ADVANCED CALCULUS

MA1132

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3. Multiple Integrals

In Chapter 1, we generalized the calculus you were familiar with from your earlier studies to the case where the codomain of the functions was $\mathbb{R}^2$ or $\mathbb{R}^3$. Then in Chapter 2, our generalization was in a different direction. There we looked at functions which had domain $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^n$, but restricted ourselves to studying differential calculus. In this chapter we will be looking again at functions which have $\mathbb{R}^2$ or $\mathbb{R}^3$ as their domains (our work can be generalized reasonably easily in theory to $\mathbb{R}^n$, but in practice, the calculations would become very tricky), but this time we will be studying integral calculus.

The difficulty in integrating such functions is not in finding an indefinite integral (this is really no more difficult than for functions of one variable), it is in finding the upper and lower limits over which we integrate. If we are working with functions with domain $\mathbb{R}$, then the upper and lower limits will be numbers, or perhaps a parameter. However in this chapter, the limits can be numbers or functions of one variable (for functions with domain $\mathbb{R}^2$) or numbers, functions of one variable or functions of two variables (for functions with domain $\mathbb{R}^3$).

Accordingly, we will start off in Section 3.1 with the simplest sort of region we can integrate over (rectangles) and then gradually look at more complicated regions as we proceed through the chapter.

3.1. **Double Integrals.**

As was the case with functions of one variable, integration of functions of two variables can be introduced using Riemann sums. The only difference is that instead of partitioning an interval into smaller intervals, we partition rectangles into smaller rectangles. However MA1132 is a calculus course rather than an analysis course, so we will concentrate on calculating integrals.

Let us first consider a rectangular region R that lies on and inside the lines $x = a$, $x = b$, $y = c$ and $y = d$, where a, b, c and d are real numbers with $a < b$ and $c < d$. In this case we denote the integral of a function $f(x, y)$ over the region R by

$$\iint_R f(x, y)\, dA. \tag{1}$$

As you would expect, this integral gives the signed volume described by the function over R. Of course if we want to find an actual volume, then we would need to split the integral up into an integral (or possibly integrals) over a region where the function is positive and an integral (or integrals) over a region where the function is negative. If these are not rectangles, then we will have to use techniques from later in the chapter to find these integrals.

With functions of one variable, we could find the average value of a function over an interval by dividing the integral over this interval by the length of the interval. For a function of two variables, the corresponding result is that we can find its average value over a region by dividing its integral over the region by the area of the region.

The technique we use to find the integral in Equation (1) is similar to the one we use when performing partial differentiation. When we are integrating in the x direction, we treat all the occurrences of y as a constant, and similarly for x when we are integrating in the y direction. Such an integral in the x direction is denoted $\int_a^b f(x, y)\, dx$ and such an integral in the y direction is denoted $\int_c^d f(x, y)\, dy$.

With this notation, we can also define repeated integrals as follows.

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx \tag{2}$$

and

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy. \tag{3}$$

Remark 3.1.1.

(a) Note that we integrate with respect to the 'inner integral' first.
(b) Some texts write these integrals as

$$\int_a^b \int_c^d dy\, dx\, f(x,y) \quad \text{or} \quad \int_c^d \int_a^b dx\, dy\, f(x,y),$$

respectively

At this stage, it could be that the integrals in Equation (2) and (3) are not equal to each other, or perhaps one or both of them is not equal to the integral over the region defined using Riemann sums. In fact, provided f is continuous, we don't need to worry.

Theorem 3.1.2 (Fubini's Theorem)**.** *Let R be the rectangle defined by the inequalities $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant d$. Then, if f is continuous on R,*

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Remark 3.1.3. Depending on the function f, it may be much easier to use Equation (2) or Equation 3 to find the integral. Deciding which variable to integrate with respect to is an important skill to develop, and it will become more and more important as we move through the chapter.

To finish this section, we note that the sum and multiple rules carry over to double integrals. If f and g are functions of two variables, then

$$\iint_R f(x,y) + g(x,y)\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA,$$

and, if c is a constant, then

$$\iint_R cf(x,y)\, dA = c \iint_R f(x,y)\, dA.$$

3.2. **Double Integrals over Nonrectangular Regions.**

As the title suggests, in this section, we will extend our repertoire to integrals over non-rectangular regions. As you can imagine, the general case is very complicated, so we will restrict out attention to two basic sort of regions, and we start by defining these.

