Advanced Calculus
MA1132
Exercises 6 Solutions

1. Given that the functions $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$ and $f(u, v, w)$ are all differentiable, show that if we regard f as a function of x, y and z, then

$$\nabla f = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w.$$

Solution:

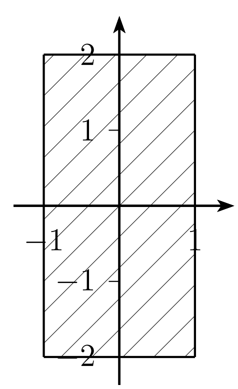
Here we use three applications of the chain rule:

$$\begin{aligned}
\nabla f =& \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\
=& \left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}\right)\mathbf{j} \\
&+ \left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}\right)\mathbf{k} \\
=& \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k}\right) \\
&+ \frac{\partial f}{\partial w}\left(\frac{\partial w}{\partial x}\mathbf{i} + \frac{\partial w}{\partial y}\mathbf{j} + \frac{\partial w}{\partial z}\mathbf{k}\right) \\
=& \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w.
\end{aligned}$$

2. Find the maximum and minimum values of the function $f(x, y) = x^3 + x^2 - x - y^3 - y^2 + y$ on and inside the rectangle bounded by the lines $x = -1$, $x = 1$, $y = -2$ and $y = 2$.

Solution: We first look for critical points in the interior of the rectangle.
Now $\nabla f = (3x^2 + 2x - 1, -3y^2 - 2y + 1) = (0, 0)$
implies that

$$\begin{aligned}
3x^2 + 2x - 1 &= 2 \qquad (1) \\
-3y^2 - 2y + 1 &= 0 \qquad (2)
\end{aligned}$$



Then (1) implies

$$(3x - 1)(x + 1) = 0 \Longrightarrow x = \frac{1}{3} \text{ or } x = -1.$$

and (2) implies

$$(3y - 1)(-y - 1) = 0 \Longrightarrow y = \frac{1}{3} \text{ or } y = -1.$$

Since these values of x and y can be chosen independently, we have the four critical points $\left(\frac{1}{3}, \frac{1}{3}\right)$, $\left(\frac{1}{3}, -1\right)$, $\left(-1, \frac{1}{3}\right)$ and $(-1, -1)$. The first two of these are inside the rectangle and the other two we will find again when we check the boundaries. To do this we have to check the four sides separately.

On $x = 1$, $-2 \leqslant y \leqslant 2$.
In this case we can express f as a function of y:

$$f(y) = 1 + 1 - 1 - y^3 - y^2 + y = -y^3 - y^2 + y + 1.$$

Hence

$$\begin{aligned} f'(y) = -3y^2 - 2y + 1 = 0 &\Longrightarrow (3y - 1)(-y - 1) = 0 \\ &\Longrightarrow y = \frac{1}{3} \text{ or } y = -1. \end{aligned}$$

So we have to check $\left(1, \frac{1}{3}\right)$ and $(1, -1)$ as well as the endpoints $(1, -2)$ and $(1, 2)$.

On $x = -1$, $-2 \leqslant y \leqslant 2$.
Here we will again express f as a function of y:

$$f(y) = -1 + 1 + 1 - y^3 = y^2 + y = -y^3 - y^2 + y + 1,$$

which is the same as when $x = 1$, so again we get $y = \frac{1}{3}$ or $y = -1$. Thus we have to check $\left(-1, \frac{1}{3}\right)$ and $(-1, -1)$ (these are the critical points we found above) as well as the endpoints $(-1, -2)$ and $(-1, 2)$. Note we have now checked all the endpoints, so I won't mention them below.

