

“Experimental Data Processing”

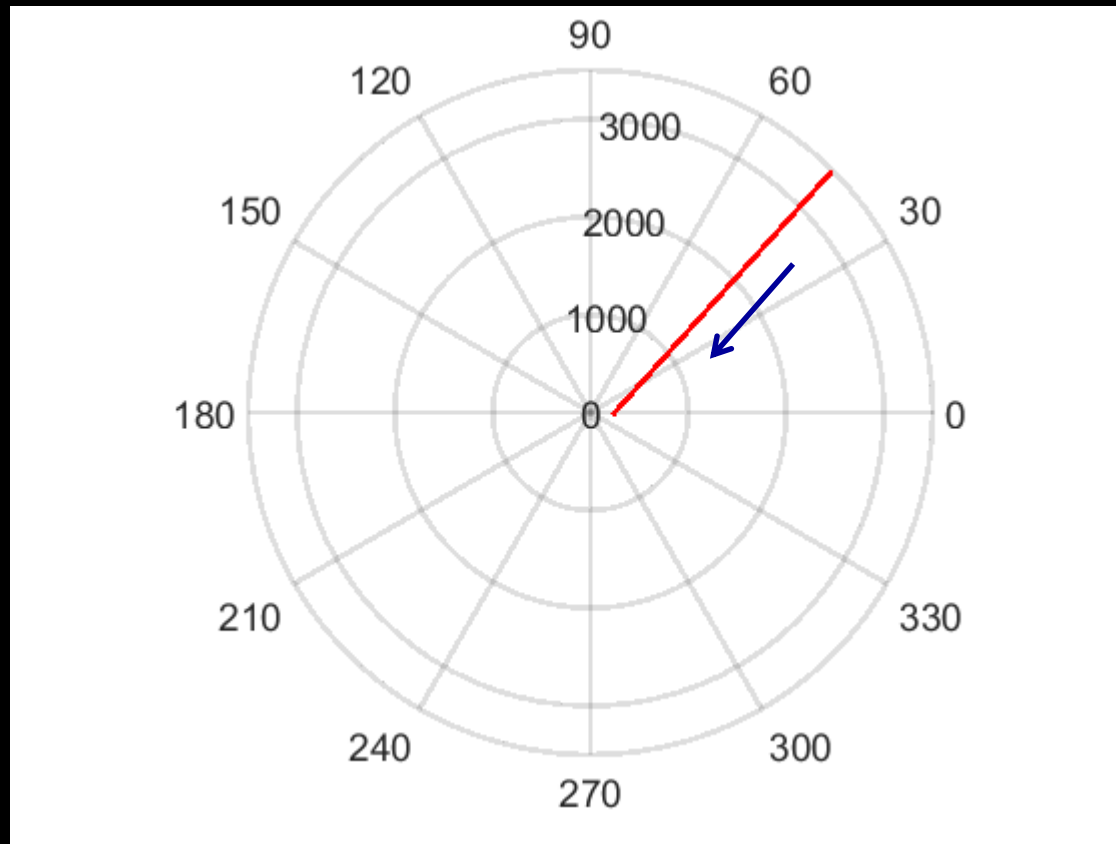
Development of tracking filter of a moving object
when measurements and motion model are in different
coordinate systems

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Measurements of navigation parameters are available in polar coordinate system



