Advanced Calculus
MA1132
Homework Assignment 3

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SOLUTIONS

1. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = (x-1)^2 + y^2$ subject to the constraint $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

   Solutions:

   We first note that the set described by the constraint is closed and bounded (it is an ellipse), so we will get a maximum and a minimum.

   Let $g(x, y) = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 - 1$.

   Then $\nabla g = \left(\frac{2x}{9}, \frac{y}{2}\right) \neq (0, 0)$, since if we had $\left(\frac{2x}{9}, \frac{y}{2}\right) = (0, 0)$, we would have $x = y = 0$ and this does not satisfy $g(x, y) = 0$. Thus we are justified in using Lagrange multipliers.

   Now $\nabla f = (2x - 2, 2y)$, so $\nabla f = \lambda \nabla g$ yields

   $$2x - 2 = \frac{2\lambda x}{9} \tag{1}$$
   $$2y = \frac{\lambda y}{2} \tag{2}$$

   Next, Equation (26) yields $4y = \lambda y$ and there are two cases to consider.

   Case 1: $y = 0$.

   In this case the constraint equation implies $\left(\frac{x}{3}\right)^2 = 1$, so it follows that $x = \pm 3$. Since Equation (25) is also satisfied if we let $\lambda = \frac{9(1-x)}{x}$, we have the two points $(3, 0)$ and $(-3, 0)$.

   Case 2: $y \neq 0$.

   In this case $4y = \lambda y \Rightarrow \lambda = 4$ and then Equation (25) yields

   $$2x - 2 - \frac{8x}{9} = 0 \quad \Rightarrow \quad \frac{10x}{9} = 2 \quad \Rightarrow \quad x = \frac{9}{5}.$$

   Substituting this into the constraint equation gives

   $$\left(\frac{3}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \quad \Rightarrow \quad \left(\frac{y}{2}\right)^2 = \frac{16}{25} \quad \Rightarrow \quad \frac{y}{2} = \pm\frac{4}{5} \quad \Rightarrow \quad y = \pm\frac{8}{5}.$$

Thus we also have the two points $\left(\frac{9}{5}, \frac{8}{5}\right)$ and $\left(\frac{9}{5}, -\frac{8}{5}\right)$.

We now check the value of f at the four points we have found.

$$
\begin{aligned}
f(3,0) &= 2^2 = 4 \\
f(-3,0) &= (-4)^2 = 16 \\
f\left(\frac{9}{5}, \frac{8}{5}\right) &= \left(\frac{4}{5}\right)^2 + \left(\frac{8}{5}\right)^2 = \frac{16+64}{25} = \frac{80}{25} = \frac{16}{5} \\
f\left(\frac{9}{5}, -\frac{8}{5}\right) &= \left(\frac{4}{5}\right)^2 + \left(-\frac{8}{5}\right)^2 = \frac{16+64}{25} = \frac{80}{25} = \frac{16}{5}.
\end{aligned}
$$

Hence, subject to the constraint $g(x,y) = 0$, f attains its maximum of 16 at $(-3,0)$ and its minimum of $\frac{16}{5}$ at $\left(\frac{9}{5}, \frac{8}{5}\right)$ and $\left(\frac{9}{5}, -\frac{8}{5}\right)$.

2. Consider the intersection of the surfaces

$$
z = \sqrt{a^2 - x^2 - y^2}\,, \quad \text{and} \quad \frac{x^2}{b^2} + \frac{y^2}{c^2} = 1\,, \qquad a > b > c\,.
$$

(a) What is the surface $z = \sqrt{a^2 - x^2 - y^2}$? Sketch the surface $z = \sqrt{a^2 - x^2 - y^2}$ and its projection onto the xy plane for $a = 3$.

(b) What is the surface $\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1$? Sketch the surface $\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1$ and its projection onto the xy plane for $b = 2$, $c = 1$.

(c) Use Lagrange multipliers to find the coordinates of the points on the intersection which have the maximum z-coordinate and the minimum z-coordinate.

Solution :

(a) It is the upper semi-sphere of radius a. Its projection onto the xy plane for $a = 3$ is a circle of radius 3.

(b) It is an elliptic cylinder. Its projection onto the xy plane for $b = 2$, $c = 1$ is an ellipse with semi-axis 2 and 1.

