dot{\vec r}(t)\,dt + \frac{m}{2}\int_{t_1}^{t_2}\delta\dot{\vec r}(t)^2\,dt \\ &= -m\int_{t_1}^{t_2}\ddot{\vec r}(t)\cdot\delta\vec r(t)\,dt + \frac{m}{2}\int_{t_1}^{t_2}\delta\dot{\vec r}(t)^2\,dt \ge 0\,,\end{aligned} \tag{1.20}$$

where we integrated by parts and used $\delta\vec r(t_1) = \delta\vec r(t_2) = 0$. The part linear in $\delta\vec r$ is denoted by δS

$$\delta S = -m\int_{t_1}^{t_2}\ddot{\vec r}(t)\cdot\delta\vec r(t)\,dt\,, \tag{1.21}$$

and its integrand must vanish for any $\delta\vec r$ because if it does not vanish then both δS and ΔS can be made negative by choosing for example $\delta\vec r(t) = \epsilon\, m\,\ddot{\vec r}(t)$ where $\epsilon > 0$ is small enough so that $|\delta S|$ is greater than the second term in (1.20). Thus, one gets the equations

$$\ddot{\vec r} = \vec 0 \quad\Longrightarrow\quad \dot{\vec r} = \vec v = \text{const}\,, \tag{1.22}$$

which are the eom for a freely moving particle, and therefore $L = \frac{m}{2}\,\dot{\vec r}^{\,2} = \frac{m}{2}\,\vec v^{\,2}$ is its Lagrangian. We also see that if $\ddot{\vec r} = \vec 0$ then

$$\Delta S = \frac{m}{2}\int_{t_1}^{t_2}\delta\dot{\vec r}(t)^2\,dt\,, \tag{1.23}$$

and it is positive only if $m > 0$. The constant m is the mass of the particle.

Example 2. Consider a particle with the action

$$S = \int_{t_1}^{t_2}\big(\frac{m}{2}\dot{\vec r}^{\,2} + \vec F(t)\cdot\vec r\big)\,dt\,, \quad L(\vec r,\dot{\vec r},t) = \frac{m}{2}\dot{\vec r}^{\,2} + \vec F(t)\cdot\vec r\,, \tag{1.24}$$

where $\vec{F}(t)$ is independent of $\vec{r}, \dot{\vec{r}}$. Then δS, the part linear in $\delta\vec{r}$, is

$$\delta S = \int_{t_1}^{t_2} \big(- m\ddot{\vec{r}} \cdot \delta\vec{r} + \vec{F}(t) \cdot \delta\vec{r} \big)\, dt = \int_{t_1}^{t_2} \big(- m\,\ddot{\vec{r}} + \vec{F}(t) \big) \cdot \delta\vec{r}\, dt\,, \tag{1.25}$$

and it again must vanish. Thus, one gets the equations of motion

$$\delta S = 0 \quad\Longrightarrow\quad m\ddot{\vec{r}} = \vec{F}(t) \quad \text{this is second Newton's law}\,, \tag{1.26}$$

which are the eom for a particle moving under the influence of the force $\vec{F}$.

Example 3. Consider a system of N particles with the action

$$S = \int_{t_1}^{t_2} \big(\sum_{a=1}^{N} \frac{m_a}{2} \dot{\vec{r}}_a^{\,2} - U(\vec{r}_1, \ldots, \vec{r}_N, t) \big)\, dt\,, \quad L(\vec{r}, \dot{\vec{r}}, t) = \sum_{a=1}^{N} \frac{m_a}{2} \dot{\vec{r}}_a^{\,2} - U(\vec{r}_1, \ldots, \vec{r}_N, t)\,. \tag{1.27}$$

where $U(\vec{r}_1, \ldots, \vec{r}_N, t)$ is a differentiable function of $\vec{r}$. Then δS is

$$\delta S = \int_{t_1}^{t_2} \big(- \sum_{a=1}^{N} m_a \ddot{\vec{r}}_a \cdot \delta\vec{r}_a - \sum_{a=1}^{N} \frac{\partial U}{\partial \vec{r}_a} \cdot \delta\vec{r}_a \big)\, dt = - \int_{t_1}^{t_2} \sum_{a=1}^{N} \big(m_a \ddot{\vec{r}}_a + \sum_{a=1}^{N} \frac{\partial U}{\partial \vec{r}_a} \big) \cdot \delta\vec{r}_a\, dt\,, \tag{1.28}$$

and its vanishing gives the equations of motion

$$\delta S = 0 \quad\Longrightarrow\quad m_a \ddot{\vec{r}}_a = - \sum_{a=1}^{N} \frac{\partial U}{\partial \vec{r}_a} = \vec{F}_a\,. \tag{1.29}$$

This is second Newton's law for a system of N particles. The function U is the potential energy of the system. We see that the Lagrangian of this system can be written as

$$L = T - U\,, \quad T = \sum_{a=1}^{N} \frac{m_a}{2} \dot{\vec{r}}_a^{\,2}\,, \quad U = U(\vec{r}_1, \ldots, \vec{r}_N, t)\,, \tag{1.30}$$

where T is the kinetic energy.

1.2.5 Lagrange's equations

The eom for a general system are derived in a similar way. First consider a system with one dof. According to (1.17),

$$\Delta S = \int_{t_1}^{t_2} \Delta L(q, \dot{q}, t)\, dt \ge 0\,, \tag{1.31}$$

where

$$\Delta L(q, \dot{q}, t) = L(q + \delta q, \dot{q} + \delta\dot{q}, t) - L(q, \dot{q}, t) = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \mathcal{O}(\delta q^2)\,. \tag{1.32}$$

Thus, the variation δS, which is the part of ΔS linear in $\delta q, \delta\dot{q}$, is

$$\delta S = \int_{t_1}^{t_2} \big(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \big)\, dt\,, \tag{1.33}$$

and the necessary condition for S to have an extremum is that the variation δS vanishes. Thus, the weak form (S has a critical "point" but not necessarily a minimum) of Hamilton's principle is

$$\delta S = \int_{t_1}^{t_2} \delta L(q, \dot{q}, t)\, dt = 0\,, \quad \delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\,. \tag{1.34}$$

Since $\delta\dot{q} = \frac{d}{dt}\delta q$, we get

$$\delta S = \frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} (\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}})\delta q\, dt = \int_{t_1}^{t_2} (\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}})\delta q\, dt\,, \tag{1.35}$$

where $\frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2} = 0$ because $\delta q(t_1) = \delta q(t_2) = 0$. Thus the eom are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial q} = 0\,, \quad v = \dot{q}\,. \tag{1.36}$$

For a system with s dof the s functions $q^i(t)$ must be varied independently and we get s eom

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial q^i} = 0\,, \quad v^i = \dot{q}^i\,, \quad i = 1, \ldots, s\,. \tag{1.37}$$

These are Lagrange's equations. It is a set of s second-order differential equations for s unknown functions $q^i(t)$. The general solution contains $2s$ arbitrary constants. To fix them and find the motion of the system, one specifies the initial conditions, i.e. the coordinates and velocities at some given instant.

The quantity

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i} \tag{1.38}$$

is called the (generalised) momentum conjugate to the coordinate q^i. By using momenta, the eom take the form

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q^i}\,, \quad p_i = \frac{\partial L}{\partial \dot{q}^i}\,, \quad i = 1, \ldots, s\,. \tag{1.39}$$

The equation $p_i = \frac{\partial L}{\partial \dot{q}^i}$ can be used to express $\dot{q}$ as functions of q and p. Then, the motion of the system is defined by the values of coordinates and momenta at a given instant of time. This is Hamiltonian mechanics to be discussed in the second half of the course.

Example. Find eom of a system with s dof and the Lagrangian

$$L = \frac{1}{2} g_{ij}(q)\dot{q}^i\dot{q}^j\,, \tag{1.40}$$

where $g_{ij} = g_{ji}$ is a second rank tensor which depends only on q's, and we sum over repeated indices. Geometrically, g_{ij} is the metric tensor on the configuration space of the system, and, according to Hamilton's principle, it must be positive definite. The distance between two infinitesimally close points in the configuration space is called the line element, and is defined by

$$ds^2 = g_{ij}(q)dq^i dq^j\,. \tag{1.41}$$

To find the eom we calculate

$$\frac{\partial L}{\partial \dot{q}^i} = g_{ij}\dot{q}^j\,, \quad \frac{\partial L}{\partial q^i} = \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k\,. \tag{1.42}$$

Thus, the eom are

$$\frac{d}{dt}\big(g_{ij}\dot{q}^j\big) - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k = 0\,, \tag{1.43}$$

or after a simplification

$$g_{ij}\ddot{q}^{\,j} + \frac{1}{2}\Big(\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}\Big)\dot{q}^j\dot{q}^k = 0\,. \tag{1.44}$$

Introducing *the Christoffel symbols*

$$\Gamma^i_{jk} = \frac{1}{2}g^{in}\Big(\frac{\partial g_{nj}}{\partial q^k} + \frac{\partial g_{nk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^n}\Big)\,, \tag{1.45}$$

where g^{ij} is the inverse of g_{ij}: $g^{ik}g_{kj} = \delta^i_j$, one gets

$$\ddot{q}^{\,i} + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0\,. \tag{1.46}$$

These are the equations for geodesics (the shortest curve between two points) in the configuration space.

It is of interest to see how the metric tensor g_{ij} transforms under a change of generalised coordinates. Let

$$q^i \to Q^i : \ q^i = q^i(Q^1, \ldots, Q^s) = q^i(Q)\,, \quad i = 1, \ldots, s\,, \tag{1.47}$$

be a nondegenerate transformation of coordinates. Then, the velocity changes as

$$\dot{q}^i \to \dot{Q}^i : \ \dot{q}^i = \frac{\partial q^i}{\partial Q^k}\dot{Q}^k\,, \quad i = 1, \ldots, s\,. \tag{1.48}$$

This is the transformation rule under a change of coordinates of a vector.

Since the Lagrangian should not change, we get

$$L = \frac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j = \frac{1}{2}g_{ij}(q)\frac{\partial q^i}{\partial Q^k}\dot{Q}^k\frac{\partial q^j}{\partial Q^l}\dot{Q}^l = \frac{1}{2}G_{kl}(Q)\dot{Q}^k\dot{Q}^l\,. \tag{1.49}$$

Thus, under a change of coordinates the metric tensor transforms as

$$g_{kl}(q) \to G_{kl}(Q) : \ G_{kl}(Q) = \frac{\partial q^i}{\partial Q^k}\frac{\partial q^j}{\partial Q^l}g_{ij}(q)\,, \quad k, l = 1, \ldots, s\,. \tag{1.50}$$

This is the transformation rule for a second-rank tensor of type $(0, 2)$.

Notice also that a co-vector (a component of one-form) transforms as

$$p_k \to P_k : \ P_k = \frac{\partial q^i}{\partial Q^k}p_i\,, \tag{1.51}$$

and the scalar product

$$p_i\dot{q}^i \tag{1.52}$$

is invariant under a change of coordinates.

1.2.6 Properties of Lagrangian

Additivity of Lagrangian

If a system consists of two subsystems A and B then in the limit where the interaction between the parts may be neglected the Lagrangian L of the system is equal to the sum of Lagrangians of the subsystems A and B

$$\lim_{\text{no interaction between A and B}} L = L_A + L_B \,. \tag{1.53}$$

Multiplicativity of Lagrangian

Multiplication of L of a system by a constant does not change the eom:

$$L \to gL \;\Rightarrow\; \text{eom} \to \text{eom}.$$

However, the Lagrangians of different isolated mechanical systems cannot be multiplied by different constants due to the additivity property. The possibility of multiplying Lagrangians of all systems by the same constant reflects the natural arbitrariness in the choice of the unit of measurement.

Total derivative freedom

Two Lagrangians differing by a total derivative with respect to time of some function $\Lambda(q,t)$ of coordinates and time (no velocities!) lead to the same eom, and therefore describe the same mechanical system. Let

$$\tilde{L}(q,\dot{q},t) = L(q,\dot{q},t) + \frac{d}{dt}\Lambda(q,t)\,. \tag{1.54}$$

The actions are related as

$$\begin{aligned}\tilde{S} = \int_{t_1}^{t_2} \tilde{L}(q,\dot{q},t)dt &= \int_{t_1}^{t_2} L(q,\dot{q},t)dt + \int_{t_1}^{t_2} \frac{d}{dt}\Lambda(q,t)dt \\ &= S + \Lambda(q_{(2)},t_2) - \Lambda(q_{(1)},t_1)\,.\end{aligned} \tag{1.55}$$

Thus,

$$\delta\tilde{S} = \delta S\,, \quad \text{if } \delta q_{(1)} = \delta q_{(2)} = 0\,, \tag{1.56}$$

and therefore S and $\tilde{S}$ give the same eom.

Example. Find eom of a system with s dof and the Lagrangian

$$L = \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j + b_{ij}\dot{q}^i q^j - U(q)\,, \tag{1.57}$$

where $g_{ij} = g_{ji}$ and b_{ij} are constants, and we sum over repeated indices.

We calculate

$$\frac{\partial L}{\partial \dot{q}^i} = g_{ij}\dot{q}^j + b_{ij}q^j\,, \quad \frac{\partial L}{\partial q^i} = b_{ji}\dot{q}^j - \frac{\partial U}{\partial q^i}\,. \tag{1.58}$$

Thus the eom are

$$\frac{d}{dt}\big(g_{ij}\dot{q}^j + b_{ij}q^j\big) - b_{ji}\dot{q}^j + \frac{\partial U}{\partial q^i} = 0\,, \tag{1.59}$$

or after a simplification

$$g_{ij}\ddot{q}^{\,j} + (b_{ij} - b_{ji})\dot{q}^j + \frac{\partial U}{\partial q^i} = 0\,. \tag{1.60}$$

The eom depend only on the anti-symmetric part of b_{ij}. The reason is that the symmetric part of b_{ij} contributes a total time derivative to L. Indeed,

$$\begin{aligned} b_{ij}\dot{q}^i q^j &= \frac{1}{2}(b_{ij} - b_{ji})\dot{q}^i q^j + \frac{1}{2}(b_{ij} + b_{ji})\dot{q}^i q^j \\ &= \frac{1}{2}(b_{ij} - b_{ji})\dot{q}^i q^j + \frac{1}{4}\frac{d}{dt}\big((b_{ij} + b_{ji})q^i q^j\big)\,. \end{aligned} \tag{1.61}$$

So, without loss of generality one can consider b_{ij} to be anti-symmetric. In 3-space the term with b_{ij} describes the interaction of a particle with the magnetic field b_{ij}.

1.2.7 Lagrangian of a constrained system

Consider a system A with generalised coordinates Q^a, $a = 1, \ldots, s + r$ and the Lagrangian

$$L_A = L_A(Q^1, \ldots, Q^{s+r}, \dot{Q}^1, \ldots, \dot{Q}^{s+r}, t)\,. \tag{1.62}$$

Let us now restrict the possible configurations of the system by imposing a set of r independent holonomic constraints

$$C_\alpha(Q^1, \ldots, Q^{s+r}, t) = 0\,, \quad \alpha = 1, \ldots, r\,. \tag{1.63}$$

The constraints reduce the number of dof of A to s. The general solution of the constraints equations (1.63) can be written in the form

$$Q^a = F^a(q^1, \ldots, q^s, t)\,, \quad a = 1, \ldots, s + r\,. \tag{1.64}$$

It depends on s independent parameters q^i which can be chosen as the generalised coordinates of the reduced system.

To analyse the motion of the reduced system we need to know the Lagrangian $L = L(q, \dot{q}, t)$ of the reduced system.

A natural proposal for L is that it is just equal to L_A evaluated on the solution (1.64) of the constraints equations. Taking into account that

$$\dot{Q}^a = \frac{\partial F^a(q^1, \ldots, q^s, t)}{\partial q^i}\dot{q}^i + \frac{\partial F^a(q^1, \ldots, q^s, t)}{\partial t}\,, \tag{1.65}$$

one finds

$$L = L_A(F(q,t), \frac{\partial F(q,t)}{\partial q}\dot{q} + \frac{\partial F(q,t)}{\partial t}, t)\,. \tag{1.66}$$

where $F(q,t)$ denotes the set of $F^1(q,t), \ldots, F^{s+r}(q,t)$, and $\frac{\partial F(q,t)}{\partial q}\dot{q} + \frac{\partial F(q,t)}{\partial t}$ denotes the set of $\frac{\partial F^1(q,t)}{\partial q^i}\dot{q}^i + \frac{\partial F^1(q,t)}{\partial t}, \ldots, \frac{\partial F^{s+r}(q,t)}{\partial q^i}\dot{q}^i + \frac{\partial F^{s+r}(q,t)}{\partial t}$.

Example 1. L of a simple planar pendulum in a uniform gravitational field.

