MA1125 – Calculus
Homework #1 solutions

1. Find the domain and the range of the function f which is defined by

$$f(x) = \frac{3-2x}{5-3x}.$$

The domain consists of all points $x \neq 5/3$. To find the range, we note that

$$\begin{aligned} y = \frac{3-2x}{5-3x} \quad &\Longleftrightarrow \quad 5y - 3xy = 3 - 2x \quad \Longleftrightarrow \quad 2x - 3xy = 3 - 5y \\ &\Longleftrightarrow \quad x(2-3y) = 3 - 5y \quad \Longleftrightarrow \quad x = \frac{3-5y}{2-3y}. \end{aligned}$$

The rightmost formula determines the value of x that satisfies $y = f(x)$. Since the formula makes sense for any number $y \neq 2/3$, the range consists of all numbers $y \neq 2/3$.

2. Find the domain and the range of the function f which is defined by

$$f(x) = \frac{\sqrt{2x-1}}{x}.$$

When it comes to the domain, we need to have $x \neq 0$ and $2x - 1 \geq 0$. This gives $x \geq 1/2$ and the domain is $[1/2, +\infty)$. Since x is non-negative, the same is true for $y = f(x)$ and

$$y^2 = \frac{2x-1}{x^2} \quad \Longleftrightarrow \quad y^2x^2 = 2x - 1 \quad \Longleftrightarrow \quad y^2x^2 - 2x + 1 = 0.$$

If it happens that $y = 0$, then $x = 1/2$. If it happens that $y \neq 0$, then the last equation is quadratic in x and one may use the quadratic formula to conclude that

$$x = \frac{2 \pm \sqrt{4-4y^2}}{2y^2} = \frac{1 \pm \sqrt{1-y^2}}{y^2}.$$

This leads to the restriction $1 - y^2 \geq 0$, which gives $y^2 \leq 1$ and thus $-1 \leq y \leq 1$. Since y is also non-negative, however, the range of the given function is merely $[0, 1]$.

3. Show that the function $f\colon (0,1) \to (1,\infty)$ is bijective in the case that

$$f(x) = \frac{1+x}{1-x}.$$

To show that the given function is injective, we note that

$$\begin{aligned} \frac{1+x_1}{1-x_1} = \frac{1+x_2}{1-x_2} \quad &\Longrightarrow \quad 1 - x_2 + x_1 - x_1x_2 = 1 - x_1 + x_2 - x_1x_2 \\ &\Longrightarrow \quad 2x_1 = 2x_2 \quad \Longrightarrow \quad x_1 = x_2. \end{aligned}$$

To show that the given function is surjective, we note that

$$y = \frac{1+x}{1-x} \quad \Longleftrightarrow \quad y - xy = 1 + x \quad \Longleftrightarrow \quad y - 1 = xy + x \quad \Longleftrightarrow \quad x = \frac{y-1}{y+1}.$$

The rightmost formula determines the value of x such that $y = f(x)$ and we need to check that $0 < x < 1$ if and only if $y > 1$. When $y > 1$, we have $y + 1 > y - 1 > 0$, so $0 < x < 1$. When $0 < x < 1$, we have $0 < 1 - x < 1 + x$ and this gives $y > 1$, as needed.

4. Express the following polynomials as the product of linear factors.

$$f(x) = 3x^3 - 2x^2 - 7x - 2, \qquad g(x) = x^3 + x^2 - \frac{7x}{4} + \frac{1}{2}.$$

When it comes to $f(x)$, the possible rational roots are $\pm 1, \pm 2, \pm 1/3, \pm 2/3$. Checking these possibilities, one finds that $x = -1$, $x = 2$ and $x = -1/3$ are all roots. According to the factor theorem, each of $x + 1$, $x - 2$ and $x + 1/3$ is thus a factor and one has

$$f(x) = 3(x+1)(x-2)(x+1/3) = (x+1)(x-2)(3x+1).$$

When it comes to $g(x)$, let us first clear denominators and write

$$4g(x) = 4x^3 + 4x^2 - 7x + 2.$$

The only possible rational roots are $\pm 1, \pm 2, \pm 1/2, \pm 1/4$. Checking these possibilities, one finds that only $x = -2$ and $x = 1/2$ are roots. This gives two of the factors and then the third can be found using division of polynomials. More precisely, one has

$$4g(x) = (x+2)(4x^2 - 4x + 1) = (x+2)(2x-1)^2 \quad \Longrightarrow \quad g(x) = \frac{1}{4}(x+2)(2x-1)^2.$$

5. Determine all angles $0 \leq \theta \leq 2\pi$ such that $2\sin^2\theta + 9\sin\theta = 5$.

Letting $x = \sin\theta$ for convenience, one finds that $2x^2 + 9x - 5 = 0$ and

$$x = \frac{-9 \pm \sqrt{81 + 4 \cdot 10}}{4} = \frac{-9 \pm 11}{4} \quad \Longrightarrow \quad x = \frac{1}{2}, -5.$$

Since $x = \sin\theta$ must lie between -1 and 1, the only relevant solution is $x = \sin\theta = \frac{1}{2}$. In view of the graph of the sine function, there should be two angles $0 \leq \theta \leq 2\pi$ that satisfy this condition. The first one is $\theta_1 = \frac{\pi}{6}$ and the second one is $\theta_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.