Advanced Calculus
MA1132
Exercises 7 Solutions

1. Find the integral of the function $f(x,y) = 4xye^{x^2+y^2}$ over the rectangle
$$\{(x,y) \in \mathbb{R}^2 \colon 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 3\}.$$

   Solution:

   Here it doesn't matter which variable we integrate with respect to first.
$$\begin{aligned}
\int_0^3 \int_0^2 4xye^{x^2+y^2}\, dx\, dy &= \int_0^3 \left[2ye^{x^2+y^2}\right]_0^2 dy \quad \text{by inspection} \\
&= \int_0^3 2ye^{4+y^2} - 2ye^{y^2}\, dy \\
&= \left[e^{4+y^2} - e^{y^2}\right]_0^3 \quad \text{by inspection} \\
&= e^{13} - e^9 - (e^4 - e^0) \\
&= e^{13} - e^9 - e^4 + 1.
\end{aligned}$$

2. Find the integral of the function $f(x,y) = 4xy^2 + 4x^3 + \dfrac{28}{3}$ over the rectangle
$$\{(x,y) \in \mathbb{R}^2 \colon -2 \leqslant x \leqslant 0, 0 \leqslant y \leqslant 1\}.$$
   What does the result tell us about the signed volume of the region bounded between the rectangle and $f(x,y)$? What does it tell us about the unsigned volume (meaning the absolute volume of this region, ignoring that part of it might be below $z = 0$).

   Solution:

   It is easier to integrate first with respect to y. The integral to evaluate is
$$\int_{-2}^0 \int_0^1 \left(4xy^2 + 4x^3 + \frac{28}{3}\right) dydx = \int_{-2}^0 \left[\frac{4}{3}xy^3 + 4x^3y + \frac{28}{3}y\right]_{y=0}^{y=1} dx = \int_{-2}^0 \frac{4}{3}x + 4x^3 + \frac{28}{3} dx$$
   Integrating with respect to x gives
$$\left[\frac{4}{6}x^2 + x^4 + \frac{28}{3}x\right] \int_{x=-2}^{x=0} = -\left(\frac{8}{3} + 16 - \frac{56}{3}\right) = 0.$$
   Because the double integral is defined $\iint_R f(x,y)dA = \iint_{R^+} f(x,y)dA - \iint_{R^-} |f(x,y)|dA$ where R^+ is the part of R where $f \geq 0$ and R^{-1} is the part of R where $f < 0$, we can interpret the integraton resulting in zero to mean that equal amounts of the volume bounded by the rectangle and the given function are above and below the $z = 0$ plane. In terms of absolute volume, we can conclude that the volume of the bounded region V satisfies
$$V = 2 \iint_{R^+} f(x,y)dA.$$

3. Evaluate

$$\int_0^1 \int_{\sqrt[4]{x}}^1 \sqrt{1-y^5}\, dy\, dx.$$

Solution:

In this case it is much easier to integrate with respect to x first.
Observe that the inner integral limits tell us that $\sqrt[4]{x} \le y \le 1 \implies x \le y^4 \le 1$. From the outer integral limits, we see that $0 \le x \le 1$. Thus, we can write $0 \le x \le y^4$. The fixed integration limits $0 \le x \le 1 \implies 0 \le \sqrt[4]{x} \le 1$ can also be used to observe that $\sqrt[4]{x} \le y \le 1 \implies 0 \le y \le 1$. We see then that the integral is

$$\begin{aligned}
&\int_0^1 \int_0^{y^4} \sqrt{1-y^5}\, dx\, dy \\
&\quad = \int_0^1 \left[x\sqrt{1-y^5} \right]_0^{y^4} dy \\
&\quad = \int_0^1 y^4 \sqrt{1-y^5}\, dy \\
&\quad = \left[-\frac{2}{15}(1-y^5)^{\frac{3}{2}} \right]_0^1 \text{ by inspection} \\
&\quad = -\frac{2}{15}(0)^{\frac{3}{2}} - \left(-\frac{2}{15}(1-0)^{\frac{3}{2}} \right) \\
&\quad = \frac{2}{15}.
\end{aligned}$$

4. Evaluate the double integral

$$I = \iint_R \frac{1}{x+y} dx dy\,,$$

where R is the region enclosed by the lines $y = 2$, $y = x$, and the hyperbola $xy = 1$.

Solution: The region R is shown below

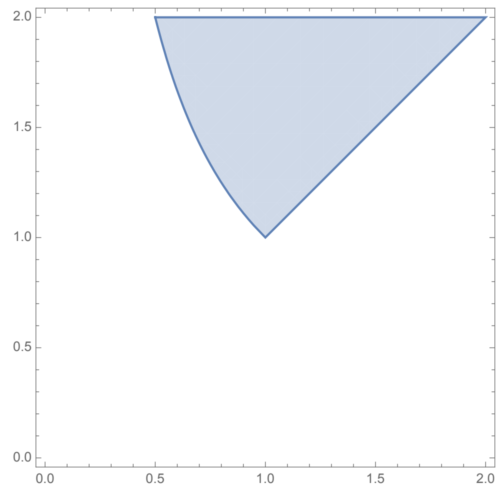
We consider R as a region of type II. Then I is given by

