Skoltech

Skolkovo Institute of Science and Technology

"Experimental Data Processing"

Topic 1
"Introduction to statistical analysis"

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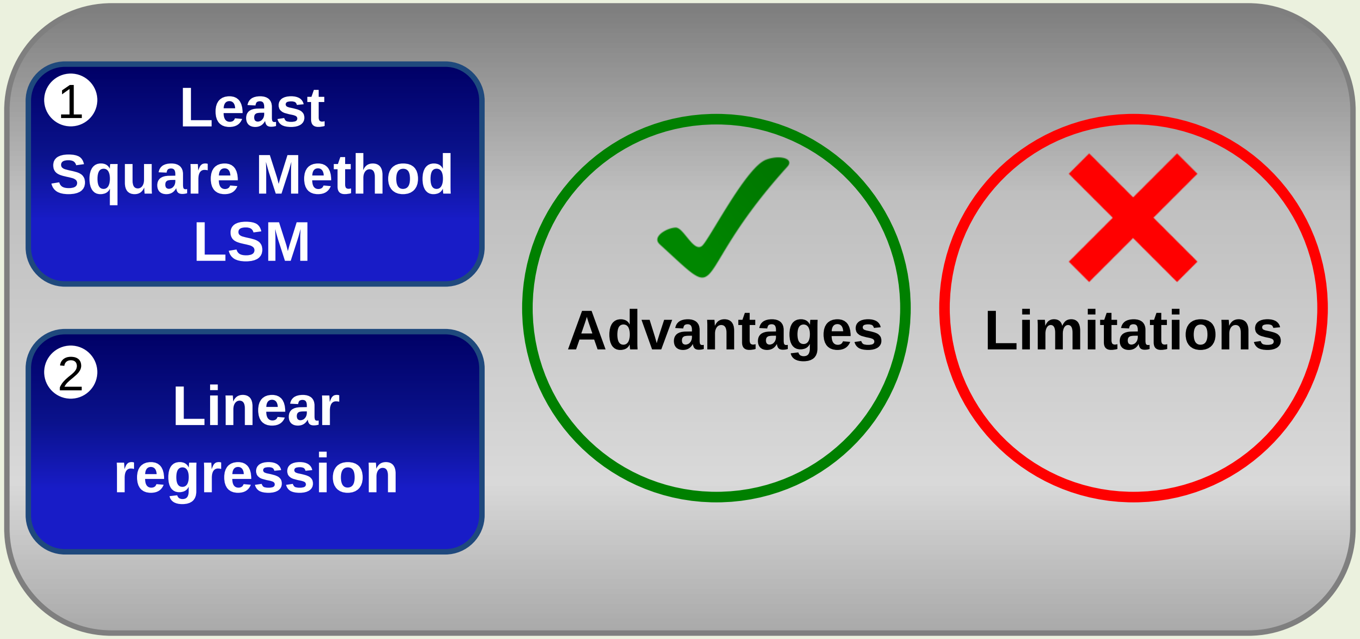
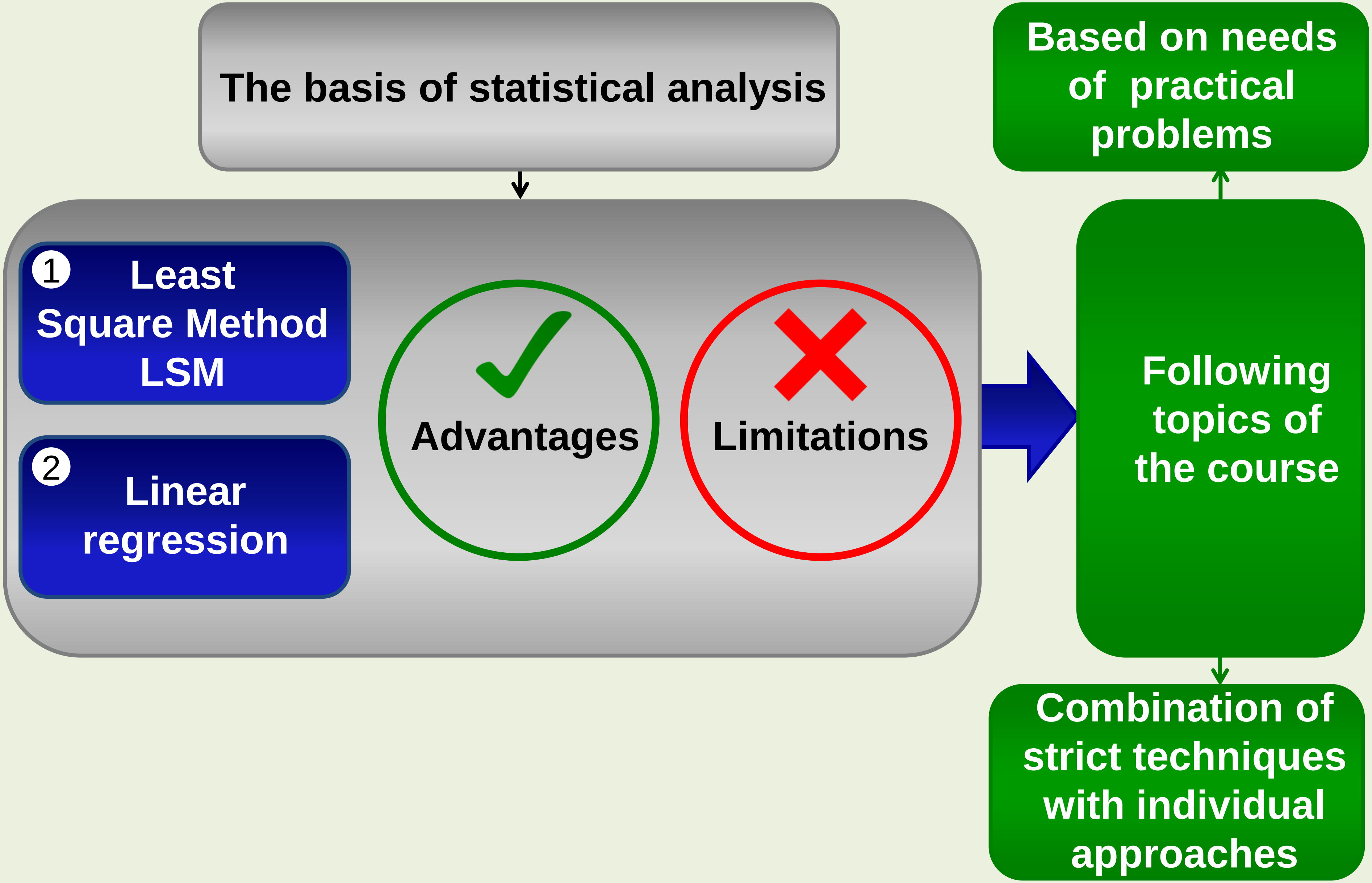
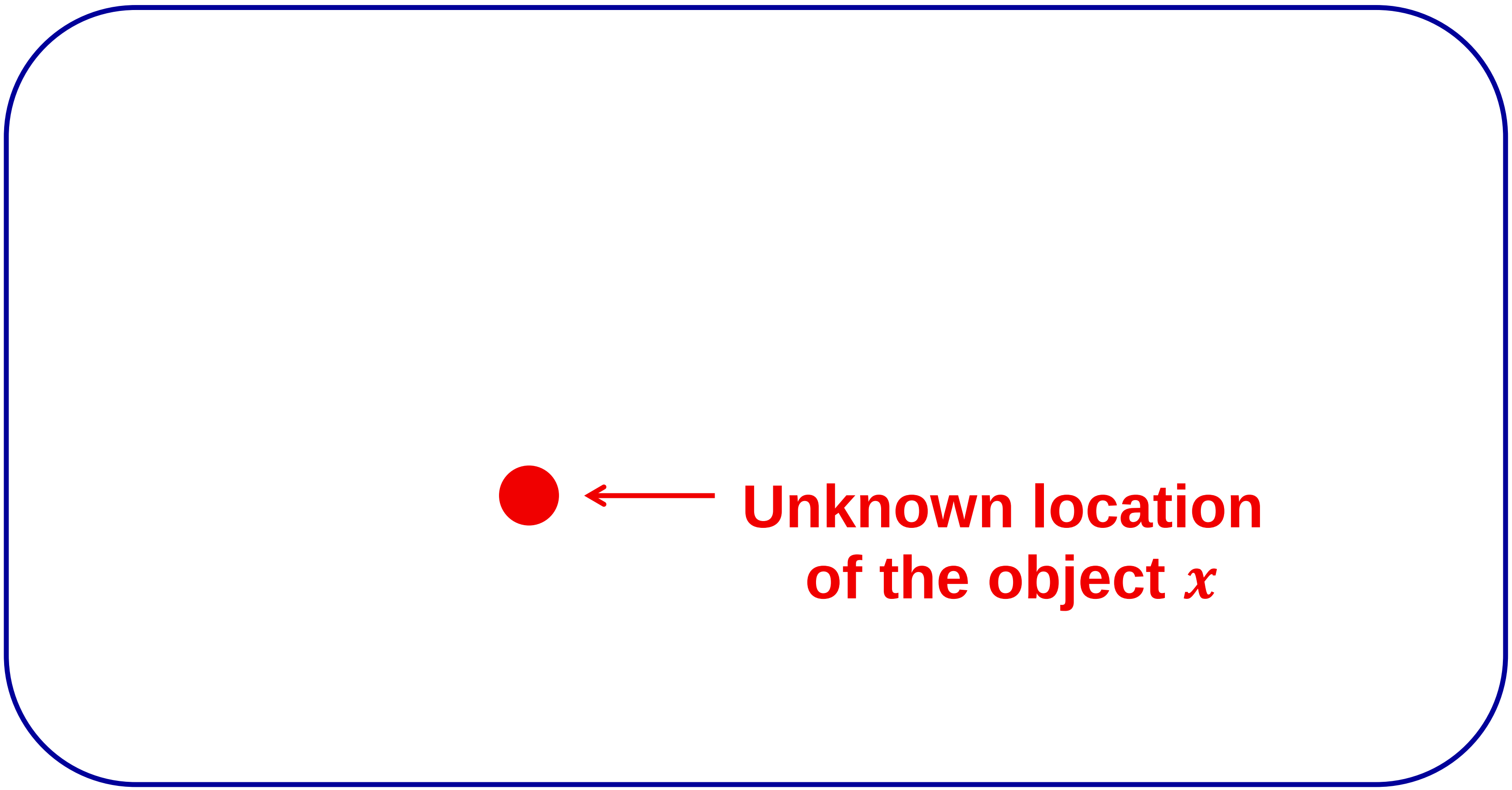
The basis of statistical analysis