Definition 3.2.1.

(a) A *type I region* is a region bounded on the left and right by the vertical lines $x = a$ and $x = b$ and bounded below and above by the continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \leqslant g_2(x)$ for $a \leqslant x \leqslant b$.
(b) A *type II region* is a region bounded below and above by the horizontal lines $y = c$ and $y = d$ and bounded to the left and the right by the continuous curves $x = h_1(y)$ and $x = h_2(y)$, where $h_1(y) \leqslant h_2(y)$ for $c \leqslant y \leqslant d$.

Remark 3.2.2. While it may seem that this is a very limited range of regions, in fact, by combining various Type I and Type II regions, we can define quite an impressive range of possible regions. It can be shown that if R_1 and R_2 are regions which intersect on a set of 'measure zero' (in $\mathbb{R}^2$ this is a set with zero area, for example a curve) and if $R = R_1 \cup R_2$, then

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA.$$

To enable us to actually calculate integrals, we have the following theorem.

Theorem 3.2.3.

(a) If R is a type I region bounded on the left and right by the vertical lines $x = a$ and $x = b$ and bounded below and above by the continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \leqslant g_2(x)$ for $a \leqslant x \leqslant b$, and on which $f(x,y)$ is continuous, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

(b) If R is a type II region bounded below and above by the horizontal lines $y = c$ and $y = d$ and bounded to the left and the right by the continuous curves $x = h_1(y)$ and $x = h_2(y)$, where $h_1(y) \leqslant h_2(y)$ for $c \leqslant y \leqslant d$, and on which $f(x,y)$ is continuous, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Remark 3.2.4.

(a) The notation $\displaystyle\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$ means that when we have integrated with respect to y, we substitute the function $g_2(x)$ for y in the resulting expression and subtract what we get when we substitute the function $g_1(x)$ for y.

(b) Until you become proficient at finding these integrals, it can help to write them as $\displaystyle\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$, for example, since this helps to remind you which variable you should be substituting for.

(c) Sometimes it can be much easier to integrate over a type I region than a type II region, and vice-versa. Indeed sometimes it might only be possible to integrate over one sort of region (at least without using numerical methods). For example, suppose we want to integrate $f(x,y) = e^{x^2}$ over a region, then we have to integrate with respect to y first since it is not possible to find a definite integral of f with respect to x. However, if we integrate with respect to y first, then depending on the limits for y, we may end up with something we can integrate with respect to x, for example $f(x) = 2xe^{x^2}$. One skill it will be important for you to develop will be to start with the limits for a Type I or II region, and find limits for the opposite sort of region, since this will enable you to choose

which variable you want to integrate with respect to first. We will look at some examples of this in the worked problems.

Before we end this section, we note that as well as using them to find volumes over a region, double integrals can be used to find the area *of* a region. To do this, we simply integrate the function $f(x, y) = 1$ over the region.

3.3. **Double Integrals in Polar Coordinates.**

As you have probably found when studying geometry, Cartesian coordinates are good for some sort of problems, while for other sort of problems, polar coordinates are better. Polar coordinates usually come into their own when there are circles involved in some way. A similar situation holds when finding integrals, and in this section we will look at how polar coordinates can be used to our advantage.

However there are a couple of hurdles to cross before we can proceed. In Definition 3.2.1 in the previous section, we defined Type I and Type II regions, and the first thing we will do is to give the corresponding definition for regions defined using polar coordinates.

Definition 3.3.1. A simple polar region in a polar coordinate system is a region that is enclosed between two rays, $\theta = a$ and $\theta = b$, and two continuous curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

$$a \leqslant b, \quad b - a \leqslant 2\pi \quad \text{and} \quad r_1(\theta) \leqslant r_2(\theta).$$

The boundary defined by $r_1$is called the *inner boundary* and the boundary defined by r_2 is called the *outer boundary*.

