4.5 Local minima and maxima

Definition 4.22 – Local minimum and maximum

We say that f has a local minimum at x_0, if there is an interval I around x_0 such that

$$f(x_0) \leq f(x) \quad \text{for all } x \in I.$$

We say that f has a local maximum at x_0, if there is an interval I around x_0 such that

$$f(x_0) \geq f(x) \quad \text{for all } x \in I.$$

Theorem 4.23 – First derivative test

Suppose that f is differentiable on some interval around the point x_0.

(a) If f' changes from being negative to being positive at the point x_0, then f changes from being decreasing to being increasing, so f has a local minimum at x_0.

(b) If f' changes from being positive to being negative at the point x_0, then f changes from being increasing to being decreasing, so f has a local maximum at x_0.

Theorem 4.24 – Second derivative test

Suppose that f is twice differentiable at the point x_0.

(a) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a local minimum at x_0.

(b) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a local maximum at x_0.

Example 4.25 We use the second derivative test to find the local minima/maxima of

$$f(x) = x^3 + 3x^2 - 9x.$$

To find the points at which the first derivative is zero, we note that

$$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x - 1)(x + 3).$$

The solutions of $f'(x) = 0$ are thus $x = 1$ and $x = -3$. When it comes to the former,

$$f''(x) = 6x + 6 \quad \Longrightarrow \quad f''(1) = 6 + 6 = 12,$$

so f attains a local minimum at the point $x = 1$. When it comes to the latter,

$$f''(x) = 6x + 6 \quad \Longrightarrow \quad f''(-3) = -18 + 6 = -12,$$

so f attains a local maximum at the point $x = -3$. $\square$

Example 4.26 We use the first derivative test to find the local minima/maxima of

$$f(x) = \frac{3x+4}{x^2+1}.$$

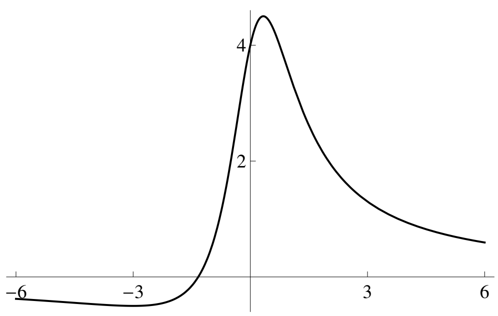
According to the quotient rule, the derivative of this function is given by

$$f'(x) = \frac{3(x^2+1) - 2x(3x+4)}{(x^2+1)^2} = \frac{-3x^2 - 8x + 3}{(x^2+1)^2}.$$

The quadratic in the numerator has roots $x_1 = 1/3$ and $x_2 = -3$, so one may also write

$$f'(x) = -\frac{3(x-x_1)(x-x_2)}{(x^2+1)^2} = \frac{(1-3x)(x+3)}{(x^2+1)^2}.$$

We now determine the sign of $f'(x)$ using the table below. When $x = -3$, the derivative changes from being negative to being positive, so f has a local minimum. When $x = 1/3$, the derivative changes from being positive to being negative, so f has a local maximum. $\square$



	-3	$1/3$	
$1-3x$	$+$	$+$	$-$
$x+3$	$-$	$+$	$+$
$f'(x)$	$-$	$+$	$-$

Figure 4.6: The graph of $f(x) = \dfrac{3x+4}{x^2+1}$.

Example 4.27 We use the second derivative test to find the local minima/maxima of

$$f(x) = x^4 - 4x^2 + 3.$$

To find the points at which the first derivative is zero, we note that

$$f'(x) = 4x^3 - 8x = 4x(x^2-2) = 4x(x-\sqrt{2})(x+\sqrt{2}).$$

The solutions of $f'(x) = 0$ are thus $x = 0$ and $x = \pm\sqrt{2}$. When it comes to the first point,

$$f''(x) = 12x^2 - 8 \quad \Longrightarrow \quad f''(0) = -8,$$

so f attains a local maximum at the point $x = 0$. When it comes to the other two points,

$$f''(x) = 12x^2 - 8 \quad \Longrightarrow \quad f''(\pm\sqrt{2}) = 12 \cdot 2 - 8 = 16,$$

so f attains a local minimum at each of the points $x = \pm\sqrt{2}$. $\square$

4.6 Global minima and maxima

Definition 4.28 – Global minimum and maximum

Consider a function f with domain A and let $x_0 \in A$ be a given point.

