ADVANCED CALCULUS

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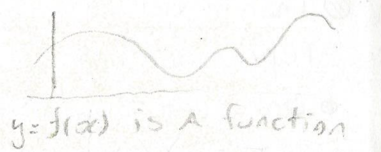
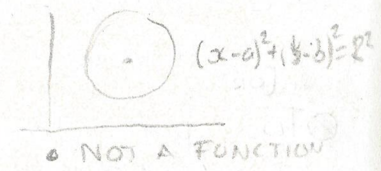
MA1132

VECTOR-VALUED FUNCTIONS - PARAMETRIC CURVES

- $y = f(x)$ a function
- $(x-a)^2 + (y-b)^2 = R^2$ circle
  - Not A FUNCTION
  - IS A PARAMETRIC CURVE

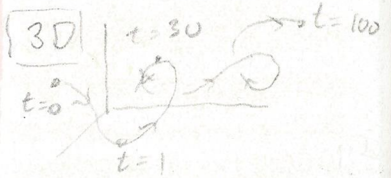
    represented by $x = a + R\cos t$, $y = b + R\sin t$





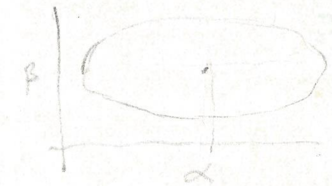
- IN GENERAL, a 2D parametric curve, has the form: $x=f(t)$ $y=g(t)$

  A 3D curve: $x=f(t)$ $y=g(t)$ $z=h(t)$



Ex 1: $\dfrac{(x-\alpha)^2}{a^2} + \dfrac{(y-\beta)^2}{\beta^2} = 1$ Elipse (α, β)

PARAMETRIC: $x = \alpha + a\sin t$ $y = \beta + b\sin t$



Ex 2: $\dfrac{(x-\alpha)^2}{a^2} - \dfrac{(y-\beta)^2}{b^2} = 1$ Hyperbola

Gives: $x = \alpha + a\sinh t$ $y = \beta + b\cosh t$

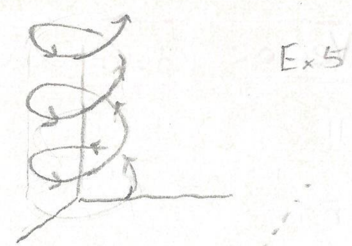
Ex 3: $y = ax^2$ parabola $\Rightarrow x = t, y = at^2$

Ex 4: $y^2 = ax \Leftrightarrow \frac{1}{a}y^2 = x \Rightarrow \boxed{x = \frac{1}{a}t^2 \ \& \ y = t}$

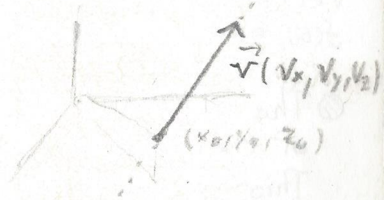
- IN GENERAL: $y = f(x) \rightarrow y = f(t)$ $x = t$

  $x = f(y) \rightarrow x = f(t)$ $y = t$

Ex 5: $x = R\cos t$ $y = R\sin t$ $z = vt$ $(v>0)$
- upwards spiral



Ex 6: $x = x_0 + V_x t$ $y = y_0 + V_y t$ $z = z_0 + V_z t$
- a line in 3D



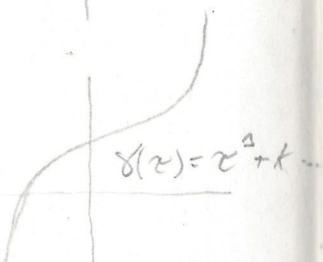
Ex 7: $x = x_0 + V_x\left(-\frac{t}{2}\right)$ $y = y_0 + V_y\left(-\frac{t}{2}\right)$ $z = z_0 + V_z\left(-\frac{t}{2}\right)$
- the same line but oriented differently and "slower"

- can also replace w/ t^3 and still OK • t^2 only gives positive

Parametric Equations

- There exists *infinitely many parametric Eq.s* representing a curve C.
  - one can make a change of parameter $t = \gamma(\tau)$, where $\gamma(\tau)$ is a *one-to-one function* on the interval $\alpha \leq \tau \leq \beta$ *of interest* [OR
- Two parameterisations are <u>equivalent</u> if $\gamma'(\tau) \neq 0$ for $\alpha \leq \tau \leq \beta$
  - FURTHERMORE, *if* $\gamma'(\tau) > 0$, then the two curves have the same orientation

$\gamma(\tau) = \tau^3$



Ex: $x = \cos t$, $y = \sin t$ - *circle anticlockwise* ↺

(i) let us look at $\gamma_1(\tau) = -\tau$

- This gives us $\gamma_1'(\tau) = -1$ ~~≠ 0 ⇒ equivalent~~

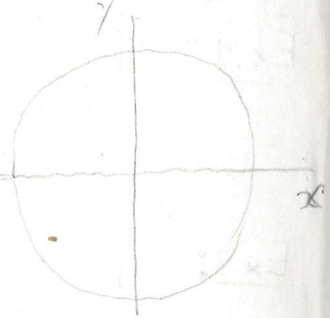
  $|-1| \neq 0 \Rightarrow$ EQUIVALENT TO PREVIOUS *function*

  $-1 < 0 \Rightarrow$ *opposite orientation (clockwise ↻)*

  $x = \cos \tau \qquad y = \sin -\tau$

(ii) Now let us look at $\gamma(\tau) = \tau - \frac{\pi}{2} \Rightarrow x = \sin \tau$, $y = \cos \tau$

$\gamma_2'(\tau) = 1 \neq 0$ = EQUIVALENT and same orientation



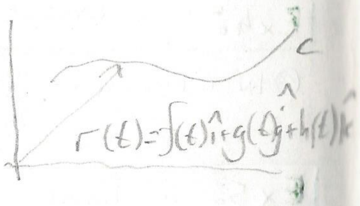
Vector-Valued Functions

IF we have par. eqs. $x = f(t), y = g(t), z = h(t)$, $\vec{r} = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$

$\vec{r}$ is a function of $t \in \mathbb{R}$ and it is vector-valued of one real variable, then $f(t), g(t), h(t)$ are called <u>*component functions*</u>.

- The DOMAIN of $\vec{r}(t)$ is the *intersection* of the domains of x, y, z.

  This is called the <u>*natural domain*</u> *of* $\vec{r}(t)$



Ex: $\vec{r}(t) = \sqrt{t-1}\,\hat{i} + \ln(2-t)\hat{j} + e^{2t}\hat{k}$

$t \geq 1 \qquad t < 2 \qquad t \in \mathbb{R}$

$\Rightarrow \{t \geq 1\} \cap \{t < 2\}$

$D(\vec{r}) = [1, 2)$

Vector Form of a Line Segment

If $\vec{r_0}$ is a *position vector* w/ initial point at $P_0 = 0$ and a terminal point at $P_1 = (x_1, y_1, z_1)$.

Then, the line through P_0P_1 is <u>*parallel to* $\vec{v}$</u>

$\Rightarrow \vec{r} = \vec{r_0} + \vec{v}t$

so $\vec{r}(0) = \vec{r_0}$, $\vec{r}(1) = \vec{r_1} = \vec{r_0} + \vec{v}$

$$\boxed{\vec{r}(t) = \vec{r_0} + t(\vec{r_1} - \vec{r_0})}$$

This is called the *two-point form of a line*

also written as: $\vec{r}(t) = (1-t)r_0 + t\vec{r_1}$

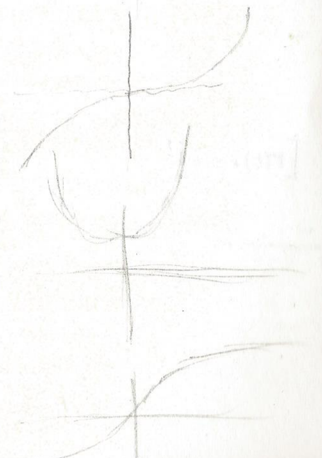
- IF $0 \leq t \leq 1$, then (*) represents the line segment from $\vec{r_0}$ to $\vec{r_1}$

Hyperbolic Functions

$$\sinh(t) = \frac{e^t - e^{-t}}{2} = i\sin(it)$$

$$\cosh(t) = \frac{e^t + e^{-t}}{2} = \cos(it)$$

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)}$$



Calculus of Vector-Valued Functions

$$\lim_{t\to a} \vec{r}(t) = \left(\lim_{t\to a} x(t)\right)\hat{i} + \left(\lim_{t\to a} y(t)\right)\hat{j} + \left(\lim_{t\to a} z(t)\right)\hat{k}$$

- Limit exists for $\vec{r}$ if it exists for x, y, z

  $\vec{r}(t)$ is continuous if x, y, z are continuous

Derivative of Vector-Valued Functions

$$\vec{r}\,'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$$

$$= \lim_{\varepsilon\to 0}\left(\frac{x(t+\varepsilon)-x(t)}{\varepsilon}\right)\hat{i} + \lim_{\varepsilon\to 0}\left(\frac{y(t+\varepsilon)-y(t)}{\varepsilon}\right)\hat{j} + \lim_{\varepsilon\to 0}\left(\frac{z(t+\varepsilon)-z(t)}{\varepsilon}\right)\hat{k}$$

$$= \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\left[(x(t+\varepsilon)-x(t))\hat{i} + (y(t+\varepsilon)-y(t))\hat{j} + (z(t+\varepsilon)-z(t))\hat{k}\right]\right)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[r(t+\varepsilon) - r(t)\right]$$

Derivative Rules:

$$\frac{d}{dt}\left[a\vec{r}_1(t) + b\vec{r}_2(t)\right] = a\vec{r}_1\,'(t) + b\vec{r}_2\,'(t)$$

$$\frac{d}{dt}\left[\vec{f}(t)\vec{r}(t)\right] = \vec{f}\,'(t)\vec{r}(t) + \vec{f}(t)\vec{r}\,'(t)$$

CROSS AND DOT PRODUCT

$$[r(t)\cdot s(t)]' = r(t)\cdot s'(t) + r'(t)\cdot s(t)$$

$$= \left[x_r(t)\cdot x_s'(t) + x_r'(t)\cdot x_s(t)\right]\hat{i} + \cdots$$

provided both have the same dimension for their codomain.
Generalisable to infinite spaces (called "INNER PRODUCT")

$$[r(t)\times s(t)]' = r(t)\times s'(t) + r'(t)\times s(t)$$

provided the codomain of r and s is $\mathbb{R}^3$ or $\mathbb{R}^7$

Calculus of Vector-Valued Functions: Integration, $t = y(t)$

THM: $r: [a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^n$)

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\hat{i} + \left(\int_a^b y(t)\,dt\right)\hat{j}$$

THM:

$$\int_a^b k\vec{r}(t)\,dt = k\int_a^b \vec{r}(t)\,dt$$

$$\int_a^b \vec{r}(t) + \vec{s}(t)\,dt = \int_a^b \vec{r}(t)\,dt + \int_a^b \vec{s}(t)\,dt$$

- **ANTIDERIVATIVE** of r is any R s.t. we have $R'(t) = r(t)$
- **INDEFINITE INTEGRAL** of r is

$$\int r(t)\,dt = R(t) + c$$

where R is any ANTIDERIVATIVE, C is a general vector

- It follows that if $X' = x$, $Y' = y$ that $R = (X(t)\ Y(t))$
- FUNDAMENTAL THEOREM OF CALCULUS still holds:

$$\left(\int r(t)\,dt\right)' = r(t) \quad \text{and} \quad \int r'(t)\,dt = r(t) + c$$

- multiple and sum rules still hold
- Lastly, for a real-valued function:

THM:

$$\int_a^b r(t)\,dt = \left|R(t)\right|_a^b = R(b) - R(a)$$

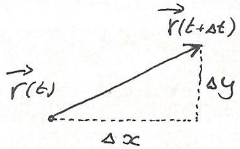
Change of Parameter: Arc-Length

- $\vec{r}(t), \quad t = g(\tau) \Rightarrow \vec{r}(t) = \vec{r}(g(\tau))$

$$s(t) = \int_{t_0}^{t} |r'(t)|\,dt$$

- we try to get arc length

$\Delta t = t_{i+1} - t_i \qquad \lim_{n \to \infty} \Delta t = 0$



$\Delta r = \sqrt{\Delta x^2 + \Delta y^2}$ (PYTHAGORAS)

$$\frac{\Delta r}{\Delta t} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$$\sum \frac{\Delta r}{\Delta t} = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$$\sum \Delta r = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$$

$$s = \int ds = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \quad \text{ARCLENGTH}$$

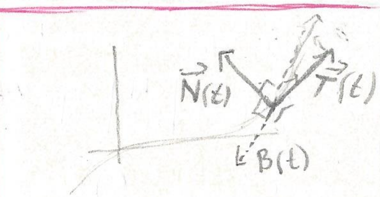
- $r(s)$ arc-length parameterisation is sometimes very useful

$r(t)$

$s = \int \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}\,dt \rightarrow$ t in terms of $s \rightarrow t(s)$

$r(t) \rightarrow \boxed{r(t(s))}$

Unit Tangent, Normal, Binormal Vectors



Unit Tangent Vector:

$\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ is the unit tangent vector

Unit Normal Vector:

$\vec{N}(t) = \dfrac{\vec{T}'t}{|T'(t)|} = \dfrac{\vec{r}''(t)}{|\vec{r}''(t)|}$ is unit "PRINCIPAL" Normal Vector

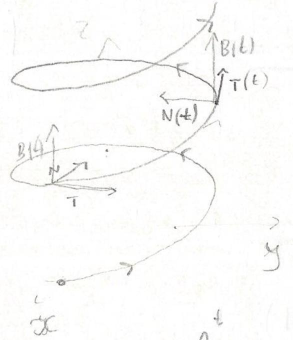
(THOUGH THERE ARE INFINITELY MANY, this ONE HAS THE most PHYSICAL USE (PRINCIPAL))

Unit Binormal Vector:

$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ if r is smooth

- size already 1: $|B(t)| = |T(t)| \cdot |N(t)| \cos 90° = 1$

Ex: $\vec{r}(t) = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + a t\,\hat{k} \quad (a > 0)$



$$\vec{T}(t) = \frac{-a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath} + a\,\hat{k}}{\sqrt{a^2(\sin^2 t + \cos^2 t) + a^2}} =$$

$$= \frac{a}{\sqrt{2a^2}}\left(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}\right) = \frac{1}{\sqrt{2}}\left(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} + \hat{k}\right)$$

$$\vec{T}'(t) = \frac{1}{\sqrt{2}}\left(-\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}\right) \qquad |T'(t)| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \vec{N}(t) = -\cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{pmatrix} -\frac{\sin t}{\sqrt{2}} \\ \frac{\cos t}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \times \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$$

$$\vec{B}(t) = \left(\frac{\sin^2 t}{\sqrt{2}} + \frac{\cos^2 t}{\sqrt{2}}\right)\hat{k} + \left(-\frac{\cos t}{\sqrt{2}}\right)\hat{\jmath} + \left(\frac{\sin t}{\sqrt{2}}\right)\hat{\imath}$$

$$\vec{B}(t) = \frac{\sin t}{\sqrt{2}}\,\hat{\imath} - \frac{\cos t}{\sqrt{2}}\,\hat{\jmath} + \frac{1}{\sqrt{2}}\,\hat{k}$$

$|r'(t)| = a\sqrt{2} \Rightarrow s(t) = \int_0^t a\sqrt{2}\,du = at\sqrt{2}$

$\Rightarrow s = at\sqrt{2} \Rightarrow \boxed{t = s/a\sqrt{2}}$

$$\vec{r}(s) = \begin{pmatrix} a\cos\frac{s}{\sqrt{2}a} \\ a\sin\frac{s}{a\sqrt{2}} \\ \frac{at}{a\sqrt{2}} \end{pmatrix} \Rightarrow \vec{r}'(s) = \begin{pmatrix} \frac{a}{a\sqrt{2}}\left(-\sin\frac{s}{a\sqrt{2}}\right) \\ \frac{a}{a\sqrt{2}}\cos\left(\frac{s}{a\sqrt{2}}\right) \\ 1/\sqrt{2} \end{pmatrix}$$

$$\Rightarrow T(s) = \vec{r}'(s) = -\frac{\sin}{\sqrt{2}}\left(\frac{s}{a\sqrt{2}}\right)\hat{\imath} + \frac{\cos}{\sqrt{2}}\left(\frac{s}{a\sqrt{2}}\right)\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k}$$

$$\vec{T}'(s) = \frac{-\cos}{2a}\left(\frac{s}{a\sqrt{2}}\right)\hat{\imath} - \frac{\sin}{2a}\left(\frac{s}{a\sqrt{2}}\right)\hat{\jmath}$$

$$|T'(s)| = \frac{1}{2a}$$

$$N(s) = -\cos\left(\frac{s}{a\sqrt{2}}\right)\hat{\imath} - \sin\left(\frac{s}{a\sqrt{2}}\right)\hat{\jmath}$$

$$B(s) = \frac{\sin}{\sqrt{2}}\left(\frac{s}{a\sqrt{2}}\right)\hat{\imath} - \frac{\cos}{\sqrt{2}}\left(\frac{s}{a\sqrt{2}}\right) + \frac{\hat{k}}{\sqrt{2}}$$

In Arc-Length Parameterisation

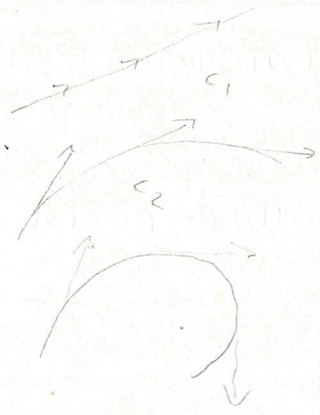
$$\vec{T}(s) = \vec{r}'(s) \qquad \vec{N}(s) = \frac{r''(s)}{\|r''(s)\|} \qquad \vec{B}(s) = \frac{\vec{r}'(s) \times \vec{r}''(s)}{|\vec{r}''(s)|}$$

CURVATURE

- We define curvature to be

$$\mathscr{K}(s) = \left|\frac{dT}{ds}\right| = |r''(s)|$$

$$\Rightarrow \vec{T}'(s) = \mathscr{K}(s)\,\vec{N}(s)$$



Ex: circle: $\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$ $\quad 0 \le t \le 2\pi$