The other problem is the area element. In Cartesian coordinates it is simply $dx\,dy$. However in polar coordinates it is slightly more complicated (and it will get even more complicated when we move on to integrals over three variables). Here is the result.

Theorem 3.3.2. *Let R be a simple polar region whose boundaries are the rays $\theta = a$ and $\theta = b$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$. If $f(r, \theta)$ is continuous on R, then*

$$\iint_R f(r, \theta)\, dA = \int_a^b \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta)\, r dr d\theta.$$

Remark 3.3.3.

(a) Note the extra r that appears in the area element.
(b) Sometimes we might want to convert an integral in Cartesian coordinates to an integral in polar coordinates, or vice versa. In this case we not only have to change the limits of integration and the variables, we also have to remember to change $dx\,dy$ into $r\,dr\,d\theta$, or vice versa.

3.4. **Surface Areas; Parametric Surfaces.**

As the title suggests, in this section we will use double integrals to find the areas of surfaces, both where the surfaces are given by the graph of a function, and also when the surfaces are given parametrically.

The first theorem deals with the case where the surface is part of the graph of a function of two variables.

Theorem 3.4.1. *Suppose that f is a function of two variables with continuous first order partial derivatives on the interior of a region R. Then the area of the surface of the graph of f over R is*

$$\iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA.$$

Remark 3.4.2. Remember that if we decide to use polar variables, then as well as substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have to replace the area element with $r\, dr\, d\theta$.

Often it can happen that a surface is more easily described using parametric equations, and indeed sometimes it can be virtually impossible to describe such surfaces using the graph of a function, so we also want to have a theorem that deals with this case.

Before we can state the theorem, we need to do a bit of preparatory work. If we want to parametrize a (one dimensional) curve lying in three dimensions, then we saw in Chapter 1 that we need a vector valued function of one variable of the form

$$\begin{aligned} \mathbf{r}\colon \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto \mathbf{r}(t) = (x(t), y(t), z(t)). \end{aligned}$$

Similarly, if we want to parametrize a (two dimensional) surface lying in three dimensions, then we need a vector-valued function of two variables of the form

$$\begin{aligned} &\mathbf{r}\colon \mathbb{R}^2 \to \mathbb{R}^3 \\ &(u, v) \mapsto \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)). \end{aligned}$$

Remark 3.4.3.

(a) Of course, for a particular surface, we may have to restrict the domain to a subset of $\mathbb{R}^2$.

(b) If we do have an explicit function $f(x, y)$ whose graph we want to parametrize, then we can always do it by taking

$$x = u, \quad y = v, \quad z = f(u, v).$$

Instead of Cartesian coordinates, we can also use cylindrical coordinates, and in this case the parametrization is

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = f(r\cos(\theta), r\sin(\theta)).$$

Sometimes, for example for a sphere, it can also be possible to use spherical coordinates. In this case we substitute $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$ and $z = \rho\cos(\phi)$. This only works if one of the variables ρ, θ or ϕ vanishes for a particular surface though.

(c) A lot of surfaces can be thought of as rotations of a curve about one of the coordinate axes, and in this case, it is easy to parametrize them. For example, if we want to rotate the graph of the function $f(x)$ about the x axis, then a suitable parametrization is

$$x = u, \quad y = f(u)\cos(v), \quad z = f(u)\sin(v),$$

where v is the angle of rotation about the x-axis.

In MA1132, we won't study vector-valued functions of several variables in any detail, but we do need the following.

Definition 3.4.4. If $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ is a vector-valued function of two variables, then its partial derivatives with respect to u and v given by

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

We are now in a position to state the theorem which gives the surface area of a parametric surface.

Theorem 3.4.5. *Suppose that a surface is defined parametrically by a function* $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$. *Then the area of the surface of the graph of* f *defined by a region* R *in the* uv*-plane is given by*

$$\iint_R \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA,$$

provided that $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0}$ *for all points in the interior of* R.

Remark 3.4.6.

(a) The vector $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}$ gives the direction of a normal to the surface at a point, so we can easily find the equation of a tangent plane to a surface using this.

