Taylor series

In physics, one often needs to approximate functions. The most common approximation is the **Taylor series**, which approximates a function in terms of a polynomial around $x = a$:

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{n!} \cdot \left. \frac{d^n f}{dx^n} \right|_{x=a} \cdot (x-a)^n$$

$$f(x) = f(a) + \frac{df(a)}{dx}(x-a) + \frac{1}{2}\frac{df^2(a)}{dx^2}(x-a)^2 + ..$$

(Note: $0! = 1$)

Taylor series

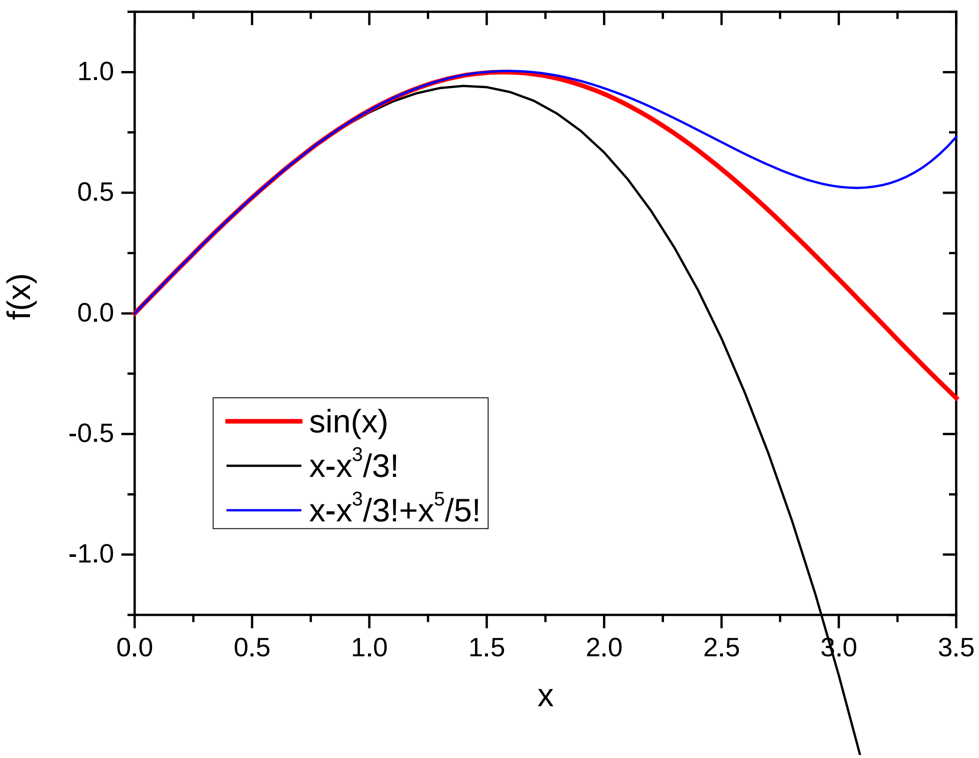
In the case where $a = 0$, the series is called **MacLaurin series:**

$$f(x) = f(0) + \frac{df(0)}{dx}x + \frac{1}{2}\frac{df^2(0)}{dx^2}x^2 + ..$$

Example: $f(x) = \sin(x)$. Expand around $a = 0$:

$$f(x) = \frac{1}{0!}\sin(0) + \frac{1}{1!}\cos(0)\cdot x - \frac{1}{2!}\sin(0)\cdot x^2 - \frac{1}{3!}\cos(0)\cdot x^3 + ..$$
$$= x - \frac{1}{6}\mathrm{x}^3 + ..$$

MacLaurin series of sin(x)



Only good approximation for small x. Keeping higher order terms in the series will increase the range of x where the series is a good approximation.