According to the consideration in section 1.1.3, L_A is equal to

$$L_A = T_A - U_A = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgx\,, \tag{1.67}$$

where the x-axis points in the direction of the gravitational field. The solution of the constraint $x^2 + y^2 - l^2 = 0$ is

$$x = l\cos\phi\,, \quad y = l\sin\phi\,, \quad \dot{x} = -l\dot{\phi}\sin\phi\,, \quad \dot{y} = l\dot{\phi}\cos\phi\,. \tag{1.68}$$

Substituting the solution into L_A, one gets

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi\,. \tag{1.69}$$

The eom of the pendulum is

$$\ddot{\phi} + \frac{g}{l}\sin\phi = 0\,. \tag{1.70}$$

Example 2. L of a simple pendulum in 3-space in a uniform gravitational field. According to the consideration in section 1.1.3, L_A is equal to

$$L_A = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz\,, \tag{1.71}$$

where the z-axis points in the direction of the gravitational field. The solution of the constraint $C(x, y, z) = x^2 + y^2 + z^2 - l^2 = 0$ is

$$\begin{aligned} &x = l\cos\varphi\sin\theta\,, \quad y = l\sin\varphi\sin\theta\,, \quad z = l\cos\theta\,, \\ &\dot{x} = -l\dot{\varphi}\sin\varphi\sin\theta + l\dot{\theta}\cos\varphi\cos\theta\,,\ \dot{y} = l\dot{\varphi}\cos\varphi\sin\theta + l\dot{\theta}\sin\varphi\cos\theta\,,\ \dot{z} = -l\dot{\theta}\sin\theta\,. \end{aligned} \tag{1.72}$$

Substituting the solution into L_A, one gets

$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + mgl\cos\theta\,, \tag{1.73}$$

and the eom

$$\begin{aligned} &\frac{d}{dt}(\sin^2\theta\dot{\varphi}) = 0 \ \Rightarrow\ \sin^2\theta\dot{\varphi} = \text{const}\,, \\ &\ddot{\theta} - \frac{1}{2}\dot{\varphi}^2\sin 2\theta + \frac{g}{l}\sin\theta = 0\,. \end{aligned} \tag{1.74}$$

Example 3. L of a coplanar double pendulum. According to the consideration in section 1.1.3, L_A is equal to

$$L_A = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + m_1gx_1 + m_2gx_2\,, \tag{1.75}$$

where the x-axis points in the direction of the gravitational field. The solution of the constraints

$$x_1^2 + y_1^2 - l_1^2 = 0\,, \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 - l_2^2 = 0\,, \tag{1.76}$$

is

$$x_1 = l_1\cos\phi_1\,, \quad y_1 = l_1\sin\phi_1\,, \quad x_2 - x_1 = l_2\cos\phi_2\,, \quad y_2 - y_1 = l_2\sin\phi_2\,. \tag{1.77}$$

Substituting the solution into L_A, one gets

$$\begin{aligned} L = &\frac{1}{2}(m_1 + m_2)l_1^2\dot{\phi}_1^2 + \frac{1}{2}m_2l_2^2\dot{\phi}_2^2 + m_2l_1l_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) \\ &+ (m_1 + m_2)gl_1\cos\phi_1 + m_2gl_2\cos\phi_2\,. \end{aligned} \tag{1.78}$$

Lagrange multipliers

Sometimes it is more convenient to analyse the motion of a constrained system in terms of the original coordinates Q^a. Then one uses a different approach called the method of Lagrange's multipliers. This is a generalisation of the standard Lagrange's method for finding an extremum of a function whose coordinates are subject to a set of constraints.

For each constraint C_α one introduces an auxiliary coordinate λ^α, and considers the following Lagrangian

$$L_{\text{aux}}(Q, \dot{Q}, \lambda, t) = L_A(Q, \dot{Q}, t) + \lambda^\alpha C_\alpha(Q, t)\,. \tag{1.79}$$

Then, the motion of the constrained system between $Q_{(1)} \equiv Q(t_1)$ at the instant t_1 and $Q_{(2)} \equiv Q(t_2)$ at the instant t_2 (both satisfying the constraints) is such that the action

$$S_{\text{aux}} = \int_{t_1}^{t_2} L_{\text{aux}}(Q, \dot{Q}, \lambda, t)dt\,, \tag{1.80}$$

takes the least possible value.

Clearly, one gets the usual Lagrange's eom from S_{aux}

$$\begin{aligned} &\frac{d}{dt}\frac{\partial L_{\text{aux}}}{\partial \dot{Q}^a} = \frac{\partial L_{\text{aux}}}{\partial Q^a} \;\Rightarrow\; \frac{d}{dt}\frac{\partial L_A}{\partial \dot{Q}^a} = \frac{\partial L_A}{\partial Q^a} + \lambda^\alpha \frac{\partial C_\alpha(Q,t)}{\partial Q^\alpha}\,, \quad a = 1, \dots, s + r\,, \\ &\frac{d}{dt}\frac{\partial L_{\text{aux}}}{\partial \dot{\lambda}^\alpha} = \frac{\partial L_{\text{aux}}}{\partial \lambda^a} \;\Rightarrow\; C_\alpha(Q, t) = 0\,, \quad \alpha = 1, \dots, r\,. \end{aligned} \tag{1.81}$$

Thus, the eom which follow from a variation of the Lagrange multipliers are just the constraints. One can show that if one solves the constraints and substitutes the solution to the equations for Q^a in (1.81) then the resulting eom for q^i coincide with the eom derived from the Lagrangian (1.66) for the reduced system.

1.3 Lagrangians of various systems

The Lagrangian describing a mechanical system depends on the choice of a frame of reference. Usually, a frame of reference can be chosen in which space is *homogeneous* and *isotropic* and time is *homogeneous*. Such a frame of reference is called *inertial*.

Let us discuss how to describe the properties mathematically. Consider a system of N particles with radius vectors $\vec{r}_a$, $a = 1, \dots, N$, and the Lagrangian $L(\vec{r}, \vec{v}, t)$ where $\vec{r}$ and $\vec{v} = \dot{\vec{r}}$ denote the sets of radius vectors and velocities.

The homogeneity of space

It means that the eom have the same form at any point of space, or, in other words, the eom are invariant under a simultaneous shift of all the radius vectors $\vec{r}_a$ by a constant vector

$$\vec{r}_a \to \vec{r}_a + \vec{\epsilon}\,. \tag{1.82}$$

Thus, if space is homogeneous then the Lagrangian L may depend only on the differences $\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$ and the velocities which remain unchanged under the shift of space coordinates: $L = L(\vec{r}_{ab}, \vec{v}, t)$.

The homogeneity of time

Similarly, it means that the eom have the same form at any point of time, or, in other words, the eom are invariant under a shift of time by a constant

$$t \to t + \varepsilon\,. \tag{1.83}$$

Thus, if time is homogeneous then the Lagrangian L may not explicitly depend on time: $L = L(\vec{r}, \vec{v})$.

The isotropy of space

It means that the eom have the same form in any direction of space, or, in other words, the eom are invariant under a simultaneous rotation of all the radius vectors $\vec{r}_a$ about the origin

$$x_{ai} \to A_{ij} x_{aj}\,, \quad i,j = 1,\ldots,d\,, \quad \vec{r}_a = (x_{a1}, \ldots x_{ad})\,, \tag{1.84}$$

where A is an orthogonal $d \times d$ matrix with the unit determinant: $A_{ik}A_{jk} = \delta_{ij}$, $\det A = 1$. The velocities obviously transform in the same way. Since rotations preserve distances and angles, if space is isotropic then the Lagrangian L may depend only on the norms of radius vectors and velocities $|\vec{r}_a|$, $|\vec{v}_a|$, the norms of the differences $|\vec{r}_{ab}| = |\vec{r}_a - \vec{r}_b|$, $|\vec{v}_{ab}| = |\vec{v}_a - \vec{v}_b|$, and scalar products $\vec{r}_a \cdot \vec{v}_b$: $L = L(|\vec{r}_a|, |\vec{r}_{ab}|, \vec{r}_a \cdot \vec{v}_b, |\vec{v}_a|, |\vec{v}_{ab}|, t)$.

Combining the results obtained we see that if space is homogeneous and isotropic and time is homogeneous then the Lagrangian of a system of N particles must have the form

$$L = L(|\vec{r}_{ab}|, |\vec{v}_a|, |\vec{v}_{ab}|)\,. \tag{1.85}$$

1.3.1 Particle in homogeneous and isotropic space and time

As the first application of the consideration above, let us consider just a single particle. Then, the Lagrangian is

$$L = L(v^2)\,, \tag{1.86}$$

where instead of $|\vec{v}|$ we assumed that L depends on its square: $v^2 = |\vec{v}|^2$. The eom which follow from this Lagrangian are

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \vec{v}}\Big) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \vec{v}} = \text{const} \quad \Rightarrow \quad \vec{v} = \text{const}\,. \tag{1.87}$$

Thus, a single particle always moves with a constant velocity in homogeneous and isotropic space and time, or equivalently in an inertial frame of reference.

To determine the functional dependence of L on v one has to use a *relativity principle* which states that all inertial frames of reference are equivalent. This implies that the eom derived from the Lagrangian describing the system in one inertial frame are the same as the eom derived from the Lagrangian describing the same system in another inertial frame. Mathematically, this means that any relativity principle must be supplemented with transformation rules which relate coordinates and times in two different frames of references K and K'.

In the case of *Galileo's relativity principle* the rules are called *Galilean transformations*, and they take the form[3]

$$t = t'\,, \quad \vec{r} = \vec{r}\,' + \vec{V} t\,, \;\; \Rightarrow \;\; \vec{v} = \vec{v}\,' + \vec{V}\,, \tag{1.88}$$

[3] In the case of *Einstein's relativity principle* the rules are called *Lorentz transformations*, and they will be discussed in the second part of the course.

where $\vec{V}$ is the velocity of K' measured in K. The eom will be the same if the Lagrangians L' and L differ by the total time derivative of a function of $\vec{r}$ and t. Considering an infinitesimal Galilean transformation

$$\vec{v} = \vec{v}\,' + \vec{\epsilon}\,, \tag{1.89}$$

and expanding $L' = L(v'^2)$ in powers of ϵ keeping only the linear term in ϵ, one gets

$$\begin{aligned} L(v'^2) &= L(v^2 - 2\vec{v}\cdot\vec{\epsilon} + \vec{\epsilon}^{\,2}) = L(v^2) - \frac{\partial L}{\partial v^2}\,2\vec{v}\cdot\vec{\epsilon} + \mathcal{O}(\epsilon^2) \\ &= L(v^2) - 2\frac{\partial L}{\partial v^2}\frac{d}{dt}(\vec{r}\cdot\vec{\epsilon}) + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{1.90}$$

It is obvious that the second term on the second line is a total time derivative only if $\frac{\partial L}{\partial v^2}$ =const. Thus,

$$L = \frac{1}{2}mv^2\,, \tag{1.91}$$

where the quantity m is the mass of the particle which, as was discussed earlier, must be positive. It is easy to check that L' and L given by (1.91) differ by a total time derivative for a finite Galilean transformation.

For a system of noninteracting particles the additivity property of the Lagrangian leads to

$$L = \sum_a \frac{1}{2}m_a v_a^2\,. \tag{1.92}$$

It is equal to the kinetic energy of the system. Due to the multiplicativity property of L only ratios of the masses are physically meaningful.

To find the Lagrangian of a particle in other coordinate systems it is useful to notice that

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{ds^2}{dt^2}\,, \tag{1.93}$$

where ds is the element of arc or the line element in a given coordinate system.

In Cartesian coordinates

$$\begin{aligned} &ds^2 = dx^2 + dy^2 + dz^2 = g_{ij}dq^i dq^j\,, \\ &q^1 = x\,,\ q^2 = y\,,\ q^3 = z\,,\ (g_{ij}) = \mathrm{diag}\,(1,1,1)\,, \end{aligned} \tag{1.94}$$

where g_{ij} are the components of the metric tensor of $\mathbb{R}^3$ in Cartesian coordinates.

In cylindrical coordinates

$$x = r\cos\phi\,,\quad y = r\sin\phi\,,\quad z = z\,, \tag{1.95}$$

one finds

$$\begin{aligned} &ds^2 = dr^2 + r^2 d\phi^2 + dz^2 = g_{ij}dq^i dq^j\,, \\ &q^1 = r\,,\ q^2 = \phi\,,\ q^3 = z\,,\ (g_{ij}) = \mathrm{diag}\,(1,r^2,1)\,, \end{aligned} \tag{1.96}$$

where g_{ij} are the components of the metric tensor of $\mathbb{R}^3$ in cylindrical coordinates.

In spherical coordinates

$$x = r\cos\varphi\sin\theta\,,\quad y = r\sin\varphi\sin\theta\,,\quad z = r\cos\theta\,, \tag{1.97}$$

one finds

$$
\begin{aligned}
&ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 = g_{ij} dq^i dq^j \,, \\
&q^1 = r\,,\ q^2 = \theta\,,\ q^3 = \phi\,,\ (g_{ij}) = \mathrm{diag}\,(1, r^2, r^2\sin^2\theta)\,,
\end{aligned}
\tag{1.98}
$$

where g_{ij} are the components of the metric tensor of $\mathbb{R}^3$ in spherical coordinates.

Finally, for arbitrary generalised coordinates q^i

$$
x = f_1(q^1, q^2, q^3)\,, \quad y = f_2(q^1, q^2, q^3)\,, \quad z = f_3(q^1, q^2, q^3)\,, \tag{1.99}
$$

one finds

$$
ds^2 = g_{ij}(q) dq^i dq^j\,, \quad g_{ij}(q) = \frac{\partial f_a}{\partial q^i}\frac{\partial f_a}{\partial q^j}\,, \tag{1.100}
$$

and

$$
L = \frac{1}{2} m\, g_{ij}(q) \dot{q}^i \dot{q}^j\,, \tag{1.101}
$$

where we sum over repeated indices, and g_{ij} are the components of the metric tensor of $\mathbb{R}^3$ expressed in terms of q^i. The Lagrangian of a system of N noninteracting particles in d-dimensional space has the same form in terms of generalised coordinates q^i, $i = 1, \ldots, d\,N$, and the quantity m is fixed by a choice of the unit of mass for the system.

1.3.2 System of particles subject to Galileo's principle

Consider a system of N particles in homogeneous and isotropic space and time, and assume that Galileo's relativity principle is valid. Then, the Lagrangian of the system in Cartesian coordinates has the form

$$
L = \sum_{a=1}^{N} \frac{1}{2} m_a v_a^2 - U(|\vec{r}_{ab}|)\,, \tag{1.102}
$$

where the function $U(|\vec{r}_{ab}|) \equiv U(|\vec{r}_{12}|, |\vec{r}_{13}|, \ldots, |\vec{r}_{N-1,N}|)$ is called the *potential energy* of the system, and it may depend on the differences $|\vec{r}_{ab}|$, $a, b = 1, \ldots, N$. The Lagrangian eom can be easily derived

$$
m_a \ddot{\vec{r}}_a = -\frac{\partial U}{\partial \vec{r}_a} = \vec{F}_a\,. \tag{1.103}
$$

These are Newton's equations, and $\vec{F}_a$ is the force on the ath particle which depends only on distances between particles. Since the force depends on the particles coordinates any change in the position of any particle instantaneously effects all other particles, and therefore the interactions are instantaneously propagated. Note also that the Lagrangian (1.102) is invariant under the time reversal: if t is replaced by $-t$ the Lagrangian and therefore the eom are unchanged. This means that all motions in homogeneous and isotropic space and time which obey Galileo's relativity principle are reversible.

1.3.3 Closed system of particles

Consider a system of N particles with the following Lagrangian

$$
L = \sum_{a=1}^{N} \frac{1}{2} m_a v_a^2 - U(\vec{r})\,, \tag{1.104}
$$

where the function $U(\vec{r}) \equiv U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ is again the *potential energy* of the system, but now it may depend not only on $|\vec{r}_{ab}|$ but also on the radius vectors $\vec{r}_a$. Obviously, this system of particles does not obey Galileo's relativity principle, and the Lagrangian describes the system in a special reference frame. Still, the Lagrangian can be thought of as a limiting case of the Lagrangian (1.102) for a system of $N+M$ particles where the $N+1$, $N+2, \ldots, N+M$-th particles are infinitely massive. According to (1.103), if $m_a = \infty$ then the velocity of ath particle must be constant. Assuming that the velocities of all the infinitely massive particles are the same, one can choose the inertial frame so that their common velocity is zero. Then, the Lagrangian (1.102) reduces to

$$L = \sum_{a=1}^{N} \frac{1}{2} m_a v_a^2 - U(|\vec{r}_{ab}|, |\vec{r}_{a\beta}|)\,, \tag{1.105}$$

where $\vec{r}_{a\beta} = \vec{r}_a - \vec{r}_\beta$, $\beta = N+1, \ldots, N+M$, and $\vec{r}_\beta$ are the constant radius vectors of the infinitely massive particles. By choosing properly $\vec{r}_\beta$ and the number of infinitely massive particles M (which may be infinite too), one can represent any function $U(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$ in the form $U(|\vec{r}_{ab}|, |\vec{r}_{a\beta}|)$.