(a) We say that f has a global minimum at x_0, if $f(x_0) \leq f(x)$ for all $x \in A$.

(b) We say that f has a global maximum at x_0, if $f(x_0) \geq f(x)$ for all $x \in A$.

Theorem 4.29 – Unique change of sign

Suppose that f is differentiable and f' changes sign exactly once. If f' changes from being negative to being positive at x_0, then f has a global minimum at x_0. If f' changes from being positive to being negative at x_0, then f has a global maximum at x_0.

Theorem 4.30 – Continuous functions on finite intervals

Suppose that f is continuous on the finite interval $[a, b]$. Then f attains both a global minimum and a global maximum. In fact, these may only occur at the endpoints a, b, the points at which $f'(x)$ is zero and the points at which $f'(x)$ does not exist.

Example 4.31 We find the global minimum/maximum values that are attained by

$$f(x) = x^3 - 3x, \qquad 0 \leq x \leq 2.$$

This function is differentiable at all points and its derivative is given by

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1).$$

Thus, the global minimum/maximum values may only occur at the points

$$x = -1, \qquad x = 1, \qquad x = 0, \qquad x = 2.$$

We exclude the leftmost point, as it does not lie in the given interval, and we compute

$$f(1) = 1^3 - 3 = -2, \qquad f(0) = 0, \qquad f(2) = 2^3 - 3 \cdot 2 = 2.$$

This means that the minimum value is $f(1) = -2$ and the maximum value is $f(2) = 2$. □

Example 4.32 We use Theorem 4.29 in order to establish the inequality

$$xe^{-x} \leq e^{-1} \quad \text{for all } x.$$

Consider the function f that is defined by $f(x) = xe^{-x}$. As one can easily check,

$$f'(x) = e^{-x} + x \cdot (e^{-x}) \cdot (-x)' = e^{-x} - xe^{-x} = (1 - x)e^{-x}.$$

This implies that $f'(x)$ is positive when $x < 1$ and $f'(x)$ is negative when $x > 1$. Thus, f attains a global maximum at the point $x = 1$ and one has $f(x) \leq f(1) = e^{-1}$ for all x. □

Example 4.33 We find the global minimum and maximum values that are attained by

$$f(x) = \sin x + \cos x, \qquad 0 \leq x \leq 2\pi.$$

This function is differentiable at all points and its derivative is given by

$$f'(x) = \cos x - \sin x.$$

To say that $f'(x) = 0$ is to say that $\sin x = \cos x$ and this is true precisely when $\tan x = 1$. Thus, the only points at which the minimum/maximum values may occur are the points

$$x = 0, \qquad x = 2\pi, \qquad x = \pi/4, \qquad x = 5\pi/4.$$

As one can easily check, the corresponding values of $f(x)$ are

$$f(0) = f(2\pi) = 1, \qquad f(\pi/4) = \sqrt{2}, \qquad f(5\pi/4) = -\sqrt{2}.$$

Thus, the minimum value is $f(5\pi/4) = -\sqrt{2}$ and the maximum value is $f(\pi/4) = \sqrt{2}$. $\square$

Example 4.34 We use Theorem 4.29 in order to establish the inequality

$$e^x \geq x + 1 \quad \text{for all } x.$$

Consider the function f that is defined by $f(x) = e^x - x - 1$. Its derivative is

$$f'(x) = e^x - 1 = e^x - e^0,$$

so it is negative when $x < 0$ and it is positive when $x > 0$. This implies that f attains a global minimum at the point $x = 0$. In particular, one has $f(x) \geq f(0) = 0$ for all x. $\square$

Example 4.35 We find the global minimum and maximum values that are attained by

$$f(x) = x\sqrt{4 - x^2}, \qquad -2 \leq x \leq 2.$$

Using both the product rule and the chain rule, one may easily check that

$$f'(x) = \sqrt{4 - x^2} + x \cdot \frac{-2x}{2\sqrt{4 - x^2}} = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = \frac{4 - 2x^2}{\sqrt{4 - x^2}}.$$

Thus, the only points at which the minimum/maximum values may occur are the points

$$x = -2, \qquad x = 2, \qquad x = -\sqrt{2}, \qquad x = \sqrt{2}.$$

The corresponding values that are attained by $f(x)$ are given by

$$f(-2) = f(2) = 0, \qquad f(-\sqrt{2}) = -2, \qquad f(\sqrt{2}) = 2.$$

This makes $f(-\sqrt{2}) = -2$ the minimum value and $f(\sqrt{2}) = 2$ the maximum value. $\square$

4.7 Optimisation

- There are several standard problems that ask for the minimum/maximum value of a function which represents a physical quantity such as length, area and volume.
- To solve these problems, one introduces the variables of interest and expresses them in terms of a single variable x. This variable is not usually arbitrary, as length, area and volume must be non-negative. One must thus determine the exact restrictions on x.

