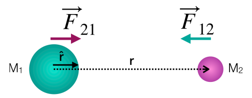
Lecture 3: Tides and The laws of planetary motion

Read: Ch 23.5 of “Astronomy: a Physical Perspective” (M. Kutner)
Ch. 13.5 of “University Physics” (Young & Freedman);
Ch. 4 of “The Cosmic Perspective” (Bennett et al.)

Prof Aline Vidotto

Quick recap of last lecture

The universal law of gravitation



Magnitude of the force:

$$F = \frac{GM_1M_2}{r^2}$$

In vector form:

$$\vec{F}_{12} = -\frac{GM_1M_2}{r^2}\hat{r}$$

Pointing against $\hat{r}$ — The unit vector along $\vec{r}$

The gravitational potential energy

From the definition of work:

$$\Delta U := U_2 - U_1 = -W$$

the potential energy is

$$U = -\frac{Gm_1m_2}{r}$$

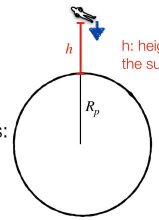
• When the object moves away from the Earth, r increases and U becomes less negative (i.e., U increases). And vice-versa.

The acceleration of gravity

$$g = \frac{F_g}{m}$$

At Earth, acceleration decreases at higher altitudes:

$$g = \frac{GM_E}{(R_E + h)^2}$$



h: height above the surface

Properties of circular orbits:

• Velocity

$$v_{\rm orb} = \sqrt{\frac{GM}{r}}$$

• Period

$$P = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

• Total mechanical energy:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

• Escape velocity

$$v_{\rm esc} = \sqrt{\frac{2GM_E}{R_E}}$$

What we will cover today…

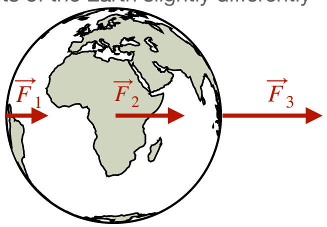
Goal: understand how tides are created (in the Earth and in other astronomical objects) and the laws of planetary motion

1. Tidal forces
2. Kepler’s laws
   - 1st law
   - 2nd law
   - 3rd law

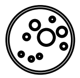
1. Tidal forces

How does gravity cause tides? (1)

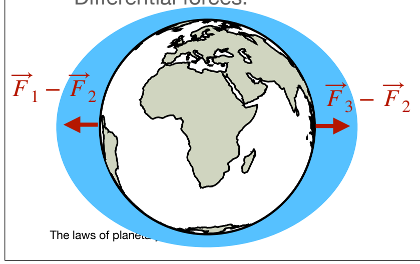
- Gravity attracts Earth and moon towards each other, but it affects different parts of the Earth slightly differently



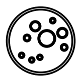
Moon's gravitational attraction is greater on the side that faces the moon (F_3) than on the opposite side (F_1)



- Differential forces:

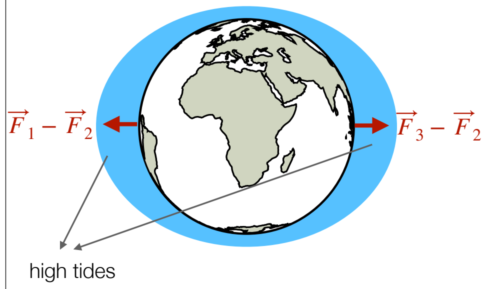


Difference in Moon's gravitational pull raises **two** tidal bulges, both toward and away from the Moon.
The effect is greatest in Earth's oceans: liquid moves around more easily.



