Laboratory work 5
Tracking of a moving object which trajectory is disturbed by random acceleration

Performance - Monday, October 9, 2018
Due to submit a performance report – Thursday, October 11, 2018, 23:59 p.m.

The objective of this laboratory work is to develop standard Kalman filter for tracking a moving object which trajectory is disturbed by random acceleration. Important outcome of this exercise is getting deeper understanding of Kalman filter parameters and their role in estimation. Students will analyze the sensitivity of estimations to choice of non-optimal parameters and dependence on initial conditions.

This laboratory work is performed in the class by students as in teams of 3 on October 9, 2018 and the team will submit one document reporting about the performance till October 11, 2018. Within your group, you may discuss all issues openly, and discuss and debate until you reach a consensus.

Here is the recommended procedure:

1. Generate a true trajectory X_i of an object motion disturbed by normally distributed random acceleration

$$x_i = x_{i-1} + V_{i-1}T + \frac{a_{i-1}T^2}{2}$$
$$V_i = V_{i-1} + a_{i-1}T$$

   Size of trajectory is 200 points.
   Initial conditions: $x_1 = 5; V_1 = 1; T = 1$
   Variance of noise a_i, $\sigma_a^2 = 0.2^2$

2. Generate measurements z_i of the coordinate x_i

$$z_i = x_i + \eta_i$$

   η_i –normally distributed random noise with zero mathematical expectation and variance $\sigma_\eta^2 = 20^2$.

3. Present the system at state space taking into account that only measurements of coordinate x_i are available

$$X_i = \Phi X_{i-1} + G a_{i-1}$$
$$z_i = \mathrm{H}_i X_i \quad + \quad \eta_i$$

   Here $\boldsymbol{X_i}$ - state vector, that describes full state of the system (coordinate x_i and velocity V_i);
   $\boldsymbol{\Phi}$ – transition matrix that relates X_i and X_{i-1};
   $\boldsymbol{G}$ – input matrix, that determines how random acceleration a_i affects state vector;
   $\boldsymbol{z_i}$ – measurements of coordinate x_i
   $\boldsymbol{H}$ – observation matrix

4. Develop Kalman filter algorithm to estimate state vector X_i (extrapolation and filtration) Consult charts from lecture **Topic_3_Optimal approximation at state space.pdf**

Use initial conditions

Initial filtered estimate $X_0 = \begin{vmatrix} 2 \\ 0 \end{vmatrix}$

Initial filtration error covariance matrix

$$P_{0,0} = \begin{vmatrix} 10000 & 0 \\ 0 & 10000 \end{vmatrix}$$

Useful hints

To calculate covariance matrix Q of state noise Ga_{i-1} that is used in Kalman filter algorithm to determine prediction error covariance matrix use following equation

$$Q = E[\,(Ga_{i-1})(Ga_{i-1})^T] = \\ = GE[\,\mathrm{a}_{i-1}^2]G^T = \\ = G\sigma_a^2 G^T = GG^T\sigma_a^2$$

To calculate covariance matrix R of measurements noise η_i that is used in Kalman filter algorithm to determine filter gain use following recommendation: Dimension of covariance matrix R is determined by a number of state vector components that are measured. In this particular case, only coordinate x_i is measured. Thus

$$R = \sigma_\eta^2$$

5. Plot results including true trajectory, measurements, filtered estimates of state vector X_i. Run filter several times to see that estimation results are different with every new trajectory.

6. Plot filter gain K over the whole filtration interval.

   To analyze filtration error covariance matrix $P_{i,i}$ over observation period, please also make another plot of square root of its first diagonal element corresponding to standard deviation of estimation error of coordinate x_i.

   Verify whether filter gain K and filtration error covariance matrix become constant very quickly. It means that in conditions of a trajectory disturbed by random noise we cannot estimate more than established limit of accuracy due to uncertainty.

7. Add to the code extrapolation on $m = 7$ steps ahead on every time step.

$$X_{i+m-1,i} = \Phi_{i+m-1,i} X_{i,i}$$

   Here

$$\Phi_{i+m-1,i} = \Phi_{i+m-1,i+m-2}\Phi_{i+m-2,i+m-3}\cdots\Phi_{i+2,i+1}\Phi_{i+1,i}$$

   For example

$$X_{7,1} = \Phi_{7,1}X_{1,1}$$
$$\Phi_{7,1} = \Phi_{7,6}\Phi_{6,5}\Phi_{5,4}\Phi_{4,3}\Phi_{3,2}\Phi_{2,1}$$

8. Make $M = 500$ runs of filter and estimate dynamics of mean-squared error of estimation over observation interval. Please calculate this error for filtered estimate of coordinate $x_{i,i}$ and its forecasting (extrapolation) m steps ahead $x_{i+m-1,i}$.