In particular if $N=1$ and $M=1$, and $\vec{r}_2 = \vec{0}$, one gets

$$L = \frac{1}{2} m v^2 - U(|\vec{r}|)\,, \tag{1.106}$$

which is the Lagrangian of a particle moving in a central field.

Another interesting case is described by the Lagrangian

$$L = \frac{1}{2} m v^2 + \vec{F} \cdot \vec{r}\,, \tag{1.107}$$

where $\vec{F}$ is a constant force. This Lagrangian is for a particle moving in a uniform field. It can be obtained from (1.102) if the infinitely heavy particles form an infinite wall and interact with the "light" particle with a Coulomb potential.

1.3.4 System A moving in a given external field due to B

This is a generalisation of a closed system of particles. Consider systems A and B interacting with each other. Then, according to (1.102), the Lagrangian for the combined system $A+B$ can be written in the form

$$\begin{gathered} L_{A+B} = L_A + L_B + L_{\text{interaction}}\,, \\ L_A = T_A(v_A) - U(\vec{r}_A)\,, \quad L_B = T_B(v_B) - U(\vec{r}_B)\,, \quad L_{\text{interaction}} = -U(\vec{r}_A, \vec{r}_B)\,. \end{gathered} \tag{1.108}$$

If the particles of the system B are much heavier than those of A then one can neglect the back-reaction of A on B. Then, the trajectories of the particles of B follow from its Lagrangian L_B. Fixing some initial conditions for the particles of B one can solve their Lagrange's eom and find $\vec{r}_B = \vec{r}_B(t)$ as some definite functions of time. Substituting these functions into L_{A+B}, we see that it is equivalent to

$$L = T_A(v_A) - U(\vec{r}_A) - U(\vec{r}_A, \vec{r}_B(t))\,, \tag{1.109}$$

because they differ by a function of time only which can be always written as a time derivative of its anti-derivative. The Lagrangian L shows that we can think of A as being in an external time-dependent field created by B. Note, that the potential energy of the external field may depend explicitly on time.

Chapter 2

CONSERVATION LAWS

For any mechanical system there are functions of coordinates, velocities and time which remain constant during the motion. Such a function, $I(q, \dot{q}, t)$, is called an *integral of motion*: $\frac{d}{dt} I(q, \dot{q}, t) = 0$. Solving the eom one finds

$$q^i = q^i(t, c^1, \ldots, c^{2s}) \text{ and } \dot{q}^i = \dot{q}^i(t, c^1, \ldots, c^{2s}) \,,$$

where the constants c^i can be fixed by the initial conditions. One can think of the solution as a system of $2s$ equations on c^i. Solving these equations, one finds c^i as some functions of q, $\dot{q}$ and t. They obviously do not change during the motion, and therefore they are integrals of motion. Since any integral of motion is a function of c^i and t, there are $2s$ independent integrals.

If the system is closed, i.e. its eom are invariant under shifts of time (homogeneity of time), then the solution of eom depends on $t - t_0$ and $2s - 1$ other constants

$$q^i = q^i(t - t_0, c^1, \ldots, c^{2s-1}) \text{ and } \dot{q}^i = \dot{q}^i(t - t_0, c^1, \ldots, c^{2s-1}) \,,$$

One can use one of the equations to express $t - t_0$ as a function of q, $\dot{q}$ and c. Substituting $t - t_0$ into the remaining expressions one can then express the $2s - 1$ constants c^i as some functions of q and $\dot{q}$ without any explicit dependence of t. Thus, for a closed system there are $2s - 1$ integrals which depend on q and $\dot{q}$ only.

The most important integrals are *additive*: if in some limit the system is composed of several noninteracting parts then the value of an additive integral is equal to the sum of its values for these parts. For example consider a system which at $t = -\infty$ is composed of noninteracting subsystems A and B, while at $t = +\infty$ it is composed of noninteracting subsystems A' and B'. Then, an additive integral I for the system satisfies the relation

$$I = I_A + I_B = I_{A'} + I_{B'} \,, \tag{2.1}$$

which imposes essential restrictions on possible motions of the system. The quantities represented by additive integrals are said to be *conserved*. It appears that each conserved quantity is related to a particular continuous symmetry of a mechanical system, e.g. energy is related to homogeneity of time, momentum to homogeneity of space, and angular momentum to isotropy of space. This relation is made manifest by Emmy Noether's theorem, formulated in 1918.

2.1 Noether's theorem

Consider an arbitrary transformation of coordinates $q^i \to \tilde{q}^i = \tilde{q}^i(q, \dot{q}, t)$. The coordinates q^i satisfy their Lagrange's eom while the transformed coordinates $\tilde{q}^i$ in general satisfy different eom because the form of the Lagrangian may not be preserved by the transformation. A transformation which maps a solution of the eom to another solution of the same eom is called a *symmetry* of the mechanical system. Since an inverse symmetry transformation is a symmetry, and a composition of two symmetry transformations is a symmetry too, the set of all symmetry transformations of a mechanical system forms a *group*.

A transformation of coordinates is called *continuous* if it depends on a parameter which can take any value in some interval. For example a shift of a coordinate by a constant, or a rotation of a radius vector through an angle are continuous transformations. The zero value of the parameter usually corresponds to the identity transformation.

Any known continuous symmetry transformation is analytic with respect to its parameter (can be expanded in power series in the parameter). Thus, the set of all continuous symmetry transformations forms a *Lie group*. According to Noether's theorem, continuous symmetry transformations are in one-to-one correspondence with conserved quantities. Here, we only discuss how a continuous symmetry transformation leads to a conservation law. It is sufficient to consider infinitesimal transformations close to the identity transformation so that their parameters are infinitesimally small.

As we discussed, if the Lagrangian under a transformation changes by a total time derivative of a function of coordinates and time then the eom for the original and transformed coordinates are the same. Therefore, such a transformation is a symmetry. Moreover, it is clear that it must only depend on coordinates and time. We will formulate Noether's theorem only for transformations which only depend on coordinates and time.

Noether's theorem. Consider a mechanical system with s dof. Let an infinitesimal transformation of coordinates

$$q^i \to \tilde{q}^i = q^i + \delta q^i \,, \quad \delta q^i = \epsilon\, \zeta^i(q,t)\,, \quad i = 1, \dots, s\,, \tag{2.2}$$

depend only on q and t, and let the Lagrangian of the system transform under this transformation as

$$L(q, \dot{q}, t) \to L(\tilde{q}, \dot{\tilde{q}}, t) = L(q, \dot{q}, t) + \epsilon \frac{d}{dt}\Lambda(q,t)\,. \tag{2.3}$$

Then, the transformation is a symmetry, and the quantity

$$J = \frac{\partial L}{\partial \dot{q}^i}\zeta^i - \Lambda \tag{2.4}$$

is conserved, i.e.

$$\frac{d}{dt} J = 0\,, \tag{2.5}$$

if q^i satisfy the eom.

Proof. We just take J and differentiate it with respect to time

$$\begin{aligned}\frac{d}{dt}J &= \frac{d}{dt}(\frac{\partial L}{\partial \dot q^i})\zeta^i + \frac{\partial L}{\partial \dot q^i}\frac{d}{dt}\zeta^i - \frac{d}{dt}\Lambda \\ &= \frac{d}{dt}(\frac{\partial L}{\partial \dot q^i})\zeta^i + \frac{\partial L}{\partial \dot q^i}\dot\zeta^i - (\frac{\partial L}{\partial q^i}\zeta^i + \frac{\partial L}{\partial \dot q^i}\dot\zeta^i) \\ &= \Big(\frac{d}{dt}(\frac{\partial L}{\partial \dot q^i}) - \frac{\partial L}{\partial q^i}\Big)\zeta^i = 0\,,\end{aligned} \tag{2.6}$$

where the last line vanishes due to Lagrange's eom, and in the second line we used that under any variation of q

$$L(\tilde q, \dot{\tilde q}, t) - L(q, \dot q, t) = \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot q^i}\delta \dot q^i\,, \tag{2.7}$$

and therefore

$$\frac{d}{dt}\Lambda = \frac{\partial L}{\partial q^i}\zeta^i + \frac{\partial L}{\partial \dot q^i}\dot\zeta^i\,. \tag{2.8}$$

Remark 1. Since the momentum is $p_i = \partial L/\partial \dot q^i$, the conserved integral can be written in the form

$$J = p_i\zeta^i - \Lambda\,. \tag{2.9}$$

Thus, if $\delta\Lambda = 0$ then the conserved integral in terms of p and q has a universal form independent of the form of the Lagrangian.

Remark 2. Multiplying J by ϵ, one gets

$$J\epsilon = \frac{\partial L}{\partial \dot q^i}\delta q^i - \Lambda\epsilon\,. \tag{2.10}$$

This form admits a straightforward generalisation to the case where a symmetry transformation $q^i \to q^i + \delta q^i\,,\ \delta q^i = \epsilon^\alpha\,\zeta^i_\alpha(q,t)$ depends on several parameters ϵ^α, so it is a composition of one-parameter transformations. Then, for each α there is a conserved quantity which can be extracted from the formula

$$J_\alpha\epsilon^\alpha = \frac{\partial L}{\partial \dot q^i}\delta q^i - \Lambda_\alpha\epsilon^\alpha\,, \tag{2.11}$$

where as usual we sum over repeated indices.

Remark 3. One can easily see that the quantity $J = \frac{\partial L}{\partial \dot q^i}\zeta^i - \Lambda$ is conserved even if ζ^i and Λ have any dependence on q, $\dot q$, and so on. The difficult part is then to show that the corresponding transformation is a symmetry, i.e. it indeed maps a a solution of the eom to another solution of the same eom. We will discuss it later on the example of the energy conservation.

2.2 Applications of Noether's theorem

2.2.1 Cyclic coordinate

Consider a system with s dof described by a Lagrangian which has no explicit dependence on one of the coordinates, say, the last one q^s

$$L = L(q^1, \ldots, q^{s-1}, \dot q^1\ \ldots, \dot q^{s-1}, \dot q^s, t)\,. \tag{2.12}$$

Such a coordinate is called *cyclic*. Obviously, the Lagrangian is invariant under a shift of the cyclic coordinate by a constant

$$q^s \to q^s + \epsilon\,, \quad q^1 \to q^1\,, \ldots, q^{s-1} \to q^{s-1} \;\Rightarrow\; L \to L\,. \tag{2.13}$$

Therefore,

$$\delta q^s = \epsilon\,, \quad \delta q^1 = 0\,, \ldots, \delta q^{s-1} = 0\,, \quad \Lambda = 0\,, \tag{2.14}$$

and the following quantity is conserved

$$J\epsilon = \frac{\partial L}{\partial \dot q^i}\delta q^i = \frac{\partial L}{\partial \dot q^s}\epsilon \;\Rightarrow J = \frac{\partial L}{\partial \dot q^s} = p_s\,. \tag{2.15}$$

Thus, the momentum conjugate to a cyclic coordinate is conserved.

2.2.2 Linear momentum $\leftrightarrow$ homogeneity of space

If the space is homogeneous then the Lagrangian of a system is invariant under a simultaneous shift of all radius-vectors by a constant vector

$$\vec r_a \to \vec r_a + \vec\epsilon, \quad \delta\vec r_a = \vec\epsilon, \quad L \to L \;\Rightarrow \sum_a \frac{\partial L}{\partial \vec r_a} = 0\,, \tag{2.16}$$

and the following vector quantity is conserved

$$\vec P\cdot\vec\epsilon = \frac{\partial L}{\partial \dot{\vec r}_a}\cdot\delta\vec r_a = \sum_a \frac{\partial L}{\partial \dot{\vec r}_a}\cdot\vec\epsilon \;\Rightarrow \vec P = \sum_a \frac{\partial L}{\partial \dot{\vec r}_a} = \sum_a \vec p_a\,. \tag{2.17}$$

The total (linear) momentum is called the momentum of the system, and it is conserved if the space is homogeneous. The momentum is obviously additive, and moreover, it is equal to the sum of momenta $\vec p_a$ of individual particles even if the interaction between them cannot be neglected.

The condition $\sum_a \frac{\partial L}{\partial \vec r_a} = 0$ has a simple physical meaning. Since the force $\vec F_a$ acting on the ath particle is equal to $\frac{\partial L}{\partial \vec r_a}$, the condition just states that in homogeneous space the sum of the forces on all the particles is zero

$$\sum_a \vec F_a = 0\,. \tag{2.18}$$

For a system of two particles one gets $\vec F_1 + \vec F_2 = 0$ which is Newton's third law: the equality of action and reaction.

Centre of mass

Consider a system with the Lagrangian

$$L = \frac12 m_a \dot{\vec r}_a^{\,2} - U(\vec r_{ab}, t)\,. \tag{2.19}$$

The momentum $\vec P = \sum_a m_a\vec v_a$ of this system is conserved. It is measured in a frame K, and in a frame K' moving with the velicity $\vec V$ relative to K, the momentum $\vec P'$ is

$$\vec P' = \sum_a m_a \vec v_a' = \sum_a m_a\vec v_a - \sum_a m_a\vec V = \vec P - \mu\vec V\,, \tag{2.20}$$

where $\mu = \sum_a m_a$ is the total mass of the system. Thus, if

$$\vec{V} = \frac{\vec{P}}{\mu} = \frac{\sum_a m_a \vec{v}_a}{\sum_a m_a}\,, \tag{2.21}$$

then the total momentum of the system in K' is zero. This is the velocity of the system as a whole. It can be also written in the form

$$\vec{V} = \frac{d\vec{R}}{dt}\,, \quad \vec{R} = \frac{\sum_a m_a \vec{r}_a}{\sum_a m_a}\,, \tag{2.22}$$

where $\vec{R}$ is the radius-vector of the centre of mass of the system. The conservation of momentum implies that the centre of mass moves uniformly in a straight line.

Let us introduce new coordinates $\vec{R}$, $\vec{\rho}_a$ by the formula

$$\vec{r}_a = \vec{\rho}_a + \vec{R}\,. \tag{2.23}$$

The coordinates $\vec{\rho}_a$ are not independent. They are subject to the constraint

$$\sum_a m_a \vec{\rho}_a = 0 \;\Rightarrow\; \sum_a m_a \dot{\vec{\rho}}_a = 0\,. \tag{2.24}$$

Expressing L in terms of the new coordinates, one finds

$$\begin{aligned}
L &= \frac{1}{2}\sum_a m_a(\dot{\vec{\rho}}_a + \dot{\vec{R}})^2 - U(\vec{\rho}_{ab}, t) \\
&= \frac{1}{2}\sum_a m_a \dot{\vec{\rho}}_a^{\,2} + \sum_a m_a \dot{\vec{\rho}}_a \cdot \dot{\vec{R}} + \frac{1}{2}\sum_a m_a \dot{\vec{R}}^2 - U(\vec{\rho}_{ab}, t) \\
&= \frac{1}{2}\mu \dot{\vec{R}}^2 + \frac{1}{2}\sum_a m_a \dot{\vec{\rho}}_a^{\,2} - U(\vec{\rho}_{ab}, t)\,.
\end{aligned} \tag{2.25}$$

We see that L is independent of $\vec{R}$, and therefore the radius-vector of the centre of mass is cyclic. Therefore, its momentum, which is the momentum of the system, is conserved.

2.2.3 Angular momentum $\leftrightarrow$ isotropy of space

If the space is isotropic then the Lagrangian of a system is invariant under a simultaneous rotation of all radius-vectors about any line through the origin. Any rotation in 3-space can be characterised by a vector $\vec{\phi}$ whose direction coincides with the direction of rotation (being that of a right-handed screw driven along $\vec{\phi}$) and whose norm $\phi = |\vec{\phi}|$ is equal to the angle of rotation.

Any radius-vector $\vec{r}_a = (x_{a1}, x_{a2}, x_{a3})$ transforms as follows under an infinitesimal rotation $\delta\vec{\phi}$

$$\vec{r}_a \to \vec{r}_a + \delta\vec{r}_a\,, \quad \delta\vec{r}_a = \delta\vec{\phi} \times \vec{r}_a\,, \tag{2.26}$$

In terms of the coordinates x_{ai}, $i = 1, 2, 3$ the formula takes the form

$$\delta x_{ai} = \varepsilon_{ijk}\epsilon_j x_{ak}\,, \quad \epsilon_i \equiv \delta\phi_i\,, \tag{2.27}$$

where we sum over repeated indices, and ε_{ijk} is the anti-symmetric tensor, also called the Levi-Cevita symbol: $\varepsilon_{123} = 1$, $\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji}$.

