JF PY1T10 Special Relativity

Lecture 12:

Energy / Momentum Invariant & Lorentz Transformation of Forces

Summary of Lecture 11

Elastic Scattering of Identical Particles:

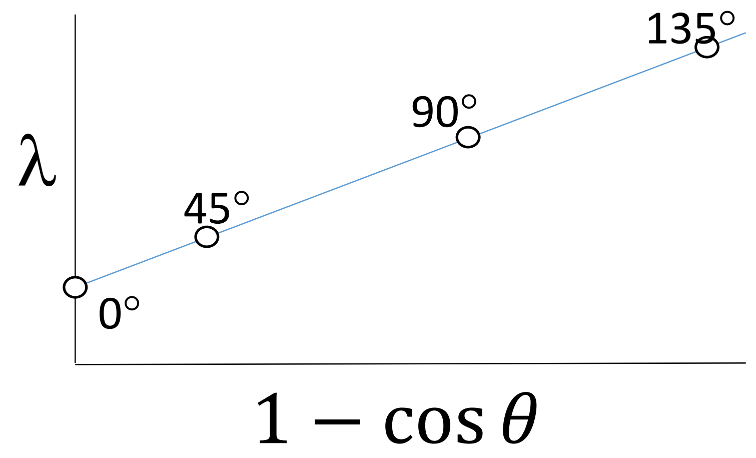
$$\theta = \cos^{-1}\left(\frac{K_1}{4E_0 + K_1}\right) < 90^\circ$$

Compton Effect:

First experimental evidence that light behaved as a particle (photon)

Collision between a photon and a nearly-free electron.

$$\lambda_2 - \lambda_1 = \frac{h(1-\cos\theta)}{m_0 c}$$



Energy Momentum Invariant

Single Particle:

$$p = \gamma m_0 v \, , E = \gamma m_0 c^2$$

Related by: $E^2 - c^2 p^2 = E_0^2 = m_0^2 c^4$

If particle has rest energy E_0 (i.e. total energy E_0 as measured in frame where $p = 0$), the E and p as measured <u>in</u> <u>any</u> <u>other</u> frame can be combined to form an invariant quantity:

$$E^2 - c^2 p^2 = E_0^2$$

Holds for any frame.

$$E^2 - c^2 p^2 = E'^2 - c^2 p'^2 = E_0^2$$

Energy Momentum Invariant

This can also be applied to a collection of particles:

As measured in a given frame:

E $=$ sum of energies of particles = $E_1 + E_2 + E_3 + \cdots$
$\vec{p} =$ vector sum of momenta = $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots$

Then:

$$E^2 - c^2\vec{p}^2 = E_0^2$$

Where E_0 is the total energy of all particles measured in a frame where $|\vec{p}| = 0$

N.B. Particles may not all be at rest in this frame $\therefore E_0 \neq E_{0_1} + E_{0_2} + \cdots$

Energy Momentum Invariant

Consider the production of an anti-proton in a proton-proton collsion:

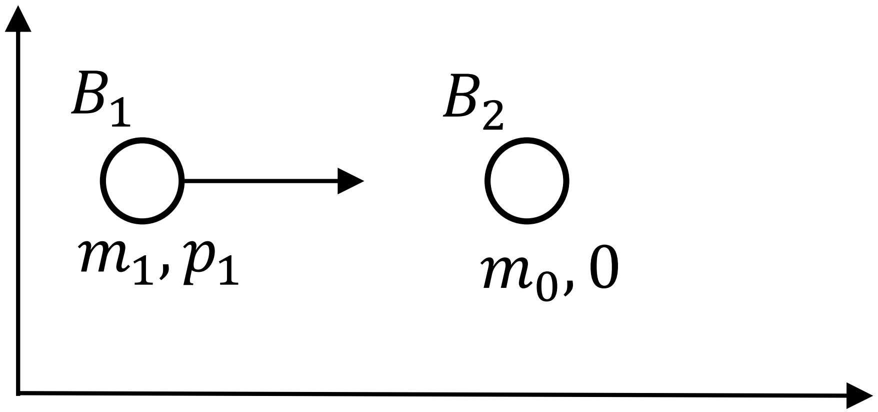
$$B_1 + B_2 \rightarrow B_1 + B_2 + \bar{B} + B_3$$

Anti-proton
(negative charge)

Suppose target proton B_2 is initially at rest in the lab frame.

