SPECIAL RELATIVITY

PY1T10

SPECIAL RELATIVITY INTRO

NUALA CAFFREY

- PUBLISHED 1905 BY ALBERT EINSTEIN "On the ELECTRODYNAMICS OF MOVING BODIES"
- IN THE LIMIT OF *Low Velocity* v
  SPECIAL RELATIVITY ⟶ CLASSICAL MECHANICS
- DIFFERENT BEHAVIOUR WHEN ~~[illegible]~~ $v \sim c$ *speed of light*

__ODDITIES:__
- Events that are simultaneous to one observer may *not* be simultaneous to ~~an~~ another
- observers moving relative to one-another measuring a time interval or length may not get the same result

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- SPECIAL RELATIVITY is only an accurate description at *inertial reference frames*

> An INERTIAL REFERENCE FRAME is a frame in which Newton's 1st Law holds

- Idea GOES BACK TO GALILEO:
  "If you are in a boat below deck, there is no experiment you can do to know you are moving if the boat can move smoothly"
- IF a COORDINATE SYSTEM (x, y, z, t) is moving at a constant velocity w.r.t another inertial reference frame, then the COORDINATE SYSTEM can *also* be considered an *INERTIAL REFERENCE FRAME*

GALILEAN TRANSFORMATIONS

$x = x' + vt \qquad y = y' \qquad z = z' \qquad t = t'$

$dx = dx' + v\,dt \qquad dy = dy' \qquad dz = dz' \qquad dt = dt'$

$\frac{dx}{dt} = \frac{dx'}{dt} + \frac{v\,dt}{dt} \qquad \frac{dy}{dt} = \frac{dy'}{dt} \qquad \frac{dz}{dt} = \frac{dz'}{dt} \qquad dt$

$u_x = u_x' + v \qquad u_y = u_y' \qquad u_z = u_z'$

$\Rightarrow \vec{u} = \vec{u}' + \vec{v}$

$\vec{a} = \frac{du}{dt} = \frac{du'}{dt'} = a'$

$F = ma$

$F' = m'a' \xrightarrow[a' = a]{m' = m} F = F'$

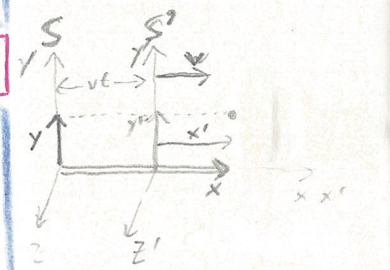
$K = \int_1^2 \vec{F} \cdot d\vec{r} = \frac{m}{2}(v_2^2 - v_1^2)$

RELATIVITIES:

GALILEAN
- Valid at low velocities
- Approximation of SPECIAL Relativity

SPECIAL
- valid at all velocities even as $v \sim c$
- not valid in strong gravitational fields or NON-INERTIAL REFERENCE FRAMES
- APPROXIMATION OF:

GENERAL
- valid at all velocities
- describes gravity
- not yet compatible w/ current quantum mechanics theories



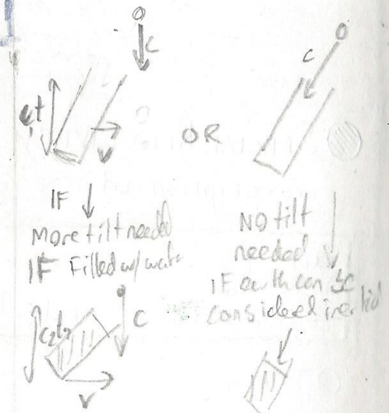
LIGHT

Hot topic for debate: wave vs. PARTICLE
- Physicists believed all waves need a medium
  ⇒ Invented LUMINIFEROUS Ether
  - velocity of light would depend on motion w.r.t motion through ether

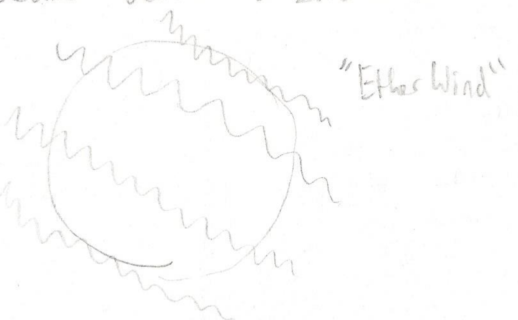
By this: same v in both directions ⇒ we are stationary
different v in 2 directions ⇒ we are moving

Fizeau Experiment & Airy (water in telescope)
- It was noticed that as the earth moved, the angle to point up at a star changed. "steller aberration"
- stars look highest when earth was moving away fastest, NOT when closest

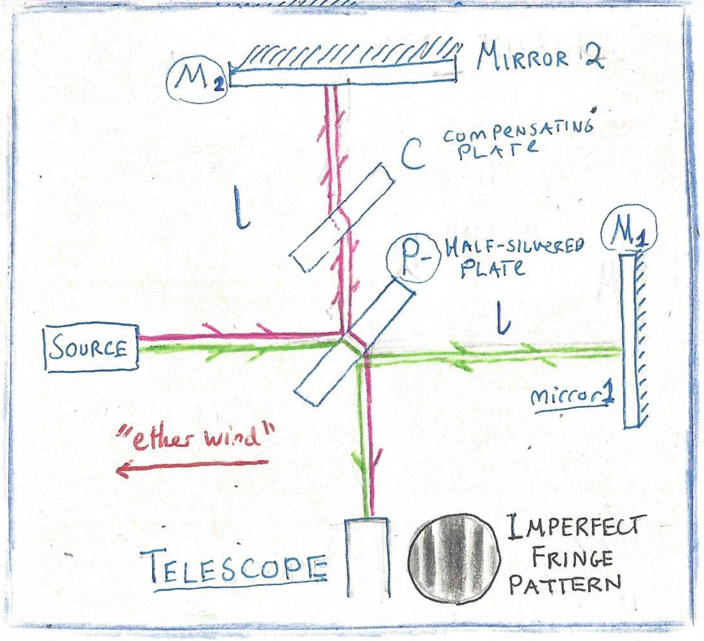
- Tried to measure velocity of earth relative to photon by filling telescope w/ water
- Found that no tilt was needed
  → This was explained with "Ether drag"



o should be able to detect an "Ether Wind"



MICHELSON-MORLEY EXPERIMENT



P = Glass Plate w/ semi-transparent coating on it's front face, at 45% to source beam

M_1, M_2 = Two mirrors

C = Compensating plate to ensure both beams travel through the same amount of glass

Whole interferometer was floated in a bath of mercury
- Change in length of an arm would lead to phase shift in lights
  → WITH CARE, can measure small fraction of a wavelength due to shifting fringes

Let t_1 = time it takes to go from $P \to M_1 \to P$

t_2 = time it takes to go from $P \to M_2 \to P$

and suppose the setup begins w/ the ether moving in the M_1P direction

let $\Delta T = t_1 - t_2$

IF ΔT changes then the fringes change
A way to measure the speed of "ether wind"

MICHELSON - MORLEY

ALONG L_1 (P, M_1)

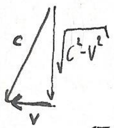
$$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c+v} = \frac{l_1(c+v) + l_1(c-v)}{(c-v)(c+v)} = \frac{2l_1 c}{c^2-v^2}$$

$$t_1 = \frac{\frac{1}{c^2}}{\frac{1}{c^2}} \frac{2l_1 c}{(c^2-v^2)} = 2\frac{\frac{l_1}{c}\cdot\left(1+\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)\left(1+\frac{v^2}{c^2}\right)} = \frac{2\frac{l_1}{c}\left(1+\frac{v^2}{c^2}\right)}{1-\frac{v^4}{c^2}}$$

- consider $1-\frac{v^4}{c^4}$ to be neglegable

$$\boxed{t_1 = 2\frac{l_1}{c}\left(1+\frac{v^2}{c^2}\right)}$$

ALONG L_2 must also account for the "ether"



$$t_2 = \frac{l_2}{\sqrt{c^2-v^2}} + \frac{l_2}{\sqrt{c^2-v^2}} = \frac{2l_2}{c\sqrt{1-\frac{v^2}{c^2}}}$$

- TAYLOR EXPANSION OF $(1+x)^m$

$$\boxed{(1+x)^m = 1 + \frac{mx}{1!} + \frac{m(m-1)x^2}{2!} + \cdots}$$

so $t_2 = \frac{2l_2}{c}\left(1-\left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}} \approx \frac{2l_2}{c}\left[1-\frac{1}{2}\left(\frac{v^2}{c^2}\right)+\frac{1}{2(2)}\left(\frac{v^4}{c^4}\right)\ldots\right]$

Neglegable

$$\boxed{t_2 = 2\frac{l_2}{c}\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)}$$

$$\Delta T = 2\left(\frac{l_2-l_1}{c}\right) + \frac{\cancel{2}l_2 v^2}{\cancel{2}c^3} - \frac{2l_1 v^2}{c^3}$$

- IF we rotate it s.t. the second arm is against the ether wind, then the times get reversed, so

$$\Delta T' = 2\frac{(L_2-L_1)}{c} + \frac{\cancel{2}L_1 v^2}{\cancel{2}c^3} - \frac{2L_2 v^2}{c^3}$$

- So the TOTAL CHANGE IN ΔT is: $\boxed{\frac{L_1 v^2}{c^3} + \frac{L_2 v^2}{c^3}}$
- a time delay of $\frac{\lambda}{c}$ would cause the fringes to shift ~~by~~ 1 fringe
- difference in time delay will shift by $N = (l_1+l_2)\frac{v^2}{c^2}\cdot\frac{1}{\lambda}$
- $\boxed{N = \frac{(l_1+l_2)v^2}{\lambda c^3}}$ Known: $l \approx 11\,m$, $v = 3\times10^4\,ms^{-1}$, $\lambda = 6\times10^{-7}$, $c = 3\times10^8$

→ THEORETICAL EXPECTED SHIFT: 0.40m
→ EXPERIMENTAL RESULT: 0.01m

$$N = \frac{2l\,v^2}{\lambda c^3}$$

EXPECTED: 0.40m
REALITY: ~0.01m

- M-M Experiment proves no-ether
- SCIENTISTS were not ready to accept this
- IT was suggested that the motion of the earth caused a shortening of the arm of the interferometer by exactly the amount required to elimenate the fringe SHIFT BY A FACTOR of $\sqrt{1-\frac{v^2}{c^2}}$
- GEORGE FRANCIS FITZGERALT 1889:

"We know that electric forces are affected by the motion of electrified bodies relative to the ether, and it seems a not-improbable supposition that the molecular forces are affected by the motion and that the size of the body alters consequentially."