Calculating the tidal force

- Tidal effects are caused by the difference between the gravitational forces on opposite sides of an object



- The change in gravitational force ΔF_{tid} in going from r to r+Δr is

$$\Delta F_{\rm tid} = \frac{dF}{dr}\Delta r$$

- Given that the gravitational force is $F = \frac{GmM}{r^2}$

then

$$\frac{dF}{dr} = \frac{-2GmM}{r^3}$$

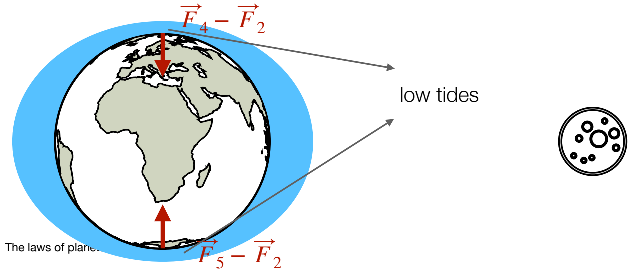
- The tidal force is

$$\Delta F_{\rm tid} = \frac{-2GmM}{r^3}\Delta r$$

Tidal effects fall as $1/r^3$, faster than the $1/r^2$ fall-off of the gravitational force itself.

How does gravity cause tides? (2)

- Differential forces:



Example: tidal effects on the Earth

- Compare the effects of the tidal effects caused on the Earth by the Sun and by the Moon

Moon-Earth $\quad \Delta F_{\rm tid,ME} = \frac{-2GM_{\rm moon}M_E}{r_{\rm ME}^3}\Delta r$

Sun-Earth $\quad \Delta F_{\rm tid,SE} = \frac{-2GM_{\rm sun}M_E}{r_{\rm SE}^3}\Delta r$

$M_{\rm sun} = 2\times 10^{30}$ kg
$M_E = 6\times 10^{24}$ kg
$M_{\rm moon} = 7\times 10^{22}$ kg
$r_{\rm SE} = 1{\rm au} = 1.5\times 10^8$ km
$r_{\rm ME} = 3.85\times 10^5$ km

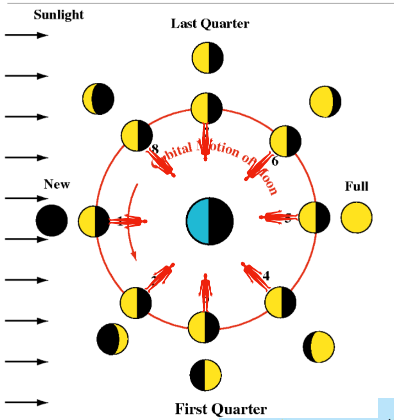
$$\frac{\Delta F_{\rm tid,SE}}{\Delta F_{\rm tid,ME}} = \frac{\frac{\cancel{-2G}M_{\rm sun}\cancel{M_E}}{r_{SE}^3}\cancel{\Delta r}}{\frac{\cancel{-2G}M_{\rm moon}\cancel{M_E}}{r_{\rm ME}^3}\cancel{\Delta r}} = \frac{M_{\rm sun}}{M_{\rm moon}}\frac{r_{\rm ME}^3}{r_{\rm SE}^3}$$

$$= \frac{2\times 10^{30}}{7\times 10^{22}}\left(\frac{3.85\times 10^5\times 10^3}{1.5\times 10^8\times 10^3}\right)^3$$

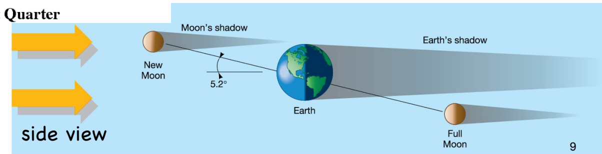
$$\simeq 0.48$$

- The Sun also raises tidal bulges on the Earth, but its tidal force is ~ half that of the moon on the Earth.
- Because of moon's orbit, the Sun-moon-Earth position changes, affecting tides.

Sun-Earth-moon position: the phases of the moon

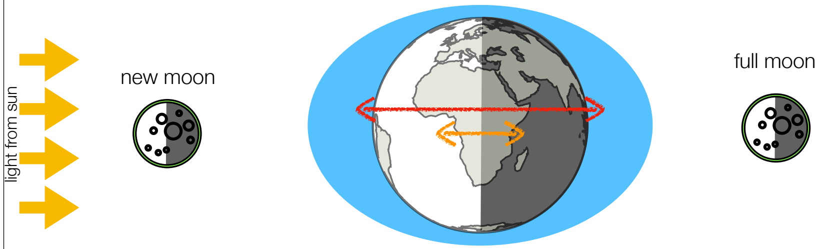


- At any given time, half of the moon is illuminated by sunlight
  - Inner images: actual position and illumination of the moon
- Phase depends on the Sun-Earth-Moon angle:
  - Outer images: appearance of the moon as seen from Earth
- Synchronous rotation: moon rotates with the same period of the orbit, we always see the same face
  - No part of the Moon is in perpetual darkness



