Quantum Physics PY1T20/PYU11P20

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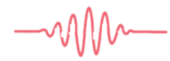
Lecture 6: Uncertainty principle

- The Uncertainty Principle
- Analysis in terms of waves
- Thought-experiments
- - microscope
- - single slit and 2-slit diffraction
- “Practical” applications
- - propagation of wave group
- - minimum energy of confinement
- Alternative *E*-*t* form

The Uncertainty Principle



Suppose we have a wavegroup. Where is this particle?: somewhere within the length of the wave group? ***Most probably*** in the middle, where $|\psi|^2$ is greatest. To be more precise, need *narrower* wave group!

Problem:
the wavelength of a narrow wave group is poorly defined! "not enough oscillations to measure λ accurately".
Therefore, using $p = h/\lambda$, momentum is poorly defined…..
…need a wider wave group!

Problem:
Now p is better defined but the <u>position</u> of the particle is not!

The Uncertainty Principle

Heisenberg (1927): *"It is impossible to know the exact position and exact momentum of an object at the same time".*

99+?% of physicists now: "*It is impossible for an object to have an exact position and exact momentum at the same time*".

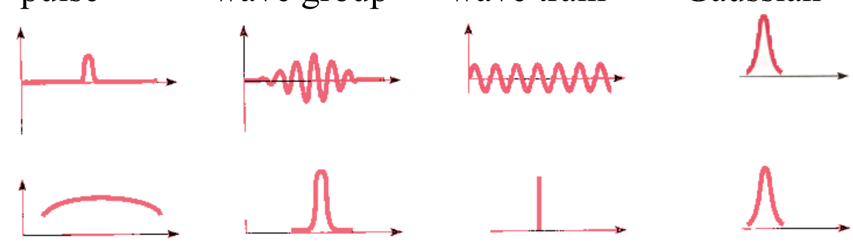
In general, there is an uncertainty in position (Δx)
and in momentum (Δp);
Δx and Δp are "inversely" related:
reduce Δx (shorten the wave group), find Δp increases
reduce Δp (lengthen the wave group), then Δx increase

Is this quantifiable? Yes : their <u>product</u> cannot be less than a certain minimum!

Uncertainty Principle analysed

- Think about the wavefunction $\psi(x)$ as a sum of harmonic waves $\psi(x) = \sum_k g_k e^{ikx}$
- Or, as the wavevector is continuous, as an integral of harmonic waves $\psi(x) = \int_{-\infty}^{\infty} g(k) e^{ikx} dk$

"pulse" "wave group" "wave train" "Gaussian"



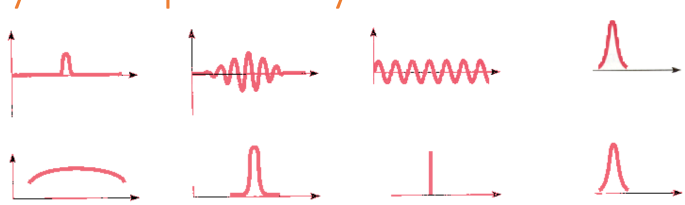
Forms of wave in space

Corresponding spectra g(k) *(Fourier transforms)*

"wave train" has a <u>single</u> value of *k*
"wave group" has a <u>narrow</u> range of *k*
"pulse" has a <u>broad</u> range of *k*
the characteristic half-widths Δx (top) and Δk (bottom) inversely related;

<u>minimum</u> value of the product $\Delta x . \Delta k$ is the "Gaussian" case!

Uncertainty Principle analysed



<u>minimum</u> value of the product $\Delta x \Delta k$ is the "Gaussian" case!

$$Gaussian \quad : \Delta x . \Delta k = 1/2 \quad \Rightarrow \; in \; general \quad : \Delta x . \Delta k \geq 1/2$$

$$Now \;\; p = \frac{h}{\lambda} = \hbar k \;\; so \;\; \Delta p = \hbar \Delta k$$

$$\Delta x \Delta k \geq \frac{1}{2} \Rightarrow \boxed{\Delta x \Delta p \geq \frac{\hbar}{2} = \frac{h}{4\pi}}$$

"Heisenberg's uncertainty principle"

Uncertainty Principle in practice

- What does this mean in practice?
- It is a statement about the possible forms of wavefunction – and hence the sort of spread we get if we measure x in a particular wavefunction, or measure p in that same wavefunction.
- How can we determine Δx? If we measure we just get one x, with some probability. "Spread??"

- Can we go ahead and measure position again? No! Wavefunction collapse so we'd get $\Delta x = 0$
- We need many copies of the wavefunction, and to measure position in them. E.g. we might measure the position of the electron in the ground state of hydrogen – on different atoms. Each time we'll get some value, and the spread of these values is Δx. If we do the measurements of p instead (on more atoms), we get a spread Δp .

