JF PY1T10 Special Relativity

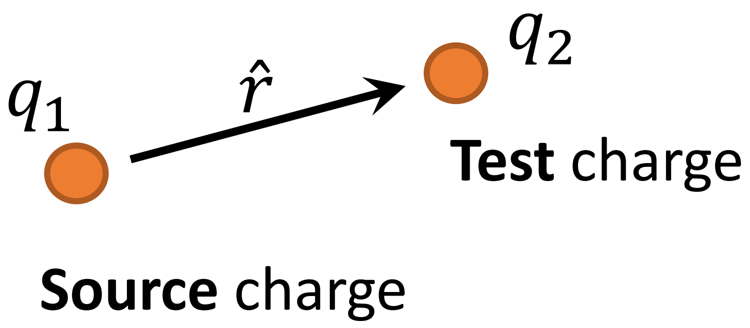
Lecture 13:

Special Relativity, Electricity and Magnetism

Electric Force

Coulomb's Law:

Force which as *stationary* charge q_1 exerts on a *stationary* charge q_2:



$\hat{r}$ = unit vector, $q_1 \rightarrow q_2$

Force on q_2:

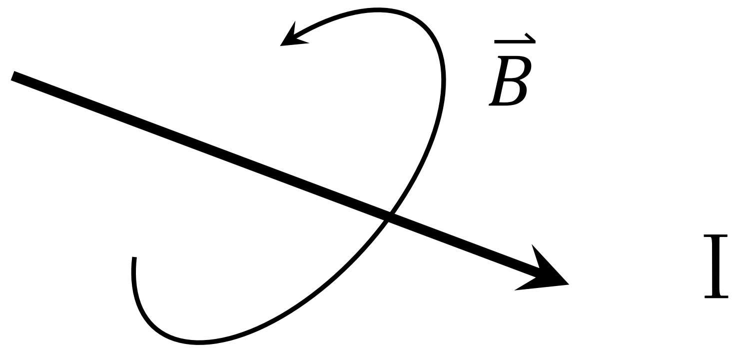
$$F = k\frac{q_1 q_2}{r^2}\hat{r}$$

where $k = \frac{1}{4\pi\varepsilon_0}$, $\varepsilon_0 = 8.85 \text{ x } 10^{-12}\mathrm{Fm}^{-1}$

Coulomb's law also holds when q_2 is moving, <u>as long as</u> q_1 <u>is stationary</u>.

Magnetic Field ($\vec{B}$)

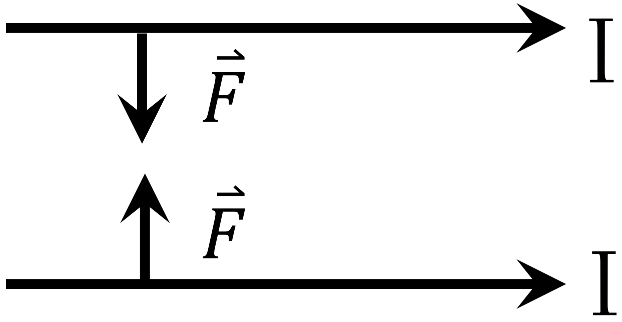
$\vec{B}$-field is generated by a moving charge, e.g. a current in a wire



Charge moving with velocity $\vec{u}$ in $\vec{B}$ is subject to magnetic force:

$$\vec{F}_{mag} = q\vec{u} \times \vec{B}$$

This equation defines $\vec{B}$ in terms of a force which is proportional to velocity and charge.



Maxwell's Equations (1864)

Defined relations between $\vec{E}$ and $\vec{B}$ for stationary and moving charges.

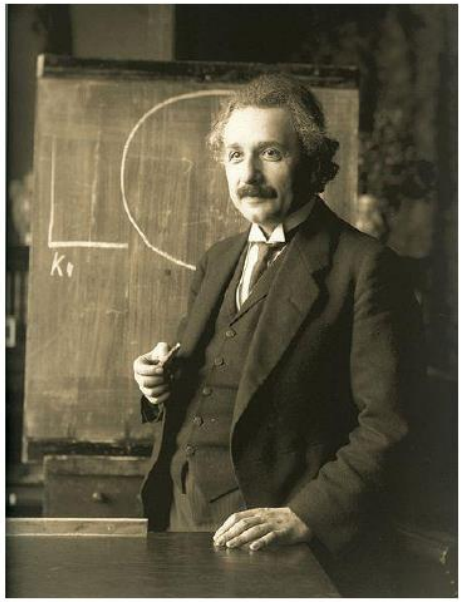
Developed before SR but still:

- Correct as $v \to c$
- Describes electromagnetic (e.m.) phenomena in *any* inertial frame

But $\vec{E}$ and $\vec{B}$ are <u>not</u> the same in different inertial frames.

