JF PY1T10 Special Relativity

Lecture 9:

Relativistic Dynamics

Relativistic Dynamics

So far we have considered measurements of:

- Position
- Time
- Velocity

These are **kinematics.**

Now we go on to consider:

- Momentum
- Mass
- Force

These are **dynamics.**

Newton's Laws & Classical Dynamics

Newton's Laws:

$$\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p}$$

and

$$\boldsymbol{p} = m\boldsymbol{v}$$

If we assume mass is independent of velocity, then:

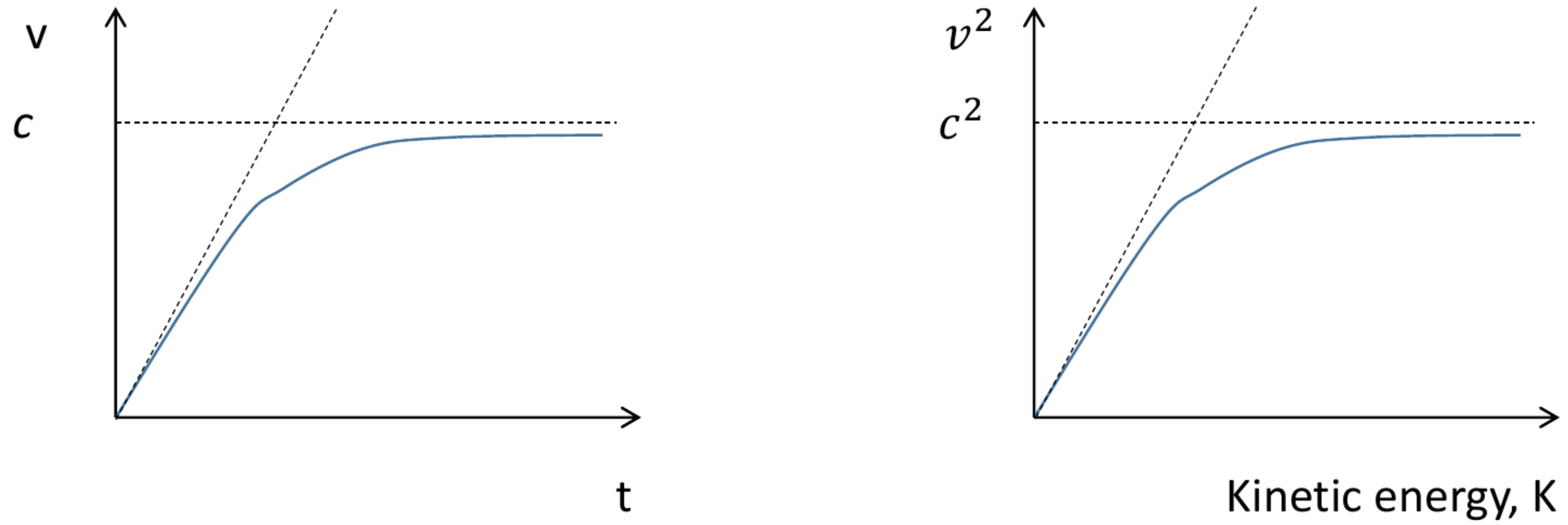
$$\boldsymbol{F} = m\boldsymbol{a}$$

$\boldsymbol{F} = m\boldsymbol{a}$ implies that, under a constant $\boldsymbol{F}$, $\boldsymbol{v}$ will increase indefinitely.

But will it?

Newton's Laws & Classical Dynamics

Can test this. Consider an electron. Because of its small mass relative to its charge, we can accelerate it to very high speeds (in an electric field):



Remember:
$KE = ½\, mv^2$

Instead of increasing in proportion to the KE, v^2 asymptotically approaches a limit, c^2.

i.e. You can give the particle as much energy as you please, but you cannot give it an arbitrarily high speed. This is limited by *c*.

Classical Dynamics

v approaches c, but does not reach it.

If m is constant, then $v \rightarrow c$ (const.)

Therefore $p \rightarrow constant$. Even though $F > 0$.

