Chapter 6

Techniques of integration

6.1 Integration by parts

Theorem 6.1 – Integration by parts

If f, g are differentiable functions and their derivatives f', g' are continuous, then

$$\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx. \tag{6.1}$$

- It is quite common, and also convenient, to express the last equation in the form

  $$\int u\, dv = uv - \int v\, du. \tag{6.2}$$

  This alternative version arises by letting $u = f(x)$ and $v = g(x)$ for simplicity.

- Some typical integrals that may be computed using integration by parts are

  $$\int p(x) \cdot e^{ax}\, dx, \qquad \int p(x) \cdot \sin(ax)\, dx, \qquad \int p(x) \cdot \cos(ax)\, dx,$$

  where $p(x)$ is a polynomial and $a \neq 0$ is a given constant. In each of these cases, one needs to integrate by parts n times, where n is the degree of the polynomial.

- Some other integrals for which integration by parts might be needed are those that involve $\sin^{-1}$, $\tan^{-1}$ or ln. These are functions whose derivatives are much simpler, so they are all natural choices for the variable u that appears in equation (6.2).

Example 6.2 We use integration by parts to compute the integral

$$\int xe^x\, dx.$$

Letting $u = x$ and $dv = e^x\, dx$, we find that $du = dx$ and $v = e^x$. It easily follows that

$$\int xe^x\, dx = uv - \int v\, du = xe^x - \int e^x\, dx = xe^x - e^x + C. \qquad \square$$

Example 6.3 We use integration by parts to compute the integral

$$\int x\cos(2x)\,dx.$$

In this case, we take $u = x$ and $dv = \cos(2x)\,dx$. Since $du = dx$ and $v = \frac{1}{2}\sin(2x)$, we get

$$\int x\cos(2x)\,dx = \frac{x}{2}\,\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx = \frac{x}{2}\,\sin(2x) + \frac{1}{4}\,\cos(2x) + C. \qquad \square$$

Example 6.4 We use integration by parts to compute the integral

$$\int x^2 \ln x\,dx.$$

Letting $u = \ln x$ and $dv = x^2\,dx$, we find that $du = \frac{1}{x}\,dx$ and $v = \frac{x^3}{3}$. This implies that

$$\int x^2\ln x\,dx = \frac{x^3}{3}\,\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C. \qquad \square$$

Example 6.5 We use a double integration by parts to compute the integral

$$\int e^{ax}\sin(bx)\,dx, \qquad b \neq 0.$$

If we let $u = e^{ax}$ and $dv = \sin(bx)\,dx$, then $du = ae^{ax}\,dx$ and $v = -\frac{1}{b}\cos(bx)$, so

$$\int e^{ax}\sin(bx)\,dx = -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b}\int e^{ax}\cos(bx)\,dx.$$

Next, we take $u = e^{ax}$ and $dv = \cos(bx)\,dx$. Since $du = ae^{ax}\,dx$ and $v = \frac{1}{b}\sin(bx)$, one has

$$\begin{aligned}\int e^{ax}\sin(bx)\,dx &= -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b}\left[\frac{1}{b}\,e^{ax}\sin(bx) - \frac{a}{b}\int e^{ax}\sin(bx)\,dx\right]\\ &= -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b^2}\,e^{ax}\sin(bx) - \frac{a^2}{b^2}\int e^{ax}\sin(bx)\,dx.\end{aligned}$$

Here, the rightmost integral is the same as the leftmost integral, so we actually have

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\sin(bx)\,dx = -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b^2}\,e^{ax}\sin(bx).$$

Once we now multiply the last equation by $b^2/(a^2+b^2)$, we may finally conclude that

$$\int e^{ax}\sin(bx)\,dx = -\frac{be^{ax}\cos(bx)}{a^2+b^2} + \frac{ae^{ax}\sin(bx)}{a^2+b^2} + C. \qquad \square$$

6.2 Integration by substitution

- A very useful tool for simplifying integrals is provided by the formula

$$\int g(f(x)) \cdot f'(x)\, dx = \int g(u)\, du. \tag{6.3}$$

- Here, the idea is to introduce a variable $u = f(x)$ to replace the leftmost integral by another integral which is much simpler and also expressible in terms of u alone.