What energy must B_1 have in order for the reaction to occur?

S frame:



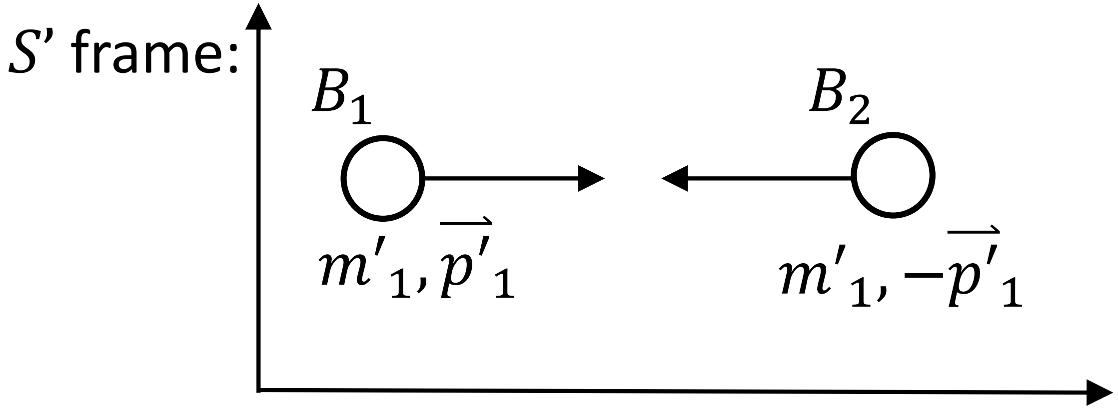
In S:

$$E = (m_1 + m_0)c^2$$

$$\vec{P} = \vec{p}_1$$

Energy Momentum Invariant

Move to the S' frame – where the sum of all momenta $= \ 0$



In S':

$$\sum \vec{p}'_i$$

$$E' = 2m'_1c^2$$

$$\overrightarrow{P'} = 0$$

Energy Momentum Invariant

What energy must B_1 have in order for the reaction to occur?

The minimum energy will result in a situation where all particles are at rest after the collision:

i.e. in S’: $\overrightarrow{P'} = 0$ and $E' = 4m_0c^2$

Using the old way: <u>Conservation of energy</u>

$$2m'_1c^2 = 4m_0c^2$$

$$\Rightarrow m'_1 = 2m_0$$

$$\Rightarrow \frac{m'_1}{m_0} = 2 = \gamma$$

This gives v’. From this we can work out v in S (lab frame)

But let us try using the $E^2 - c^2\vec{p}^2$ invariant.

Energy Momentum Invariant

$$E^2 - \ (cp)^2 = E'^2 - \ (cp)'^2$$

$$(m_1c^2 + m_0c^2)^2 - cp_1^2 = (4m_0c^2)^2 - 0$$

$$(m_1c^2)^2 + 2m_1m_2c^4 + (m_0c^2)^2 - cp_1^2 = 16(m_0c^2)^2$$

But for the single particle B_1: $(m_1c^2)^2 - (cp_1)^2 = (m_0c^2)^2$

$$\Rightarrow 2m_1m_2c^2 + 2(m_0c^2)^2 = 16(m_0c^2)^2$$

$$2(m_1c^2)(m_0c^2) + 2(m_0c^2)^2 = 16(m_0c^2)^2$$

$$2m_1c^2 + 2m_0c^2 = 16\, m_0c^2$$

$$m_1c^2 = 7m_0c^2$$

$$\Rightarrow E_{min} = m_1c^2 = 7m_0c^2$$

Minimum K.E. of $B_1 = 7m_0c^2 - m_0c^2 = 6m_0c^2$

Energy Momentum Invariant

Minimum K.E. of $B_1 = 7m_0c^2 - m_0c^2 = 6m_0c^2$

For a proton, $m_0c^2 = 0.938 \text{ GeV}$

Therefore, the minimum K.E. is 5.62 GeV

For this reason, the University of California Bevatron was designed to deliver protons of ~ 6 GeV kinetic energy into a fixed target.

