Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 10
Each set of homework questions is worth 100 marks

Problem 1. Any rotation matrix G belonging to the Lie group $SO(3)$ can be written as the following product

$$G = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ϕ, θ and ψ are the Euler angles.

Introduce the following matrix

$$J = \frac{dG}{dt} G^{-1},$$

where G^{-1} and G^T are the inverse and transpose matrices, respectively.

(a) Show that the matrix J belongs to the Lie algebra $so(3)$, that is it is skew-symmetric: $J^T = -J$.

Answer: We have

$$J^T = \big(\frac{dG}{dt} G^{-1}\big)^T = \big(\frac{dG}{dt} G^T\big)^T = G\frac{dG^T}{dt} = \frac{d}{dt}(GG^T) - \frac{dG}{dt}G^T = -J\,.$$

(b) Show that the components Ω_i of the angular velocity vector $\vec{\Omega}$ along the moving axes x_1, x_2, x_3 are expressed through the components J_{ij} as follows

$$\Omega_i = \frac{1}{2}\epsilon_{ijk} J_{jk}\,,$$

where ϵ_{ijk} is the skew-symmetric tensor.

Answer: Let us introduce

$$G_{12}(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_{23}(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix}. \tag{0.1}$$

Then, we find $G = G_{12}(\psi)G_{23}(\theta)G_{12}(\phi)$, and

$$\begin{aligned} J = \frac{dG}{dt}G^T = \frac{dG_{12}(\psi)}{dt}G_{12}(\psi)^T + G_{12}(\psi)\frac{dG_{23}(\theta)}{dt}G_{23}(\theta)^T G_{12}(\psi)^T \\ + G_{12}(\psi)G_{23}(\theta)\frac{dG_{12}(\phi)}{dt}G_{12}(\phi)^T G_{23}^T(\theta) G_{12}^T(\psi)\,. \end{aligned} \tag{0.2}$$

Then we have

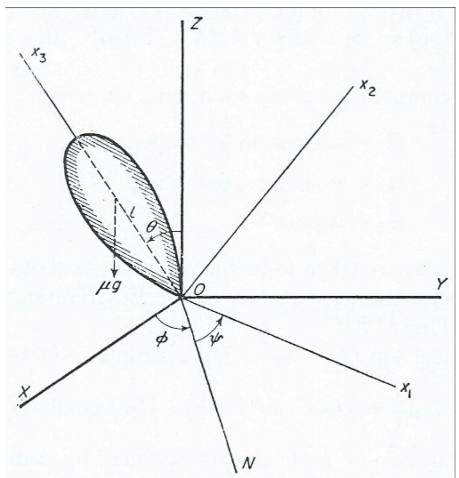
$$\frac{dG_{12}(\varphi)}{dt}G_{12}(\varphi)^T = \begin{pmatrix} 0 & \dot\varphi & 0 \\ -\dot\varphi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{dG_{23}(\theta)}{dt}G_{23}(\theta)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot\theta \\ 0 & -\dot\theta \end{pmatrix}.$$

By using these formulae, we get (do the calculation!)

$$
\begin{aligned}
J_{12} &= \Omega_3 = \dot{\phi}\cos\theta + \dot{\psi}\,, \\
J_{23} &= \Omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\,, \\
J_{31} &= \Omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\,.
\end{aligned}
\tag{0.3}
$$

Problem 2. Consider the Lagrangian of a heavy symmetric top whose lowest point is fixed (Lagrange's top)

$$L = \frac{1}{2}(I_1 + ml^2)(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - mgl\cos\theta\,.$$



(a) Which of the coordinates of the top are cyclic? Find the integrals of motion corresponding to the cyclic angles, and relate them to the angular momentum of the top.

Answer: The angles ϕ and ψ are cyclic. We take the common origin of the moving and fixed systems of coordinates at the fixed point O of the top, and the Z-axis vertical. Then we get

$$p_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = M_3 = constant$$

$$p_\phi = (\tilde{I}_1\sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta = M_z = constant\,,$$

where $\tilde{I}_1 = I_1 + ml^2$.

(b) Use the integrals of the motion to reduce the problem of the motion of the top to a one-dimensional one.

Answer: From these equations we find

$$\dot{\phi} = \frac{M_z - M_3\cos\theta}{\tilde{I}_1\sin^2\theta}\,, \quad \dot{\psi} = \frac{M_3}{I_3} - \cos\theta\frac{M_z - M_3\cos\theta}{\tilde{I}_1\sin^2\theta}\,.$$

The energy

$$E = \frac{1}{2}(I_1 + ml^2)(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + mgl\cos\theta$$

is also conserved.