Example 4.36 Out of all rectangles of perimeter 40, which one has the largest area? To answer this question, we denote by x, y the two sides of the rectangle and we note that

$$2x + 2y = 40 \quad \Longrightarrow \quad x + y = 20 \quad \Longrightarrow \quad y = 20 - x.$$

To maximise the area A of the rectangle, we first express it in terms of x, namely

$$A = xy = x(20 - x) = 20x - x^2.$$

Next, we determine the restrictions on x. Since the lengths x, y must be non-negative, we need to have $x \geq 0$ and $y = 20 - x \geq 0$. Thus, we are looking for the maximum value of

$$A(x) = 20x - x^2, \qquad 0 \leq x \leq 20.$$

Since $A'(x) = 20 - 2x$, the only points at which the maximum value may occur are the points

$$x = 0, \qquad x = 20, \qquad x = 10.$$

The corresponding values that are attained by $A(x)$ are given by

$$A(0) = A(20) = 0, \qquad A(10) = 200 - 100 = 100.$$

Thus, the largest area arises when $x = y = 10$, in which case the rectangle is a square. □

Example 4.37 If a right triangle has a hypotenuse of length 6, how large can its area be? In this case, we let x, y be the other two sides and we use Pythagoras' theorem to get

$$x^2 + y^2 = 6^2 \quad \Longrightarrow \quad y^2 = 36 - x^2 \quad \Longrightarrow \quad y = \sqrt{36 - x^2}.$$

Eliminating y, one may express the area of the right triangle in the form

$$A = \frac{xy}{2} = \frac{x\sqrt{36 - x^2}}{2}.$$

Since A becomes maximum if and only if A^2 becomes maximum, it suffices to maximise

$$f(x) = \frac{x^2(36 - x^2)}{4} = 9x^2 - \frac{x^4}{4}, \qquad 0 \leq x \leq 6.$$

The derivative of this function is easy to compute and one has

$$f'(x) = 18x - x^3 = x(18 - x^2).$$

Thus, the only points at which the maximum value may occur are the points

$$x = 0, \qquad x = 6, \qquad x = \sqrt{18}.$$

The corresponding values that are attained by $f(x)$ are given by

$$f(0) = f(6) = 0, \qquad f(\sqrt{18}) = \frac{18 \cdot 18}{4} = 9^2.$$

In particular, the maximum value is $f(\sqrt{18}) = 9^2$ and the largest possible area is 9. □

Example 4.38 A cylinder is formed when a rectangle is rotated around one of its sides. If the rectangle has perimeter 6, then how large can the volume of the cylinder be? Let x be the side of the rectangle which lies along the line of rotation and let y be the other side. Then x becomes the height of the cylinder and y becomes the radius, so the volume of the cylinder is given by $V = \pi y^2 x$. Since $2x + 2y = 6$ by assumption, one may write

$$V(y) = \pi y^2 x = \pi y^2 (3 - y) = 3\pi y^2 - \pi y^3.$$

To ensure that the lengths x, y are non-negative, we need to assume that $0 \leq y \leq 3$. Since

$$V'(y) = 6\pi y - 3\pi y^2 = 3\pi y(2 - y),$$

the only points at which the maximum value may occur are the points

$$y = 0, \qquad y = 3, \qquad y = 2.$$

Since $V(0) = V(3) = 0$ and $V(2) = 4\pi$, the largest possible volume is $V(2) = 4\pi$. □

Example 4.39 We find the point on the line $y = 2x + 1$ which is closest to $A(3, 6)$. In this case, we need to minimise the distance d between (x, y) and $(3, 6)$, namely

$$d = \sqrt{(x - 3)^2 + (y - 6)^2} = \sqrt{(x - 3)^2 + (2x - 5)^2}.$$

Since d becomes minimum if and only if d^2 becomes minimum, it suffices to minimise

$$f(x) = (x - 3)^2 + (2x - 5)^2.$$

The derivative of this function is easily found to be

$$f'(x) = 2(x - 3) + 2(2x - 5) \cdot 2 = 2(x - 3 + 4x - 10) = 2(5x - 13).$$

Thus, $f'(x)$ is negative when $x < 13/5$ and positive when $x > 13/5$. This implies that f has a global minimum at $x = 13/5$, so the closest point on the line is $(13/5, 27/5)$. □