- The H-U principle is that, in this 'measurements on an ensemble', $\Delta x \Delta p \geq \frac{\hbar}{2} = \frac{h}{4\pi}$

- Alternative: collect data on one atom, but wait between measurements so it returns to the ground state.

- Normally uncertainties are defined as standard deviations: Δx is the standard deviation of the data you would get by measuring the position of the particle, starting in the same wavefunction every time.

Uncertainty and Error?

Uncertainty looks like some sort of "experimental error" - **It is not!**

Experimental error can be arbitrarily reduced by better experiment.

But ultimately experiments measure quantities which, in quantum physics, are themselves random $\therefore$ no matter how good your experiment, you will always get a spread of values when you repeat it.

This fundamental limit is quantified by the uncertainty principle $\boxed{\Delta x \Delta p \geq \hbar/2}$

which is a property of the wave nature of matter.

Note central role of Planck's constant h or $\hbar = h/(2\pi)$

(later) will see that $\hbar$ is basic unit of angular momentum

Uncertainty and Wavefunction Collapse

"A measurement of one of x or p alters the value of the other"

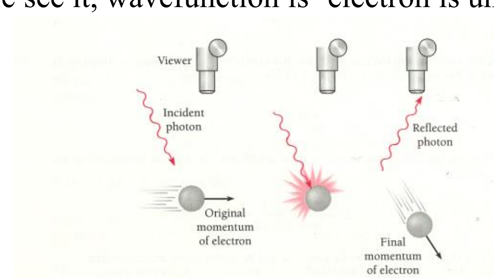
This statement combines *two* principles: wavefunction collapse and the uncertainty principle.

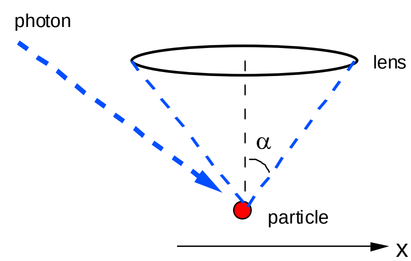
Uncertainty in 'wave' experiments?
- Microscope and Diffraction

Thought-Experiment (microscope)

NB we are now applying uncertainty principle to the wavefunction created by the 'collapse'

Use optical microscope to find particle (e.g. electron) position.
"see" the electron by scattering a photon <u>into the lens.</u>
After we see it, wavefunction is 'electron is under lens' with some Δx (=resolution).



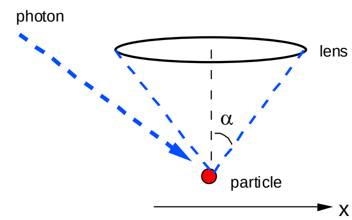


…..<u>anywhere</u> within the lens angle 2α.
Photon momentum ($p=h/\lambda$) change causes recoil of electron!

Along <u>horizontal</u>, change ranges from $-p\sin\alpha$ to $+p\sin\alpha$
i.e. range in photon momentum $\Delta p = 2p\sin\alpha = 2(h/\lambda)\sin\alpha$
…….which becomes the uncertainty in particle momentum

Thought-Experiment (microscope)

Uncertainty in particle position associated with "diffraction limit" : minimum separation of points is



$\Delta x = \lambda / \sin \alpha$

$\Delta x . \Delta p = (\lambda / \sin \alpha)(2(h/ \lambda) \sin \alpha) = 2h$

$$\Delta x . \Delta p \geq h/2$$

How could we "improve" microscope?

by decreasing λ : decreasing Δx but increasing Δp?
by decreasing α : decreasing Δp but increasing Δx?

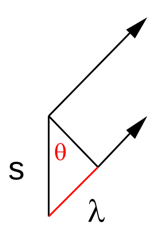
Quantum concept of photon is ***intrinsic***:

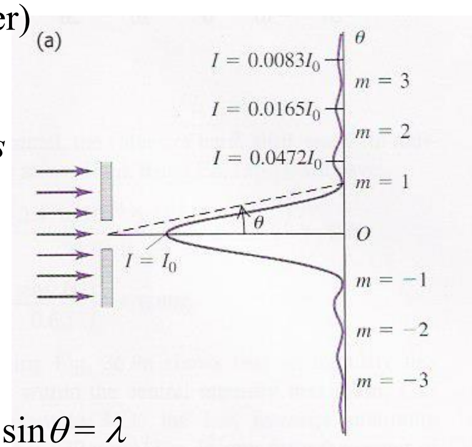
Classically, could decrease Δx without increasing Δp
(lower intensity and wait?)