   Hint how to do:

   Calculate squared deviation of true coordinate x_i from its estimation $\hat{x}_{i,i}$ for every run over the whole observation interval N=200.

$$Error^{Run}(i) = \left(x_i - \hat{x}_{i,i}\right)^2$$
Run – number of run;
$i = 3, \dots, N$ - observation interval
(please start error calculation from step $i = 3$);
$Run = 1, \dots, M$ - number of runs;

Find average value of $Error^{Run}(i)$ over M runs for every step i and calculate its square root

$$Final\_Error(i) = \sqrt{\frac{1}{M-1}\sum_{Run=1}^{M} Error^{Run}(i)}$$

Plot final error and check when it becomes almost constant and estimation accuracy doesn't increase anymore. At this moment filter becomes stationary and in practice this constant filter gain can be used in the algorithm instead of calculating filter gain at every time step.

9. Compare mean-squared error of filtered estimate of coordinate $x_{i,i}$ with standard deviation of measurement errors. Make conclusions about effectiveness of filtration.

10. Make $M = 500$ runs again, but with more accurate initial filtration error covariance matrix

$$P_{0,0} = \begin{vmatrix} 100 & 0 \\ 0 & 100 \end{vmatrix}$$

Calculate mean-squared error of filtered estimate of coordinate $x_{i,i}$ again as in item 8. Compare estimation results for both variants of initial $P_{0,0}$.

Please analyze how the accuracy of initial conditions $P_{0,0}$ affects the estimation results? When the choice of initial conditions doesn't affect the estimation results?

11. Compare calculation errors of estimation $P_{i,i}$ provided Kalman filter algorithm with true estimation errors.

Hint how to do:
Make a plot of two curves:
a) Final error (true estimation error) obtained over $M = 500$ runs according to item 8.
b) Filtration error covariance matrix $P_{i,i}$ (calculation error provided by Kalman filter) Please use square root of the first diagonal element of $P_{i,i}$ that corresponds to standard deviation of estimation error of coordinate x_i. It doesn't depend on runs, for every run it is the same, as it depends only on model parameters $\Phi, \mathrm{H}, \mathrm{Q}, \mathrm{R}$.

Verify if calculation errors of estimation correspond to true estimation errors.

12. Run filter for deterministic trajectory (no random disturbance). It can be easily done by indicating in the code that variance of state noise equals to zero $\sigma_a^2 = 0$.
Make $M = 500$ runs and verify
1) Filter gain approaches to zero
2) Both true estimation errors defined according to item 8 and calculation errors $P_{i,i}$ (square root of the first diagonal element of $P_{i,i}$ that corresponds to standard deviation of estimation error of coordinate x_i) also approach to zero.

This means that in conditions of motion without any random disturbances, estimation error approaches to zero and filter switches off from measurements (new measurements almost do not adjust estimates).

13. Verify what happens if you use deterministic model of motion, but in fact motion is disturbed by random acceleration. In other words you don't take into account the covariance matrix Q of state noise $w_i = Ga_i$ in assimilation algorithm.

    Hint how to do
    Generate a trajectory with variance of state noise $\sigma_a^2 = 0.2^2$
    Use $Q = 0$ in Kalman filter algorithm.

    a) Make $M = 500$ runs of filter and estimate dynamics of mean-squared error of estimation over observation interval. Please calculate this error for both filtered estimate of coordinate $x_{i,i}$ and its forecasting (extrapolation) m steps ahead $x_{i+m-1,i}$. Make conclusions about estimation in conditions of neglecting state noise in Kalman filter algorithm.
    b) Compare calculation errors of estimation $P_{i.i}$ (only estimation error of coordinate x_i) provided Kalman filter algorithm with true estimation errors. Make conclusions.

14. Analyze how the relationship between state and measurement noise $\frac{\sigma_w^2}{\sigma_\eta^2}$ affect time when filter gain become almost constant and estimation accuracy doesn't increase anymore.

    Generate a trajectory with variance of state noise $\sigma_a^2 = 1$ and compare estimation results with that when $\sigma_a^2 = 0.2^2$. Make conclusions.

15. Analyze sensitivity of filter to underestimated non-optimal filter gain K
    Generate a trajectory with variance of state noise $\sigma_a^2 = 0.2^2$.
    Use initial filtered estimate $X_0 = \begin{vmatrix} 100 \\ 5 \end{vmatrix}$
    Run filter for two conditions
    a) Calculate optimal filter gain according to Kalman filter equations and calculate mean-squared error of filtered estimate of coordinate $x_{i.i}$ for $M = 500$ runs.
    b) Run filter with underestimated filter gain K.
       Use steady-state value of optimal filter gain divided by 5 $K = \frac{K_{steady-state}}{5}$
       Calculate mean-squared error of filtered estimate of coordinate $x_{i.i}$ for in this conditions for $M = 500$ runs and compare with estimation results obtained in (a).

16. General conclusions.

17. Prepare performance report and submit to Canvas:
    Performance report should include 2 documents:
    1) A report (PDF) with performance of all the items listed above
    2) Code
       The code should be commented. It should include:
       -Title of the laboratory work;
       -The names of a team, indication of Skoltech, and date;
       -Main procedures also should be commented, for example
       %13-month running mean