This is <u>inconsistent</u> with $\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p}$

To solve this, we need to assume that the mass of an object depends on its velocity.:

$$m = m(\boldsymbol{v})$$

Need to find the form of this relationship.

Classical Dynamics

Consider an elastic collision between two particles:

Assume there are two experimenters – one in *S*, the other in *S′*.

In *S*: The experimenter is at rest. They project a particle **A** along the y-axis with a speed u_0 (as measured in *S*)

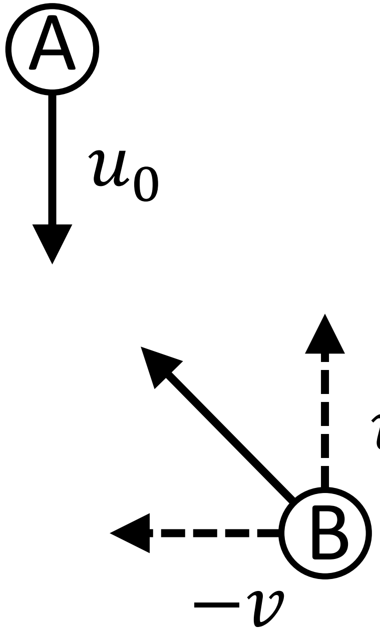
In *S′*: The experimenter is at rest. They project a particle **B** along the y-axis with a speed $-u_0$ (as measured in *S′*)

u_0 is small but *S* and *S′* have a very large relative velocity to each other along x.

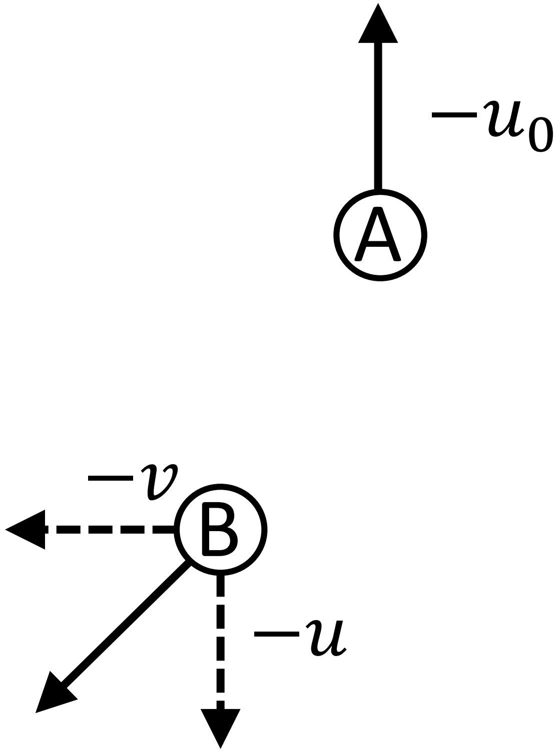
Classical Dynamics

① **In *S*:**

Before collision:



After collision:



Classical Dynamics

Look at y-components:

y-component of **A** = $+u_0$ before collision

= $-u_0$ after collision

y-component of **B** = $+u$ before collision

= $-u$ after collision

<u>What is u as measured in *S*?</u>

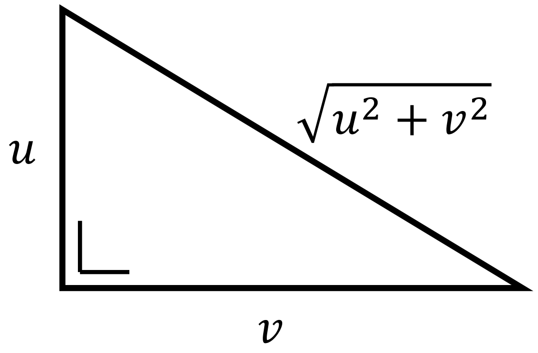
$u_y = \frac{u'_y/\gamma}{1+ {vu_x}/{c^2}}$ (In S', u'_x of **B** is 0.), $\qquad u = {u_0}/{\gamma} = u_0\sqrt{1-\frac{v^2}{c^2}}$

Classical Dynamics

② **In S':**

The situation is symmetric, **A** and **B** are interchanged and the sign of v is reversed.