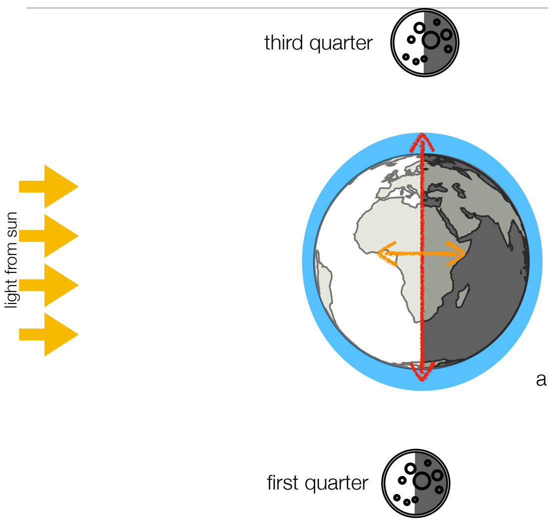
Tides and lunar phases

- When the moon is either full or new, the Sun-Moon-Earth are ~ aligned
  - tidal bulges raised in Earth by the moon and the Sun reinforces each other
  - tides are largest



Tidal forces from the SUN and the MOON work together

Tides and lunar phases

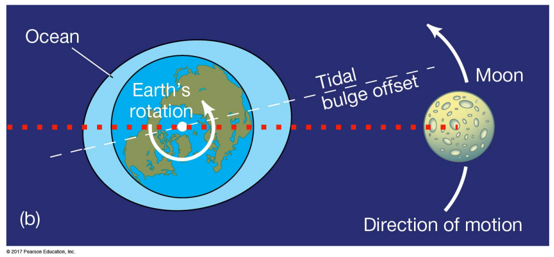


- When the moon is either in its 1st or 3rd quarter, tidal effects of the moon and the sun partially cancel each other
  - the tides are smallest

Tidal forces from the SUN and the MOON work against each together

Tidal locking: orbital synchronisation

1. The Earth does not respond instantaneously to tidal effects: Earth's rotation causes the bulge in the side near the Moon to get ahead of the Earth-moon line.

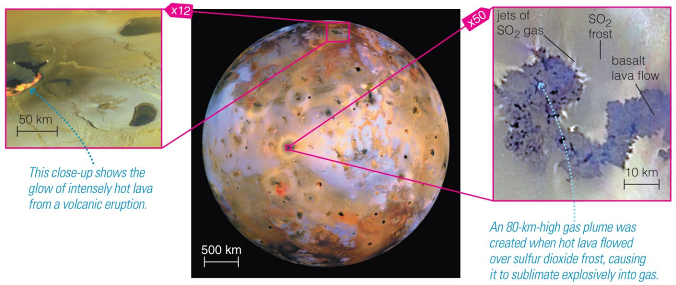


3. A slower rotating Earth, causes the Moon's orbital distance to increase (total angular momentum of the Earth-Moon system does not change). The moon distance increases ~4cm per century

2. The Moon's gravity tries to pull the misaligned bulge back into line, slowing Earth's rotation.

- This will continue until Earth rotates synchronously with the Moon: same side of Earth always pointing towards the Moon.
- Tidal forces has already caused the moon to be in synchronous orbit; tidal friction caused it to "lock" in synchronous rotation