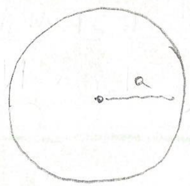
$$s(t) = \int_0^t a\,dt = at \Rightarrow s = at \Rightarrow t = \frac{s}{a}.$$

$$\vec{r}(s) = a\cos\left(\frac{s}{a}\right)\hat{i} + a\sin\left(\frac{s}{a}\right)\hat{j}$$

$$\vec{r}'(s) = -\sin\left(\frac{s}{a}\right)\hat{i} + \cos\left(\frac{s}{a}\right)\hat{j} = \vec{T}(s)$$

$$T'(s) = -\frac{\cos(s/a)}{a}\hat{i} - \frac{\sin(s/a)}{a}\hat{j}$$

$$|T'(s)| = \mathscr{K}(s) = 1/a \qquad \text{so } \frac{1}{\mathscr{K}} = r$$



Ex: LINE $\quad \vec{r}(s) = \vec{r}_0 + s\vec{v}$

$$T(s) = r'(s) = \vec{v}$$

$$T'(s) = r''(s) = 0 \Rightarrow \mathscr{K}(s) = 0$$

1) $\mathscr{K} = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\cdot\frac{dt}{ds}\right| = \left|\frac{T'(t)}{r'(t)}\right|$

2) $\vec{r}'(t) = |\vec{r}'(t)|\,\vec{T}(t) \Rightarrow$

$$\Rightarrow r''(t) = \frac{d}{dt}\left(\vec{r}'(t)\right)T(t) + |\vec{r}'(t)|\frac{d}{dt}\left(T(t)\right)$$

$$T'(t) = |T'(t)|\,N(t)$$

$$|T'(t)| = \mathscr{K}(t)\cdot|\vec{r}'(t)|$$

$$T'(t) = \mathscr{K}(t)\cdot|\vec{r}'(t)|\,N(t)$$

$$\Rightarrow r''(t) = |\vec{r}'(t)|\,\vec{T}(t) + \mathscr{K}(t)|\vec{r}'(t)|^2\,\vec{N}(t)$$

- we multiply $T(t)$ by $r''(t)$

$$\frac{r'(t)}{|r'(t)|}\times r''(t) = \cancel{\vec{T}(t)\times|r'(t)|\vec{T}(t)}^{\,0} + \vec{T}(t)\times\mathscr{K}(t)|r'(t)|^2 N(t)$$

$$|T(t)\times r''(t)| = |\mathscr{K}(t)|r'(t)|^2 B(t)| \Rightarrow \boxed{\mathscr{K}(t) = \frac{|r'(t)\times r''(t)|}{|r'(t)|^3}}$$

$$\mathscr{K}(t) = \frac{|r'(t)\times r''(t)|}{|r'(t)|^3}$$

RADIUS OF CURVATURE

- If a curve C has non-zero curvature at a point P, then we call the circle of the radius $\rho = \frac{1}{\mathscr{K}}$ TANGENT to C at P the OSCULATING CIRCLE or CIRCLE of CURVATURE.
- ρ is called the radius of curvature at P

IN 2-DIM: $T(\phi) = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j} \Rightarrow$

$$\Rightarrow T'(\phi) = \frac{dT}{ds}\cdot\frac{ds}{d\varphi}$$

$$\mathscr{K}(s) = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{d\varphi}\right|\cdot\left|\frac{d\varphi}{ds}\right| = \left|\cancel{-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}}^{\,1}\right|\cdot\left|\frac{d\varphi}{ds}\right| = \left|\frac{d\varphi}{ds}\right|$$

MOTION ALONG A CURVE

- $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{ds}{dt}\vec{T}(t)$ $\qquad$ Here, $\frac{ds}{dt}$ = speed

- $\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2s}{dt^2}\vec{T}(t) + \frac{ds}{dt}\vec{T}'(t)$

$$= \frac{d^2s}{dt^2}\vec{T}(t) + \mathscr{K}(t)\left|\frac{ds}{dt}\right|^2 N(t)$$

so $\vec{a}(t) = a_T\,\vec{T}(t) + a_N\,\vec{N}(t)$

a_T: TANGENTIAL SCALAR OF ACCELERATION

a_N: NORMAL SCALAR OF ACCELERATION

- DISTANCE TRAVELLED between t_1 and t_2

$$\Delta\vec{r} = \int_{t_1}^{t_2}\left|\frac{d\vec{r}}{dt}\right|dt = \int_{t_1}^{t_2}|v(t)|\,dt = \vec{r}(t_2) - \vec{r}(t_1)$$

$$a_T = |\vec{T}\cdot\vec{a}|$$

$$a_N = |\vec{T}\times\vec{a}|$$

SERRET FRENET FORMULAE

① $\frac{d}{ds}\vec{T} = \kappa(s)\vec{N}(s)$

② $\frac{d}{ds}\vec{N} = \tau(s)\vec{B}(s) - \kappa(s)\vec{T}(s)$

③ $\frac{d}{ds}\vec{B} = -\tau(s)\vec{N}(s)$

where $\kappa(s) = \|r''(s)\|$ curvature

$\tau(s) = N'(s) \cdot B(s)$ TORSION

PROOF:

① $\frac{d}{ds}T = \frac{d}{ds}r'(s) = r''(s)$ BUT $N(s) = \frac{r''(s)}{\|r''(s)\|}$

$\Rightarrow r''(s) = \|r''(s)\| N(s)$ $\Rightarrow \boxed{\frac{d}{ds}T = \kappa N}$

③ $\frac{d}{ds}B = \frac{d}{ds}(T \times N) = \frac{d}{ds}T \times N + T \times \frac{d}{ds}N$

$= \tau \cancel{N \times N} + T \times \frac{d}{ds}N$

• rewrite $\frac{d}{ds}N$ IN TERMS OF ITS COMPONENTS

$= T \times \Big(\underbrace{\left(\frac{d}{ds}N \cdot T\right)\hat{T}}_{T \times T = 0} + \underbrace{\left(\frac{dN}{ds} \cdot N\right)\hat{N}}_{\text{FIXED SIZE} \Rightarrow \text{ZERO}} + \underbrace{\left(\frac{dN}{ds} \cdot B\right)\hat{B}}_{\text{IS LEFT}} \Big)$

$\frac{d}{ds}B = \left(\frac{dN}{ds} \cdot B\right) \hat{T} \times \hat{B} = -\left(\frac{dN}{ds} \cdot B\right)\hat{N} = \boxed{-\tau \hat{N}}$

② $\frac{d}{ds}N = \frac{d}{ds}(B \times T) = \frac{d}{ds}B \times T + B \times \frac{d}{ds}T$

$= -\tau N \times T + B \times \kappa N$

$= \tau B - \kappa T$

CHAPTER: PARTIAL DERIVATIVES

FUNCTIONS OF 2 OR MORE VARIABLES

i.e: formulas w/ 2 or more inputs

eG: $A_\Delta = \frac{1}{2}\text{base} \times \text{height}_\perp$ $\quad V_\square = A_\Delta \cdot h = \frac{1}{2} bh \cdot H$

Arithmetic Average $\bar{x}$ of n real numbers

$\bar{x} = 1/n (x_1 + x_2 + \dots + x_n) = 1/n \sum_{j=1}^{n} x_j$

- IN THESE EXAMPLES, WE HAVE
  - A is a function of 2 variables $(b, h) \to A(b,h)$
  - V is of 3 $(b,h,H) \to V(b,h,H)$
  - $\bar{x}$ is of n variables $(x_1, \dots x_n) \mapsto \bar{x}(x_1, \dots, x_n)$
- WE WRITE: $h(x_1, \dots, x_n)$ and $P = (\hat{x}_1, \hat{x}_2 \dots x_n)$, $P \in \mathbb{R}^n$

  $h: P \mapsto u \in \mathbb{R}$
- INDEPENDANT VARIABLES CAN BE RESTRICTED TO A SET D. THIS IS THE DOMAIN.

NATURAL DOMAIN: of $h(\cdot\cdot)$ is the set of ALL $P \in \mathbb{R}^n$ s.t. $h(P) \in \mathbb{R}$

EX: $g(x,y) = \sqrt{x+y}$

Natural Domain is $\{(x,y) \mid x+y \geq 0\}$

The GRAPH: of a function $f(x,y)$ in the xyz space is defined to be the set of points $(x, y, f(x,y))$, where $(x,y) \in D$.

IN GENERAL, this surface is 3 Dimensional

Let $z = f(x,y)$. What's the GRAPH?

Ex: $z = 0 \to$ GRAPH IS THE xy PLANE

$z = 1 \to$ GRAPH IS PLANE THROUGH $(0,0,1) \parallel$ to xy plane

$z = x \to$ line $z=x$ extended into the y

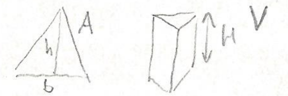
$z = ax + by + c \leftrightarrow$ linear example of x&y GIVES A PLANE

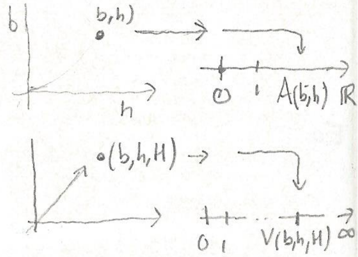
VECTOR VALUED FUNCTION

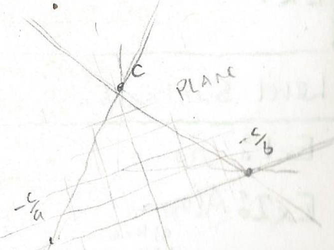
SCALAR → VECTOR

HERE

VECTOR → SCALAR







Ex 6 $Z = \sqrt{1 - x^2 - y^2}$ $\quad D(f) = \{ \binom{x}{y} : x^2 + y^2 < 1 \}$

disk bounded by a circle

$Z^2 = 1 - x^2 - y^2$

$Z^2 + x^2 + y^2 = 1 \quad \Rightarrow Z \geq 0 \quad$ SPHERE POSITIVE PART

Ex 6: $Z = -\sqrt{x^2+y^2} \rightarrow D(f) = \mathbb{R}^2$

$Z^2 = x^2 + y^2 \quad Z \leq 0$

CIRCULAR CONE - NEGATIVE PART

Ex 7: $Z = \sqrt{1-x^2} \Rightarrow \boxed{Z \geq 0}$ $\quad D(f) = \{ \binom{x}{y} : x^2 \leq 1 \}$

$Z^2 = 1 - x^2 \qquad \{ \binom{x}{y} : -1 \leq x \leq 1 \}$

$\boxed{Z^2 + x^2 = 1}$

Ex 8: $Z = \sqrt{x^2+y^2-1}$ $\quad D(f) = \{ x^2 + y^2 > 1 \}$

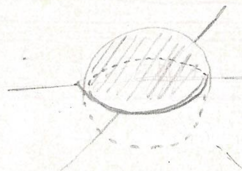
$Z^2 = x^2 + y^2 - 1$

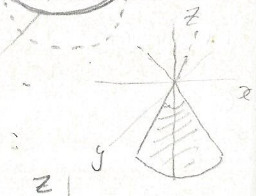
$x^2 + y^2 - z^2 = 1 \quad$ *hyperboloid of 1 sheet*

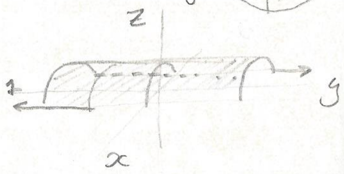
Ex 9: $Z = \sqrt{x^2+y^2+1}$ $\quad D(f) = \cancel{\{x^2+y^2 \geq 1\}}\ \mathbb{R}^2$

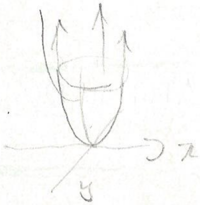
$Z^2 - x^2 - y^2 = 1$

Ex 10: $Z^2 = x^2 + y^2 \leftarrow$ *Paraboloid*









Level Curves / Surfaces

- Let $z = f(x, y)$.

Then the level curve for $z = k$ are the points of intersection in between the graphs of $z = f(x, y)$ and $z = k$

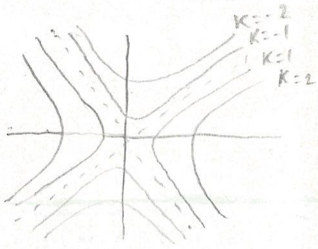
$\Updownarrow$

graph of $f(x, y) = k$

"contour" plot

Ex 1 $f(x,y) = y^2 - x^2 \qquad y^2 - x^2 = K$

Hyperbolic PARABALOID



Level Surfaces: 3D INTERSECTION OF 4D GRAPHS

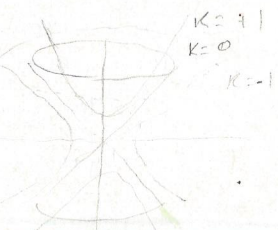
Ex 1: $f(x,y,z) = x^2 + y^2 + z^2 \Rightarrow x^2 + y^2 + z^2 = k \geq 0$

Ex 2: $A(x,y,z) = z^2 - x^2 - y^2 \Rightarrow z^2 - x^2 - y^2 = 0$ $\quad$ CONE

a) $k = 0 \rightarrow z^2 = x^2 + y^2$

b) $k = d^2 > 0 \rightarrow z^2 - x^2 - y^2 = d^2 \quad$ Hyperboloid of 2 sheets

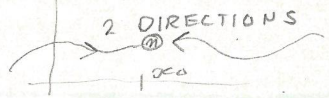
c) $k = -d^2 < 0 \rightarrow x^2 + y^2 - z^2 = d^2 \quad$ Hyperboloid of 1 sheet



Limits & Continuity

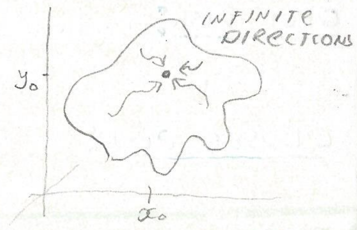
Def: $\lim_{n \to \infty} x_n = L$ means: for any $\varepsilon > 0$, there exists N_ε s.t. $|L - x_n| < \varepsilon$ for all $n > N_\varepsilon$

- $\lim_{x \to x_0} f(x)$ exists iff $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$



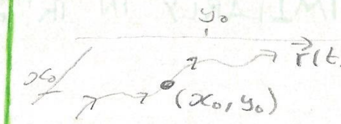
- What is $\lim_{(x,y) \to (x_0, y_0)} f(x)$?

$\rightarrow$ Limits along specific curves



DEF: suppose $\vec{r}(t) = (x(t), y(t))$, with $\vec{r}(t) = (x_0, y_0)$ and $\vec{r}$ is smooth parameterises C

THEN: $\lim_{\substack{\binom{x}{y} \to \binom{x_0}{y_0} \\ \text{Along } C}} f(x,y) = \lim_{t \to t_0} f(\vec{r}(t))$ $\quad$ TWO SIDED LIMIT



- n- dimensions: $\lim_{\substack{(x_1, \dots x_n) \to (x_{0_1}, \dots, x_{0_n}) \\ \text{Along } C}} f(x_1, \dots x_n)$

where C is parameterised by a curve $\vec{r}(t) = (x_1(t), \dots\ x_n(t))$

$\vec{r}(t_0) = (x_{0_1}, \dots\ x_{0n})$

Becomes $\lim_{t \to t_0^+} f(x_1(t), \dots, x_n(t))$

$f: \mathbb{R}^n \to \mathbb{R} \qquad \vec{r}: \mathbb{R} \to \mathbb{R}^n \qquad f \circ \vec{r} = \mathbb{R} \to \mathbb{R}$

FOR SIMPLICITY, WRITE VECTORS: $\lim_{\substack{\vec{x} \to \vec{a} \\ \text{Along } c}} f(\vec{x}) = \lim_{t \to t_0^+} f(\vec{r}(t))$

Ex: $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\quad$ NOT DEFINED AT $(0,0)$

a) LIMIT Along x-AXIS: $\vec{r}(t) = (t, 0)$, $f(\vec{r}(t)) = f(t,0) = 0$

b) y-Axis: SAME $\quad \Rightarrow \lim f(\vec{r}(t)) = 0$

c) $y = x$, $\vec{r}(t) = (t,t)$, $f(\vec{r}(t)) = \dfrac{t^2}{2t^2} \rightarrow \frac{1}{2}$

d) $y = -x$, $\vec{r}(t) = (t,-t)$, $f(\vec{r}(t)) = \dfrac{-t^2}{2t^2} \rightarrow -\frac{1}{2}$

e) $y = x^2$, $\vec{r}(t) = (t, t^2)$, $f(\vec{r}(t)) = t/1+t^2 \rightarrow 0$

- LIMITS NOT CONSISTENT $\Rightarrow$ LIMIT DOES NOT EXIST

OPEN/CLOSED DISKS/BALLS

DEF: Let C be a circle of radius $\delta > 0$ be centered at (a,b). Then we call the points enclosed by C the disk/ball of radius δ centered at (a,b)

→ This in the n-dimensional analogue of an interval

OPEN DISK: Centered at (a,b) of radius δ

$$B_\delta(a,b) = \{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : (x-a)^2 + (y-b)^2 < \delta \}$$

CLOSED DISK: ALL points in $B_\delta(a,b)$ $\cup$ Points on Circle

so $\{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : (x-\delta)^2 + (y-\delta)^2 \leq \delta \}$

SIMILARLY IN $\mathbb{R}^n$:

$$S_\delta^{n-1}(\vec{a}) = \left\{ \left(\begin{smallmatrix} x_1 \\ \vdots \\ x_n \end{smallmatrix}\right) : (x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 = \delta^2 \right\}$$

a SPHERE (n-DIMENSIONAL) centered at $\vec{a} = \left(\begin{smallmatrix} a_1 \\ \vdots \\ a_n \end{smallmatrix}\right)$

OPEN BALL: $B_\delta(\vec{a}) = \{ \left(\begin{smallmatrix} x_1 \\ \vdots \\ x_n \end{smallmatrix}\right) : (x_1 - a)^2 + \cdots + (x_n - a_n)^2 < \delta \}$

CLOSED BALL: $B_\delta(\vec{a}) \cup S_\delta^{n-1}(\vec{a})$

NOTIONS OF OPEN & CLOSED

- IF $D \subset \mathbb{R}^n$ is a set of points, then: $p = (a_1, \ldots, a_n)$ is an *interior point* of D IF there exists an open ball containing P // $B_\delta(P)$ s.t: $B_\delta(P) = D$
- P is a boundary point of D IF *every* open ball $B_\delta(P)$ [i.e: $\forall \delta > 0$] contains points INSIDE D and NOT INSIDE D
- SET OF ALL INTERIOR POINTS is called the INTERIOR OF D

SET OF ALL BOUNDARY POINTS CALLED BOUNDARY OF D

D IS CLOSED ⇒ CONTAINS ALL IT'S BOUNDARY POINTS

D IS OPEN = CONTAINS NONE OF IT'S BOUNDARY POINTS

→ $\mathbb{R}^n$ is open & closed

GENERAL LIMITS OF FUNCTIONS OF MANY VARIABLES

DEF: Let f be a function of n variables, and assume it is defined at all points on some ball centered at $\vec{P} = \left(\begin{smallmatrix} a_1 \\ \vdots \\ a_n \end{smallmatrix}\right)$, except maybe at $\vec{P}$.