If the Lagrangian is invariant under this transformation then

$$M_i \epsilon_i = \sum_a \frac{\partial L}{\partial \dot{x}_{ai}} \delta x_{ai} = \sum_a \frac{\partial L}{\partial \dot{x}_{ai}} \varepsilon_{ijk} \epsilon_j x_{ak} = \sum_a \varepsilon_{ijk} x_{aj} p_{ak} \epsilon_i \, . \tag{2.28}$$

Thus

$$M_i = \sum_a \varepsilon_{ijk} x_{aj} p_{ak} = \sum_a (\vec{r}_a \times \vec{p}_a)_i \;\Rightarrow\; \vec{M} = \sum_a \vec{r}_a \times \vec{p}_a \, . \tag{2.29}$$

is conserved. It is called the *angular momentum* of the system.

Properties of the angular momentum

1. Angular momentum is additive even if the interactions are not negligible (same the linear momentum).

2. Its value depends on the choice of origin

$$\begin{aligned} &\vec{r}_a = \vec{r}_a^{\,\prime} + \vec{b} \, , \\ &\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a = \sum_a \vec{r}_a \times \vec{p}_a^{\,\prime} + \vec{b} \times \sum_a \vec{p}_a = \vec{M}' + \vec{b} \times \vec{P} \, , \end{aligned} \tag{2.30}$$

   except when the system is at rest as a whole, $\vec{P} = 0$.

3. $\vec{M}$ and $\vec{M}'$ in frames K and K' are related as follows. Assume the origins coincide at a given instant, and therefore

$$\begin{aligned} &\vec{r}_a = \vec{r}_a^{\,\prime} \, , \quad \vec{v}_a = \vec{v}_a^{\,\prime} + \vec{V} \, , \\ &\vec{M} = \sum_a \vec{r}_a \times m_a \vec{v}_a = \sum_a \vec{r}_a \times m_a \vec{v}_a^{\,\prime} + \sum_a m_a \vec{r}_a \times \vec{V} = \vec{M}' + \mu \vec{R} \times \vec{V} \, , \end{aligned}$$

   If K' is that in which the system is at rest as a whole, then $\vec{V}$ is the velocity of its centre of mass relative to K, and $\mu \vec{V} = \vec{P}$, and therefore $\vec{M} = \vec{M}' + \vec{R} \times \vec{P}$. Thus, $\vec{M}$ consists of its *"intrinsic angular momentum"* in a frame in which it is at rest, and $\vec{R} \times \vec{P}$ due to its motion as a whole.

4. If an external field is *symmetric about an axis* then the angular momentum along the axis is conserved. The angular momentum must be defined relative to an origin lying on the axis.

   (a) In the case of a *centrally symmetric field* or *central field* the potential depends only on the distance from some particular point (the centre of the field). Then, the system is invariant under a rotation about any axis passing through the centre, and therefore $\vec{M}$ is conserved provided it is defined with respect to the centre of the field.

   (b) If the field pointing in the z-direction is homogeneous then M_z is conserved whichever point is taken as the origin.

5. The component of $\vec{M}$ along any axis, say the x_3-axis, can be found as

$$M_3 = \sum_a \frac{\partial L}{\partial \dot{\phi}_a} = \sum_a p_{\phi_a} \,, \tag{2.31}$$

where ϕ_a is the polar angle of the ath particle in cylindrical coordinates

$$x_{a1} = r_a \cos\phi_a \,, \quad x_{a2} = r_a \sin\phi_a \,.$$

Indeed, the Lagrangian is

$$L = \frac{1}{2}\sum_a m_a(\dot{r}_a^2 + r_a^2\dot{\phi}_a^2 + \dot{z}_a^2) - U \,, \tag{2.32}$$

and

$$M_3 = \sum_a m_a(x_{a1}\dot{x}_{a2} - x_{a2}\dot{x}_{a1}) = \sum_a m_a r_a^2 \dot{\phi}_a \,. \tag{2.33}$$

Thus, the M_3 component of the angular momentum of the ath particle is the (generalised) momentum p_{ϕ_a} conjugate to the angle coordinate ϕ_a.

2.2.4 Angular momentum in d-dimensional space

Rotations in d-dimensional space are characterised by real orthogonal matrices A with the unit determinant[1] which form the special orthogonal group $SO(d)$:

$$A \cdot A^T = A^T \cdot A = I \,, \quad \det A = 1 \,.$$

The components x_i, $i = 1, \ldots, d$ of any radius-vector $\vec{r} = (x_1, \ldots, x_d)$ transform as follows

$$x_i \to x_i' = A_{ij}x_j \,. \tag{2.34}$$

To find an infinitesimal transformation we represent

$$A = I + \epsilon \,, \tag{2.35}$$

where the entries ϵ_{ij} of the matrix ϵ are infinitesimally small. Since A is orthogonal the matrix ϵ is anti-symmetric

$$(I+\epsilon)(I+\epsilon^T) = I + \epsilon + \epsilon^T + \mathcal{O}(\epsilon^2) = I \;\Rightarrow\; \epsilon + \epsilon^T = 0 \,. \tag{2.36}$$

We say that the vector space of anti-symmetric matrices is the Lie algebra of the special orthogonal group $SO(d)$.

Thus, infinitesimal rotations in d-space have the form

$$x_i \to x_i' = x_i + \epsilon_{ij}x_j \;\Rightarrow\; \delta x_i = \epsilon_{ij}x_j \,, \quad \epsilon_{ji} = -\epsilon_{ij} \,, \tag{2.37}$$

where we sum over repeated indices. The parameters ϵ_{ij} have simple meaning – they are the angles of rotation in the x_ix_j-plane. Indeed, let us assume that only ϵ_{12} and $\epsilon_{21} = -\epsilon_{12}$ do not vanish. Then, one gets

$$\delta x_1 = \epsilon_{1j}x_j = \epsilon_{12}x_2 \,, \quad \delta x_2 = \epsilon_{2j}x_j = -\epsilon_{12}x_1 \,, \quad \delta x_i = 0 \,, \quad i = 2, 3, ..., d \,. \tag{2.38}$$

[1] An orthogonal matrix with $\det A = -1$ describes a composition of a rotation with an odd number of reflections, e.g. $x_1 \to -x_1$.

These formulae describe the infinitesimal rotation in the x_1x_2-plane through the angle ϵ_{21}.

Assuming now that the Lagrangian is invariant under rotations, one gets the angular momentum

$$\frac{1}{2}J_{ij}\epsilon_{ij} = \sum_a \frac{\partial L}{\partial \dot{x}_{ai}}\delta x_{ai} = \sum_a p_{ai}\epsilon_{ij}x_{aj} = \frac{1}{2}\sum_a (p_{ai}x_{aj} - p_{aj}x_{ai})\epsilon_{ij}\,, \tag{2.39}$$

where J_{ij} are anti-symmetric, and we multiplied the lhs by 1/2 to avoid the double counting. Thus, in d-dimensional space the angular momentum is given by

$$J_{ij} = \sum_a (p_{ai}x_{aj} - p_{aj}x_{ai})\,. \tag{2.40}$$

Finally, let us mention that in 3-dimensional space the relation of ϵ_{ij} with ϵ_i follows from (2.27)

$$\epsilon_{ij} = \epsilon_{ikj}\epsilon_k = -\epsilon_{ijk}\epsilon_k \;\Rightarrow\; \epsilon_i = -\frac{1}{2}\epsilon_{ijk}\epsilon_{jk}\,, \tag{2.41}$$

and the relation between J_{ij} and M_i is similar

$$J_{ij} = -\epsilon_{ijk}M_k \;\Rightarrow\; M_i = -\frac{1}{2}\epsilon_{ijk}J_{jk}\,. \tag{2.42}$$

2.2.5 Particle in a magnetic field

Consider a particle in a magnetic field described by the Lagrangian

$$L = \frac{1}{2}m\dot{x}_i^2 \,+\, b_{ij}\dot{x}_i x_j\,, \tag{2.43}$$

where $b_{ij} = -b_{ji}$ are constants, and we sum over repeated indices.

The eom are invariant under a constant shift of coordinates

$$x_i \to x_i + \epsilon_i\,, \quad \delta x_i = \epsilon_i\,, \tag{2.44}$$

because L changes by a total time derivative

$$L \to L + b_{ij}\dot{x}_i\epsilon_j\,, \quad \delta L = \frac{d}{dt}b_{ij}x_i\epsilon_j \;\Rightarrow \Lambda_i\epsilon_i = b_{ij}x_i\epsilon_j\,. \tag{2.45}$$

Thus, the corresponding conserved quantities are

$$J_i\epsilon_i = \frac{\partial L}{\partial \dot{x}_i}\delta x_i - b_{ij}x_i\epsilon_j = (m\dot{x}_i + b_{ij}x_j)\epsilon_i - b_{ij}x_i\epsilon_j = (m\dot{x}_i + 2b_{ij}x_j)\epsilon_i\,, \tag{2.46}$$

or

$$J_i = m\dot{x}_i + 2b_{ij}x_j = \text{const}\,. \tag{2.47}$$

2.2.6 Particle in a gravitational field

Consider a particle in a gravitational field described by the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 \,+\, mgx\,, \tag{2.48}$$

Under a shift of x: $x \to x' = x + \epsilon$, the Lagrangian changes by a total time derivative

$$L \to L + mg\epsilon \,, \quad \delta L = mg\epsilon = \frac{d}{dt}(mgt\epsilon) \;\Rightarrow\; \Lambda\epsilon = mgt\epsilon \,. \tag{2.49}$$

Thus, the shift of x is a symmetry, and the corresponding conserved quantity is

$$J\epsilon = \frac{\partial L}{\partial \dot{x}}\delta x - mgt\epsilon = (m\dot{x} - mgt)\epsilon \;\Rightarrow\; J = m\dot{x} - mgt \,. \tag{2.50}$$

2.3 Conservation of energy $\leftrightarrow$ homogeneity of time

In all the examples we've considered an infinitesimal symmetry transformation depended only on q. There are, however, situations where it depends also on $\dot{q}$, and, therefore, the transformed eom naively are third-order differential equations. In reality for a finite transformation the transformed equations are second-order differential equations and the appearance of third derivative terms is an artefact of the expansion in the infinitesimal parameter.

As an example, let the time be homogeneous, and let us therefore consider a system with the Lagrangian which has no explicit time dependence. Then, the eom are obviously invariant under a shift of time: $t \to t' = t + \epsilon$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \;\leftrightarrow\; \frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}^i} = \frac{\partial \tilde{L}}{\partial \tilde{q}^i} \,, \tag{2.51}$$

where

$$\tilde{q}^i(t) = q^i(t+\epsilon) \,, \quad \tilde{L} = L(\tilde{q}, \dot{\tilde{q}}) \,. \tag{2.52}$$

On the other hand, infinitesimal transformations of q and L are given by

$$\begin{aligned} &q^i(t) \to \tilde{q}^i(t) = q^i(t) + \dot{q}^i(t)\epsilon \;\Rightarrow\; \delta q^i(t) = \dot{q}^i(t)\epsilon \,, \\ &L \to \tilde{L} = L(q + \dot{q}\epsilon, \dot{q} + \ddot{q}\,\epsilon) = L(q,\dot{q}) + \frac{\partial L}{\partial q^i}\dot{q}^i\epsilon + \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^{\,i}\epsilon \\ &\qquad = L(q,\dot{q}) + \frac{d}{dt}L(q,\dot{q})\epsilon \;\Rightarrow\; \Lambda = L \,. \end{aligned} \tag{2.53}$$

Now, according to Remark 3 in section 2.1, the corresponding conserved quantity is

$$E = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L = p_i\dot{q}^i - L \,, \tag{2.54}$$

which is the energy of the system. Mechanical system whose energy is conserved are called *conservative* systems.

In particular if

$$L = T(q,\dot{q}) - U(q) = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j - U(q) \,, \tag{2.55}$$

then

$$E = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L = g_{ij}\dot{q}^i\dot{q}^j - \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j + U(q) = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j + U(q) = T + U \,, \tag{2.56}$$

Thus, it is the sum of the kinetic and potential energies.

If we have a system of particles in homogeneous and isotropic space-time, then, as was discussed,

$$L = \sum_a \frac{1}{2} m_a v_a^2 - U(|\vec{r}_{ab}|)\,, \tag{2.57}$$

and

$$E = \sum_a \frac{1}{2} m_a v_a^2 + U(|\vec{r}_{ab}|)\,, \tag{2.58}$$

The value of the energy obviously depends on the choice of a frame. The relation between E in K and E' in K' can be easily found

$$\begin{aligned} E &= \sum_a \frac{1}{2} m_a v_a^2 + U = \sum_a \frac{1}{2} m_a (\vec{v}_a' + \vec{V})^2 + U \\ &= \sum_a \frac{1}{2} m_a v_a'^2 + \sum_a m_a \vec{v}_a' \cdot \vec{V} + \frac{1}{2}\mu V^2 + U \\ &= E' + \vec{V}\cdot\vec{P}' + \frac{1}{2}\mu V^2\,. \end{aligned} \tag{2.59}$$

We see that if the centre of mass is at rest in K' then $\vec{P}' = 0$ and $E' = E_{\text{internal}}$, where E_{internal} is the *internal* energy, that is the energy of the system in K' where it is at rest as a whole. Thus, the total energy of a system moving as a whole with speed V is given by the sum of the internal energy and the kinetic energy of its centre of mass

$$E = E_{\text{internal}} + \frac{1}{2}\mu V^2\,. \tag{2.60}$$

Chapter 3

SMALL OSCILLATIONS

Small or linear or harmonic oscillations of a system about a position of stable equilibrium are very common. This is the most important type of motion because it allows for a complete analytic description and serves as a starting point of perturbative study of anharmonic or nonlinear oscillations.

3.1 Free oscillations in one dimension

Consider a system with one degree of freedom and the following Lagrangian

$$L = \frac{1}{2}a(q)\dot{q}^2 - U(q)\,. \tag{3.1}$$

Let us assume that at $q = q_0$ the system is at stable equilibrium, and that $U''(q_0) \neq 0$. Then $U'(q_0) = 0$, $U''(q_0) > 0$, and $a(q_0) > 0$. Expanding L up to quadratic order in q and $\dot{q}$, we find

$$L = \frac{1}{2}a(q_0)\dot{q}^2 - U(q_0) - \frac{1}{2}U''(q_0)(q - q_0)^2 + \cdots\,. \tag{3.2}$$

Denoting

$$x = q - q_0\,, \quad m = a(q_0) > 0\,, \quad k = U''(q_0) > 0\,, \tag{3.3}$$

and omitting the constant $U(q_0)$, we get the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\,, \tag{3.4}$$

which describes a system called a one-dimensional oscillator. It executes harmonic or linear oscillations. The eom is the following linear *homogeneous* differential equation

$$m\ddot{x} + kx = 0 \quad \Longrightarrow \quad \ddot{x} + \omega^2 x = 0\,. \tag{3.5}$$

Here

$$\omega = \sqrt{\frac{k}{m}}\,, \tag{3.6}$$

is called the *frequency* of the oscillations because the general solution of this equation is a superposition of the two independent solutions $\cos\omega t$ and $\sin\omega t$

$$x = c_1\cos\omega t + c_2\sin\omega t = a\cos(\omega t + \alpha)\,, \tag{3.7}$$

where

$$a = \sqrt{c_1 + c_2}\,, \quad \tan\alpha = -\frac{c_2}{c_1}\,. \tag{3.8}$$

Here a is the *amplitude* of the harmonic oscillations, $\omega t + \alpha$ is their *phase*, and α is the *initial phase*. The frequency ω is determined b the properties of the system, and it is a fundamental characteristic of the harmonic oscillations independent of the initial conditions of the motion.

The energy is conserved and is equal to

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m(\dot{x}^2 + \omega^2 x^2) = \frac{1}{2}m\omega^2 a^2\,, \tag{3.9}$$

and it is proportional to the square of the amplitude.

Let us note that the general solution (3.7) can be written as the real part of $Ae^{i\omega t}$

$$x = \Re(Ae^{i\omega t})\,, \quad A = ae^{i\alpha}\,, \tag{3.10}$$

where A is the *complex amplitude*: $|A| = a$, $\arg(A) = \alpha$.

3.2 Forced oscillations

Let us now place a harmonic oscillator in a time-dependent external field

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - U_{\rm e}(x,t)\,, \tag{3.11}$$

and let us assume that the motion of the system under the influence of the external field is finite and its maximum displacement from $x = 0$ is small. Oscillations of such a system are called *forced*. Expanding the external potential in powers of x, we get

$$U_{\rm e}(x,t) = U_{\rm e}(0,t) + U_{\rm e}'(0,t)x + \frac{1}{2}U_{\rm e}''(0,t)x^2 + \cdots\,. \tag{3.12}$$

The first term is a function of time and therefore can be omitted, then $U_{\rm e}'(0,t) = -F(t)$ where $F(t)$ is the external force acting on the system in the equilibrium position, and, finally, $U_{\rm e}''(0,t) \equiv k_{\rm e}(t)$ can be thought of as a time-dependent spring constant which leads to a time-dependent frequency. We assume that $k_{\rm e}(t)$ can be neglected.[1] Then, the Lagrangian takes the form

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + F(t)x\,. \tag{3.13}$$

Since we do not want the external force to change the equilibrium position we should assume that $F(t)$ is small, and that either $F(t) \to 0$ as $t \to \pm\infty$, or $F(t)$ is periodic and its average over the period is 0.