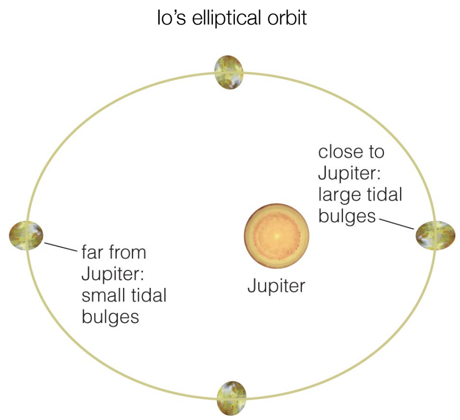
Another example of tides: Jupiter’s satellite Io



- Io is the most volcanically active body in the solar system. Most of the black, brown and red spots on the surface are recently active volcanic features. White and yellow are sulphur dioxide and sulphur deposits, respectively, from volcanic gases.
- Tides are the cause of the extreme heating of Io

Tidal heating

- Jupiter’s mass makes tidal force far larger than the tidal force the Earth exerts on the Moon.
- Io is in synchronous rotation (orbital period = rotation period), but due to elliptical orbit, orbital speed varies, causing matter to oscillate through the tidal elongation.
- Io is “tidally heated”



Conceptual question

At what lunar phase would the variation between high and low tides be greatest?

(a) new
(b) first quarter
(c) full
(d) third quarter
(e) both new and full
(f) both first and third quarters

Conceptual question

At what lunar phase would the variation between high and low tides be greatest?

(a) new
(b) first quarter
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(d) third quarter
(e) both new and full
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Explanation: At new and full Moon phases, the Sun and Moon combine to stretch the Earth and its oceans even more.
We see *higher high tides* and *lower low tides.*

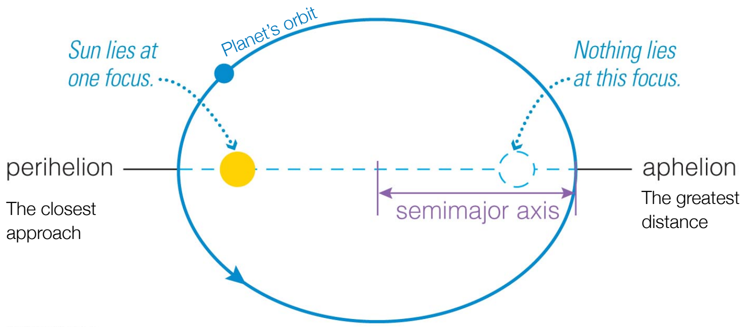
2. Kepler's laws of orbits: the first law

Kepler's laws

1. Planetary orbits are **ellipses**, with the Sun at one focus.
2. As a planet moves around its orbit, it sweeps out equal areas in equal times
3. Square of period of planet's orbital motion is proportional to cube of semi-major axis.

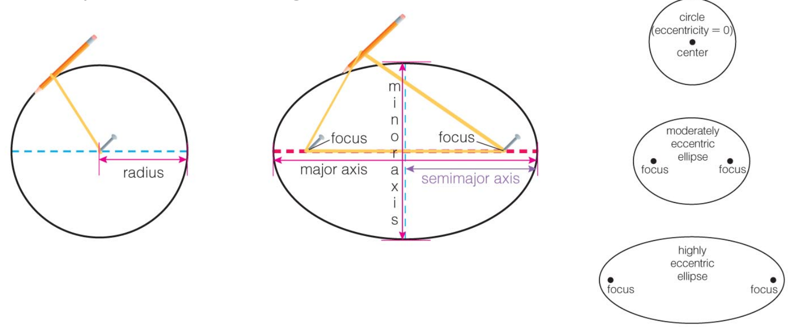
Kepler's first law

1. Planetary orbits are **ellipses**, with the Sun at one focus.



What is an ellipse?

- An ellipse looks like an elongated circle.

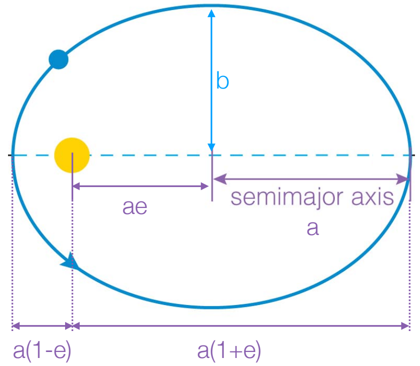


a Drawing a circle with a string of fixed length.

b Drawing an ellipse with a string of fixed length.

c Eccentricity describes how much an ellipse deviates from a perfect circle.