Taylor expansion of functions with two or more independent variables

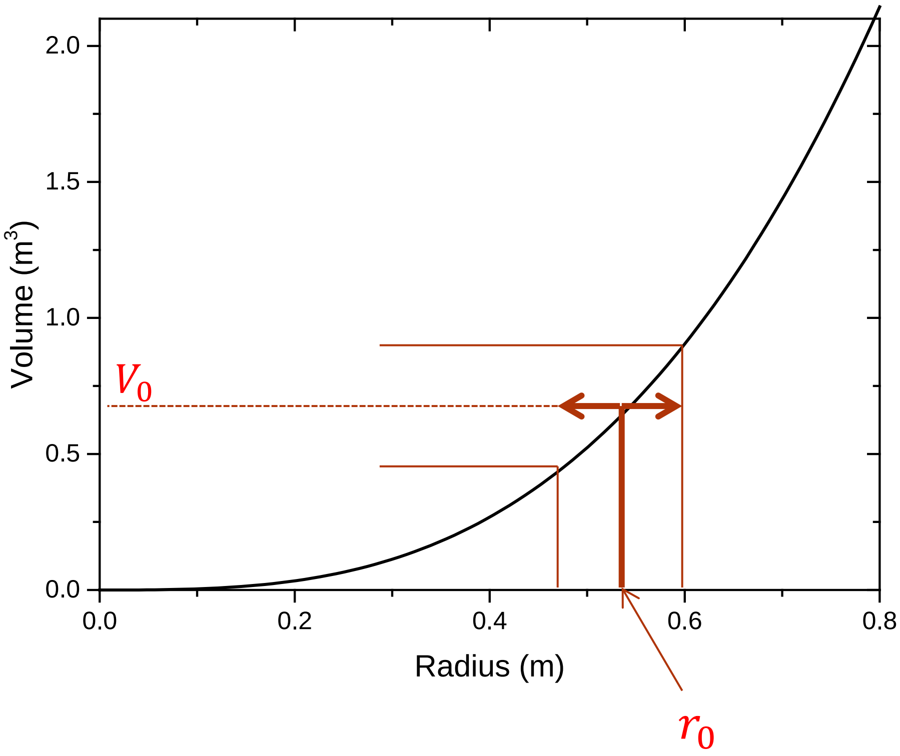
Taylor expansion can be generalised for functions with two or more variables. Here, we expand $f(x,y)$ around (x_0, y_0) :

$$f(x,y) = f(x_0, y_0\ ) + \frac{\partial f}{\partial x}(x_i - x_0) + \frac{\partial f}{\partial y}(y_i - y_0) + ..$$

The partial derivatives are evaluated at (x_0, y_0)

Error propagation of one variable

Example: The radius r_0 of a sphere is measured with an uncertainty Δr. What is the uncertainty ΔV in the volume $V_0 = \frac{4}{3}\pi r_0^3$?



Expand $V(r)$ around r_0 to first order:

$$V(r) = V(r_0) + \frac{dV(r_0)}{dr}(\mathrm{r} - \mathrm{r}_0)$$

$$V(r) - V(r_0) = 4\pi r_0^2(\mathrm{r} - \mathrm{r}_0)$$

$$\Delta V = 4\pi r_0^2 \Delta r$$

$$\frac{\Delta V}{V} = \frac{4\pi r^2 \Delta r}{V} = \frac{4\pi r^2 \Delta r}{\frac{4}{3}\pi r^3}$$

$$\frac{\Delta V}{V_0} = 3\,\frac{\Delta r}{r_0}$$

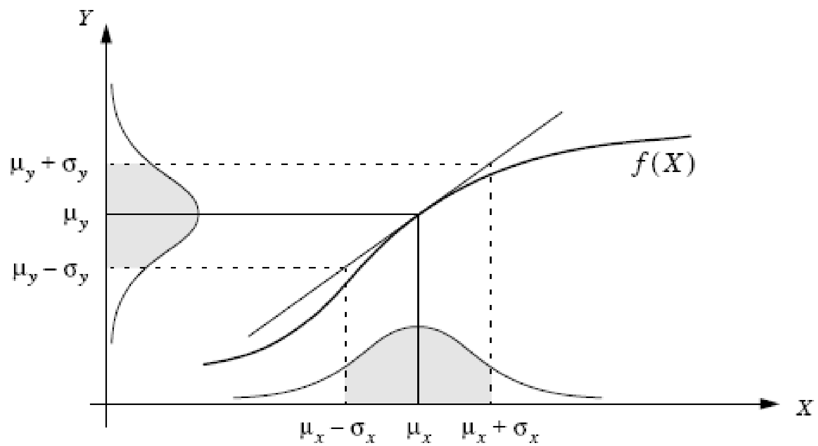
A 1% error in the radius leads to a 3% error in the Volume