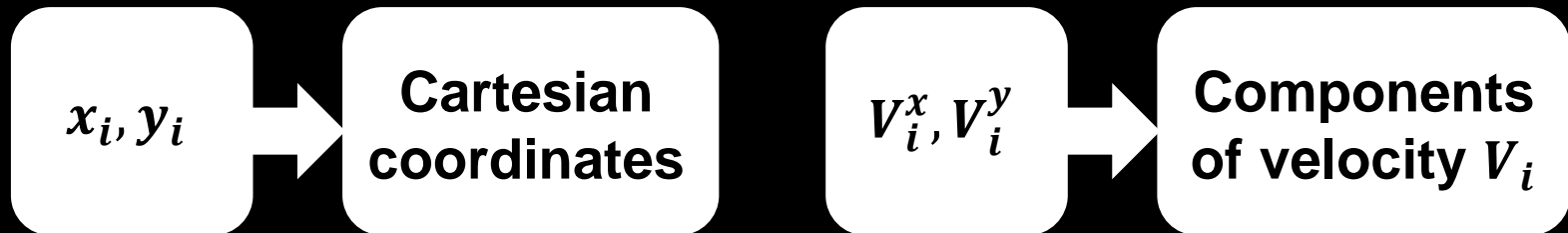
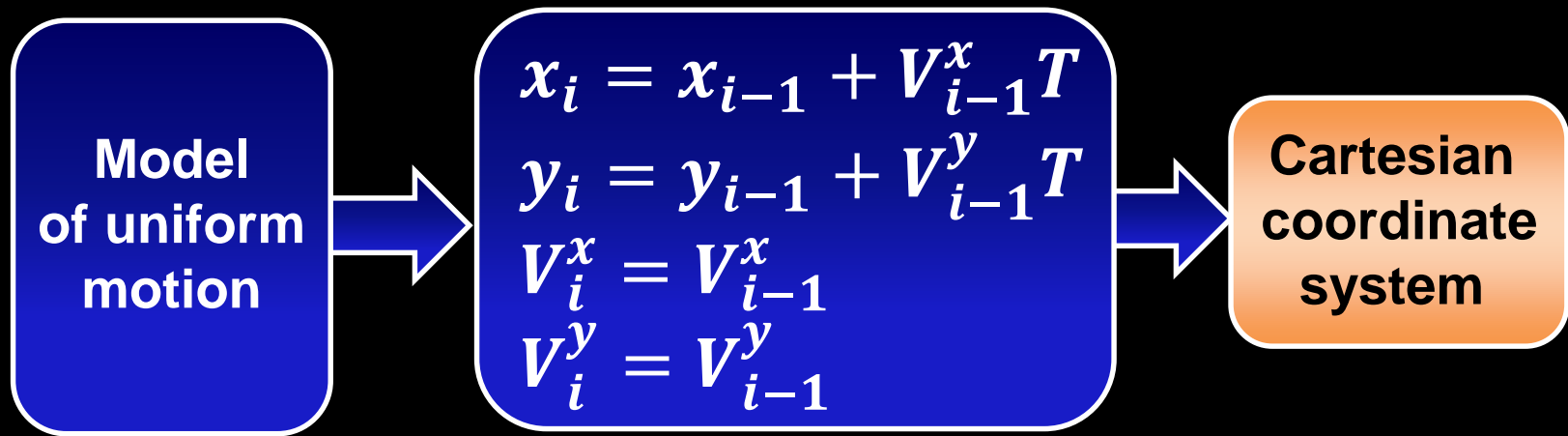
Range
 D

Distance from
an observer to
a moving object

Azimuth
 β

Angle between direction
of North and direction
of a moving object

Motion model is in Cartesian coordinate system



$$V_i = \sqrt{(V_i^x)^2 + (V_i^y)^2}$$

State-space model, state equation

State
equation



$$X_i = \Phi_{i,i-1} X_{i-1}$$

State
vector



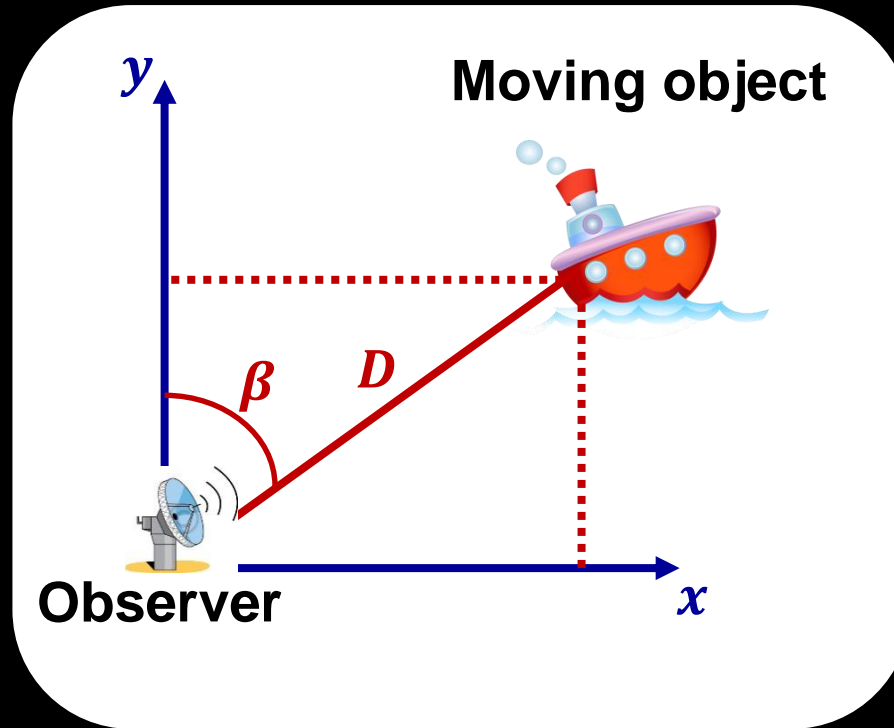
$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \end{bmatrix}$$