Transform theoretical knowledge into useful skills

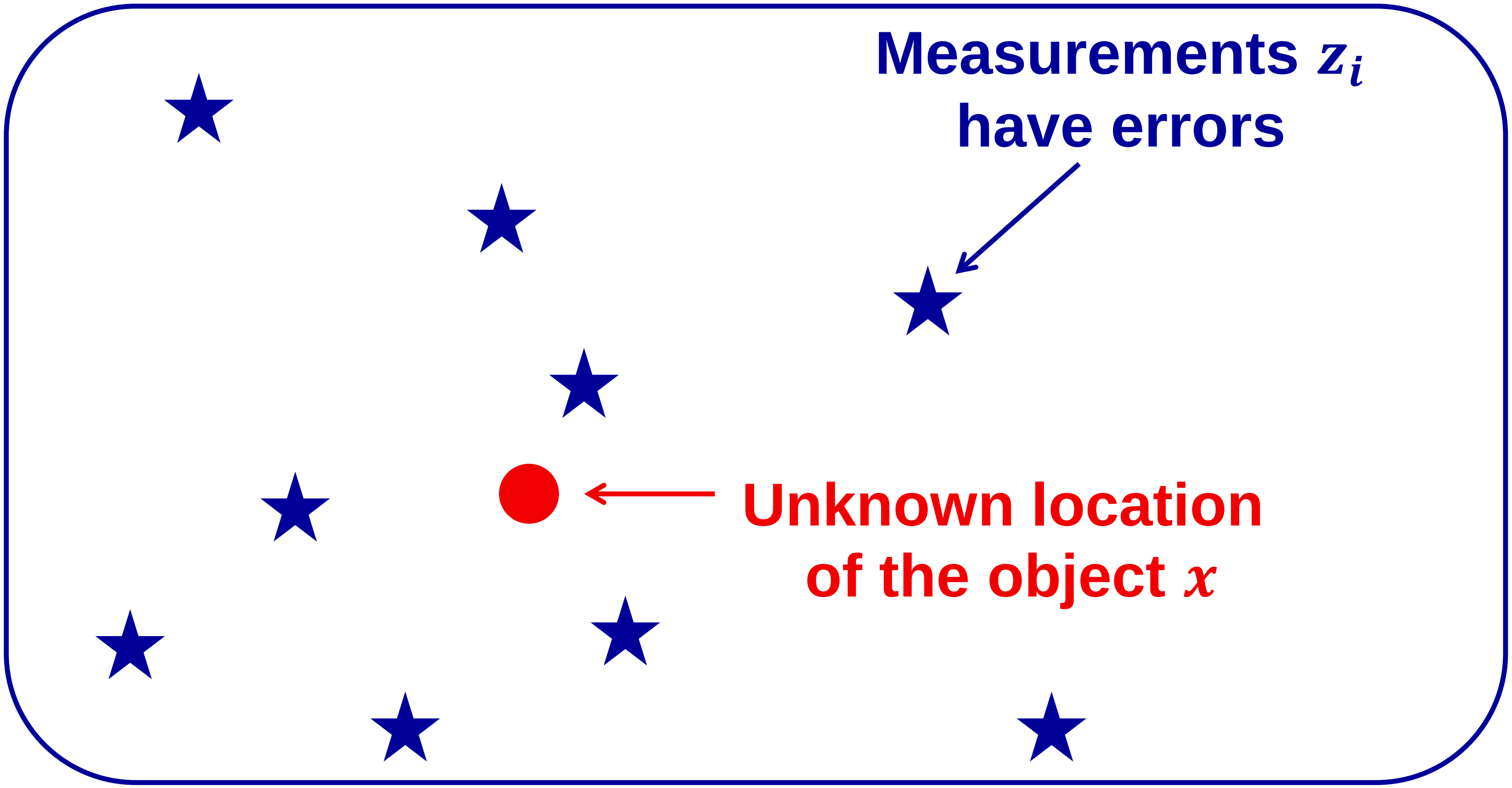


Estimate the location of an unmoving object



LSM. Example 1

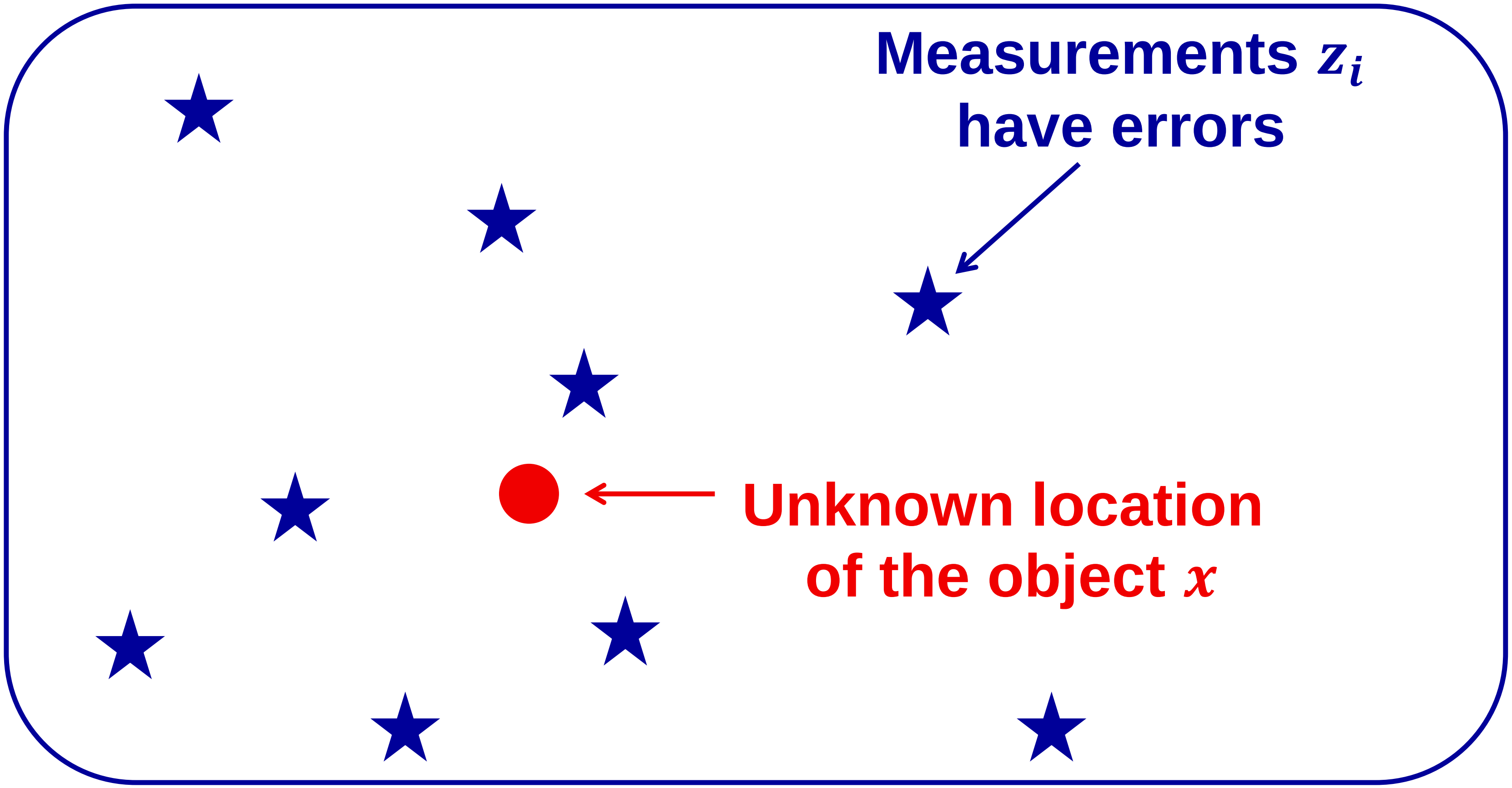
Estimate the location of an unmoving object



LSM. Example 1

Estimate the location of an unmoving object

First use common sense!