We write: $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ if given any $\varepsilon > 0$ we can find $\delta_\varepsilon > 0$ s.t. $f(\vec{x})$ satisfies $|f(\vec{x}) - L| < \varepsilon$ for all points $\vec{x} \in B_{\delta_\varepsilon}(\vec{a})$

Thm: $\boxed{a}$ IF $f(\vec{x}) \xrightarrow{\vec{x} \to \vec{a}} L$ then $f(\vec{x}) \xrightarrow{\vec{x} \to \vec{a}} L$ along any smooth curve according to definition

$\boxed{b}$ IF $\lim_{\vec{x} \to \vec{a}} f(\vec{x})$ fails to exist OR IS DIFFERENT ALONG TWO CURVES C_1, C_2, then THE LIMIT DOES NOT EXIST ACCORDING TO ε-δ definition

CONTINUITY

DEF: A function $f(\vec{x})$ is continuous at $\vec{a}$ IF $f(\vec{a}) = \lim_{\vec{x} \to \vec{a}} f(\vec{x})$

DEF: IF f IS CONTINUOUS at all points in D, then we say f is continuous on D

Thm: $\boxed{1}$ SUM/DIFFERENCE/PRODUCT OF CONTINUOUS FUNCTIONS IS CONTINUOUS

$\boxed{2}$ COMPOSITION OF CONT. IS CONT.

$\boxed{3}$ QUOTIENT OF CONT. IS CONT. EXCEPT maybe where denominator is 0

PARTIAL DIFFERENTIATION

DEF: The PARTIAL DERIVATIVE of $f(x,y)$ w.r.t x at (a,b) is

$$f_x(a,b) = \frac{d}{dx} f(x,b)\big|_{x=a}$$

w.r.t y at (a,b) is $f_y(a,b) = \frac{d}{dy}(f(a,y))$

$$f_{xx}(x,y) = \frac{d}{dx} f_x(x,y) = \frac{\delta}{\delta x} f($$

$$f_{xy}(x,y) = \frac{d}{dy} f_x(x,y)$$

- $f(x,y)$ →
  - $f_x(x,y)$
    - f_{xx}
      - f_{xxx}
      - f_{xxy}
    - f_{xy}
      - f_{xyx}
      - f_{xyy}
  - $f_y(x,y)$
    - f_{yx}
      - f_{yxx}
      - f_{yxy}
    - f_{yy}
      - f_{yyx}
      - f_{yyy}

PARTIAL DERIVATIVES

$Z = f(x,y)$ – changes in ~~x~~ as x varies, y fixed
OR as y varies, x changes

e.g: $F = f(m,r)$, m = mass, r = distance to sun
FORCE OF GRAVITATIONAL ATTRACTION OF AN OBJECT TO THE SUN

PARTIAL DERIVATIVE of f (or z) with respect to x at (a,b) is

$$f_x(a,b) = \frac{d}{dx} f(x,b)\Big|_{x=a} = \frac{\delta}{\delta x} f(x,y)$$

... of f w.r.t y at (a,b) is

$$f_y(a,b) = \frac{d}{dy}\left(f(a,y)\right)\Big|_{y=b} = \frac{\delta}{\delta y} f(x,y)$$

$$f_x(x,y) = \frac{\delta}{\delta x}\left[f(x,y)\right] = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y(x,y) = \frac{\delta}{\delta x}\left[f(x,y)\right] = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Ex: $f(x,y) = Z = \sqrt{1 + 2x^2 \cos 3y}$

$$f_x(x,y) = \frac{\cancel{2} \cdot 2x \cos 3y}{\cancel{2}\sqrt{1 + 2x^2 \cos 3y}}$$

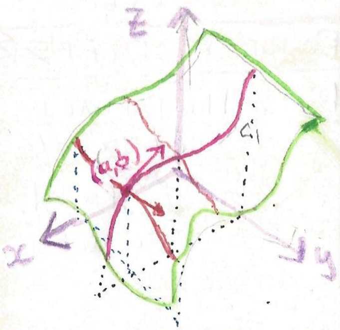
$$f_y(x,y) = \frac{-3 \cdot \cancel{2} x^2 \sin 3y}{\cancel{2}\sqrt{1 + 2x^2 \cos 3y}}$$

PARTIAL DERIVATIVES AS RATE OF CHANGE

- C_1 is the curve of intersection of the surface $Z = f(x,y)$ and the plane $y = b$

⇒ $f_x(a,b)$ is the slope of the tangent to C_1 at the point (a,b)

⇒ $f_y(a,b)$ is the slope of the tangent to C_2 at the point (a,b)



- $f_x(a,b)$ and $f_y(a,b)$ will be called the slopes of the surface in the x-direction and y-direction (respectively) at (a,b)

- Existance of partial derivatives is NOT related to continuity.

Ex: $f(x,y) = \begin{cases} xy/(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

IF $(x,y) \neq 0$:

$$f_x(x,y) = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{x^3 - xy^2}{(x^2+y^2)^2}$$

FOR $(0,0)$:

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{0}{\Delta x^2} - 0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{0}{\Delta y^2} - 0}{\Delta y} = 0$$

- WE SHOWED BEFORE THAT THIS FUNCTION IS NOT CONTINUOUS AT $(0,0)$ BUT we see THAT THE DERIVATIVE IS well DEFINED

PARTIAL DERIVATIVES OF FUNCTIONS OF MANY VARIABLES

$w = f(x,y,z)$: $f_x(x,y,z)$ $\delta w/\delta x$, $f_y(x,y,z)$ $\delta w/\delta y$, $f_z(x,y,z)$ $\delta w/\delta z$

$u = f(x_1, x_2, \dots, x_n)$: $\frac{\delta f}{\delta x_1}, \dots, \frac{\delta f_i}{x_i}, \dots, \frac{\delta f}{x_n}$

Ex: $r(x_1, x_2, \dots, x_n) = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

$$\frac{\delta r}{\delta x_i} = \frac{2 x_i}{2 \cdot \sqrt{x_1^2 + \dots + x_n^2}} = \frac{x_i}{r} \Rightarrow \boxed{\frac{\delta r}{\delta x_i} = \frac{x_i}{r}}$$

$$u(x_1, \dots x_n) = \sum_{i=1}^{n} \frac{1}{2}(x_{i+1} - x_i)^2$$

$$\frac{\delta}{\delta x_1} u = -(x_2 - x_1)$$

$$\frac{\delta}{\delta x_i} u = 0 + \dots + (x_j - x_{j-1}) - (x_{j+1} - x_j) = 2x_j - x_{j-1} - x_{j+1}$$

Ex: $f(x,y) = \sqrt{\cos 2x + 2x\sin 3y}$

$$\frac{\delta f}{\delta x} = \frac{-2\sin 2x + 2\sin 3y}{2\sqrt{\cos 2x + 2x\sin 3y}} = \frac{-2\sin 2x + 2\sin 3y}{2f(x,y)}$$

$$\frac{\delta f}{\delta y} = \frac{6x\cos 3y}{2\sqrt{\cos\cdots}} = \frac{3x\cos 3y}{f(x,y)}$$

$$\frac{\delta f}{\delta x^2} = \frac{-2\cos 2x}{f} - \frac{f_x}{f^2}(-\sin 2x + \sin 3y)$$

$$\frac{\delta}{\delta y}\left(\frac{f}{\delta x}\right) = \frac{3\cos 3y}{f} - \frac{f_y}{f^2}(-2\sin 2x + \sin 3y)$$

EQUALITY OF MIXED PARTIALS

THM: Let f be a function of 2 variables.
If f_{xy} and f_{yx} are continuous on the same open disc D.
THEN: $f_{xy} = f_{yx}$ on D

The Wave Equation:

- string of length L stretched taut between points $0 \le x \le l$
- Let u represent displacement of the string.

$u = u(x,t)$

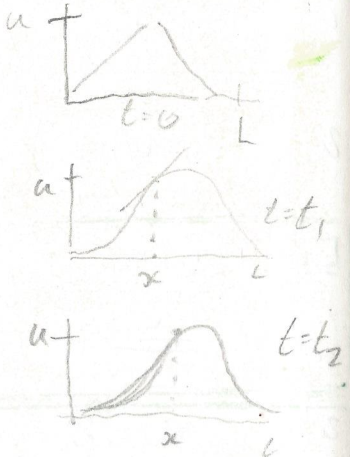
For Fixed t: $\frac{\delta u}{\delta x}$ represents slope of the string at x and sign of $\frac{\delta u}{\delta x}$ tells us whether concave up/down

For Fixed x: $u(x,t)$ is displacement in time OF STRING AT POINT x.

$\frac{\delta u}{\delta t}$ = the velocity $\quad \frac{\delta^2 u}{\delta x^2}$ = the acceleration

FOR SMALL OSCILLATIONS & AMPLITUDES, IT APPEARS THAT $u(x,t)$ SATISFIES 1D wave Eqtn:

$$\frac{\delta^2 u}{\delta t^2} = c^2\frac{\delta^2 u}{\delta x^2}$$ PARTIAL DIFFERENTIAL EQTN (PDE)



$$\frac{\delta^2 u}{\delta t^2} = c^2\frac{\delta^2 u}{\delta x^2}$$ 1D WAVE EQTN.

Ex: Show that $u(x,t) = e^{-(x-ct)^2}$ is a solution of the 1D wave equation.

$$\frac{\delta u}{\delta t} = e^{-(x-ct)^2}(-2(x-ct)(c)) = 2c(x-ct)e^{-(x-ct)^2}$$

$$\frac{\delta u}{\delta t^2} = -2c^2e^{-(x-ct)^2} + 4c^2(x-ct)^2e^{-(x-ct)^2}$$

$$\frac{\delta u}{\delta x} = -2(x-ct)e^{-(x-ct)^2}$$

$$\frac{\delta u}{\delta x^2} = -2e^{-(x-ct)^2} + 4(x-ct)^2e^{-(x-ct)^2}$$

THIS IS CALLED A GENERAL SOLUTION

TAYLOR EXPANSIONS

$\boxed{1D}$ $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f'(a)(x-a)^2 + \cdots + \frac{1}{n!}f'(a)(x-a)^n$

$$= \sum_{i=0}^{\infty} f^{(i)}(a)\cdot\frac{1}{i!}\cdot(x-a)^i$$

$\boxed{2D}$ $f(x,y) = f(a,y) + f_x(a,y)(x-a) + \cdots + \frac{1}{n!}\frac{\delta^n f}{\delta x^n}(a,y)(x-a)^n + \cdots$

$$= \sum_{i=0}^{\infty}\frac{1}{i!}\frac{\delta^i f}{\delta x^i}(a,y)(x-a)^i$$

sum of functions of y
→ TAYLOR EXPAND AGAIN

BUT: $f(a,y) = f(a,b) + \frac{\delta f}{\delta y}(a,b) + \cdots + \frac{1}{m!}\frac{\delta^m f}{\delta y^m}(a,b)(y-b)^m + \cdots$

$f_x(a,y)(x-a) = \frac{\delta f}{\delta x}(a,b)(x-a) + \frac{\delta f}{\delta y\delta x}(a,b)(x-a)(y-b)$

$\vdots \quad + \frac{\delta f}{\delta y^2\delta x}(a,b)\frac{(x-a)(y-b)^2}{2} + \cdots + \frac{\delta f}{\delta y^m\delta x}(a,b)\cdot\frac{1}{m!}(y-b)^m(x-a) + \cdots$

$$\Rightarrow \boxed{f(x,y) = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{1}{k!\,i!}\frac{\delta^{k+i}f}{\delta x^i\delta y^k}(a,b)(x-a)^i(y-b)^k}$$

IN GENERAL FOR $f(x_1, x_2, \cdots x_n) = f(\vec{x})$

$$f(\vec{x}) = \sum_{k_1,k_2\cdots k_n=0}^{\infty}\left(\frac{1}{k_1!\,k_2!\cdots k_n!}\right)\left(\frac{\delta^{k_1+k_2+\cdots+k_n}f(\vec{a})}{\delta x_1^{k_1}\delta x_2^{k_2}\cdots\delta x_n^{k_n}}\right) \times (x_1-a_1)^{k_1}(x_2-a_2)^{k_2}\cdots(x_n-a_n)^{k_n}$$

New NOTATION: $\frac{\delta f}{\delta x_i} = \delta_i f \;\|\; \frac{\delta^2 f}{\delta x_i\delta x_j} = \delta_i\delta_j f$

$$\boxed{f(\vec{x}) = \sum_{k_1\cdots k_n=0}^{\infty}\frac{1}{k_1!\cdots k_n!}\delta_1^{k_1}\delta_2^{k_2}\cdots\delta_n^{k_n}f(\vec{a})(x_1-a_1)^{k_1}\cdots(x_n-a_n)^{k_n}}$$

Chain Rule:

$y = f(x(t)) \Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$z = f(x, y), \quad x = x(t), \; y = y(t)$

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))$$

Look at $x(t+\Delta t) = x(t) + x'(t)\Delta t + \underbrace{O(\Delta t^2)}_{\text{approaches } 0 \text{ as } \Delta t \to 0}$

$\approx x(t) + x'(t)\Delta t$

$y(t+\Delta t) = y(t) + y'(t)\Delta t + O(\Delta t^2) \approx y(t) + y'(t)\Delta t$

$O(\cdot)$ – BIG "OH" NOTATION, shows just the power of the term

$$\Rightarrow \frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{f(x(t) + x'(t)\Delta t, y(t) + y'(t)\Delta t) - f(x(t), y(t))}{\Delta t}$$

- TAYLOR EXPAND THE FIRST f TERM

$$f(x,y) = f(a,b) + \frac{\delta}{\delta x} f(a,b)(x-a) + \cdots O\left(\max((x-a)^2, (y-b)^2)\right)$$
$$+ \frac{\delta}{\delta y} f(a,b)(x-a) + \cdots$$

$$\Rightarrow \frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\cancel{f(x(t), y(t))} + f_x(x(t), y(t))\,x'(t)\Delta t - \cancel{f(x(t), y(t))} + f_y(x(t), y(t))\,y'(t)\Delta t}{\Delta t}$$

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{f_x(x(t), y(t))\,x'(t)\Delta t + f_y(x(t), y(t))\,y'(t)\Delta t}{\Delta t}$$

$$\frac{dz}{dt} = f_x(x(t), y(t))\,x'(t) + f_y(x(t), y(t))\,y'(t)$$

- SIMILARLY FOR n VARIABLES

$z = f(x_1, x_2, \dots x_n), \quad x_i = x_i(t)$

$$\boxed{\frac{dz}{dt} = \sum_{i=1}^{n} \frac{dz}{dx_i} \cdot \frac{dx_i}{dt}}$$

Implicit Differentiation

- Any function $y = f(x)$ can be defined implicitly as a solution of an equation $F(x, y) = 0$

○ IF $F(x,y)$ is continuous at (x_0, y_0) and is "smooth enough", and $\delta_y F(x,y) \neq 0$, then $F(x,y)$ has a unique solution in some open disc D centered at (x_0, y_0) and ~~there~~ exists an $f(x)$ s.t. $y = f(x)$ on D

in D: $0 = F(x,y) = F(x, f(x))$

$$\frac{d}{dx}[0] = \frac{d}{dx}[F(x, f(x))]$$

$$0 = \frac{dF}{dx} + \frac{dF}{dy} \cdot \frac{dy}{dx}$$

$$-\frac{dF/dx}{dF/dy} = \frac{dy}{dx}$$

Ex: $ye^y + y^2 e^x - 1 = 0$

$F(x,y) = ye^y + y^2 e^x - 1 \Rightarrow \frac{\delta F}{\delta y} = ye^y + e^y + 2ye^x \neq 0$

$\frac{\delta F}{\delta x} = y^2 e^x$

$$\Rightarrow \frac{dy}{dx} = \frac{-y^2 e^x}{ye^y + e^y + 2ye^x}$$

Chain Rule for Partial Derivatives

- Let $z = f(x,y) \quad x = x(u,v) \quad y = y(u,v)$

$\Rightarrow z = f(x(u,v), y(u,v)) = z(u,v)$

$$\frac{\delta z}{\delta u} = \lim_{\Delta u \to 0} \frac{f(x(u+\Delta u, v), y(u+\Delta u, v)) - f(x(u,v), y(u,v))}{\Delta u}$$

$$= \frac{\delta z}{\delta x} \cdot \frac{\delta x}{\delta u} + \frac{\delta z}{\delta y} \cdot \frac{\delta y}{\delta u}$$

MOST GENERAL CHAIN RULE

$\omega = f(v_1, v_2, \dots v_n)$ & $v_i = v_i(t_1, t_2 \dots t_m)$

An: $\omega(t_1, t_2, \dots, t_m) = f[v_1(t_1, \dots, t_m), \dots, v_n(t_1, \dots, t_m)]$

$$\frac{\delta \omega}{\delta t} = \sum_{i=1}^{n} \frac{\delta \omega}{\delta v_i} \cdot \frac{\delta v_i}{\delta t_j} = \frac{\delta \omega}{\delta v_i} \cdot \frac{\delta v_i}{\delta t_j}$$

← "DUMMY" INDEX

SUMMATION OVER i USING EINSTEIN NOTATION

TAYLOR'S THEOREM RECAP (1 DIM)

- Let $k \geq 1$ and $f: \mathbb{R} \to \mathbb{R}$ be k-times differentiable at a point $a \in \mathbb{R}$.

$\Rightarrow \exists\ h_k: \mathbb{R} \to \mathbb{R}$ s.t:

$$f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + h_k(x)(x-a)^k$$

k terms + error h_k

- WITH $\lim_{x \to a} h_k(x) = 0$

LANGUAGE FOR N-DIMENSIONS

- MULTI INDEX: $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n$

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

THUS: $D^\alpha f(\vec{a}) = \frac{\delta^{\alpha_1}}{\delta x_1^{\alpha_1}} \cdot \frac{\delta^{\alpha_2}}{\delta x_2^{\alpha_2}} \cdots \frac{\delta^{\alpha_n}}{\delta x_n^{\alpha_n}} f(\vec{a})$

$$(\vec{x} - \vec{a})^\alpha = (x_1 - a_1)^{\alpha_1}(x_2 - a_2)^{\alpha_2} \cdots (x_n - a_n)^{\alpha_n}$$

TAYLOR'S THEOREM (N-DIM)

- Let $f: \mathbb{R}^n \to \mathbb{R}$ be k-times differentiable [ALL MIXED DERIVATIVES UP TO ORDER k EXIST] at the point $\vec{a} \in \mathbb{R}^n$.

THEN: there exists $h_\alpha\ \mathbb{R}^n \to \mathbb{R}$ s.t:

$$\boxed{f(\vec{x}) = \sum_{|\alpha| \leq k} \frac{D^\alpha f(\vec{a})}{\alpha!}(\vec{x} - \vec{a})^\alpha + \sum_{|\alpha| = k} h_\alpha(\vec{x})(\vec{x} - \vec{a})^\alpha}$$

WITH $h_\alpha(\vec{x}) \xrightarrow[\vec{x} \to 0]{} 0$

DIFFERENTIABILITY, DIFFERENTIALS & LOCAL LINEARITY

$$\Delta f = f(a + \Delta x) - f(a) \approx f'(a)\Delta x$$

IF WE HAVE Δx IS "small enough"

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) = 0$$

TRUE IF f is differentiable at the point a

- FOR $f(x,y)$, WE DENOTE by Δf (the "increment" of f) the change in value of $f(x,y)$ as (x,y) varies from some initial position (a,b) to $(a+\Delta a,\ b+\Delta b)$

i.e: $\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b)$

- WE CAN APPROXIMATE Δf:

$$\Delta f = f(a+\Delta x, b+\Delta y) - f(a, b+\Delta y) + f(a, b+\Delta y) - f(a,b)$$

$$\approx f_x(a, b+\Delta y)\Delta x + f_y(a,b)\Delta y$$

$$\approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$$

IF f_x is continuous on y.

