Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 1
Each set of homework questions is worth 100 marks

Problem 1. Consider a particle moving in a D-dimensional flat space with the following Lagrangian

$$L = \frac{1}{2} m v_i^2 - U(r)\,, \quad r = \sqrt{x_i^2}\,, \quad m > 0 \text{ is a constant}\,,$$

where $U(r)$ is the "Mexican hat" (or Higgs) potential

$$U(r) = \frac{k^2}{4g} - \frac{k}{2} r^2 + \frac{g}{4} r^4\,, \quad k > 0\,,\ g > 0\,, \quad k\,,\ g \text{ are constants}\,,$$

Here and in what follows the summation over the repeated indices is assumed. That means

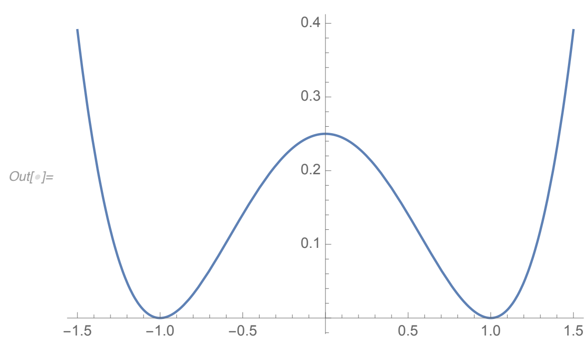
$$a_i b_i \equiv \sum_{i=1}^{D} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_D b_D\,, \quad a_i^2 \equiv a_i a_i$$

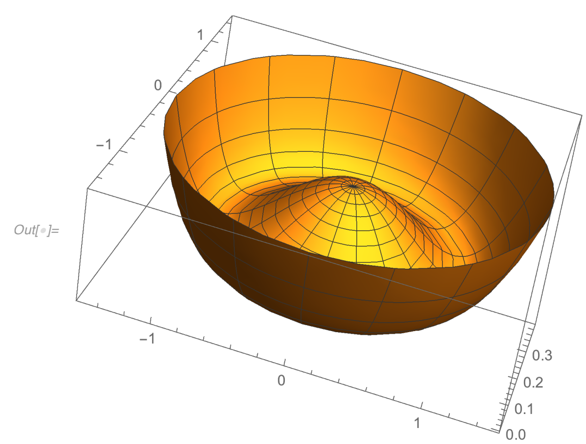
for any sets of objects $a_1, \dots, a_D$ and $b_1, \dots, b_D$.

(a) Find the absolute minimum and a local maximum of the potential.

Use Mathematica to plot the potential for $D = 1\,,\,2,\ k = 1,\ g = 1.$

Answer: The absolute minimum at $r = \sqrt{k/g}$ is equal to 0 while the local maximum at $r = 0$ is $\frac{k^2}{4g}$. The plot is below





(b) Find equations of motion (eom) of the particle.

Answer

$$\begin{aligned} &\frac{\partial L}{\partial v_i} = m v_i\,, \quad \frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial r}\frac{\partial r}{\partial x_i} = (kr - g r^3)\,\frac{x_i}{r} \ \Rightarrow \\ &m \ddot{x}_i = (k - g r^2) r\,\frac{x_i}{r}\,. \end{aligned} \tag{0.1}$$

(c) Prove that L is invariant under the $O(D)$ group of rotations and reflections of the coordinates x_i:

$$x_i \to \mathcal{O}_{ij} x_j \,, \quad \textit{summation over } j \;! \,,$$

where $\mathcal{O}_{ij}$ is an orthogonal D by D matrix, that is for any indices i and j

$$\mathcal{O}_{ik}\mathcal{O}_{jk} = \delta_{ij} \,, \quad \textit{summation over } k \;!$$

and δ_{ij} is the Kronecker delta: $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

Answer: To prove that L is invariant under the $SO(D)$ group of rotations it is sufficient to prove that r^2 and v_i^2 are invariant. We have

$$r^2 = x_i x_i \to \mathcal{O}_{ij} x_j \mathcal{O}_{ik} x_k = \mathcal{O}_{ij}\mathcal{O}_{ik} x_j x_k = \delta_{jk} x_j x_k = x_i^2 = r^2 \,, \tag{0.2}$$

and similarly for v_i^2.