Example 6.6 We use an appropriate substitution to compute the integral

$$\int \frac{(\ln x)^3}{x}\, dx.$$

If we take $u = \ln x$, then we have $du = \frac{1}{x}\, dx$ and this is easily seen to imply that

$$\int \frac{(\ln x)^3}{x}\, dx = \int u^3\, du = \frac{1}{4}\, u^4 + C = \frac{1}{4}\, (\ln x)^4 + C. \qquad \square$$

Example 6.7 We use integration by substitution to compute the integral

$$\int x^2(x^3+6)^4\, dx.$$

In this case, the choice $u = x^3 + 6$ is suitable because $du = 3x^2\, dx$ and this gives

$$\int x^2(x^3+6)^4\, dx = \frac{1}{3}\int u^4\, du = \frac{1}{15}\, u^5 + C = \frac{1}{15}\, (x^3+6)^5 + C. \qquad \square$$

Example 6.8 We use an appropriate substitution to compute the integral

$$\int \frac{2x+7}{(x+2)^2}\, dx.$$

In this case, we take $u = x+2$ to merely simplify the denominator. Since $du = dx$, we get

$$\begin{aligned}\int \frac{2x+7}{(x+2)^2}\, dx &= \int \frac{2(u-2)+7}{u^2}\, du = \int 2u^{-1}\, du + \int 3u^{-2}\, du \\ &= 2\ln|u| - 3u^{-1} + C = 2\ln|x+2| - \frac{3}{x+2} + C. \qquad \square\end{aligned}$$

Example 6.9 We use integration by substitution to compute the integral

$$\int \cos\sqrt{x}\, dx.$$

If we let $u = \sqrt{x}$ to simplify the square root, then $x = u^2$ and $dx = 2u\, du$, so

$$\int \cos\sqrt{x}\, dx = \int \cos u \cdot 2u\, du = 2\int u\cos u\, du.$$

Next, we need to integrate by parts. Letting $dv = \cos u\, du$ and $v = \sin u$, we find that

$$\begin{aligned}\int \cos\sqrt{x}\, dx = 2u\sin u - 2\int \sin u\, du &= 2u\sin u + 2\cos u + C \\ &= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C. \qquad \square\end{aligned}$$

6.3 Reduction formulas

- A reduction formula expresses an integral I_n that depends on some integer n in terms of another integral I_m that involves a smaller integer m. If one repeatedly applies this formula, one may then express I_n in terms of a much simpler integral.

Example 6.10 We use integration by parts to establish the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cdot\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx. \tag{6.4}$$

If we take $dv = \sin x\,dx$, then we have $v = -\cos x$ and we may integrate by parts with

$$u = \sin^{n-1} x, \qquad du = (n-1)\sin^{n-2} x\cdot\cos x.$$

Using the fact that $\sin^2 x + \cos^2 x = 1$, one may thus conclude that

$$\begin{aligned}\int \sin^n x\,dx &= -\sin^{n-1} x\cdot\cos x + (n-1)\int \sin^{n-2} x\cdot\cos^2 x\,dx\\ &= -\sin^{n-1} x\cdot\cos x + (n-1)\int \sin^{n-2} x\cdot(1-\sin^2 x)\,dx\\ &= -\sin^{n-1} x\cdot\cos x + (n-1)\int \sin^{n-2} x\,dx + (1-n)\int \sin^n x\,dx.\end{aligned}$$

Here, the rightmost integral coincides with the original integral on the left. Once we now rearrange terms, we end up with n copies of the integral and equation (6.4) follows. □