LSM. Example 1

Estimate the location of an unmoving object

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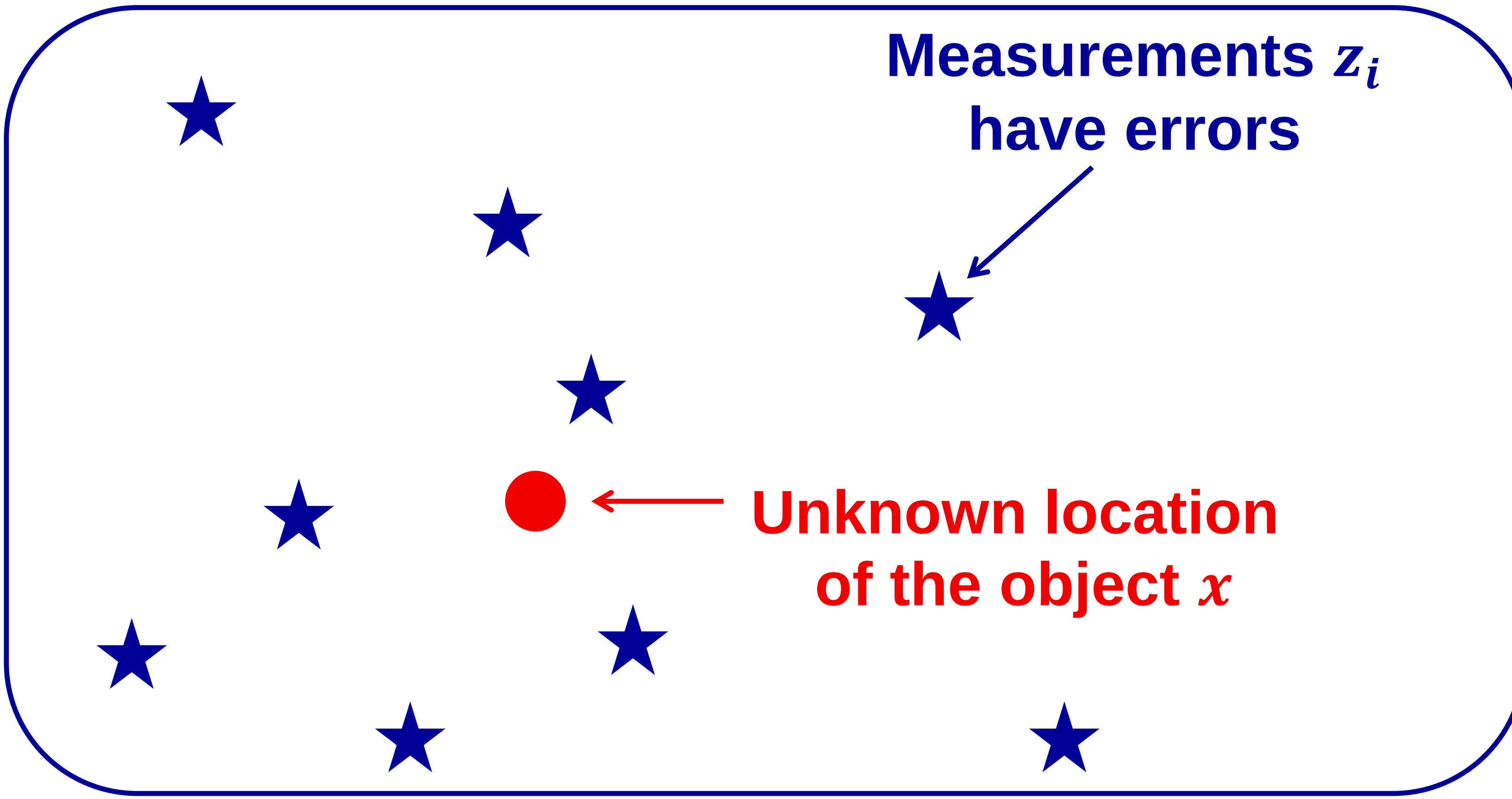


LSM. Example 1

Solution using common sense: to average measurements

Estimate the location of an unmoving object

Let's find the theoretical ground
of this solution using
least-square method (LSM)

Solution using common sense:
to average measurements

LSM. Example 1

Estimate the location of an unmoving object

1 Let's minimize functional J

$$J = \frac{1}{N}\sum_{i=1}^{N}(z_i - x)^2$$

z_i – measurements
x – object location
N – number of measurements

Solution using common sense: to average measurements

LSM. Example 1

Estimate the location of an unmoving object

1 Let's minimize functional J

$$J = \frac{1}{N}\sum_{i=1}^{N}(z_i - x)^2$$

z_i – measurements
x – object location
N – number of measurements

2 First derivative in respect to x

$$\frac{dJ}{dx} = -\frac{2}{N}\sum_{i=1}^{N}(z_i - x) = 0$$

Solution using common sense: to average measurements

LSM. Example 1

Estimate the location of an unmoving object

① Let's minimize functional J

$$J = \frac{1}{N}\sum_{i=1}^{N}(z_i - x)^2$$

z_i – measurements
x – object location
N – number of measurements

② First derivative in respect to x

$$\frac{dJ}{dx} = -\frac{2}{N}\sum_{i=1}^{N}(z_i - x) = 0$$

③ The estimate of location

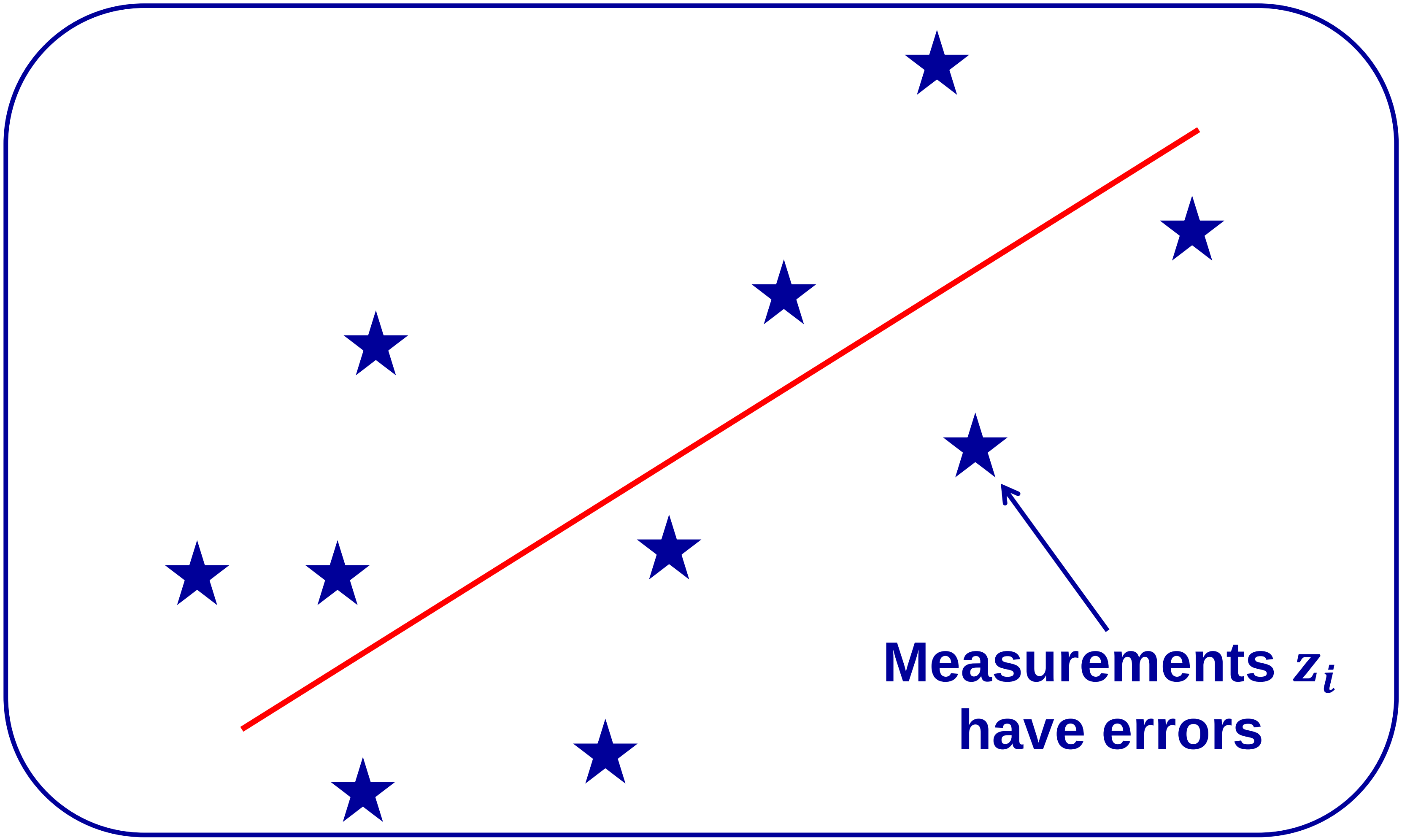
$$\hat{x} = \frac{1}{N}\sum_{i=1}^{N} z_i$$

LSM provides BLUE estimate: Best Linear Unbiased Estimator

Solution using common sense: to average measurements

LSM. Example 1

Estimate the location of a moving object



LSM. Example 2

Uniform and linear movement

Estimate the location of a moving object

1

Motion equation

$$x_i = x_0 + V \cdot iT$$

x_0 – initial position
V – velocity
i – time step
T – time interval between measurements

LSM. Example 2

Uniform and linear movement

Estimate the location of a moving object

1 Motion equation

$$x_i = x_0 + V \cdot iT$$

x_0 – initial position
V – velocity
i – time step
T – time interval between measurements

2 Let's minimize functional J

$$J = \sum_{i=1}^{N} (z_i - x_0 - V \cdot iT)^2$$

LSM. Example 2

Uniform and linear movement

Estimate the location of a moving object

1 **Motion equation**

$$x_i = x_0 + V \cdot iT$$

x_0 – initial position
V – velocity
j – time step
T – time interval between measurements