What is an ellipse?



- semi-major axis: a
- semi-minor axis: b
- eccentricity: e

- The {x,y} coordinates of an ellipse is described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- The distance from the centre of the ellipse to a focus is ae.
- Homework: From the figure, demonstrate that

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

- In polar coordinates {r,θ}, the equation of the ellipse is

$$\frac{a(1-e^2)}{r} = 1 + e\cos\theta$$

Example: properties of an elliptical orbit

- A comet has a perihelion distance of 1AU and an aphelion distance of 7AU. Determine the semi-major axis, the eccentricity and the semi-minor axis of the ellipse.

From the figure, we have for the aphelion

$$a(1+e)=7$$

and for the perihelion

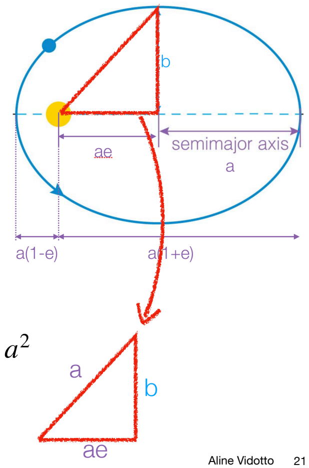
$$a(1-e)=1$$

$$\div \quad \frac{(1+e)}{(1-e)}=\frac{7}{1}$$

$$\boxed{e=\frac{6}{8}=0.75}$$

$$a(1-0.75)=1$$

$$\boxed{a=4\text{AU}}$$



$$(ae)^2+b^2=a^2$$

$$b^2=a^2-(ae)^2$$

$$=4^2-(4\times0.75)^2$$

$$\boxed{b=2.64\text{AU}}$$

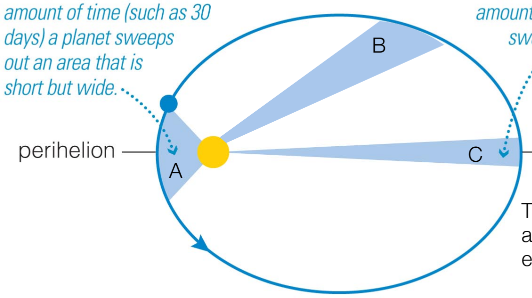
3. Kepler's laws of orbits: the second law

Kepler's laws

1. Planetary orbits are **ellipses**, with the Sun at one focus.
2. As a planet moves around its orbit, it sweeps out equal areas in equal times
3. Square of period of planet's orbital motion is proportional to cube of semi-major axis.

Kepler's second law

2. As a planet moves around its orbit, it sweeps out equal areas in equal times



This means that a planet travels faster when it is nearer to the Sun and slower when it is farther from the Sun.

What keeps a planet rotating and orbiting the Sun? A: conservation of angular momentum

- Angular momentum = mass x velocity x radius $\vec{l}=\vec{r}\times\vec{p}=\vec{r}\times m\vec{v}$

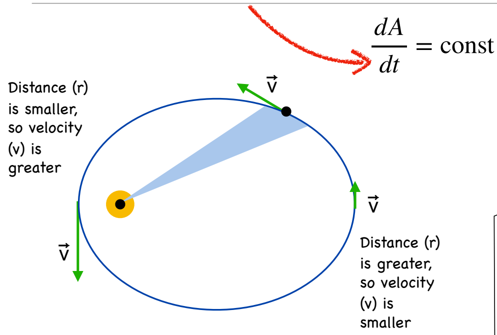
$$l=mv_{\perp}r=\text{const}$$

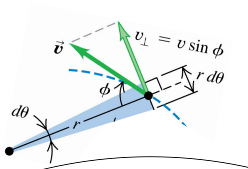
Angular momentum conservation also explains why objects rotate faster as they shrink in radius

- The angular momentum of an object cannot change unless an external <u>twisting</u> <u>force</u> (torque) is acting on it: $\vec{\tau}=\vec{r}\times\vec{F}$
  - Earth experiences <u>no</u> torque as it orbits the Sun, because $\vec{F}_g \parallel \vec{r}$.
  - So its orbit will continue indefinitely.