Eliminating $\dot{\phi}$ and $\dot{\psi}$, we find the effective one-dimensional system

$$\tilde{E} = \frac{1}{2}\tilde{I}_1\dot{\theta}^2 + U_{\text{eff}}(\theta)\,,$$

where

$$\tilde{E} = E - \frac{M_3^2}{2I_3} - mgl\,, \quad U_{\text{eff}}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2\tilde{I}_1\sin^2\theta} - mgl(1-\cos\theta)\,.$$

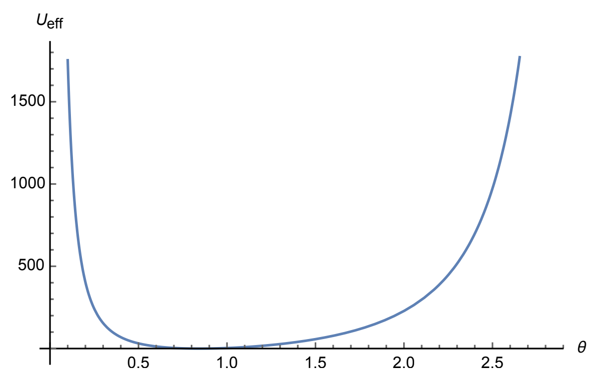
(c) Find the effective potential and the effective Lagrangian.

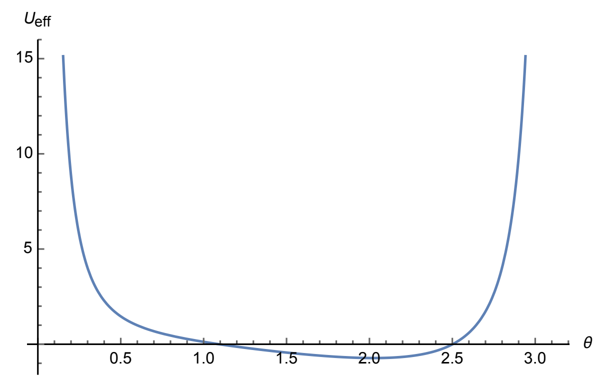
Use Mathematica to plot the effective potential for $m = g = l = \tilde{I}_1 = 1$, and i) $M_z = 12$, $M_3 = 18$; ii) $M_z = 1$, $M_3 = 1/6$. Explain the plots.

Answer: They are

$$L_{\text{eff}} = \frac{1}{2}\tilde{I}_1\dot{\theta}^2 - U_{\text{eff}}(\theta)\,, \quad U_{\text{eff}}(\theta) = \frac{(M_z - M_3\cos\theta)^2}{2\tilde{I}_1\sin^2\theta} - mgl(1-\cos\theta)\,.$$

The effective potential for the two cases is shown below.





The effective potential drawn on the left plot has very large M_z and M_3, and as a result the minimum of the potential is at $\theta_0 < \pi/2$.

(d) Let the top rotate about the vertical axis. Explain why such a rotation is possible only for $M_z = M_3$.

Answer: If the top rotates about the vertical axis then $\theta = 0$. Since the effective potential for small theta is repulsive

$$U_{\text{eff}}(\theta) = \frac{(M_z - M_3)^2}{2\tilde{I}_1\theta^2} + \mathcal{O}(1)\,, \tag{0.4}$$

$\theta = 0$ can be only if $M_z = M_3$.

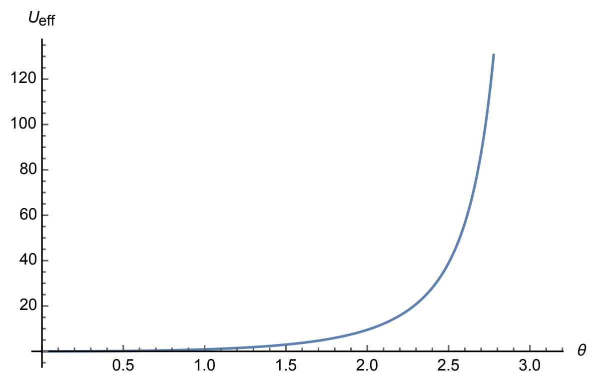
(e) Simplify the effective potential for $M_z = M_3$, and use Mathematica to plot the effective potential for $m = g = l = \tilde{I}_1 = 1$, and i) $M_z = M_3 = 3$; ii) $M_z = M_3 = 1/2$.