- contraction hypothesis correct but for the wrong reason.
- called "LORENTZ-FITZGERALD" CONTRACTION

EINSTEIN'S REASONING

- Viewed problem of light wave vs. particle as an error in basic dynamic principles.
  → Does not need to fit into old models
- Argued the speed of light must be universal

POSTULATES

1) ALL INERTIAL FRAMES ARE EQUIVALENT
2) SPEED OF LIGHT IN EPTY SPACE ALWAYS HAS THE SAME VALUE

P2 - CONSISTENT WITH M-M EXPERIMENT, but But M-M experiment does not prove P2

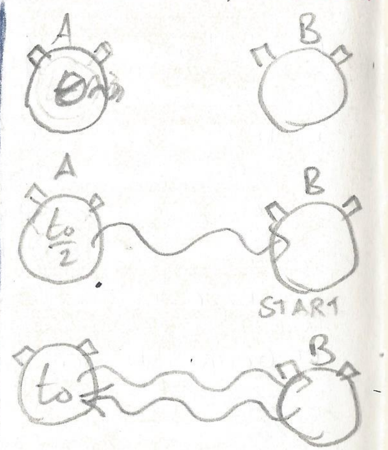
POSTULATES NOT CONSISTENT under Galilean transformations/Newtonian Mechanics

- NEW TRANSFORMATIONS NEEDED

LORENTZ TRANSFORMATION & Synchronised DEF

- We must define a specific procedure for synchronizing clocks (counts periodic events)
- Suppose we have 2 clocks: A and B
  BY definition, time to travel from A to B by a photon is the same as the time to travel from B to A

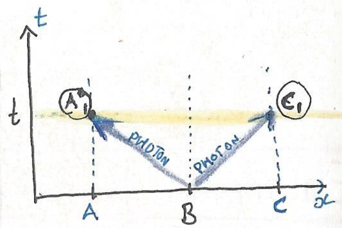
Suppose
Clock A shoots out a photon at time $t = 0$, then the photon hits B at a time $t = \frac{t_0}{2}$ and comes back to A at a time $t = t_0$



- Under this definition, simultinaity is now *relative*, not absolute

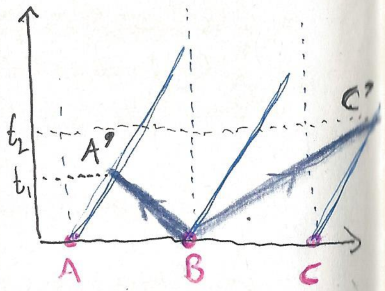
CASE 1: suppose A, B, C at rest, evenly spaced at $t = 0$

- B sends out signal at $t = 0$
  SIGNAL described as (initial position) ± (distance travelled)
  $$x = x_B \pm ct$$
- The ARRIVALS at A and C of the signal is given by the intersection of the WORLDLINES of A and C w/ the signals, at points $\boxed{A_1 \& C_1}$



- Simultinaity at A & C defined by the line $A_1 C_1$ parallel to the x axis, and both points possess the same value of t

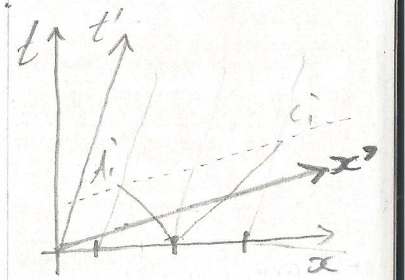
CASE 2: suppose A, B, C at rest in the frame S' BUT S' is moving a speed V along x axis.
- In th S reference frame, these world lines are now inclined
- the signals is still described as $x = x_B \pm ct$
- Arrives at A' and C' no longer simultaneous
  - SIGNAL REACHES A before C at times $t_1 < t_2$, but still simultaneous in frame S



SIMULTINAITY DEPENDS ON THE REFERENCE FRAME

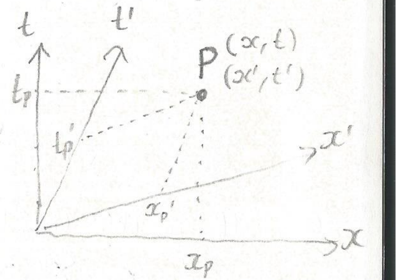
LORENTZ TRANSFORMATIONS

- We look for a transformation that gives a speed of light independent of speed of the source or the reciever
- Suppose we draw a t' axis parallel to the inclined world lines (as A, B, C will be Stationary in S')
  - it coincides w/ $t = 0$ at $t' = 0$



- NOW, THE Line $A_1' C_1'$ defines SIMULTANEITY at A and C in S'
  → THIS gives us another coordinate axis x' WHICH IS parallel to the $A_1' C_1'$ line, which crosses $t' = 0$ at $x' = 0$
  (as positions along a line defining simultineity have the same time value)
  → THIS x' axis gives all the points along the line of simultaneity where $t' = 0$
  [ie: sending a signal anywhere on this line can be vertified to be simultaneous to A B or C]

- Suppose we have an event occur at a time & place P
  - we TRY TO RELATE IT'S coordinates w.r.t S and S'
  - we TRY TO FIND $x = ax' + bt'$
    $x' = ax - bt$
  - now we substitute points that we know, eg:
    ORIGINS: S moves at a velocity $-V$ w.r.t S'
    origin of S is $x = 0 = ax' + bt'$
    $-bt' = ax'$ $\quad \frac{x'}{t'} = \frac{-b}{a} = -V$
    so we get $\boxed{\frac{+b}{a} = V}$
    S' moves at velocity $+V$ w.r.t S
    $x' = 0 = ax - bt \Rightarrow \frac{x}{t} = \boxed{\frac{b}{a} = V}$



LORENTZ TRANSFORMATIONS

- now we have $x = ax' + bt'$, $x' = ax - bt$; $\frac{b}{a} = v \Rightarrow b = av$

so we get
$$\boxed{\begin{aligned} x &= a(x' + vt') \\ x' &= a(x - vt) \end{aligned}}$$

- Now consider a light signal from the origin at $x = x' = 0$ moving in the positive x direction.

$$\boxed{x = ct} \qquad \boxed{x' = ct'}$$

○ PUTTING THIS INTO OUR PREVIOUS equations:

$$x = a(x' + vt') \rightarrow ct = a(ct' + vt')$$
$$x' = a(x - vt) \rightarrow ct' = a(ct - vt)$$

THIS GIVES:
$$at'(c+v) = ct$$
$$ct' = at(c-v).$$

DEVIDING the two equations, we get

$$\frac{a(c+v)}{c} = \frac{c}{a(c-v)}$$

$$\Rightarrow a^2(c^2 - v^2) = c^2$$

$$\Rightarrow a^2 = \frac{c^2}{c^2 - v^2} \cdot \frac{\frac{1}{c^2}}{\frac{1}{c^2}} = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow \boxed{a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \text{ its denotation}}$$

- THIS GIVES US THAT:

$$\boxed{\begin{aligned} x &= \gamma(x' + vt') \\ x' &= \gamma(x - vt) \end{aligned}}$$

○ SAME AS gallilean, except for $\gamma \geq 1$

- By rearranging these, one also gets:

$$\boxed{\begin{aligned} t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}}$$

LORENTZ TRANSFORMATIONS

- To FIND the time transformations:

$$x = \gamma x' + \gamma vt' \rightarrow \gamma vt' = x - \gamma x'$$
$$x' = \gamma x - \gamma vt \rightarrow \gamma vt = x' - \gamma x$$

○ multiplying the second by γ, we get:

$$\gamma vt' = x - \gamma x'$$
$$\gamma^2 vt = \gamma x' - \gamma^2 x$$

$$\Rightarrow \boxed{\gamma vt' + \gamma^2 vt = x - \gamma^2 x} \text{ By addition}$$

$$\gamma vt' = \gamma^2 vt + (1 - \gamma^2)x$$

$$\boxed{t' = \frac{\gamma^2 vt + (1 - \gamma^2)x}{\gamma v}}$$

$$t' = \gamma t + \frac{1}{v}\left(\frac{1}{\gamma} - \gamma\right)x$$

$$t' = \gamma\left(t + \frac{1}{v}\left(\frac{1}{\gamma^2} - 1\right)x\right)$$

$$t' = \gamma\left(t + \frac{x}{v}\left(\cancel{1} - \frac{v^2}{c^2} - \cancel{1}\right)\right)$$

$$\boxed{t' = \gamma\left(t - \frac{xv}{c^2}\right)}$$

AND SIMILARLY:

$$\boxed{t = \gamma\left(t' + \frac{x'v}{c^2}\right)}$$

- as space is the same in all directions, (ie: isotropic) all displacements $\perp$ to the direction defined by relative motion are equivalent: $\boxed{y = y'}$ & $\boxed{z = z'}$

○ Lorentz discovered to account for null result with the existance of an inertial frame w.r.t the ether.