On $y = 2$, $-1 \leqslant x \leqslant 1$.
In this case we will express f as a function of x:

$$f(x) = x^3 + x^2 - x - 8 - 4 + 2 = x^3 + x^2 - x - 10.$$

Hence

$$f'(x) = 3x^2 + 2x - 1 = 0 \Longrightarrow (3x - 1)(x + 1) = 0 \Longrightarrow x = \frac{1}{3} \text{ or } x = -1.$$

Thus we have to check $\left(\frac{1}{3}, 2\right)$ (we have already found $(-1, 2)$ above).

On $y = -2$, $-1 \leqslant x \leqslant 1$.
Compared to when $y = 2$, the only change in f is in the constant term. Since this doesn't affect the derivative we again get $x = \frac{1}{3}$ or $x = -1$, so we have to check

$\left(\frac{1}{3}, -2\right)$ (we have already found $(-1,-2)$ above).

We now check f at the twelve points we have found.

$$\begin{aligned}
f\left(\frac{1}{3}, \frac{1}{3}\right) &= \frac{1}{27} + \frac{1}{9} - \frac{1}{3} - \frac{1}{27} - \frac{1}{9} + \frac{1}{3} = 0 \\
f\left(\frac{1}{3}, -1\right) &= \frac{1}{27} + \frac{1}{9} - \frac{1}{3} + 1 - 1 - 1 = -\frac{32}{27} \\
f\left(1, \frac{1}{3}\right) &= 1 + 1 - 1 - \frac{1}{27} - \frac{1}{9} + \frac{1}{3} = \frac{32}{27} \\
f(1,-1) &= 1 + 1 - 1 + 1 - 1 - 1 = 0 \\
f(1,-2) &= 1 + 1 - 1 + 8 - 4 - 2 = 3 \\
f(1,2) &= 1 + 1 - 1 - 8 - 4 + 2 = -9 \\
f\left(-1, \frac{1}{3}\right) &= -1 + 1 + 1 - \frac{1}{27} - \frac{1}{9} + \frac{1}{3} = \frac{32}{27} \\
f(-1,-1) &= -1 + 1 + 1 + 1 - 1 + 1 = 2 \\
f(-1,-2) &= -1 + 1 + 1 + 8 - 4 - 2 = 3 \\
f(-1,2) &= -1 + 1 + 1 - 8 - 4 + 2 = -9 \\
f\left(\frac{1}{3}, 2\right) &= \frac{1}{27} + \frac{1}{9} - \frac{1}{3} - 8 - 4 + 2 = -\frac{275}{27} \\
f\left(\frac{1}{3}, -2\right) &= \frac{1}{27} + \frac{1}{9} - \frac{1}{3} + 8 - 4 - 2 = \frac{49}{27}
\end{aligned}$$

Hence, on and inside the rectangle, f attains its maximum of 3 at $(1,-2)$ and $(-1,-2)$, and its minimum of $-\frac{275}{27}$ at $\left(\frac{1}{3}, 2\right)$.

3. Use the method of Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y) = (x-1)^2 + y^2$, subject to the constraint $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$.

   Solution: We first note that the set described by the constraint is closed and bounded (it is an ellipse), so we will get a maximum and a minimum.

   Let $g(x,y) = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 - 1$.

   Then $\nabla g = \left(\frac{2x}{9}, \frac{y}{2}\right) \neq (0,0)$, since if we did have $\left(\frac{2x}{9}, \frac{y}{2}\right) = (0,0)$, we would have $x = y = 0$ and this does not satisfy $g(x,y) = 0$.

   Thus we are justified in using Lagrange multipliers.

   Now $\nabla f = \lambda \nabla g$ yields

$$2x - 2 = \frac{2\lambda x}{9} \tag{3}$$

$$2y = \frac{\lambda y}{2} \tag{4}$$

Then (4) yields $4y = \lambda y$ and there are two cases to consider.

Case 1: $y = 0$.

In this case the constraint equation implies $\left(\frac{x}{3}\right)^2 = 1$, so it follows that $x = \pm 3$.

Since (3) is also satisfied if we let $\lambda = \dfrac{9(1-x)}{x}$, we have the two points $(3,0)$ and $(-3,0)$.