IS THIS APPROXIMATION ANY GOOD?

$$\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y \quad (?)$$

IF Δx, Δy are close to 0, we want that:

$$\Delta f - [f_x(a,b)\Delta x + f_y(a,b)\Delta y] << \sqrt{\Delta x^2 + \Delta y^2}$$

"much smaller" than the distance of $(\Delta x, \Delta y)$ from $(0,0)$

i.e: THE DISTANCE BETWEEN $\binom{a}{b}$ and $\binom{a+\Delta x}{b+\Delta y}$

WE CAN ENFORCE THIS BY REQUIRING:

$$(*) \lim_{\binom{\Delta x}{\Delta y}\to 0} \frac{\Delta f - [f_x(a,b)\Delta x + f_y(a,b)\Delta y]}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

- By Definition, a function $f: \mathbb{R}^2 \to \mathbb{R}$ is said to be differentiable at (a,b) IF: $f_x(a,b)$ & $f_y(a,b)$ exist, and thus the statement $(*)$ holds

- IN GENERAL: $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\vec{a}$ IF: $\frac{\delta f}{\delta x_i}$ exist for $i = 1, \dots, n$

$$\Rightarrow \lim_{\Delta \vec{x} \to \vec{0}} \frac{\Delta f - \sum_{i=1}^{n} \frac{\delta f}{\delta x_i}(\vec{a})\Delta x_i}{\sqrt{\sum_{i=1}^{n} x_i^2}}$$

- DEF: f is DIFFERENTIABLE in R, IF f is differentiable at all points in a certain region R

THM: IF f is differentiable at $\vec{a}$, then it is continuous at $\vec{a}$ also.

PROOF: CALCULATION USING DEFINITION OF CONTINUITY

THM: IF ALL FIRST ORDER PARTIAL DERIVATIVES OF f exist, ~~AND~~ ARE CONTINUOUS AT $\vec{a}$, THEN f IS DIFFERENTIABLE AT $\vec{a}$

DIFFERENTIALS: IF $z = f(x,y)$ differentiable @ $\binom{a}{b}$:

- Let $dz = f_x\binom{a}{b}dx + f_y\binom{a}{b}dy$ denote a new function $dz(dx, dy)$.
- THIS IS CALLED THE TOTAL DIFFERENTIAL OF z AT $\binom{a}{b}$
- SIMILARLY FOR $u = f(\vec{x})$:

$$du = \sum_{i=1}^{n} \frac{df}{dx_i}(\vec{a})\, dx_i$$

LOCAL LINEAR APPROXIMATIONS

$$\Delta f \approx dz(\Delta x, \Delta y)$$

LOCAL LINEAR APPROXIMATION: If f is differentiable at $\vec{a}$ Then $L(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{N} \frac{\delta f}{\delta x_i}(\vec{a})(x_i - a_i)$ is called the ~~local~~ linear approximation of f

Ex: $f(x,y) = \sqrt{x^2+y^2}$ $\quad (a,b) = (3,4)$

- FIND the approximate value of $\sqrt{3.04^2 + 3.98^2}$

$f_x(x,y) = \frac{x}{f(x,y)} \Rightarrow f_x(3,4) = \frac{3}{5}$ $\quad \boxed{f(3,4) = 5}$

$f_y(x,y) = \frac{y}{f(x,y)} \Rightarrow f_y(3,4) = \frac{4}{5}$

$L(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$

$f(3.04, 3.98) \approx L(3.04, 3.98) = 5.008$

- ACTUAL VALUE IS 5.00819 (difference is $.00019$)

$\sqrt{\Delta x^2 + \Delta y^2} = 0.045 << 0.00019$

DIRECTIONAL DERIVATIVES & GRADIENTS

DIRECTIONAL DERIVATIVE = Generalisation of Partial derivative

RATE OF CHANGE of $f(x,y)$ w.r.t a direction (a,b)

- Let $\vec{u} = u_1\hat{i} + u_2\hat{j}$ be a vector with initial point at (a,b)
  - THIS defines a line passing through (a,b) in the xy-plane.
  - PARAMETRISE THIS LINE: $x(s) = x_0 + s u_1$
    $y(s) = y_0 + s u_2$

where s denotes the arc-length parametrisation

DEF:
DIRECTIONAL DERIVATIVE of $f(x,y)$ in the direction of $\vec{u}$, $|\vec{u}| = 1$ at the point (a,b) is defined as

$$D_{\vec{u}} f(a,b) = \frac{d}{ds}\Big[f(a+su_1, b+su_2)\Big]_{s=0}$$

$$= f_x(a,b)u_1 + f_y(a,b)u_2$$

- GEOMETRICALLY: $D_{\vec{u}} f(a,b)$ is the slope of the surface in the direction $\vec{u}$
- IN GENERAL: FOR $z = f(x_1, x_2 \dots x_n)$

$$\vec{u} = \sum_{i=1}^{n} u_i \vec{e}_i \qquad |\vec{u}| = 1$$

$$D_{\vec{u}} f(\vec{a}) = \frac{d}{ds}\Big[f(a_1 + su_1), f(a_2+su_2), \dots, f(a_n + su_n)\Big]$$

$$= \frac{\delta}{\delta x_1} f\, u_1 + \frac{\delta f}{\delta x_2} u_2 + \dots + \frac{\delta f}{\delta x_n} u_n$$

IMPORTANT: $\vec{u}$ should be unit length

DIRECTIONAL DERIVATIVE EXAMPLE, GRADIENTS

EXAMPLE: $f(x,y,z) = x^2 y - yz^3 + z$

WE WANT: $D_{\vec{u}} f(1,-2,0)$ in the direction $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$|\vec{a}| = \sqrt{4+1+4} = 3 \Rightarrow \vec{u} = \tfrac{2}{3}\hat{i} + \tfrac{1}{3}\hat{j} - \tfrac{2}{3}\hat{k}$$

$$f_x\begin{pmatrix}x\\y\\z\end{pmatrix} = 2x,y \Rightarrow f_x\begin{pmatrix}1\\-2\\0\end{pmatrix} = -4$$

$$f_y\begin{pmatrix}x\\y\\z\end{pmatrix} = x^2 - z^3 \Rightarrow f_y\begin{pmatrix}1\\-2\\0\end{pmatrix} = 1$$

$$f_z\begin{pmatrix}x\\y\\z\end{pmatrix} = 1 - 3yz^2 \Rightarrow f_y\begin{pmatrix}1\\-2\\0\end{pmatrix} = 1$$

THUS: $D_{\vec{u}} f(1,-2,0) = (-4)\frac{2}{3} + (1)\frac{1}{3} + \frac{2}{3} = -3$

THE GRADIENT

- In general: $D_{\vec{u}} f(x_1, \dots, x_n) = \begin{pmatrix}\delta_{x_1} f(\vec{x})\\ \vdots \\ \delta_{x_n} f(\vec{x})\end{pmatrix} \cdot \vec{u}$

DEF: The *GRADIENT* of $f(\vec{x})$ is

$$\vec{\nabla} f(\vec{x}) = \frac{\delta f}{\delta x_1}(\vec{x}) e_1 + \dots + \frac{\delta f}{\delta x_n}(\vec{x}) e_n$$

$$= \sum_{i=1}^{n} \frac{\delta f}{\delta x_i}(\vec{x}) e_i$$

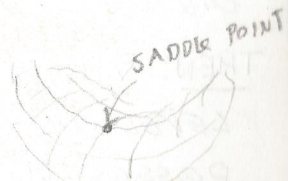
PROPERTIES OF THE GRADIENT:

$$D_{\vec{u}} f(\vec{x}) = \vec{\nabla} f(\vec{x}) \cdot \vec{u} = \overbrace{|\vec{u}|}^{1} |\vec{\nabla} f| \cos\theta_{\vec{u},\nabla f}$$

$$= |\vec{\nabla} f| \cos\theta$$

- maximum occurs when:
  (I) $\theta = 0$ (II) max slope @ (a,b) is $|\vec{\nabla} f\binom{a}{b}|$
- IF $\vec{\nabla} f = 0 \Rightarrow D_{\vec{u}} f = 0$

USUALLY OCCURS WHEN: (a,b) is relative min
(a,b) is relative max
(a,b) is SADDLE POINT



The Gradient & Level Curves

- **Gradients are normal to any level curves or surfaces**

- Level curve $c: f(x,y)=k$
  - → suppose c can be smoothly parametrized as $x=x(s)$, $y=y(s)$ (s = arc length parametrisation)
  - → The unit tangent vector $\vec{T}(s)=T$ can be described as $\vec{T}(s)=\frac{dx}{ds}\hat{i}+\frac{dy}{ds}\hat{j}$

$\Rightarrow f(x,y)$ is constant on c in the direction of $\vec{T}(s)$

$\Rightarrow D_{\vec{T}} f(x,y) = \emptyset$

PROOF: $D_{\vec{T}} f(x,y) = \vec{\nabla} f \cdot \vec{T}$

$$= \left(\frac{\delta f}{\delta x}\hat{i}+\frac{\delta f}{\delta y}\hat{j}\right)\cdot\left(\frac{\delta x}{\delta s}\hat{i}+\frac{\delta y}{\delta s}\hat{j}\right)$$

$$D_{\vec{T}} f\binom{x}{y} = \frac{\delta f}{\delta x}\hat{i}\cdot\frac{\delta x}{\delta s}\hat{i} + \frac{\delta f}{\delta x}\hat{i}\cdot\frac{\delta y}{\delta s}\hat{j} + \frac{\delta f}{\delta y}\hat{j}\cdot\frac{\delta y}{\delta s}\hat{i} + \frac{\delta f}{\delta y}\hat{j}\cdot\frac{\delta y}{\delta s}\hat{j}$$

- BUT $\hat{i}\cdot\hat{i}=1$ $\quad \hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{i}=\emptyset$ $\quad \hat{j}\cdot\hat{j}=1$

$$D_{\vec{T}} f\binom{x}{b} = \frac{\delta f}{\delta x}\frac{\delta x}{\delta s} + \frac{\delta f}{\delta y}\frac{\delta y}{\delta s} = \frac{d}{ds}\Big(f(x(s),y(s))\Big)=\emptyset$$

$\Rightarrow \vec{\nabla} f\cdot\vec{T} = \emptyset$

$\Rightarrow \vec{\nabla} f \perp \vec{T}$

THM: Assume that $f(x,y)$ has continuous first-order partial derivatives in an open disc around $\binom{a}{b}$, & $\vec{\nabla} f\binom{a}{b} \neq \emptyset$

THEN: $\vec{\nabla} f\binom{a}{b}$ is NORMAL TO the LEVEL CURVE of $f(x,y)$ that passes through (a,b)

- **For level surfaces σ**

$$\sigma: F(x,y,z)=k$$

- Let a curve $c \in \sigma$ be represented by:

$$x=x(s) \qquad y=y(s) \qquad z=z(s)$$

where s is the arc-length parametrisation

- Consider the unit tangent vector $\vec{T}$

$$\vec{T}(s) = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

- THEN: $D_{\vec{T}} F = \vec{\nabla} F\cdot T$

$$\Rightarrow D_{\vec{T}} F = \left(\frac{\delta F}{\delta x}\hat{i}+\frac{\delta F}{\delta y}\hat{j}+\frac{\delta F}{\delta z}\hat{k}\right)\cdot\left(\frac{\delta x}{\delta s}\hat{i}+\frac{\delta y}{\delta s}\hat{j}+\frac{\delta z}{\delta s}\hat{k}\right)$$

BUT $\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$ and $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=\emptyset$

$$\Rightarrow D_{\vec{T}} F = \frac{\delta F}{\delta x}\frac{\delta x}{\delta s} + \frac{\delta F}{\delta y}\cdot\frac{\delta y}{\delta s} + \frac{\delta F}{\delta z}\cdot\frac{\delta z}{\delta s} = \frac{d}{ds}F\begin{pmatrix}x(s)\\y(s)\\z(s)\end{pmatrix}=\emptyset$$

$\Rightarrow \vec{\nabla} F$ is perpendicular to $\vec{T}$

- **In General**

IF $F\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=k$, $\vec{r}=\begin{pmatrix}x_1(s)\\ \vdots\\ x_n(s)\end{pmatrix}$ is the arc-length parametrisation of c on σ THEN:

$$\vec{T}(s) = \sum_{i=1}^{n}\frac{dx_i}{ds}\hat{e}_i$$

$$D_T F = \vec{\nabla} F\cdot\vec{T} = \left(\sum_{i=1}^{n}\frac{\delta f}{\delta x_i}\hat{e}_i\right)\cdot\left(\sum_{i=1}^{n}\frac{\delta x_i}{\delta s}\hat{e}_i\right) = \left[\sum_{i=1}^{n}\frac{\delta f}{\delta x_i}\cdot\frac{\delta x_i}{\delta s}\right]$$

$$D_T F = \frac{d}{ds}\Big(F(\vec{r}(s))\Big) = \emptyset \Rightarrow \vec{\nabla} f \perp \vec{T}$$

$\Rightarrow \nabla$

Tangent Planes & Normal Vectors

- **Consider** $\sigma : f\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$

Assume $\begin{cases} \vec{\nabla} f\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix} \neq 0 \\ \text{All first order partial derivatives Continuous} \end{cases}$

- Near a point P_0, equation $f(x,y,z)=0$ defines a surface.
  - Let $f_z\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix} \neq 0$

$\Rightarrow$ IT IS NON-ZERO IN A NEIGHBOURHOOD (disc around of (x,y))

For $|z-z_0| \ll 1$, we have the approximation:

$$0 = f\begin{pmatrix}x\\y\\z\end{pmatrix} \approx f\begin{pmatrix}x\\y\\z_0\end{pmatrix} + f_z\begin{pmatrix}x\\y\\z_0\end{pmatrix}(z-z_0) \approx 0$$

$$\Rightarrow \text{so} \quad z \approx z_0 - \frac{f(x,y,z_0)}{f_z(x,y,z_0)}$$

Suppose curve $C \in \sigma$ described as:

$r(t) = (x(t), y(t), z(t)) \qquad r'(t) = (x'(t), y'(t), z'(t))$

Since $C \in \sigma$:

$$F(x(t), y(t), z(t)) = 0$$

$$\frac{d}{dt}F(x(t), y(t), z(t)) = 0$$

$$\Rightarrow 0 = \frac{\delta F}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta f}{\delta y}\frac{dy}{\delta t} + \frac{\delta f}{\delta z}\frac{\delta z}{\delta t} = \vec{\nabla} f \cdot r'(t) = 0$$

- If $\vec{\nabla} f \neq 0$ at P_0

$$\vec{\nabla} f \perp \vec{r}'$$

- This is true for all C through P_0

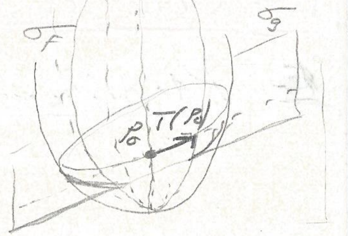
$\Rightarrow$ we define the tangent plane as the plane with the normal vector $\vec{\nabla} f(P_0)$

normal $\boxed{\vec{n} = \vec{\nabla} f(x_0, y_0, z_0)}$

Equation of tangent plane is: $\boxed{\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0}$

Intersection of Two Surfaces

$\boxed{\sigma_F : F\begin{pmatrix}x\\y\\z\end{pmatrix} = 0} \qquad \boxed{\sigma_G : G\begin{pmatrix}x\\y\\z\end{pmatrix} = 0}$



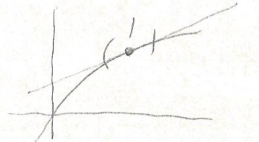
They intersect in general at a curve C

If $\vec{P}_0$ is on C: $\quad \vec{\nabla} F(P_0) \perp \sigma_F$
$\qquad \vec{\nabla} G(P_0) \perp \sigma_G$

- We can find a smooth parametrization of C:

$$\nabla F(P_0) \times \vec{\nabla} G(P_0) = \vec{T}(P_0)$$

Tangent Planes of Surfaces at Points



$\sigma : F(x,y,z) = 0$

$\vec{\nabla} F(P_0) = \vec{n}$ normal vector

$$\begin{aligned}\vec{n} \cdot (\vec{r} - \vec{r}_0) = {} & F_x(P_0)(x - x_0) \\ & + F_y(P_0)(y - y_0) \\ & + F_z(P_0)(z - z_0) = 0\end{aligned}$$

Def: the line through P_0 parallel to $\vec{n}$ is perpendicular to the tangent plane at P_0, is called the NORMAL LINE or "NORMAL" to the surface $f(x,y,z)=0$ at P_0

Parametrically: $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x_0 + F_x(t)\\ y_0 + F_y(t)\\ z_0 + F_z(t)\end{pmatrix}$

Ex: $\sigma : 4x^2 + y^2 + z^2 = 18 \qquad P_0 = (2,1,1)$
$(4(2^2) + 1^2 + 1^2 = 18$ so $P_0 \in \sigma)$

$$\vec{\nabla} F = \begin{pmatrix}8x\\2y\\2z\end{pmatrix} = 8x\hat{i} + 2y\hat{j} + 2y\hat{k}$$

$$\vec{\nabla} F(P_0) = \begin{pmatrix}16\\2\\2\end{pmatrix}$$

Tangent plane: $16(x-2) + 2(y-1) + 2(z-1) = 0$

$\Leftrightarrow \boxed{8y + \cancel{2}x + z = 18}$

Intersection of Planes Example

Ex: *Paraboloid* $z = x^2 + y^2$

$$\iff z - x^2 - y^2 = 0$$

$$G(x,y,z) = 3x^2 + 2y^2 + z^2 - 9 = 0$$

$$\Rightarrow \vec\nabla F = 2x\hat{i} - 2y\hat{j} + \hat{k}$$

$$\vec\nabla G = 6x\hat{i} + 4y\hat{j} + 2z\hat{k}$$

- Look at $P_0 = (1,1,2)$

Check it is on both: $2 - 1 - 1 = 0$

$3 + 2 + 4 - 9 = 0$

$$\vec\nabla F(P_0) = -2\hat{i} - 2\hat{j} + \hat{k}$$

$$\vec\nabla G(P_0) = 6\hat{i} + 4\hat{j} + 4\hat{k}$$

TANGENT: $T(P_0) = \vec\nabla F(P_0) \times \vec\nabla G(P_0) + P_0$

$$T(P_0): \begin{pmatrix}-2\\-2\\1\end{pmatrix} \times \begin{pmatrix}6\\4\\4\end{pmatrix} = \begin{pmatrix}-8-4\\6+8\\-8+12\end{pmatrix} = \begin{pmatrix}-12\\14\\4\end{pmatrix}$$

$$\boxed{T(P_0) = \begin{pmatrix}1\\1\\2\end{pmatrix} + \begin{pmatrix}-12\\14\\4\end{pmatrix}t}$$

