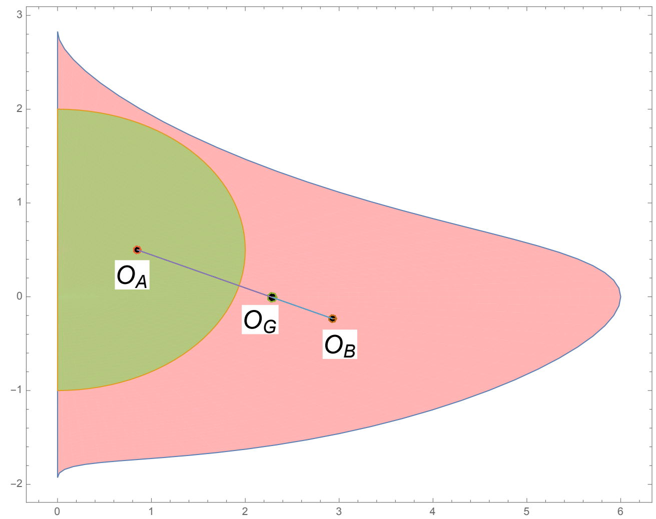
Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 9
Each set of homework questions is worth 100 marks

Problem 1. Let the rigid body G be a composition of two rigid bodies A and B, see the picture below. Express the inertia tensor $I_{ik}^{(A)}$ of the rigid body A defined with respect to its centre of mass O_A through the inertia tensors $I_{ik}^{(G)}$ and $I_{ik}^{(B)}$ of the rigid bodies G and B. Assume that the location of the centres of mass O_G, O_A, O_B, and the masses of A and B are known.



Answer: Due to the additivity property, the inertia tensor $I_{ik}^{(G)}$ of the rigid body G defined with respect to its centre of mass O_G is given by

$$I_{ik}^{(G)} = I_{ik}^{(A)}(\vec{a}_A) + I_{ik}^{(B)}(\vec{a}_B)\,, \tag{0.1}$$

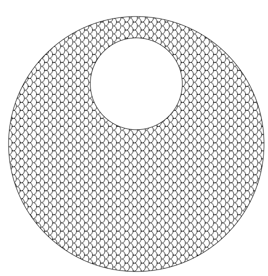
where $I_{ik}^{(A)}(\vec{a}_A)$ is the inertia tensor $I_{ik}^{(A)}$ of the rigid body A defined with respect to O_G and $\vec{a}_A$ is the vector from O_G to O_A, and similarly for B. Note that $\vec{a}_A$ and $\vec{a}_B$ are collinear. Expressing $I_{ik}^{(A)}(\vec{a}_A)$ and $I_{ik}^{(B)}(\vec{a}_B)$ as

$$I_{ik}^{(A)}(\vec{a}_A) = I_{ik}^{(A)} + m_A(a_A^2\delta_{ik} - a_{Ai}a_{Ak})\,, \quad I_{ik}^{(B)}(\vec{a}_B) = I_{ik}^{(B)} + m_B(a_B^2\delta_{ik} - a_{Bi}a_{Bk})\,, \tag{0.2}$$

we get

$$I_{ik}^{(A)} = I_{ik}^{(G)} - I_{ik}^{(B)} - m_B(a_B^2\delta_{ik} - a_{Bi}a_{Bk}) - m_A(a_A^2\delta_{ik} - a_{Ai}a_{Ak})\,. \tag{0.3}$$

Problem 2. Let the rigid body G be a homogeneous solid cylinder of radius R and of height H. Let the rigid body A be obtained by cutting out from G a cylinder B of radius r whose axes of symmetry is parallel to the axes of G. The distance between the axis of G and the cylinder B is $a \leq R - r$.



(a) Find the principal moments of inertia of the rigid body A.

Answer: If we fill in the hole then we get the solid cylinder G which is the composition of A and the solid cylinder B. We choose the x-axis to coincide with the axis of G, the z-axis to go through the centres of mass of G and B (and therefore A). Then, the coordinates of the centres of mass of G and B are

$$x_G = 0\,, \quad y_G = 0\,, \quad z_G = 0\,, \quad x_B = 0\,, \quad y_B = 0\,, \quad z_B = a\,, \tag{0.4}$$

while the coordinates of the centre of mass of A are

$$x_A = 0\,, \quad y_A = 0\,, \quad z_A = \frac{m_G z_G - m_B z_B}{m_A} = -\frac{m_B z_B}{m_G - m_B} = -\frac{r^2 a}{R^2 - r^2}\,. \tag{0.5}$$

Taking into account that the moments of inertia of the solid cylinder G of radius R and height L (with our choice of axis) are

$$I_x = \rho \int_G (y^2 + z^2) dV = \rho \int_{-L/2}^{L/2} \int_0^{2\pi} \int_0^R r^3 dr d\phi dx = \rho \frac{R^4}{4} 2\pi L = \frac{1}{2} m_G R^2\,, \tag{0.6}$$

$$\begin{aligned} I_y = I_z = \rho \int_G (x^2 + z^2) dV &= \rho \int_{-L/2}^{L/2} \int_0^{2\pi} \int_0^R x^2 r dr d\phi dx + \rho \int_{-L/2}^{L/2} \int_0^{2\pi} \int_0^R r^3 \cos^2\phi dr d\phi dx \\ &= \rho \frac{R^2}{2} 2\pi \frac{L^3}{12} + \rho \frac{R^4}{4} \pi L = \frac{1}{12} m_G (3R^2 + L^2)\,, \end{aligned} \tag{0.7}$$