Example 4.40 Out of all rectangles of area $a > 0$, which one has the smallest perimeter? To answer this question, we let x, y be the two sides of the rectangle and we note that $xy = a$. The perimeter of the rectangle is $2x + 2y$ and this can be expressed in the form

$$P(x) = 2x + 2y = 2x + \frac{2a}{x}.$$

The only restriction on x is that $x > 0$, while the derivative of $P(x)$ is

$$P'(x) = 2 - \frac{2a}{x^2} = \frac{2(x^2 - a)}{x^2}.$$

This gives $P'(x) < 0$ when $0 < x < \sqrt{a}$ and $P'(x) > 0$ when $x > \sqrt{a}$, so $P(x)$ attains its minimum when $x = y = \sqrt{a}$. The rectangle of smallest perimeter is thus a square. □

4.8 Related rates

- Suppose that two or more variables are related by some equation. Then one may use implicit differentiation to see that their derivatives are related as well.
- This situation arises frequently when quantities such as length, area and volume are varying with time. We shall thus mainly focus on functions of time t.

Example 4.41 If the radius of a circle is increasing at the rate of 1 cm/sec, how fast is the area of the circle changing when the radius is 3 cm? In this case, the variables of interest are the radius $r(t)$ and the area $A(t)$. They are related by the usual formula

$$A(t) = \pi r(t)^2$$

and one may differentiate both sides of this equation to find that

$$A'(t) = \pi \cdot 2r(t) \cdot r'(t).$$

At the given moment, $r'(t) = 1$ and $r(t) = 3$, so it easily follows that $A'(t) = 6\pi$. $\square$

Example 4.42 A ladder that is 10 ft long is resting against a wall. If its base starts sliding along the floor at the rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the base is 6 ft away from the wall? To solve this problem, let x be the horizontal distance between the base and the wall, and let y be the vertical distance between the top of the ladder and the floor. According to Pythagoras' theorem, one must have

$$x(t)^2 + y(t)^2 = 10^2 \quad \Longrightarrow \quad 2x(t)x'(t) + 2y(t)y'(t) = 0.$$

At the given moment, $x'(t) = 1$ and $x(t) = 6$, so the last equation gives

$$y(t)y'(t) = -x(t)x'(t) = -6 \quad \Longrightarrow \quad y'(t) = -\frac{6}{y(t)}.$$

Using Pythagoras' theorem to determine the remaining variable $y(t)$, we conclude that

$$y(t) = \sqrt{10^2 - x(t)^2} = \sqrt{10^2 - 6^2} = 8 \quad \Longrightarrow \quad y'(t) = -\frac{6}{8} = -\frac{3}{4}. \qquad \square$$

Example 4.43 A girl flies a kite at a constant height of 30 meters above her hand and the wind is carrying the kite horizontally at a rate of 2 m/sec. How fast must she let out the string when the kite is 50 meters away from her? Let $x(t)$ be the horizontal distance between the girl and the kite, and let $z(t)$ be the length of the string. We must then have

$$x(t)^2 + 30^2 = z(t)^2 \quad \Longrightarrow \quad 2x(t)x'(t) = 2z(t)z'(t).$$

Since $x'(t) = 2$ and $z(t) = 50$ by assumption, it easily follows that

$$z'(t) = \frac{x(t)x'(t)}{z(t)} = \frac{2\sqrt{50^2 - 30^2}}{50} = \frac{2 \cdot 40}{50} = \frac{8}{5}. \qquad \square$$

4.9 Linear approximation

Definition 4.44 – Linear approximation

Given a function f which is differentiable at the point x_0, we say that

$$L(x) = f'(x_0) \cdot (x - x_0) + f(x_0) \tag{4.8}$$

is the tangent line approximation or linear approximation of f at the point x_0.