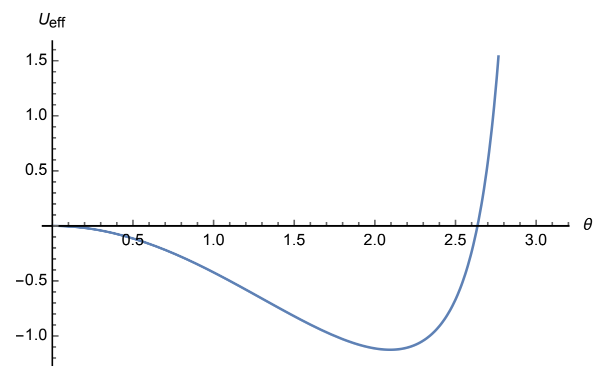
Explain the plots.

Answer: If $M_z = M_3$ we use $1 - cos\theta = 2\sin^2\frac{\theta}{2}$, $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ to get

$$U_{\text{eff}}(\theta) = \frac{M_3^2}{2\tilde{I}_1}\tan^2\frac{\theta}{2} - mgl(1-\cos\theta)\,. \tag{0.5}$$

The effective potential vanishes at $\theta = 0$, and is shown below for the two cases.





For large enough M_3 the potential has a global minimum at $\theta = 0$ and the rotation about the vertical axis is stable. For small M_3 the potential has a local maximum at $\theta = 0$ and the rotation about the vertical axis is unstable.

(f) Find the exact condition for the rotation about the vertical axis to be stable.

Answer: Expanding the effective potential for $M_z = M_3$ up to quadratic order in θ, we get

$$U_{\text{eff}}(\theta) = \frac{1}{2}(\frac{M_3^2}{4\tilde{I}_1} - mgl)\theta^2 + \mathcal{O}(\theta^4)\,. \tag{0.6}$$

The equilibrium at $\theta = 0$ is stable if $U''_{\text{eff}}(0) > 0$, and therefore

$$\frac{M_3^2}{4\tilde{I}_1} > mgl\,. \tag{0.7}$$

(g) Find the frequency of small oscillations in the θ-direction if the top was shifted from the equilibrium position. What type of motion does the top undergo?

Answer: The effective Lagrangian for small θ is

$$L_{\text{eff}} = \frac{1}{2}\tilde{I}_1\dot{\theta}^2 - \frac{1}{2}(\frac{M_3^2}{4\tilde{I}_1} - mgl)\theta^2\,, \tag{0.8}$$

and the frequency is

$$\omega^2 = \frac{M_3^2}{4\tilde{I}_1^2} - \frac{mgl}{\tilde{I}_1}\,. \tag{0.9}$$

The top oscillates through the vertical because $\dot{\theta} \neq 0$ for $\theta = 0$ and at the same time rotates about the vertical because $\dot{\phi} \neq 0$ for $\theta = 0$.

Problem 3. A symmetric top with a fixed centre of mass experiences a constant torque $\vec{K}$. Use the Euler angles to find the angular velocity of the top as a function of time. The initial angular momentum is proportional to $\vec{K}$: $\vec{M}(0) = \alpha\vec{K}$, where α is a constant.

Answer: Let the Z-axis be in the direction of the constant torque $\vec{K}$ so that $\vec{K} = \{0, 0, K\}$. We have

$$\dot{\vec{M}} = \vec{K} \quad \Rightarrow \quad \vec{M} = \vec{K}t + \vec{M}(0) = \vec{K}(t + \alpha)\,. \tag{0.10}$$

Thus, the Z-axis is in the direction of the angular momentum $\vec{M}$, and, therefore,

$$M_X = 0\,, \quad M_Y = 0\,, \quad M_Z = K(t + \alpha)\,. \tag{0.11}$$

The projections of $\vec{M}$ onto the axes x_1, x_2, x_3 are

$$\begin{cases} M_1 &= M_Z \sin\theta \sin\psi = I_1\Omega_1 = I_1(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi) \\ M_2 &= M_Z \sin\theta \cos\psi = I_1\Omega_2 = I_1(\dot{\phi}\sin\theta\sin\psi - \dot{\theta}\sin\psi) \\ M_3 &= M_Z \cos\theta = I_3\Omega_3 = I_3(\dot{\phi}\cos\theta + \dot{\psi}) \end{cases} \tag{0.12}$$

The first two equations give

$$\dot{\theta} = 0\,, \quad \dot{\phi} = \frac{M_z}{I_1} \quad \Rightarrow \quad \theta = const\,, \quad \dot{\phi} = \frac{K}{I_1}(t + \alpha)\,. \tag{0.13}$$

Thus, the angle between the axis of the top and $\vec{M}$ (or $\vec{K}$) is constant, and $\dot{\phi} = \frac{K}{I_1}(t+\alpha)$ is the angular velocity of precession. Finally,

$$\Omega_3 = \frac{M_Z\cos\theta}{I_3} = \frac{K\cos\theta}{I_3}(t + \alpha) \tag{0.14}$$

is the angular velocity with which the top rotates about its own axis.