Thought-Experiment (single-slit)

Remove "complication" of photon and electron by single-slit diffraction (of either)

Slit width (s) is the uncertainty in position: $\Delta x = s$





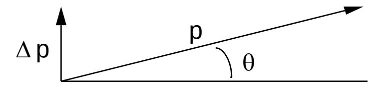
At 1st diffraction minimum: $s \sin\theta = \lambda$

Therefore, $\Delta x = s = \lambda / \sin \theta = h/p \sin \theta$

Thought-Experiment (single-slit)

Electron (or photon) arriving within central maximum must be deflected through angle range 0 to θ: this means uncertainty in tranverse momentum :

If momentum is p, then, $\Delta p = p\sin\,\theta$



$\Delta x.\ \Delta p = (h/p\sin\theta)(p\sin\theta) = h$

Again, this wave/particle physics is consistent with u.p. $\Delta x.\Delta p \geq h/2$

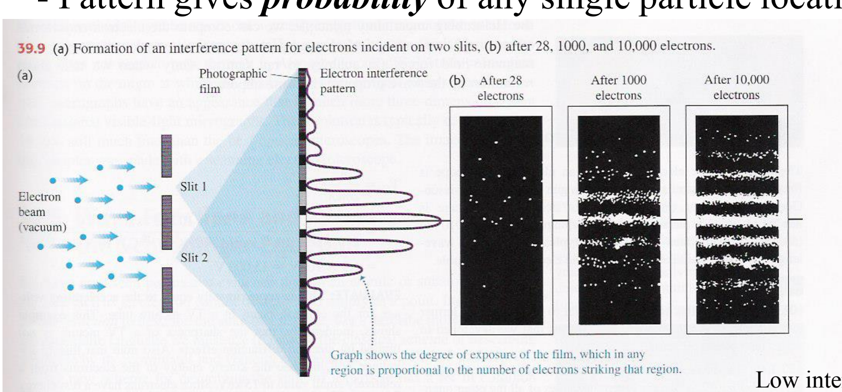
*Equivalent analysis of **Young's (Two) Slits** using 1st maximum, Where slit <u>separation</u> is the uncertainty in position (exercise)*

Q: "which slit does the particle (or photon) go through?" !!

Two-slit experiment

Observe:

- Close one slit *(i.e. the particle must go through the other)*
  $\Rightarrow$lose the 2-slit diffraction pattern!
- Single particle causes single point of scintillation
  $\Rightarrow$ pattern results from addition of many particles!
- Pattern gives ***probability*** of any single particle location



39.9 (a) Formation of an interference pattern for electrons incident on two slits, (b) after 28, 1000, and 10,000 electrons.

G.I.Taylor
Low intensity beam

Two slit experiment - summary

(1) Both slits required to give pattern, even for single particle

(2) Single particle arrives at single point. i.e. "explores" all regions available *(see 1)*, but occupies only one point when actually "measured"

(3) Arrival of individual particle conforms to statistical pattern of diffraction (complementarity).

(4) Average over many particles gives standard diffraction pattern. (complementarity)

Key features of quantum mechanics!

"Practical" applications of UP: propagation of a wave group?

Establish particle position to an uncertainty Δx_o at time zero: what is uncertainty Δx_t at <u>later</u> time t?

UP implies $\Delta p \geq h/2\Delta x_0$ and $p = mv$

So $\Delta v = \Delta p/m \geq h/2m\Delta x_0$

uncertainty in velocity implies uncertainty in position at time t

$$\Delta x_t = \Delta v.t \geq \hbar t / 2m\Delta x_o$$

$\Delta x_t \propto t$: uncertainty in position increases with time (dispersion)

$\Delta x_t \propto 1/\Delta x_o$: "more you know now, less you know later"

Application of UP :
minimum energy of confinement

Rough estimate *KE* of electron in hydrogen atom

(in full, later lecture)

$\Delta x \sim$ radius of H atom = 5.3 x 10^{-11} m
$\Delta p \geq h/4\pi \, \Delta x = 1$ x 10^{-24} kg m s^{-1}

Treat electron as non-relativistic, $KE = p^2/2m_o$
where $p \sim \Delta p$ at least:

$$KE \geq \left(\frac{\Delta p^2}{2m_o}\right) = \frac{(1x10^{-24})^2}{(2)(9.1x10^{-31})} = 5.4x10^{-19}\ J = 3.4\ eV$$

(see later lecture: KE=13.6 eV so correct order of magnitude)

An energy-time 'uncertainty':

$\Delta x. \Delta p \geq h/4\pi$ related to <u>spatial</u> extent needed to measure λ

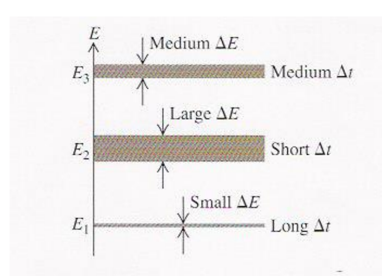
What about the <u>temporal</u> extent needed to measure λ *(or f)*?

- at least one period?

Estimate: $\Delta f. \Delta t \geq 1$ $E=hf$

$\Delta E = h\Delta f$ So $\Delta E \Delta t \geq$ h

(correct maths gives) $\Delta E \Delta t \geq h/2$



Eg. ΔE is the spectral "width"
of optical emission lines,
where Δt is "lifetime" of transition

(see atomic transitions, later)

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