Error propagation of one variable

In general, the uncertainty in a function of one variable $f(x)$ at $x = x_0$ is given by

$$\Delta f = \left|\frac{df}{dx}\right|_{x=x_0} \Delta x$$

This is equivalent to a Taylor expansion to first order around x_0



What about a function of 2 independent variables $f(u, v)$?

Error propagation of two or more variables

Can we just use the partial differential (i.e. Taylor expansion to first order)? e.g. $f(x, y)$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

No! This overestimates the error as there can be partial cancellation of errors in x and y (here, dx and dy reflect both positive and negative deviations from the mean).

Proper formula is Gauss's law of error propagation:

$$df = \sqrt{\left(\frac{\partial f}{\partial x} dx\right)^2 + \left(\frac{\partial f}{\partial y} dy\right)^2}$$

Derivation of Gauss' law of error propagation

Suppose we have a function $f(x,y)$, where we measure the two independent variables x and y several times and obtain N pairs of data $[(x_1, y_1), (x_2, y_2) .., (x_N, y_N)]$.

From this data set we can calculate the mean and variance of both x_i and y_i separately in the usual way: $\langle x \rangle, \sigma_x^2$ and $\langle y \rangle, \sigma_y^2$

We can also calculate the mean and corresponding variance of f:

$$\langle f \rangle = \frac{1}{N} \sum_{i=1}^{N} f(x_i, y_i)$$

$$\sigma_f^2 = \frac{1}{N} \sum_{i=1}^{N} (f(x_i, y_i) - \langle f \rangle)^2$$

In the following, we will establish a relation between σ_f^2 and the variances of its independent variables σ_x^2 and σ_y^2.

Derivation of Gauss' law of error propagation

Assuming that all the x_i and y_i's are close to their respective means $\langle x \rangle$ and $\langle y \rangle$, we can **Taylor expand** each $f(x_i, y_i) \equiv f_i$ around $f(\langle x \rangle, \langle y \rangle)$:

$$f_i \approx f(\langle x \rangle, \langle y \rangle) + \left.\frac{\partial f}{\partial x}\right|_{x=\langle x \rangle, y=\langle y \rangle} (x_i - \langle x \rangle) + \left.\frac{\partial f}{\partial y}\right|_{x=\langle x \rangle, y=\langle y \rangle} (y_i - \langle y \rangle) + ..$$

Can neglect higher order terms in the expansion since both $(x_i - \langle x \rangle)$ and $(x_i - \langle x \rangle)$ are assumed to be small.

Note that each derivative is evaluated at $(\langle x \rangle, \langle y \rangle)$. In the following we assume that all partial derivatives are evaluated at that point and drop the vertical evaluation bar.

Derivation of Gauss' law of error propagation

Let's calculate the mean of $f(x, y)$ first:

$$\langle f \rangle = \frac{1}{N} \sum_{i=1}^{N} f(x_i, y_i)$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} \left(f(\langle x \rangle, \langle y \rangle) + \frac{\partial f}{\partial x}(x_i - \langle x \rangle) + \frac{\partial f}{\partial y}(y_i - \langle y \rangle) \right)$$

where we substituted the Taylor expansion from the previous slide.

$$\langle f \rangle = \frac{1}{N}(N \cdot f(\langle x \rangle, \langle y \rangle) + 0 + 0) = f(\langle x \rangle, \langle y \rangle)$$