3.2.1 Inhomogeneous linear differential equations

The eom is the following linear *inhomogeneous* differential equation

$$\ddot{x} + \omega^2 x = f(t)\,, \quad f(t) = \frac{F(t)}{m}\,, \quad \omega = \sqrt{\frac{k}{m}}\,. \tag{3.14}$$

[1] Oscillatory systems with time-dependent parameters (mass and spring constant) exhibit variuos interesting phenomena such as parametric resonance.

Let us assume that we know a particular solution x_1 of this inhomogeneous equation. Then, it is easy to see that if x is another solution then the difference $x - x_1$ satisfies the corresponding homogeneous equation. Thus, the general solution of this inhomogeneous equation is

$$x = x_0 + x_1 \,, \tag{3.15}$$

where x_0 is the general solution of the homogeneous equation, and x_1 is any particular solution of the inhomogeneous equation. We know that $x_0 = a\cos(\omega t + \alpha)$, and let us discuss how one can find a particular solution.

Let us introduce an auxiliary complex variable

$$\xi = \dot{x} + i\omega x \,. \tag{3.16}$$

Since x is real, it is equal to the imaginary part of ξ divided by ω

$$x = \frac{1}{\omega}\Im(\xi) \,. \tag{3.17}$$

It is easy to check that if x satisfies eom (3.14), then ξ satisfies the following first-order linear differential equation

$$\dot{\xi} - i\omega\xi = f(t) \,. \tag{3.18}$$

To solve the equation let us use the following ansatz

$$\xi = C(t)e^{i\omega t} \,, \tag{3.19}$$

where $C(t)$ is a unknown function. Substituting the ansatz in the equation (3.18), we find

$$\frac{dC}{dt} = e^{-i\omega t} f(t) \,, \tag{3.20}$$

and therefore

$$C(t) = C_0 + \int_{t_0}^{t} e^{-i\omega\tau} f(\tau)d\tau \quad \Longrightarrow \quad \xi(t) = C_0 e^{i\omega t} + \int_{t_0}^{t} e^{i\omega(t-\tau)} f(\tau)d\tau \,, \tag{3.21}$$

where t_0 is any constant, and $C_0 = C(t_0)$. Thus, setting $C_0 = 0$, we find the following particular solution x_1 of the inhomogeneous equation (3.14)[2]

$$x_1(t) = \frac{1}{\omega}\Im\Big(\int_{t_0}^{t} e^{i\omega(t-\tau)} f(\tau)d\tau\Big) = \frac{1}{\omega}\int_{t_0}^{t} \sin(\omega(t-\tau))f(\tau)d\tau \,, \tag{3.22}$$

where we used that f and ω are real. Note that the particular solution satisfies $x_1(t_0) = 0$. The general solution of the homogeneous equation (3.14), therefore, is

$$x(t) = a\cos(\omega t + \alpha) + \frac{1}{\omega}\int_{t_0}^{t} \sin(\omega(t-\tau))f(\tau)d\tau \,. \tag{3.23}$$

The constants a, α, t_0 are fixed by the initial conditions. For example, since the energy is not conserved, one can ask how much energy is transmitted to the system during all time assuming

[2]Keeping C_0 arbitrary gives the general solution to (3.14).

its initial energy to be 0. Then, it is natural to choose $t_0 = -\infty$, and therefore $a = 0$ because the system was at rest initially. The solution of eom (3.14), therefore, is

$$x(t) = \frac{1}{\omega}\int_{-\infty}^{t} \sin(\omega(t-\tau))f(\tau)d\tau\,, \tag{3.24}$$

the corresponding ξ is

$$\xi(t) = \int_{-\infty}^{t} e^{i\omega(t-\tau)}f(\tau)d\tau\,, \tag{3.25}$$

and the energy transferred is

$$E(\infty) = \frac{1}{2}m(\dot{x}^2+\omega^2x^2) = \frac{1}{2}m|\xi(\infty)|^2 = \frac{1}{2}m\left|\int_{-\infty}^{\infty} e^{-i\omega t}f(t)dt\right|^2 = \frac{1}{2}m\left|\tilde{f}(\omega)\right|^2 = \frac{1}{2m}\left|\tilde{F}(\omega)\right|^2\,, \tag{3.26}$$

where $\tilde{F}(\omega)$ is the Fourier component of the force $F(t)$. If the force acts only during some period of time $t_0 - \Delta \le t \le t_0 - \Delta$ the energy transferred can be written as

$$E(\infty) = \frac{1}{2m}\left|\int_{t_0-\Delta}^{t_0+\Delta} e^{-i\omega t}F(t)dt\right|^2 = \frac{1}{2m}\left|\int_{-\Delta}^{\Delta} e^{-i\omega\tau}F(\tau+t_0)d\tau\right|^2\,. \tag{3.27}$$

Thus if the time interval 2Δ is short in comparison with $1/\omega$, i.e. $\omega\Delta << 1$ then the formula can be approximated by

$$E(\infty) \approx \frac{1}{2m}\left(\int_{-\Delta}^{\Delta} F(\tau+t_0)d\tau\right)^2 = \frac{1}{2m}\left(\int_{-\infty}^{\infty} F(t)dt\right)^2 = \frac{p^2}{2m}\,. \tag{3.28}$$

This formula shows that a force of short duration gives the system a momentum $p = \int_{-\infty}^{\infty} F(t)dt$ without causing a visible displacement.

3.2.2 Green's function

Sometimes one is interested in a particular solution of (3.14) which satisfies the vanishing *boundary* conditions

$$x(t_0) = x(t_1) = 0\,, \quad t_0 < t_1\,, \tag{3.29}$$

at some given instants of time t_0, t_1. The corresponding solution can be written in the form

$$x(t) = \int_{t_0}^{t_1} G(t,\tau)f(\tau)d\tau\,, \tag{3.30}$$

where $G(t,\tau)$ is called Green's function, and it satisfies the following conditions

$$\frac{\partial^2 G(t,\tau)}{\partial t^2} + \omega^2 G(t,\tau) = \delta(t-\tau)\,, \quad G(t_0,\tau) = G(t_1,\tau) = 0\,. \tag{3.31}$$

Here $\delta(t-\tau)$ is Dirac's delta-function which satisfies the defining relation

$$\int_{t_0}^{t_1} \delta(t-\tau)h(\tau)d\tau = h(t)\,, \quad t_0 < t < t_1\,, \tag{3.32}$$

for any function h continuous on the interval $[t_0, t_1]$. Clearly, $x(t)$ given by (3.30) satisfies (3.14) if $G(t,\tau)$ satisfies (3.31).

To find $G(t,\tau)$ we can regard τ as a parameter. Then, for $t_0 \le t < \tau$ and for $\tau < t \le t_1$ Green's function satisfies the homogeneous equation, and therefore one can look for a solution of the form

$$G(t,\tau) = G_{t<\tau}(t,\tau)\theta(\tau - t) + G_{t>\tau}(t,\tau)\theta(t-\tau)\,, \tag{3.33}$$

where θ is the Heaviside θ-function

$$\theta(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases} . \tag{3.34}$$

The two functions $G_{t<\tau}(t,\tau)$ and $G_{t>\tau}(t,\tau)$ must satisfy

$$\begin{aligned}
&\frac{\partial^2 G_{t<\tau}(t,\tau)}{\partial t^2} + \omega^2 G_{t<\tau}(t,\tau) = 0\,, \quad G_{t<\tau}(t_0,\tau) = 0\,, \\
&\frac{\partial^2 G_{t>\tau}(t,\tau)}{\partial t^2} + \omega^2 G_{t>\tau}(t,\tau) = 0\,, \quad G_{t>\tau}(t_1,\tau) = 0\,, \\
&G_{t<\tau}(\tau,\tau) = G_{t>\tau}(\tau,\tau)\,, \\
&\Big(\frac{\partial G_{t>\tau}(t,\tau)}{\partial t} - \frac{\partial G_{t<\tau}(t,\tau)}{\partial t}\Big)\Big|_{t=\tau} = 1\,,
\end{aligned} \tag{3.35}$$

where the third and fourth conditions guarantee that Green's function satisfy (3.31).

The first two equations give

$$G_{t<\tau}(t,\tau) = A\sin\omega(t-t_0)\,, \quad G_{t>\tau}(t,\tau) = B\sin\omega(t-t_1)\,. \tag{3.36}$$

The third and fourth conditions then give

$$\begin{aligned}
&A\sin\omega(\tau - t_0) - B\sin\omega(\tau - t_1) = 0\,, \\
&B\cos\omega(\tau - t_1) - A\cos\omega(\tau - t_0) = \frac{1}{\omega}\,.
\end{aligned} \tag{3.37}$$

Solving these equations one finds

$$A = \frac{1}{\omega}\,\frac{\sin\omega(\tau - t_1)}{\sin\omega(t_1 - t_0)}\,, \quad B = \frac{1}{\omega}\,\frac{\sin\omega(\tau - t_0)}{\sin\omega(t_1 - t_0)}\,, \tag{3.38}$$

and therefore

$$G(t,\tau) = \frac{\sin\omega(t-t_0)\sin\omega(\tau - t_1)}{\omega\sin\omega(t_1 - t_0)}\theta(\tau - t) + \frac{\sin\omega(t-t_1)\sin\omega(\tau - t_0)}{\omega\sin\omega(t_1 - t_0)}\theta(t-\tau)\,. \tag{3.39}$$

3.2.3 Periodic external force

Let us analyse forced oscillations under the action of a periodic external force. Since any periodic function can be expanded in a Fourier series it is sufficient to consider the simplest force of this type

$$F(t) = F_0\cos(\gamma t + \beta)\,. \tag{3.40}$$

The eom is

$$\ddot{x} + \omega^2 x = f\cos(\gamma t + \beta)\,, \quad f = \frac{F_0}{m}\,, \quad \omega = \sqrt{\frac{k}{m}}\,. \tag{3.41}$$

We choose $t_0 = 0$ in (3.42). Then, the particular solution is

$$\begin{aligned} x_1(t) &= \frac{f}{\omega}\int_0^t \sin(\omega(t-\tau))\cos(\gamma\tau+\beta)d\tau \\ &= -\frac{f}{2\omega}\int_0^t \big(\sin((\omega-\gamma)\tau-\omega t-\beta)+\sin((\gamma+\omega)\tau-\omega t+\beta)\big)d\tau\,. \end{aligned} \tag{3.42}$$

To compute this integral we need to consider the two cases: *i*) $\gamma \neq \omega$, and *i*) $\gamma = \omega$.

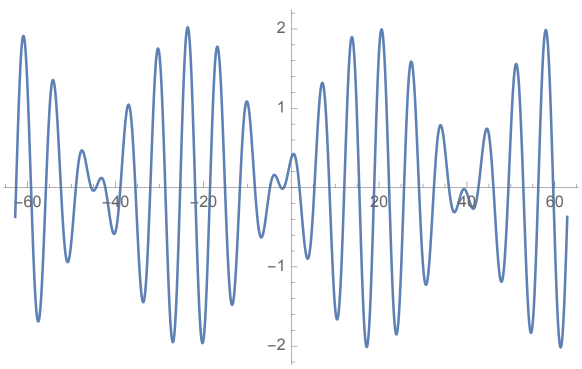
If $\gamma \neq \omega$ we get

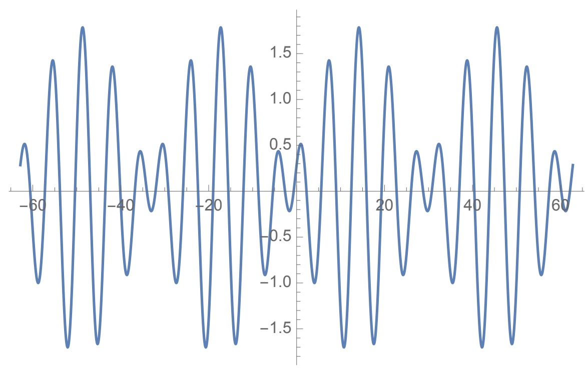
$$\begin{aligned} x_1(t) &= \frac{f}{2\omega}\Big(\frac{1}{\omega-\gamma}\cos((\omega-\gamma)\tau-\omega t-\beta)+\frac{1}{\omega+\gamma}\cos(\omega((\gamma+\omega)\tau-\omega t+\beta)\Big)\Big|_0^t \\ &= \frac{f}{\omega^2-\gamma^2}\cos(\gamma t+\beta)-\frac{f}{2\omega}\Big(\frac{1}{\omega-\gamma}\cos(\omega t+\beta)+\frac{1}{\omega+\gamma}\cos(\omega t-\beta)\Big)\,. \end{aligned} \tag{3.43}$$

Since the last term in (3.43) is a solution to the homogeneous equation, we can choose the first term as a particular solution to (3.41), and therefore the general solution to (3.41) is

$$x(t) = a\cos(\omega t+\alpha)+\frac{f}{\omega^2-\gamma^2}\cos(\gamma t+\beta)\,. \tag{3.44}$$

Thus the motion of the system is a superposition of two oscillations. It is not a periodic motion unless the frequencies ω and γ are commensurable, i.e. ω/γ is a rational number, see the pictures below





The solution (3.44) cannot be used if $\gamma = \omega$ when *resonance* occurs. In this case computing the integral in (3.42), we get

$$x_1(t) = \frac{f}{2\omega}\int_0^t \big(\sin(\omega t+\beta)-\sin(2\omega\tau-\omega t+\beta)\big)d\tau = \frac{f}{2\omega}t\sin(\omega t+\beta)\,, \tag{3.45}$$

and therefore the general solution is

$$x(t) = a\cos(\omega t+\alpha)+\frac{f}{2\omega}t\sin(\omega t+\beta) = a(t)\cos(\omega t+\alpha(t))\,. \tag{3.46}$$

Obviously, in resonance the amplitude $a(t)$ of oscillations increases linearly with the time (at large t), and therefore the approximation of small oscillations will be broken eventually.

Let us analyse the motion of the system near resonance when

$$\gamma = \omega+\epsilon\,, \quad \epsilon << \omega\,. \tag{3.47}$$

To this end it is convenient to write the solution (3.44) in the complex form

$$x(t) = \Re\Big(Ae^{i\omega t} + Be^{i\gamma t}\Big) = \Re\Big((A + Be^{i\epsilon t})e^{i\omega t}\Big)\,, \tag{3.48}$$

where

$$A = ae^{i\alpha}\,, \quad B = be^{i\beta}\,, \quad b = \frac{f}{\omega^2 - \gamma^2}\,. \tag{3.49}$$

Since $\epsilon/\omega << 1$, the quantity

$$C(t) = A + Be^{i\epsilon t}\,, \tag{3.50}$$

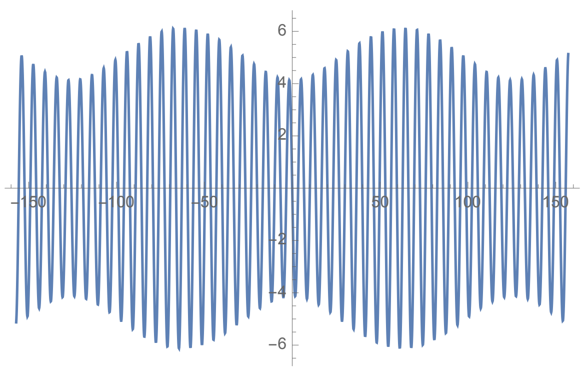
which may be regarded as a time-dependent complex amplitude of oscillations, changes slowly over the period $2\pi/\omega$ of $e^{i\omega t}$, and the motion near resonance may be regarded as small oscillations of the variable amplitude. The absolute value of C is

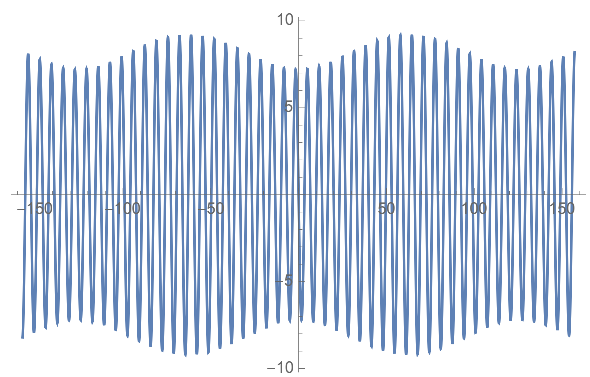
$$|C(t)|^2 = a^2 + b^2 + 2ab\cos(\epsilon t + \beta - \alpha)\,, \tag{3.51}$$

and therefore it varies with frequency ϵ between[3]

$$\big|a - |b|\big| \le |C(t)| \le a + |b|\,. \tag{3.52}$$

This phenomenon is called *beats*, see the pictures below





3.3 Oscillations of systems with several dof

Consider a system with s degrees of freedom and the following Lagrangian

$$L = \frac{1}{2}g_{jn}(q)\dot q^j\dot q^n - U(q)\,. \tag{3.53}$$

Let us assume that at $q^j = q_0^j$, $j = 1,\dots,s$ the system is at stable equilibrium, and that the matrix of second-partial derivatives $\big(\frac{\partial^2 U(q_0)}{\partial q^j\partial q^n}\big)$ is nondegenerate. Then $\frac{\partial U(q_0)}{\partial q^j} = 0$, $j = 1,\dots,s$, and $\big(\frac{\partial^2 U(q_0)}{\partial q^j\partial q^n}\big)$ and $\big(g_{jn}(q_0)\big)$ are positive-definite. Expanding L up to quadratic order in q and $\dot q$, we find

$$L = \frac{1}{2}g_{jn}(q_0)\dot q^j\dot q^n - U(q_0) - \frac{1}{2}\frac{\partial^2 U(q_0)}{\partial q^j\partial q^n}(q^j - q_0^j)(q^n - q_0^n) + \cdots\,. \tag{3.54}$$

[3]Note that b may be negative.

