MA1125 – Calculus
Homework #5 solutions

1. Show that the polynomial $f(x) = x^3 - 4x^2 - 3x + 1$ has exactly one root in $(0, 2)$.

Being a polynomial, f is continuous on the interval $[0, 2]$ and we also have

$$f(0) = 1, \qquad f(2) = 8 - 16 - 6 + 1 = -13.$$

Since $f(0)$ and $f(2)$ have opposite signs, f must have a root that lies in $(0, 2)$. To show it is unique, suppose that f has two roots in $(0, 2)$. Then f' must have a root in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 3x^2 - 8x - 3 = (3x + 1)(x - 3).$$

Since f' has no roots in $(0, 2)$, we conclude that f has exactly one root in $(0, 2)$.

2. Suppose that $0 < a < b$. Use the mean value theorem to show that

$$1 - \frac{a}{b} < \ln b - \ln a < \frac{b}{a} - 1.$$

Since $f(x) = \ln x$ is differentiable with $f'(x) = 1/x$, the mean value theorem gives

$$\frac{\ln b - \ln a}{b - a} = f'(c) = \frac{1}{c}$$

for some point $a < c < b$. Using this fact to estimate the right hand side, one finds that

$$\frac{1}{b} < \frac{1}{c} < \frac{1}{a} \quad \Longrightarrow \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a} \quad \Longrightarrow \quad 1 - \frac{a}{b} < \ln b - \ln a < \frac{b}{a} - 1.$$

3. Compute each of the following limits.

$$L_1 = \lim_{x \to 3} \frac{2x^3 - 8x^2 + 7x - 3}{3x^3 - 8x^2 - x - 6}, \qquad L_2 = \lim_{x \to \infty} \frac{x^2}{e^x}, \qquad L_3 = \lim_{x \to 0} (e^x + x)^{1/x}.$$

The first limit has the form $0/0$, so one may use L'Hôpital's rule to find that

$$L_1 = \lim_{x \to 3} \frac{6x^2 - 16x + 7}{9x^2 - 16x - 1} = \frac{54 - 48 + 7}{81 - 48 - 1} = \frac{13}{32}.$$

The second limit has the form ∞/∞ and one may apply L'Hôpital's rule twice to get

$$L_2 = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0.$$

The third limit involves a non-constant exponent which can be eliminated by writing

$$\ln L_3 = \ln \lim_{x\to 0} (e^x + x)^{1/x} = \lim_{x\to 0} \ln(e^x + x)^{1/x} = \lim_{x\to 0} \frac{\ln(e^x + x)}{x}.$$

This gives a limit of the form 0/0, so one may use L'Hôpital's rule to find that

$$\ln L_3 = \lim_{x\to 0} \frac{e^x + 1}{e^x + x} = \frac{1+1}{1+0} = 2.$$

Since $\ln L_3 = 2$, the original limit L_3 is then equal to $L_3 = e^{\ln L_3} = e^2$.

4. On which intervals is f increasing? On which intervals is it concave up?

$$f(x) = \frac{x}{x^2+3}.$$

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = \frac{x^2 + 3 - 2x \cdot x}{(x^2+3)^2} = \frac{3 - x^2}{(x^2+3)^2}.$$

Since the denominator is always positive, $f(x)$ is increasing if and only if

$$3 - x^2 > 0 \iff x^2 < 3 \iff -\sqrt{3} < x < \sqrt{3}.$$

To say that $f(x)$ is concave up is to say that $f''(x) > 0$. In this case, we have

$$\begin{aligned} f''(x) &= \frac{-2x(x^2+3)^2 - 2(x^2+3)\cdot 2x \cdot (3-x^2)}{(x^2+3)^4} \\ &= \frac{-2x(x^2+3) - 4x(3-x^2)}{(x^2+3)^3} \\ &= -\frac{2x(x^2+3+6-2x^2)}{(x^2+3)^3} = -\frac{2x(3-x)(3+x)}{(x^2+3)^3}. \end{aligned}$$

To determine the sign of this expression, one needs to find the sign of each of the factors. According to the table below, $f(x)$ is concave up if and only if $x \in (-3, 0) \cup (3, +\infty)$.

	-3	0	3	
$-2x$	$+$	$+$	$-$	$-$
$3-x$	$+$	$+$	$+$	$-$
$3+x$	$-$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$	$+$

5. Find the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down. Use this information to sketch the graph of f.

$$f(x) = \frac{(x-1)^2}{x^2+1}.$$

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = \frac{2(x-1)(x^2+1) - 2x \cdot (x-1)^2}{(x^2+1)^2} = \frac{2(x-1)(x+1)}{(x^2+1)^2} = \frac{2(x^2-1)}{(x^2+1)^2}.$$

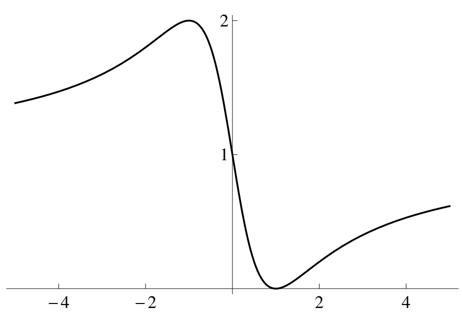
Since the denominator is always positive, $f(x)$ is increasing if and only if

$$x^2 - 1 > 0 \iff x^2 > 1 \iff x \in (-\infty, -1) \cup (1, +\infty).$$

To say that $f(x)$ is concave up is to say that $f''(x) > 0$. In this case, we have

$$\begin{aligned} f''(x) &= \frac{4x \cdot (x^2+1)^2 - 2(x^2+1) \cdot 2x \cdot 2(x^2-1)}{(x^2+1)^4} \\ &= \frac{4x(x^2+1) - 8x(x^2-1)}{(x^2+1)^3} = \frac{4x(3-x^2)}{(x^2+1)^3}. \end{aligned}$$

To determine the sign of this expression, one needs to find the sign of each of the factors. According to the table below, $f(x)$ is concave up if and only if $x \in (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$.



	$-\sqrt{3}$		0		$\sqrt{3}$	
$4x$	$-$		$-$		$+$	$+$
$3-x^2$	$-$		$+$		$+$	$-$
$f''(x)$	$+$		$-$		$+$	$-$

Figure 1: The graph of $f(x) = \dfrac{(x-1)^2}{x^2+1}$.