Problem 2. Consider a system with s degrees of freedom and the Lagrangian

$$L = \frac{1}{2} m_{ij} \dot{q}^i \dot{q}^j \, + \, b_{ij} \dot{q}^i q^j - \frac{1}{2} k_{ij} e^{q^i + q^j} \,, \tag{0.3}$$

where m_{ij}, k_{ij} and b_{ij} are constants, and we sum over repeated indices.

(a) Explain why without loss of generality for any i and j one can assume that $m_{ij} = m_{ji}$ and $k_{ij} = k_{ji}$ that is the matrices (m_{ij}) and (k_{ij}) with the entries m_{ij} and k_{ij}, respectively, are symmetric. Explain why (m_{ij}) is positive definite.

Answer: One can assume that $m_{ij} = m_{ji}$ and $k_{ij} = k_{ji}$ because L does not depend on their anti-symmetric parts

$$\frac{1}{2} m_{ij} \dot{q}^i \dot{q}^j - \frac{1}{2} k_{ij} e^{q^i + q^j} = \frac{1}{4}(m_{ij} + m_{ji}) \dot{q}^i \dot{q}^j - \frac{1}{4}(k_{ij} + k_{ji}) e^{q^i + q^j} \,. \tag{0.4}$$

According to Hamilton's principle, (m_{ij}) must be positive definite otherwise the action would have no minimum.

(b) Find equations of motion of this system.

Answer: We get

$$\frac{\partial L}{\partial \dot{q}^i} = m_{ij} \dot{q}^j + b_{ij} q^j \,, \quad \frac{\partial L}{\partial q^i} = b_{ji} \dot{q}^j - k_{ij} e^{q^i + q^j} \,. \tag{0.5}$$

Thus the eom are

$$\frac{d}{dt}\big(m_{ij} \dot{q}^j + b_{ij} q^j\big) - \, b_{ji} \dot{q}^j + k_{ij} e^{q^i + q^j} = 0 \,, \tag{0.6}$$

or after a simplification

$$m_{ij} \ddot{q}^{\,j} + \, (b_{ij} - b_{ji}) \dot{q}^j + k_{ij} e^{q^i + q^j} = 0 \,. \tag{0.7}$$

(c) Explain why the equations of motion depend only on the anti-symmetric part of b_{ij}.

Answer: The reason is that the symmetric part of b_{ij} contributes a total time derivative to L. Indeed,

$$\begin{aligned} b_{ij}\dot{q}^i q^j &= \frac{1}{2}(b_{ij} - b_{ji})\dot{q}^i q^j + \frac{1}{2}(b_{ij} + b_{ji})\dot{q}^i q^j \\ &= \frac{1}{2}(b_{ij} - b_{ji})\dot{q}^i q^j + \frac{1}{4}\frac{d}{dt}\big((b_{ij} + b_{ji})q^i q^j\big)\,. \end{aligned} \tag{0.8}$$

Problem 3. Consider a relativistic charged particle of rest mass m and charge e moving in constant uniform electric $\vec{E} = \{E_1, E_2, E_3\}$ and magnetic $\vec{B} = \{B_1, B_2, B_3\}$ fields described by the following Lagrangian (c is the speed of light)

$$L = -mc^2\sqrt{1 - \frac{v_i^2}{c^2}} + e\,x_i E_i + \frac{e}{2c}\,\epsilon_{ijk}\,B_i x_j v_k\,, \quad v_i^2 \equiv v_i v_i\,,$$

where ϵ_{ijk} is the antisymmetric tensor with $\epsilon_{123} = 1$.

(a) Find the momentum $\vec{p}$ of the particle as a function of its velocity $\vec{v}$. What is the component of the momentum along the x_3-axis?