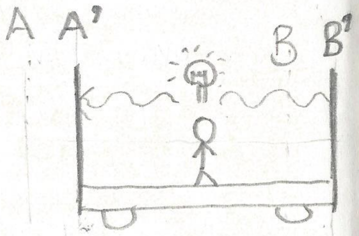
○ Einstein discovered the equations differently w/o any ether.

Consequences & Examples

Consequences:

- There is *no universal time between frames*
- Moving clocks appear to run slow, yet keep time as accurately as any other clock
- Objects moving at high speeds seem to contract along the direction of motion
- Objects seem to gain mass at high speeds

EXAMPLE: Person on trolley of length 2L in middle flicks on lantern. trolley $\xrightarrow{\text{moving}}$ $V\,ms^{-1}$



- IN S' reference frame of person:

time to reach A' is $\boxed{t_a' = \frac{L}{c}}$ and B' is $\boxed{t_b' = \frac{L}{c}} = T$

simultaneous

- IN OUTSIDE S reference frame:

$$t_a = \gamma\left(t_a' - \frac{Vx_a}{c^2}\right) \quad \text{but } x_a = (\cancel{L}) = cT$$

$$t_a = \gamma\left(T - \frac{VcT}{c^2}\right) = \gamma T\left(1 - \frac{V}{c}\right)$$

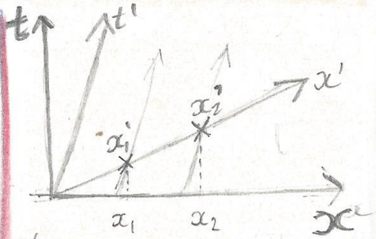
$$t_a = \frac{\sqrt{\left(1 - \frac{V}{c}\right)^2}}{\sqrt{1 - \frac{V^2}{c^2}}}\, T = \sqrt{\frac{1 - \frac{V}{c}}{1 + \frac{V}{c}}}\, T < T$$

For B, V is $(-V) \Rightarrow$ $t_b = \sqrt{\frac{\left(1 + \frac{V}{c}\right)}{\left(1 - \frac{V}{c}\right)}}\, T > T$

- NO LONGER SIMULTANEOUS

Length Contraction & Time Dilation

- We define **rest length** (L_0) as the length in an inertial frame where the object is at rest



- so to find its length, we suppose that both ends of the object are at the same time in that frame: $\boxed{t' = \text{a constant}}$ (note: $t \neq$ constant)
- The body is at rest in S'

so $\boxed{L_0 = x_1'(t') - x_2'(t')}$

- IN the S reference frame we get

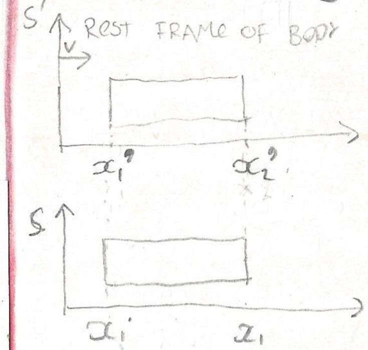
$\boxed{L = x_1(t_1) - x_2(t_2)}$

- Lorentz Transforming $L_0 = x_1'(t') - x_2'(t')$ we get

$$L_0 = \gamma(x_1 + vt) - \gamma(x_2 + vt)$$

$$L_0 = \gamma(x_1' - x_2')$$

$$* \quad \boxed{L_0 = \gamma L}$$

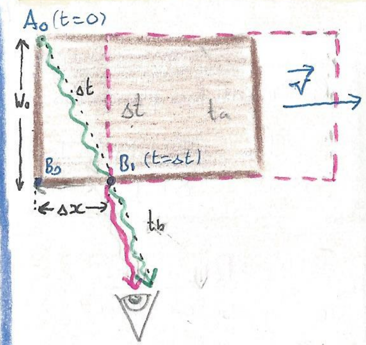


$$\boxed{L_0 = \gamma L}$$

- Note that we must make a distinction:

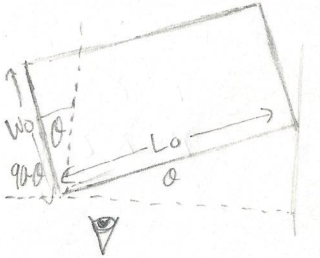
Observing: photons leave object simultaneously

Seeing: photons arrive at observer simultaneously



Observe contracted block

See Rotated Block



- this is because it takes time for photons to reach you
- For example, if we have a block as shown on the right travelling w/ velocity v.

Looking at the photon ~~that~~ leaves the block at A_0 ($t=0$) it takes time to even reach the bottom side. by the time it does (Δt), the block would have travelled a distance Δx.

we say $\Delta t = \frac{W_0}{c}$ $\frac{\text{height of the block}}{\text{speed of light}}$

Then $\Delta x = v\Delta t = \frac{vW_0}{c}$

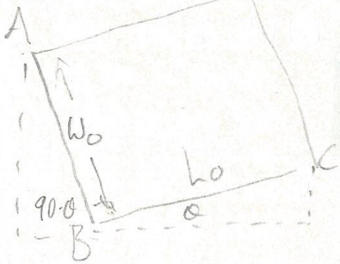
- A photon leaving the block at $t=0$ would have reached the observer far earlier than the one from A_0
- A photon leaving the block at $t = \Delta t$ from B_1 would arrive at the same time
- we also see the edge of the board contracted by $\frac{L_0}{\gamma}$

- so due to the time delay, the observer sees two different effects:

I) The viewer sees a forshortened/closer view of AB (the back of the block)

II) The viewer sees a contracted bottom of the block BC

$$\boxed{A'B' = V\frac{W_0}{c}} \qquad \boxed{B'C' = \frac{L_0}{\gamma} = L_0\sqrt{1-\frac{v^2}{c^2}}}$$



- now suppose the board was rotated by an angle θ as shown. Then the projected lengths would be

$$A'B' = W_0 \cos(90-\theta) = W_0 \sin\theta$$
$$B'C' = L_0\cos\theta$$

- If we let $\boxed{\sin\theta = \frac{v}{c}}$ we see that the situation becomes analagous, as $\boxed{\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{v^2}{c^2}}}$

we get $\boxed{A'B' = W_0\frac{v}{c}}$ $\boxed{B'C' = L_0\sqrt{1-\frac{v^2}{c^2}}}$ as before

- We call this phenomenon Terell / Penrose Rotation & conclude:
  - We OBSERVE that the board is contracted
  - We SEE that the board is rotated

TIME DILATION

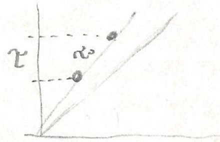
- suppose we have an event occur in the same position in a reference frame S' moving at a velocity v w.r.t S. the event occurs at times t_1' and t_2'

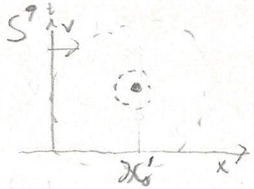
we let $\boxed{\tau_0 = t_2' - t_1'}$ "PROPER TIME"

Then: $\tau = t_2 - t_1 = \gamma(t_2' + \frac{vx_1'}{c}) - \gamma(t_1' + \frac{vx_1'}{c})$

$\tau = \gamma(t_2' - t_1')$ BUT $t_2' - t_1' = \tau_0$

* $\boxed{\tau = \gamma\tau_0}$





$$\boxed{\tau = \gamma\tau_0}$$

- Time dilation means that moving clocks run slow. this means
  - People appear to age more slowly.
  - Applies to ANY clock = any periodically occuring event eg: Atomic clock, pendulum, mechanical clock...

1941 Rossi & Hall Experiment

- observed muon decay as they went through atmosphere
- number of muons given by $\boxed{N(t) = N_0 e^{\frac{\ln 2 \cdot t}{T_{1/2}}}}$
- normal decay rate of muons: 2.2×10^{-6} seconds
- They measured number of muons at top of mountain & at sea level

ON EARTH

Distance = D_0

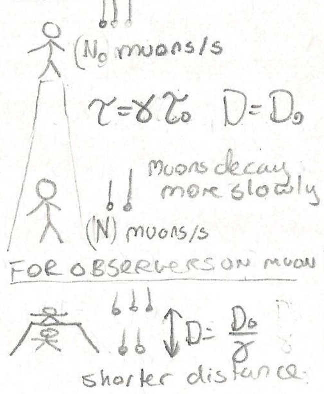
Life of muon = $\tau = \gamma\tau_0 >> \tau_0$

ON Meson

Distance = $D = \frac{D_0}{\gamma} << D_0$

Life of muon = τ_0

- THIS VERIFIED TIME DILATION



- Note: we must make a distinction again:

PROPER TIME = Time between two events that OCCUR AT THE SAME POINT (single clock at rest in one frame)

IMPROPER TIME = SPACIALLY SEPARATED CLOCKS IN the other frame

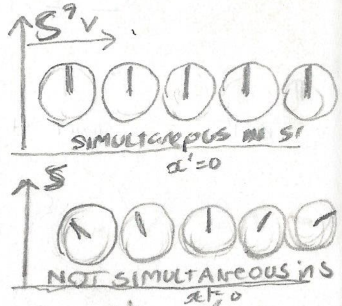
- We must make this distinction, because as we know, the Lorentz transform on time depends on position

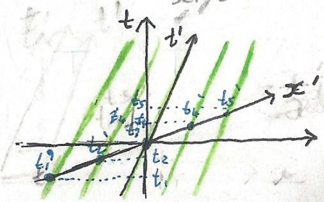
$$t = \gamma\left(t' + \frac{vx}{c^2}\right)$$

- For example, suppose we have 5 synchronized clocks in the S' reference fram that are evenly spaced out.
- in S, we get a progressive error

we know $(x_1', t_1') \dots (x_5', t_5')$ $\boxed{t_1' = t_2' = \dots = t_5'}$

$\boxed{\text{BUT} \quad t_1 \neq t_2 \neq \dots \neq t_5}$





Doppler Effect for Light

Doppler Effect = phenomenon where the measured frequency f (or wavelength λ) is changed by relative motion of the source or the observer