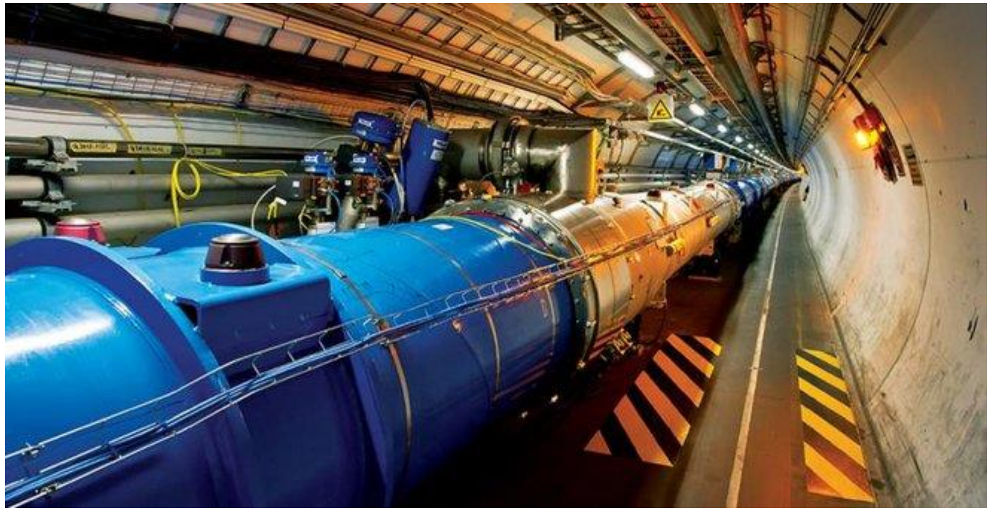
The antiproton was discovered there in 1955, resulting in the 1959 Nobel Prize in physics for Segrè and Chamberlain.

Energy Momentum Invariant

Better to have a colliding beam of protons:

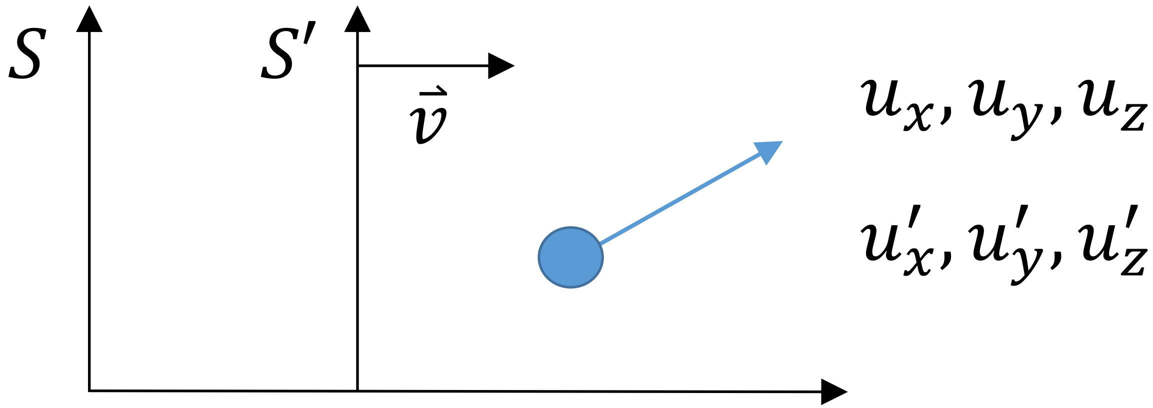
1983: Colliding p and $\bar{p}$ at CERN to create W and Z bosons, which carry the weak nuclear force.