Transition
matrix



$$\Phi_{i,i-1} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation from polar to Cartesian coordinate system

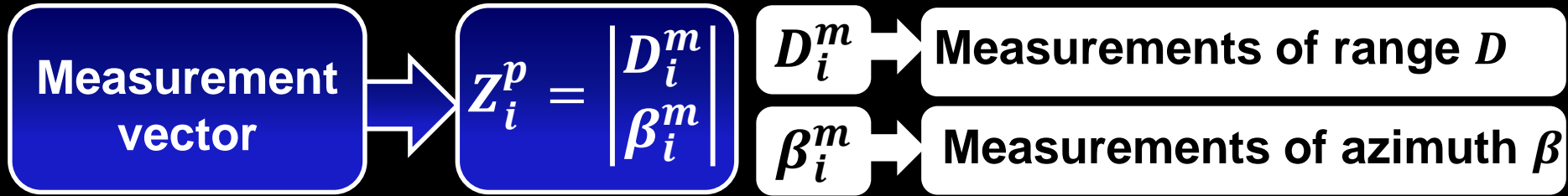


$$D = \sqrt{x^2 + y^2}$$

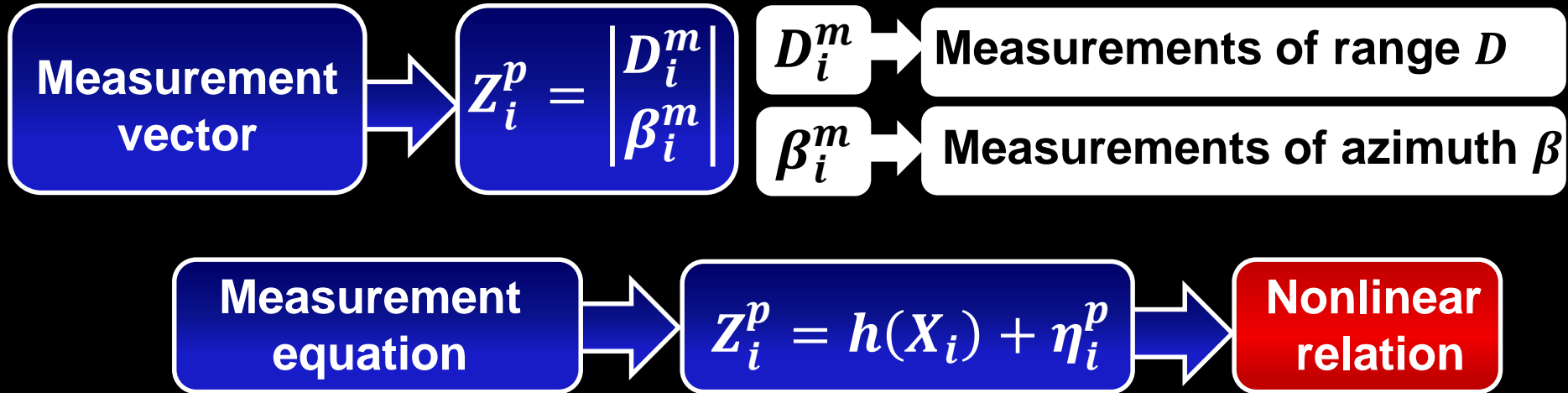
$$\beta = \arctg\left(\frac{x}{y}\right)$$