(b) The unit vector $\dfrac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left\| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right\|}$ (provided that $\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0}$) is called the *principal unit normal vector* to the surface at that point and is often denoted by $\mathbf{n}$ or $\mathbf{n}(u, v)$.

3.5. **Triple Integrals.**

When we moved from integrating functions of one variable to integrating functions of two variables, the main difficulty we encountered was in describing the region we were integrating over, the actual integration was not any harder, and indeed sometimes it turned out to be easier since we had freedom to decide which variable to integrate with respect to first. The same will be true of the move from integrating functions of two variables to functions of three variables.

In fact things are a bit more complicated for functions of three variables since, while in two dimensions, the only common coordinate systems are Cartesian and polar, in three dimensions, we have Cartesian, cylindrical polar and spherical polar coordinate systems to contend with. In this section, we will just look at integrating with respect to the Cartesian system, in the next section we will look at cylindrical polar and spherical polar coordinate systems, and in the final section, we will look at how to convert between these coordinate systems.

Even using Cartesian coordinates, the regions are more complicated. For regions in $\mathbb{R}^2$, we only looked at Type I and Type II regions, where we had constant limits for one of the variables and a function for the other variable. However for regions in $\mathbb{R}^3$, there can potentially be functions for two variables, and indeed for one of the variables, the functions can be functions of two variables.

However, let us start off with things that are the same or similar for functions of three variables as they were for functions of two variables.

As was the case with functions of one and two variables, integration of functions of three variables can be introduced using Riemann sums. This time we would divide volume elements into smaller volume elements. However we won't do this in MA1132.

Let us first consider a rectangular box G that lies on and inside the lines $x = a$, $x = b$, $y = c$, $y = d$, $z = f$ and $z = g$, where a, b, c, d, f and g are real numbers with $a < b$, $c < d$ and $f < g$. In this case we denote the integral of a function $f(x, y, z)$ over the region G by

$$\iiint_G f(x, y, z)\, dV.$$

Perhaps a physical interpretation is not quite as obvious for functions of three variables, as it was for functions of two variables. For functions of two variables, we considered the integrals as volumes in three dimensions. However if we try that for functions of three variables, we end up needing four dimensions, which is not very helpful. However there are still a couple of interpretations that you may find useful. Firstly, if we integrate the function $f(x, y, z) = 1$ over a (three dimensional) region G, we get the volume G, and also if a function $f(x, y, z)$ represents the density of a material, then integrating over a (three dimensional) region G gives the total amount of material in the region. Of course if we divide the integral by the volume of the region, we get the average density of the material, or equivalently, the average value of the function.

As was the case with regions in two dimensions, we can split the region up and add the integrals: If G_1 and G_2 are three dimensional regions which intersect on a set of 'measure zero' (in $\mathbb{R}^3$ this is a set with zero volume, for example a two dimensional surface) and if $G = G_1 \cup G_2$, then

$$\iiint_G f(x,y,z)\,dV = \iiint_{G_1} f(x,y,z)\,dV + \iiint_{G_2} f(x,y,z)\,dV.$$

Another thing that carries over is Fubini's Theorem.

Theorem 3.5.1 (Fubini's Theorem)**.** *Let G be the rectangular box defined by the inequalities $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ and $f \leqslant z \leqslant g$. Then, if f is continuous on G,*

$$\iiint_G f(x,y,z)\,dV = \int_a^b \int_c^d \int_f^g f(x,y,z)\,dz\,dy\,dx.$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

As you might expect, the sum and multiple rules also carry over to triple integrals. If f and g are functions of three variables, then

$$\iiint_G f(x,y,z) + g(x,y,z)\,dV = \iiint_G f(x,y,z)\,dV + \iiint_G g(x,y,z)\,dV,$$

and, if c is a constant, then

$$\iiint_G cf(x,y,z)\,dV = c\iiint_G f(x,y,z)\,dV.$$

Having covered integrating over rectangular boxes, we now move onto more general regions. We start with a definition.

Definition 3.5.2. Suppose a solid G is bounded above by a surface $z = g_2(x,y)$ and below by a surface $z = g_1(x,y)$ and that the projection of the solid onto the xy-plane is a Type I or a Type II region R. If in addition, $g_1(x,y)$ and $g_2(x,y)$ are continuous on R and that $g_1(x,y) \leqslant g_2(x,y)$ on R, then we say that G is a *simple xy-solid*.