- Consider longtitudinal case:
  - relative motion along line joining source & observer
  - 1^{st} crest emitted at $t = 0$
  - 2^{nd} crest emitted at $t = T$
- we know $f = \frac{1}{T}$ and $t' = \gamma(t - \frac{vx}{c^2})$
- we can relate x_1 and x_2 to the speed of light/the position of the photon & the position of the body

$$x_1 = ct_1 = x_0 + vt_1$$
$$x_2 = c(t_2 - T) = x_0 + vt_2$$

- These give us that

$$ct_1 - vt_1 = x_0 \Rightarrow \boxed{t_1 = \frac{x_0}{c-v}}$$
$$ct_2 - vt_2 = x_0 + cT \Rightarrow \boxed{t_2 = \frac{x_0 + cT}{c-v}}$$

- Taking the difference between the two:

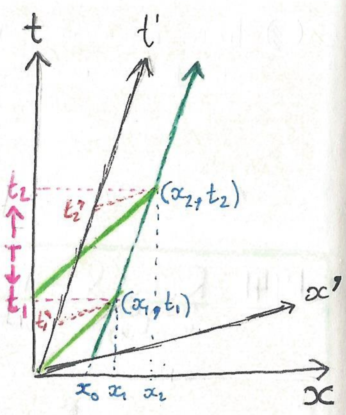
$$\boxed{t_2 - t_1 = \frac{cT}{c-v}}$$

- we can now Lorentz transform this:

$$\gamma(t_2' - \frac{vx_2'}{c^2}) - \gamma(t_1' - \frac{vx_1'}{c^2}) = \frac{cT}{c-v}$$
$$\Rightarrow \boxed{\gamma(t_2' - t_1') = \frac{cT}{c-v} + \gamma\left(\frac{vx_2'}{c^2} - \frac{vx_1'}{c^2}\right)}$$

- But we assume the moving body is at rest in the S' reference frame $\Rightarrow \boxed{x_2' = x_1'}$

$$\Rightarrow \cancel{\gamma}\ t_2' - t_1' = \frac{cT}{c-v} \cdot \frac{1}{\gamma} = \frac{T}{1-\frac{v}{c}} \cdot \frac{1}{\gamma}$$
$$\Rightarrow t_2' - t_1' = \frac{T}{\sqrt{(1-\frac{v}{c})^2}} \cdot \sqrt{(1-\frac{v}{c})(1+\frac{v}{c})}$$
$$\Rightarrow \boxed{T' = T_0 \frac{\sqrt{1+\frac{v}{c}}}{\sqrt{1-\frac{v}{c}}}}$$



$x_0' = x_1' = x_2'$ is constant as the body is at rest is S'

We Get: (letting $\beta = \frac{v}{c}$)

$$T' = T_0 \sqrt{\frac{1+\beta}{1-\beta}}$$
$$\lambda' = \lambda \sqrt{\frac{1+\beta}{1-\beta}}$$
$$f' = f \sqrt{\frac{1-\beta}{1+\beta}}$$
$$\beta = \frac{(\frac{\lambda'}{\lambda})^2 - 1}{(\frac{\lambda'}{\lambda})^2 + 1}$$

- IF $\beta \ll 1$ can also approximate

- Note moving towards you, you would:
  - OBSERVE a clock moving slow
  - SEE a clock moving fast

- Redshift of far away GALAXIES

Hubble $v = H_0 d$ — v = velocity of galaxy, H_0 = a constant, d = distance from us

Redshift "z"

$$z = \frac{\lambda - \lambda_0}{\lambda_0} \Rightarrow (z+1)^2 = \frac{\beta + 1}{\beta - 1}$$

Transformation of Velocity

- Suppose we have a body moving in S'
- Let us confine motion to the xy plane for simplicity

we get (in S'): $\boxed{u_x' = \frac{dx'}{dt'}}$ and $\boxed{u_y' = \frac{dy'}{dt'}}$

- We know:

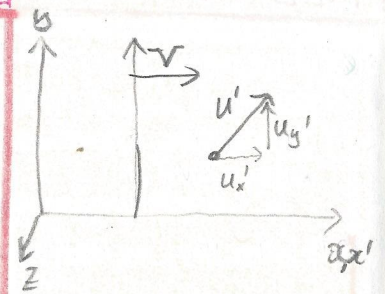
$$x = \gamma(x' + vt') \qquad y = y' \qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

- we take the derivative of each w.r.t t'

$$\boxed{\frac{dx}{dt'} = \gamma\left(\frac{dx'}{dt'} + v\frac{dt'}{dt'}\right)} \quad \boxed{\frac{dy}{dt'} = \frac{dy'}{dt'}} \quad \boxed{\frac{dt}{dt'} = \gamma\left(\frac{dt'}{dt'} + \frac{v}{c^2}\frac{dx'}{dt'}\right)}$$
$$\boxed{dx = \gamma(u_x' + v)dt'} \quad \boxed{dy = dy'} \quad \boxed{dt = \gamma(1 + \frac{v}{c^2}u_x')dt'}$$

- using this, we can now get u_x and u_y

$$\frac{dx}{dt} = u_x = \frac{\gamma(u_x' + v)dt'}{\gamma(1 + \frac{vu_x'}{c^2})dt'} = \frac{u_x' + v}{1 + \frac{vu_x}{c^2}}$$
$$\frac{dy}{dt} = u_y = \frac{\cancel{\gamma}\ dy'}{\gamma(1 + \frac{vu_x}{c^2})dt'} = \frac{u_y'}{\gamma} \cdot \frac{1}{1 + \frac{vu_x'}{c^2}}$$



- To get u_x' from u_x, just set v negative

$$u_x = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}}$$
$$u_y = \frac{u_y'/\gamma}{1 + \frac{vu_x'}{c^2}}$$

Ex: Rocket traveling at $0.5c$ launches missile at $0.5c$

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = \frac{0.5c + 0.5c}{1 + (0.5)(0.5)} = \frac{1}{1.25}c = \boxed{0.8c}$$

Ex: Photon traveling at c in S': $u_x = \frac{v+c}{1+\frac{vc}{c^2}} = \frac{v+c}{1+\frac{v}{c}} = c\,\frac{v+c}{c+v} = \boxed{c}$ STILL c

- consistent w/ assumptions

The Headlight Effect

- Suppose a source of light is travelling at velocity v in S' parallel to x relative to an observer in S
- suppose it is a point source radiating uniformly in all directions in S'
- what direction would a photon travel in S?

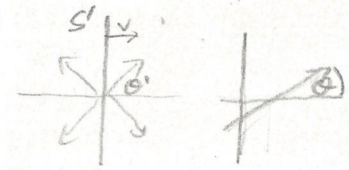
- Use Lorentz transformation: $u_x = \frac{u_x' + v}{1 + \frac{vu_x}{c^2}}$

BY SPLITTING PHOTON x & y components

$$u_x = c\cos\theta \qquad u_y = c\sin\theta$$
$$u_x' = c\cos\theta' \qquad u_y' = c\sin\theta'$$

$$\Rightarrow \gamma(\cos\theta) = \frac{c\cos\theta' + v}{1 + \frac{v\cos\theta\, c}{c^2}} \qquad \Rightarrow c\sin\theta = \frac{c\sin\theta' + v}{\gamma(1 + \frac{vc\sin\theta'}{c^2})} =$$

$$\Rightarrow \boxed{\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}} \qquad \sin\theta = \frac{\sin\theta'}{\gamma(1 + \beta\sin\theta')}$$



$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

IF $\theta' \in (0°, 90°)$

$\theta < \theta'$

more concentrated Beam

Relativistic Dynamics – Mass, Momentum

- In Newtonian mechanics: $F=\frac{d}{dt}p$, $p=mv$,
  - If mass is constant $F=ma$
    this implies v can increase indefinitely = wrong
  - $KE=\frac{1}{2}mv^2$
    ⇒ INFINITE energy would give infinite velocity = wrong
- To solve this, we need to assume mass depends on velocity
- CONSIDER AN ELASTIC COLLISION BETWEEN 2 PARTICLES:
  - Look at it from S and S' perspectives
  - S' is moving at a speed $V \gg u_0$ w.r.t S

IN S: $u_{y'} = \dfrac{\frac{u_y}{\gamma}}{1-\frac{Vu_x}{c^2}}$ But no u_x exists, $u_x=0$, so $u_y' = \frac{u_y}{\gamma} = \boxed{\frac{u_0}{\gamma}=u_y'}$

IS S': same but reversed: $\boxed{u_y'=\frac{u_0}{\gamma}}$

- This gives 2 possibilities for mass:

$$m_1 = m(u_0) \quad \text{or} \quad m_2 = m\left(\sqrt{V^2+u_y'^2}\right) = mV$$

By the conservation of momentum:

IN S: (y axis momentum) $m_{u_0}u_0 - m_V u_y' = -m_{u_0}u_0 + m_V u_y'$

$$2m_1u_0 = 2m_2u_y'$$

$$\frac{m_2}{m_1}=\frac{u_0}{u_y'} \quad \text{but } u_y'=\frac{u_0}{\gamma}$$

$$\Rightarrow \boxed{\frac{m_2}{m_1}=\gamma} \quad \text{let}$$

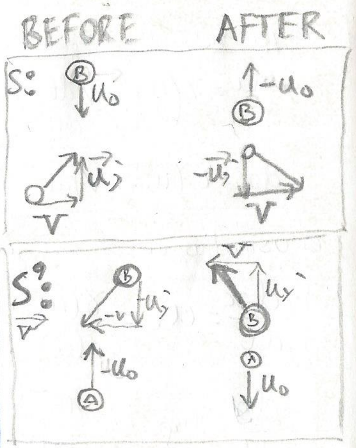
- now let $u_0\to 0$: then $\boxed{m(u_0)=m(0)=m_0 \text{ (rest mass)}}$
- By letting $u_0\to 0$, we get $V\to v$

