



Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin

Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

JF Mathematics

JF Theoretical Physics

JF TSM Mathematics

Trinity Term 2016

MA1132 - Advanced Calculus

Friday, May 27

EXAM HALL

9.30 — 11.30

Dr. Sergey Frolov

Instructions to Candidates:

ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

Each question is worth 25 marks. Show the details of your work.

1. The equations of motion of a system of n particles are given by

$$m_i \ddot{x}_i = -\frac{\partial U(x_1, \dots, x_n)}{\partial x_i}, \quad \ddot{x}_i = \frac{d^2 x_i}{dt^2}, \quad i = 1, 2, \dots, n,$$

where m_i is the mass and x_i is the coordinate of the i -th particle, and $U(x_1, \dots, x_n)$ is the potential energy of the system.

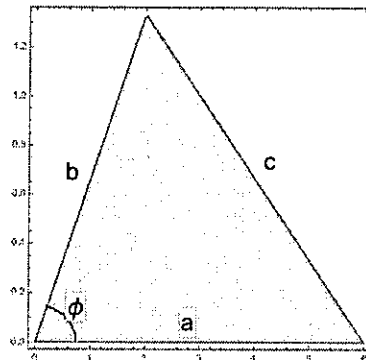
Consider a system of n particles with the potential

$$U(x_1, \dots, x_n) = \sum_{i=1}^{n-1} \cosh(\kappa(x_{i+1} - x_i)) + \sum_{i,j=1}^n \frac{\lambda}{4} (x_i - x_j)^4.$$

- (a) 4 marks. Find the equations of motion of the first particle (x_1).
 - (b) 4 marks. Find the equations of motion of the second particle (x_2).
 - (c) 4 marks. Find the equations of motion of the last particle (x_n).
 - (d) 7 marks. Find the equations of motion of the i -th particle (x_i) for $1 < i < n$.
 - (e) 6 marks. Write the equations of motion of the i -th particle (x_i) for $1 \leq i \leq n$ by using the Kronecker delta δ_{ij} .
2. Consider the surface

$$z = f(x, y) = \ln \left(\frac{1}{2} e^{2/3} \sqrt[3]{8x^2 - 6xy^2 - y^3 + 32 - 12 \sin(2x - y)} \right).$$

- (a) 11 marks. Find an equation for the tangent plane to the surface at the point $P = (1, 2, z_0)$ where $z_0 = f(1, 2)$.
- (b) 4 marks. Find points of intersection of the tangent plane with the x -, y - and z -axes.
- (c) 2 marks. Sketch the tangent plane, and show the point $P = (1, 2, z_0)$ on it.
- (d) 6 marks. Find parametric equations for the normal line to the surface at the point $P = (1, 2, z_0)$.
- (e) 2 marks. Sketch the normal line to the surface at the point $P = (1, 2, z_0)$.



3. Consider a triangle which has the sides a, b, c and the angle between the sides a and b is ϕ , see the picture.

- (a) 1 mark. What is the perimeter of the triangle?
- (b) 3 mark. What is the area of the triangle?
- (c) 6 marks. What is the constraint which relates a, b, c and ϕ ?
- (d) 15 marks. Show that a triangle with fixed perimeter has maximum area if it is equilateral.

4. Consider the sum of the integrals

$$\int_0^1 \int_0^{\sqrt{3}x} f(x, y) dy dx + \int_1^2 \int_0^{\sqrt{4-x^2}} f(x, y) dy dx.$$

- (a) 7 marks. Determine and sketch the integration region R .
- (b) 4 marks. Write the sum as one repeated integral by reversing the order of integration.
- (c) 4 marks. Find the area of the region R .
- (d) 5 marks. Find the x -coordinate of the centroid of the region R .
- (e) 5 marks. Find the y -coordinate of the centroid of the region R .

5. (a) 1 mark. Express rectangular coordinates in terms of spherical coordinates, and draw the corresponding picture.
- (b) Consider the solid G bounded above by the surface $z^2 = x^2 + y^2$ and below by the surface $z = -\sqrt{4 - x^2 - y^2}$.
- 1 mark. What is the surface $z^2 = x^2 + y^2$?
 - 1 mark. What is the surface $z = -\sqrt{4 - x^2 - y^2}$?
 - 2 marks. Sketch the solid G .
 - 7 marks. Use a triple integral and spherical coordinates to compute the volume V of the solid G .
 - 9 marks. Use a triple integral and spherical coordinates to find the mass M of the solid G if its density is

$$\delta(x, y, z) = \frac{1 - e^{-2\sqrt{x^2 + y^2 + z^2}}}{\sqrt{x^2 + y^2 + z^2}}.$$

- 4 marks. What is the density of the solid G at the origin: $\delta(0, 0, 0) = ?$

Useful Formulae

1. Let $\mathbf{r}(t)$ be a vector function with values in \mathbb{R}^3 : $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.
 - (a) The unit normal vector is $\mathbf{N}(t) = \frac{\frac{d\mathbf{T}}{dt}}{|\frac{d\mathbf{T}}{dt}|}$.
 - (b) The unit binormal vector is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.
 - (c) The curvature of C is $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.
2. Let σ be a surface in \mathbb{R}^3 : $z = f(x, y)$
 - (a) The slope k_x of the surface in the x -direction at (x_0, y_0) is $k_x = \frac{\partial z}{\partial x}(x_0, y_0)$.
 - (b) The slope k_y of the surface in the y -direction at (x_0, y_0) is $k_y = \frac{\partial z}{\partial y}(x_0, y_0)$.
 - (c) The equation for the tangent plane to the surface at the point $P = (x_0, y_0, z_0)$ is $z = z_0 + k_x(x - x_0) + k_y(y - y_0)$.
 - (d) Parametric equations for the normal line to the surface at $P = (x_0, y_0, z_0)$ are $\mathbf{r}(t) = \mathbf{r}_0 + t(-k_x\mathbf{i} - k_y\mathbf{j} + \mathbf{k})$, $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$.
 - (e) The mass of the lamina with the density $\delta(x, y, z)$ that is the portion of the surface that is above a region R in the xy -plane is
$$M = \iint_{\sigma} \delta(x, y, z) dS = \iint_R \delta(x, y, z) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA.$$
3. Let R be a plain lamina with density $\delta(x, y)$.
 - (a) Its mass is equal to $M = \iint_R \delta(x, y) dA$.
 - (b) The x -coordinate of its centre of gravity is equal to $x_{cg} = \frac{1}{M} \iint_R x \delta(x, y) dA$.
 - (c) The y -coordinate of its centre of gravity is equal to $y_{cg} = \frac{1}{M} \iint_R y \delta(x, y) dA$.
4. The volume element in spherical coordinates is $dV = r^2 \sin \phi dr d\phi d\theta$.
5. Let a region R_{xy} in the xy -plane be mapped to a region R_{uv} in the uv -plane under the change of variables $u = u(x, y)$, $v = v(x, y)$.
 - (a) The magnitude of the Jacobian of the change is $\left| \frac{\partial(u, v)}{\partial(x, y)} \right| = \left| \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right|$.
 - (b) The integral over R_{xy} is $\iint_{R_{xy}} f(x, y) dA_{xy} = \iint_{R_{uv}} f(x(u, v), y(u, v)) \left| \frac{\partial(u, v)}{\partial(x, y)} \right|^{-1} dA_{uv}$.