Advanced Calculus
MA1132
Exercises 5 Solutions

1. (a) Find the total differential of the function $f(x, y, z) = \dfrac{xyz}{x + y + z}$ at a point (a, b, c).

   (b) Use your answer to Part (a) to estimate $f(-1.04, -1.98, 3.97)$ using the total differential at $f(-1, -2, 4)$ and compare it with the actual value.

   Solution:

   (a) We have

$$\frac{\partial f}{\partial x} = \frac{yz(x + y + z) - 1(xyz)}{(x + y + z)^2} = \frac{yz(y + z)}{(x + y + z)^2},$$
$$\frac{\partial f}{\partial y} = \frac{xz(x + y + z) - 1(xyz)}{(x + y + z)^2} = \frac{xz(x + z)}{(x + y + z)^2}$$

   and

$$\frac{\partial f}{\partial z} = \frac{xy(x + y + z) - 1(xyz)}{(x + y + z)^2} = \frac{xy(x + y)}{(x + y + z)^2}.$$

   Hence, the total differential at (a, b, c) is given by

$$\begin{aligned} df(dx, dy, dz) &= \frac{\partial f}{\partial x}(a, b, c)dx + \frac{\partial f}{\partial y}(a, b, c)dy + \frac{\partial f}{\partial z}(a, b, c)dz \\ &= \frac{bc(b + c)}{(a + b + c)^2}dx + \frac{ac(a + c)}{(a + b + c)^2}dy + \frac{ab(a + b)}{(a + b + c)^2}dz. \end{aligned}$$

   (b) First we find the total differential at $(a, b, c) = (-1, -2, 4)$.

$$\begin{aligned} df(dx, dy, dz) &= \frac{-2(4)(-2 + 4)}{(-1 - 2 + 4)^2}dx + \frac{-1(4)(-1 + 4)}{(-1 - 2 + 4)^2}dy + \frac{-1(-2)(-1 - 2)}{(-1 - 2 + 4)^2}dz \\ &= -16dx - 12dy - 6dz. \end{aligned}$$

   Since $-1.04 - (-1) = -0.04$, $-1.98 - (-2) = 0.02$ and $3.97 - 4 = -0.03$, an approximation is given by

$$df(-0.04, 0.02, -0.03) = -16(-0.04) - 12(0.02) - 6(-0.03) = 0.58.$$

   The actual value is

$$f(-1.04, -1.98, 3.97) - f(-1, -2, 4) \simeq 8.605 - 8 = 0.605,$$

   so again we have quite a reasonable approximation.

2. Let $f(x, y, z) = \tan^{-1}\left(\dfrac{x}{y + z}\right)$.

(a) Find the directional derivative of f at the point $(4,2,2)$ in the direction $(2,3,4)$.

(b) Find the unit vectors in the directions in which f is increasing/decreasing most rapidly at the point $(4,2,2)$, and give the rate of increase and decrease, respectively.

Solution:

(a) First we need to find a unit vector $\mathbf{u}$. We have

$$\|(2,3,4)\| = \sqrt{2^2+3^2+4^2} = \sqrt{29},$$

so $\mathbf{u} = \left(\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)$.

If we let $\mathbf{a} = (4,2,2)$, then

$$D_{\mathbf{u}} f(\mathbf{a}) = f_x(\mathbf{a})\left(\frac{2}{\sqrt{29}}\right) + f_y(\mathbf{a})\left(\frac{3}{\sqrt{29}}\right) + f_z(\mathbf{a})\left(\frac{4}{\sqrt{29}}\right).$$

Now

$$f_x = \frac{1}{y+z}\cdot\frac{1}{1+\left(\frac{x}{y+z}\right)^2}, \quad \text{so} \quad f_x(\mathbf{a}) = \frac{1}{2+2}\cdot\frac{1}{1+\left(\frac{4}{2+2}\right)^2} = \frac{1}{8},$$

$$f_y = -\frac{x}{(y+z)^2}\cdot\frac{1}{1+\left(\frac{x}{y+z}\right)^2}, \quad \text{so} \quad f_y(\mathbf{a}) = -\frac{4}{(2+2)^2}\cdot\frac{1}{1+\left(\frac{4}{2+2}\right)^2} = -\frac{1}{8},$$

$$f_z = -\frac{x}{(y+z)^2}\cdot\frac{1}{1+\left(\frac{x}{y+z}\right)^2}, \quad \text{so} \quad f_z(\mathbf{a}) = -\frac{4}{(2+2)^2}\cdot\frac{1}{1+\left(\frac{4}{2+2}\right)^2} = -\frac{1}{8}.$$

Hence

$$D_{\mathbf{u}} f(\mathbf{a}) = \frac{1}{8}\left(\frac{2}{\sqrt{29}}\right) - \frac{1}{8}\left(\frac{3}{\sqrt{29}}\right) - \frac{1}{8}\left(\frac{4}{\sqrt{29}}\right) = -\frac{5}{8\sqrt{29}}.$$

(b) The directions in which f is increasing/decreasing most rapidly at $\mathbf{a}$ are given by $\nabla f(\mathbf{a})$ and $-\nabla f(\mathbf{a})$.

