MA1125 – Calculus
Homework #4 solutions

1. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = \tan(e^x) + e^{\sec x}, \qquad y = \cos(\sin^2(\ln x)).$$

When it comes to the first function, one may use the chain rule to get

$$y' = \sec^2(e^x) \cdot (e^x)' + e^{\sec x} \cdot (\sec x)' = e^x \sec^2(e^x) + e^{\sec x} \sec x \tan x.$$

When it comes to the second function, one similarly finds that

$$\begin{aligned} y' &= -\sin(\sin^2(\ln x)) \cdot [\sin^2(\ln x)]' \\ &= -\sin(\sin^2(\ln x)) \cdot 2\sin(\ln x) \cdot [\sin(\ln x)]' \\ &= -\sin(\sin^2(\ln x)) \cdot 2\sin(\ln x) \cdot \cos(\ln x) \cdot 1/x. \end{aligned}$$

2. Compute the derivative $y' = \frac{dy}{dx}$ in the case that $y^2 \sin x + x^2 \sin y = x^2 y$.

Differentiating both sides of the given equation, one finds that

$$2yy' \sin x + y^2 \cos x + 2x \sin y + x^2 y' \cos y = 2xy + x^2 y'.$$

We now collect the terms that contain y' on the left hand side and we get

$$(2y \sin x + x^2 \cos y - x^2) y' = 2xy - y^2 \cos x - 2x \sin y.$$

Solving this equation for y', one may thus conclude that

$$y' = \frac{2xy - y^2 \cos x - 2x \sin y}{2y \sin x + x^2 \cos y - x^2}.$$

3. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = e^{\sin x} \cdot \cos(e^x), \qquad y = (x \cdot \tan x)^{\ln x}.$$

When it comes to the first function, we use the product rule and the chain rule to get

$$y' = e^{\sin x} \cos x \cdot \cos(e^x) - e^{\sin x} \cdot e^x \sin(e^x).$$

When it comes to the second function, logarithmic differentiation gives

$$\begin{aligned} \ln y = \ln x \cdot \ln(x \tan x) \quad &\Longrightarrow \quad \frac{y'}{y} = \frac{1}{x} \cdot \ln(x \tan x) + \frac{\ln x}{x \tan x} \cdot (x \tan x)' \\ &\Longrightarrow \quad \frac{y'}{y} = \frac{\ln(x \tan x)}{x} + \frac{\ln x}{x \tan x} \cdot (\tan x + x \sec^2 x) \\ &\Longrightarrow \quad y' = (x \cdot \tan x)^{\ln x} \cdot \left(\frac{\ln(x \tan x)}{x} + \frac{\ln x}{x} + \frac{\ln x}{\sin x \cos x} \right). \end{aligned}$$

4. Compute the derivative $f'(x_0)$ in the case that

$$f(x) = \frac{(x^2+3)^2 \cdot x^{\ln x} \cdot e^{4-4x}}{\sqrt{e^{2x-2}+3}}, \qquad x_0 = 1.$$

First, we use logarithmic differentiation to determine $f'(x)$. In this case, we have

$$\begin{aligned} \ln f(x) &= \ln(x^2+3)^2 + \ln x^{\ln x} + \ln e^{4-4x} - \ln(e^{2x-2}+3)^{1/2} \\ &= 2\ln(x^2+3) + (\ln x)^2 + 4 - 4x - \frac{1}{2}\ln(e^{2x-2}+3). \end{aligned}$$

Differentiating both sides of this equation, one easily finds that

$$\frac{f'(x)}{f(x)} = \frac{2\cdot 2x}{x^2+3} + \frac{2\ln x}{x} - 4 - \frac{2e^{2x-2}}{2(e^{2x-2}+3)}.$$

To compute the derivative $f'(1)$, one may then substitute $x = 1$ to conclude that

$$\frac{f'(1)}{f(1)} = \frac{4}{4} + 2\ln 1 - 4 - \frac{2}{8} = -\frac{13}{4} \quad\Longrightarrow\quad f'(1) = -\frac{13}{4}\cdot 8 = -26.$$

5. Compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$y = \tan^{-1} u, \qquad u = \sqrt{2z^3+1}, \qquad z = \frac{x^2-3}{x^2+1}.$$

Differentiating the given equations, one easily finds that

$$\frac{dy}{du} = \frac{1}{u^2+1}, \qquad \frac{du}{dz} = \frac{6z^2}{2\sqrt{2z^3+1}}, \qquad \frac{dz}{dx} = \frac{2x(x^2+1) - 2x(x^2-3)}{(x^2+1)^2} = \frac{8x}{(x^2+1)^2}.$$

According to the chain rule, the derivative $\frac{dy}{dx}$ is the product of these factors, namely

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dz}\frac{dz}{dx} = \frac{1}{u^2+1}\cdot\frac{3z^2}{\sqrt{2z^3+1}}\cdot\frac{8x}{(x^2+1)^2}.$$