Example 6.11 We use a reduction formula to compute the integral I_3 in the case that

$$I_n = \int x^n e^{2x}\,dx.$$

If we take $u = x^n$ and $dv = e^{2x}\,dx$, then $du = nx^{n-1}\,dx$ and $v = \frac{1}{2}e^{2x}$, so one has

$$I_n = \frac{1}{2}x^n e^{2x} - \frac{n}{2}\int x^{n-1}e^{2x}\,dx = \frac{1}{2}x^n e^{2x} - \frac{n}{2}\cdot I_{n-1}. \tag{6.5}$$

We now apply the last formula repeatedly to determine I_3. According to the formula,

$$\begin{aligned}I_3 &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\cdot I_2 = \frac{1}{2}x^3e^{2x} - \frac{3}{2}\cdot\left[\frac{1}{2}x^2e^{2x} - I_1\right]\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\cdot\left[\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\cdot I_0\right]\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\int e^{2x}\,dx\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C.\end{aligned}$$

□

Example 6.12 We use integration by parts to establish the reduction formula

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\cdot \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx. \tag{6.6}$$

In this case, we note that $(\tan x)' = \sec^2 x$ and we write the given integral as

$$\int \sec^n x\,dx = \int \sec^{n-2} x\cdot \sec^2 x\,dx.$$

If we take $dv = \sec^2 x\,dx$, then we have $v = \tan x$ and we may integrate by parts with

$$u = \sec^{n-2} x, \qquad du = (n-2)\sec^{n-3} x\cdot \sec x\tan x = (n-2)\sec^{n-2} x\cdot \tan x.$$

Using the fact that $1 + \tan^2 x = \sec^2 x$, one may thus establish the identity

$$\begin{aligned}\int \sec^n x\,dx &= \sec^{n-2} x\cdot \tan x - (n-2)\int \sec^{n-2} x\cdot \tan^2 x\,dx\\ &= \sec^{n-2} x\cdot \tan x - (n-2)\int \sec^{n-2} x\cdot (\sec^2 x - 1)\,dx\\ &= \sec^{n-2} x\cdot \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx.\end{aligned}$$

Since the integral on the left hand side also appears on the right hand side, this gives

$$(n-1)\int \sec^n x\,dx = \sec^{n-2} x\cdot \tan x + (n-2)\int \sec^{n-2} x\,dx.$$

In particular, the reduction formula (6.6) follows by dividing both sides with $n-1$. □

Example 6.13 Let $a \neq 0$ be some given constant and consider the integral

$$I_n = \int \frac{dx}{(x^2+a)^n} = \int (x^2+a)^{-n}\,dx.$$

If we take $u = (x^2+a)^{-n}$ and $dv = dx$, then we may integrate by parts to find that

$$I_n = x(x^2+a)^{-n} + n\int x(x^2+a)^{-n-1}\cdot 2x\,dx.$$

Let us now rearrange terms and express the last equation in the form

$$\begin{aligned}I_n &= x(x^2+a)^{-n} + 2n\int \frac{x^2+a-a}{(x^2+a)^{n+1}}\,dx\\ &= x(x^2+a)^{-n} + 2n\int \frac{dx}{(x^2+a)^n} - 2na\int \frac{dx}{(x^2+a)^{n+1}}.\end{aligned}$$

The integrals on the right hand side have the same form as the original integral, so

$$I_n = x(x^2+a)^{-n} + 2n\cdot I_n - 2na\cdot I_{n+1}.$$

Rearranging terms once again, one may thus establish the reduction formula

$$2na\cdot I_{n+1} = (2n-1)\cdot I_n + x(x^2+a)^{-n}.$$ □

6.4 Trigonometric integrals

Theorem 6.14 – Powers of sine and cosine

Consider the integral $\int \sin^m x \cdot \cos^n x \, dx$ for any non-negative integers m, n.

(a) When n is odd, one may compute this integral using the substitution $u = \sin x$.

(b) When m is odd, one may compute this integral using the substitution $u = \cos x$.