$$\begin{aligned}x &= D \sin \beta \\ y &= D \cos \beta\end{aligned}$$

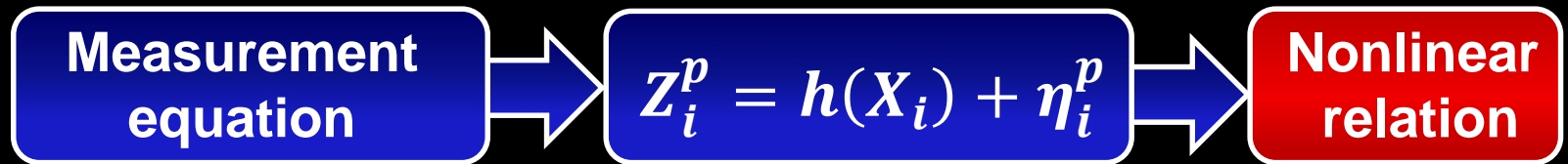
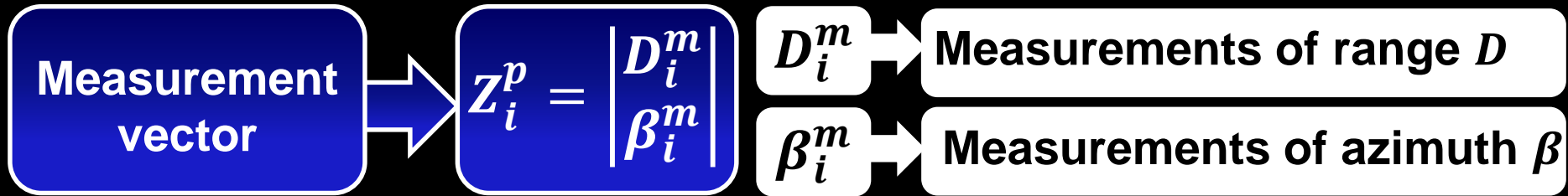
State-space model, measurement equation



State-space model, measurement equation

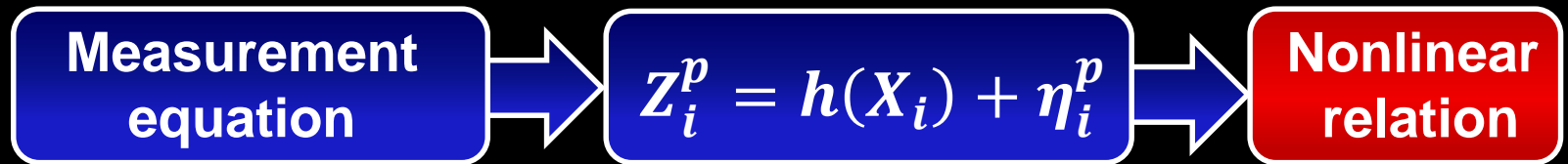
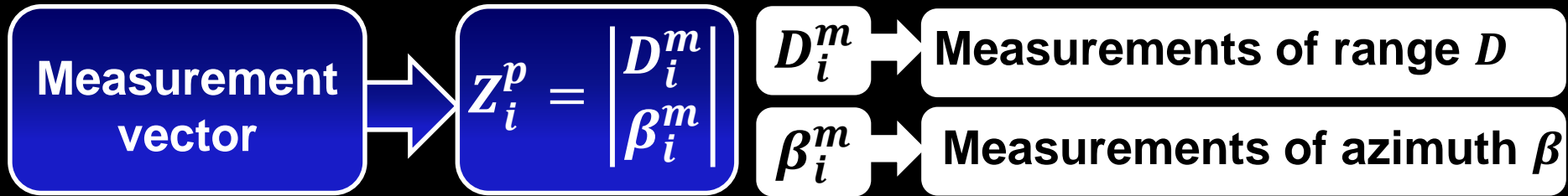


State-space model, measurement equation



$$Z_i^p = \begin{bmatrix} D_i^m \\ \beta_i^m \end{bmatrix} = \begin{bmatrix} D_i \\ \beta_i \end{bmatrix} + \eta_i^p \Rightarrow Z_i^p = \begin{bmatrix} \sqrt{x_i^2 + y_i^2} \\ \arctg\left(\frac{x_i}{y_i}\right) \end{bmatrix} + \eta_i^p$$

State-space model, measurement equation



$$Z_i^p = \begin{bmatrix} D_i^m \\ \beta_i^m \end{bmatrix} = \begin{bmatrix} D_i \\ \beta_i \end{bmatrix} + \eta_i^p \Rightarrow Z_i^p = \begin{bmatrix} \sqrt{x_i^2 + y_i^2} \\ \arctg\left(\frac{x_i}{y_i}\right) \end{bmatrix} + \eta_i^p$$



Noises η_i^D, η_i^β are uncorrelated with each other and have variances $\sigma_D^2, \sigma_\beta^2$

Measurement equation

From nonlinear to linear equation

Transform polar measurements D_i^m and β_i^m to Cartesian coordinates

$$\begin{aligned}x_i^m &= D_i^m \sin \beta_i^m \\y_i^m &= D_i^m \cos \beta_i^m\end{aligned}$$

Measurement vector

$$Z_i^p = \begin{bmatrix} D_i^m \\ \beta_i^m \end{bmatrix}$$

Pseudo-measurement vector

$$Z_i^c = \begin{bmatrix} x_i^m \\ y_i^m \end{bmatrix}$$

Measurement equation

From nonlinear to linear equation

Transform polar measurements D_i^m and β_i^m to Cartesian coordinates

$$\begin{aligned} x_i^m &= D_i^m \sin \beta_i^m \\ y_i^m &= D_i^m \cos \beta_i^m \end{aligned}$$

