Advanced Calculus
MA1132
Tutorial Exercises 2

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To be completed before and during tutorials of Friday, 8. February

1. Find an arc length parametrization of the curve
$$\mathbf{r}(t) = \sin(e^t)\mathbf{i} + \cos(e^t)\mathbf{j} + \sqrt{3}e^t\mathbf{k},$$
starting at the point $(\sin(1), \cos(1), \sqrt{3})$ and going in the same direction as the original curve.

2. For the curve
$$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}, \quad t \in \mathbb{R},$$
   (a) Find $\mathbf{T}(0)$, $\mathbf{N}(0)$ and $\mathbf{B}(0)$, using the definitions.
   (b) Find $\mathbf{B}(0)$ using the formula in Remark 1.4.6(e).

3. For the curve
$$\mathbf{r}(t) = e^t\sin(t)\mathbf{i} + e^t\cos(t)\mathbf{j} + 3\mathbf{k}, \quad t \in \mathbb{R},$$
   (a) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$ and $\mathbf{B}(t)$, using the definitions.
   (b) Find $\mathbf{B}(t)$ using the formula in Remark 1.4.6(e).

4. Consider the vector function (with values in $\mathbb{R}^3$)
$$\mathbf{r}(t) = \ln(3 - \sqrt{t})\,\mathbf{i} + (1 + \sqrt{t})\,\mathbf{j} + \frac{(3 - \sqrt{t})^2}{4}\,\mathbf{k} \tag{1}$$
   (a) Find the arc length of the graph of $\mathbf{r}(t)$ if $1 \le t \le 4$.
   (b) Find a negative change of parameter from t to s where s is an arc length parameter of the curve having $\mathbf{r}(4)$ as its reference point. It is sufficient to find s as a function of t.

5. Consider parabolic coordinates (μ, ν)
$$x = \mu\,\nu\,, \quad y = \frac{1}{2}(\mu^2 - \nu^2)\,. \tag{2}$$
   (a) Recall that if a parabola is described by the equation $y = a(x - h)^2 + k$, then its vertex is at (h, k) and its focus is at $(h, k + 1/(4a))$.
   Show that if you hold ν constant, the resulting curves form confocal parabolae that open upwards (i.e., towards $+y$), while holding μ constant results in curves which are confocal parabolae that open downwards (i.e., towards $-y$). Where are the foci of all these parabolae located?

(b) Show that in parabolic coordinates a curve given by the parametric equations $\mu = \mu(t)$, $\nu = \nu(t)$ for $a \le t \le b$ has arc length

$$L = \int_a^b \sqrt{(\mu^2 + \nu^2)\left(\left(\frac{d\mu}{dt}\right)^2 + \left(\frac{d\nu}{dt}\right)^2\right)}\, dt\,. \tag{3}$$

6. Consider the vector function

$$\mathbf{r}(t) = e^{-t}\,\mathbf{i} + e^{-t}\cos t\,\mathbf{j} - e^{-t}\sin t\,\mathbf{k}\,. \tag{4}$$

(a) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$, at $t = 0$.

(b) Find equations for the **TN**-plane at $t = 0$.

(c) Find equations for the **NB**-plane at $t = 0$.

(d) Find equations for the **TB**-plane at $t = 0$.