

Advanced Mechanics 241, Spring 2008
Scholarship Examination Questions

Part I. Questions

1. Formulate Hamilton's principle (the principle of least action) and use it to derive Lagrange's equations of motion for a system with s degrees of freedom. What is called the number of degrees of freedom? What are the properties of the Lagrangian of a mechanical system? What is the Lagrangian for a free particle in relativistic mechanics?
2. Formulate Noether's theorem and use it to derive the conservation laws of energy, momentum and angular momentum of a classical mechanical system with s degrees of freedom. What symmetries are responsible for the existence of the conservation laws?
3. Discuss the motion in one dimension and in a central field. What is a turning point? What is a potential well? If the motion is finite how can one find the period of oscillations. How to find the equation of the path of a particle in a central field. What types of paths exist?
4. Discuss Kepler's problem in an attractive field. What is the Runge-Lenz vector?
5. How to find the deflection angle of a particle scattered by a central field? What is the impact parameter? Find the deflection angle of a particle moving in a repulsive Coulomb field.
6. Discuss small oscillations of a system with s degrees of freedom. What is the characteristic equation? What are the characteristic (or normal, or eigen) frequencies of the system? What are normal coordinates?
7. Discuss forced oscillations of a one-dimensional harmonic oscillator if the external force is

$$F(t) = f \cos(\gamma t + \beta).$$

What is resonance?

8. Discuss damped oscillations of a one-dimensional harmonic oscillator.
9. Discuss forced oscillations of a one-dimensional harmonic oscillator under friction if the external force is

$$F(t) = f \cos(\gamma t + \beta).$$

10. Express the kinetic energy of a rigid body as a sum of the kinetic energy of the translational motion and the kinetic energy of the rotation. Define the inertia tensor. What are principal axes of inertia? What are principal moments of inertia? What is a symmetrical top? What is a spherical top?
11. Derive the expression for the angular momentum of a rigid body in terms of its inertia tensor and its angular velocity. Define the inertia tensor. What are principal axes of inertia? What are principal moments of inertia? What is a symmetrical top? What is a spherical top? Discuss the free motion of a symmetric top? What is the regular precession of a symmetric top?
12. Determine the frequency of small oscillations of a compound pendulum (a rigid body swinging about a fixed horizontal axis in a gravitational field).

13. Derive the equations of motion of a rigid body in a fixed system of reference. What is the total torque?
14. What are the Euler angles? What is the line of nodes? Represent a general rotation matrix as a product of three simple rotation matrices. Express the components of the angular velocity vector $\boldsymbol{\Omega}$ along the moving axes x_1, x_2, x_3 in terms of the Euler angles and their derivatives. Determine the free motion of a symmetric top by using the Euler angles.
15. Reduce to quadratures the problem of the motion of a heavy symmetrical top whose lowest point is fixed (Lagrange's top). Use the Euler angles. What angles are cyclic? What is nutation? What types of motion are possible?
15. Derive Euler's equations, that is the equations of motion of a rigid body in a moving system of reference. Use them to analyze the free motion of a symmetrical top.
16. **This problem will not be given this year.** Find the Lagrangian describing the motion of a particle in an arbitrary non-inertial frame of reference. What is the Coriolis force? What is the centrifugal force? What is centrifugal energy?
17. Discuss Legendre's transformation, and apply it to a system with the following Lagrangian

$$L = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j - U(q) .$$

18. Define the Poisson bracket of two functions on the phase space and discuss its properties. Formulate and prove Poisson's theorem.
19. Define canonical transformations. How can the generating functions S_i be used? Show that canonical transformations preserve the Poisson brackets. How can a function $V(p, q)$ be used to generate a canonical transformation?
20. Discuss Liouville's theorem.
21. Discuss the action as a function of the coordinates and time, and show that it is a generating function of a canonical transformation.
22. Discuss the Hamilton-Jacobi equation and separation of variables. When can the spherical coordinates be used to separate variables? Use the Hamilton-Jacobi equation to analyze the motion of a harmonic oscillator.
23. Discuss adiabatic invariants. Find the adiabatic invariant for a harmonic oscillator.
24. Discuss action/angle variables. Find action-angle variables I and α for a harmonic oscillator as functions of p and x . Find p and x as functions of action-angle variables I and α for a harmonic oscillator.
25. Discuss conditionally periodic motion.
26. What is the action of a free relativistic particle? What are the equations of motion of a relativistic particle if an external force \mathbf{F} acts on it? What is the momentum of a free relativistic particle? What is the energy of a free relativistic particle? How are they related?