Find the velocity $\vec{v}$ of the particle as a function of $\vec{p}$. What is the component of the velocity along the x_2-axis?

Answer: We denote $v^2 = v_i^2$, $p^2 = p_i^2$, and get

$$\begin{aligned} p_k &= \frac{\partial L}{\partial v_k} = \frac{m v_k}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{2c}\,\epsilon_{ijk}\,B_i x_j \;\Rightarrow\; \vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{2c}\,\vec{B}\times\vec{r}\,, \\ p_3 &= \frac{m v_3}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{2c}\,\epsilon_{ij3}\,B_i x_j = \frac{m v_3}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{2c}(B_1 x_2 - B_2 x_1)\,. \end{aligned} \tag{0.9}$$

From the formula above we find

$$(\vec{p} - \frac{e}{2c}\,\vec{B}\times\vec{r})^2 = \frac{m^2 v^2}{1 - \frac{v^2}{c^2}} \;\Rightarrow\; \frac{v^2}{c^2} = \frac{p^2}{m^2c^2 + p^2} \;\Rightarrow\; \sqrt{1 - \frac{v^2}{c^2}} = \frac{c\,m}{\sqrt{m^2c^2 + (\vec{p} - \frac{e}{2c}\,\vec{B}\times\vec{r})^2}}\,. \tag{0.10}$$

Thus,

$$\begin{aligned} \vec{v} &= \frac{c(\vec{p} - \frac{e}{2c}\,\vec{B}\times\vec{r})}{\sqrt{m^2c^2 + (\vec{p} - \frac{e}{2c}\,\vec{B}\times\vec{r})^2}} = \frac{1}{\sqrt{1 + \frac{(\vec{p} - \frac{e}{2c}\,\vec{B}\times\vec{r})^2}{m^2c^2}}}\,\frac{\vec{p} - \frac{e}{2c}\,\vec{B}\times\vec{r}}{m}\,, \\ v_2 &= \frac{1}{\sqrt{1 + \frac{(\vec{p} - \frac{e}{2c}\,\vec{B}\times\vec{r})^2}{m^2c^2}}}\,\frac{p_2 - \frac{e}{2c}\,(B_3 x_1 - B_1 x_3)}{m}\,. \end{aligned} \tag{0.11}$$

(b) Find eom of the particle.

Answer: We have already found $\frac{\partial L}{\partial v_i}$, and calculating $\frac{\partial L}{\partial x_i}$, we get

$$\begin{aligned}
&\frac{\partial L}{\partial v_i} = \frac{m v_i}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{e}{2c}\,\epsilon_{ijk}\,B_j x_k\,, \quad \frac{\partial L}{\partial x_i} = e\,E_i - \frac{e}{2c}\,\epsilon_{ijk}\,B_j v_k \;\Rightarrow \\
&\frac{d}{dt}\frac{m v_i}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{e}{2c}\,\epsilon_{ijk}\,B_j v_k = e\,E_i - \frac{e}{2c}\,\epsilon_{ijk}\,B_j v_k \;\Rightarrow \\
&\frac{d}{dt}\frac{m v_i}{\sqrt{1-\frac{v^2}{c^2}}} = e\,E_i - \frac{e}{c}\,\epsilon_{ijk}\,B_j v_k \;\Rightarrow \\
&\frac{m\dot v_i}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{m v_i v_j \dot v_j}{c^2(1-\frac{v^2}{c^2})^{3/2}} = e\,E_i - \frac{e}{c}\,\epsilon_{ijk}\,B_j v_k\,.
\end{aligned} \tag{0.12}$$

(c) Show that in the absence of the electric field, $\vec{E}=0$, the speed of the particle is constant.

Answer: Multiplying the eom by v_i and taking the sum over i, one gets

$$\frac{m}{2(1-\frac{v^2}{c^2})^{3/2}}\frac{d}{dt}v^2 = e\,E_i v_i\,. \tag{0.13}$$

Thus if $\vec{E}=0$ then $v^2 =$constant.