Case 2: $y \neq 0$.

In this case $4y = \lambda y \Rightarrow \lambda = 4$ and then (3) yields

$$2x - 2 - \frac{8x}{9} = 0 \Longrightarrow \frac{10x}{9} = 2 \Longrightarrow x = \frac{9}{5}.$$

Substituting this into the constraint equation gives

$$\left(\frac{3}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \Longrightarrow \left(\frac{y}{2}\right)^2 = \frac{16}{25} \Longrightarrow \frac{y}{2} = \pm\frac{4}{5} \Longrightarrow y = \pm\frac{8}{5}.$$

Thus we also have the two points $\left(\frac{9}{5}, \frac{8}{5}\right)$ and $\left(\frac{9}{5}, -\frac{8}{5}\right)$.

We now check the value of f at the four points we have found.

$$\begin{aligned}
f(3,0) &= 2^2 = 4 \\
f(-3,0) &= (-4)^2 = 16 \\
f\left(\frac{9}{5}, \frac{8}{5}\right) &= \left(\frac{4}{5}\right)^2 + \left(\frac{8}{5}\right)^2 = \frac{16+64}{25} = \frac{80}{25} = \frac{16}{5} \\
f\left(\frac{9}{5}, -\frac{8}{5}\right) &= \left(\frac{4}{5}\right)^2 + \left(-\frac{8}{5}\right)^2 = \frac{16+64}{25} = \frac{80}{25} = \frac{16}{5}.
\end{aligned}$$

Hence, subject to the constraint $g(x,y) = 0$, f attains its maximum of 16 at $(-3,0)$, and its minimum of $\frac{16}{5}$ at $\left(\frac{9}{5}, \frac{8}{5}\right)$ and $\left(\frac{9}{5}, -\frac{8}{5}\right)$.

4. Consider the function
$$f(x,y) = x^4 - x^2 y + y^2 - 3y + 4$$

Locate all relative maxima, relative minima, and saddle points, if any.

Solution: We first find all critical points

$$f_x(x,y) = 4x^3 - 2xy = 0\,, \quad f_y(x,y) = -x^2 + 2y - 3 = 0\,.$$

From the second equation we find y in terms of x

$$y = \frac{x^2}{2} + \frac{3}{2}\,,$$

and substituting it to the first equation, we derive the following equation for x

$$3x^3 - 3x = 0\,.$$

There are three solutions to this equation

$$x = 0\,,\ x = -1\,,\ x = 1\,,$$

and, therefore, three critical points

$$(x = 0\,,\ y = \frac{3}{2})\,, \quad (x = -1\,,\ y = 2)\,, \quad (x = 1\,,\ y = 2)\,.$$

Computing the values of f at critical points, we get

$$f(0, \frac{3}{2}) = \frac{7}{4}\,, \quad f(-1, 2) = 1\,, \quad f(1, 2) = 1\,.$$

To find out if they are maximum, minimum or saddle points we use the second derivative test. To this end we compute

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 12x^2 - 2y\,,\ \frac{\partial^2 f}{\partial y^2}(x, y) = 2\,,\ \frac{\partial^2 f}{\partial x \partial y}(x, y) = -2x\,,$$

and

$$D(x, y) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 20x^2 - 4y\,,$$

Computing D and $\frac{\partial^2 f}{\partial x^2}$ for the three critical points, we get

$$D(0, \frac{3}{2}) = -6\,, \quad \frac{\partial^2 f}{\partial x^2}(0, \frac{3}{2}) = -3\,,$$

and therefore $(0\,,\ \frac{3}{2})$ is a saddle point.