$\boxed{m(v)=\gamma m_0}$ which agrees w/ newtonian when $\gamma\approx 1$ ($v \ll c$)

- now to get momentum: $p=mv$

$$\boxed{p=\gamma m_0 v}$$

Kinematics	Dynamics
Position	Mass
Time	Momentum
Velocity	Force



$$m=\gamma m_0$$

$$p=\gamma m_0 V$$

Relativistic Dynamics – Energy

- Let us inspect this new measure for mass: by finding how much it increases at different velocities:

$$m=\gamma m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

- the difference in mass between it at rest and at v is

$$\Delta m = m - m_0 = m_0(\gamma-1)$$

- we can look at the binomial expansion of $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$ w/ $(1+x)^m = 1+mx+\frac{m(m-1)x^2}{2!}\dots$

$$\text{so } \gamma = 1 + \left(-\tfrac{1}{2}\right)\left(-\tfrac{v^2}{c^2}\right) + \left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(\tfrac{1}{2!}\right)\left(\tfrac{v^4}{c^4}\right)$$

$$= 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4}\dots$$

$$\text{so } (\gamma-1) = \frac{1}{2}\frac{v^2}{c^2} + \frac{3v^4}{8c^4} =$$

$$\Delta m = m_0\left(\frac{1}{2}\frac{v^2}{c^2}+\frac{3v^4}{8c^4}\dots\right)$$

$$\Delta mc^2 = \underbrace{\frac{m_0v^2}{2}}_{\text{THIS PART LOOKS JUST LIKE NEWTONIAN KINETIC ENERGY}}\left[1+\frac{3v^2}{4c^2}+\dots\right]$$

- This hints that increasing the energy of the body increases it's mass
- Lets look at it's energy:

$$E=\int \vec{F}\cdot d\vec{x}$$

To do this though, we would first need to look at its force:

$$F=\frac{d}{dt}\vec{p}$$

Knowing that $p=\gamma m_0 V$

Relativistic Dynamics: Force Energy

$$F = \frac{d}{dt}p = \frac{d}{dt}\gamma m_0 v = m_0\left(v\frac{d}{dt}(\gamma) + \gamma\frac{d}{dt}(v)\right)$$

- we find $\frac{d}{dt}\gamma = \frac{d}{dv}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\cdot\frac{dv}{dt}$

$$= -\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\cdot\left(-\frac{2v}{c^2}\right)\cdot\frac{dv}{dt}$$

$$\frac{d}{dt}\gamma = \frac{\gamma^3 v}{c^2}\cdot\frac{dv}{dt}$$

- so we get $F = m_0\left(\frac{\gamma^3 v^2}{c^2}+1\right)\frac{dv}{dt} = m_0\gamma\left(\frac{\gamma^2 v^2}{c^2}+1\right)\frac{dv}{dt}$

But: $\frac{\gamma^2 v^2}{c^2}+1 = \frac{v^2}{c^2}\cdot\frac{1}{\left(1-\frac{v^2}{c^2}\right)}+1 = \frac{v^2}{c^2-v^2}+\frac{c^2-v^2}{c^2-v^2} = \frac{c^2}{c^2-v^2} = \frac{1}{1-\frac{v^2}{c^2}} = \gamma^2$

And so, we get $\boxed{\vec{F}_x = \gamma^3 m_0 \frac{dv}{dt}}$ *

$$\vec{F} = \gamma^3 m_0 \frac{dv}{dt}$$

$$\frac{d\gamma}{dt} = \frac{\gamma^3 v}{c^2}\cdot\frac{dv}{dt}$$

$$\frac{\gamma^2 v^2}{c^2}+1 = \gamma^2$$

Now to get Kinetic Energy: $E = \int F\,dx$

$$E = \int \gamma^3 m_0 \frac{dv}{dt}dx$$

But we know $\frac{d}{dt}\gamma = \frac{\gamma^3 v}{c^2}\cdot\frac{dv}{dt}$

$$\Rightarrow \boxed{\frac{dv}{dt} = \frac{c^2}{\gamma^3 v}\cdot\frac{d\gamma}{dt}}$$

$$\Rightarrow E = \int \cancel{\gamma^3} m_0 \cdot \frac{c^2}{\cancel{\gamma^3} v}\frac{d\gamma}{dt}\cdot dx$$

But $\frac{dx}{dt} = v$ so $\boxed{\frac{1}{v}\frac{dx}{dt} = 1}$

$$\Rightarrow E = \int_1^{\gamma} m_0 c^2 d\gamma = \gamma m_0 c^2 - m_0 c^2$$

$$\boxed{\text{Kinetic Energy} = (\gamma-1)m_0c^2}$$ *

$$K.E. = (\gamma-1)m_0c^2$$

So, $KE = mc^2 - m_0c^2$

$$\Rightarrow \boxed{mc^2 = KE + m_0c^2}$$

- we can call m_0c^2 the rest mass energy

TOTAL ENERGY = KINETIC ENERGY + REST MASS

$\Rightarrow$
- Energy is conserved
- Energy has mass
- Mass is conserved (though not necessarily rest mass)
- Mass & energy are different physical quantities

Relativistic Dynamics

So we know: $E = \gamma m_0 c^2$

$$p = \gamma m_0 v$$

A useful relation we can find is

$$E^2 - p^2c^2 = \gamma^2 m_0^2 c^4 - \gamma^2 m_0^2 c^2 v^2$$

$$= \gamma^2 m_0^2 c^4\left(1-\frac{v^2}{c^2}\right)$$

$$= \frac{m_0^2c^4}{\left(1-\frac{v^2}{c^2}\right)}\cdot\left(1-\frac{v^2}{c^2}\right)$$

so: $\boxed{E^2 - p^2c^2 = m_0^2c^4}$ An invariant

Both are same for all observers

$E^2 - p^2c^2 = m_0^2c^4$.

AN INVARIANT ACCROSS REFERENCE FRAMES

Photon Momentum

Speed of light is c for all observers

since $\boxed{m = \gamma m_0}$ $\Rightarrow m_0 = 0$

$$\boxed{m_0 = m\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

- USING $E^2 - p^2c^2 = m_0^2c^4 = 0$, we get

we get $E^2 = p^2c^2$

$$\boxed{E = pc}$$

But $E = hf$

$c = f\lambda \Rightarrow \frac{c}{f} = \lambda$

$$hf = pc$$

$$h = \frac{pc}{f} = p\lambda$$

$$\boxed{p = \frac{h}{\lambda}}$$

INELASTIC COLLISIONS: MASS ⟷ ENERGY

Suppose we have two bodies each of rest mass m_0 moving towards eachother with a speed v. They collide and merge into a new mass, M_0.

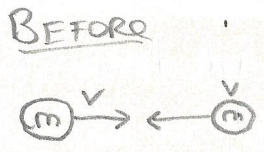
BY CONSERVATION OF MOMENTUM: $mv + m(-v) = M(0)$

BY CONSERVATION OF ENERGY: $\boxed{mc^2 + mc^2 = M_0c^2}$

Thus we get $2mc^2 = M_0c^2$

$$2(m_0c^2 + KE) = M_0c^2$$
$$2KE = M_0c^2 - 2m_0c^2$$
$$2KE = c^2(M_0 - 2m_0)$$

- As the kinetic energy is non-zero, $M_0 - 2m_0$ is non zero
- This means $\boxed{M_0 > 2m_0}$
- thus, we could have new particles if KE > Mass of a new particle [PAID DLC™ ONLY €50 TO UNLOCK™]



$P = m(v) + m(-v) = 0$

Experiment: Pair-Production - Anderson 1932

FIRST DISCOVERED energy from a gamma ray photon being converted into an electron + positron pair

- CHARGE IS CONSERVED

$$\gamma \rightarrow e^- + e^+$$

- Both have same Mass:

$$M_{e^+} = M_{e^-} \equiv M_e$$

- MINIMUM energy: $E_{min} = 2(M_0e)c^2$

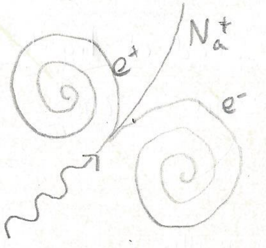
$$E_{min} = 2(9.1\times10^{31})c^2 = 1.64\times10^{-13}J$$
$$E_{min} = 1.02\times10^6 eV = \boxed{1.02 MeV}$$

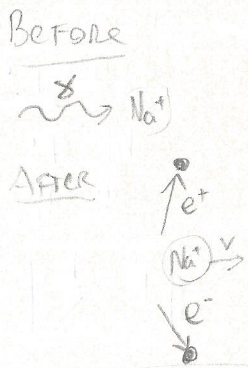
However, Momentum must be conserved. as learned in mechanics, we can find a frame of reference s.t. the e^+ and e^- have a total momentum of ZERO

BUT the momentum of the photon is ~~not~~ not zero.