③ It is an elastic collision, so the speed doesn't change. It is either u_0 or $\sqrt{u^2+v^2}$



So there are only two possible values for m:

$m(u_0)$ or $m(\sqrt{u^2+v^2}) \equiv m(V)$

Classical Dynamics

Conservation of momentum (in y, as measured in *S*)

$$m(u_0)u_0 - m(V)u = -m(u_0) + m(V)u$$

$$2m(u_0)u_0 = 2m(V)u$$

$$\frac{m(V)}{m(u_0)} = \frac{u_0}{u} \qquad \text{Eq. ①}$$

Let $u_0 \to 0$. Then $m(u_0) = m_0$ (the rest mass)

Also $u_0 \ll \mathrm{v}$.

$\Rightarrow u \ll v$ as $u = {}^{u_0}/_{\gamma}$

$\Rightarrow V \cong v$

Classical Dynamics

So from, Eq. ① and letting $u_0 \to 0$

$$\boxed{\begin{gathered} m(v) = \gamma m_0 \\[2em] m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \end{gathered}}$$

Then:

$$\boldsymbol{p} = m(\boldsymbol{v})\boldsymbol{v} = \gamma m_0 \boldsymbol{v}$$

Classical Dynamics

Let us consider this expression for inertial mass more closely:

$$m(v) = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Use the binomial expansion:

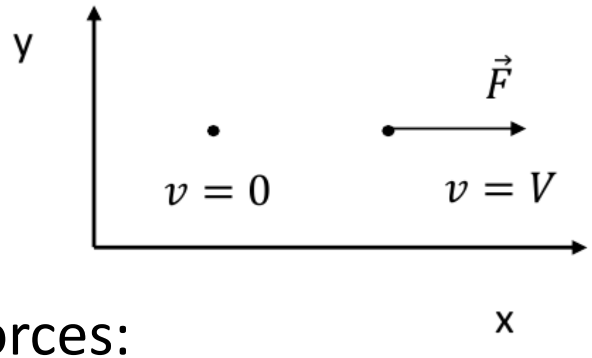
$$m = m_0 \left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots \right]$$

$$\Delta m = m - m_0 = \frac{1}{2}\frac{m_0 v^2}{c^2}\left[1 + \frac{3}{4}\frac{v^2}{c^2} + \cdots \right]$$

$$\Delta m c^2 = \underbrace{\frac{1}{2} m_0 v^2}\left[1 + \frac{3}{4}\frac{v^2}{c^2} + \cdots \right]$$

This is just classical KE.

Relativistic Dynamics



A change in kinetic energy corresponds to the work done by external forces:

$$dE = F\,dx$$

$$F = \frac{d}{dt}p = \frac{d}{dt}(\gamma m_0 v) = m_0\gamma\frac{dv}{dt} + m_0 v\frac{d\gamma}{dt} \qquad \text{Eq. ①}$$

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv}\frac{dv}{dt}$$

$$\frac{d\gamma}{dv} = \frac{d}{dv}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} = \frac{\gamma^3 v}{c^2}$$

$$\frac{d\gamma}{dt} = \frac{\gamma^3 v}{c^2}\frac{dv}{dt} \qquad \text{Eq. ②}$$

Relativistic Dynamics

Put this into Eq. ①:

$$F = m_0\gamma \frac{dv}{dt}\left(1 + \frac{\gamma^2 v^2}{c^2}\right)$$

But $\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}}, \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$

$$F = m_0\gamma \frac{dv}{dt}\left(1 + \left(1 - \frac{1}{\gamma^2}\right)\gamma^2\right)$$

$$F = m_0\gamma \frac{dv}{dt}(1 + \gamma^2 - 1)$$

$$F = m_0\gamma^3 \frac{dv}{dt}$$

Relativistic Dynamics

Use Eq. ②:

$$F = m_0 \gamma^3 \frac{c^2}{\gamma^3 v} \frac{d\gamma}{dt} = m_0 \frac{c^2}{v} \frac{d\gamma}{dt} \qquad \text{Eq. ③}$$