Einstein's Insight

" What led me more or less directly to the special theory of relativity was the conviction that the electromotive force acting on a body moving in a magnetic field was nothing else than an electric field "



<u>Both at rest:</u>

q_1 — $\vec{F}_{elec}$ only

q_2 — $\vec{E}$ only

<u>Both moving:</u>

$q_1 \longrightarrow$ $\vec{F}_{elec} + \vec{F}_{mag}$

$q_2 \longrightarrow$ $\vec{E}$ and $\vec{B}$

What appears as $\vec{B}$ in one frame is nothing else but $\vec{E}$ when viewed from another frame.

Use S.R. to specify this link between $\vec{E}$ and $\vec{B}$

Einstein's Insight

Consider frame in which source charge is at rest.

Coulomb's law tells us the force on test charge q_2.

Now: Find force on q_2 in another frame using transformation of force.

Identify $\vec{B}$ from component of force on q_2 which <u>depends</u> on velocity of q_2

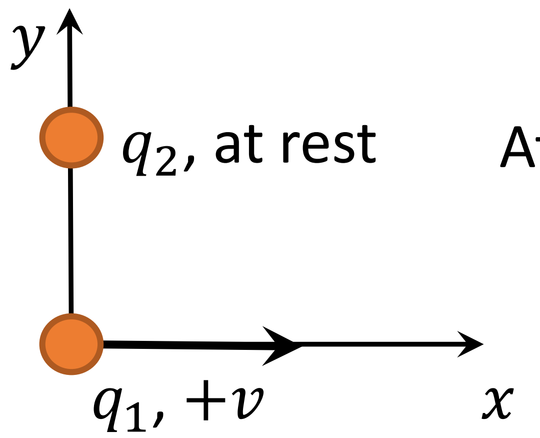
Identify $\vec{E}$ from component of force on q_2 which is <u>independent</u> of velocity of q_2

Invariance of charge: Charge not changed by motion of carrier

[Evidence: Take neutral atoms where $\sum q_i = 0$. Use heat to speed up the e^-.
Still $\sum q_i = 0$. Atoms are neutral even though electrons in different orbits have different velocities.]

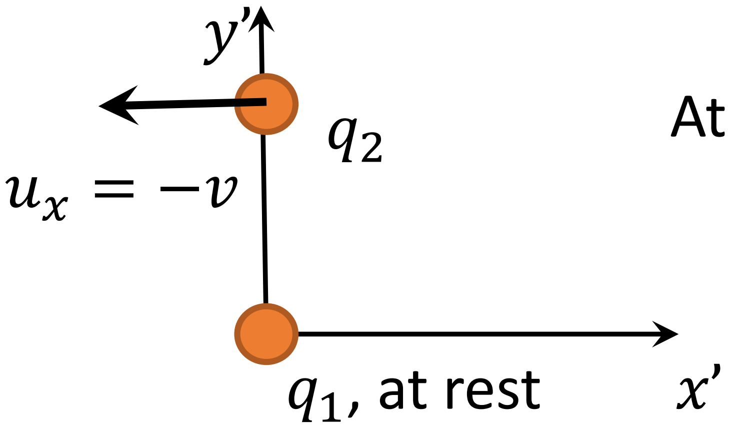
Force on a stationary test charge q_2 due to moving source q_1

Source q_1 with $v \,||\, x$:



At $t = 0, q_2$ is at $(0, y, 0)$

Shift to S', in which q_1 is now at rest:



At $t' = 0, q_2$ is at $(0, y', 0)$

In S', as source is at rest, we can apply Coulomb's Law:

$$F'_y = \frac{kq_1q_2}{y'^2}, F'_y = F'_z \, = 0$$

Force on a stationary test charge q_2 due to moving source q_1

Transform back to inertial frame S:

$$F_y = \frac{F'_y}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)} = \frac{F'_y}{\gamma\left(1 - \frac{v^2}{c^2}\right)} = \gamma F'_y$$

$F_x = F_z = 0$ (as $u' = u'_x$ and $F' = F'_y$, $\therefore\ \overrightarrow{F'}.\overrightarrow{u'} = 0$