2 **Let's minimize functional J**

$$J = \sum_{i=1}^{N} (z_i - x_0 - V \cdot iT)^2$$

3 **The estimate of location**

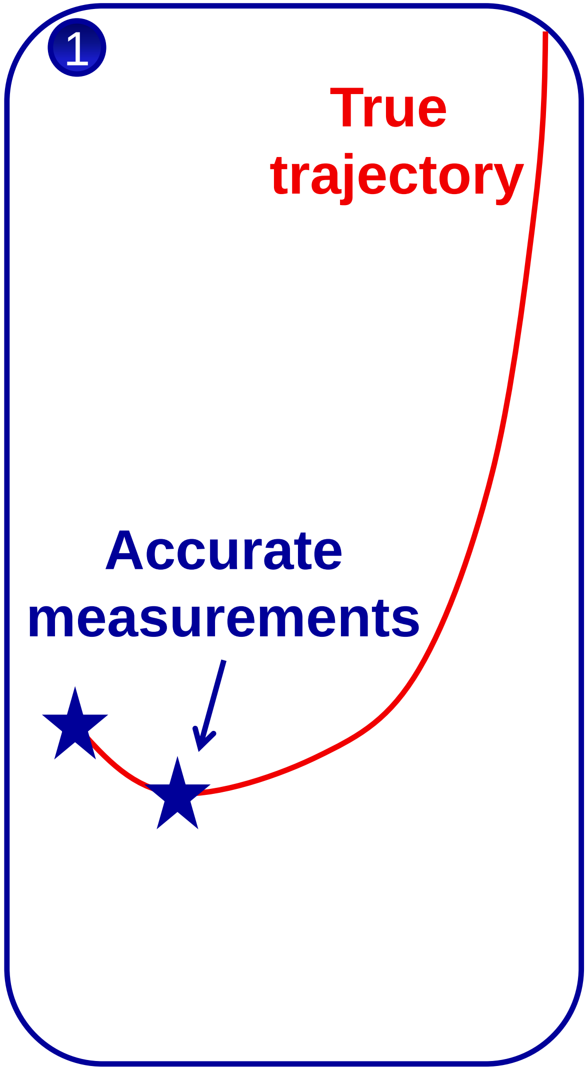
$$\frac{dJ}{dx_0} = 0; \ \frac{dJ}{dV} = 0$$

From these equations we find x_0 and V

LSM. Example 2

Uniform and linear movement

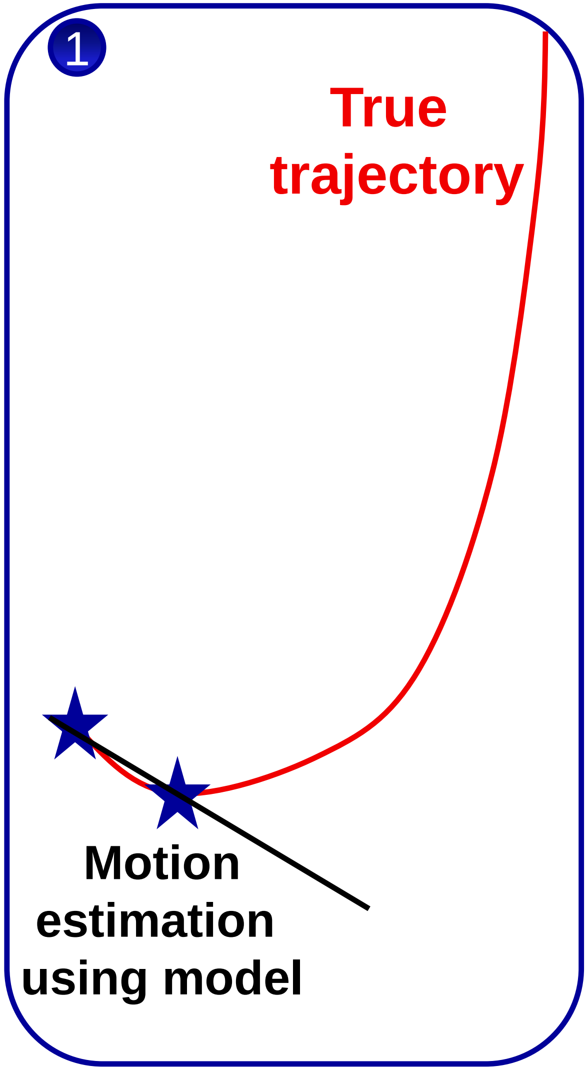
What happens if we assume the object to move along the line, but in fact it moves along the parabola?



LSM. Example 3

Unknown object dynamics

What happens if we assume the object to move along the line, but in fact it moves along the parabola?



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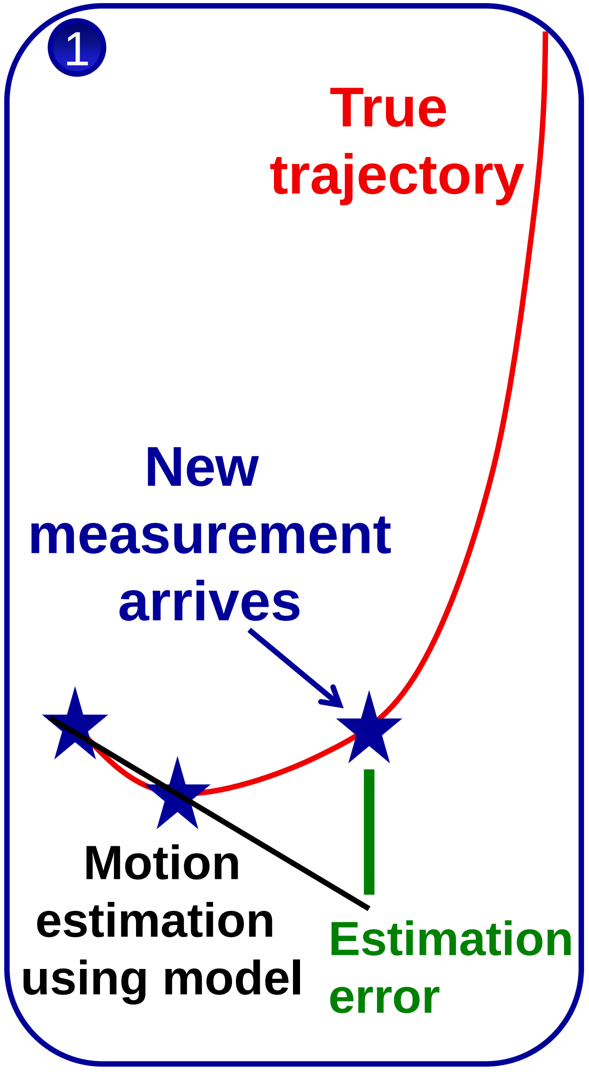

LSM. Example 3

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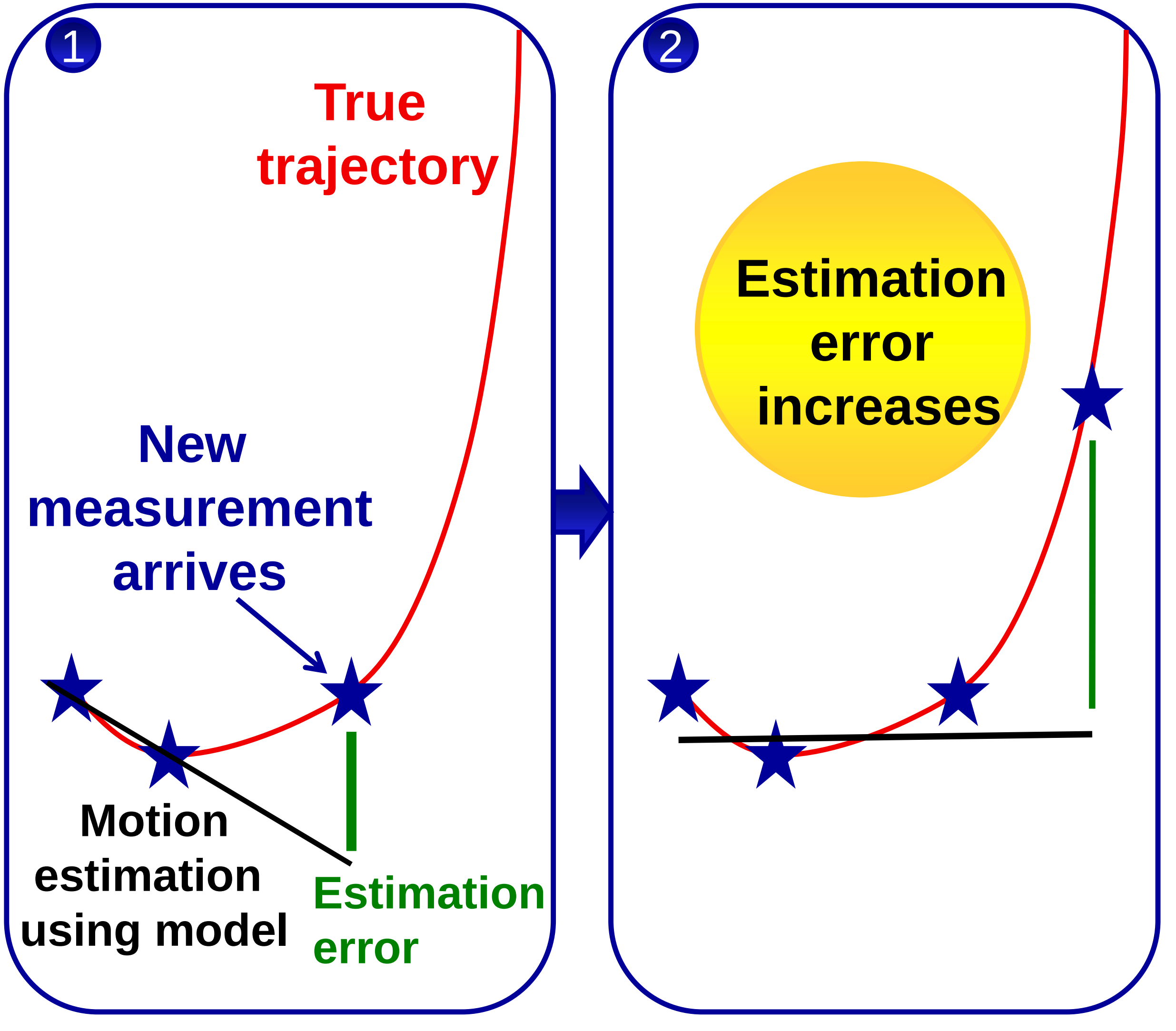