Using Part (a), $\nabla f(\mathbf{a}) = \left(\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}\right)$ and $-\nabla f(\mathbf{a}) = \left(-\frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$.

Since $\left\|\left(\frac{1}{8}, -\frac{1}{8}, -\frac{1}{8}\right)\right\| = \sqrt{\left(\frac{1}{8}\right)^2 + \left(-\frac{1}{8}\right)^2 + \left(-\frac{1}{8}\right)^2} = \sqrt{\frac{3}{64}} = \frac{\sqrt{3}}{8}$, unit vectors in the directions in which f is increasing and decreasing most rapidly are $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, respectively.

The maximum rate of increase is $\frac{\sqrt{3}}{8}$ and the maximum rate of decrease is $-\frac{\sqrt{3}}{8}$.

3. Given that $z = 3x^2 - y^2$, find all the points at which $\|\nabla z\| = 6$.

Solution:

We have $\nabla z = (6x, -2y)$, so that

$$\begin{aligned}\|\nabla z\| = 6 &\Rightarrow \|(6x, -2y)\| = 6\\ &\Rightarrow \sqrt{36x^2 + 4y^2} = 6\\ &\Rightarrow 36x^2 + 4y^2 = 36\\ &\Rightarrow 9x^2 + y^2 = 9.\end{aligned}$$

Thus the set of all points at which $\|\nabla z\| = 6$ is the ellipse $\{(x, y) \in \mathbb{R}^2 \colon 9x^2 + y^2 = 9\}$.

4. Given that $z = 3x + y^2$, find $\nabla\|\nabla z\|$ at the point $(5, 2)$.

   Solution:

   We have

$$\begin{aligned}\nabla\|\nabla z\| &= \nabla\|(3, 2y)\|\\ &= \nabla\sqrt{9 + 4y^2}\\ &= \left(0, \frac{4y}{\sqrt{9 + 4y^2}}\right).\end{aligned}$$

   Hence

$$\nabla\|\nabla z\|(5, 2) = \left(0, \frac{8}{5}\right).$$

5. Find an equation for the tangent plane and a parametric equation for the normal line to the graph of the function $f(x, y) = \sqrt{x} + \sqrt{y}$ at the point $(4, 9)$.

   Solution:

   We first form the function $g(x, y, z) = z - \sqrt{x} - \sqrt{y}$.
   Now $f(4, 9) = 5$ and $g(4, 9, 5) = 0$.
   Thus we we want to find an equation for the tangent plane and a parametric equation for the normal line to the level surface $g(x, y, z) = 0$ at the point $(4, 9, 5)$.
   Now $\nabla g = \left(-\frac{1}{2\sqrt{x}}, -\frac{1}{2\sqrt{y}}, 1\right)$, so that $\nabla g(4, 9, 5) = \left(-\frac{1}{4}, -\frac{1}{6}, 1\right)$ is the direction of the normal line.
   Hence the parametric equation of the normal line is given by

$$(x, y, z) = (4, 9, 5) + t\left(-\frac{1}{4}, -\frac{1}{6}, 1\right), \quad t \in \mathbb{R},$$

   or equivalently,

$$(x, y, z) = (4, 9, 5) + t\,(3, 2, -12), \quad t \in \mathbb{R}.$$

   The tangent plane is given by

$$\begin{aligned}\nabla g(4, 9, 5) \cdot (x - 4, y - 9, z - 5) = 0 &\Rightarrow -\frac{1}{4}(x - 4) - \frac{1}{6}(y - 9) + 1(z - 5) = 0\\ &\Rightarrow -\frac{1}{4}x - \frac{1}{6}y + z = \frac{5}{2}\\ &\Rightarrow 3x + 2y - 12z = -30.\end{aligned}$$

6. Find the parametric equation of the tangent line to the curve of intersection of the cone $z = \sqrt{x^2+y^2}$ and the plane $x+2y+2z=20$ at the point $(4,3,5)$

   Solution:

   We first find the directions of the normals to each surface.

   If $f(x,y,z) = z - \sqrt{x^2+y^2}$, then $\nabla f = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right)$, so that $\nabla f(4,3,5) = \left(-\frac{4}{5}, -\frac{3}{5}, 1\right)$

   If $g(x,y,z) = x+2y+2z$, then $\nabla g = (1,2,2)$.

