TRINITY COLLEGE DUBLIN
THE UNIVERSITY OF DUBLIN

School of Mathematics

JF Mathematics
JF Theoretical Physics
SF TSM — Mathematics

Trinity Term 2015

1212: Linear Algebra II

Friday, May 15   **Luce Upper**   **9.30 — 11.30**

Prof. V. Dotsenko

Instructions to Candidates:

ATTEMPT **ALL** QUESTIONS

The number of marks you can get for a complete solution to any individual question is written next to the question. A complete solution to a question includes coherent explanations of answers you give.

Unless otherwise specified, you may use all statements proved in class without proof; when using some statement, you should formulate it clearly, e.g. "in class, we proved that a symmetric matrix with real entries has an orthonormal basis of eigenvectors".

Materials Permitted for this Examination:

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) (5 points) Write down the definition of a linear transformation of a vector space, and of the rank of a linear transformation.

   (b) (5 points) Write down the definition of an eigenvector of a linear transformation, and of an eigenvalue of a linear transformation. Explain how to find all eigenvalues of the given linear transformation.

   (c) (5 points) Write down the definition of a bilinear form on a real vector space. Which symmetric bilinear forms are said to be positive definite?

2. (a) (15 points) Let V be a vector space. Show that for every two linear transformations $A\colon V \to V$ and $B\colon V \to V$ we have

$$\mathrm{rk}(AB) \le \mathrm{rk}(A) \quad \text{and} \quad \mathrm{rk}(AB) \le \mathrm{rk}(B).$$

   (b) (10 points) Show that if A is invertible, then $\mathrm{rk}(AB) = \mathrm{rk}(B)$, and give an example showing that this equality might hold even if A is not invertible.

3. (a) (20 points) Determine the Jordan decomposition (the normal form matrix and some Jordan basis) for the matrix

$$A = \begin{pmatrix} 9 & 5 & 2 \\ -16 & -9 & -4 \\ 2 & 1 & 1 \end{pmatrix}$$

   (b) (15 points) Use the Jordan form and a Jordan basis that you computed to find a closed formula for A^n (that is, a formula expressing individual entries of that matrix as arithmetic expressions depending on n).

4. (25 points) Consider a quadratic form Q on the space $\mathbb{R}^3$ by the formula

$$Q(xe_1 + ye_2 + ze_3) = (20 + 4a)x^2 + 12(1 + a)xz + 6y^2 + 3z^2.$$

Find all values of the parameter a for which this quadratic form is positive definite.