The following theorem gives a method for integrating over simple xy-regions.

Theorem 3.5.3. *Let G be a simple xy-solid with upper surface $z = g_2(x,y)$ and lower surface $z = g_1(x,y)$, and let R be the projection of G onto the xy-plane. If $f(x,y,z)$ is continuous on G, then*

$$\iiint_G f(x,y,z)\,dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right] dA.$$

Remark 3.5.4. As you might expect, we can give analogous definitions of a simple xz-solid and a simple yz-solid, leading to corresponding versions of Theorem 3.5.3. If you are given the volume G, then you have to decide which version of the theorem to use, and this will often also depend on the function f.

3.6. **Triple Integrals in Cylindrical and Spherical Coordinates.**

As we saw in Section 3.3, it can often be much easier to perform a double integral using polar coordinates rather than Cartesian coordinates. The situation is similar for triple integrals, and indeed, the difference in difficulty depending on which coordinate system is chosen can be even greater.

As usual, we will not go into the details of the Riemann integration, we will focus on performing the integration.

Let us start with integration using cylindrical coordinates.

Theorem 3.6.1. *Let G be a solid region whose upper surface is given by the equation $z = g_2(r, \theta)$ and whose lower surface is given by the equation $z = g_1(r, \theta)$. If the projection of G onto the $r\theta$-plane is a simple polar region R and if $f(r, \theta, z)$ is continuous on G, then*

$$\iiint_G f(r, \theta, z)\, dV = \iint_R \left[\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z)\, dz \right] dA.$$

If in addition , R is a simple polar region bounded by the rays $\theta = \theta_1$ and $\theta = \theta_2$, with $\theta_1 < \theta_2$, and by the curves $r = r_1(\theta)$ and $r = r_2(\theta)$, with $r_1(\theta) < r_2(\theta)$, then

$$\iiint_G f(r, \theta, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z)\, r\, dz\, dr\, d\theta. \tag{4}$$

Remark 3.6.2.

(a) Note that the volume element in Equation (4) is $r\, dz\, dr\, d\theta$.
(b) It follows that if we are given an integral of a function in terms of Cartesian coordinates, then to change the integral to one in terms of cylindrical coordinates, then we have to:
   - Find the new limits expressed in terms of cylindrical coordinates.
   - Substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into the rule of the function.
   - Change the volume element from $dx\, dy\, dz$ to $r\, dz\, dr\, d\theta$.

(c) It is of course possible to integrate in a different order.

For spherical coordinates it would take us quite a bit of effort to state a theorem corresponding to Theorem 3.6.1. This is since in Theorem 3.6.1, we could project from the z direction onto a simple polar region, but there is no equivalent projection for spherical coordinates.

However the main thing is that we are able to perform the integral, and the formula is as follows.

$$\iiint_G f(\rho, \theta, \phi)\, dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho, \theta, \phi)\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta. \tag{5}$$

Remark 3.6.3.

(a) Note that the volume element in Equation (5) is $\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$.

(b) It follows that if we are given an integral of a function in terms of Cartesian coordinates, then to change the integral to one in terms of spherical coordinates, then we have to:
- Find the new limits expressed in terms of spherical coordinates.
- Substitute $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$ and $z = \rho\cos(\phi)$ into the rule of the function.
- Change the volume element from $dx\,dy\,dz$ to $\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$.

(c) It is of course possible to integrate in a different order.

3.7. **Change of Variables in Multiple Integrals; Jacobians.**

When dealing with double and triple integrals, we have often found the integral 'by inspection'. However sometimes it might not be obvious what the integral is, so in this final section, we will generalize the technique of integration by substitution to integrals of several variables.

Before we embark on this generalization, let us have another look at integration by substitution for functions of one variable, and look at it in a slightly different way to the one you are probably used to.

Let us first consider the special case where we are making the substitution $x = g(u)$ and g is differentiable and either strictly increasing or decreasing on the interval $[a, b]$. In both these cases g has an inverse and

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,du.$$

Now if g is strictly increasing then $g^{-1}(b) > g^{-1}(a)$, and the lower limit is less than the upper limit.