Also, $y' = y$ (by construction)

$$\Rightarrow F_y = \gamma \frac{k q_1 q_2}{y^2}$$

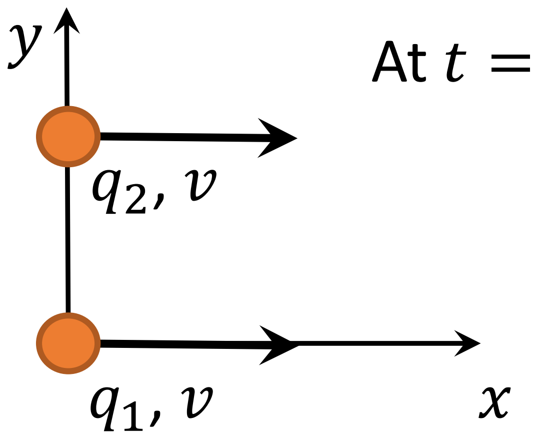
This is Coulomb's law, now modified for moving source charge.

Note: This is for a specific case when the line from the moving source charge to the stationary test charge is $\perp$ to direction of motion of the source charge. It can be generalized for any point.

Force on a moving test charge q_2 due to moving source q_1

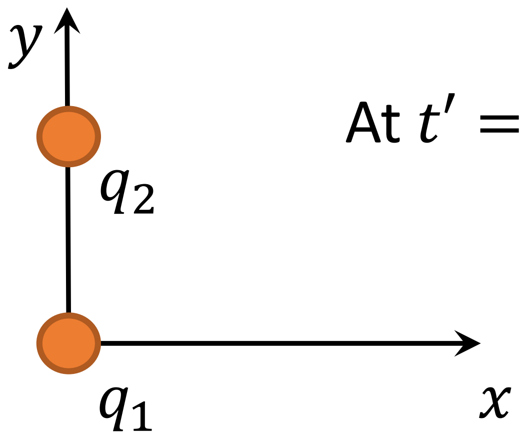
Now assume both source q_1 and test q_2 are moving with velocity $v \parallel x$ at $t = 0$

***S* frame:**



At $t = 0, q_2$ is at $(0, y, 0)$

***S*' frame:** Both q_1 and q_2 are at rest.



At $t' = 0, q_2$ is at $(0, y', 0)$

In S', as source is at rest, we can apply Coulomb's Law:

$$F'_y = \frac{kq_1q_2}{y'^2}, F'_y = F'_z = 0, \overrightarrow{u'} = 0$$

Force on a moving test charge q_2 due to moving source q_1

Again, transform back into S:

$$F_y = \frac{F'_y}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)} = \frac{1}{\gamma} F'_y, \qquad F_x = F_z = 0$$

$$F_y = \frac{1}{\gamma} \frac{k q_1 q_2}{y^2} \text{ (as } \overrightarrow{u'} = 0\text{)}$$

Compare with previous situation (where q_2 was stationary, and q_1 was moving. We had:

$$F_y = \gamma \frac{k q_1 q_2}{y^2}$$

Magnetism

Therefore, we have an <u>extra</u> force on q_2 due to its velocity in S. This is the **magnetic force** on q_2.

$$F_{mag} = \frac{kq_1q_2}{y^2}\left(\frac{1}{\gamma} - \gamma\right)$$

$$= -\frac{v^2}{c^2}\gamma\left(\frac{kq_1q_2}{y^2}\right)$$

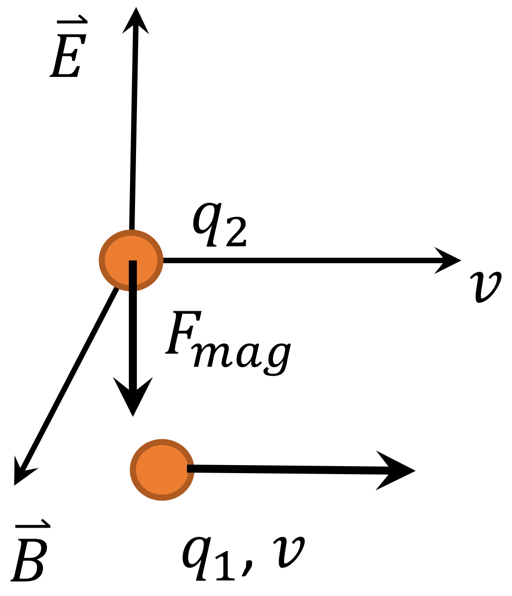
$$\boxed{F_{mag} = -\frac{v^2}{c^2}F_{elec}}$$

F_{mag} is given by: F_{mag} = $q_2\vec{v} \times \vec{B}$

If $\vec{v} \perp \vec{B}$: $F_{mag} = q_2 vB$

What direction is $\vec{B}$?