2nd law: Planet sweeps out equal areas in equal times

$$\frac{dA}{dt} = \text{const}$$





- Area of the blue triangle is

$$dA = \frac{r\,rd\theta}{2}$$

- Rate at which area is swept is

$$\frac{dA}{dt} = \frac{1}{2}\frac{r\,rd\theta}{dt} \propto \frac{d\theta}{dt}r^2 \qquad (2)$$

- From (1) and (2), we conclude that rate at which area is swept is a constant.

Demonstration of Kepler's 2nd law:

$$\left.\begin{aligned} l &= mv_\perp r = \text{const} \\ v_\perp &:= \frac{\Delta s}{\Delta t} = \frac{rd\theta}{dt} \end{aligned}\right\} \quad \frac{d\theta}{dt}r^2 = \text{const} \qquad (1)$$

4. Kepler's laws of orbits: the third law

Kepler's laws

1. Planetary orbits are **ellipses**, with the Sun at one focus.
2. As a planet moves around its orbit, it sweeps out equal areas in equal times
3. Square of period of planet's orbital motion is proportional to cube of semi-major axis.

Kepler's third law

3. Square of period of planet's orbital motion is proportional to cube of semi-major axis.

$$P^2 \propto a^3 \quad \Rightarrow \quad P^2 = ka^3$$

- semi-major axis: a
- period: P
- k: constant

- If we choose the year as the unit of time and the AU as the unit of distance, then k=1.

$$\left(\frac{P}{1\text{yr}}\right)^2 = k\left(\frac{a}{1\text{AU}}\right)^3 \quad \Rightarrow \quad \left(\frac{P}{1\text{yr}}\right)^2 = \left(\frac{a}{1\text{AU}}\right)^3$$

Attention! k=1 only for a planet orbiting our sun, with period given in years, and orbital distance in AU.

Verification of Kepler's third law ↓

Planet	Orbital Semimajor Axis, a (AU)	Orbital Period, P (Earth Years)	Orbital Eccentricity, e	P^2/a^3
Mercury	0.387	0.241	0.206	1.002
Venus	0.723	0.615	0.007	1.001
Earth	1.000	1.000	0.017	1.000
Mars	1.524	1.881	0.093	1.000
Jupiter	5.203	11.86	0.048	0.999
Saturn	9.537	29.42	0.054	0.998
Uranus	19.19	83.75	0.047	0.993
Neptune	30.07	163.7	0.009	0.986

Example: asteroid Pallas

- The asteroid Pallas has an orbital period of 4.62 years and an orbital eccentricity of 0.233. Find the semi-major axis of its orbit

$$P^2 = ka^3 \xrightarrow{\text{for the solar system, we can write}} \left(\frac{P}{1\text{yr}}\right)^2 = \left(\frac{a}{1\text{AU}}\right)^3$$

$$\frac{a}{1\text{AU}} = \left(\frac{P}{1\text{yr}}\right)^{2/3}$$

$$\frac{a}{1\text{AU}} = \left(\frac{4.62\,\cancel{\text{yr}}}{1\cancel{\text{yr}}}\right)^{2/3} = 2.77$$

$$\longrightarrow a = 2.77\ \text{AU}$$

Note: period does not depend on eccentricity e. An asteroid in an elongated elliptical orbit with semi-major axis a will have the same orbital period as a planet in a circular orbit of radius a.