Max & Min of Functions of 2 Variables

Def: A function is said to have a relative max/min at a point $P_0 = \begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix}$ IF THERE EXISTS A BALL CENTERED AT P_0 s.t.:

max: $f(P_0) \geq f(P)$ min: $f(P_0) \leq f(P)$

FOR ALL P IN THE BALL ~~FROM~~ WHICH ARE IN THE DOMAIN OF f

MIN: $f(P_0) \leq f(P)$ IN BALL

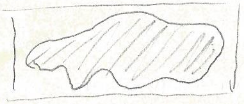
MAX: $f(P_0) \geq f(P)$ IN BALL

ABSOLUTE IF BALL IS DOMAIN

RELATIVE IF BALL $\subset$ DOMAIN

- IF f has a relative maximum OR MINIMUM (@B) Then we say f has a relative Extremum at B
- ABSOLUTE MAX/MIN → ABSOLUTE EXTREMUM

Bounded Set: any set that can be contained in some rectangle/cuboid



Extreme Value Theorem

- IF f is continuous on a closed and bounded set R, then f has BOTH: an absolute maximum & an absolute minimum on the set

Finding Relative Extrema

IF f has a relative extremum at P_0 & all first order partial derivatives exist

THEN: $\partial_{x_1} f(P_0) = \partial_{x_2} f(P_0) = \dots = \partial_{x_n} f(P_0) = 0$

Critical Point: point of f at which $\partial_{x_i} f(P_0) = 0 \;\; \forall\, i = 1, 2, \dots, n$

Def: Saddle Point: POINT ON SURFACE $z = f(x,y)$ has saddle point at (a,b) IF there exists two distinct vertical planes passing through $\begin{pmatrix}a\\b\end{pmatrix}$ s.t. the curve of intersection between: one plane and the surface has a maximum the other plane and the other surface has a minimum

Second Partial Test: $D = f_{xx}\begin{pmatrix}a\\b\end{pmatrix} f_{yy}\begin{pmatrix}a\\b\end{pmatrix} - f_{xy}^2\begin{pmatrix}a\\b\end{pmatrix}$

a) $D > 0$ $f_{xx}\begin{pmatrix}a\\b\end{pmatrix} > 0 \Rightarrow$ rel min

b) $D > 0$ $f_{xx}\begin{pmatrix}a\\b\end{pmatrix} < 0 \Rightarrow$ rel max

c) $D < 0 \Rightarrow$ SADDLE POINT

d) $D = 0 \Rightarrow$ No CONCLUSION

THE SECOND PARTIALS TEST

- Let f be a function of 2 variables with second-order partial derivatives which are continuous in some disc centered at $P_0 = \binom{a}{b}$ a critical point
- Let $D = f_{xx}(P_0)\, f_{yy}(P_0) - f_{xy}^2(P_0)$

(I) $D > 0$ & (a) $f_{xx}(P_0) > 0 \Rightarrow$ Relative Minimum
(b) $f_{xx}(P_0) < 0 \Rightarrow$ Relative Maximum

(II) $D < 0 \Rightarrow f$ HAS SADDLE POINT

(III) $D = 0 \Rightarrow$ NO CONCLUSIONS DRAWN

JUSTIFICATION FOR $D = 0$

- We expand $f(x,y)$ @ P_0 w/ TAYLOR SERIES

$$f\binom{x}{y} = f(P_0) + f_x(P_0)(x-a) + \tfrac{1}{2} f_{xx}(P_0)(x-a)^2 + f_y(P_0)(y-b) + \tfrac{1}{2} f_{yy}(P_0)(y-b)^2 + f_{xy}(P_0)(x-a)(y-a) + O\left[(x-a)^3, (x-a)^2(y-b), (x-a)(y-b)^2, (y-b)^3\right]$$

- BUT P_0 IS A CRITICAL POINT, SO:

- But P_0 IS A CRITICAL POINT

SO: $$f\binom{x}{y} = f(P_0) + \tfrac{1}{2} f_{xx}(P_0)(x-a)^2 + f_{xy}(P_0)(x-a)(y-b) + f_{yy}(P_0)(y-y)^2$$
$$\approx f(P_0) + \tfrac{1}{2} f_{xx}\Delta x^2 + f_{xy}\Delta x \Delta y + \tfrac{1}{2} f_{yy}\Delta y^2$$
$$= f(P_0) + \tfrac{1}{2} f_{xx}\left(\Delta x^2 + 2\frac{f_{xy}}{f_{xx}}\Delta x \Delta y\right) + f_{yy}\Delta y^2$$
$$= f(P_0) + \tfrac{1}{2} f_{xx}\left(\Delta x^2 + 2\frac{f_{xy}}{f_{xx}}\Delta x \Delta y + \left(\frac{f_{xy}}{f_{xx}}\Delta y\right)^2\right) - \frac{1}{2}\frac{f_{xy}^2}{f_{xx}}\Delta y^2 + \tfrac{1}{2} f_{yy}\Delta y^2$$
$$= f(P_0) + \tfrac{1}{2} f_{xx}\left(\Delta x + \frac{f_{xx}}{f_{yy}}\Delta y\right)^2 - \Delta y^2\left(\frac{f_{xy}^2}{f_{xx}} - f_{yy}\right)$$

SECOND PARTIALS TEST – GENERALISED

- $$D = f_{xx} f_{yy} - f_{xy}^2$$
$$= f_{xx} f_{yy} - f_{xy} f_{yx}$$
$$= \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \det H(P_0)$$

- We call this matrix the **HESSIAN MATRIX**

- We know $f_{xy} = f_{yx}$ which means that it is a symmetric matrix
$\Rightarrow$ eigenvalues are <u>real</u>

- <u>SO WHAT ARE THE EIGENVALUES OF THE MATRIX?</u>

$$\det \begin{bmatrix} h_{11} - \lambda & h_{12} \\ h_{12} & h_{22} - \lambda \end{bmatrix} = 0 = (h_{11} - \lambda)(h_{22} - \lambda) - h_{12}^2$$

$$\Rightarrow h_{11} h_{22} - h_{11}\lambda - h_{22}\lambda + \lambda^2 - h_{12}^2 = 0$$

$$\Rightarrow \lambda^2 - (f_{xx} + f_{yy})\lambda + (f_{xx} f_{yy} - f_{xy}^2) = 0$$

$$\Rightarrow \lambda^2 - (f_{xx} + f_{yy})\lambda + D = 0$$

$$\lambda = \frac{f_{xx} + f_{yy}}{2} \pm \frac{\sqrt{(f_{xx} + f_{yy})^2 - 4D}}{2}$$

<u>CASES:</u>

a) $D > 0$ and $(f_{xx} + f_{yy}) > 0 \Rightarrow \lambda_\pm > 0$

b) $D > 0$ and $(f_{xx} + f_{yy}) < 0 \Rightarrow \lambda_\pm < 0$

c) $D < 0 \Rightarrow$ ONE POSITIVE EIGENVALUE, ONE NEGATIVE

d) $D = 0 \Rightarrow$ ONE EIGENVALUE IS ZERO

But $D = f_{xx} f_{yy} - f_{xy}^2 > 0$
$\Rightarrow D + f_{xy}^2 = f_{xx} f_{yy} > 0$

<u>SO EITHER:</u> $\left.\begin{array}{l} f_{xx}, f_{yy} > 0 \\ f_{xy}, f_{yy} < 0 \end{array}\right\}$ ONLY NEED TO CHECK 1

MULTIDIMENSIONAL CASE $f(x_1, \dots, x_n)$

THE HESSIAN:

$$H(P_0) = \begin{bmatrix} \delta_1\delta_1 f & \delta_1\delta_2 f & \cdots & \delta_1\delta_n f \\ \delta_2\delta_1 f & \delta_2\delta_2 f & & \vdots \\ \vdots & & & \vdots \\ \delta_n\delta_1 f & \cdots & & \delta_n\delta_n f \end{bmatrix}$$

- IF ALL 2^{nd} partial derivatives are CONTINUOUS IN A DISC AROUND THE CRITICAL POINT P_0:

a) IF $H(P_0)$ is *positive definite*, THEN f has a local minimum at P_0

$\lambda_1, \dots \lambda_n > 0$ ⇒ MINIMUM

b) IF $H(P_0)$ is *negative definite*, THEN f has a local maximum at P_0

$\lambda_1, \dots \lambda_n < 0$ ⇒ MAXIMUM

c) IF $H(P_0)$ is *indefinite* THEN P_0 is a saddle point of f

$\lambda_i < 0$, $\lambda_j > 0$ ⇒ SADDLE Point

d) IF $H(P_0)$ is SINGULAR THEN NO CONCLUSIONS CAN BE DRAWN

$\lambda_i = 0$ ⇒ Give up & CRY

HOW TO COMPUTE ABSOLUTE EXTREMA IN A BOUNDED DOMAIN R

1. FIND ALL CRITICAL POINTS of f THAT LIE IN THE INTERIOR OF R
2. FIND ALL BOUNDARY POINTS AT WHICH AN EXTREMA COULD OCCUR
3. EVALUATE f @ all these POINTS TO SEE WHICH IS THE MAX / MIN

ABSOLUTE EXTREMA COMPUTING Ex

Ex: $f\binom{x}{y} = 3x + 6y - 3xy - 7$ on R: the Δ $\binom{0}{0}, \binom{0}{3}, \binom{5}{0}$

$f_x = 3 - 3y$ || $f_{xx} = 0$

$f_y = 6 - 3x$ || $f_{yy} = 0$

$f_{xy} = f_{yx} = -3 \Rightarrow \boxed{D = -9}$

$f_x = 0 \Rightarrow y = 1$, $f_y = 0 \Rightarrow x = 2$ } CRITICAL POINT $(2, 1)$

2) CONSIDER BOUNDARIES:

a) $x \in [0,5]$, $y = 0$

$f\binom{x}{0} = 3x - 7 \Rightarrow$ MIN AT $x = 0$, MAX AT $x = 5$ $\Rightarrow f\binom{0}{0} = -7$, $f\binom{5}{0} = 8$

b) $x = 0$, $y \in [0,3]$

$f(0,y) = 6y - 7 \Rightarrow$ MIN AT $y = 0$, MAX AT $y = 3$ $\Rightarrow f\binom{0}{0} = -7$, $f\binom{0}{3} = 11$

c) LINE SEGMENT $\binom{0}{3} \to \binom{5}{0}$

$y = -\frac{3}{5}x + 5$ $\quad x \in [0,5]$

$g(x) = f(x, -\frac{3}{5}x + 3) = 3x + 6(-\frac{3}{5}x + 3) - 3x(-\frac{3}{5}x + 3) - 7$

$= \frac{9}{5}x^2 - \frac{48}{5}x + 11$

$g'(x) = \frac{18}{5}x - \frac{48}{5} = 0 \Rightarrow \boxed{x = \frac{8}{3}} \Rightarrow \boxed{y = \frac{7}{5}}$

$f(\frac{8}{3}, \frac{7}{5}) = -\frac{9}{5}$

3) Look AT ALL POSSIBILITIES:

$\binom{x}{y}$	$\binom{0}{0}$	$\binom{5}{0}$	$\binom{0}{3}$	$\binom{8/3}{7/5}$	$\binom{2}{1}$
f	-7	8	11	-9/5	-1

WE FIND: MINIMUM AT (0,0) OF -7

MAXIMUM AT (0,3) OF 11

LAGRANGE MULTIPLIERS

- USEFUL FOR EXTREMAL PROBLEMS THAT HAVE A CONSTRAINT. $(P \in \mathbb{R}^n)$

$\rightarrow$ FIND min/max of $f(P)$

SUBJECT TO $g_i(P) = 0, \quad i = 1, 2, \dots, n$

EXAMPLE PROBLEM: m = cost of materials
L = labour hours cost

revenue $h(m, L) = 200 m^{3/2} L^2$

CONSTRAINT: BUDGET = €200 000

FIND maximum ~~Relative~~ $h(m, L)$

SUBJECT to L (hourly wage) ~~+~~ m (materials cost) $= 200k$

- FOR 2 VARIABLE FUNCTIONS

$f(x, y)$ and $g(x, y) = 0$

$$(*) \quad \vec{\nabla} f\binom{a}{b} = \lambda \vec{\nabla} g\binom{a}{b}$$

SOLVE $(*)$ FOR ALL POINTS (a, b)

- FIRST OF ALL, WE PARAMETRIZE $g\binom{x}{y} = 0$ IN TERMS OF $x(t)$ and $y(t)$
- THEN WE FIND $\frac{d}{dt} f\binom{x(t)}{y(t)} = \frac{\delta F}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\cdot\frac{dy}{dt} = 0$

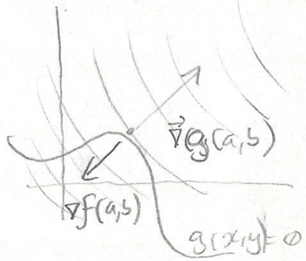
$$\Rightarrow \vec{\nabla} f \cdot \underbrace{(x'(t)\hat{\imath} + y'(t)\hat{\jmath})}_{} = 0$$

$$\Rightarrow \vec{\nabla} f \cdot \vec{T}(t) = 0$$

$$\Rightarrow \vec{\nabla} f \perp \vec{T}(t)$$

BUT WE ALSO KNOW THAT $\vec{\nabla} g \perp \vec{T}(t)$

SO: $\boxed{\vec{\nabla} f = \lambda \vec{\nabla} g}$



THM: f, g two functions with continuous 1st ORDER PARTIAL DERIVATIVES ON some open set containing the constraint curve $g\binom{x}{y} = 0$.
Assume $\vec{\nabla} g \neq 0$ ON THIS CURVE

○ IF f has a CONSTRAINT-LOCAL EXTREMUM, then it occurs AT (a, b) on the curve $g(x, y) = 0$ at which:

$\vec{\nabla} f(a, b)$ and $\vec{\nabla} g(a, b)$ are parallel

$$\Leftrightarrow \exists \lambda \text{ s.t. } \nabla f\binom{a}{b} = \lambda g\binom{a}{b}$$

- $F\begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} = f(x, y) - \lambda g(x, y)$

$$\frac{\delta F}{\delta \lambda} = -g(x, y)$$

$$\frac{\delta F}{\delta x} = \frac{\delta f}{\delta x} - \lambda \frac{\delta g}{\delta x}$$

$$\frac{\delta F}{\delta y} = \frac{\delta f}{\delta y} - \lambda \frac{\delta g}{\delta y}$$

EXAMPLE: FIND ALL POINTS ON SPHERE:

$$x^2 + y^2 + z^2 = 36$$

that are closest & furthest from $(1, 2, 2)$

$d\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x-1)^2 + (y-2)^2 + (z-2)^2$ subject to

$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2 - 36 = 0$

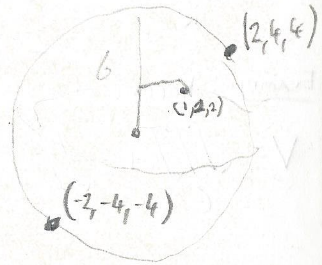
$$\vec{\nabla} f = 2(x-1)\hat{\imath} + 2(y-2)\hat{\jmath} + 2(z-2)\hat{k}$$

$$\vec{\nabla} g = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$\Rightarrow \boxed{x - 1 = \lambda x} \quad \boxed{y - 1 = \lambda y} \quad \boxed{z - 1 = \lambda z}$

$\Rightarrow \boxed{x = \frac{1}{1-\lambda}} \quad \boxed{y = \frac{2}{1-\lambda}} \quad \boxed{z = \frac{2}{1-\lambda}}$

$\Rightarrow \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{2}{1-\lambda}\right)^2 + \left(\frac{2}{1-\lambda}\right)^2 = 36 \Rightarrow \boxed{\lambda = \frac{1}{2}} \text{ OR } \boxed{\frac{3}{2}}$



$(x_1, y_1, z_1) = (-2, -4, -4)$

$(x_2, y_2, z_2) = (2, 4, 4)$

$f\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = 81$ MAX

$f\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = 9$ MIN

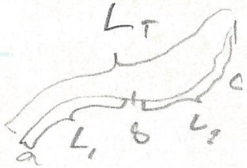
Multiple Integrals

- Suppose we have a quantity $Q(\sigma)$ where Q IS A FUNCTION OF σ

$\Rightarrow$ FOR ANY σ, THERE EXISTS ONLY 1 $Q(\sigma)$

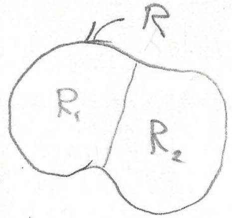
ADDITIONALLY: IF WE HAVE two NON-INTERSECTING objects σ_1 & σ_2 with $Q_1 = Q(\sigma_1)$ $Q_2 = Q(\sigma)$

IF $\sigma = \sigma_1 + \sigma_2$,
THEN $Q(\sigma) = Q(\sigma_1) + Q(\sigma_2)$



EXAMPLE OF A WIRE: $L_T = L_1 + L_2$

$$L = \int_a^c |r'(t)|\,dt = \int_a^b |r'(t)|\,dt + \int_b^c |r'(t)|\,dt$$



EXAMPLE OF AREA:

we have a CURVE WITH AREA A

$$\left.\begin{array}{l} Area(R_1) = A_1 \\ Area(R_2) = A_2 \end{array}\right\} A = A_1 + A_2$$

$$A = \iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA$$

EXAMPLE: MASS OF A FLAT OBJECT w/ density $f(x,y)$ $\frac{\text{mass}}{\text{unit area}}$

$$M = M_1 + M_2$$

$$M = \iint_R f(x,y)\,dA = \overbrace{\iint_{R_1} f(x,y)\,dA}^{M_1} + \overbrace{\iint_{R_2} f(x,y)\,dA}^{M_2}$$

EXAMPLE: VOLUME OF A SOLID G. $V = V_1 + V_2$

$$V = \iiint_G dV = \iiint_{G_1} dV + \iiint_{G_2} dV$$

How to Define Multiple Integrals

- DEVIDE σ INTO n small objects

σ_i: where $i = 1, 2, \ldots, n$

- SINCE $Q = Q(\sigma) = \sum_{i=1}^{N} Q(\sigma_i) = \sum_{i=1}^{N} Q_i$

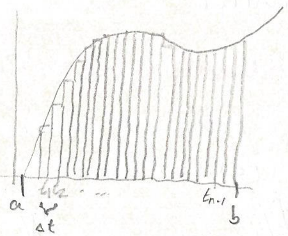
- Assume σ_i's are chosen s.t. we know that:

$Q_i \approx Q_i^*$ WITH HIGH ACCURACY

$$Q \approx \sum_{i=1}^{n} Q_i^*$$

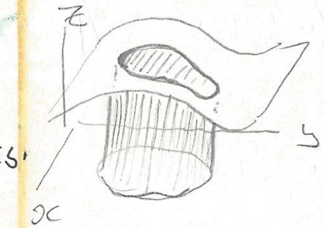
We define $Q = \lim_{n\to\infty} \sum_{i=1}^{n} Q_i^* =$ "$\int_\sigma q(P)\,d\sigma$" "RIEMAN INTEGRATION"

$$A = \int f(x)\,dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{N} f(t_i)\Delta t_i, \text{ where } \sum_{i=1}^{n} \Delta t_i = |b-a|$$



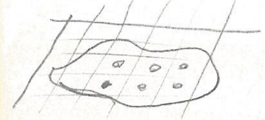
- IN THE VOLUME PROBLEMS

FIND THE volume of a solid enclosed between the surface $f(x,y) = z$ and the region R where $f(x,y) \geq 0$ for $(x,y) \in R$, f CONTINUOUS on R



→ ① DEVIDE REGION R INTO SUBRECTANGLES USING lines parallel to x-axis & y-axis

② EXCLUDE ANY RECTANGLES which have parts outside of R



③ CHOOSE ANY POINT IN THE RECTANGLE (x_k^*, y_k^*) in σ_k

④ Volume of σ_k rectanguler solid is:

$$A_k = \text{base AREA} \times \text{height} = \Delta x_k \Delta y_k \cdot f(x_k^*, y_k^*)$$

$$V \approx \sum_{k=1}^{n} \Delta x_k \Delta y_k f(x_k^*, y_k^*)$$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta x_k \Delta y_k = \iint_R f(x,y)\,dx\,dy$$

$$V = \lim_{n\to\infty} \sum f(x_k^*, y_k^*)\Delta x_k \Delta y_k = \iint_R f(x,y)\,dx\,dy$$

- IF $f(x,y)$ represents density in a region R:

  f IS MASS/UNIT AREA $\quad dx\,dy = \text{AREA} \to \text{MASS}$

- IF $f(x,y)$ IS NOT ALWAYS POSITIVE IN R:

$$\iint_R f(x,y)\,dx\,dy = \iint_{\substack{R\\ f>0}} f(x,y)\,dx\,dy - \underbrace{\iint_{\substack{R\\ f\le 0}} \|f(x,y)\|\,dx\,dy}_{\text{SIGNED VOLUME}}$$

→ CALLED A DOUBLE INTEGRAL of f over R

EVALUATING DOUBLE INTEGRALS

- TO EVALUATE DOUBLE INTEGRALS, WE INTEGRATE SUCCESSIVELY w.r.t both variables

$$\int_c^d\!\!\int_a^b f(x,y)\,dx\,dy = \int_c^d \Big[\underbrace{\int_a^b f(x,y)\,dx}_{\text{"CONSTANT"}}\Big] dy \overset{?}{=} \int_c^d \Big[\underbrace{\int_a^b f(x,y)\,dy}_{\text{"x CONSTANT"}}\Big] dx$$

"x CONSTANT" "y CONSTANT"

SOMETIMES TRUE

Ex: $f(x,y) = 40 - 2xy$

$$\iint_{v_1}^{v_2} f\binom{x}{y}\,dx\,dy \qquad v_1 = \binom{1}{2} \quad v_2 = \binom{3}{4}$$

FIRST WAY:

$$\int f\,dx = \int_1^3 40 - 2xy = 40x - x^2y\Big|_{x=1}^3 = 80 - 8y$$

$$\int_2^4 80 - 8y\,dy = 80y - 4y^2\Big|_{y=2}^4 = \boxed{112}$$

SECOND WAY:

$$\int_2^4 f\,dy = \int_2^4 40 - 2xy = 40y - xy^2\Big|_{y=2}^4 = 80 - 12x$$

$$\int_1^3 80 - 12x\,dx = 80 - 6x^2\Big|_{x=1}^3 = \boxed{112}$$

$$\left.\begin{aligned}\int_c^d\!\!\int_a^b f\binom{x}{y}\,dx\,dy &= \int_c^d\Big[\int_a^b f\binom{x}{y}\,dx\Big]dy\\ \int_a^b\!\!\int_c^d f\binom{x}{y}\,dy\,dx &= \int_a^b\Big[\int_c^d f\binom{x}{y}\,dy\Big]dx\end{aligned}\right\}\ \text{ITERATED INTEGRALS}$$

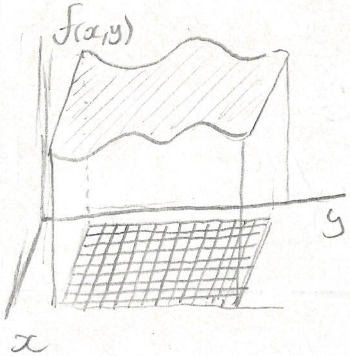
FUBINI'S THEOREM

- Let R be the rectangle defined by the inequalities: $a \le x \le b$, $c \le y \le d$

- IF $f(x,y)$ IS CONTINUOUS ON R

THEN: $$\iint_R f\binom{x}{y}\,dA = \int_c^d\!\!\int_a^b f\binom{x}{y}\,dx\,dy = \int_a^b\!\!\int_c^d f\binom{x}{y}\,dy\,dx$$

"PROOF": $$\iint f\binom{x}{y}\,dA \approx \sum_{K_1,K_2=1}^{n} f\left(x^*_{(K_1,K_2)}, y^*_{(K_1,K_2)}\right)\Delta x_{K_1K_2}\Delta y_{K_1K_2}$$



- chose (x_K^*, y_K^*) to be the center point of each sub-rectangle

SO: $x^*_{(K_1,K_2)} = x_{K_1} \quad y^*_{(K_1,K_2)} = y^*_{K_2}$

- CHOOSE Δx Δy to have the same size

$$\Rightarrow \iint_R f(x,y)\,dA \approx \sum_{K_2=1}^{n_y}\left(\sum_{K_1=1}^{n_x} f(x^*_{K_1}, y^*_{K_2})\Delta x\right)\Delta y$$

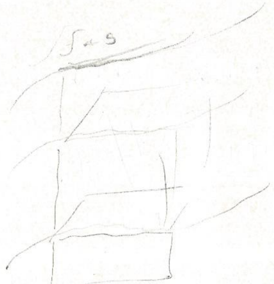
$$= \lim_{n_x\to\infty}\lim_{n_y\to\infty}\sum_{K_2=1}^{n_y}\left(\sum_{K_1=1}^{n_x} f\binom{x^*_{K_1}}{y^*_{K_2}}\Delta x\right)\Delta y$$

$$= \lim_{n_y\to\infty}\sum_{K_2=1}^{n_y}\lim_{n_x\to\infty}\left(\sum_{K_1=1}^{n_x} f\binom{x^*_{K_1}}{y^*_{K_2}}\Delta x\right)\Delta y = \iint f\binom{x}{y}\,dx\,dy$$

$$= \lim_{n_x\to\infty}\sum\left(\lim_{n_y\to\infty}\sum f\binom{x^*_{K_1}}{y^*_{K_2}}\Delta y\right)\Delta y = \iint f\binom{x}{y}\,dy\,dx$$

Ex: V is the solid paraboloid bounded by above by $z = 1+x^2+y^2$
by below by $R = [0,1]\times[0,2]$

$$V = \iint_R (1+x^2+y^2)\,dA$$
$$= \int_0^2\int_0^1 1+x^2+y^2\,dx\,dy$$
$$= \int_0^2 \left(x + \frac{x^3}{3} + y^2 x\right]_{x=0}^{1} dy = \int_0^2 \frac{4}{3} + y^2\,dy$$
$$= \left[\frac{4}{3}y + \frac{y^3}{3}\right]_0^2 = \boxed{\frac{16}{3}} \text{ ANS}$$

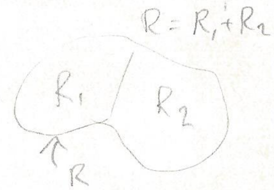


PROPERTIES OF DOUBLE INTEGRALS

① LINEAR:

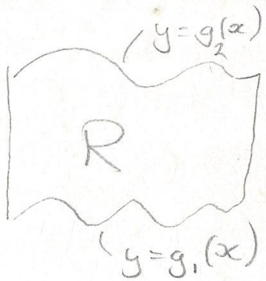
$$\iint af\binom{x}{y} + bg\binom{x}{y}\,dx\,dy = a\iint f\binom{x}{y}dx\,dy + b\iint g\binom{x}{y}dx\,dy$$

② ADDITIVITY:



$$\iint_R f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA$$

NON-RECTANGULAR REGIONS

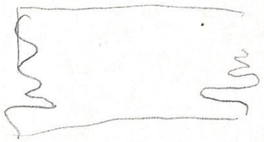


$$\int_a^b\int_c^d f\binom{x}{y}\,dy\,dx = \int_a^b\left[\int_c^d f\binom{x}{y}\,dy\right]dx$$

- suppose the non-rectangular functions can have the y values be described in terms of x

$y_1 = g_1(x) = c \qquad g_2(x) = d = y_2\,,\ g_2(x) \geq g_1(x)$

TYPE I: $\displaystyle\int_a^b\left[\int_{g_1(x)}^{g_2(x)} f\binom{x}{y}dy\right]dx$



- suppose this time $h_1(y) = a \qquad h_2(y) = b$

TYPE II: $\displaystyle\int_c^d\left[\int_{h_1(y)}^{h_2(y)} f\binom{x}{y}dx\right]dy$

NON-RECTANGULAR REGIONS - THEOREM

THEOREM:

a) If R is a type I region on which $f\binom{x}{y}$ continuous

THEN $\displaystyle\iint f\,dA = \int_a^b\left[\int_{g_1(x)}^{g_2(x)} f\binom{x}{y}dA\right.$

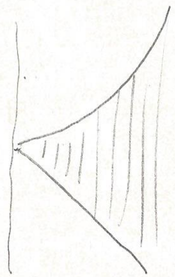
b) If R is a type II region on which $f\binom{x}{y}$ is continuous

$$\iint f\,dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f\binom{x}{y}\,dA$$

- PROOF IS SAME AS

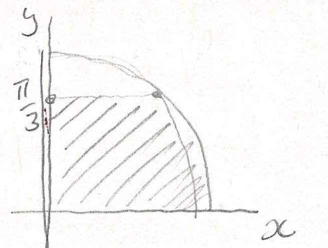
Ex: Sketch the integration region for the iterated integrals and calculate:

(i) $$\int_0^1\int_{-x}^{x^2} xy^2\,dy\,dx$$



$$= \int_0^1\left[\frac{x}{3}y^3\right]_{y=-x}^{x^2} dx = \frac{1}{3}\int_0^1 x^7 + x^4\,dx$$
$$= \frac{1}{3}\left[\frac{x^8}{8} + \frac{x^5}{5}\right]_0^1 = \frac{13}{120}$$

(ii) $$\int_0^{\frac{\pi}{3}}\int_0^{\cos y} x\sin y\,dx\,dy$$



$$= \int_0^{\frac{\pi}{3}}\left[+\frac{x^2}{2}\sin y\right]_0^{\cos y} dy = \int_0^{\frac{\pi}{3}} \frac{\cos^2 y\sin y}{2}\,dy$$
$$= -\int_1^{\frac{1}{2}} \frac{1}{2}u^2\,du \qquad u = \cos y \quad du = -\sin y\,dy$$
$$= \int_{\frac{1}{2}}^1 \frac{1}{2}u^2\,du = \left.\frac{u^3}{6}\right|_{\frac{1}{2}}^1 = \frac{1}{6}\left(\frac{7}{8}\right) = \frac{7}{48}$$

FURTHER EXAMPLES OF DOUBLE INTEGRATION

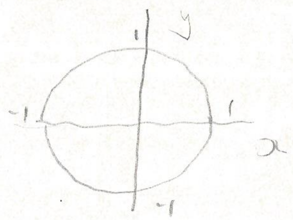
Ex: $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y^2 x^4 \, dy \, dx$

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$y^2 + x^2 = 1$

- GETTING INTEGRAL IS BETTER REVERSED



Ex: $f(x,y) = 2x - y^2$

R IS TRIANGLE: $\binom{-2}{3}, \binom{0}{1}, \binom{2}{3}$

- EVALUATE AS A TYPE I INTEGRAL

$$g_1(x) = \begin{cases} -x+1 & \text{IF } x \le 0 \\ x+1 & \text{IF } x > 0 \end{cases}$$

$$g_2(x) = 3$$

$$\iint_R f\,dA = \int_{-2}^{2}\int_{g_1(x)}^{3} f\,dA = \int_{-2}^{0}\int_{-x+1}^{3} f\,dA + \int_{0}^{2}\int_{x+1}^{3} f\,dA$$

$$= \int_{-2}^{0}\left[2xy - \frac{y^3}{3}\right]_{-x+1}^{3} dx + \int_{0}^{2}\left[2xy - \frac{y^3}{3}\right]_{x+1}^{3} dx$$

$$= \int_{-2}^{0}\left(2x(3+x-1) - \frac{1}{3}(27-(1-x)^3\right) dx + \int$$

SUBSTITUTE $x \to -x$, $dx \to -dx$

$$+ \int_{0}^{2}\left(2x(3-x-1) - \frac{1}{3}(27-(x+1)^3)\right) dx$$

$$= -\int_{2}^{0}\left(-2x(3-x-1) - \frac{1}{3}(27-(1+x)^3)\right) dx$$

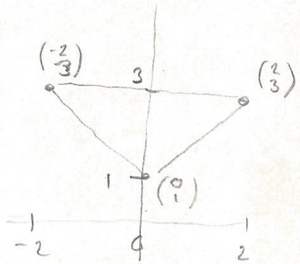
$$+ \int_{0}^{2} 2x(3-x-1) - \frac{1}{3}(27-(1+x)^3) dx$$

$$= \int_{0}^{2} 0 - \frac{2}{3}(27-(1+x)^3) dx = \frac{-68}{3}$$

- CAN ALSO EVALUATE AS A TYPE II

$$\int_{1}^{3}\int_{-y+1}^{y-1} 2x - y^3 \, dx\,dy = \int_{1}^{3}\left[x^2 - y^3 x\right]_{-y+1}^{y-1} dx = \int_{1}^{3} -2y^3 + 2y^2 \, dy$$

$$= \left[-\frac{1}{2}y^4 + \frac{2}{3}y^3\right]_1^3 = -\frac{81}{2} + \frac{54}{3} + \frac{1}{2} - \frac{2}{3} = \frac{-68}{3}$$



REVERSEING INTEGRATION ORDER

$$\int_{0}^{2}\int_{\frac{y}{2}}^{1} e^{x^2} \, dy \, dx \qquad 0 < y < 2x$$

AREA AS A DOUBLE INTEGRAL

- Let $f\binom{x}{y} = z = 1$ and we want to integrate it over a region R. We are calculating the volume of a right CYLINDER WITH BASE AREA AS EQUAL TO THE AREA OF R

SO: $V = A \cdot h = A = \iint_R 1 \, dA$

EXAMPLE: R IS THE REGION ENCLOSED IN BETWEEN $y = x^m$ $y = x^n$ on $x \in [0,1]$

- We assume that $m > n$

$$\Rightarrow A = \iint_R dA = \int_0^1 \int_{x^m}^{x^n} dy \, dx = \int_0^1 x^n - x^m dx$$

$$A = \left[\frac{x^{n+1}}{n+1} + \frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{n+1} - \frac{1}{m+1}$$

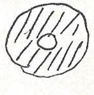
DOUBLE INTEGRALS IN POLAR COORDINATES

- FOR CONVENTION, WE ASSUME THAT:
  - i) $\alpha \leq \beta$
  - ii) $\beta - \alpha \leq 2\pi \qquad \beta \leq \alpha + 2\pi$
  - iii) $0 \leq r_1(\theta) \leq r_2(\theta)$

- LOOKING AT THE AREA OF A CIRCLE

$$\pi r^2 = 2\pi r \cdot \frac{r}{2} = \frac{1}{2}\int_0^{2\pi} [f(\theta)]^2 d\theta \text{ where } f(\theta) = r^2$$

ANNULUS: $A = \pi r_2^2 - \pi r_1^2 = \frac{1}{2}\int_0^{2\pi} \underbrace{f_2(\theta)^2}_{r_2} - \underbrace{f_1(\theta)^2}_{r_1} d\theta$



SECTOR: $A = \frac{1}{2}\int_{\theta_2}^{\theta_1} r^2 d\theta = (\theta_2 - \theta_1) r^2$

POLAR RECTANGLE: $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r_2^2 - r_1^2) d\theta = \frac{1}{2}(\theta_2 - \theta_1)(r_2^2 - r_1^2)$

- **NOW SUPPOSE**:

Δr is uniform and $\Delta\theta$ is uniform

$r_{k+1} = r_k + \Delta k \qquad \theta_{k+1} = \theta_k + \Delta\theta$

$$A_i = \frac{1}{2}\Delta\theta\left[\left(r_k^* + \frac{\Delta k}{2}\right)^2 - \left(r_k^* - \frac{\Delta k}{2}\right)^2\right]$$

$$= r_i^* \Delta r \Delta\theta$$

$V =$ sum of $f\binom{r}{\theta}$ times A_i

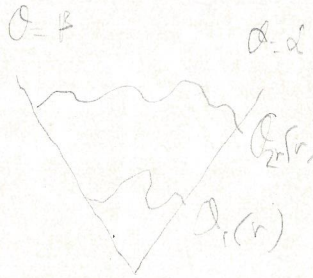
$= \iint_R f\binom{r}{\theta} dA$

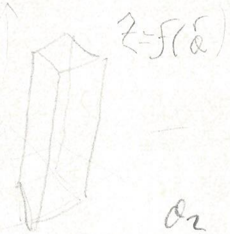
$= \lim_{n\to\infty} \sum_{k=1}^{n} \Delta A_k \cdot f(r_k^*, \theta_k^*)$

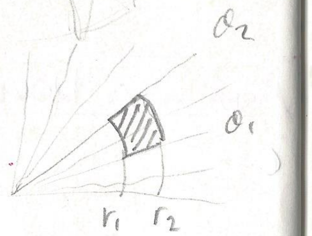
$= \lim_{n\to\infty} \sum_{k=1}^{n} r_k^* \Delta r_k \Delta\theta_k f(r_k^*, \theta_k^*)$

$\Rightarrow V = \iint_R f(r,\theta) dA = \iint_R f(r,\theta) r\, dr\, d\theta$

$V = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f\binom{r}{\theta} r\, dr\, d\theta$







THEOREM

- IF R is a simple polar region with boundaries $\theta_1 = \alpha \quad \theta_2 = \beta \quad \alpha < \beta$