(c) When m, n are even, one may use the half-angle formulas to simplify the integral.

Theorem 6.15 – Powers of secant and tangent

Consider the integral $\int \sec^m x \cdot \tan^n x \, dx$ for any non-negative integers m, n.

(a) When n is odd, one may compute this integral using the substitution $u = \sec x$.

(b) When m is even, one may compute this integral using the substitution $u = \tan x$.

(c) When m is odd and n is even, one may reduce the integrand to powers of $\sec x$.

- The three cases that arise in Theorem 6.14 are closely related to the identities

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad \sin^2 x + \cos^2 x = 1.$$

  If one uses the substitution $u = \sin x$, then one may express any even power of cosine in terms of u^2, but also needs a copy of cosine for $du = \cos x \, dx$. This yields an odd number of cosines, so the substitution $u = \sin x$ will only help when n is odd.

- The last case that arises in Theorem 6.14 requires the half-angle formulas

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}, \qquad \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}. \tag{6.7}$$

  These formulas are helpful for reducing the even powers of sine and cosine.

- The three cases that arise in Theorem 6.15 are closely related to the identities

$$(\sec x)' = \sec x \tan x, \qquad (\tan x)' = \sec^2 x, \qquad 1 + \tan^2 x = \sec^2 x.$$

  These imply that an odd number of tangents is needed to substitute $u = \sec x$, while an even number of secants is needed to substitute $u = \tan x$.

Example 6.16 We use the substitution $u = \sin x$ to compute the integral

$$\int \sin^4 x \cdot \cos^5 x \, dx.$$

In this case, we have $du = \cos x\,dx$ and also $\sin^2 x + \cos^2 x = 1$, so

$$\begin{aligned}\int \sin^4 x \cdot \cos^5 x\,dx &= \int \sin^4 x \cdot (1-\sin^2 x)^2 \cdot \cos x\,dx = \int u^4(1-u^2)^2\,du \\ &= \int u^4(1-2u^2+u^4)\,du = \int (u^4 - 2u^6 + u^8)\,du \\ &= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C. \qquad \square\end{aligned}$$

Example 6.17 We use the half-angle formulas to simplify and compute the integral

$$\int \sin^2 x \cdot \cos^2 x\,dx.$$

Since the exponents are both even, one needs to express the integrand in the form

$$\begin{aligned}\sin^2 x \cdot \cos^2 x &= \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} = \frac{1}{4} \cdot \left[1 - \cos^2(2x)\right] \\ &= \frac{1}{4} \cdot \left[1 - \frac{1+\cos(4x)}{2}\right] = \frac{1}{8} \cdot [1 - \cos(4x)]\,.\end{aligned}$$

Once we now integrate both sides of this equation, we may easily conclude that

$$\int \sin^2 x \cdot \cos^2 x\,dx = \frac{1}{8}\left[x - \frac{\sin(4x)}{4}\right] + C = \frac{x}{8} - \frac{\sin(4x)}{32} + C. \qquad \square$$

Example 6.18 We use an appropriate substitution to compute the integral

$$\int \sec^4 x \cdot \tan^2 x\,dx.$$

If we let $u = \tan x$, then $du = \sec^2 x\,dx$ and also $\sec^2 x = 1 + \tan^2 x = 1 + u^2$, so one has

$$\begin{aligned}\int \sec^4 x \cdot \tan^2 x\,dx &= \int \sec^2 x \cdot \tan^2 x \cdot \sec^2 x\,dx = \int (1+u^2)\cdot u^2\,du \\ &= \int (u^2 + u^4)\,du = \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C. \qquad \square\end{aligned}$$

Example 6.19 We use an appropriate substitution to compute the integral

$$\int \frac{\sin^3 x}{\cos^8 x}\,dx.$$

Since the cosine appears in the denominator, it is better to first simplify and write

$$\int \frac{\sin^3 x}{\cos^8 x}\,dx = \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^5 x}\,dx = \int \tan^3 x \cdot \sec^5 x\,dx.$$