Kinetic Energy = $\int_0^x F\, dx = \int_0^t F\, v\, dt$

[Using Eq. ③: $F = m_0 \frac{c^2}{v} \frac{d\gamma}{dt} \Rightarrow F\, v\, dt = m_0 c^2\, d\gamma$]

$$KE = m_0 c^2 \int_1^\gamma d\gamma$$

$$KE = m_0 c^2\, (\gamma - 1)$$

where the upper limit of $\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}$

Change in limits:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\gamma(t = 0) = \gamma\big(v(t = 0)\big) = \gamma(v = 0) = 1$$

$$\gamma(t) \equiv \gamma(V)$$

Relativistic Dynamics

So:

$$\boxed{\begin{gathered} F = m_0\gamma^3 \frac{dv}{dt} \\ KE = m_0c^2\,(\gamma - 1) \end{gathered}}$$

The kinetic energy in a particular reference frame can be defined as the difference between the energy an object has when it is moving and at rest:

$$KE = mc^2 - m_0c^2$$

or

$$mc^2 = KE + m_0c^2$$

Relativistic Dynamics

Call mc^2 the total energy E.

Then

$$E = mc^2 = KE + m_0c^2$$

So:

The total energy = kinetic energy + rest mass

- Energy is conserved
- Energy has mass
- Mass is conserved (not necessarily the rest mass)
- Mass and energy are different physical quantities.

A Useful Relation

$$p = \gamma m_0 v$$
$$E = \gamma m_0 c^2$$
$$\therefore E^2 - p^2c^2 = \gamma^2 m_0^2 c^4 - \gamma^2 m_0^2 v^2 c^2 = m_0^2 c^4 \gamma^2 \left(1 - \frac{v^2}{c^2}\right)$$
$$E^2 - p^2c^2 = m_0^2 c^4$$

But, m_0 and c are invariant under a Lorentz Transform:

$\Rightarrow E^2 - p^2c^2$ is also invariant under a L.T.

$\Rightarrow E^2 - p^2c^2$ is the same for all observers!

Summary

$$p = mv$$

$$E = mc^2 = KE + m_0c^2$$

$$m = \gamma m_0$$

$$E^2 - p^2c^2 = m_0^2c^4$$

Photons

$v = c$ for all observers

Since $m = \gamma m_0$ and $m_0 = m\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

$\Rightarrow m_0 = 0$

Using $E^2 - p^2c^2 = m_0^2c^4 = 0$

$\Rightarrow E^2 = p^2c^2$

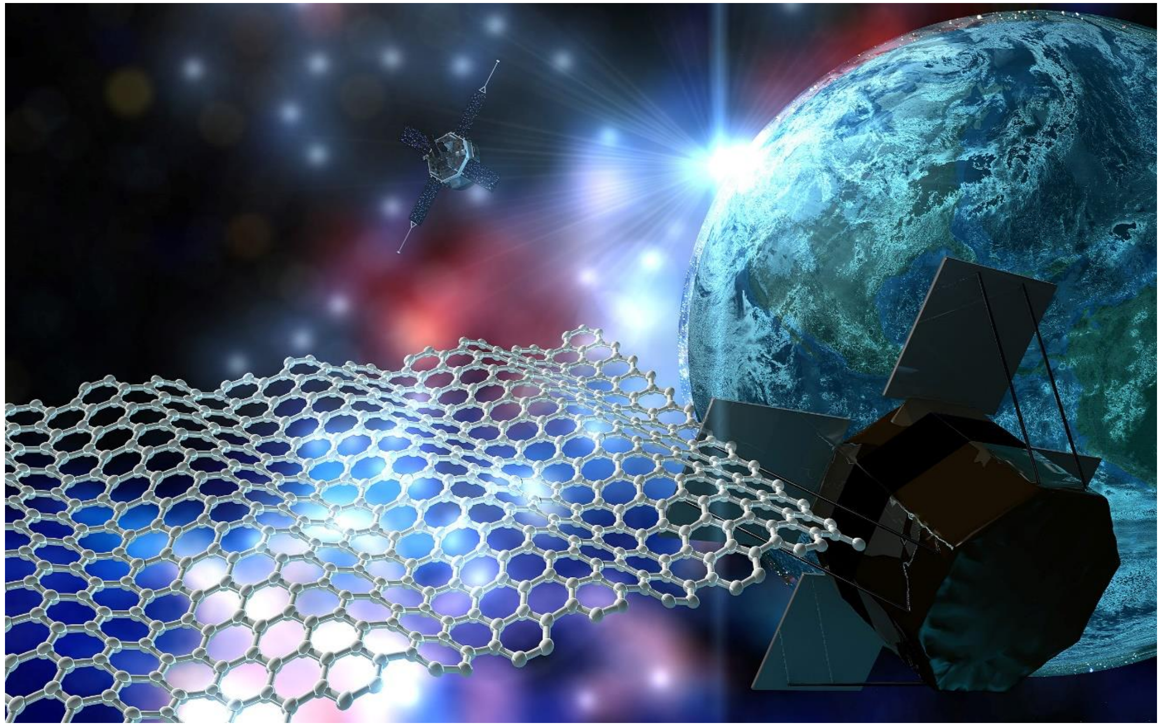
$\Rightarrow p = \frac{E}{c} = \frac{h\nu}{c}$