On the other hand, if g is strictly decreasing then $g^{-1}(b) < g^{-1}(a)$, and the lower limit is greater than the upper limit. In this case we can write

$$\int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,du = \int_{g^{-1}(b)}^{g^{-1}(a)} f(g(u))(-g'(u))\,du = \int_{g^{-1}(b)}^{g^{-1}(a)} f(g(u))|g'(u)|\,du,$$

the second equality holding since g is decreasing, so that $g'(u) < 0$. In this case note that the upper limit of integration is greater than the lower limit of integration.

So in either case we can write

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(u))|g'(u)|\,du,$$

where $\alpha < \beta$. Note that $|g'(u)|$ is actually the absolute value of the determinant of the 1×1 matrix $[g'(u)] = \left[\dfrac{dg}{du}\right]$. This matrix is also known as the *Jacobian* of the function, so before we look at the generalization of integration by substitution, let us define the Jacobian for a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Definition 3.7.1. Suppose that the function

$$\begin{aligned} \mathbf{f}\colon \mathbb{R}^n &\to \mathbb{R}^m \\ (x_1, \ldots, x_n) &\mapsto (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \end{aligned}$$

is partially differentiable with respect to $x_1, \ldots, x_n$, then its *Jacobian* is defined to be

$$\mathbf{J_f} = \left[\frac{\partial \mathbf{f}}{\partial x_1}, \ldots, \frac{\partial \mathbf{f}}{\partial x_m}\right] = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_m} \end{bmatrix} \tag{6}$$

Remark 3.7.2.

(a) In this section, we will only use the Jacobian for mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ and from $\mathbb{R}^3$ to $\mathbb{R}^3$, but I thought it might be useful to state the definition for the general case.

(b) Depending on the application, the Jacobian is sometimes defined as the transpose of the matrix in Equation (6), and indeed we did this in Section 2.4.

(c) The Jacobian is a $n \times m$ matrix whose entries are functions. At a particular point, $(x_1, \ldots, x_n)$, the matrix $\mathbf{J_f}(x_1, \ldots, x_n)$ is a $n \times m$ matrix whose entries are numbers.

(d) The Jacobian is also sometimes denoted by $\dfrac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)}$, and this is the notation we will use for the rest of the chapter, with f_1, f_2 and f_3 replaced by x, y and z, and x_1, x_2 and x_3 replaced by u, v and w.

We are now in a position to give the formula that will enable us to integrate by substitution. The formulae for the two dimensional case and the three dimensional case are pretty similar, but we will state them separately.

Theorem 3.7.3 (Change of variables formula for double integrals)**.** *If the transformation $x = x(u, v)$, $y = y(u, v)$ maps the region S in the uv-plane onto the region R in the xy-plane, and if the determinant of the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$ is non-zero and does not change sign on S, then with appropriate restrictions on the transformation and the regions, then*

$$\iint_R f(x, y)\, dA_{xy} = \iint_S f(x(u, v), y(u, v)) \left|\det\left(\frac{\partial(x, y)}{\partial(u, v)}\right)\right| dA_{uv}.$$

Remark 3.7.4. In Section 3.3, we stated that the area element in polar coordinates is $r\, dr\, d\theta$. We can see that this now follows directly from the fact that the determinant of the Jacobian of the transformation from Cartesian coordinates to polar

coordinates is

$$\begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

Now for the equivalent theorem for triple integrals.

Theorem 3.7.5 (Change of variables formula for triple integrals)**.** *If the transformation $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ maps the volume S in the uvw-space onto the region R in the xyz-space, and if the determinant of the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is non-zero and does not change sign on S, then with appropriate restrictions on the transformation and the regions, then*

$$\iiint_R f(x, y, z)\, dA_{xyz}$$
$$= \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\det\left(\frac{\partial(x, y, z)}{\partial(u, v.w)}\right)\right| dA_{uvw}.$$

Remark 3.7.6. The area elements for cylindrical and spherical coordinates can also be obtained using Jacobians.