$r = r_1(\theta) \quad r = r_2(\theta)$

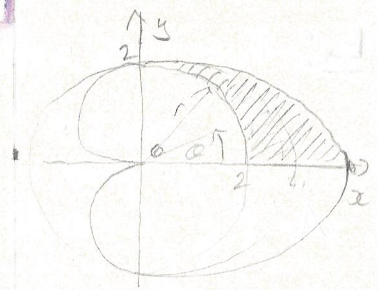
AND $f(r,\theta)$ is continuous on R

THEN $\iint_R f\binom{r}{\theta} dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f\binom{r}{\theta} r\, dr\, d\theta$

Ex $\iint_R \sin\theta \, dA$ where R is the region in the FIRST QUADRANT THAT IS OUTSIDE

OUTSIDE OF THE CIRCLE $r = 2$

INSIDE OF THE CARDOID $r = 2(1+\cos\theta)$



SO: $\alpha = 0 \quad \beta = \frac{\pi}{2}$

$$\int_0^{\pi/2}\int_2^{2(1+\cos\theta)} \sin\theta\, r\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{1}{2} r^2 \sin\theta\right]_2^{2+\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\sin\theta\left[\cancel{4} + 4\cos\theta + \cos^2\theta - \cancel{4}\right] d\theta$$

$$= \int_0^{\pi/2} 2\sin\theta\cos\theta + \cos^2\theta \sin\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin 2\theta \, d\theta + \int_1^0 -\frac{t^2}{2} dt$$

$t = \cos\theta$
$dt = -\sin\theta d\theta$
$t_0 = 1$
$t_1 = 0$

$$= -\frac{\cos 2\theta}{2}\Big|_0^{\frac{\pi}{2}} + \frac{t^3}{6}\Big|_0^1$$

$$= +\frac{1}{4} - \left(-\frac{1}{4}\right) + \frac{1}{6} = \frac{1}{2} + \frac{1}{6} \qquad \frac{8}{3}$$

Area of R

- Area of $R = \iint_R dA \qquad R = \left\{ \begin{pmatrix} r \\ \theta \end{pmatrix} : \alpha < \theta < \beta,\ r_1(\theta) < r < r_2(\theta) \right\}$

$$= \int_\alpha^\beta \int_{r_1\theta}^{r_2\theta} r\,dr\,d\theta = \int_\alpha^\beta \left[\tfrac{1}{2} r^2\right]_{r_1(\theta)}^{r_2(\theta)} d\theta =$$

$$= \frac{1}{2}\int_\alpha^\beta r_2(\theta)^2 - r_1(\theta)^2\,d\theta$$

Ex: Find the area of the 3 "Petal Rose"

$$r = \cos 3\theta$$

$$A = \frac{3}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos 3\theta} r\,dr\,d\theta = \frac{3}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3\theta\,d\theta$$

$$A = \frac{3}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1+\cos 6\theta}{2}\,d\theta$$

$$= \frac{3}{4}\left(\theta + \frac{1}{6}\sin 6\theta\right)_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{3}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = \boxed{\frac{\pi}{4}}$$

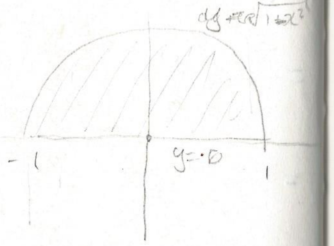
Converting from Cartesian to Polar

$$\iint_R f\begin{pmatrix} x \\ y \end{pmatrix} dA = \iint_R f\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} r\,dr\,d\theta$$

- and express region in terms of $r(\theta)$ functions

Ex: $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx \qquad \boxed{x^2+y^2=r^2}$

$y = \sqrt{1-x^2} \Rightarrow x^2+y^2=1$



$$= \int_0^\pi \int_0^1 (r^3)\,r\,dr\,d\theta$$

Bounds:

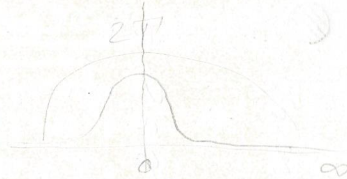
$0 \leq \theta \leq \pi$

$0 \leq r \leq 1$

$$= \int_0^\pi \left(\frac{r^5}{5}\right)_0^1 d\theta = \int_0^\pi \frac{1}{5}\,d\theta = \boxed{\frac{\pi}{5}}$$

Further Examples of Double Integrals

Ex: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \left[\int_{-\infty}^{\infty} e^{-x^2}dx \cdot \int_{-\infty}^{\infty} e^{-y^2}dy\right]^{\frac{1}{2}}$ TRICK



$$\rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\cdot r\,dr\,d\theta$$

$$t = r^2 \qquad dt = 2r\,dr$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} e^{-t}\,dt\,d\theta =$$

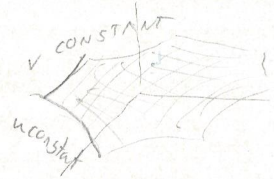
$$= \frac{1}{2}\int_0^{2\pi} (-e^{-\infty}) - (-e^0)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1\,d\theta = \frac{1}{2}(2\pi) = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

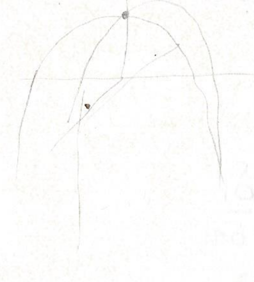
Parametric Structures/surface Area

- A curve: $x = x(t),\ y = y(t),\ z = z(t)$
- Imagine: $\boxed{x = x(u,v)}\ \boxed{y = y(u,v)}\ \boxed{z = z(u,v)}$



We map $\underset{\mathbb{R}^2}{\begin{pmatrix} u \\ v \end{pmatrix}} \rightarrow \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \in \mathbb{R}^3$

Example: Parabaloid $z = 4 - x^2 - y^2$



(1) $x = u,\ y = v,\ z = 4 - u^2 - v^2 \quad \begin{pmatrix} u \\ v \\ 4-u^2-v^2 \end{pmatrix}$

(2) $\boxed{\begin{array}{l} x = x\begin{pmatrix} r \\ \theta \end{pmatrix} \\ = r\cos\theta \end{array}}\ \boxed{\begin{array}{l} y = y\begin{pmatrix} r \\ \theta \end{pmatrix} \\ = r\sin\theta \end{array}}\ \boxed{\begin{array}{l} z = z\begin{pmatrix} r \\ \theta \end{pmatrix} \\ = 4 - r^2 \end{array}} \quad \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 4-r^2 \end{pmatrix}$

Example: Sphere $x^2+y^2+z^2=a^2$

Cylindrical coordinates

(1) $x = r\cos\theta \quad y = r\sin\theta \quad z^2 = a^2 - r^2,\ z = \sqrt{a^2-r^2} \quad \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ \sqrt{a^2-r^2} \end{pmatrix}$

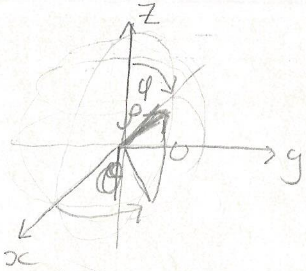
PARAMETRIC STRUCTURES

SPHERICAL COORDINATES:

$$x, y, z \rightarrow \rho = \sqrt{x^2+y^2+z^2} \quad \rho \geq 0$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad 0 \leq \theta \leq 2\pi$$
$$\varphi = \frac{\pi}{2} - \tan^{-1}\left(\frac{z}{x^2+y^2}\right) \quad 0 \leq \varphi \leq \pi$$

$$\rightarrow \begin{aligned} x &= a \sin\varphi \cos\theta \\ y &= a \sin\varphi \sin\theta \\ z &= a \cos\varphi \end{aligned}$$

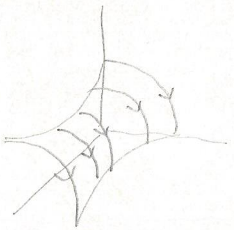
- For a sphere of radius a, we have a dependence on only the angles φ and θ



Representing a Surface of Revolution:

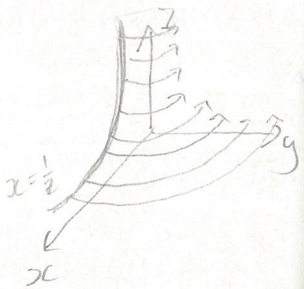
- surface generated by revolving the curve $y=f(x)$ around the x-axis

$x = u \quad y = f(u)\cos v \quad z = f(u)\sin v$



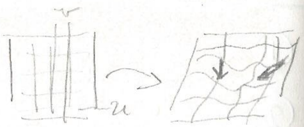
EXAMPLE: $x = \frac{1}{z}$ rotate about z axis

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{z=u} \begin{pmatrix} \cos v \cdot \frac{1}{u} \\ \sin v \cdot \frac{1}{u} \\ u \end{pmatrix} \quad u, v \in \mathbb{R}$$



Vector-Valued function of 2 Variables

$$\vec{r}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \rightarrow \frac{\delta \vec{r}}{\delta u}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \delta_u \vec{x} \\ \delta_u \vec{y} \\ \delta_u \vec{z} \end{pmatrix} \quad \frac{\delta \vec{r}}{\delta v} = \begin{pmatrix} \delta_v \vec{x} \\ \delta_v \vec{y} \\ \delta_v \vec{z} \end{pmatrix}$$



$$\frac{\delta \vec{r}}{\delta u} = \frac{\delta x}{\delta u}\hat{i} + \frac{\delta y}{\delta u}\hat{j} + \frac{\delta z}{\delta u}\hat{k}$$

$$\frac{\delta \vec{r}}{\delta v} = \frac{\delta x}{\delta v}\hat{i} + \frac{\delta y}{\delta v}\hat{j} + \frac{\delta z}{\delta v}\hat{k}$$

TANGENT PLANES TO PARAMETRIC SURFACE

- Let σ be a parametric surface in 3D space

DEF: A PLANE is tangent to σ at P_0 if a line through P_0 lies in the plane IF AND ONLY it is a tangent line at P_0 to a curve on σ

$\sigma : \vec{\sigma}(u,v) \quad P_0 = (a, b, c)$

$$a = x(u_0, v_0)$$
$$b = y(u_0, v_0)$$
$$c = z(u_0, v_0)$$

IF $\frac{\delta r}{\delta u}(P_0) \neq 0 \Rightarrow$ it is tangent to constant u curve on σ at P_0

IF $\frac{\delta r}{\delta v}(P_0) \neq 0 \Rightarrow$

$$\vec{n} = \frac{\frac{\delta \vec{r}}{\delta u} \times \frac{\delta r}{\delta v}}{\left|\frac{\delta \vec{r}}{\delta u} \times \frac{\delta \vec{r}}{\delta v}\right|}$$ IS THE PRINCIPAL UNIT VECTOR TO THE SURFACE σ AT P_0

EXAMPLE: $x = uv \quad y = u \quad z = v^2$

FIND TANGENT PLANE AT $u=2, v=1$

$$\delta_u \vec{r} = v\hat{i} + 1\hat{j} + 0 \rightarrow (2,1,0)$$
$$\delta_v \vec{r} = u\hat{i} + 0 + 2v\hat{j} \rightarrow (-1,0,4)$$
$$\vec{n}(P_0) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = (1,$$
$$\delta_u \vec{r} \times \delta_v \vec{r} = -2v^2\hat{j} - u\hat{k} -$$

$$\iint_R \left|\frac{\delta r}{\delta u} \times \frac{\delta r}{\delta v}\right| dA = \text{AREA}$$

Area of Parimetric Surface

- suppose we have a parametric survace, of u, v in the region R
  - Area of a region R_K

$$\Delta A_K = \Delta u_K \Delta v_K$$

This can be transfered to a region of the curve, which we approximate as a parallelogram.

- Area of a parallelogram ΔS_K

$$\Delta S_K = \left\| \frac{\delta \vec{r}}{\delta u} \Delta u_K \times \frac{\delta \vec{r}}{\delta v} \Delta v_K \right\|$$

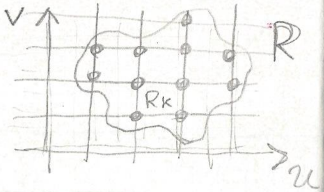
$$= \left\| \frac{\delta \vec{r}}{\delta u} \times \frac{\delta \vec{r}}{\delta v} \right\| \Delta u_K \Delta v_K$$

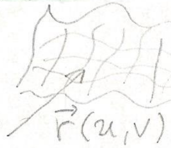
$$= \left\| \frac{\delta \vec{r}}{\delta u} \times \frac{\delta \vec{r}}{\delta v} \right\| \Delta A_K$$

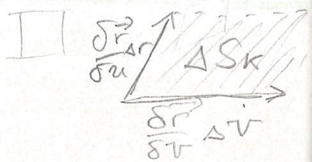
- Taking the sum of these

$$S = \lim_{n \to \infty} \sum_{K=1}^{n} \left\| \frac{\delta \vec{r}}{\delta u} \times \frac{\delta \vec{r}}{\delta v} \right\| \Delta A_K$$

$$S = \cancel{\lim_{n\to\infty}} \iint_R \left\| \frac{\delta \vec{r}}{\delta u} \times \frac{\delta \vec{r}}{\delta v} \right\| dA$$







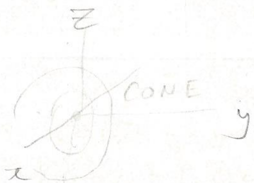
Example: $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ u\cos v \\ u \sin v \end{pmatrix}$ R: $0 \leq u \leq 2$, $0 \leq v \leq 2\pi$

$$\frac{\delta}{\delta u} \vec{r} = \begin{pmatrix} 1 \\ \cos v \\ \sin v \end{pmatrix}$$

$$\frac{\delta}{\delta v} \vec{r} = \begin{pmatrix} 0 \\ -u \sin v \\ u \cos v \end{pmatrix} \Rightarrow \frac{\delta \vec{r}}{\delta u} \times \frac{\delta \vec{r}}{\delta v} = \begin{pmatrix} u \\ -u\cos v \\ -u \sin v \end{pmatrix}$$

$$S = \int_0^{2\pi} \int_0^2 \sqrt{u^2 + u^2\cos^2 v + u^2\sin^2 v} \, du \, dv$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{2}\, u \, du \, dv = \int_0^{2\pi} 2\sqrt{2} \, dv = \boxed{4\pi\sqrt{2}}$$



Surface Integral

$x = u \qquad y = v \qquad z = f(u,v)$

$$\vec{r}(u,v) = u\hat{i} + v\hat{j} + f(u,v)\hat{k}$$

$$\frac{\delta}{\delta u}\vec{r} = 1\hat{i} + 0 + \frac{\delta f}{\delta u}\hat{k} = \hat{i} + \frac{\delta f}{\delta x}\hat{k}$$

$$\frac{\delta}{\delta v}\vec{r} = 0 + \hat{j} + \frac{\delta f}{\delta v}\hat{k} = \hat{j} + \frac{\delta f}{\delta y}\hat{k}$$

$$\Rightarrow \frac{\delta \vec{r}}{\delta u} \times \frac{\delta \vec{r}}{\delta v} = -\frac{\delta z}{\delta x}\hat{i} - \frac{\delta z}{\delta y}\hat{j} + 1\hat{k}$$

$$\left\| \frac{\delta \vec{r}}{\delta u} \times \frac{\delta \vec{r}}{\delta v} \right\| = \sqrt{1 + \left(\frac{\delta z}{\delta x}\right)^2 + \left(\frac{\delta z}{\delta y}\right)^2}$$

$$\Rightarrow S = \iint_R \sqrt{1 + \left(\frac{\delta z}{\delta x}\right)^2 + \left(\frac{\delta z}{\delta y}\right)^2} \, dA$$

Example: Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ found below $z = 1$

$$S = \iint_R \sqrt{1 + 4x^2 + 4y^2} \, dA$$

$$= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2} \, r \, dr \, d\theta$$

$$= 2\pi \cdot \frac{1}{8} \cdot \frac{2}{3} (1+4t)^{\frac{3}{2}} \Big|_0^1 = \frac{\pi}{6}(3\sqrt{3} - 1)$$

TRIPLE INTEGRALS

- MASS OF G PROBLEM

→ WE HAVE A CLOSED SOLID REGION IN THE x, y, z COORDINATE SYSTEM

→ WE ASSUME IT IS *finite* i.e: it can be enclosed in a box

WE DEFINE THE *triple integral over G* just as we had done in previous cases:

- DEVIDE THE REGION INTO "CUBES"
- IN THE k^{th} "cube"/volume, choose (x_k^*, y_k^*, z_k^*)

VOLUME $V_K = \Delta x_K \Delta y_K \Delta z_K$

$$\iiint_G f(x,y,z)\,dA = \lim_{h\to\infty} \sum f(x_K^*, y_K^*, z_K^*)\,\Delta V_K$$

PROPERTIES:

① $\iiint_G (cf + dg)\,dV = c\iiint_G f\,dV + d\iiint_G g\,dV$

② $\iiint_G f\,dV = \iiint_{G_1} f\,dV + \iiint_{G_2} f\,dV$

IF THE INTERSECTION OF G_1 and G_2 IS ONLY POINTS, LINES OF 2D SURFACES

- IF $f(x,y,z) \geq 0$ ON G

THEN it can be described as it's density at each point

AND $f_{density} \Rightarrow f(x,y,z) \cdot \frac{\text{MASS}}{\text{VOLUME}}$

$$\text{mass} = \iiint_G f_{density}\begin{pmatrix}x\\y\\z\end{pmatrix} dV$$

EVALUATING TRIPLE INTEGRALS

- Suppose we have a rectangular region G

$$G = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} : a \leq x \leq b;\ c \leq y \leq d;\ e \leq z \leq f \right\}$$

$$\iiint_G f\begin{pmatrix}x\\y\\z\end{pmatrix} dV = \int_a^b\int_c^d\int_e^f f\begin{pmatrix}x\\y\\z\end{pmatrix} dz\,dy\,dx$$

- ORDER DOES NOT MATTER AS LONG AS THE FUNCTION f IS CONTINUOUS IN G

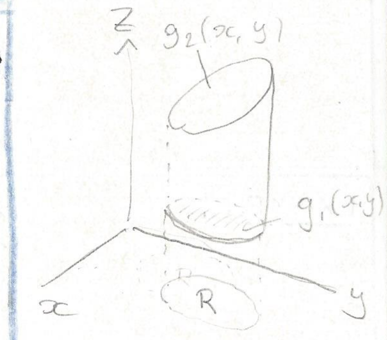
INTEGRATING OVER MORE GENERAL REGIONS

- Suppose G is bounded from:

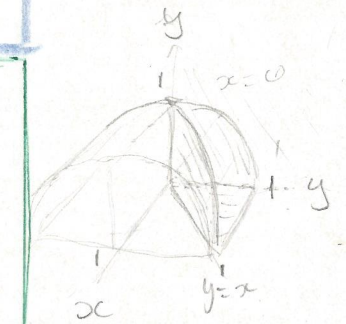
below by $z = g_1(x,y)$
above by $z = g_2(x,y)$