- The linear approximation is merely the linear function that best approximates $f(x)$ near the point x_0. In fact, the points x which are sufficiently close to x_0 satisfy

$$\frac{f(x) - f(x_0)}{x - x_0} \approx f'(x_0) \quad \Longrightarrow \quad f(x) \approx f'(x_0) \cdot (x - x_0) + f(x_0).$$

Example 4.45 The linear approximation to $f(x) = \sin x$ at the point $x_0 = 0$ is given by

$$L(x) = f'(0) \cdot (x - 0) + f(0).$$

Since $f(0) = \sin 0 = 0$ and $f'(0) = \cos 0 = 1$, the linear approximation is thus $L(x) = x$. One may use this approximation to argue that $\sin x \approx x$ for all small enough x. □

Example 4.46 We find the linear approximation to $f(x)$ at the point x_0 in the case that

$$f(x) = \frac{4x^2 - 5x - 1}{x + 1}, \qquad x_0 = 1.$$

To find the derivative of $f(x)$ at the given point, we use the quotient rule to get

$$f'(x) = \frac{(8x - 5) \cdot (x + 1) - (4x^2 - 5x - 1)}{(x + 1)^2} \quad \Longrightarrow \quad f'(1) = \frac{3 \cdot 2 + 2}{4} = 2.$$

Since $f(1) = -2/2 = -1$, the linear approximation is $L(x) = 2(x - 1) - 1 = 2x - 3$. □

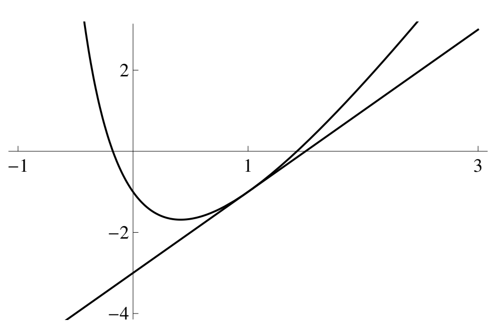


Figure 4.7: The graph of $f(x) = \dfrac{4x^2 - 5x - 1}{x + 1}$ and its tangent line at $x = 1$.

4.10 Newton's method

- Newton's method is a standard approach for approximating the roots of an equation of the form $f(x) = 0$. The method does not always work, but it works quite often.
- The idea is to start with an initial guess x_1, find the tangent line to f at that point and determine the point x_2 at which the line meets the x-axis. One may then use x_2 as a second guess and proceed in this manner to obtain the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for each } n \geq 1. \tag{4.9}$$

Example 4.47 We use Newton's method to approximate $\sqrt{2}$. In this case, we need to approximate the positive root of $f(x) = x^2 - 2$ and equation (4.9) has the form

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = x_n - \frac{x_n}{2} + \frac{1}{x_n} = \frac{x_n}{2} + \frac{1}{x_n}.$$

Starting with $x_1 = 1$ as our initial guess, we apply this equation repeatedly to get

$$\begin{array}{lll} x_1 = 1, & x_2 = 1.5, & x_3 = 1.41666667, \\ x_4 = 1.41421569, & x_5 = 1.41421356, & x_6 = 1.41421356. \end{array}$$

Based on these calculations, we find that $\sqrt{2}$ is approximately 1.41421 within five decimal places. In fact, the same conclusion may be reached by taking $x_1 = 2$, for instance. If one starts with $x_1 = -1$, then Newton's method leads to $-\sqrt{2}$, the other root of f. □

Example 4.48 Consider the polynomial $f(x) = x^3 - 3x + 1$ which is continuous with

$$f(0) = 1, \qquad f(1) = 1 - 3 + 1 = -1.$$

Since $f(0)$ and $f(1)$ have opposite signs, f has a root that lies in $(0, 1)$. In fact, this root is unique by Rolle's theorem because $f'(x) = 3(x^2 - 1)$ has no roots in the given interval. Let us now use Newton's method to approximate the unique root. Equation (4.9) gives

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}$$

and the fraction is not defined when $x_n^2 = 1$. Starting with the initial guess $x_1 = 0$, we get

$$\begin{array}{lll} x_1 = 0, & x_2 = 0.33333333, & x_3 = 0.34722222, \\ x_4 = 0.34729635, & x_5 = 0.34729635, & x_6 = 0.34729635. \end{array}$$

This gives an approximation of the root which is accurate to eight decimal places. □

Example 4.49 We show that Newton's method fails in the case that $f(x) = x^{1/3}$. If one uses equation (4.9) to approximate the root of $f(x) = 0$, one finds that

$$x_{n+1} = x_n - \frac{x_n^{1/3}}{x_n^{-2/3} \cdot 1/3} = x_n - 3x_n = -2x_n.$$

For instance, the initial guess $x_1 = 1$ gives $x_2 = -2$, $x_3 = 4$ and so on. The points x_n are thus getting larger in absolute value and they fail to approach a limiting value. □