(c) To find $z_{\max}$ nd $z_{\min}$ we use the Lagrange multiplier method, and get the equations

$$
-\frac{x}{z} = 2\Lambda\frac{x}{b^2}\,, \quad -\frac{y}{z} = 2\Lambda\frac{y}{c^2}\,, \quad z = \sqrt{a^2 - x^2 - y^2}\,, \quad \frac{x^2}{b^2} + \frac{y^2}{c^2} = 1\,. \tag{3}
$$

Since $b \neq c$ these equations have four solutions

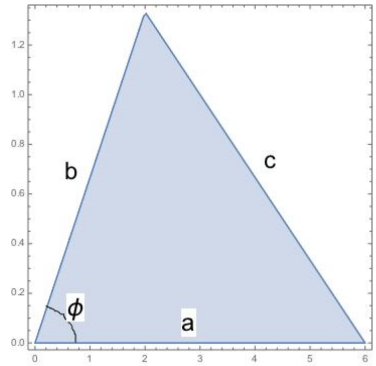
1),2) $x = 0\,,\ y = \pm c\,,\ z_{\max} = \sqrt{a^2 - c^2}$.

3),4) $x = \pm b\,,\ y = 0\,,\ z_{\min} = \sqrt{a^2 - b^2}$.

3. Show that a triangle with fixed area has minimum perimiter if it is equilateral.

Solution: Let the triangle has the sides a, b, c and the angle between the sides a and b is ϕ, see the picture. Then, the perimeter is

$$
P(a,b,c,\phi) = a + b + c\,, \tag{4}
$$



and we have the area constraint

$$A(a,b,c,\phi) = \frac{1}{2}ab\sin\phi - s = 0\,, \quad s \text{ is constant}\,, \tag{5}$$

and the constraint which relates a, b, c and ϕ is

$$\Phi(a,b,c,\phi) = a^2 + b^2 - 2ab\cos\phi - c^2 = 0\,. \tag{6}$$

We introduce two Lagrange multipliers λ_A and λ_Φ, and get the equations

$$\begin{aligned}
1 &= \lambda_A \frac{1}{2} b\sin\phi + \lambda_\Phi(2a - 2b\cos\phi)\,, \\
1 &= \lambda_A \frac{1}{2} a\sin\phi + \lambda_\Phi(2b - 2a\cos\phi)\,, \\
1 &= \lambda_\Phi(-2c) \quad \Longrightarrow \quad \lambda_\Phi = -\frac{1}{2c}\,, \\
0 &= \lambda_A \frac{1}{2} ab\cos\phi + \lambda_\Phi(2ab\sin\phi) \quad \Longrightarrow \quad \lambda_A = \frac{2}{c}\tan\phi\,.
\end{aligned} \tag{7}$$

Substituting λ_P and λ_Φ into the first two equations one gets

$$\begin{aligned}
1 &= -\frac{a}{c} + \frac{b}{c}\frac{1}{\cos\phi} \quad \Longrightarrow \quad b = (c+a)\cos\phi\,, \\
1 &= -\frac{b}{c} + \frac{a}{c}\frac{1}{\cos\phi} \quad \Longrightarrow \quad a = (c+b)\cos\phi\,.
\end{aligned} \tag{8}$$

Subtracting the first equation from the second one, one gets

$$a - b = -(a-b)\cos\phi \quad \Longrightarrow \quad a = b\,, \quad \cos\phi = \frac{a}{a+c}\,, \tag{9}$$

because $0 < \phi < \pi$. Substituting the found values in Φ, one gets

$$2a^2(1 - \frac{a}{a+c}) - c^2 = 0 \quad \Longrightarrow \quad 2a^2 = c(a+c) \quad \Longrightarrow \quad c = a\,. \tag{10}$$

Thus, $a = b = c$, and the triangle is equilateral.

4. What is the volume of the largest n-dimensional box with edges parallel to the coordinate axes that fits inside the n-dimensional ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1\,. \tag{11}$$

Solution: The volume of a box with edges parallel to the coordinate axes that fits inside the ellipsoid is

$$V(x_1, \ldots, x_n) = 2^n x_1 \cdots x_n\,, \tag{12}$$

where $x_i > 0$ are coordinates of the vertex of the box in the first "octant". The constraint is

$$g(x_1, \ldots x_n) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} - 1 = 0\,. \tag{13}$$