Part II. Problems.

1. Find the Lagrangian of a particle moving on the surface of a paraboloid $z = a(x^2 + y^2)$ in a uniform gravitational field $\mathbf{F} = \{F_x, F_y, F_z\}$. What is the number of degrees of freedom of this system?
2. Use Noether's theorem to find conserved charges, J_{ij} , corresponding to the rotational symmetry of the Lagrangian of a particle moving in a D-dimensional flat space

$$L = \frac{1}{2}mv_i^2 - \frac{\kappa}{r}, \quad r = \sqrt{x_i^2}.$$

How many independent charges are there? Compute the Poisson brackets of the charges with the momenta p_k , coordinates x_k and themselves, that is

$$\{J_{ij}, p_k\}, \quad \{J_{ij}, x_k\}, \quad \{J_{ij}, J_{kl}\}.$$

Specify the formulas for the $D = 3$ case. What is the relation of J_{ij} for $D = 3$ with the angular momentum \mathbf{M} ?

3. Find the one-dimensional particle motion in the trigonometric potential

$$U(x) = V \tan^2(ax), \quad V > 0.$$

4. Integrate (reduce to quadrature) the equations of motion for a particle moving on the surface of a paraboloid $z = a(x^2 + y^2)$ in a uniform gravitational field $\mathbf{F} = \{0, 0, F_z\}$.
5. Integrate the equations of motion for a particle in a central field

$$U = -\frac{\alpha}{r^2}, \quad \alpha > 0$$

6. Find the frequency of oscillations of a particle of mass m which is free to move along the parabola $y = -ax^2$ and is attached to an ideal spring whose other end is fixed at a point A with coordinates $(0, l)$. A force F is required to extend the spring to length l .
7. Determine the forced oscillations and find the energy acquired by an oscillator experiencing a force

$$F(t) = \begin{cases} F_0 e^{\alpha t} & t < 0 \\ F_0 e^{-\alpha t} & t > 0 \end{cases}$$

The initial energy as $t \rightarrow -\infty$ is $E_0 = 0$.

8. Find the normal coordinates and frequencies of a system of three equal masses constrained to move on a circle connected to each other by identical ideal springs.
9. Determine the motion of an oscillator due to an external force

$$F(t) = \begin{cases} F_0 e^{\alpha t} & t < 0 \\ F_0 e^{-\alpha t} & t > 0 \end{cases}$$

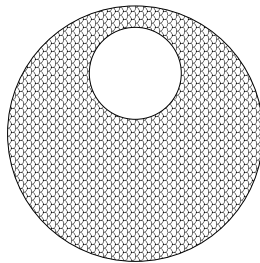
in the presence of friction.

10. Find the principal moments of inertia for the following homogeneous bodies: (a) a thin rod of length l , (b) a sphere of radius R , (c) a circular cylinder of radius R and height h , (d) a

rectangular parallelepiped of sides a , b , and c , (e) a circular cone of height h and base radius R , (f) an ellipsoid of semiaxes a , b , and c .

Find the frequency of small oscillations of a rigid body (for each of them) about a horizontal axis passing through the point of the body farthest from the centre of mass and perpendicular to the line connecting the point and the centre of mass.

11. Find the principal moments of inertia of a homogeneous ball of radius R which has a cavity in the shape of a sphere of radius r . The distance between the centres of the ball and the sphere is $a \leq R - r$.



Find the frequency of small oscillations of the rigid body about a horizontal axis perpendicular to the line connecting the centres of the ball and the sphere and passing through the center of (i) the ball. (ii) the cavity.

12. A symmetric top with a fixed center of mass experiences a constant torque \mathbf{K} . Use the Euler angles to find the angular velocity of the top as a function of time. The initial angular velocity is equal to 0: $\boldsymbol{\Omega}(0) = 0$.

13. A heavy symmetrical top whose lowest point is fixed rotates about a vertical axis. Find
(i) The condition for the rotation to be stable.

(ii) The frequency of small oscillations in the θ -direction if the top was shifted from the equilibrium position.