LSM. Example 3

Unknown object dynamics

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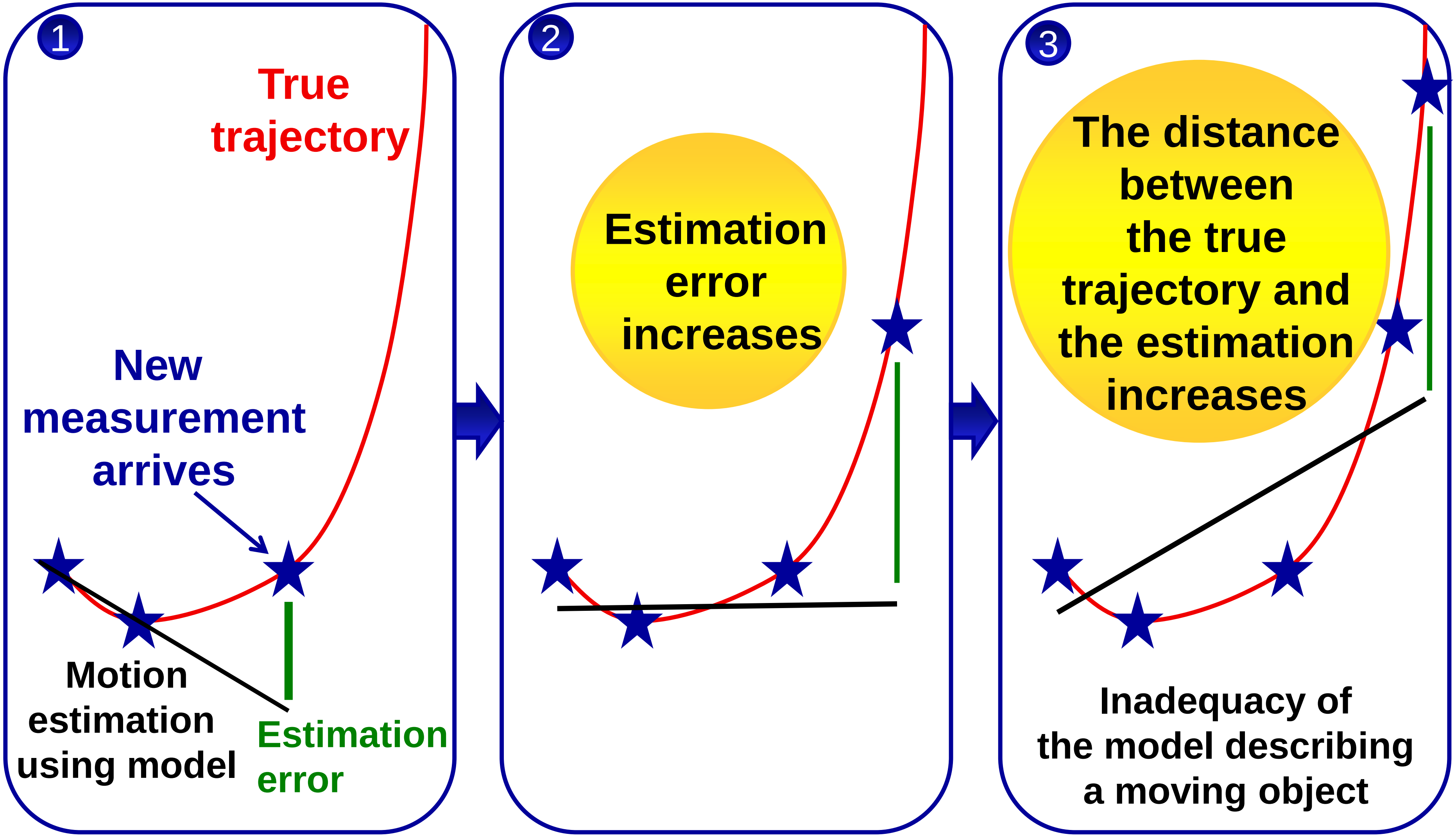


LSM. Example 3

Unknown object dynamics

Conclusion. Advantages and limitations of LSM applications

Process dynamics is known

Advantages

LSM provides BLUE estimate: Best Linear Unbiased Estimator

Process dynamics is unknown

Limitations

The LSM method leads to divergence and loses its practical value

Conclusion. Advantages and limitations of LSM applications

Process dynamics is known

Advantages

LSM provides BLUE estimate: Best Linear Unbiased Estimator

Process dynamics is unknown

Limitations

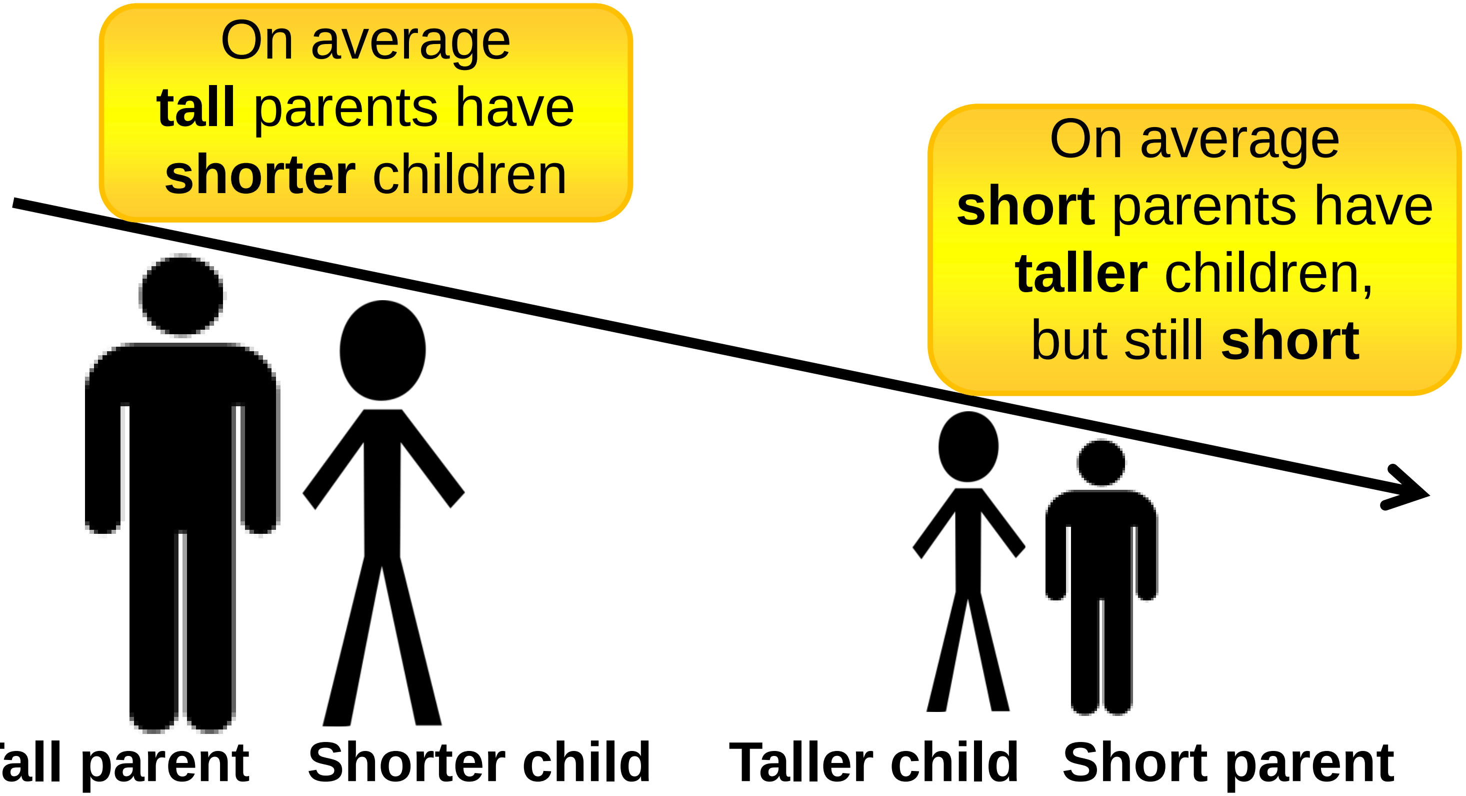
The LSM method leads to divergence and loses its practical value

Therefore in parallel to optimal methods the robust quasi optimal methods are developed

Such methods can be applied in conditions of uncertainty of process dynamics

Following topics of the course

Is there any relationship between the height of parents and children?

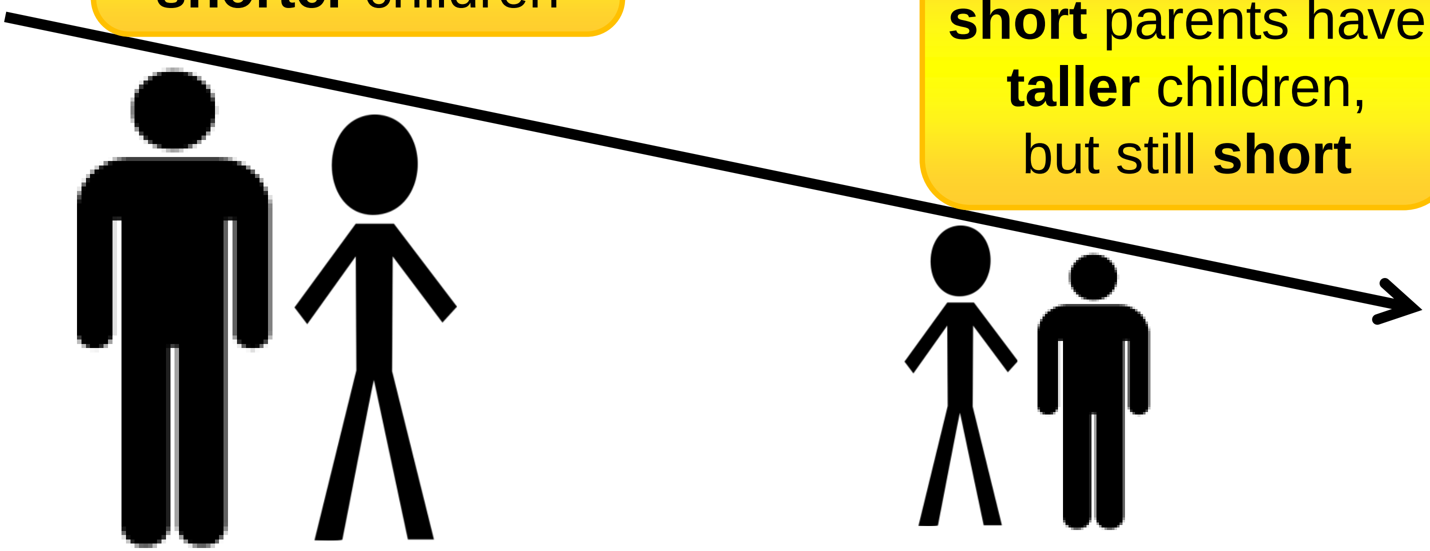


Sir Francis Galton, 1822 –1911

Is there any relationship between the height of parents and children?