Denoting

$$x_j = q^j - q_0^j\,, \quad m_{jn} = g_{jn}(q_0)\,, \quad k_{jn} = \frac{\partial^2 U(q_0)}{\partial q^j \partial q^n}\,, \tag{3.55}$$

and omitting the constant $U(q_0)$, we get the Lagrangian

$$L = \frac{1}{2} m_{jn} \dot{x}_j \dot{x}_n - \frac{1}{2} k_{jn} x_j x_n\,, \tag{3.56}$$

which describes a system of coupled oscillators. Its coordinates execute superpositions of harmonic oscillations with different frequencies as we shall see in a moment.

3.3.1 Normal frequencies and coordinates from eom

The eom is the following set of s linear homogeneous differential equations

$$m_{jn} \ddot{x}_n + k_{jn} x_n = 0\,, \quad j = 1, 2, \ldots, s\,, \tag{3.57}$$

and, as usual, we sum over the repeated index n. To solve the equations let us use the ansatz[4]

$$x_j = A_j e^{i\omega t}\,, \quad j = 1, 2, \ldots, s\,, \tag{3.58}$$

where A_j and ω are some constants to be determined. Substituting (3.58) in the eom (3.57), we get a set of linear homogeneous *algebraic* equations

$$\big(- \omega^2 m_{jn} + k_{jn} \big) A_n = 0\,, \quad j = 1, 2, \ldots, s\,. \tag{3.59}$$

These equations have a nontrivial solution only if the determinant of the matrix $\big(-\omega^2 m_{jn} + k_{jn}\big)$ vanishes

$$\det \big(- \omega^2 m_{jn} + k_{jn} \big) = 0\,. \tag{3.60}$$

The equation (3.60) is called the *characteristic* equation. Its lhs is a polynomial of degree s in ω^2 which is called the *characteristic* polynomial, and its roots ω_α^2, $\alpha = 1, \ldots, s$ are called the *characteristic frequencies* or *eigenfrequencies* of the mechanical system.[5] Since we have assumed that at $x_j = 0$ the system is in stable equilibrium all eigenfrequencies must be real, and therefore ω_α^2 must be positive. To prove it we multiply (3.59) by $\bar{A}_j$, and take a sum over j

$$\sum_{j,n} \big(- \omega^2 m_{jn} + k_{jn} \big) A_n \bar{A}_j = 0 \quad \Longrightarrow \quad \omega^2 = \frac{\sum_{j,n} k_{jn} A_n \bar{A}_j}{\sum_{j,n} m_{jn} A_n \bar{A}_j}\,. \tag{3.61}$$

Since the matrices (k_{jn}) and (m_{jn}) are positive-definite[6] both the numerator and denominator in (3.61) are positive as can be seen by representing A_j as the sum of its real and imaginary parts: $A_j = a_j + i b_j$. Then

$$\sum_{j,n} k_{jn} A_n \bar{A}_j = \sum_{j,n} k_{jn} (a_n + i b_n)(a_j - i b_j) = \sum_{j,n} k_{jn} (a_n a_j + b_n b_j) > 0\,, \tag{3.62}$$

[4]To be precise x_j are given by the real part of (3.58).

[5]For systems with additional symmetries some of these characteristic frequencies may coincide.

[6]Let us mention that if (k_{jn}) is not positive-definite but it is nondegenerate then some $\omega_\alpha^2 < 0$ and the system is unstable. The corresponding modes are called *tachyonic*. If (k_{jn}) is degenerate then some $\omega_\alpha^2 = 0$ and the centre of mass of the system can move with constant velocities in the corresponding directions which are called *flat*.

where we used that (k_{jn}) is symmetric: $k_{nj} = k_{jn}$.

Having found ω_α we substitute each of them in (3.59) and find the corresponding $A_n^{(\alpha)}$. If all ω_α are different then the rank of $\big(-\omega_\alpha^2 m_{jn} + k_{jn}\big)$ is equal to $s-1$, and the solution $A_n^{(\alpha)}$ is proportional to the minors $\Delta_{n\alpha}$ of $\big(-\omega_\alpha^2 m_{jn} + k_{jn}\big)$: $A_n^{(\alpha)} \sim \Delta_{n\alpha}$. The general solution is then the superposition of all these solutions

$$x_n = \Re \sum_{\alpha=1}^{s} \Delta_{n\alpha} C_\alpha e^{i\omega_\alpha t} = \sum_{\alpha=1}^{s} \Delta_{n\alpha}\Theta_\alpha\,, \quad n = 1, 2, \ldots, s\,, \tag{3.63}$$

where

$$\Theta_\alpha = \Re\big(C_\alpha e^{i\omega_\alpha t}\big)\,, \quad \alpha = 1, 2, \ldots, s\,. \tag{3.64}$$

It is a superposition of s simple periodic oscillations $\Theta_1, \Theta_2, \ldots, \Theta_s$ with arbitrary amplitudes and phases but definite frequencies determined by the properties of the mechanical system.

The equations $x_n = \sum_{\alpha=1}^{s} \Delta_{n\alpha}\Theta_\alpha$ can be considered as a set of equations for s new generalised coordinates Θ_α which are called the *normal coordinates*. According to (3.64), they execute simple periodic oscillations called *normal oscillations* of the system, and satisfy the equations

$$\ddot\Theta_\alpha + \omega_\alpha^2 \Theta_\alpha = 0\,, \quad \alpha = 1, 2, \ldots, s\,. \tag{3.65}$$

Thus, in terms of Θ_α the eom become a set of decoupled harmonic oscillator equations.

It is clear that the Lagrangian which leads to eom (3.65) must be a sum of the harmonic oscillators Lagrangians

$$L = \frac{1}{2}\sum_\alpha m_\alpha(\dot\Theta_\alpha^2 - \omega_\alpha^2\Theta_\alpha^2)\,, \quad m_\alpha > 0\,. \tag{3.66}$$

This form of L means that the change of coordinates $x_n = \sum_{\alpha=1}^{s} \Delta_{n\alpha}\Theta_\alpha$ simultaneously diagonalises both the kinetic and potential quadratic forms.

It is also convenient to rescale the normal coordinates Θ_α

$$\Theta_\alpha = \frac{1}{\sqrt{m_\alpha}} Q_\alpha\,, \quad \alpha = 1, 2, \ldots, s\,, \tag{3.67}$$

so that L takes the form

$$L = \frac{1}{2}\sum_{\alpha=1}^{s}(\dot Q_\alpha^2 - \omega_\alpha^2 Q_\alpha^2)\,. \tag{3.68}$$

In particular this form makes obvious that in the *degenerate* case where some of the normal frequencies are equal[7] the mechanical system has additional rotational symmetry. Indeed assuming that the first r frequencies are equal, the Lagrangian (3.68) becomes

$$L = \frac{1}{2}\sum_{\alpha=1}^{r}(\dot Q_\alpha^2 - \omega_1^2 Q_\alpha^2) + \frac{1}{2}\sum_{\alpha=r+1}^{s}(\dot Q_\alpha^2 - \omega_\alpha^2 Q_\alpha^2)\,. \tag{3.69}$$

It is clear that it is invariant under $SO(r)$ which rotates the coordinates $Q_1, Q_2, \ldots, Q_r$.

[7] In this case the coefficients $\Delta_{n\alpha}$ in (3.63) are not the minors because they vanish. They are independent solutions to (3.59).

The analysis of forced oscillations also simplifies by using the normal coordinates. The Lagrangian

$$L = \frac{1}{2} m_{jn} \dot{x}_j \dot{x}_n - \frac{1}{2} k_{jn} x_j x_n + F_j(t) x_j \,, \quad (\textit{sum over repeated indices!})\,, \tag{3.70}$$

in the normal coordinates takes the form

$$L = \sum_{\alpha=1}^{s} \left(\frac{1}{2} \dot{Q}_\alpha^2 - \frac{1}{2} \omega_\alpha^2 Q_\alpha^2 + f_\alpha(t) Q_\alpha \right), \tag{3.71}$$

where

$$f_\alpha(t) = \sum_{j=1}^{s} \frac{1}{\sqrt{m_\alpha}} F_j(t) \Delta_{j\alpha} \,. \tag{3.72}$$

The eom are

$$\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = f_\alpha(t)\,, \tag{3.73}$$

which are s independent equations for unknown functions $Q_\alpha(t)$. Thus, in normal coordinates the motion of a system of s coupled oscillators reduces to the motion of one-dimensional oscillator.

3.3.2 Normal frequencies and coordinates from Lagrangian

It is not necessary to solve eom to find the normal frequencies and coordinates. one can just use the Lagrangian and a chain of simple coordinate transformations. We begin by rewriting the Lagrangian L (3.70) as follows

$$L = \frac{1}{2} m_{jn} \dot{x}_j \dot{x}_n - \frac{1}{2} k_{jn} x_j x_n + F_j(t) x_j = \frac{1}{2} \dot{X}^T M \dot{X} - \frac{1}{2} X^T K X + F^T X \,. \tag{3.74}$$

Here we used the usual matrix product, and the following columns, rows and matrices

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{pmatrix}, \; F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_s \end{pmatrix}, \; M = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1s} \\ m_{21} & m_{22} & \dots & m_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ m_{s1} & m_{s2} & \dots & m_{ss} \end{pmatrix}, \; K = \begin{pmatrix} k_{11} & k_{12} & \dots & k_{1s} \\ k_{21} & k_{22} & \dots & k_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ k_{s1} & k_{s2} & \dots & k_{ss} \end{pmatrix}, \tag{3.75}$$

and A^T for any A denotes the matrix transposed to A (i.e. X^T is the column $(x_1, \dots, x_s)$). Both M and K are real symmetric. As we know from Hamilton's principle, M must be positive-definite. If $X = 0$ is stable equilibrium K is positive-definite too. For the sake of generality we consider arbitrary symmetric K and discuss what happens if it is not positive-definite at the end of this subsection.

We want to bring L to the form (3.71). It can be done in the following three steps

1. Diagonalise M by using an orthogonal matrix A: $A^T A = I$

$$M = A \mu A^T \,, \quad \mu = \operatorname{diag}(\mu_1, \dots, \mu_s)\,, \quad \mu_i > 0\,, \quad i = 1, \dots, s\,. \tag{3.76}$$

   L takes the form

$$L = \frac{1}{2} \dot{X}^T A \mu A^T \dot{X} - \frac{1}{2} X^T K X + F^T X\,, \tag{3.77}$$

and it is obvious that the transformation

$$A^T X = Y \iff X = AY\,, \tag{3.78}$$

brings it to the form

$$L = \frac{1}{2}\dot{Y}^T \mu \dot{Y} - \frac{1}{2} Y^T A^T K A Y + F^T A Y\,, \tag{3.79}$$

2. Rescale Y so that $\dot{Y}^T \mu \dot{Y} = \dot{Z}^T \dot{Z}$

$$Z = \sqrt{\mu}\, Y \iff Y = \frac{1}{\sqrt{\mu}} Z\,, \quad \sqrt{\mu} = \mathrm{diag}\,(\sqrt{\mu_1}, \ldots, \sqrt{\mu_s})\,. \tag{3.80}$$

In components $z_i = \sqrt{\mu_i}\, y_i$, $i = 1, \ldots, s$. The rescaling brings L to the form

$$\begin{aligned} L &= \frac{1}{2}\dot{Z}^T \dot{Z} - \frac{1}{2} Z^T \frac{1}{\sqrt{\mu}} A^T K A \frac{1}{\sqrt{\mu}} Z + F^T A \frac{1}{\sqrt{\mu}} Z \\ &= \frac{1}{2}\dot{Z}^T \dot{Z} - \frac{1}{2} Z^T \widetilde{K} Z + F^T A \frac{1}{\sqrt{\mu}} Z\,, \end{aligned} \tag{3.81}$$

where

$$\widetilde{K} = \frac{1}{\sqrt{\mu}} A^T K A \frac{1}{\sqrt{\mu}}\,, \tag{3.82}$$

is symmetric: $\widetilde{K}^T = \widetilde{K}$.

3. Diagonalise $\widetilde{K}$ by using an orthogonal matrix B: $B^T B = I$

$$\widetilde{K} = B \kappa B^T\,, \quad \kappa = \mathrm{diag}\,(\kappa_1, \ldots, \kappa_s)\,. \tag{3.83}$$

L takes the form

$$L = \frac{1}{2}\dot{Z}^T \dot{Z} - \frac{1}{2} Z^T B \kappa B^T Z + F^T A \frac{1}{\sqrt{\mu}} Z\,, \tag{3.84}$$

and it is obvious that the transformation

$$B^T Z = Q \iff Z = BQ\,, \tag{3.85}$$

brings it to the desired form

$$\begin{aligned} L &= \frac{1}{2}\dot{Q}^T B^T B \dot{Q} - \frac{1}{2} Q^T \kappa Q + F^T A \frac{1}{\sqrt{\mu}} B Q \\ &= \frac{1}{2}\dot{Q}^T \dot{Q} - \frac{1}{2} Q^T \kappa Q + \widetilde{F}^T Q \\ &= \sum_{\alpha=1}^{s} \left(\frac{1}{2}\dot{Q}_\alpha^2 - \frac{1}{2}\kappa_\alpha Q_\alpha^2 + f_\alpha(t) Q_\alpha \right), \end{aligned} \tag{3.86}$$

where

$$\widetilde{F}^T = F^T A \frac{1}{\sqrt{\mu}} B = (f_1(t), \ldots, f_s(t))\,. \tag{3.87}$$

Combining the transformations we also get he relations between X and the normal coordinates Q

$$X = A \frac{1}{\sqrt{\mu}} B Q \iff Q = B^T \sqrt{\mu}\, A^T X\,. \tag{3.88}$$

Few comments are in order

- If K is positive-definite then all $\kappa_\alpha > 0$, and normal frequencies are given by $\omega_\alpha^2 = \kappa_\alpha$.
- If K is degenerate, i.e. $\det K = 0$ then some of κ_α are equal to zero. Let for example $\kappa_1 = 0$. In the absence of the external force $f_1(t) = 0$, the normal coordinate Q_1 is cyclic, and therefore the momentum p_1 conjugate to Q_1 is conserved. Thus the centre of mass of the system moves with a constant velocity in the Q_1 direction which is called flat direction. If all the other κ_α are positive then in the reference frame where the velocity is 0 the system undergoes the usual oscillatory motion in the remaining directions.
- If some κ_α are negative, say $\kappa_1 < 0$, then the system is unstable, and the corresponding normal coordinate Q_1 is called *tachyonic*. In the absence of the external force $f_1(t) = 0$, the general solution of eom
$$\ddot{Q}_1 + \kappa_1 Q_1 = 0\,, \tag{3.89}$$
is
$$Q_1 = C_- e^{-\sqrt{-\kappa_1}\,t} + C_+ e^{+\sqrt{-\kappa_1}\,t}\,, \tag{3.90}$$
and therefore unless we tune the initial conditions so that $C_+ = 0$, the normal coordinate increases exponentially fast. The system is still unstable in the past.