Magnetism



If $\vec{B} \perp \vec{v}$ (as expected for the magnetic field due to current in direction of $\vec{v}$ of q_1) then:

F_{mag} is in the opposite direction to F_{elec}

This is consistent with $F_{mag} = -\frac{v^2}{c^2} F_{elec}$

$$|qvB| = \left| \frac{v^2}{c^2} \gamma \left(\frac{k q_1 q_2}{y^2} \right) \right|$$

But $F_{elec} = q_2 E$ where $E = \gamma \frac{k q_1}{y^2}$

$$\Rightarrow \; B = \frac{1}{c^2} v E$$

Also, the figure suggests $\vec{B}$ is parallel to $\vec{v} \times \vec{E}$:

Thus we can write:$\vec{B} = \frac{1}{c^2} \vec{v} \times \vec{E}$, where $\vec{E} = \gamma \frac{k q_1}{y^2} \hat{n}$ and $k = \frac{1}{4\pi\varepsilon_0}$

$\vec{B}$ depends on the velocity of q_1

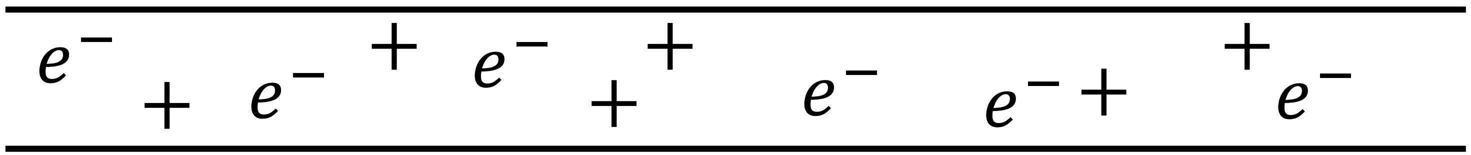
Magnetic Force

$$\left|F_{mag}\right| = \frac{v^2}{c^2}\left|F_{elec}\right|$$

Usually $F_{elec} \gg F_{mag}$ so we cannot easily observe F_{mag}.

If we get rid of F_{elec} we should be able to observe F_{mag}.
We can do this by neutralizing with positive charge.

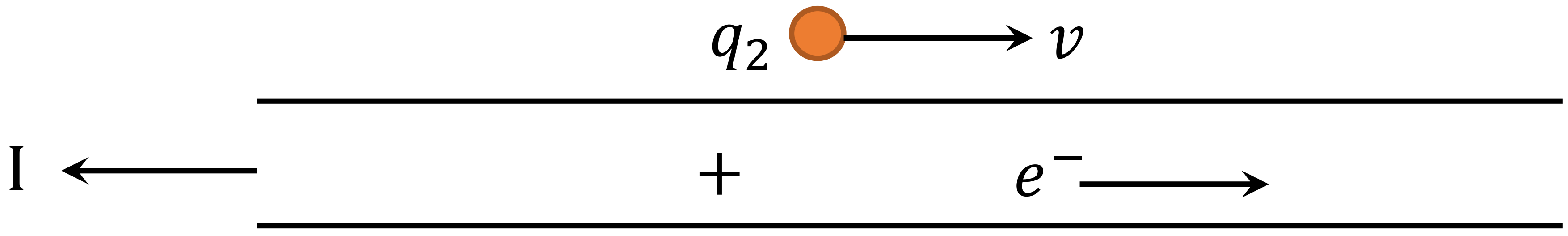
Consider arrangement where $F_{elec} = 0$, e.g. electron current in metal wire where the positive ions are at rest. Overall the wire is neutral – the positive and negative charges cancel.



e^- move with drift velocity, v

Magnetic Force

Consider a moving test charge, outside the wire, with charge q_2, and moving || to wire the same velocity, v:

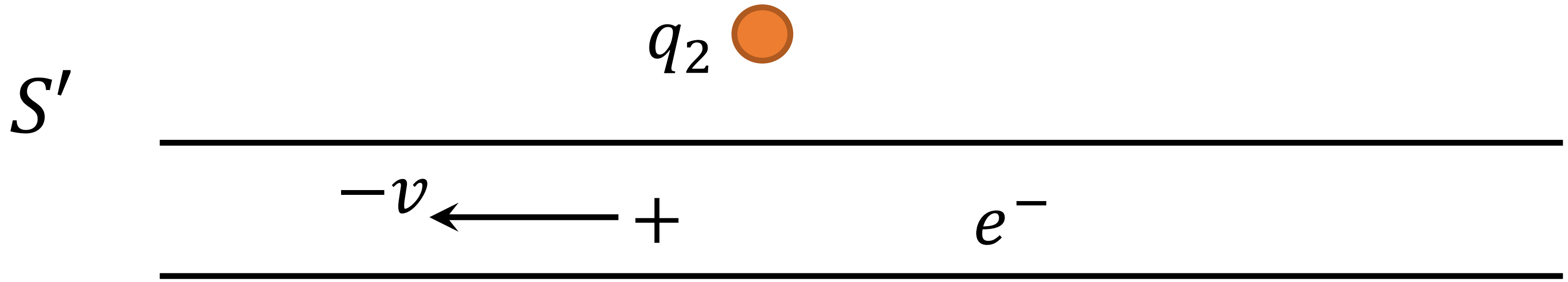


The net $\vec{E}$ at $q_2 = 0$ (the electric fields from the electrons and positive ions cancel).

If the test charge was stationary, it would therefore feel no force.

As it is moving, it will feel a magnetic field due to the moving electrons.

Magnetic Force



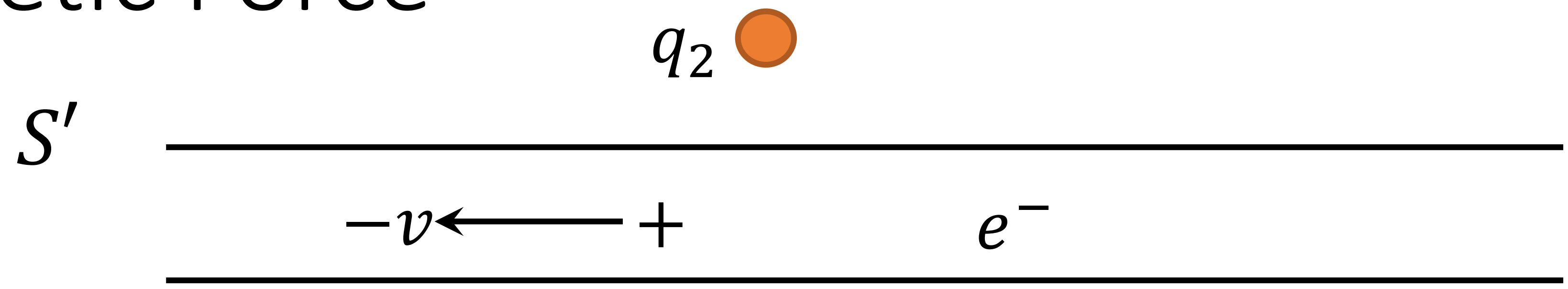
Jump to a reference frame S' moving with velocity v w.r.t S. Now q_2 is stationary, the e^- are stationary, and the positive ions are moving with velocity $-v$.

In S', the distance between electrons will have increased [by factor $^1/_\gamma$], but the distance between ions will have decreased [by factor γ] (using Lorentz contraction).

They are no longer equal and opposite! ! Have a net charge density of $D' = D(\gamma - {^1/_\gamma})$.

In S', the test charge is stationary, and the force on it, F', depends only on the electric field of both positive and negative charges.

Magnetic Force



The net force on q_2 is:

$$F'_y = \frac{2kD'}{r} q_2 = \frac{2kD}{r}\left(\gamma - \frac{1}{\gamma}\right) q_2$$

Transform back to S:

$$F_y = \frac{F'_y}{\gamma} = \frac{2kD}{r}\left(1 - \frac{1}{\gamma^2}\right) q_2 = \frac{v^2}{c^2}\left(\frac{2kD}{r}\right) q_2 = \frac{v^2}{c^2} E q_2$$

Since there is no net electric force in S, this force F_y must be entirely the magnetic force.

So the magnetic field observed in S is related to the electric field observed in S'.

The electric field in S' arises from Lorentz contraction effects!

Magnetic Force

In a typical copper wire, the drift velocity is:

$$v = 6 \text{ x } 10^{-2}\text{cm/sec} = 0.6\text{mm/sec}$$

i.e., $\frac{v}{c} = 2 \text{ x } 10^{-12}$

The magnetic field can be ascribed to the effect of seemingly negligible relativistic contraction effects at low speeds!