The maximum of V is therefore given by

$$2^n \frac{x_1 \cdots x_n}{x_k} = \frac{V}{x_k} = \lambda \frac{2x_k}{a_k^2}\,, \quad k = 1, \ldots, n\,, \tag{14}$$

and therefore

$$2\lambda \frac{x_k^2}{a_k^2} = V \quad \Longrightarrow \quad 2\lambda = nV \quad \Longrightarrow \quad x_k = \frac{a_k}{\sqrt{n}}\,, \tag{15}$$

where we summed over k and used the constraint. Thus, the maximum volume V is

$$V = \left(\frac{2}{\sqrt{n}}\right)^n a_1 \cdots a_n\,. \tag{16}$$

5. Find the integral of the function $f(x, y) = 4xye^{x^2+y^2}$ over the rectangle

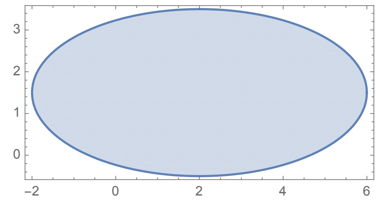
$$\{(x, y) \in \mathbb{R}^2 \colon 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 3\}.$$

Soluton:

Here it doesn't matter which variable we integrate with respect to first.

$$\begin{aligned}
\int_0^3 \int_0^2 4xye^{x^2+y^2}\, dx\, dy &= \int_0^3 \left[2ye^{x^2+y^2}\right]_0^2 dy \quad \text{by inspection} \\
&= \int_0^3 2ye^{4+y^2} - 2ye^{y^2}\, dy \\
&= \left[e^{4+y^2} - e^{y^2}\right]_0^3 \quad \text{by inspection} \\
&= e^{13} - e^9 - (e^4 - e^0) \\
&= e^{13} - e^9 - e^4 + 1.
\end{aligned}$$

6. Sketch the integration region R and reverse the order of integration



(a)

$$\int_{-1/2}^{7/2}\int_{2-\sqrt{7+12y-4y^2}}^{2+\sqrt{7+12y-4y^2}} f(x,y)dxdy \tag{17}$$

Solution: The region R is shown below

It is found by noting that

$$2-\sqrt{7+12y-4y^2}\le x\le 2+\sqrt{7+12y-4y^2} \implies (x-2)^2\le 7+12y-4y^2$$
$$\implies (x-2)^2+4(y-\frac{3}{2})^2\le 16 \implies \frac{(x-2)^2}{16}+\frac{(y-\frac{3}{2})^2}{4}\le 1\,. \tag{18}$$

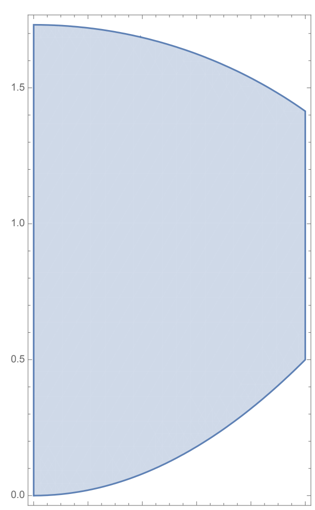
Thus it is a closed region bounded by an ellipse centred at $(2,3/2)$ with semi-axis 4 and 2. Reversing the order of integration one gets

$$\int_{-1/2}^{7/2}\int_{2-\sqrt{7+12y-4y^2}}^{2+\sqrt{7+12y-4y^2}} f(x,y)dxdy = \int_{-2}^{6}\int_{\frac{3}{2}-\sqrt{3+x-\frac{x^2}{4}}}^{\frac{3}{2}+\sqrt{3+x-\frac{x^2}{4}}} f(x,y)dydx\,. \tag{19}$$

(b)

$$\int_0^1\int_{x^2/2}^{\sqrt{3-x^2}} f(x,y)dydx \tag{20}$$

Solution: The region R is shown below Reversing the order of integration one gets



the sum of three repeated integrals

$$\int_0^1 \int_{x^2/2}^{\sqrt{3-x^2}} f(x,y)dydx$$
$$= \int_0^{1/2} \int_0^{\sqrt{2y}} f(x,y)dxdy + \int_{1/2}^{\sqrt{2}} \int_0^1 f(x,y)dxdy + \int_{\sqrt{2}}^{\sqrt{3}} \int_0^{\sqrt{3-y^2}} f(x,y)dxdy\,. \tag{21}$$

7. Prove the Dirichlet formula

$$\int_a^b \int_a^x f(x,y)dydx = \int_a^b \int_y^b f(x,y)dxdy\,, \tag{22}$$

and use it to prove that

$$\int_a^x \int_a^{t_1} (t_1-t)^{n-1} f(t)dtdt_1 = \frac{1}{n}\int_a^x (x-t)^n f(t)dt\,. \tag{23}$$

Solution: The Dirichlet formula follows from the fact that the repeated integrals are equal to the double integral over the triangle enclosed by the lines $y=a$, $x=b$, $y=x$.