Let us now recall how to diagonalise a symmetric matrix. Let S be an $s \times s$ symmetric matrix, and let $D = \mathrm{diag}\,(d_1, \ldots, d_s)$ be the diagonal matrix of its eigenvalues. Then there is an orthogonal matrix A such that $S = ADA^T$. Let us view A as a collection of s columns A_i and A^T as a collection of s rows A_i^T

$$A = \begin{pmatrix} | & | & \ldots & | \\ A_1 & A_2 & \ldots & A_s \\ | & | & \ldots & | \end{pmatrix}\,,\ A_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{si} \end{pmatrix}\,,\quad A^T = \begin{pmatrix} \text{---} & A_1^T & \text{---} \\ \text{---} & A_2^T & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & A_s^T & \text{---} \end{pmatrix}\,. \tag{3.91}$$

Now multiplying the equation $S = ADA^T$ by A from the right we get

$$SA = AD \;\Rightarrow\; SA_i = d_i A_i\,. \tag{3.92}$$

Thus for each $i = 1, \ldots, s$ the column A_i is an eigenvector of S with the eigenvalue d_i. Since A is orthogonal the columns A_i are orthonormal: $A^T A = I \;\Rightarrow\; A_i^T A_j = \delta_{ij}$. Therefore finding A reduces to finding a set of orthonormal eigenvectors of S. Let us also note that by using (3.91) we get the *spectral decomposition* of S

$$\begin{aligned} S = ADA^T &= \begin{pmatrix} | & | & \ldots & | \\ A_1 & A_2 & \ldots & A_s \\ | & | & \ldots & | \end{pmatrix} \begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & d_s \end{pmatrix} \begin{pmatrix} \text{---} & A_1^T & \text{---} \\ \text{---} & A_2^T & \text{---} \\ \vdots & \vdots & \vdots \\ \text{---} & A_s^T & \text{---} \end{pmatrix} \\ &= d_1 A_1 A_1^T + d_2 A_2 A_2^T + \cdots d_s A_s A_s^T = \sum_{i=1}^{s} d_i P_i\,, \quad P_i = A_i A_i^T\,, \end{aligned} \tag{3.93}$$

where P_i are the projection operators which satisfy

$$\sum_{i=1}^{s} P_i = I\,, \quad P_i P_j = \delta_{ij} P_j\,, \quad \textit{no summation over } j\,. \tag{3.94}$$

Example

Let S be equal to[8]

$$S = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} . \tag{3.95}$$

First we find the eigenvalues of S by solving the characteristic equation

$$\begin{aligned} \det(S - dI) = \begin{vmatrix} 2-d & -1 & 0 \\ -1 & 2-d & -1 \\ 0 & -1 & 2-d \end{vmatrix} &= (2-d)((2-d)^2 - 1) - (2-d) \\ = (2-d)(d^2 - 4d + 2) = 0 \,. & \end{aligned} \tag{3.96}$$

Thus, the eigenvalues of S are

$$d_1 = 2 - \sqrt{2}\,, \quad d_2 = 2\,, \quad d_3 = 2 + \sqrt{2}\,. \tag{3.97}$$

1. Consider $d_1 = 2 - \sqrt{2}$

$$(S - d_1 I)A_1 = \begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = 0 \quad \Rightarrow \quad a_{21} = \sqrt{2}\, a_{11}\,, \; a_{31} = a_{11}\,. \tag{3.98}$$

The normalisation condition then gives

$$A_1^T A_1 = 1 \quad \Rightarrow \quad 4a_{11}^2 = 1 \quad \Rightarrow \quad a_{11} = \pm\frac{1}{2}\,. \tag{3.99}$$

Choosing the plus sign we get

$$A_1^T = \left(\frac{1}{2}\,, \frac{1}{\sqrt{2}}\,, \frac{1}{2}\right) . \tag{3.100}$$

2. Consider $d_2 = 2$

$$(S - d_2 I)A_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = 0 \quad \Rightarrow \quad a_{22} = 0\,, \; a_{32} = -a_{12}\,. \tag{3.101}$$

The normalisation condition then gives

$$A_2^T A_2 = 1 \quad \Rightarrow \quad 2a_{12}^2 = 1 \quad \Rightarrow \quad a_{12} = \pm\frac{1}{\sqrt{2}}\,. \tag{3.102}$$

Choosing the plus sign we get

$$A_2^T = \left(\frac{1}{\sqrt{2}}\,, 0\,, -\frac{1}{\sqrt{2}}\right) . \tag{3.103}$$

Obviously, it is orthogonal to A_1.

[8]Note that S is the Cartan matrix of the Lie algebra A_3 which is the Lie algebra of the Lie group $SL(4, \mathbb{C})$.

3. Consider $d_3 = 2+\sqrt{2}$

$$(S-d_3 I)A_3 = \begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = 0 \quad \Rightarrow \quad a_{23} = -\sqrt{2}\, a_{13}\,,\ a_{33} = a_{13}\,. \tag{3.104}$$

The normalisation condition then gives

$$A_3^T A_3 = 1 \quad \Rightarrow \quad 4a_{13}^2 = 1 \quad \Rightarrow \quad a_{13} = \pm\frac{1}{2}\,. \tag{3.105}$$

Choosing the plus sign we get

$$A_3^T = \left(\frac{1}{2}\,,\, -\frac{1}{\sqrt{2}}\,,\, \frac{1}{2}\right)\,. \tag{3.106}$$

Obviously, it is orthogonal to A_1 and A_2.

Thus,

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}\,, \quad D = \begin{pmatrix} 2-\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix}\,. \tag{3.107}$$

3.4 Anharmonic oscillations

Expanding $a_{jn}(q)$ and $U(q)$ in (3.53) in powers of $x_j = q^j - q_0^j$, and keeping higher-order terms in x_j, one gets a Lagrangian which describes *anharmonic* or *nonlinear* oscillations.

3.4.1 Anharmonic oscillations in one dimension

The method we use is a version of the Poincaré-Lindstedt expansion method which can be applied to some multi-dimensional cases.

Consider a system with the Lagrangian (3.1) which at $q = q_0$ is at stable equilibrium. Then $U'(q_0) = 0$, $m = a(q_0) > 0$, $k = U''(q_0) > 0$. Let us introduce a new coordinate $x = x(q)$ (similar to an arc length parameter) such that $x = 0$ corresponds to $q = q_0$, and

$$a(q)\dot{q}^2 = m\dot{x}^2\,. \tag{3.108}$$

Then the Lagrangian (3.1) takes the form

$$L = \frac{1}{2}m\dot{x}^2 - \tilde{U}(x)\,, \tag{3.109}$$

where the potential $\tilde{U}$ can be expanded in powers of x

$$\tilde{U}(x) = \tilde{U}(0) + \frac{1}{2}kx^2 + \frac{1}{3}g_3x^3 + \frac{1}{4}g_4x^4 + \cdots\,. \tag{3.110}$$

Here the constants $g_2, g_3, ...$ are proportional to derivatives of $\tilde{U}$ at $x = 0$ and are called coupling constants. Introducing

$$\omega_0 = \sqrt{\frac{k}{m}}\,, \quad \alpha = \frac{g_3}{m}\,, \quad \beta = \frac{g_4}{m}\,, \tag{3.111}$$

and omitting $\tilde{U}(0)$, one gets the Lagrangian up to fourth order in x^4

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega_0^2 x^2 - \frac{1}{3}m\alpha x^3 - \frac{1}{4}m\beta x^4 \,. \tag{3.112}$$

The eom is

$$\ddot{x} + \omega_0^2 x = -\alpha x^2 - \beta x^3 \,. \tag{3.113}$$

The motion is periodic with the period $T = 2\pi/\omega$, and therefore $x(t)$ can be expanded in a Fourier series

$$x(t) = x_0 + \sum_{k=1}^{\infty} \big(x_k \cos(k\omega t) + y_k \sin(k\omega t)\big) \,. \tag{3.114}$$

Imposing the initial condition $v(0) = \dot{x}(0) = 0$, and taking into account that with this initial condition if $x(t)$ is a solution then $x(-t)$ is a solution too, we see that all $y_k = 0$. Thus, without loss of generality we can look for a solution of the form

$$x(t) = \sum_{k=0}^{\infty} x_k \cos(k\omega t) \,. \tag{3.115}$$

The frequency ω and the Fourier coefficients x_k depend on the energy E of the system

$$E = \frac{1}{2}m\omega_0^2 a_{\rm R}^2 + \frac{1}{3}m\alpha a_{\rm R}^3 + \frac{1}{4}m\beta a_{\rm R}^4 \,, \tag{3.116}$$

where $a_R = x(0) = \sum_{k=0}^{\infty} x_k$, which we assume to be positive $a_R > 0$, is one of the amplitudes of the oscillations (the other one is $a_L = x(T/2)$). We want to analyse small corrections to harmonic oscillations at $\alpha = \beta = 0$. To give a precise meaning to the notion of smallness we need to use some dimensionless quantities. Taking into account that the dimension of α is $1/cm/sec^2$, and the dimension of β is $1/cm^2/sec^2$, we see that

$$\frac{a_R\alpha}{\omega_0^2} \quad \text{and} \quad \frac{a_R^2\beta}{\omega_0^2} \,, \tag{3.117}$$

are dimensionless, and the approximation of harmonic oscillations is applicable if $\frac{a_R\alpha}{\omega_0^2} << 1$, $\frac{a_R^2\beta}{\omega_0^2} << 1$. Then, corrections to harmonic oscillations are obtained by expanding ω and x_k in powers of $\frac{a_R\alpha}{\omega_0^2}$, $\frac{a_R^2\beta}{\omega_0^2}$. Computationally, however, it is easier to use the following expansion parameters

$$\epsilon_1 = \frac{a\alpha}{\omega_0^2} \,, \quad \epsilon_2 = \frac{a^2\beta}{\omega_0^2} \,, \tag{3.118}$$

where $a = x_1$ is the amplitude of the harmonic oscillations with frequency ω. Obviously, they are equivalent to (3.117), in particular $a_R \to a$ as $\epsilon_1, \epsilon_2 \to 0$. It is also clear that since $\epsilon_1 \sim a$ and $\epsilon_2 \sim a^2$ the order ϵ_1^2 is the same as the order ϵ_2.

To find the solution it is convenient to rescale x and t as follows

$$t = \frac{\tau}{\omega_0} \,, \quad x(t) = a\, z(\tau) \,, \tag{3.119}$$

so that both z and τ are dimensionless. Then, it is easy to check that the eom (3.113) takes the form

$$\frac{d^2 z}{d\tau^2} + z = -\epsilon_1 z^2 - \epsilon_2 z^3 \,. \tag{3.120}$$

It is also useful to represent a solution in the complex form

$$z(\tau) = \sum_{k=-\infty}^{\infty} z_k e^{ik\gamma\tau} = z_0 + 2\sum_{k=1}^{\infty} z_k \cos(k\gamma\tau)\,, \quad \forall k \quad z_{-k} = z_k\,, \quad \gamma = \frac{\omega}{\omega_0}\,, \tag{3.121}$$

where the solution we are looking for has $z_1 = \frac{1}{2} = z_{-1}$. Then, the Fourier series of $z(\tau)^2$ and $z(\tau)^3$ are given by

$$z(\tau)^2 = \sum_{l,m=-\infty}^{\infty} z_l z_m e^{i(l+m)\gamma\tau}\,, \quad z(\tau)^3 = \sum_{l,m,n=-\infty}^{\infty} z_l z_m z_n e^{i(l+m+n)\gamma\tau}\,. \tag{3.122}$$

Substituting these formulae in (3.120), we get

$$\sum_{k=-\infty}^{\infty} (1-k^2\gamma^2) z_k e^{ik\gamma\tau} = -\epsilon_1 \sum_{l,m=-\infty}^{\infty} z_l z_m e^{i(l+m)\gamma\tau} - \epsilon_2 \sum_{l,m,n=-\infty}^{\infty} z_l z_m z_n e^{i(l+m+n)\gamma\tau}\,, \tag{3.123}$$

and therefore the Fourier coefficients satisfy the equations

$$(1-k^2\gamma^2) z_k = -\epsilon_1 \sum_{l+m=k} z_l z_m - \epsilon_2 \sum_{l+m+n=k} z_l z_m z_n\,, \quad k \in \mathbb{Z}\,. \tag{3.124}$$

Since $z_{-k} = z_k$ it is sufficient to consider nonnegative k only. Let us analyse the equations.

- $k = 0$

  $$z_0 = -\epsilon_1 \sum_{l+m=0} z_l z_m - \epsilon_2 \sum_{l+m+n=0} z_l z_m z_n\,. \tag{3.125}$$

  Since $z_{-1} = z_1 = 1/2$ while all the other z_k's are at least of first order in ϵ_1 we get that the leading term in z_0 is obtained if $l = -m = \pm 1$, and therefore

  $$z_0 = -\frac{1}{2}\epsilon_1 + \mathcal{O}(\epsilon_1^3)\,. \tag{3.126}$$

- $k = 2$

  $$(1-4\gamma^2) z_2 = -\epsilon_1 \sum_{l+m=2} z_l z_m - \epsilon_2 \sum_{l+m+n=2} z_l z_m z_n\,. \tag{3.127}$$

  Since $z_{-1} = z_1 = 1/2$ while all the other z_k's are at least of first order in ϵ_1 we get that the leading term in z_2 is obtained if $l = m = 1$, and therefore

  $$z_2 = \frac{1}{12}\epsilon_1 + \mathcal{O}(\epsilon_1^3)\,. \tag{3.128}$$

- $k = 3$

  $$(1-9\gamma^2) z_3 = -\epsilon_1 \sum_{l+m=3} z_l z_m - \epsilon_2 \sum_{l+m+n=3} z_l z_m z_n\,. \tag{3.129}$$

  It is clear that the rhs is of order $\epsilon_1^2 \sim \epsilon_2$. The leading term in z_3 is obtained if in the first term on the rhs $l = 1, m = 2$ or $l = 2, m = 1$, and if in the second term $l = m = n = 1$, and therefore

  $$z_3 = \frac{1}{8}(2\epsilon_1 z_1 z_2 + \epsilon_2 z_1^3) + \mathcal{O}(\epsilon_1^4) = \frac{1}{32}(\frac{1}{3}\epsilon_1^2 + \frac{1}{2}\epsilon_2) + \mathcal{O}(\epsilon_1^4)\,. \tag{3.130}$$

- $k \geq 4$

$$(1-k^2\gamma^2)z_k = -\epsilon_1 \sum_{l+m=k} z_l z_m - \epsilon_2 \sum_{l+m+n=k} z_l z_m z_n \,. \tag{3.131}$$

The consideration above can be now easily generalised, and It is clear that the rhs, and therefore z_k, is of order ϵ_1^{k-1}. Thus, for all $k \in Z$ but $k=0$ the Fourier coefficients z_k are of order $\epsilon_1^{|k|-1}$: $z_k \sim \epsilon_1^{|k|-1}$, $k \neq 0$, $z_0 \sim \epsilon_1$. Thus, the leading term in z_k is obtained by taking sums over positive l, m, n, and can be written explicitly as

$$z_k = \frac{1}{k^2-1}\Big(\epsilon_1 \sum_{l=1}^{k-1} z_l z_{k-l} + \epsilon_2 \sum_{\substack{l,m=1\\ l+m<k}}^{k-2} z_l z_m z_{k-l-m}\Big) + \mathcal{O}(\epsilon_1^{k+1}) \,. \tag{3.132}$$

- Let us now use the z_k's to find the leading correction to γ^2. It is found from the equation for $k=1$

$$(1-\gamma^2)\frac{1}{2} = -\epsilon_1 \sum_{l+m=1} z_l z_m - \epsilon_2 \sum_{l+m+n=1} z_l z_m z_n \,. \tag{3.133}$$

The rhs is of order ϵ_1^2 the leading term in $\gamma^2 - 1$ is obtained only if $l, m, n = -1, 0, 1, 2$. Thus, we get

$$\begin{aligned} \gamma^2 &= 1 + 2\big(\epsilon_1(2z_0 z_1 + 2z_{-1}z_2) + 3\epsilon_2 z_{-1} z_1 z_1\big) + \mathcal{O}(\epsilon_1^4) \\ &= 1 + 2\big(-\frac{5}{12}\epsilon_1^2 + \frac{3}{8}\epsilon_2\big) + \mathcal{O}(\epsilon_1^4) \,, \end{aligned} \tag{3.134}$$

and therefore

$$\gamma = 1 - \frac{5}{12}\epsilon_1^2 + \frac{3}{8}\epsilon_2 + \mathcal{O}(\epsilon_1^4) \,. \tag{3.135}$$

The formulae above for z_0, z_2, z_3 and γ provide us with the leading corrections up to order ϵ_1^2 to the harmonic oscillations. Collecting all the formulae, we get

$$\begin{aligned} x(t) &= a\cos\omega t + \frac{a}{2}\epsilon_1(-1 + \frac{1}{3}\cos 2\omega t) + \frac{a}{16}(\frac{1}{3}\epsilon_1^2 + \frac{1}{2}\epsilon_2)\cos 3\omega t \,, \\ \omega &= \omega_0\gamma = \omega_0(1 - \frac{5}{12}\epsilon_1^2 + \frac{3}{8}\epsilon_2) \,. \end{aligned} \tag{3.136}$$

It is straightforward to calculate subleading corrections with the help of a computer.