2008: *Large Hadron Collider* (27 km circumference)



Two beams collide with a total kinetic energy of 14,000 GeV.
Needs a magnetic field of ~ 8.3 T
Superconducing magnets must be kept at 1.9 K.
On 4th July 2012: Higg's Boson discovered ($m_0 = 125\ \mathrm{GeV}$)

Lorentz Transformation of Energy & Momentum



In S:

$$\begin{aligned} E &= mc^2 = \gamma(u)m_0c^2 \\ \vec{p} &= m\vec{u} = \gamma(\vec{u})m_0\vec{u} \end{aligned}$$

In S':

$$\begin{aligned} E' &= m'c^2 = \gamma(u')m_0c^2 \\ \overrightarrow{p'} &= m'\overrightarrow{u'} = \gamma(\overrightarrow{u'})m_0\overrightarrow{u'} \end{aligned}$$

Can relate E, E', $\vec{p}$ and $\vec{p}'$ by the velocity transformation:

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}, u'_y = \frac{{u_y}/{\gamma}}{1 - \frac{vu_x}{c^2}}$$

Lorentz Transformation of Energy & Momentum

From French pg. 208:

$$p'_x = \gamma(v)\left[p_x - \frac{vE}{c^2}\right], \quad p'_y = p_y, \quad p'_z = p_z$$

Transformation of Forces:

$$\vec{F} = \frac{d\vec{p}}{dt}, \vec{p} = m\vec{u}$$

Viewed in another frame:

$$\overrightarrow{F'} = \frac{d\vec{p}'}{dt}$$

But $p'_x = \gamma\left(p_x - \frac{vE}{c^2}\right), p'_y = p_y, p'_z = p_z, t' = \gamma\left(t - \frac{vx}{c^2}\right)$

$$F'_x = \frac{dp'_x}{dt'} = \frac{dp'_x}{dt}\frac{dt}{dt'}$$

Lorentz Transformation of Energy & Momentum

$$F'_x = \frac{dp'_x}{dt'} = \frac{dp'_x}{dt}\frac{dt}{dt'} = \frac{\gamma\left(\frac{dp_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}\right)}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{F_x - \frac{v}{c^2}\frac{dE}{dt}}{1 - \frac{vu_x}{c^2}} \qquad ①$$

In both Newtonian and Relativistic mechanics:

$$\frac{dE}{dt} = \vec{F}.\vec{u}$$

Lorentz Transformation of Energy & Momentum

Proof:

$$E^2 = c^2p^2 + E_0^2 = c^2(\vec{p}.\vec{p}) + E_0^2$$

$$\frac{d}{dt}E^2 = \frac{d}{dt}c^2(\vec{p}.\vec{p}) + \underbrace{\frac{d}{dt}E_0^2}_{0}$$

$$2E\frac{dE}{dt} = 2c^2\vec{p}.\frac{d\vec{p}}{dt} = 2c^2\vec{p}.\vec{F}$$

$$mc^2\frac{dE}{dt} = c^2\vec{p}.\vec{F}$$

$$\Rightarrow \frac{dE}{dt} = \frac{\vec{p}}{m}.\vec{F} = \vec{u}.\vec{F}$$

Lorentz Transformation of Energy & Momentum

Going back to ①:

$$F'_x = \frac{F_x - \frac{v}{c^2}\vec{F}.\vec{u}}{1 - \frac{vu_x}{c^2}}, \qquad F'_y = \frac{F_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}, \qquad F'_z = \frac{F_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

Note: If $\vec{F}$ depends on position but not on particle velocity, then $\vec{F}'$ depends on position <u>and</u> velocity.

If $|\vec{u}| = 0$ (particle at rest in S), then $F'_x = F_x$, $F'_y = {}^{F_y}\!/_{\gamma}$, $F'_z = {}^{F_z}\!/_{\gamma}$

Kinematics: Hints for Problem Solving

1. **Conservation of Mass-Energy.** The total mass-energy is always conserved. Add up $E = \gamma m_0 c^2$ for all of the particles and it is the same before as after the collision.
2. **Conservation of Momentum.** Momentum is also conserved; but remember, this is $p = \gamma m_0 v$
3. **Invariance of $\boldsymbol{E^2 - p^2 c^2}$.** Remember that $E^2 - p^2 c^2$ is the same in all inertial frames, and it applies to the whole system as well as to individual particles.
4. **In the centre of momentum frame**, the total momentum is zero (by definition).