Note that the partial derivatives are evaluated at $(\langle x \rangle, \langle y \rangle)$ and therefore are constants. Thus

$$\frac{1}{\mathrm{N}} \sum \frac{\partial f}{\partial x}(x_i - \langle x \rangle) = \frac{\partial f}{\partial x} \frac{1}{N} \sum (x_i - \langle x \rangle) = \frac{\partial f}{\partial x} \left[\frac{1}{N} \sum x_i - \frac{1}{N} \sum \langle x \rangle \right]$$

$$= \frac{\partial f}{\partial x} \left[\langle x \rangle - \frac{1}{N} N \langle x \rangle \right] = 0$$

Derivation of Gauss' law of error propagation

The mean of f is just $\langle f \rangle = f(\langle x \rangle, \langle y \rangle)$

What about σ_f^2 ?

$$\sigma_f^2 = \frac{1}{N}\sum_{i=1}^{N}(f_i - \langle f \rangle)^2$$

Substituting the Taylor expansion for f_i we obtain

$$\frac{1}{N}\sum_{i=1}^{N}\left(f(\langle x \rangle, \langle y \rangle) + \frac{\partial f}{\partial x}(x_i - \langle x \rangle) + \frac{\partial f}{\partial y}(y_i - \langle y \rangle) - \langle f \rangle\right)^2$$

Derivation of Gauss' law of error propagation

$$\sigma_f^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\partial f}{\partial x}(x_i - \langle x\rangle) + \frac{\partial f}{\partial y}(y_i - \langle y\rangle)\right)^2$$

Multiplying out the square yields

$$\sigma_f^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\left(\frac{\partial f}{\partial x}\right)^2 (x_i - \langle x\rangle)^2 + \left(\frac{\partial f}{\partial y}\right)^2 (y_i - \langle y\rangle)^2 + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)(x_i - \langle x\rangle)(y_i - \langle y\rangle)\right)$$

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \frac{1}{N}\sum_{i=1}^{N}(x_i - \langle x\rangle)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \frac{1}{N}\sum_{i=1}^{N}(y_i - \langle y\rangle)^2 + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\frac{1}{N}\sum_{i=1}^{N}(x_i - \langle x\rangle)(y_i - \langle y\rangle)$$

Finally,

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)\sigma_{xy}$$

Derivation of Gauss' law of error propagation

The last term depends on the *covariance:*

$$\sigma_{xy} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \langle x\rangle)(y_i - \langle y\rangle)$$

If the errors in x and y are *independent* from each other and symmetrically distributed around the mean, then $\sigma_{xy} = 0$. This is true in most cases, therefore we have

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2$$

Covariance

Why does the covariance vanish for independent uncertainties?

Let's assume that the uncertainties in x and y are governed by a simple binomial distribution with two outcomes (with p=0.5 chance for each): $x = \langle x \rangle \pm \epsilon$ and likewise for y.

Sum over outcomes

In this case

$$\sigma_{xy} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \langle x \rangle)(y_i - \langle y \rangle) = \sum_{j=1}^{n} p_j^2 \left(x_j - \langle x \rangle\right)\left(y_j - \langle y \rangle\right)$$

$$= 0.5^2\left(\epsilon \cdot \epsilon + \epsilon \cdot (-\epsilon) + (-\epsilon) \cdot \epsilon + (-\epsilon)(-\epsilon)\right) = 0$$

Due to the symmetry of the distribution and their independence from each other, the uncertainties in x and y **cancel out exactly**.

Non-zero covariance usually indicates correlations. E.g. a positive deviation in x leads to a positive deviation in y.

Error propagation

The error propagation formula can be easily generalised for more than 2 variables $f(x, y, z, ..)$:

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial f}{\partial z}\right)^2 \sigma_z^2 + ..$$

Note that for one independent variable the formula reduces to the case we already considered:

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 \rightarrow \Delta f = \left|\frac{df}{dx}\right|_{x=x_0} \Delta x$$

This formula can be used to find error propagation formulae for specific cases that are common.

Multiplication by a constant

Consider $f(x) = c \cdot x$, where c is a constant.