Demonstration of Kepler's 3rd law

- semi-major axis: a=r
- period: P
- Solar mass: $M_\odot$

- Assuming circular orbits: $F_{cent} = F_g$

$$F_g = \frac{GmM_\odot}{r^2}$$

$$F_{cent} = \frac{m}{r}v^2 = \frac{m}{r}\frac{(2\pi r)^2}{P^2}$$

$$\left.\begin{matrix} F_g \\ F_{cent} \end{matrix}\right\} \quad P^2 = \frac{4\pi^2}{GM_\odot} r^3$$

Orbital velocity:

$$v = \frac{\text{circumference}}{\text{period}} = \frac{2\pi r}{P}$$

This relationship also holds for elliptical orbits

$$P^2 = \frac{4\pi^2}{GM_\odot} a^3$$

- We recognise this as Kepler's 3rd law:

$$P^2 = ka^3 \quad \text{with} \quad k = \frac{4\pi^2}{GM_\odot}$$

Attention! k=1 only for a planet orbiting our sun as the solar mass is part of the constant k, period is given in years, and orbital distance in AU.

$$\left(\frac{P}{1\text{yr}}\right)^2 = \left(\frac{a}{1\text{AU}}\right)^3$$

Why is k=1 for the solar system?

$$\frac{P^2}{a^3} = k$$

Attention! k=1 only for a planet orbiting our sun as the solar mass is part of the constant k, period is given in years, and orbital distance in AU.

Demonstration: We are going to calculate the right and left hand sides of eq above, in SI units

$$k = \frac{4\pi^2}{GM_\odot} = \frac{4\pi^2}{(6.67 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2})(2 \times 10^{30}\text{kg})} = 2.95 \times 10^{-19}\ \text{s}^2/\text{m}^3$$

For term P^2/a^3, we need *P* to be given in units of year and *a* in units of AU. The trick is:

$$\frac{P^2}{a^3} = \left(\frac{P}{(1\text{yr})}(1\text{yr})\right)^2 \left(\frac{a}{(1\text{AU})}(1\text{AU})\right)^{-3} = \left(\frac{P}{(1\text{yr})}\right)^2 \left(\frac{a}{(1\text{AU})}\right)^{-3} \frac{(1\text{yr})^2}{(1\text{AU})^3}$$

Writing the numerical constants in SI units:

$$\frac{P^2}{a^3} = \left(\frac{P}{(1\text{yr})}\right)^2 \left(\frac{a}{(1\text{AU})}\right)^{-3} \frac{(3.16 \times 10^7\text{s})^2}{(1.5 \times 10^{11}\text{m})^3}$$

$$= \left(\frac{P}{1\text{yr}}\right)^2 \left(\frac{a}{1\text{AU}}\right)^{-3} (2.95 \times 10^{-19}\text{s}^2/\text{m}^3)$$

$$k = \cancel{2.95 \times 10^{-19}\ \text{s}^2/\text{m}^3} =$$

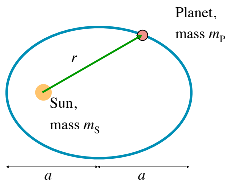
$$= \frac{P^2}{a^3} = \left(\frac{P}{1\text{yr}}\right)^2 \left(\frac{a}{1\text{AU}}\right)^{-3} \cancel{(2.95 \times 10^{-19}\text{s}^2/\text{m}^3)}$$

$$\boxed{\frac{(\frac{P}{1\text{yr}})^2}{(\frac{a}{1\text{au}})^3} = 1}$$

Conceptual question

A planet (P) is moving around the sun (S) in an elliptical orbit. As the planet moves from aphelion to perihelion, the planet's angular momentum

(a) increases at all times.
(b) decreases at all times.
(c) decreases during part of the motion and increases during the other part.
(d) increases, decreases, or remains the same during various parts of the motion.
(e) remains the same at all points between aphelion and perihelion.



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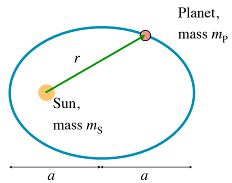
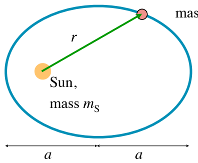


As a planet moves around an elliptical orbit, the sun exerts a force on the planet that points directly toward the sun. What is true about the torque that the sun exerts on the planet?

(a) It is constant and nonzero.
(b) It is greatest when the planet is closest to the sun.
(c) It is least (but not zero) when the planet is closest to the sun.
(d) It is zero when the planet is closest to the sun.
(e) It is zero at all times.



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