$$\begin{aligned}
I &= \int_1^2 \left[\int_{1/y}^{y} \frac{1}{x+y} dx \right] dy = \int_1^2 \ln(x+y)\big|_{x=1/y}^{x=y}\, dy = \int_1^2 \ln\left(\frac{2y^2}{1+y^2}\right) dy \\
&= \ln\left(\frac{2y^2}{1+y^2}\right) y \Big|_1^2 - \int_1^2 y\, d \ln\left(\frac{2y^2}{1+y^2}\right) = 2\ln\frac{8}{5} + \int_1^2 \frac{2}{1+y^2} dy \qquad (1) \\
&= 2\ln\frac{8}{5} + \frac{\pi}{2} - 2\arctan 2 \approx 0.296506\,.
\end{aligned}$$

5. Evaluate the double integral

$$I = \iint_R \sqrt{4x^2 - y^2} dx dy\,,$$

where R is the region enclosed by the lines $y = 0$, $y = x$, and $x = 1$.



We consider R as a region of type I. Then I is given by

$$I = \int_0^1 \left[\int_0^x \sqrt{4x^2 - y^2}\, dy\right] dx = \int_0^1 \left(\frac{y}{2}\sqrt{4x^2-y^2} + 2x^2 \arcsin\frac{y}{2x}\right)\Big|_{y=0}^{y=x} dx$$
$$= \int_0^1 \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) x^2\, dx = \frac{1}{3}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) \approx 0.637741\,. \tag{2}$$

Here the integral $\int \sqrt{4x^2 - y^2}\, dy$ can be evaluated by integrating by parts

$$\int \sqrt{4x^2-y^2}\, dy = y\sqrt{4x^2-y^2} - \int y d\sqrt{4x^2-y^2} = y\sqrt{4x^2-y^2} + \int \frac{y^2}{\sqrt{4x^2-y^2}} dy$$
$$= y\sqrt{4x^2-y^2} + \int \frac{4x^2}{\sqrt{4x^2-y^2}} dy - \int \sqrt{4x^2-y^2}\, dy\,. \tag{3}$$

Comparing the l.h.s with the r.h.s, one finds

$$\int \sqrt{4x^2-y^2}\, dy = \frac{1}{2}\left(y\sqrt{4x^2-y^2} + \int \frac{4x^2}{\sqrt{4x^2-y^2}} dy\right) = \frac{y}{2}\sqrt{4x^2-y^2} + 2x^2 \arcsin\frac{y}{2x}\,. \tag{4}$$

One can add any function of x to this expression which does not change the definite integral.

6. Sketch the integration region R and reverse the order of integration

(a)

$$\int_0^4 \int_{3x^2}^{12x} f(x,y) dy dx \tag{5}$$

Solution:

Reversing the order of integration one gets

$$\int_0^4 \int_{3x^2}^{12x} f(x,y) dy dx = \int_0^{48} \int_{y/12}^{\sqrt{y/3}} f(x,y) dx dy\,. \tag{6}$$

(b)

$$\int_{-7}^{1}\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x,y)dxdy \tag{7}$$

Solution:

It is found by noting that

$$\begin{aligned} &2-\sqrt{7-6y-y^2} \le x \le 2+\sqrt{7-6y-y^2} \implies (x-2)^2 \le 7-6y-y^2 \\ &\implies (x-2)^2+(y+3)^2 \le 16\,. \end{aligned} \tag{8}$$

Thus it is a disc of radius 4 centred at $(2,-3)$. Reversing the order of integration one gets

$$\int_{-7}^{1}\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x,y)dxdy = \int_{-2}^{6}\int_{-3-\sqrt{12+4x-x^2}}^{-3+\sqrt{12+4x-x^2}} f(x,y)dydx\,. \tag{9}$$

(c)

$$\int_{0}^{1}\int_{2x}^{3x} f(x,y)dydx \tag{10}$$

Solution:

Reversing the order of integration one gets the sum of two repeated integrals

$$\int_{0}^{1}\int_{2x}^{3x} f(x,y)dydx = \int_{0}^{2}\int_{y/3}^{y/2} f(x,y)dxdy + \int_{2}^{3}\int_{y/3}^{1} f(x,y)dxdy\,. \tag{11}$$

(d)

$$\int_{0}^{1}\int_{y^2/2}^{\sqrt{3-y^2}} f(x,y)dxdy \tag{12}$$

Solution: Reversing the order of integration one gets the sum of three repeated integrals

$$\begin{aligned} &\int_{0}^{1}\int_{y^2/2}^{\sqrt{3-y^2}} f(x,y)dxdy \\ &= \int_{0}^{1/2}\int_{0}^{\sqrt{2x}} f(x,y)dydx + \int_{1/2}^{\sqrt{2}}\int_{0}^{1} f(x,y)dydx + \int_{\sqrt{2}}^{\sqrt{3}}\int_{0}^{\sqrt{3-x^2}} f(x,y)dydx\,. \end{aligned} \tag{13}$$