Module MA2341 (Frolov), Advanced Mechanics I
Homework Sheet 1
Each set of homework questions is worth 100 marks

Problem 1. Consider a particle moving in a D-dimensional flat space with the following Lagrangian

$$L = \frac{1}{2}mv_i^2 - U(r)\,, \quad r = \sqrt{x_i^2}\,, \quad m > 0 \text{ is a constant}\,,$$

where $U(r)$ is the "Mexican hat" (or Higgs) potential

$$U(r) = \frac{k^2}{4g} - \frac{k}{2}r^2 + \frac{g}{4}r^4\,, \quad k > 0\,,\ g > 0\,, \quad k\,,\ g \text{ are constants}\,,$$

Here and in what follows the summation over the repeated indices is assumed. That means

$$a_ib_i \equiv \sum_{i=1}^{D} a_ib_i = a_1b_1 + a_2b_2 + \cdots + a_Db_D\,, \quad a_i^2 \equiv a_ia_i$$

for any sets of objects $a_1, \ldots, a_D$ and $b_1, \ldots, b_D$.

(a) Find the absolute minimum and a local maximum of the potential.
Use Mathematica to plot the potential for $D = 1$, 2, $k = 1$, $g = 1$.

(b) Find equations of motion (eom) of the particle.

(c) Prove that L is invariant under the $O(D)$ group of rotations and reflections of the coordinates x_i:

$$x_i \to \mathcal{O}_{ij}x_j\,, \quad \textit{summation over } j\ !\,,$$

where $\mathcal{O}_{ij}$ is an orthogonal D by D matrix, that is for any indices i and j

$$\mathcal{O}_{ik}\mathcal{O}_{jk} = \delta_{ij}\,, \quad \textit{summation over } k\ !$$

and δ_{ij} is the Kronecker delta: $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

Problem 2. Consider a system with s degrees of freedom and the Lagrangian

$$L = \frac{1}{2}m_{ij}\dot{q}^i\dot{q}^j + b_{ij}\dot{q}^iq^j - \frac{1}{2}k_{ij}e^{q^i+q^j}\,, \tag{0.1}$$

where m_{ij}, k_{ij} and b_{ij} are constants, and we sum over repeated indices.

(a) Explain why without loss of generality for any i and j one can assume that $m_{ij} = m_{ji}$ and $k_{ij} = k_{ji}$ that is the matrices (m_{ij}) and (k_{ij}) with the entries m_{ij} and k_{ij}, respectively, are symmetric. Explain why (m_{ij}) is positive definite.

(b) Find equations of motion of this system.

(c) Explain why the equations of motion depend only on the anti-symmetric part of b_{ij}.

Problem 3. Consider a relativistic charged particle of rest mass m and charge e moving in constant uniform electric $\vec{E} = \{E_1, E_2, E_3\}$ and magnetic $\vec{B} = \{B_1, B_2, B_3\}$ fields described by the following Lagrangian (c is the speed of light)

$$L = -mc^2\sqrt{1 - \frac{v_i^2}{c^2}} + e\,x_i E_i + \frac{e}{2c}\,\epsilon_{ijk}\,B_i x_j v_k\,, \quad v_i^2 \equiv v_i v_i\,,$$

where ϵ_{ijk} is the antisymmetric tensor with $\epsilon_{123} = 1$.

(a) Find the momentum $\vec{p}$ of the particle as a function of its velocity $\vec{v}$. What is the component of the momentum along the x_3-axis?

Find the velocity $\vec{v}$ of the particle as a function of $\vec{p}$. What is the component of the velocity along the x_2-axis?

(b) Find eom of the particle.

(c) Show that in the absence of the electric field, $\vec{E} = 0$, the speed of the particle is constant.