THIS MEANS that the process cannot occur in free space

⇒ Must occur near a nucleus





$$p_\gamma + p_{Nu} = p_{e^+} + p_{e^-} + p'_{Nu}$$

NUCLEAR FUSION & FISSION

- CONSIDER A NUCLEUS: - Mass Number A (#PROTONS + #NEUTRONS)
  - Atomic Number Z (#PROTONS)
- Rest energy of nucleus: ${}^A_Z m_0 c^2$
- BINDING ENERGY (Lost when nucleons come together): E_b

THEN: $${}^A_Z m_0 c^2 + E_b = \underbrace{Z\, m_{pro} c^2}_{\text{REST MASS OF PROTON}} + \underbrace{(A-Z) m_{neu} c^2}_{\text{REST MASS OF NEUTRON}}$$

- we call $\frac{E_b}{A}$ the binding energy per nucleon
  - GRAPHED ON THE RIGHT
- CAN GAIN ENERGY BY GOING TOWARDS STABLE Fe/Ni

FISSION - SPLITTING (A=240) → (120) + (120)

$E_b/A = 7.2 MeV/Nucleon$ $E_b/A = 8.2 MeV/Nucleon$

E_b of parent: $7.2 \times 240 = 1728 MeV$

E_b of daughters: $(2)(8.2)(120) = 1968 MeV$

ENERGY RELEASED as Kinetic Energy: $1968 - 1728 = \boxed{240 MeV}$

Ex BOMBARD $${}^{235}_{92}U + {}^1_0 n \rightarrow {}^{90}_{38}Sr + {}^{135}_{54}Xe + 10\,{}^1_0 n + \underbrace{141 MeV}_{K.E.}$$

FUSION - COMBINING $4\,{}^1_1H \rightarrow {}^4_2He$ (MORE COMPLICATED IRL)

$$m_H = 1.007825 amu$$
$$m_{He} = 4.00260 amu$$
$$\Delta m = 0.00287 amu \rightarrow \boxed{27 MeV}$$

However: protons must come within $\sim 2\times10^{-15}$ m to fuse and overcome electromagnetic P.E.

$$\boxed{P.E. = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r} = 1.2\times10^{-13} J = 0.7 MeV}$$

⇒ Protons must approach w/ comparable Kinetic Energy

→ BOLTZMAN TEMPERATURE distribution implies that there must be a very high temperature at centre of sun:

$$\boxed{T = 1.5\times10^7 Kelvin}$$

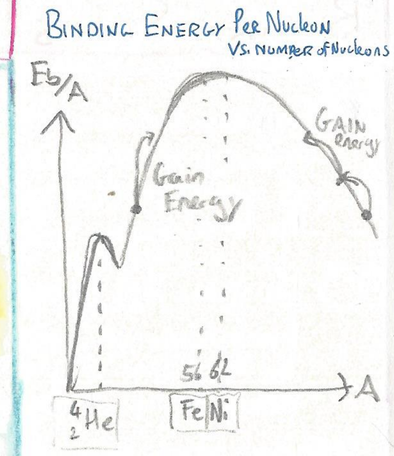
(Because $kT = 1eV$ when $T = 1.1\times10^4 K$)

IN A FUSION REACTOR: ${}^2_1H + {}^3_1H \rightarrow {}^4_2He + {}^1_0n$

this gives a $\Delta mc^2 \approx 27 MeV$ as before

- NEED TO MAKE THE GAS MIXTURE A VERY HOT PLASMA

CONFINE w/ Magnetic FIELD (ITER, Fr) or Compress w/ Lasers (HiPER or NIF)



$${}^A_Z m_0 c^2 + E_b = Z m_{pro} c^2 +$$

Binary Stars

- Suppose there are two stars m_1, m_2 orbiting around eachother, w/ a tangetal velocity $\vec{v}$. we assume (for simplicity) $m_1 = m_2$.
  The total system is moving away from us with velocity V
- We know that for hydrogen, there is a series of black bands at wavelengths that are readily absorbed by its electron shell. when we ~~dispose~~ expose white light through it and desperse the emerging light.
- For a binary system, these bands are split and slightly moved into the red (redshifted)

$$\boxed{\lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\,\lambda \approx (1+\beta)\lambda}$$ Redshift formula

- For Example: We know $\lambda_1 = 656.46\text{nm}$
  - we read a maximum distance between the shifts when

$$\boxed{\lambda_a = 656.72\text{nm}} \quad \boxed{\lambda_b = 656.86\text{nm}}$$

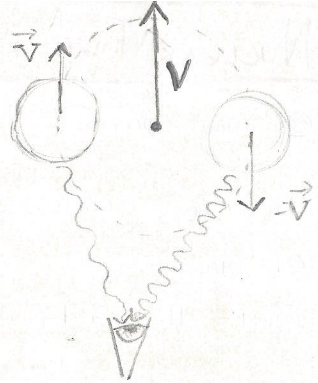
- $V_{min} = V - v = \left(\frac{\lambda_a}{\lambda_1} - 1\right)c = \boxed{119\,\text{kms}^{-1}} = V_{min}$
  ~~$V_{max} = V + v$~~
  $V_{max} = V + v = \left(\frac{\lambda_b}{\lambda_1} - 1\right)c = \boxed{183\,\text{kms}^{-1}} = V_{max}$
- $\frac{V_{min} + V_{max}}{2} = \frac{1}{2}(V - v + V + v) = V = \boxed{151\,\text{kms}^{-1}} = V$
  $\boxed{32\,\text{kms}^{-1}} = v$
- However, we also know the period of a rotation, T suppose it is $\boxed{1\text{ year} \approx 1\pi \times 10^7}$

  then: each star travels a distance $2\pi r$ in a time T

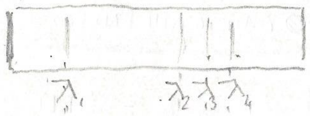
$$\boxed{\frac{2\pi r}{T} = v} \Rightarrow \boxed{r = 1.6 \times 10^{11}\,\text{m}}.$$

- Now, we can use Gravitational Force = Centripetal Force to find that:

$$\frac{GM\cancel{m}}{r^{\cancel{2}}} = \frac{\cancel{m}v^2}{\cancel{r}} \Rightarrow \boxed{m = \frac{v^2 r}{GM} = 2.48 \times 10^{30}\,\text{kg}}$$

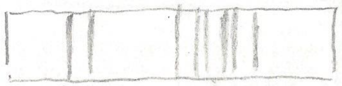


NORMAL HYDROGEN SPECTRUM



For BINARY STAR SYSTEM depending on the star location at the time:

- when ⟷ moving perpendicular to us



- when moving as shown above

THE LADDER PARADOX

ELASTIC SCATTERING OF IDENTICAL PARTICLES

Suppose we have a head-on colision between a particle m1 w/ initial speed u_1 and a particle m2 at rest.

○ FIRST CONSIDER IT UNDER Newtonian Mechanics:

By the conservation of momentum:

$$m_1\vec{u}_1 = m_1\vec{v}_1 + m_2\vec{v}_2 \quad \text{but as } m_1 = m_2$$

$$\vec{u}_1 = \vec{v}_1 + \vec{v}_2$$

By the conservation of Energy:

$$\frac{1}{2}mu_1^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

$$\boxed{u_1^2 = v_1^2 + v_2^2} \quad \text{But} \quad \boxed{\vec{u}_1 = \vec{v}_1 + \vec{v}_2}$$

$$\Rightarrow (v_1 + v_2)^2 = v_1^2 + 2\vec{v}_1\cdot\vec{v}_2 + v_2^2 = v_1^2 + v_2^2$$

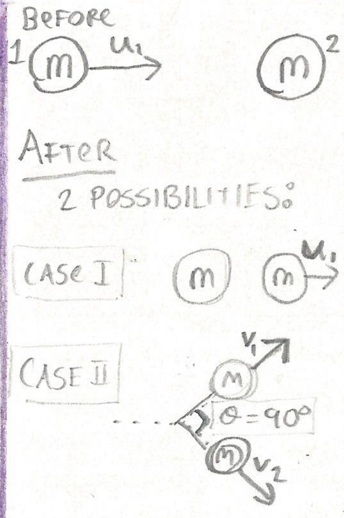
$$\Rightarrow \boxed{2\vec{v}_1\cdot\vec{v}_2 = 0}$$

This means that $|v_1||v_2|\cos\theta = 0$

which means either (a) $\cos\theta = 0 \Rightarrow \boxed{\theta = 90^\circ}$ OR

(b) $v_1 = 0 \Rightarrow \boxed{v_2 = u_1}$

(



$\Rightarrow v_1 = v_2$ to conserve momentum along the y axis

ELASTIC SCATTERING OF IDENTICAL PARTICLES

Let us consider the scattering case under relativistic scattering

① Conservation of Energy: $E_1 + E_0 = 2E_2 \Rightarrow \boxed{E_2 = \frac{E_1 + E_0}{2}}$

② Conservation of Momentum: $\boxed{p_1 = 2p_2\cos\left(\frac{\theta}{2}\right)}$

③ We introduce Kinetic Energy: $\boxed{E_1 = E_0 + k_1}$:

④ THIS AND THE INVARIANT: ARE SUFFICIENT

$$E^2 - p^2c^2 = E_0 \Rightarrow \boxed{p^2c^2 = E^2 - E_0^2}$$

WE BEGIN BY LOOKING AT THE INVARIANT BEFORE:

$$p_1^2c^2 = E_1^2 - E_0^2 = (E_0 + k_1)^2 - E_0^2 =$$

$$(2p_2\cos\tfrac{\theta}{2})^2c^2 = (\cancel{E_0^2} + 2k_1E_0 + k_1^2) - \cancel{E_0^2}$$

$$\boxed{4p_2^2\cos^2(\tfrac{\theta}{2})c^2 = k_1(2E_0 + k_1)} \quad (a)$$

We now look at the invariant after:

$$p_2^2c^2 = E_2^2 - E_0^2 = \left(\frac{E_1 + E_0}{2}\right)^2 - E_0^2 = \left(\frac{E_0 + k_1 + E_0}{2}\right)^2 - E_0^2$$

$$p_2^2c^2 = \left(E_0 + \frac{k_1}{2}\right)^2 - E_0^2 = \left(\cancel{E_0^2} + k_1E_0 + \frac{k_1^2}{4}\right) - \cancel{E_0^2}$$

$$\boxed{p_2^2c^2 = \frac{k_1}{4}(4E_0 + k_1)} \quad (b)$$

We now devide equation a by b

$$\boxed{\frac{\cancel{4}\cancel{p_2^2}\cos^2(\frac{\theta}{2})\cancel{c^2}}{\cancel{p_2^2c^2}} = \frac{\cancel{k_1}(2E_0 + k_1)}{\frac{\cancel{k_1}}{\cancel{4}}(4E_0 + k_1)}}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{2E_0 + k_1}{4E_0 + k_1}$$