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 = c^2\sigma_x^2$$

So $\Delta f = c \cdot \Delta x$

Multiplying x by a constant, changes the uncertainty by the same factor: $x \pm \Delta x \rightarrow \mathrm{c}x \pm \mathrm{c}\Delta x$

Addition and subtraction

Consider $f(x, y) = x \pm y$

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 = 1 \cdot \sigma_x^2 + (\pm 1)^2 \cdot \sigma_y^2$$

$$\sigma_f^2 = \sigma_x^2 + \sigma_y^2$$

For addition and subtraction, absolute errors add in quadrature:

$$\Delta f = \sqrt{\Delta x^2 + \Delta y^2}$$

Multiplication

Consider $f(x, y) = x \cdot y$

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 = y^2\sigma_x^2 + x^2\sigma_y^2$$

Divide by x^2y^2

$$\frac{\sigma_f^2}{x^2y^2} = \frac{\sigma_f^2}{f(x, y)^2} = \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2}$$

So

$$\frac{\Delta f^2}{f^2} = \frac{\Delta x^2}{x^2} + \frac{\Delta y^2}{y^2}$$

Division

Consider $f(x, y) = \frac{x}{y}$

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 = \left(\frac{1}{y}\right)^2 \sigma_x^2 + \left(-\frac{x}{y^2}\right)^2 \sigma_y^2$$

Divide by $f^2 = \left(\frac{x}{y}\right)^2$:

$$\frac{\sigma_f^2}{f^2} = \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2}$$

Same as for multiplication! For multiplication and division, fractional errors add in quadrature.

Powers

Consider $f(x) = x^{\pm b}$

$$\sigma_f = \left|\frac{\partial f}{\partial x}\right| \sigma_x = \left|\pm b x^{b-1}\right| \sigma_x$$

Divide by $f = x^b$:

$$\frac{\sigma_f}{f} = \left|\frac{b}{x}\right| \sigma_x$$

Exponentials and natural logarithm

Consider $f(x) = e^x$

$$\sigma_f = \left|\frac{\partial f}{\partial x}\right| \sigma_x = e^x \sigma_x$$

Divide by $f(x) = e^x$

$$\frac{\sigma_f}{f} = \left|\frac{\partial f}{\partial x}\right| \sigma_x = \sigma_x$$

Consider $f(x) = \ln(x)$

$$\sigma_f = \left|\frac{\partial f}{\partial x}\right| \sigma_x = \frac{\sigma_x}{x}$$

Summary

Addition/Subtraction:

$f = x \pm y \pm \cdots$, then $\Delta f = \sqrt{\Delta x^2 + \Delta y^2 + \cdots}$

Absolute errors add in quadrature

Multiplication:

$$f = \frac{x \cdot y \cdot ..}{u \cdot v \cdot ..}$$

then

$$\frac{\Delta f}{|f|} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2 + .. + \left(\frac{\Delta u}{u}\right)^2 + \left(\frac{\Delta v}{v}\right)^2 + ..}$$

Fractional errors add in quadrature

Summary

<u>Powers</u>

$f = x^b$, then $\frac{\Delta f}{f} = \left| b\,\frac{\Delta x}{x} \right|$

<u>Exponentials</u>

$f = e^x$, then $\frac{\Delta f}{f} = \Delta x$

<u>Natural logarithms</u>

$f = \ln(x)$, then $\Delta f = \frac{\Delta x}{x}$

Example – pendulum

Pendulum: $g = \frac{4\pi^2 l}{T^2}$

We have uncertainties both in the length l and measured period T.

$$g = \frac{4\pi^2 l}{T^2} = 4\pi^2 \frac{l}{T^2}$$

First look at error in T^2: $\Delta(T^2) = 2T \cdot \Delta T$

Fractional error in T^2: $\frac{\Delta(T^2)}{T^2} = 2\frac{\Delta T}{T}$

Example - pendulum

$\Delta(4\pi^2 l) = 4\pi^2 \Delta \mathrm{l}$

Fractional error in $4\pi^2 l$: $\frac{\Delta\left(4\pi^2 l\right)}{4\pi^2 l} = \frac{\Delta l}{l}$