AND G IS IN THE xy REGION R

$$\iiint_G f\begin{pmatrix}x\\y\\z\end{pmatrix} dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz \right] dA$$



EXAMPLE: Let G be the wedge in the first octant, that is cut from the cylinder $y^2 + z^2 \leq 1$ by the planes $y = x$ AND $x = 0$



- Let $f(x,y,z) = z$

$$\iiint_G z\,dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz \right] dA = \iint_R \left[\tfrac{1}{2} z^2 \right]_0^{\sqrt{1-y^2}} dA$$

$$= \int_0^1\int_0^y \tfrac{1}{2}(1-y^2)\,dx\,dy$$

$$= \tfrac{1}{2}\int_0^1 y - y^3\,dy = \tfrac{1}{2}\left[\tfrac{y^2}{2} - \tfrac{y^4}{4}\right]_0^1$$

ANSWER $= \frac{1}{8}$

TRIPLE INTEGRALS - EXAMPLES

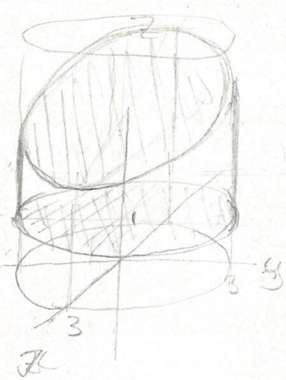
EXAMPLE: FIND the volume of a solid within the cylinder $x^2+y^2 \le 9$ AND THE PLANES $z=1$ AND $x+z=5 \Rightarrow z=5-x$

$$V = \iiint_G 1\, dV = \iint_R \left[\int_1^{5-x} dz\right] dA = \iint_R 4 - x\, dA$$

$$= \int_0^{2\pi}\int_0^3 (4 - r\cos\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{4r^2}{2} - \frac{r^3}{3}\cos\theta\right]_0^3 d\theta \;\Big|\; = \int_0^{2\pi} 18 - 9\cos\theta\, d\theta$$

ANSWER $= 36\pi$



EXAMPLE: FIND VOLUME of the solid enclosed BETWEEN $z = 5(x^2+y^2)$ and $z = 6 - 7x^2 - y^2$

TO FIND R: $5x^2+5y^2 = 6 - 7x^2 - y^2$

$\Rightarrow 12x^2 + 6y^2 = 6$

$\Rightarrow 2x^2 + y^2 = 1 \quad \Rightarrow y = \pm\sqrt{1-2x^2}$

$$V = \iiint_G dV = \iint_R \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\right] dA = \iint_R 6 - 12x^2 - 6y^2\, dA$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} 6 - 12x^2 - 6y^2\, dy\, dx$$

$\boxed{\text{CHANGE OF COORDINATES}}$ *

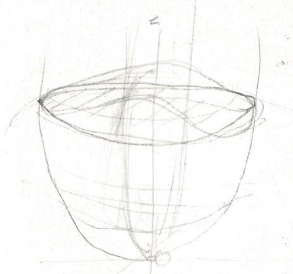
$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} 12\sqrt{1-2x^2} - 24x^2\sqrt{1-2x^2} - 2\left(2(\sqrt{1-x^2})^3\right) dx$$

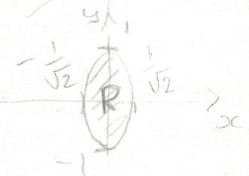
$$= 24 \int_0^{\frac{1}{\sqrt{2}}} (1-2x^2)\sqrt{1-2x^2} - \frac{1}{3}(1-2x^2)\sqrt{1-2x^2}\, dx$$

$$= 16 \int_0^{\frac{1}{\sqrt{2}}} (1-2x^2)(\sqrt{1-2x^2})\, dx$$

- LET $x = \frac{1}{\sqrt{2}}\sin t \qquad dx = \frac{1}{\sqrt{2}}\cos t\, dt$

$$\frac{16}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \cos^4 t\, dt = \qquad = \frac{3\pi}{\sqrt{2}}$$





TRIPLE INTEGRALS - EXAMPLES

EXAMPLE * w/ CHANGE OF COORDINATES

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} 6 - 12x^2 - 6y^2\, dy\, dx \xrightarrow[dx = \frac{dt}{\sqrt{2}}]{x = \frac{t}{\sqrt{2}}} \frac{6}{\sqrt{2}} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} 1 - t^2 - y^2\, dy\, dt$$

- CHANGE TO POLAR

$$\frac{6}{\sqrt{2}} \int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta$$

$$= \frac{6}{\sqrt{2}} \cdot 2\pi \cdot \left(\frac{r^2}{2} - \frac{r^4}{4}\right)_0^1 = \frac{3\pi}{\sqrt{2}}$$

- IN THE EXAMPLES, WE DO z FIRST, BUT COULD DO x OR y FIRST TOO

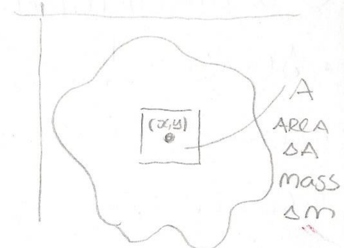
CENTROIDS / CENTRE OF GRAVITY / THM OF PAPPUS

DENSITY OF A LAMINA

- consider an idealised flat object, thin enough to be considered a 2D plane region.
  - → WE CALL THIS A LAMINA
  - → THE DENSITY of a Lamina is defined as

$$\delta\binom{x}{y} = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A} \qquad \Delta M_k = \delta\binom{x}{y}\Delta A_k$$

$\Rightarrow$ MASS $M = \iint_R \delta(x,y)\, dA$



CENTER OF GRAVITY $R(\bar{x}, \bar{y})$

s.t. $\bar{x} = \frac{1}{M}\iint_R x\,\delta(x,y)\, dA = \frac{M_y}{M}$

$\bar{y} = \frac{1}{M}\iint_R y\,\delta(x,y)\, dA = \frac{M_x}{M}$

- M_x, M_y are the <u>first moments</u> of the lamina about the x- and y- axes

EXAMPLE: FIND THE CENTER OF GRAVITY OF THE TRIANGULAR LAMINA BETWEEN $(0,0), (0,1), (1,0)$ WITH $\delta(x,y) = xy$

$$M = \iint_R xy\, dA = \frac{1}{24} \qquad M_x = M_y = \iint_R x^2 y\, dA = \frac{1}{60}$$

$$\Rightarrow \bar{x} = \bar{y} = \frac{24}{60} = \frac{1}{4} \Rightarrow \text{C.O.M.} = \left(\frac{1}{5}, \frac{1}{5}\right)$$

CENTER OF MASS OF HOMOGENOUS OBJECTS

- IF THE LAMINA HOMOGENEOUS $\Leftrightarrow \delta\binom{x}{y} =$ A CONSTANT

THE C.O.M. IS CALLED THE CENTROID of the region R

$$\bar{x} = \frac{\iint_R x\, dA}{\iint_R dA} = \frac{1}{\text{Area } R}\iint_R x\, dA \qquad \bar{y} = \frac{1}{\text{Area } R}\iint_R y\, dA$$

- FOR 3D BODIES

$$\delta(x,y,z) = \lim_{\Delta V \to 0} \frac{\Delta M_k}{\Delta V_k}$$

$$M = \iiint_G \delta(x,y,z)\, dV$$

$$\bar{x} = \frac{1}{M}\iiint x\, \delta\begin{pmatrix}x\\y\\z\end{pmatrix} dV$$

3-DIMENSIONAL OBJECTS

- $\delta(x,y,z) = \lim_{\Delta r \to 0} \frac{\Delta m}{\Delta V}$ DENSITY OF OBJECT
- MASS of $G = \iiint_G \delta\begin{pmatrix}x\\y\\z\end{pmatrix} dV$
- CENTER OF GRAVITY: $\begin{pmatrix}\bar{x}\\\bar{y}\\\bar{z}\end{pmatrix} = \frac{1}{M}\begin{pmatrix}\iiint x\, \delta\begin{pmatrix}x\\y\\z\end{pmatrix} dV\\ \iiint y\, \delta\begin{pmatrix}x\\y\\z\end{pmatrix} dV\\ \iiint z\, \delta\begin{pmatrix}x\\y\\z\end{pmatrix} dV\end{pmatrix}$

- IF $\delta(x,y,z)$ IS CONSTANT (& EQUAL TO 1)

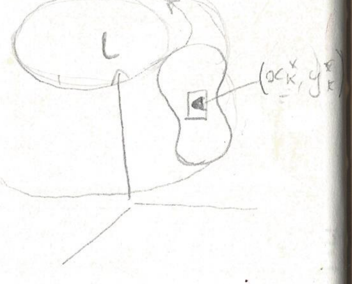
CENTROID OF G $\begin{pmatrix}\bar{x}\\\bar{y}\\\bar{z}\end{pmatrix} = \frac{1}{V}\begin{pmatrix}\iiint x\, dV\\ \iiint y\, dV\\ \iiint z\, dV\end{pmatrix}$

THEOREM OF PAPPUS:

- IF R is a bounded plane region, and L is a line that lies in the plane s.t. R is ENTIRELY ON ONE SIDE OF L,

THEN: THE VOLUME OF THE SOLID FORMED BY REVOLVING R ABOUT L IS GIVEN BY

$$V = A \cdot \begin{pmatrix}\text{distance travelled by}\\ \text{the centroid of } R\end{pmatrix}, \quad A = \text{Area of } R$$



THEOREM OF PAPPUS PROOF

PROOF: • FIRST, RIGID TRANSFORM s.t. is IN xy plane

suppose for a point (x_k^*, y_k^*) we have

$$\Delta R_k = \text{area}(R_k) = \Delta x_k \cdot \Delta y_k$$

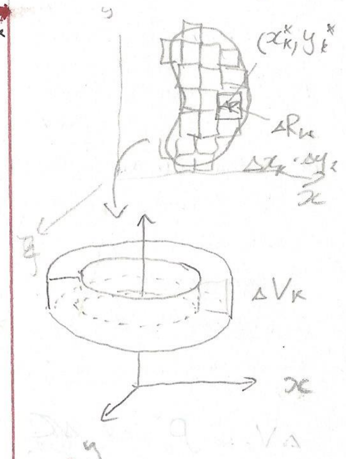
$$\Rightarrow \Delta V_k = \text{area}(R_k) \cdot 2\pi x_k^* = \Delta x_k \Delta y_k \cdot 2\pi x_k^*$$

THE VOLUME OF REVOLUTION WOULD BE

$$V \approx \sum_{i=1}^{n} \Delta x_k \Delta y_k\, 2\pi x_k^*$$

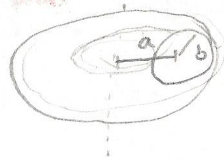
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x_k \Delta y_k\, 2\pi x_k^*$$

$$= \iint_R 2\pi x\, dA = A \cdot 2\pi \cdot \bar{x}$$



EXAMPLE: TORUS revolving circle of radius b around a radius a as shown

$$V = (\pi b^2) \times (2\pi (a))$$



TRIPLE INTEGRALS IN CYLINDRICAL COORDS

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

- DEVIDING G INTO CYLINDRICAL WEDGES

$$\iiint_G f(r,\theta,z)\, dV = \lim_{n\to\infty}\sum_{k=1}^{n} f\begin{pmatrix}r_k^*\\\theta_k^*\\z_k^*\end{pmatrix}\Delta V_k$$

$$= \iiint_G f\begin{pmatrix}r\\\theta\\z\end{pmatrix} r\, dr\, d\theta\, dz$$

$\Delta V_k = r_k^* \Delta r_k \Delta\theta_k \Delta z_k$

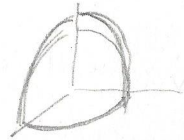
THEOREM let G be an xy SOLID

LET R BE A SIMPLE POLAR REGION.

$$\iiint_G f(r,\theta,z)\, dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

EXAMPLE:

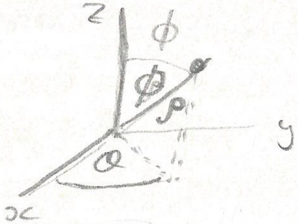
$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\, dz\, dy\, dx$$



$$\int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r^2\cos^2\theta\, r\, dz\, dr\, d\theta$$

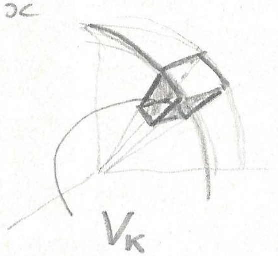
TRIPLE INTEGRALS IN SPHERICAL

$\rho \to$ CONSTANT ρ GIVES SPHERE
$\theta \to$ CONSTANT θ gives HALF-PLANE
$\varphi \to$ CONSTANT φ gives CONE (RIGHT, CIRCULAR) OR x-y PLANE OR LINE



- A SPHERICAL WEDGE OR ELEMENT of Volume is a SOLID enclosed between six surfaces:

2 spheres: $\rho = \rho_1$, $\rho = \rho_2$ $\quad \rho_1 < \rho_2$
2 HALF-PLANES: $\theta = \theta_1$, $\theta = \theta_2$ $\quad \theta_1 < \theta_2$
2 RIGHT-CIRCULAR CONES: $\phi = \phi_1$, $\phi = \phi_2$, $\phi_1 < \phi_2$



$$\Delta V_K \approx \rho_K^{*} \sin\phi_K^{*}\, \rho_K^{*} \Delta\phi_K\, \Delta\rho_K$$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\iiint_G f(\rho,\theta,\phi)\,dV = \lim_{n\to\infty} \sum_{k=1}^{n} f(\rho_K^*, \theta_K^*, \phi_K^*)\,\Delta V_K$$

$$= \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho,\theta,\phi)\,\rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

EX: FIND THE VOLUME & CENTROID OF

G BOUNDED ABOVE BY $x^2+y^2+z^2=16$
BELOW BY $z = \sqrt{x^2+y^2}$

$x^2+y^2+z^2 = 16 \to \rho = 4$
$z = \sqrt{x^2+y^2}$ CONE $\to \phi = \frac{\pi}{4}$

$$V = \iiint_G dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} \rho^2 \sin\phi \, d\rho\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} 64 \sin\phi \, d\phi\, d\theta = \dots = \frac{64}{3}\pi(2-\sqrt{2})$$

$\bar{x} = \frac{1}{V}\iiint x\, dV = 0$ due to being ANGULARLY symmetric ABOUT THE Z AXIS

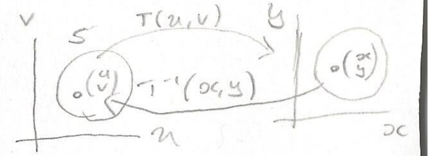
$\bar{y} = 0$

$$\bar{z} = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} \rho\cos\phi \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \frac{32\pi}{V} = \frac{3}{2(2+\sqrt{2})} \approx 2.56$$

TRANSFORMATIONS

IF T IS A TRANSFORMATION from uv space to xy space, (x,y) IS CALLED THE IMAGE
T IS ONE-TO-ONE (BIJECTIVE)
$\Rightarrow \exists\ T^{-1}$ from xy-space to uv SPACE



EX: $T(x,y)\begin{cases} x = \frac{1}{4}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases} \to T^{-1}(u,v)\begin{cases} u = 2x+y \\ v = 2x-y \end{cases}$

JACOBIANS IN 2 VARIABLES

- WE NEED TO UNDERSTAND THE RELATIONSHIP BETWEEN AREA OF A SMALL RECTANGLE in uv- and xy- SPACE

$\vec{r}\begin{pmatrix}u \\ v_0\end{pmatrix} = x\begin{pmatrix}u \\ v_0\end{pmatrix}\hat{i} + y\begin{pmatrix}u \\ v_0\end{pmatrix}\hat{j} \qquad \vec{r}\begin{pmatrix}u_0 \\ v\end{pmatrix} = x\begin{pmatrix}u_0 \\ v\end{pmatrix}\hat{i} + y\begin{pmatrix}u_0 \\ v\end{pmatrix}\hat{j}$

$\boxed{\frac{\delta r}{\delta u} = \frac{\delta x}{\delta u}\hat{i} + \frac{\delta y}{\delta u}\hat{j}} \qquad \boxed{\frac{\delta r}{\delta v} = \frac{\delta x}{\delta v}\hat{i} + \frac{\delta y}{\delta v}\hat{j}}$

$$\text{AREA} = \left|\frac{\delta r}{\delta u}\Delta u \times \frac{\delta r}{\delta v}\Delta v\right| = \underbrace{\left|\begin{matrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v}\end{matrix}\right|}_{J(u,v) \text{ THE JACOBIAN}} \Delta u \Delta v$$

DEF: IF T is the transformation from uv to xy defined by $x = x\begin{pmatrix}u \\ v\end{pmatrix}$ $y = y\begin{pmatrix}u \\ v\end{pmatrix}$, THEN THE JACOBIAN of T is defined by

$$J(u,v) = \left|\begin{matrix} \frac{\delta x}{\delta u} & \frac{\delta y}{\delta u} \\ \frac{\delta x}{\delta v} & \frac{\delta y}{\delta v}\end{matrix}\right| = \frac{\delta x}{\delta u}\frac{\delta y}{\delta v} - \frac{\delta x}{\delta v}\frac{\delta y}{\delta u}$$

IN 3D:

$$J\begin{pmatrix}u \\ v \\ w\end{pmatrix} = \left|\begin{matrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} & \frac{\delta x}{\delta w} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} & \frac{\delta y}{\delta w} \\ \frac{\delta z}{\delta u} & \frac{\delta z}{\delta v} & \frac{\delta z}{\delta w}\end{matrix}\right|$$

$$\iiint f\begin{pmatrix}x \\ y \\ z\end{pmatrix} dV_{xyz} = \iiint f\begin{pmatrix}x(u,v,w) \\ y(u,v,w) \\ z(u,v,w)\end{pmatrix} \left|J\begin{pmatrix}u \\ v \\ w\end{pmatrix}\right| dV_{uvw}$$

- CHANGE OF VARIABLE: (w/ some restrictions on T)

$$\iint_R f\begin{pmatrix}x \\ y\end{pmatrix} dA_{xy} = \iint_S f\begin{pmatrix}x(u,v) \\ y(u,v)\end{pmatrix} \left|J\begin{pmatrix}u \\ v\end{pmatrix}\right| dA_{uv}$$

- TRUE IF $J(u,v) \neq 0$, & DOES NOT CHANGE SIGN IN S

EX: $\iint \frac{x-y}{x+y}\, dA_{xy}$ w $R = \left\{ \begin{matrix} 0 \leq x-y \leq 1 \\ 1 \leq x+y \leq 3 \end{matrix} \right\}$

$\left.\begin{matrix} u = x-y \\ v = x+y \end{matrix}\right\} \to \begin{matrix} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{matrix} \qquad J\begin{pmatrix}u \\ v\end{pmatrix} = \left\| \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \right\| = \frac{-1}{2} \neq 0$

$$\int_1^3\int_0^1 \frac{v}{u} \cdot \frac{1}{2}\, dA_{uv} = \frac{1}{4}\ln 3$$