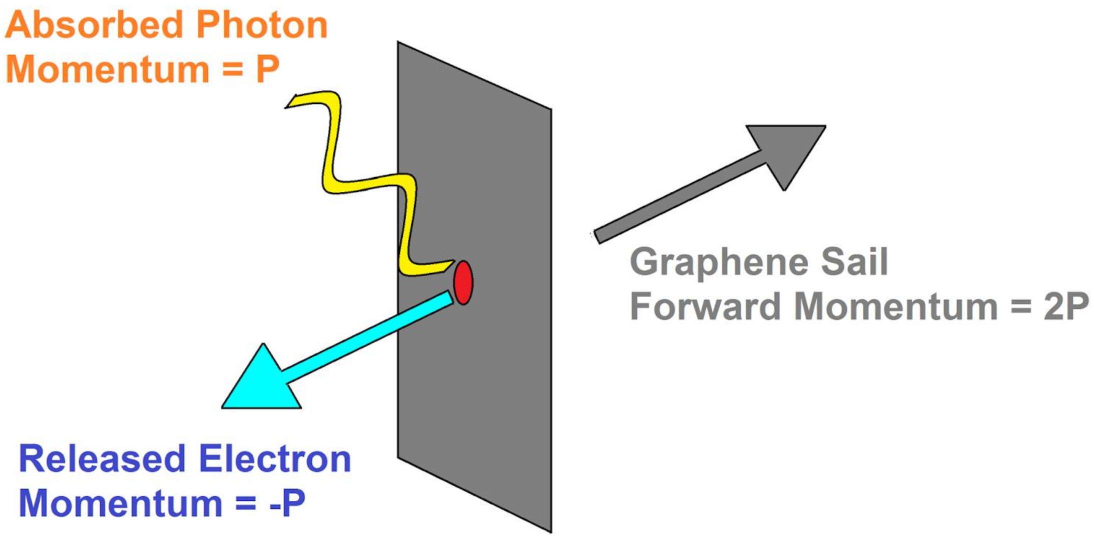
Light (photon) has momentum.

Example: laser cooling for Bose-Einstein condensation of atoms

Graphene Light Sail

Aim: Propel a space probe to the closest star system using a solar sail made from an ultralight material onto which laser light is shone from Earth.





Concept Question

A proton, with rest mass equal to $1.67 \text{ x } 10^{-27}$kg, is accelerated from rest by a constant force of $3.34 \text{ x } 10^{-11}$ N to a speed of 0.9*c*.

(i) Using the non-relativistic from of Newton's second law, calculate the distance travelled by the proton.

(ii) Using the correct, relativistic, approach, find the distance travelled.