   Now, the tangent is perpendicular to both these normals, so its direction is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4}{5} & -\frac{3}{5} & 1 \\ 1 & 2 & 2 \end{vmatrix} = \left(-\frac{16}{5}, \frac{13}{5}, -1\right).$$

   Thus the parametric equation for the tangent line to the curve of intersection is

$$(x,y,z) = (4,3,5) + t\left(-\frac{16}{5}, \frac{13}{5}, -1\right), \quad t \in \mathbb{R},$$

   or equivalently,

$$(x,y,z) = (4,3,5) + t\,(-16, 13, -5)\,, \quad t \in \mathbb{R}.$$

7. Let $\mathbf{u}_r$ be a unit vector whose counterclockwise angle from the positive x-axis is θ, and let $\mathbf{u}_\theta$ be a unit vector 90% counterclockwise from $\mathbf{u}_r$. Show that if $z = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, then

$$\nabla z = \frac{\partial z}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial z}{\partial \theta}\mathbf{u}_\theta\,.$$

Solution: We have

$$\mathbf{u}_r = \frac{\mathbf{r}}{r} = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2\,, \quad \mathbf{u}_\theta = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2\,. \tag{1}$$

   and

$$\nabla z = \frac{\partial z}{\partial x}\mathbf{e}_1 + \frac{\partial z}{\partial y}\mathbf{e}_2 = \left(\frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\mathbf{e}_1 + \left(\frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\mathbf{e}_2\,. \tag{2}$$

   Then

$$\begin{aligned} r = \sqrt{x^2+y^2} \quad &\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta\,, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\,, \\ \theta = \arctan\frac{y}{x} \quad &\Rightarrow \quad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}\,, \quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}\,. \end{aligned} \tag{3}$$

   Collecting the terms one gets the formula.

8. Consider the surface

$$z = f(x,y) = \ln\frac{\sqrt[3]{2x^2 - 3xy^2 + 3\cos(2x+3y) - 3y^3 + 18}}{2}$$

(a) Find an equation for the tangent plane to the surface at the point $P(3,-2,z_0)$ where $z_0 = f(3,-2)$.

(b) Sketch the tangent plane.

(c) Find parametric equations for the normal line to the surface at the point $P(3,-2,z_0)$.

(d) Sketch the normal line to the surface at the point $P(3,-2,z_0)$.

Show the details of your work.

Solution:

(a) We first simplify

$$z = \ln \frac{\sqrt[3]{2x^2 - 3xy^2 + 3\cos(2x+3y) - 3y^3 + 18}}{2} = \frac{1}{3}\ln\big(2x^2 - 3xy^2 + 3\cos(2x+3y) - 3y^3 + 18\big) - \ln 2\,,$$

and compute z_0

$$z_0 = z|_{x=3,y=-2} = \frac{1}{3}\ln\big(3\cos(0) + 18 + 24 - 36 + 18\big) - \ln 2 = \ln\frac{3}{2} \approx 0.405465$$

Then, we compute the partial derivatives at $P(3,-2,z_0)$

$$\frac{\partial z}{\partial x} = \frac{-6\sin(2x+3y) + 4x - 3y^2}{3\,(2x^2 - 3xy^2 + 3\cos(2x+3y) - 3y^3 + 18)} \qquad \Rightarrow \qquad \frac{\partial z}{\partial x}|_{x=3,y=-2} = 0\,.$$

$$\frac{\partial z}{\partial y} = \frac{-6xy - 9\sin(2x+3y) - 9y^2}{3\,(2x^2 - 3xy^2 + 3\cos(2x+3y) - 3y^3 + 18)} \qquad \Rightarrow \qquad \frac{\partial z}{\partial y}|_{x=3,y=-2} = 0\,.$$

The tangent plane equation is given by

$$z = z_0 + 0(x-3) + 0(y+2) = \ln\frac{3}{2}\,.$$

(b) It is a plane through the point $(3,-2,\ln\frac{3}{2})$ parallel to the xy-plane.

(c) The normal line to the surface (and the tangent plane) is given by

$$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + t\big(0\mathbf{i} + 0\mathbf{j} + \mathbf{k}\big) = 3\mathbf{i} - 2\mathbf{j} + t\mathbf{k}\,.$$

(d) It is parallel to the z-axis.

9. Show that the equation of the plane that is tangent to the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

at (x_0, y_0, z_0) can be written in the form

$$\frac{x_0 x}{a^2} + \frac{y y_0}{b^2} - \frac{z z_0}{c^2} = 1\,.$$

Solution: Consider the function $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}$. The gradient ∇F

$$\nabla F = 2\frac{x}{a^2}\mathbf{e}_1 + 2\frac{y}{b^2}\mathbf{e}_2 - 2\frac{z}{c^2}\mathbf{e}_3 \tag{4}$$

is normal to the level surfaces of F, and therefore to tangent planes to the level surfaces of F. Thus, the equation of the plane tangent to the hyperboloid at (x_0, y_0, z_0) can be written in the form

$$\frac{1}{2}(\mathbf{r} - \mathbf{r}_0) \cdot \nabla F(x_0, y_0, z_0) = 0\,. \tag{5}$$

Explicitly one gets

$$\frac{x_0 x}{a^2} + \frac{y y_0}{b^2} - \frac{z z_0}{c^2} - (\frac{x_0 x_0}{a^2} + \frac{y_0 y_0}{b^2} - \frac{z_0 z_0}{c^2}) = 0\,, \tag{6}$$

which proves the formula.