On average
tall parents have
shorter children

On average
short parents have
taller children,
but still **short**



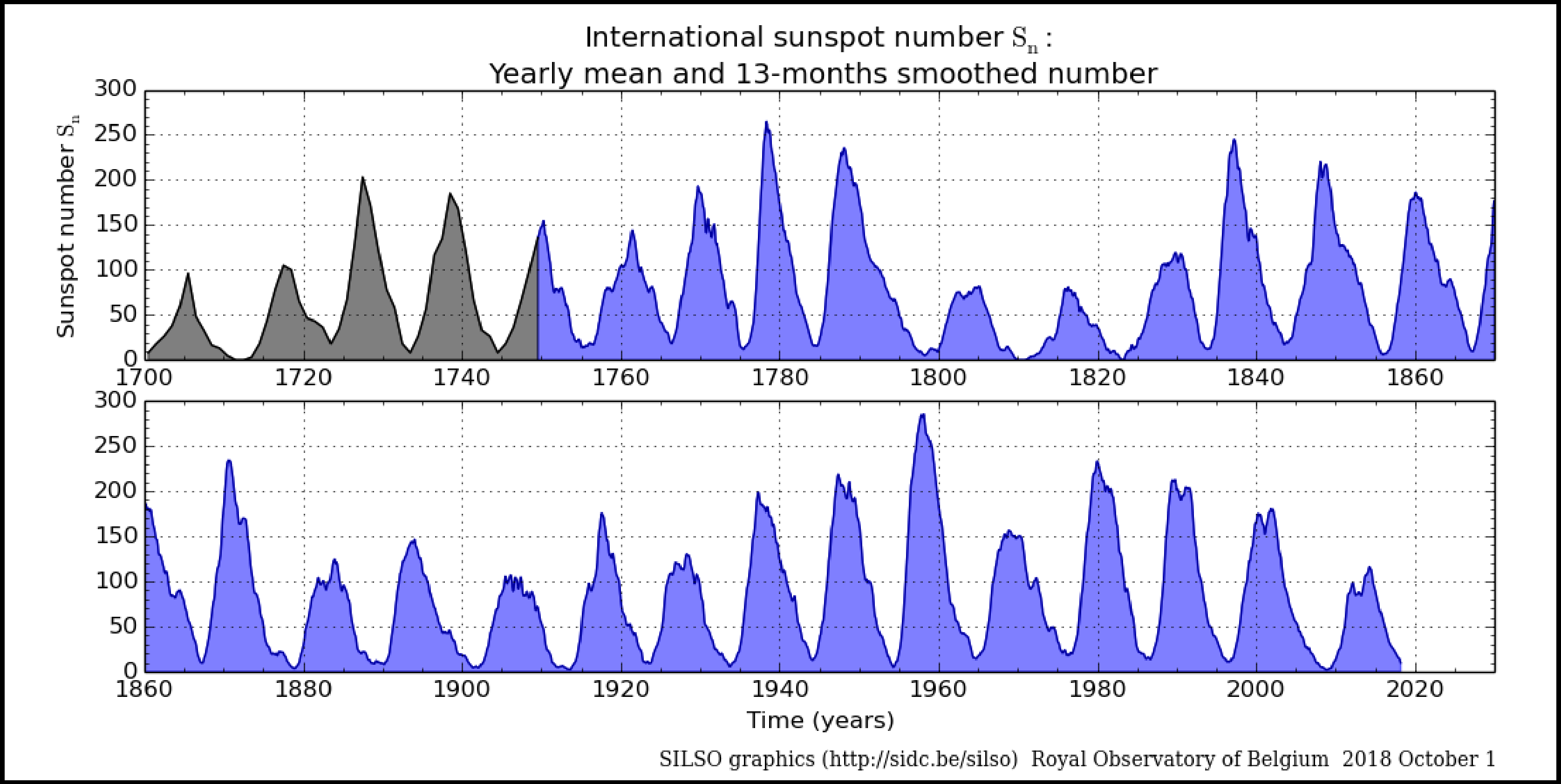
Tall parent **Shorter child** **Taller child** **Short parent**

Height of children approaches to average height of humans

REGRESSION

Sir Francis Galton, 1822 –1911

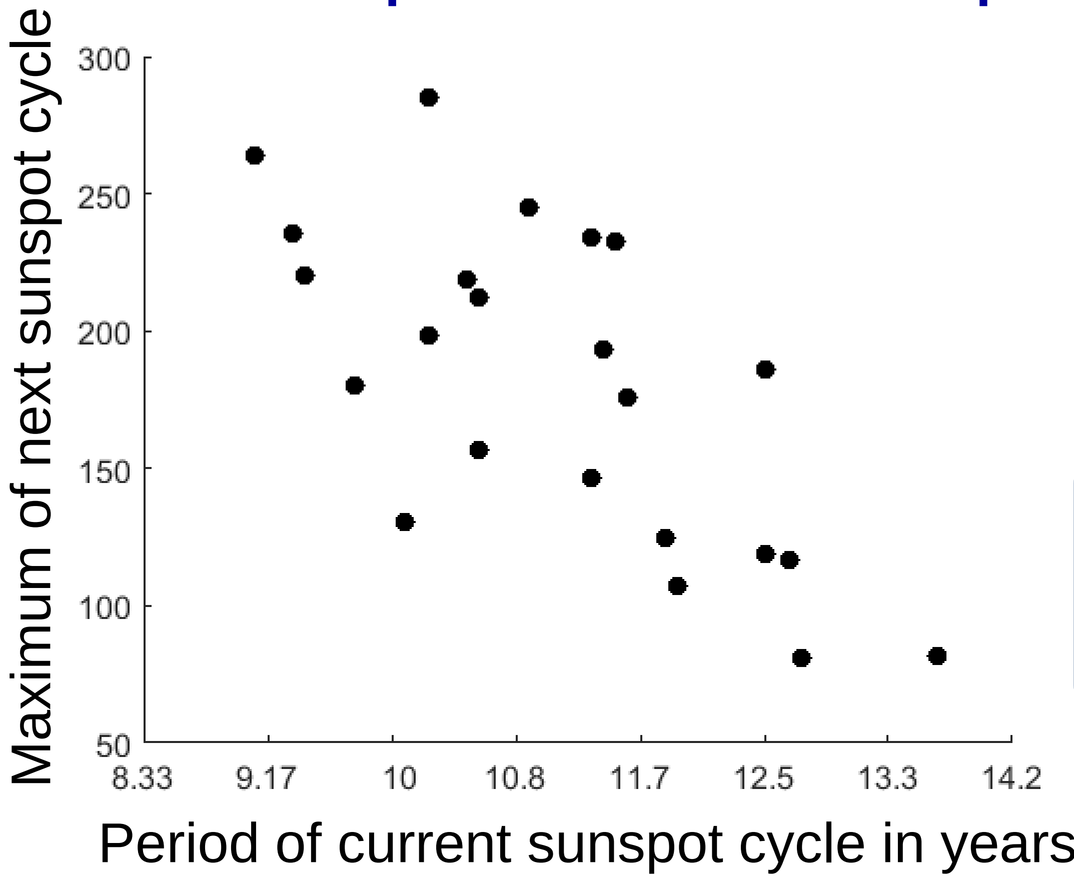
Is there any relationship between period of current sunspot cycle and the maximum of next sunspot cycle?



On average a period of sunspot cycle is 11 years. It can be shorter or greater.

Is there any relationship between period of current sunspot cycle and the maximum of next sunspot cycle?