3.4.2 Anharmonic oscillations in several dimensions

Anharmonic oscillations in several dimensions are much more complicated then the ones in one dimension, and they can lead to a chaotic behaviour of the mechanical system. In what follows we'll discuss only the case where the chaotic motion is absent.

Consider again a system with s degrees of freedom and the Lagrangian

$$L = \frac{1}{2}g_{jn}(q)\dot{q}^j\dot{q}^n - U(q) \,. \tag{3.137}$$

Let the system be at stable equilibrium at $q^j = q_0^j$, $j = 1, \ldots, s$. Then, $\frac{\partial U(q_0)}{\partial q^j} = 0$, $j = 1, \ldots, s$, and $\big(\frac{\partial^2 U(q_0)}{\partial q^j \partial q^n}\big)$ and $\big(g_{jn}(q_0)\big)$ are positive-definite. As was mentioned g_{ij} can be thought of as a metric tensor on the configuration space of the mechanical system. In the course of differential geometry it is proven that one can choose new coordinates x^j (which are also called normal coordinates) such that $x = 0$ corresponds to $q = q_0$ and

$$g_{jn}(q)\dot{q}^j\dot{q}^n = \delta_{jn}\dot{x}^j\dot{x}^n + \frac{1}{3}R_{jkln}x^kx^l\dot{x}^j\dot{x}^n + \mathcal{O}(x^3\dot{x}^2)\,, \tag{3.138}$$

where R_{jkln} are constants equal to the components of the Riemann curvature tensor in x^j coordinates at $x = 0$.

Thus, expanding L up to quartic order in x and $\dot{x}$, and switching to normal coordinates Q^α, we find

$$\begin{aligned} L = \sum_\alpha \left(\frac{1}{2}\dot{Q}_\alpha^2 - \frac{1}{2}\omega_\alpha^2 Q_\alpha^2\right) - \frac{1}{3}\sum_{\alpha,\beta,\gamma} g^{(3)}_{\alpha\beta\gamma}Q_\alpha Q_\beta Q_\gamma \\ - \frac{1}{4}\sum_{\alpha,\beta,\gamma,\rho} g^{(4)}_{\alpha\beta\gamma\rho}Q_\alpha Q_\beta Q_\gamma Q_\rho + \frac{1}{6}\sum_{\alpha,\beta,\gamma,\rho} R_{\alpha\beta\gamma\rho}\dot{Q}_\alpha Q_\beta Q_\gamma \dot{Q}_\rho\,, \end{aligned} \tag{3.139}$$

where the constants $g^{(3)}_{\alpha\beta\gamma}$ and $g^{(4)}_{\alpha\beta\gamma\rho}$ are symmetric in their indices.

To simplify the formulae let us keep only the cubic term. Then the eom are

$$\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = -g^{(3)}_{\alpha\beta\gamma}Q_\beta Q_\gamma\,, \quad \alpha = 1, \dots, s\,, \tag{3.140}$$

where we sum over β and γ. The small expansion parameters now are

$$\epsilon_{\alpha\beta\gamma} = \epsilon_{\alpha\gamma\beta} = \frac{g^{(3)}_{\alpha\beta\gamma}a_\beta a_\gamma}{\omega_\alpha^2 a_\alpha}\,, \tag{3.141}$$

where a_α are amplitudes of the harmonic oscillations $Q_\alpha = a_\alpha \cos(\omega_\alpha t + \varphi_\alpha)$.

To solve these equations perturbatively we again assume that the normal frequencies ω_α of harmonic oscillations are replaced by the exact frequencies Ω_α

$$\Omega_\alpha = \omega_\alpha(1 + \epsilon_{\beta\gamma\rho}\gamma_{\alpha,\beta\gamma\rho} + \cdots)\,, \tag{3.142}$$

where we sum over β, γ and ρ. We also assume that the only solution of $\vec{k}\cdot\vec{\Omega} = 0$ is $\vec{k} = 0$ where $\vec{k} = (k_1, \dots, k_s)$ is a vector of integers , and $\vec{\Omega} = (\Omega_1, \dots, \Omega_s)$ is a vector of frequencies. Then, we look for a solution of eom (3.140) of the form

$$Q_\alpha = \sum_{k_1,\dots,k_s} A_{\alpha,k_1k_2\cdots k_s}e^{ik_1\Omega_1 t + ik_2\Omega_2 t + \cdots + ik_s\Omega_s t} = \sum_{\vec{k}} A_{\alpha,\vec{k}}e^{i\,\vec{k}\cdot\vec{\Omega}\,t}\,. \tag{3.143}$$

The reality of Q_α then leads to

$$\bar{A}_{\alpha,-\vec{k}} = A_{\alpha,\vec{k}}\,. \tag{3.144}$$

Substituting (3.143) in the eom (3.140), one gets the equations for the coefficients $A_{\alpha,\vec{k}}$

$$(\omega_\alpha^2 - (\vec{k}\cdot\vec{\Omega})^2)A_{\alpha,\vec{k}} = -\sum_{\beta,\gamma}\sum_{\vec{l}+\vec{m}=\vec{k}} g^{(3)}_{\alpha\beta\gamma}A_{\beta,\vec{l}}A_{\gamma,\vec{m}}\,. \tag{3.145}$$

The analysis of these equations in principle follows the one-dimensional case. However, the solution depends on assumptions made about the frequencies ω_α, in particular in what follows we assume that the only solution of $\vec{k}\cdot\vec{\omega} = 0$ is $\vec{k} = 0$. Note that this condition is independent of the same condition on Ω_α because even if the frequencies Ω_α satisfy the condition the unperturbed frequencies ω_α may not. If these conditions are satisfied then as we will see the only amplitudes $A_{\alpha,\vec{k}}$ of order 1 are $A_{\alpha,\pm\vec{e}_\alpha} \equiv A^{(\pm)}_\alpha = \frac{1}{2}a_\alpha e^{\pm i\varphi_\alpha}$ where the only nonvanishing component of the basis vector $\vec{e}_\alpha$ is $(\vec{e}_\alpha)_\alpha = 1$, e.g. $\vec{e}_3 = (0, 0, 1, 0, \dots, 0)$.

- $\vec{k} = \vec{0}$

$$\omega_\alpha^2 A_{\alpha,\vec{0}} = -\sum_{\beta,\gamma} \sum_{\vec{l}+\vec{m}=\vec{0}} g^{(3)}_{\alpha\beta\gamma} A_{\beta,\vec{l}} A_{\gamma,\vec{m}} \,. \tag{3.146}$$

Since $A_{\alpha,\vec{e}_\alpha} \equiv A_\alpha$ while all the other amplitudes are at least of first order in $\epsilon_{\alpha\beta\gamma}$ we get that the leading term in $A_{\alpha,\vec{0}}$ is obtained if $\gamma = \beta$ and $\vec{l} = -\vec{m} = \pm\vec{e}_\beta$, and therefore

$$\begin{aligned} &\omega_\alpha^2 A_{\alpha,\vec{0}} = -\frac{1}{2}\sum_\beta g^{(3)}_{\alpha\beta\beta} a_\beta^2 + \mathcal{O}(\epsilon^2_{\alpha\beta\gamma}) \;\Rightarrow\; A_{\alpha,\vec{0}} = -\frac{1}{2}\sum_\beta \frac{g^{(3)}_{\alpha\beta\beta} a_\beta^2}{\omega_\alpha^2} + \mathcal{O}(\epsilon^2_{\alpha\beta\gamma}) \;\Rightarrow \\ &\quad A_{\alpha,\vec{0}} = -\frac{1}{2} a_\alpha \sum_\beta \epsilon_{\alpha\beta\beta} + \mathcal{O}(\epsilon^2_{\alpha\beta\gamma}) \,. \end{aligned} \tag{3.147}$$

Note that $A_{\alpha,\vec{0}}$ is of order $\epsilon_{\alpha\beta\gamma}$ because $\vec{0}$ can be written as a linear combination of two basis vectors $\vec{e}_\rho$ with coefficients ± 1. Consider then the general case of this type

- $\vec{k} = \varepsilon_\rho \vec{e}_\rho + \varepsilon_\mu \vec{e}_\mu,\;\; \varepsilon_\rho^2 = 1\;\forall\rho,\;\; \rho,\mu = 1,\dots,s$

$$(\omega_\alpha^2 - (\varepsilon_\rho\Omega_\rho + \varepsilon_\mu\Omega_\mu)^2) A_{\alpha,\varepsilon_\rho\vec{e}_\rho+\varepsilon_\mu\vec{e}_\mu} = -\sum_{\beta,\gamma} \sum_{\vec{l}+\vec{m}=\varepsilon_\rho\vec{e}_\rho+\varepsilon_\mu\vec{e}_\mu} g^{(3)}_{\alpha\beta\gamma} A_{\beta,\vec{l}} A_{\gamma,\vec{m}} \;\;. \tag{3.148}$$

There are two cases to consider

1. $\rho \neq \mu$
   We can have $\beta = \rho, \vec{l} = \varepsilon_\rho\vec{e}_\rho,\; \gamma = \mu, \vec{m} = \varepsilon_\mu\vec{e}_\mu$ or $\gamma = \rho, \vec{m} = \varepsilon_\rho\vec{e}_\rho,\; \beta = \mu, \vec{l} = \varepsilon_\mu\vec{e}_\mu$. Thus

$$\begin{aligned} (\omega_\alpha^2 - (\varepsilon_\rho\omega_\rho + \varepsilon_\mu\omega_\mu)^2) A_{\alpha,\varepsilon_\rho\vec{e}_\rho+\varepsilon_\mu\vec{e}_\mu} &= -\frac{1}{2} g^{(3)}_{\alpha\rho\mu} a_\rho a_\mu e^{i\varepsilon_\rho\varphi_\rho + i\varepsilon_\mu\varphi_\mu} + \mathcal{O}(\epsilon^2_{\alpha\beta\gamma}) \Rightarrow \\ A_{\alpha,\varepsilon_\rho\vec{e}_\rho+\varepsilon_\mu\vec{e}_\mu} &= -\frac{1}{2} a_\alpha \frac{\omega_\alpha^2 \epsilon_{\alpha\rho\mu}}{\omega_\alpha^2 - (\varepsilon_\rho\omega_\rho + \varepsilon_\mu\omega_\mu)^2} e^{i\varepsilon_\rho\varphi_\rho + i\varepsilon_\mu\varphi_\mu} + \mathcal{O}(\epsilon^2_{\alpha\beta\gamma}) \,. \end{aligned} \tag{3.149}$$

   Note that according to our assumption, $\omega_\alpha \neq \varepsilon_\rho\omega_\rho + \varepsilon_\mu\omega_\mu$ for any $\alpha, \rho, \mu, \varepsilon_\rho, \varepsilon_\mu$.

2. $\rho = \mu$
   We can have $\beta = \gamma = \rho, \vec{l} = \vec{m} = \varepsilon_\rho\vec{e}_\rho$. Thus

$$\begin{aligned} (\omega_\alpha^2 - 4\omega_\rho^2) A_{\alpha,2\varepsilon_\rho\vec{e}_\rho} &= -\frac{1}{4} g^{(3)}_{\alpha\rho\rho} a_\rho a_\rho e^{2i\varepsilon_\rho\varphi_\rho} + \mathcal{O}(\epsilon^2_{\alpha\beta\gamma}) \Rightarrow \\ A_{\alpha,2\varepsilon_\rho\vec{e}_\rho} &= -\frac{1}{4} a_\alpha \frac{\omega_\alpha^2 \epsilon_{\alpha\rho\rho}}{\omega_\alpha^2 - 4\omega_\rho^2} e^{2i\varepsilon_\rho\varphi_\rho} + \mathcal{O}(\epsilon^2_{\alpha\beta\gamma}) \,. \end{aligned} \tag{3.150}$$

Next we consider amplitudes of order $e^2_{\alpha\beta\gamma}$. They are given by $\vec{k}$ which are of the form $\vec{k} = \sum_{i=1}^3 \varepsilon_{\rho_i} \vec{e}_{\rho_i}$.

- $\vec{k} = \varepsilon_\rho\vec{e}_\rho,\; \rho = 1,\dots,s$

$$(\omega_\alpha^2 - \Omega_\rho^2) A_{\alpha,\varepsilon_\rho\vec{e}_\rho} = -\sum_{\beta,\gamma} \sum_{\vec{l}+\vec{m}=\varepsilon_\rho\vec{e}_\rho} g^{(3)}_{\alpha\beta\gamma} A_{\beta,\vec{l}} A_{\gamma,\vec{m}} \,. \tag{3.151}$$

There are two cases to consider

1. $\alpha \neq \rho$

   Then we take $\vec{l} = \varepsilon_\beta \vec{e}_\beta$, $\vec{m} = \varepsilon_\rho \vec{e}_\rho - \varepsilon_\beta \vec{e}_\beta$ or $\vec{m} = \varepsilon_\gamma \vec{e}_\gamma$, $\vec{l} = \varepsilon_\rho \vec{e}_\rho - \varepsilon_\gamma \vec{e}_\gamma$, and get

$$\begin{aligned}(\omega_\alpha^2 - \omega_\rho^2) A_{\alpha,\varepsilon_\rho \vec{e}_\rho} &= -\sum_{\beta,\gamma} g^{(3)}_{\alpha\beta\gamma} a_\beta A_{\gamma,\varepsilon_\rho \vec{e}_\rho - \varepsilon_\beta \vec{e}_\beta} e^{i\varepsilon_\beta \varphi_\beta} + \mathcal{O}(\epsilon^3_{\alpha\beta\gamma}) \Rightarrow \\ A_{\alpha,\varepsilon_\rho \vec{e}_\rho} &= -\sum_{\epsilon_\beta,\beta,\gamma} \frac{\omega_\alpha^2 \epsilon_{\alpha\beta\gamma}}{\omega_\alpha^2 - \omega_\rho^2} \frac{1}{a_\gamma} A_{\gamma,\varepsilon_\rho \vec{e}_\rho - \varepsilon_\beta \vec{e}_\beta} e^{i\varepsilon_\beta \varphi_\beta} + \mathcal{O}(\epsilon^3_{\alpha\beta\gamma}) \,,\end{aligned} \tag{3.152}$$

   where $A_{\gamma,\varepsilon_\rho \vec{e}_\rho - \varepsilon_\beta \vec{e}_\beta}$ are given by (3.147), (3.149), (3.150).

2. $\alpha = \rho$

   This allows one to determine the leading correction to Ω_α

$$(\omega_\alpha^2 - \Omega_\alpha^2) A_{\alpha,\varepsilon_\alpha \vec{e}_\alpha} = -\sum_{\beta,\gamma} \sum_{\vec{l}+\vec{m}=\varepsilon_\alpha \vec{e}_\alpha} g^{(3)}_{\alpha\beta\gamma} A_{\beta,\vec{l}} A_{\gamma,\vec{m}} \,. \tag{3.153}$$

   We take $\vec{l} = \varepsilon_\beta \vec{e}_\beta$, $\vec{m} = \varepsilon_\alpha \vec{e}_\alpha - \varepsilon_\beta \vec{e}_\beta$ or $\vec{m} = \varepsilon_\gamma \vec{e}_\gamma$, $\vec{l} = \varepsilon_\alpha \vec{e}_\alpha - \varepsilon_\gamma \vec{e}_\gamma$, and get

$$\begin{aligned}(1 - \frac{\Omega_\alpha^2}{\omega_\alpha^2}) \frac{1}{2} e^{i\varepsilon_\alpha \varphi_\alpha} &= -\sum_{\beta,\gamma} \epsilon_{\alpha\beta\gamma} \frac{1}{a_\gamma} A_{\gamma,\varepsilon_\alpha \vec{e}_\alpha - \varepsilon_\beta \vec{e}_\beta} e^{i\varepsilon_\beta \varphi_\beta} \;\; \Rightarrow \\ \frac{\Omega_\alpha^2}{\omega_\alpha^2} &= 1 + 2 \sum_{\varepsilon_\beta,\beta,\gamma} \epsilon_{\alpha\beta\gamma} \frac{1}{a_\gamma} A_{\gamma,\varepsilon_\alpha \vec{e}_\alpha - \varepsilon_\beta \vec{e}_\beta} e^{i\varepsilon_\beta \varphi_\beta - i\varepsilon_\alpha \varphi_\alpha} + \mathcal{O}(\epsilon^3_{\alpha\beta\gamma}) \,.\end{aligned} \tag{3.154}$$

It is clear that one can proceed and determine leading and subleading corrections to any amplitude and frequency. The resulting series may be divergent. The conditions for it to be convergent are discussed in the framework of the Kolmogorov-Arnold-Moser theory.