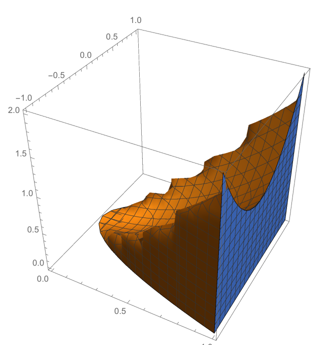
Thus, we get

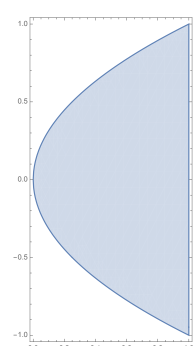
$$\begin{aligned} \int_a^x \int_a^{t_1} (t_1-t)^{n-1} f(t)dtdt_1 &= \int_a^x \int_t^x (t_1-t)^{n-1} f(t)dt_1dt \\ &= \int_a^x \frac{1}{n}(t_1-t)^n\big|_{t_1=t}^{t_1=x} f(t)dt = \frac{1}{n}\int_a^x (x-t)^n f(t)dt\,. \end{aligned} \tag{24}$$

8. Find the volume V of the solid bounded by

(a) the planes $x=1$, $z=0$, the parabolic cylinder $x-y^2=0$, and the paraboloid $z=x^2+y^2$.

Solution: The solid, and its projection R onto the xy-plane are shown below Thus,



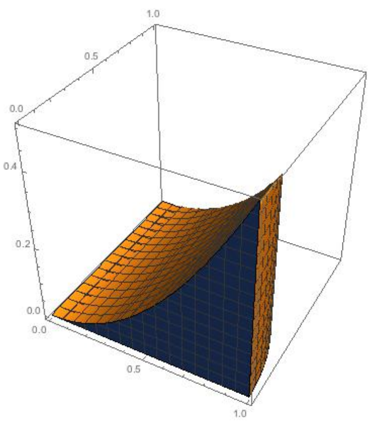


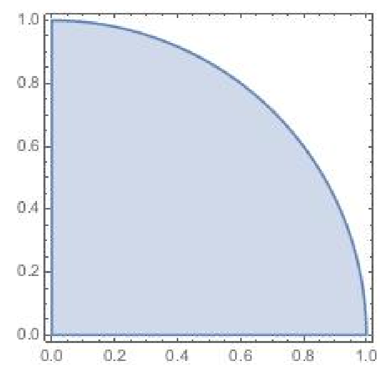
the volume is

$$V = \iint_R (x^2+y^2)dxdy = \int_{-1}^1 \int_{y^2}^1 (x^2+y^2)dxdy = \int_{-1}^1 (y^2(1-y^2) + \frac{1}{3}(1-y^6))dy$$
$$= 2\int_0^1 (y^2 - y^4 + \frac{1}{3} - \frac{1}{3}y^6)dy = 2(\frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{21}) = \frac{88}{105}\,. \tag{25}$$

(b) the planes $x = 0$, $y = 0$, $z = 0$, the cylinders $az = x^2$, $a > 0$, $x^2 + y^2 = b^2$, and located in the first octant $x \geq 0, y \geq 0, z \geq 0$.

Solution: The solid, and its projection R onto the xy-plane are shown below ($a = 2, b = 1$) Thus, the volume is





$$\begin{aligned} V &= \iint_R \frac{1}{a} x^2 dx dy = \frac{1}{a} \int_0^b \int_0^{\sqrt{b^2 - x^2}} x^2 dy dx = \frac{1}{a} \int_0^b x^2 \sqrt{b^2 - x^2} dx \\ &= \frac{b^4}{a} \int_0^1 x^2 \sqrt{1 - x^2} dx = \frac{b^4}{a} \int_0^{\pi/2} \sin^2 t \cos^2 t dt = \frac{b^4}{a} \int_0^{\pi/2} \frac{1}{4} \sin^2 2t dt \qquad (26) \\ &= \frac{b^4}{4a} \int_0^{\pi/2} \frac{1}{2} (1 - \cos 4t) dt = \frac{\pi b^4}{16a} \,. \end{aligned}$$