Fractional errors add in quadrature for ratio $4\pi^2 l / T^2$

$\frac{\Delta g}{g} = \sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(2\,\frac{\Delta T}{T}\right)^2}$

Therefore: $\Delta g = \frac{4\pi^2 l}{T^2}\sqrt{\left(\frac{\Delta l}{l}\right)^2 + \left(2\,\frac{\Delta T}{T}\right)^2}$

e.g. for $l = 92.95 \pm 0.1\ cm$ and $T = 1.936 \pm 0.004\ s$

Get $g = 9.79 \pm 0.04\ m/s^2$

Example - refraction

Measurement of refractive index of glass n.

$$n = \frac{sin\theta_i}{sin\theta_r}$$

Here θ_i is incident ray in air ($n = 1$), and θ_r is the refracted ray in glass.

Fractional error in n is given by

$$\frac{\Delta n}{n} = \sqrt{\left(\frac{\Delta sin\theta_i}{sin\theta_i}\right)^2 + \left(\frac{\Delta sin\theta_r}{sin\theta_r}\right)^2}$$

Example - refraction

Given the uncertainty in the angle $\Delta\theta$, what is $\Delta sin\theta$?

$$\Delta sin\theta = \left|\frac{d(sin\theta)}{d\theta}\right| \Delta\theta = |cos\theta| \cdot \Delta\theta$$

in radians

Thus, fractional uncertainty is

$$\frac{\Delta sin\theta}{sin\theta} = |cot\theta| \cdot \Delta\theta$$

i (deg) all ± 1	r (deg) all ± 1	$\sin i$	$\sin r$	n	$\frac{\delta \sin i}{\lvert\sin i\rvert}$	$\frac{\delta \sin r}{\lvert\sin r\rvert}$	$\frac{\delta n}{n}$
20	13	0.342	0.225	1.52	5%	8%	9%
40	23.5	0.643	0.399	1.61	2%	4%	5%

Variable transformation in PDE's

When propagating errors, the shape of the distribution does not necessarily remain Gaussian.

Consider the sphere volume $V = \frac{4}{3}\pi r^3$.

If the measurement of the radius follows a Gaussian distribution P(r) centred around r_0, what is the corresponding distribution $P(V)$?

The probability of measuring a certain radius r is $P(r)dr$. This has to equal to the probability of measuring the corresponding volume $P(V)dV$

Variable transformation in PDE's

$$P(r)dr = P(V)dV$$

Therefore

$$P(V) = P(r)\frac{dr}{dV} = \frac{P(r)}{\frac{dV}{dr}} = \frac{P(r)}{4\pi r^2}$$

$$= \frac{1}{4\pi r^2 \sigma\sqrt{2\pi}} e^{-\frac{(r-r_0)^2}{2\sigma^2}}$$

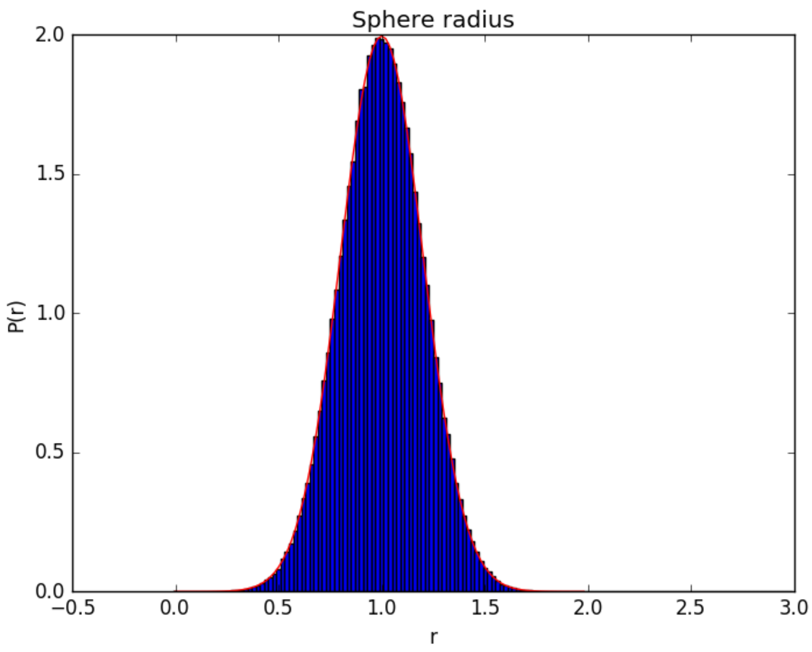
Now do a variable transformation $r = \left(\frac{3V}{4\pi}\right)^{1/3}$ and substitute above. Get