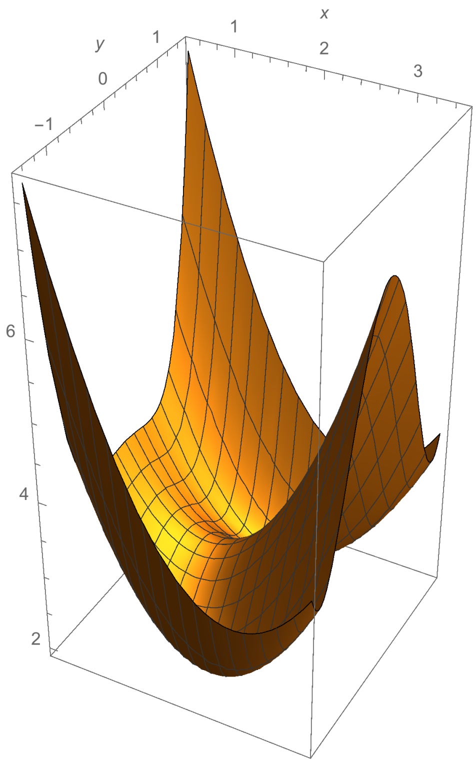
$$D(-1, 2) = 12\,, \quad \frac{\partial^2 f}{\partial x^2}(-1, 2) = 8\,,$$

and therefore $(-1, 2)$ is a relative minimum.

$$D(1, 2) = 12\,, \quad \frac{\partial^2 f}{\partial x^2}(1, 2) = 8\,,$$

and therefore $(1, 2)$ is a relative minimum too.

The graph of the function is shown below



5. Find the distance from the point (x_0, y_0, z_0) to the plane

$$ax + by + cz + d = 0\,.$$

Solution: The distance squared from the point (x_0, y_0, z_0) to a point (x, y, z) on the plane is

$$f(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\,, \tag{5}$$

and (x, y, z) are subject to the constraint

$$g(x,y,z) = ax + by + cz + d = 0\,. \tag{6}$$

The minimum of f is therefore given by

$$2(x-x_0) = \lambda a\,, \quad 2(y-y_0) = \lambda b\,, \quad 2(z-z_0) = \lambda c\,, \tag{7}$$

and therefore

$$x = x_0 + \frac{\lambda}{2}a\,, \quad y = y_0 + \frac{\lambda}{2}b\,, \quad z = z_0 + \frac{\lambda}{2}c\,. \tag{8}$$

Substituting these into the constraint equation one gets

$$ax_0 + by_0 + cz_0 + d + \frac{\lambda}{2}(a^2+b^2+c^2) = 0 \quad \Longrightarrow \quad \frac{\lambda}{2} = -\frac{ax_0+by_0+cz_0+d}{a^2+b^2+c^2}\,. \tag{9}$$

Thus, the distance ρ from the point (x_0, y_0, z_0) to the plane is

$$\rho = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}\,. \tag{10}$$

6. What is the volume of the largest n-dimensional box with edges parallel to the coordinate axes that fits inside the n-dimensional ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} = 1\,. \tag{11}$$

Solution: The volume of a box with edges parallel to the coordinate axes that fits inside the ellipsoid is

$$V(x_1, \ldots, x_n) = 2^n x_1 \cdots x_n\,, \tag{12}$$

where $x_i > 0$ are coordinates of the vertex of the box in the first "octant". The constraint is

$$g(x_1, \ldots x_n) = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} - 1 = 0\,. \tag{13}$$

The maximum of V is therefore given by

$$2^n \frac{x_1 \cdots x_n}{x_k} = \frac{V}{x_k} = \lambda \frac{2x_k}{a_k^2}\,, \qquad k = 1, \ldots, n\,, \tag{14}$$

and therefore

$$2\lambda \frac{x_k^2}{a_k^2} = V \quad \Longrightarrow \quad 2\lambda = nV \quad \Longrightarrow \quad x_k = \frac{a_k}{\sqrt{n}}\,, \tag{15}$$

where we summed over k and used the constraint. Thus, the maximum volume V is

$$V = \left(\frac{2}{\sqrt{n}}\right)^n a_1 \cdots a_n\,. \tag{16}$$