Let us take $u = \sec x$. Since $du = \sec x \tan x\,dx$ and also $u^2 = \sec^2 x = \tan^2 x + 1$, we get

$$\begin{aligned}\int \frac{\sin^3 x}{\cos^8 x}\,dx &= \int \tan^2 x \cdot \sec^4 x \cdot \sec x \tan x\,dx = \int (u^2-1)\cdot u^4\,du \\ &= \int (u^6 - u^4)\,du = \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C. \qquad \square\end{aligned}$$

6.5 Trigonometric substitutions

- Trigonometric substitutions are sometimes needed to simplify integrals that contain expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$ for some $a>0$. In each of these cases, one naturally seeks a substitution to simplify the square root.
- The three most common trigonometric substitutions are listed in the table below.

Expression	Substitution	Simplification	
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$\sqrt{a^2-x^2}=a\cos\theta,$	$dx=a\cos\theta\,d\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$\sqrt{a^2+x^2}=a\sec\theta,$	$dx=a\sec^2\theta\,d\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\sqrt{x^2-a^2}=a\lvert\tan\theta\rvert,$	$dx=a\sec\theta\tan\theta\,d\theta$

- In the first case, one has $a^2-x^2=a^2-a^2\sin^2\theta=a^2\cos^2\theta$ and $\sqrt{a^2-x^2}=a\cos\theta$. This is because $\theta=\sin^{-1}(x/a)$ lies between $-\pi/2$ and $\pi/2$, so $\cos\theta$ is non-negative.

Example 6.20 We use a trigonometric substitution to compute the integral

$$\int\frac{dx}{\sqrt{a^2-x^2}},\qquad a>0.$$

If we let $x=a\sin\theta$, then $a^2-x^2=a^2-a^2\sin^2\theta=a^2\cos^2\theta$ and also $dx=a\cos\theta\,d\theta$, so

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a\cos\theta\,d\theta}{a\cos\theta}=\int d\theta=\theta+C=\sin^{-1}\frac{x}{a}+C.\qquad\square$$

Example 6.21 We use a trigonometric substitution to compute the integral

$$\int\frac{dx}{x^2+a^2},\qquad a>0.$$

If we let $x=a\tan\theta$, then $x^2+a^2=a^2\tan^2\theta+a^2=a^2\sec^2\theta$ and also $dx=a\sec^2\theta\,d\theta$, so

$$\int\frac{dx}{x^2+a^2}=\int\frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}=\frac{1}{a}\int d\theta=\frac{1}{a}\theta+C=\frac{1}{a}\tan^{-1}\frac{x}{a}+C.\qquad\square$$

Example 6.22 We use a trigonometric substitution to compute the integral

$$\int\frac{x^2\,dx}{\sqrt{4-x^2}}.$$

If we let $x=2\sin\theta$, then $4-x^2=4-4\sin^2\theta=4\cos^2\theta$ and also $dx=2\cos\theta\,d\theta$, so

$$\begin{aligned}\int\frac{x^2\,dx}{\sqrt{4-x^2}}&=\int\frac{4\sin^2\theta\cdot2\cos\theta\,d\theta}{2\cos\theta}=\int4\sin^2\theta\,d\theta=2\int[1-\cos(2\theta)]\,d\theta\\&=2\theta-\sin(2\theta)+C=2\theta-2\sin\theta\cdot\cos\theta+C.\end{aligned}$$

It remains to express this equation in terms of $x=2\sin\theta$. Since $\theta=\sin^{-1}\frac{x}{2}$, we get

$$\int\frac{x^2\,dx}{\sqrt{4-x^2}}=2\sin^{-1}\frac{x}{2}-2\cdot\frac{x}{2}\cdot\sqrt{1-\frac{x^2}{4}}+C=2\sin^{-1}\frac{x}{2}-\frac{x}{2}\sqrt{4-x^2}+C.\qquad\square$$