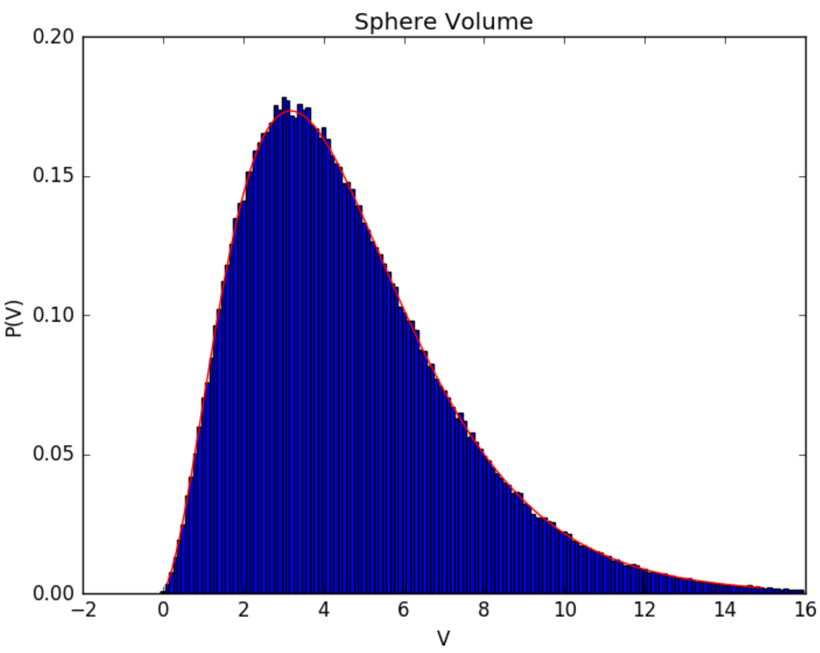
$$P(V) \sim V^{-\frac{2}{3}} e^{-a\left(V^{1/3}-b\right)^2}$$

Variable transformation in PDE’s

Radii follow normal distribution

Corresponding sphere volume doesn’t





Variable transformation

Maxwell velocity distribution of gas molecules (a, b are constants):

$$P(v) = a\, v^2 e^{-bv^2}$$

What is $P(E)$, the corresponding distribution of the kinetic energies?

$$E = \frac{1}{2} m v^2$$

$$\frac{dE}{dv} = mv$$

$P(v)dv = P(E)dE$ leads to $P(E) = \frac{P(v)}{\frac{dE}{dv}} = \frac{a}{m} v e^{-bv^2}$

Since $v = \sqrt{2E/m}$, $\quad P(E) = \frac{a}{m}\sqrt{2E/m}\, e^{-b2E/m}$

Variable transformation

Variable transformation of Probability densities has one constraint:

When going from x to y, for example, i.e.

$$P(x)dx = P(y)dy$$

$\frac{dx}{dy}$ has to be either positive or negative over the whole range of x and y, i.e. $x(y)$ (or $y(x)$) is monotonically increasing or decreasing over the whole range.

Since $\frac{dx}{dy}$ can be negative, we have in general

$$P(y) = P(x)\left|\frac{dx}{dy}\right|$$