14. Find the Hamiltonian and Hamilton's equations of motion for a system with two degrees of freedom with the following Lagrangian

$$L = \frac{m_1}{2} \dot{x}_1^2 + \frac{m_2}{2} \dot{x}_2^2 + B(\dot{x}_1 x_2 - x_1 \dot{x}_2) - U(x_1, x_2).$$

15. Find the Hamiltonian and Hamilton's equations of motion for a free relativistic particle of rest mass m_0 (c is the speed of light)

$$L = -m_0 c^2 \sqrt{1 - \frac{1}{c^2} \dot{x}_i^2}.$$

16. Let $S_1(q, Q, t)$, $S_2(q, P, t)$, $S_3(p, Q, t)$, $S_4(p, P, t)$ be the four generating functions of a canonical transformation. Express

1. p and P , and $H' - H$ as the derivatives of $S_1(q, Q, t)$.
2. p and Q , and $H' - H$ as the derivatives of $S_2(q, P, t)$.
3. q and P , and $H' - H$ as the derivatives of $S_3(p, Q, t)$.
4. q and Q , and $H' - H$ as the derivatives of $S_4(p, P, t)$.

17. Use the four generating functions $S_1(q, Q, t)$, $S_2(q, P, t)$, $S_3(p, Q, t)$, $S_4(p, P, t)$ to prove that

$$\{Q^i, Q^j\}_{p,q} = 0, \quad \{P_i, P_j\}_{p,q} = 0.$$

18. Find the explicit form (i.e. $P = P(p, q)$, $Q = Q(p, q)$) of the canonical transformation generated by the function

$$V(p, q) = \lambda p_i q^i, \quad \lambda \text{ is a constant},$$

by means of the formula

$$\begin{aligned} f(p, q) \rightarrow f(P, Q) &= f(p, q) + \{V(p, q), f(p, q)\} + \frac{1}{2}\{V, \{V, f\}\} + \cdots \\ &= f + \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{\{V, \{V, \cdots \{V, f\} \cdots\}}_n, \end{aligned}$$

where $f(p, q)$ is an arbitrary function of the phase space.

19. Find canonical transformations corresponding to the generating functions below and explain their meaning

1. $S_3(p, Q) = p_i Q^i - a^i p_i$, for some fixed vector a_i .
2. $S_2(q, P) = -q^i P_i - \epsilon_{ijk} a^i q^j P_k$, summation over i, j, k .
3. $S_3(p, Q, t) = p_i Q^i + \tau H(p, Q, t)$.
4. $S_1(q, Q) = \frac{m}{2} \frac{(Q-q)^2}{t_2-t_1}$, where m, t_1, t_2 are constants.

Consider τ and a_i to be infinitesimally small, i.e. neglect the terms containing τ^2 etc.

20. The transformations below are almost canonical. Find a simple way to modify them so that they become canonical.

$$(i) \quad P = p+q, \quad Q = 4q \left(e^{(q+p)^4} + 1 \right) + p \left(4e^{(q+p)^4} - 1 \right).$$

$$(ii) \quad P = qp, \quad Q = \ln(p^{3007} q^{2006}).$$

21. Find generating functions $S_2(q, P, t)$ producing the following canonical transformations of the six-dimensional phase space

1. Spiral motion.

22. Using the Hamilton-Jacobi equation find the trajectory and the motion of a particle in the potential

$$U(\mathbf{r}) = -Fx.$$

Note that it is a three-dimensional problem.

23. Find action-angle variables I and α for a harmonic oscillator as functions of p and x . Find p and x as functions of action-angle variables I and α for a harmonic oscillator.

24. Find the motion of a charge e in a constant uniform electric field \mathbf{E} described by the following action

$$S = \int_{t_1}^{t_2} dt \left(-mc^2 \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} + e \mathbf{r} \cdot \mathbf{E} \right).$$

Take the direction of the field as the X axis, and the momentum at $t = 0$ to be $\mathbf{p}_0 = (0, p_0, 0)$. Obtain the equation of the trajectory, and sketch its plot. Find the trajectory if the velocity of the particle is $v \ll c$.

25. Find the Hamiltonian and Hamilton's equations of motion for a charge e moving in a constant uniform magnetic field \mathbf{B} described by the following action

$$S = \int_{t_1}^{t_2} dt \left(-mc^2 \sqrt{1 - \frac{v_\alpha^2}{c^2}} + \frac{e}{2c} \epsilon_{\alpha\beta\gamma} B_\alpha x_\beta v_\gamma \right) .$$

Take the direction of the field as the Z axis. Show that the speed of the particle stays constant throughout the motion. Show that the trajectory is a helix.