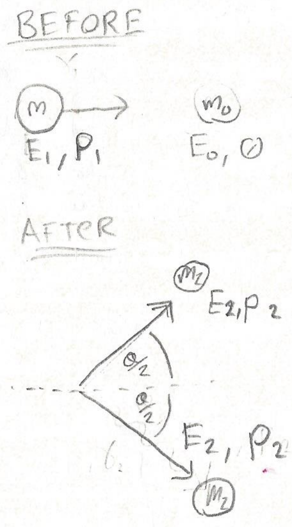
BUT $\cos^2 A = \frac{1}{2}(\cos(2A) + 1)$

$\Rightarrow \cos^2(\frac{\theta}{2}) = \frac{1}{2}(\cos\theta + 1)$

$$\frac{(\cos\theta + 1)}{2} = \frac{2E_0 + k_1}{4E_0 + k_1}$$

$$\cos\theta = \frac{4E_0 + 2k_1}{4E_0 + k_1} - 1 = \frac{4E_0 + 2k_1}{4E_0 + k_1} - \frac{4E_0 + k_1}{4E_0 + k_1}$$

$$\boxed{\cos\theta = \frac{k_1}{4E_0 + k_1}}$$



$$\cos\theta = \frac{K_1}{4E_0 + K_1}$$

Compton Scattering — Nobel Prize 1927

Suppose we have a collision of a photon γ with a free electron (nearly free, eg: loosely bound to atom)

- THE COLLISION is ELASTIC, but energy is transfered from the photon γ to the electron e
- Thus the photon would have lower f / longer λ afterwards.

suppose that initially: the photon has energy E_1 and momentum p_1
the electron has energy E_0, mass m_0, momentum 0

Suppose that afterwards: the photon has energy E_2 and momentum p_2
the electron has energy E momentum p

By conservation of ENERGY:

$$E_1 + E_0 = E + E_2$$

$$\Rightarrow \boxed{E_1 + m_0c^2 = E + E_2}$$

By conservation of momentum:

$$\vec{p_1} = \vec{p_2} + \vec{p}$$

but $\boxed{|p_1| = \frac{E_1}{c}} \Rightarrow \vec{p_1} = \frac{E_1}{c}\hat{n}_1$

$\boxed{|p_2| = \frac{E_2}{c}} \Rightarrow \vec{p_2} = \frac{E_2}{c}\hat{n}_2$

$$\Rightarrow \boxed{\frac{E_1}{c}\hat{n}_1 = \frac{E_2}{c}\hat{n}_2 + \vec{p}}$$

USING THE INVARIANT $E^2-(pc)^2 = E_0^2$:

○ $E = E_0 + KE$. but $KE = \Delta E = E_2 - E_1$

○ $\vec{p} = \vec{p_1} - \vec{p_2} = \frac{E_1}{c}\hat{n}_1 - \frac{E_2}{c}\hat{n}_2$

$$\Rightarrow E^2 - p^2c^2 = (E_0 + E_2 - E_1)^2 - \left[\frac{E_1}{\cancel{c}}\hat{n}_1 + \frac{E_2}{\cancel{c}}\hat{n}_2\right]^2 \cancel{c^2} = E_0^2$$

$$(E_0^2 + E_0E_2 - E_0E_1 + E_0E_2 + E_2^2 - E_2E_1 - E_0E_1 - E_2E_1 + E_1^2)$$

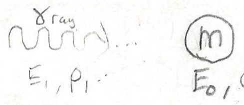
$$\Rightarrow \cancel{E_0^2} + \cancel{E_1^2} + \cancel{E_2^2} + 2E_0(E_2-E_1) - 2E_1E_2 - \cancel{E_1^2} + 2E_1E_2(\vec{n}_1\cdot\vec{n}_2) - \cancel{E_2^2} = \cancel{E_0^2}$$

$$\Rightarrow \cancel{2}E_0(E_2-E_1) = \cancel{2}E_1E_2(1-\cos\theta)$$

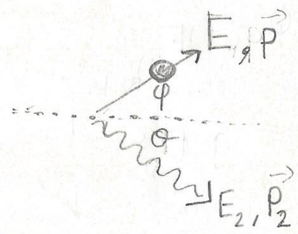
$$\Rightarrow \frac{E_2-E_1}{E_2E_1} = \frac{1}{E_2} - \frac{1}{E_1} = \frac{1-\cos\theta}{m_0c^2} \qquad \text{but } E = \frac{hc}{\lambda}$$

$$\Rightarrow \ast\ \boxed{\lambda_2 - \lambda_1 = \frac{1-\cos\theta}{m_0c}h}$$

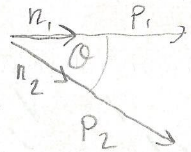
BEFORE:



AFTER



We write momentum of photon as $\frac{E}{c}$ but need direction too:



$$\hat{n}_1\cdot\hat{n}_2 = \cos\theta$$

$$\boxed{\frac{1}{E_2} - \frac{1}{E_1} = \frac{(1-\cos\theta)}{m_0c^2}}$$

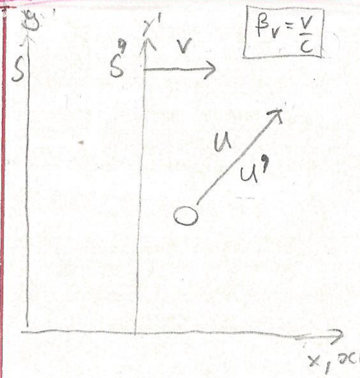
$$\boxed{\lambda_2 - \lambda_1 = \frac{h(1-\cos\theta)}{m_0c}}$$

$\frac{h}{m_0c} = \lambda_c$ Compton wavelength

(FOR e^-, $\lambda_c = 2.426$ pm)

LORENTZ TRANSFORMATION of γ, Energy, Momentum

Suppose we have a body moving relative to S and S'



WE KNOW in S:

$$\boxed{\begin{aligned} E &= \gamma_u m_0c^2 \\ p &= \gamma_u m_0 u \\ p_x &= \gamma_u m_0 u_x \\ p_y &= \gamma_u m_0 u_y \end{aligned}} \quad \beta = \frac{u}{c} \quad \beta_x = \frac{u_x}{c} \quad \beta_y = \frac{u_y}{c}$$

We know in S':

$$\boxed{\begin{aligned} E' &= \gamma_{u'} m_0c^2 \\ p' &= \gamma_{u'} m_0 u' \\ p_x' &= \gamma_{u'} m_0 u_x' \\ p_y' &= \gamma_{u'} m_0 u_y' \end{aligned}} \quad \beta' = \frac{u'}{c} \quad \beta_x' = \frac{u_x'}{c} \quad \beta_y' = \frac{u_y'}{c}$$

We also know how to transform Velocities:

$$\boxed{u_x' = \frac{u_x - V}{1-\frac{u_xV}{c^2}} \Rightarrow \beta_x' = \frac{\beta_x - \beta_v}{1-\beta_x\beta_v}} \quad \boxed{u_y' = \frac{u_y/\gamma_v}{1-\frac{u_xV}{c^2}} \Rightarrow \beta_y' = \frac{\beta_y}{\gamma_v(1-\beta_x\beta_v)}}$$

We use this to Lorentz transform $\gamma_{u'}$ back from S' to S

$$\gamma_{u'} = \frac{1}{\sqrt{1-\left(\frac{u'}{c}\right)^2}} = \frac{1}{\sqrt{1-\left(\frac{u_x'^2}{c^2}+\frac{u_y'^2}{c^2}\right)}} = 1\Big/\sqrt{1-\beta_x'^2-\beta_y'^2}$$

NOW L.T. VELOCITY

$$\gamma_{u'} = 1\Big/\sqrt{1-\left(\frac{\beta_x-\beta_v}{1-\beta_x\beta_v}\right)^2 - \left(\frac{\beta_y/\gamma_v}{1+\beta_x\beta_v}\right)^2}$$

$$\gamma_{u'} = 1\Big/\sqrt{\frac{(1-\cancel{2\beta_x\beta_v}+\beta_x^2\beta_v^2) - (\beta_x^2 - \cancel{2\beta_x\beta_v}+\beta_v^2) - (\beta_y/\gamma)^2}{(1+\beta_x\beta_v)^2}}$$

$$\gamma_{u'} = (1-\beta_x\beta_v)\Big/\sqrt{1+\beta_x^2\beta_v^2-\beta_x^2-\beta_v^2-\beta_y^2(1-\beta_v^2)}$$

$$\gamma_{u'} = (1-\beta_x\beta_v)\Big/\sqrt{(1-\beta_v^2) - \beta_x^2(1-\beta_v^2) - \beta_y^2(1-\beta_v^2)}$$

$$\gamma_{u'} = (1-\beta_x\beta_v)\Big/\sqrt{(1-\beta_v^2)(1-(\beta_x^2+\beta_y^2))}$$

$$\boxed{\gamma_{u'} = (1-\beta_x\beta_v)\,\gamma_v\gamma_u}$$

We now can use this in our energy formula

$$E' = \gamma_{u'} m_0c^2 = (\gamma_v\gamma_u(1-\beta_x\beta_v)m_0c^2)$$

$$= \gamma_v\left(\gamma_u m_0c^2 - \gamma_u m_0c^2 \cdot \frac{u_x}{c}\cdot\frac{V}{c}\right)$$

$$\boxed{E' = \gamma_v(E - p_xV)}$$

We can also transform momentum:

○ $p_x' = \gamma_{u'} m_0 u_x' = \gamma_v\gamma_u m_0\cancel{\left(1-\frac{Vu_x}{c^2}\right)}\left(\frac{u_x-V}{\cancel{1-\frac{Vu_x}{c^2}}}\right)$

$$= \gamma_v\left(\gamma_u m_0 u_x - \gamma_u m_0 c^2\frac{V}{c^2}\right)$$

$$\boxed{p_x' = \gamma_v\left(p_x - \frac{EV}{c^2}\right)}$$

○ $p_y' = \gamma_{u'} m_0 u_y' = \cancel{\gamma_v}\left(\gamma_u m_0\cancel{\left(1-\frac{Vu_x}{c^2}\right)}\right)\left(\frac{u_y}{\cancel{\gamma_v}}\cdot\frac{1}{\cancel{\left(1-\frac{Vu_x}{c^2}\right)}}\right)$

$$\boxed{p_y' = p_y}$$

$$\gamma_{u'} = \gamma_v\gamma_u\left(1-\frac{Vu_x}{c^2}\right)$$

$$E' = \gamma_v(E - p_xV)$$

$$p_x' = \gamma_v\left(p_x - \frac{EV}{c^2}\right)$$

$$p_y' = p_y$$

Lorentz Transformation of Force

- We know in S: $F = \frac{d}{dt}\vec{p}$ where $p = m\vec{u}$

  This means in S': $F' = \frac{d}{dt'}\vec{p'}$ where $p' = m'\vec{u'}$

- However, we also know:

$$p_x' = \gamma\left(p_x - \frac{vE}{c^2}\right)$$
$$p_y' = p_y$$
$$p_z' = p_z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

- Therefore we can use these to find:

$$F' = \frac{d\vec{p'}}{dt'} = \frac{d\vec{p'}}{dt} \cdot \frac{dt}{dt'} = \frac{\left(\frac{dp'}{dt}\right)}{\left(\frac{dt'}{dt}\right)}$$

- So we differentiate each of the above w.r.t t (as γ stays same, assume γ = constant, v = constant)

$$\frac{dp_x}{dt} = \gamma\left(\frac{dp_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}\right) \qquad \frac{dt'}{dt} = \gamma\left(\frac{dt}{dt} - \frac{v}{c^2}\frac{dx}{dt}\right) \qquad = \frac{\cancel{F_x - \frac{v}{c^2}\frac{dE}{dt}}}{1 - \frac{vu_x}{c^2}}$$

$$\frac{dp_y}{dt'} = \frac{dp_y'}{dt} \qquad \frac{dp_z'}{dt} = \frac{dp_z}{dt}$$

- We find F_x':

$$F_x' = \frac{\gamma\left(\frac{dP}{dt} - \frac{v}{c^2}\frac{dE}{dt}\right)}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{F - \frac{v}{c^2}\frac{dE}{dt}}{1 - \frac{vu_x}{c^2}}$$

- Now we just need to handle the dE/dt

  - we look at the invariance equation and take derivative w.r.t t

$E^2 - p^2c^2 = E_0 \Rightarrow E^2 = p^2c^2 + E_0^2$

$$\frac{d}{dt}(E^2) = \frac{d}{dt}\left(c^2(\vec{p}\cdot\vec{p})\right) + \underbrace{\frac{d}{dt}(E_0)^2}_{=0}$$

$$2E\frac{dE}{dt} = 2c^2\left(\vec{p}\cdot\frac{d\vec{p}}{dt}\right) = 2c^2\vec{p}\cdot\vec{F}$$

$$\frac{dE}{dt} = \frac{\cancel{2c^2}\, p\cdot\vec{F}}{2m\cancel{c^2}} = \frac{\vec{p}}{m}\cdot\vec{F} = \vec{u}\cdot\vec{F}$$

- We get $\boxed{\frac{dE}{dt} = u\cdot F}$

- returning back to the equations of force, we get:

$$F_x' = \frac{F_x - \frac{v}{c^2}F\cdot\vec{u}}{1 - \frac{vu_x}{c^2}} \qquad F_y' = \frac{F_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \qquad F_z' = \frac{F_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

$$\vec{F_x'} = \frac{\vec{F_x} - \frac{\vec{v}}{c^2}(\vec{F}\cdot\vec{u})}{\left(1 - \frac{vu_x}{c^2}\right)}$$

$$\vec{F_y'} = \frac{\vec{F_y}}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

$$\boxed{\frac{dE}{dt} = \vec{F}\cdot\vec{u}}$$

$dE = dF \cdot dx$

$\frac{dE}{dt} = F \cdot \frac{dx}{dt} - F \cdot du$

General Hints for Problem Solving

① Conservation of Mass-Energy

- The total mass-energy is always conserved.
- Add up $\boxed{E = \gamma m_0 c^2}$ for all the particles and it will be the same before & afterwards

② Conservation of Momentum

- momentum is also conserved along each axis.
- remember that this is $\boxed{p = \gamma m_0 v}$

③ Invariance of $E^2 - p^2c^2$

- remember that $E^2 - p^2c^2 = m_0c^2$ is the same in all reference frames
- Applies to whole system as well as individual particles

④ The centre of Momentum frame

- you can always find a frame in which the total momentum is zero

SPECIAL RELATIVITY, ELECTRICITY & MAGNETISM

- **Coulomb's Law:**

$$\boxed{F = k\frac{q_1 q_2}{r^2}\hat{r}}$$

where q_1 = source charge
q_2 = test charge
$\varepsilon_0 = 8.85\times10^{-12}\,\mathrm{Fm^{-1}}$
$k = 1/4\pi\varepsilon_0$

F = FORCE ON q_2

- HOLDS as long as q_1 is stationary

- magnetic field is generated by a moving charge (eg: a current in a wire)

$$\boxed{F_{mag} = q\vec{u}\times\vec{B}}$$

$\vec{B}$ = magnetic field

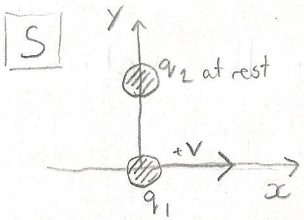
- Defines B (magnetic field)

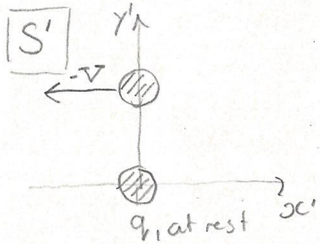
- **MAXWELL'S Equations (1864 A.D)**
  - Defined relations between $\vec{E}$ and $\vec{B}$ for stationary and moving charges
  - Developed BEFORE SPECIAL RELATIVITY BUT
    - CORRECT as $V \approx C$
    - Describes E.M. phenomena in ANY inertial frame

- **EINSTEIN:**
"what led me more or less directly to the special theory of relativity was the conviction that the electromotive force acting on a body moving in a magnetic field was nothing else than an electric field"

- Suppose we have a charge q_1 moving relative to a charge q_2 ⊥ to their displacement
- CONSIDER 2 REFERENCE FRAMES, S′ and S:

IN S: q_2 is at rest at time $t=0$: q_2 is at $(0, y, 0)$	IN S′: q_1 is at rest at time $t'=0$ q_2 is at $(0, y', 0)$





- IN S′, the source charge q_1 is at rest so we can use coulomb's Law:

$$\boxed{\vec{F_y'} = \frac{kq_1q_2}{y'^2}}$$

$\vec{F_x} = \vec{F_z} = 0$

- We can now transform this to S:

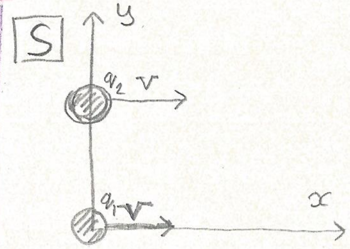
$$F_y = \frac{F_y'}{\gamma(1+\frac{VU_x}{c^2})} \xrightarrow{U_x=-V} \frac{F_y'}{\gamma(1-\frac{V^2}{c^2})} = \gamma F_y'$$

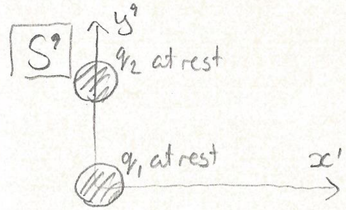
So we get coulomb's Law modified FOR A MOVING CHARGE:

$$\boxed{\vec{F_y} = \frac{\gamma kq_1q_2}{y^2}}$$

- Now suppose we have the same two charges both relatively at rest
- We shall consider the reference frames S and S′ yet again.

IN S: q_1 moving at V q_2 moving at V	in S′: q_1 at rest q_2 at rest $U_x=0$





- we know in S the force from coulomb's law:

$$\boxed{\vec{F_y'} = \frac{kq_1q_2}{y'^2}} \xrightarrow{L.T.} F_y = \frac{F_y'}{\gamma}\cdot\frac{1}{(1+\frac{VU_x'}{c^2})}$$

$$\boxed{\vec{F_y} = \frac{\vec{F_y'}}{\gamma} = \frac{1}{\gamma}\frac{kq_1q_2}{y^2}}$$

- This means we have an EXTRA FORCE on q_2 DUE TO its velocity in S: (the magnetic force)

$$F_{mag} = \frac{kq_1q_2}{y^2}\left(\frac{1}{\gamma}-\gamma\right) = \frac{kq_1q_2\gamma}{y^2}\left(\frac{1}{\gamma^2}-1\right)$$

$$\boxed{\text{BUT } \frac{1}{\gamma^2}-1 = 1-\frac{v^2}{c^2}-1 = -v^2/c^2}$$

$$\Rightarrow \boxed{F_{mag} = -\frac{v^2}{c^2}\gamma\frac{kq_1q_2}{y^2}} \boxed{= -\frac{v^2}{c^2}F_{elec}}$$