Scatter plot maximum versus period



Correlation coefficient -0.7

Greater the period of cycle – less the cycle maximum

One-dimensional linear regression

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

$$i = 1, \dots . N$$

y_i
Dependent variable
Regressand

β_0, β_1
Coefficients
β_0 - intercept
β_1 - slope

x_i
Independent variable
Regressor

ε_i
Unbiased uncorrelated Gaussian noise with constant variance

One-dimensional linear regression

1 **Equation**

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

y_i – the maximum of next sunspot cycle

x_i – the period of current sunspot cycle

Let's minimize $\sum_{i=1}^{N} \varepsilon_i^2$

β_0, β_1 unknown

2 **Let's minimize functional J**

$$J = \sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_i)^2$$

3 **Estimation of β_0, β_1**

$$\frac{dJ}{d\beta_0} = 0; \ \frac{dJ}{d\beta_1} = 0$$

Determine coefficients β_0 and β_1 using LSM

Let's minimize functional J

$$J = \sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_i)^2$$

1

$$\frac{dJ}{d\beta_0} = -2\sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_i) = 0$$

2

$$\frac{dJ}{d\beta_1} = -2\sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_i)\, x_i = 0$$

1

$$\frac{dJ}{d\beta_0} = \sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_i) = 0$$

2

$$\frac{dJ}{d\beta_1} = \sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_i)\, x_i = 0$$

Excursus: solution

Determine coefficients β_0 and β_1 using LSM

$$\frac{dJ}{d\beta_0} = \sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_i) = 0 \quad (1.1)$$

$$\frac{dJ}{d\beta_0} = \sum_{i=1}^{N} y_i - \sum_{i=1}^{N} \beta_0 - \sum_{i=1}^{N} \beta_1 x_i = 0 \quad (1.2)$$

$$\frac{dJ}{d\beta_0} = \sum_{i=1}^{N} y_i - N\beta_0 - \beta_1 \sum_{i=1}^{N} x_i = 0 \quad (1.3)$$

$$\beta_0 = \frac{1}{N}\left(\sum_{i=1}^{N} y_i - \beta_1 \sum_{i=1}^{N} x_i\right) \quad (1.4)$$

Excursus: solution

Determine coefficients β_0 and β_1 using LSM

$$\frac{dJ}{d\beta_1} = \sum_{i=1}^{N} (y_i - \beta_0 - \beta_1 x_i)\, x_i = 0 \qquad (2.1)$$

$$\frac{dJ}{d\beta_1} = \sum_{i=1}^{N} y_i x_i - \sum_{i=1}^{N} \beta_0 x_i - \sum_{i=1}^{N} \beta_1 x_i^2 = 0 \qquad (2.2)$$

$$\frac{dJ}{d\beta_1} = \sum_{i=1}^{N} y_i x_i - \beta_0 \sum_{i=1}^{N} x_i - \beta_1 \sum_{i=1}^{N} x_i^2 = 0 \qquad (2.3)$$

Let's substitute value of β_0 from Equation 1.4 to Equation 2.3

Excursus: solution

Determine coefficients β_0 and β_1 using LSM

$$\sum_{i=1}^{N} y_i x_i - \frac{1}{N}\left(\sum_{i=1}^{N} y_i - \beta_1 \sum_{i=1}^{N} x_i\right)\sum_{i=1}^{N} x_i - \beta_1 \sum_{i=1}^{N} x_i^2 = 0 \quad (2.4)$$

$$\sum_{i=1}^{N} y_i x_i - \frac{1}{N}\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i + \frac{1}{N}\beta_1\left(\sum_{i=1}^{N} x_i\right)^2 - \beta_1 \sum_{i=1}^{N} x_i^2 = 0 \quad (2.5)$$

$$\beta_1\left[\frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2 - \sum_{i=1}^{N} x_i^2\right] = \frac{1}{N}\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i x_i \quad (2.6)$$

$$\beta_1 = \left[\frac{1}{N}\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i x_i\right] \Big/ \left[\frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2 - \sum_{i=1}^{N} x_i^2\right] \quad (2.7)$$

Excursus: solution

Determine coefficients β_0 and β_1 using LSM

$$\beta_1 = \left[\frac{1}{N}\sum_{i=1}^{N} y_i \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i x_i\right] / \left[\frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)^2 - \sum_{i=1}^{N} x_i^2\right]$$

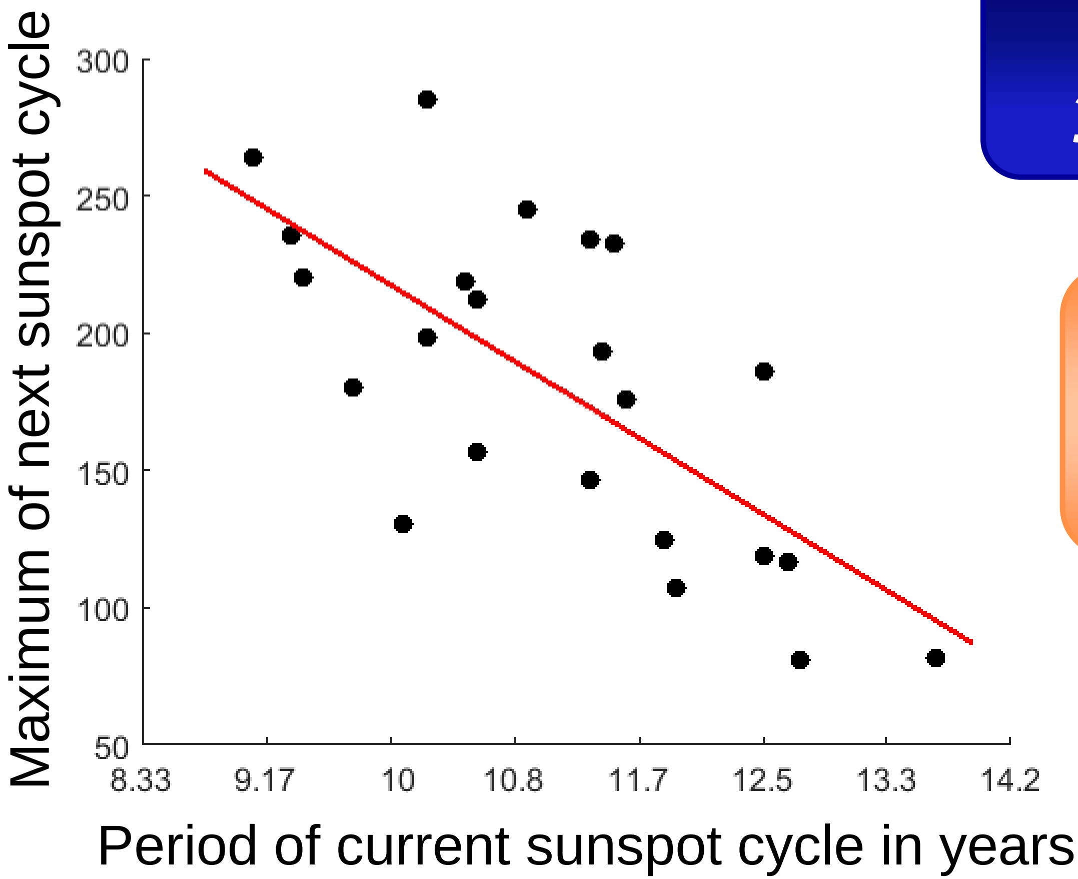
$$\beta_0 = \frac{1}{N}\left(\sum_{i=1}^{N} y_i - \beta_1 \sum_{i=1}^{N} x_i\right)$$

$\beta_0 = 551.4$

$\beta_1 = -2.8$

Excursus: solution

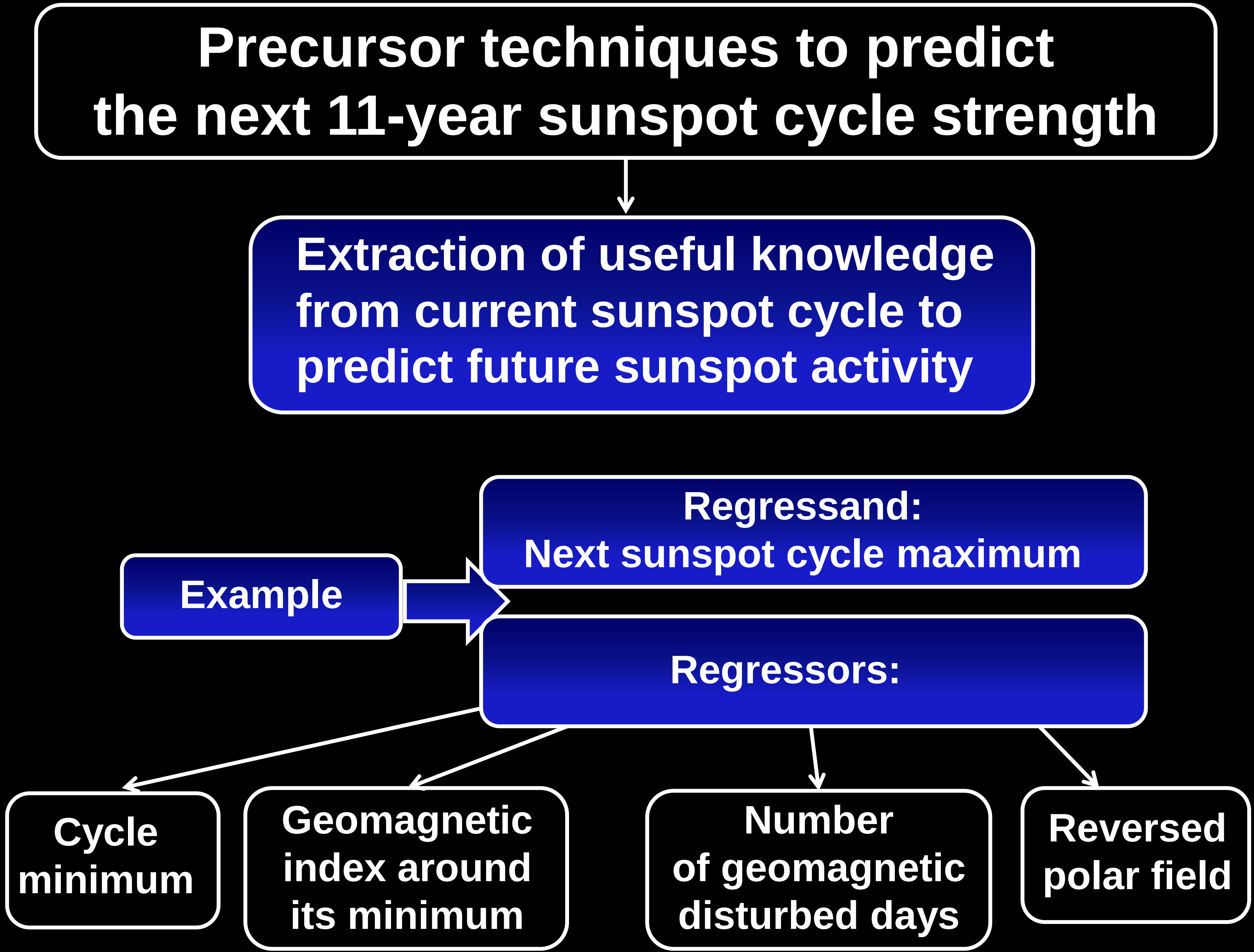
Is there any relationship between period of current sunspot cycle and the maximum of next sunspot cycle?