where ρ is the density, and using the formula from Problem 1, we get

$$\begin{aligned} I_x^{(A)} &= I_x^{(G)} - I_x^{(B)} - m_B(a_B^2 - a_{Bx} a_{Bx}) - m_A(a_A^2 - a_{Ax} a_{Ax}) \\ &= \frac{1}{2} \rho \pi L \left(R^4 - r^4\right) - \frac{\rho \pi r^2 R^2 L}{R^2 - r^2}\, a^2\,. \end{aligned} \tag{0.8}$$

$$\begin{aligned} I_y^{(A)} &= I_y^{(G)} - I_y^{(B)} - m_B(a_B^2 - a_{By} a_{By}) - m_A(a_A^2 - a_{Ay} a_{Ay}) \\ &= \frac{1}{12} \rho \pi L (R^2 - r^2) \left(L^2 + 3\left(r^2 + R^2\right)\right) - \frac{\rho \pi r^2 R^2 L}{R^2 - r^2}\, a^2\,. \end{aligned} \tag{0.9}$$

$$\begin{aligned} I_z^{(A)} &= I_z^{(G)} - I_z^{(B)} - m_B(a_B^2 - a_{Bz} a_{Bz}) - m_A(a_A^2 - a_{Az} a_{Az}) \\ &= \frac{1}{12} \rho \pi L (R^2 - r^2) \left(L^2 + 3\left(r^2 + R^2\right)\right)\,, \end{aligned} \tag{0.10}$$

(b) Find the frequency of small oscillations of the rigid body A about a horizontal axis perpendicular to the line connecting the centres of mass of G and B and passing through the centre of

(i) the cylinder G, (ii) the cylinder B.

Answer: The frequency of small oscillations of a rigid body A swinging about a fixed horizontal axis in a gravitational field is given by

$$\omega^2 = \frac{m_A g l}{m_A l^2 + I_x \cos^2\alpha + I_y \cos^2\beta + I_z \cos^2\gamma}\,, \tag{0.11}$$

where m_A is the mass of A, g is the gravity constant, l is the distance from the centre of mass of A to the horizontal axis, α, β, γ are the angles between the horizontal axis and the principle axes of inertia, and I_k are the principal moments of inertia of A.

Since the x-axis is parallel to the fixed horizontal axis, we get that $\alpha = 0\,,\ \beta = \pi/2$ and $\gamma = \pi/2$. Then, the formula simplifies

$$\omega^2 = \frac{m_A g l}{m_A l^2 + I_x^{(A)}}\,, \tag{0.12}$$

and we get

$$(i) \quad l = |z_A| \ \Rightarrow\ \omega^2 = \frac{m_A g |z_A|}{m_A z_A^2 + I_x^{(A)}}\,, \quad m_A = \rho\pi(R^2 - r^2)L\,, \tag{0.13}$$

$$(ii) \quad l = a - z_A \ \Rightarrow\ \omega^2 = \frac{m_A g (a - z_A)}{m_A (a - z_A)^2 + I_x^{(A)}}\,. \tag{0.14}$$

Problem 3. Consider the system in problem 4 of Par. 32 (Landau and Lifshitz page 103).

(a) Find the Lagrangian and equations of motion of the system.

Answer: The kinetic energy T is found in Landau-Lifshitz, p.103

$$T = \frac{1}{3} m l^2 (1 + 3\sin^2\phi)\dot{\phi}^2\,. \tag{0.15}$$

The potential energy U is given by

$$U = 2 \cdot mg \cdot \frac{1}{2} l \sin\phi = mgl\sin\phi\,. \tag{0.16}$$

Thus, the Lagrangian is

$$L = \frac{1}{3} m l^2 (1 + 3\sin^2\phi)\dot{\phi}^2 - mgl\sin\phi\,, \tag{0.17}$$

and the eom is

$$\begin{aligned} &\frac{2}{3} m l^2 (1 + 3\sin^2\phi)\ddot{\phi} + 2ml^2 sin(2\phi)\dot{\phi}^2 = ml^2\sin(2\phi)\dot{\phi}^2 - mgl\cos\phi \ \Rightarrow \\ &\frac{2}{3} m l^2 (1 + 3\sin^2\phi)\ddot{\phi} = -ml^2\sin(2\phi)\dot{\phi}^2 - mgl\cos\phi\,. \end{aligned} \tag{0.18}$$

(b) Determine the angular velocity of the rod AB the moment before it hits the ground if its initial angular velocity is $\sqrt{3g/l}$.

Answer: To find the angular velocity we use the conservation of energy

$$E = \frac{1}{3} m l^2 (1 + 3\sin^2\phi)\dot{\phi}^2 + mgl\sin\phi = \frac{1}{3} m l^2 \omega_0^2 + mgl\sin\phi_0 = \frac{1}{3} m l^2 \omega^2\,, \tag{0.19}$$

where ϕ_0 is the initial angle. So

$$\omega^2 = \frac{3g}{l}(1 + \sin\phi_0) = \frac{6g}{l}\cos^2\frac{\phi_0}{2}\,. \tag{0.20}$$

Bonus question (each bonus question is worth extra 25 marks)

A homogeneous cone of mass m, height h, and base radius r can roll without slipping on an inclined plane with its tip fixed at one point. The plane forms angle ϕ with the horizontal plane. Find the Lagrangian and equations of motion of this system, and the frequency of small oscillations.