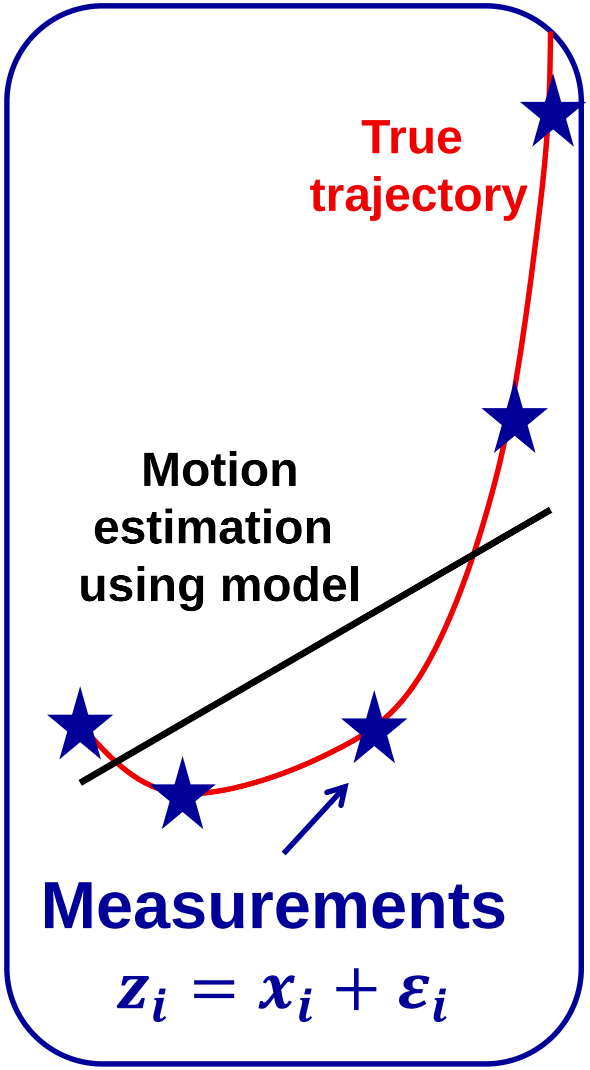
Linear regression

$$y_i = 551.4 - 2.8x_i$$

Thus, current cycle contains information on future sunspot activity


Precursor techniques to predict the next 11-year sunspot cycle strength
Extraction of useful knowledge from current sunspot cycle to predict future sunspot activity
Example
Regressand:
Next sunspot cycle maximum
Regressors:
Cycle minimum
Geomagnetic index around its minimum
Number of geomagnetic disturbed days
Reversed polar field

Example of inadequate regression model



$$x_i = x_{i-1} + VT$$

We assume the object to move uniformly

$$x_i = x_{i-1} + V_i T + \frac{aT^2}{2}$$

But in fact it is uniformly accelerated motion

$$z_i = x_0 + V \cdot iT + \varepsilon_i$$

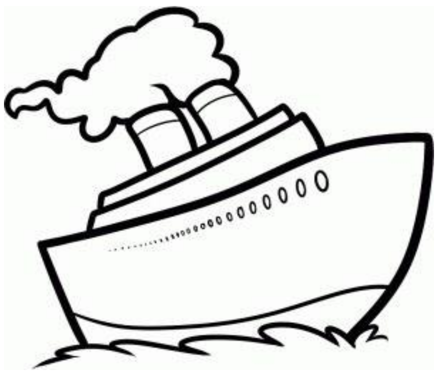
No practical value

Thus, regression model is inadequate

How to take into account the unintentional maneuver?

Motion model

$$x_i = x_{i-1} + VT$$

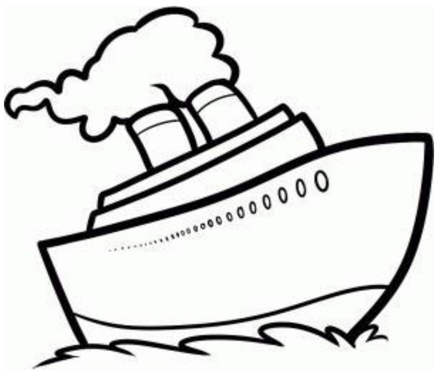


However how to take into account unintentional maneuver to track moving object (i.e., airplane turbulence, ship pitching or undercurrents)?

How to take into account the unintentional maneuver?

Motion model

$$x_i = x_{i-1} + V_i T + \frac{a_i T^2}{2}$$



However how to take into account unintentional maneuver to track moving object (i.e., airplane turbulence, ship pitching or undercurrents)?

Unintentional maneuver can be described by random acceleration a

Process noise should not be filtered

Motion model

$$x_i = x_{i-1} + V_i T + \frac{a_i T^2}{2}$$

However how to take into account unintentional maneuver to track moving object (i.e., airplane turbulence, ship pitching or undercurrents)?

Unintentional maneuver can be described by random acceleration a

$\frac{a_i T^2}{2}$

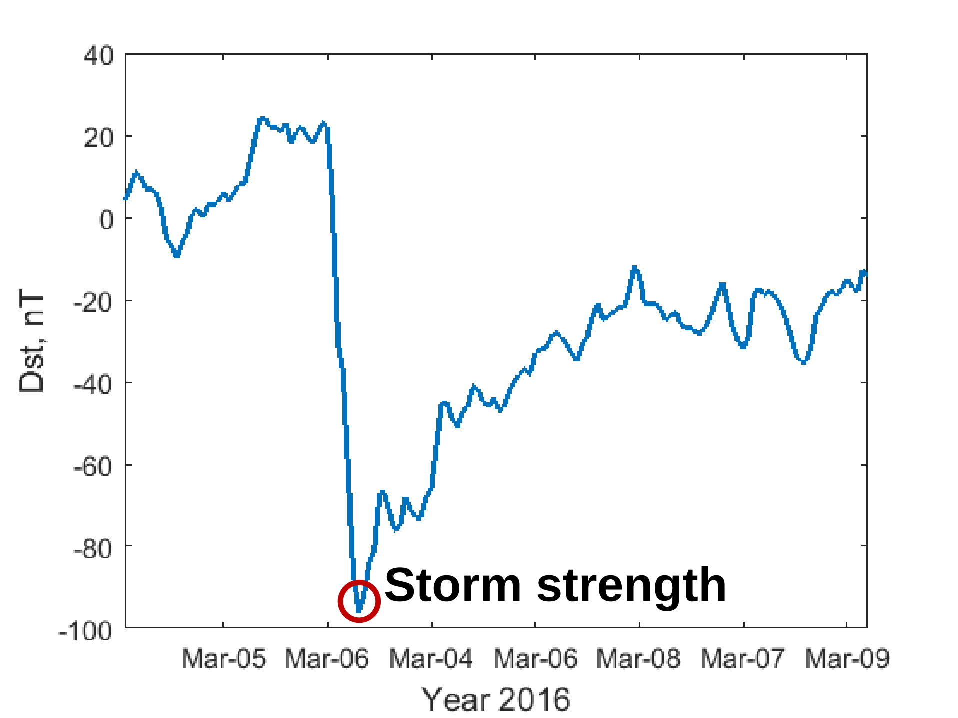
Noise intrinsic to the process itself that should not be filtered

However linear regression doesn't separate process noise and measurement noise and thus loses its practical value

Multi-dimensional linear regression

Dependence on multiple regressors

Example: Disturbance storm time (Dst) geomagnetic index



Dst is sensitive to:

1. **Solar wind speed**
2. **Southward component of Interplanetary magnetic field (IMF)**
3. **Previous Dst values**
4. **Solar wind density and pressure**

Multi-dimensional linear regression

$$y_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_{k-1} x_{i,k-1} + \varepsilon_i$$

$$i = 1, \dots, N$$

y_i
Dependent variable
Regressand

β_j
Coefficients of regression

$x_{i,j}$
Independent variable
Regressor

ε_i
Unbiased uncorrelated Gaussian noise with constant variance

Coefficients β_j are determined by LSM

$$\sum_{i=1}^{N} \varepsilon_i^2 \to min$$

Multi-dimensional linear regression

$$Y = \begin{vmatrix} y_1 \\ y_2 \\ \dots \\ y_N \end{vmatrix}$$

Vector of dependent variables

$$\beta = \begin{vmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_N \end{vmatrix}$$

Vector of coefficients

$$X = \begin{vmatrix} 1 & x_{1,1} & \dots & x_{1,k-1} \\ 1 & x_{2,1} & \dots & x_{2,k-1} \\ \dots & \dots & \dots & \dots \\ 1 & x_{N,1} & \dots & x_{N,k-1} \end{vmatrix}$$

Matrix of independent variables

$$\varepsilon = \begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \dots \\ \varepsilon_N \end{vmatrix}$$

Vector of random errors

Linear regression in matrix form

$$\boldsymbol{Y = X \cdot \beta + \varepsilon}$$

$$\boldsymbol{\hat{\beta} = \left(X^T X\right)^{-1} X^T Y}$$

Linear Regression Analysis,
G.A.F. Seber and J. Lee, Wiley, N.Y., 2003

Estimation error of coefficients β

Covariance matrix of estimation error

$$cov(\hat{\beta}) = \sigma_\varepsilon^2 \left(X^T X\right)^{-1}$$

Variance estimation of random noise

$$\sigma_\varepsilon^2 = \frac{1}{N-k}\sum_{i=1}^{N}(y_i - \hat{y}_i)^2$$

Weighted LSM

$$\sum_{i=1}^{N}\frac{\varepsilon_i^2}{\sigma_{\varepsilon_i}^2} \to min$$

Main problems of applying LSM and linear regression