Measurement vector

$$Z_i^p = \begin{bmatrix} D_i^m \\ \beta_i^m \end{bmatrix}$$

Pseudo-measurement vector

$$Z_i^c = \begin{bmatrix} x_i^m \\ y_i^m \end{bmatrix}$$

Measurement equation

$$Z_i^c = H X_i + \eta_i^c$$

Linear relation

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\eta_i^c = \begin{bmatrix} \eta_i^x \\ \eta_i^y \end{bmatrix}$$

Vector of pseudo-measurement errors of x_i^m and y_i^m

How to determine dependence of η_i^c on η_i^p ?

$$\eta_i^p = \begin{bmatrix} \eta_i^D \\ \eta_i^\beta \end{bmatrix}$$

Vector
of measurement errors
of range D_i^m and azimuth β^m

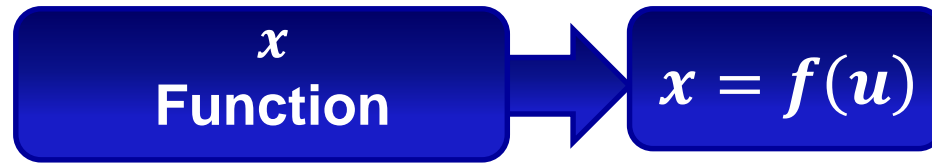
$$\begin{bmatrix} D_i^m \\ \beta_i^m \end{bmatrix} = \begin{bmatrix} D_i \\ \beta_i \end{bmatrix} + \eta_i^p$$

$$\eta_i^c = \begin{bmatrix} \eta_i^x \\ \eta_i^y \end{bmatrix}$$

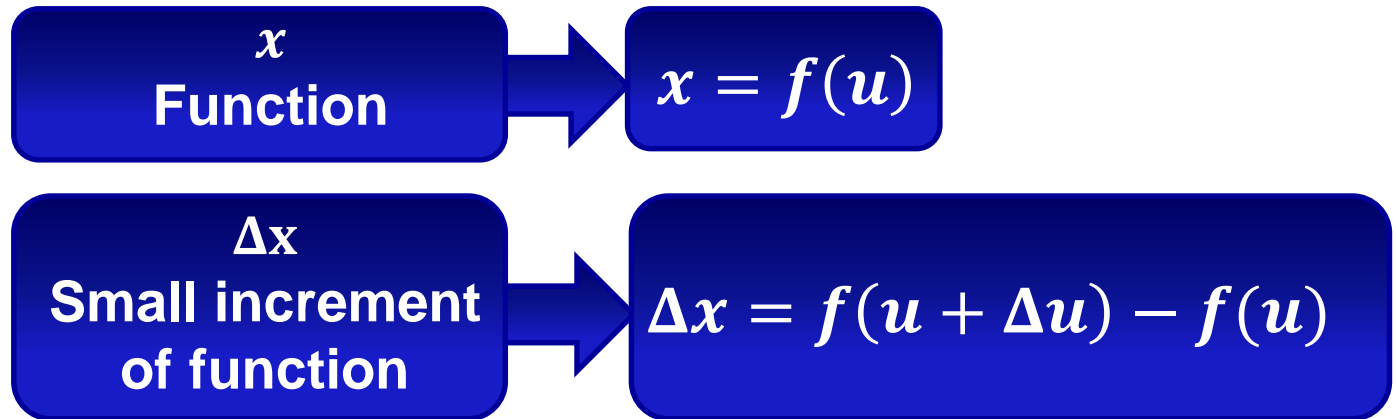
Vector
of pseudo-measurement
errors of x_i and y_i

$$\begin{bmatrix} x_i^m \\ y_i^m \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \eta_i^c$$

Full differential, example 1



Full differential, example 1



Full differential, example 1

x
Function $\rightarrow x = f(u)$

Δx
Small increment
of function $\rightarrow \Delta x = f(u + \Delta u) - f(u)$

Taylor series

$$f(u + \Delta u) = f(u) + \frac{1}{1!} \frac{dx}{du} \Delta u + \frac{1}{2!} \frac{d^2x}{du^2} (\Delta u)^2 + \frac{1}{3!} \frac{d^3x}{du^3} (\Delta u)^3 + \dots +$$

Full differential, example 1

x
Function $\rightarrow x = f(u)$

Δx
Small increment
of function $\rightarrow \Delta x = f(u + \Delta u) - f(u)$

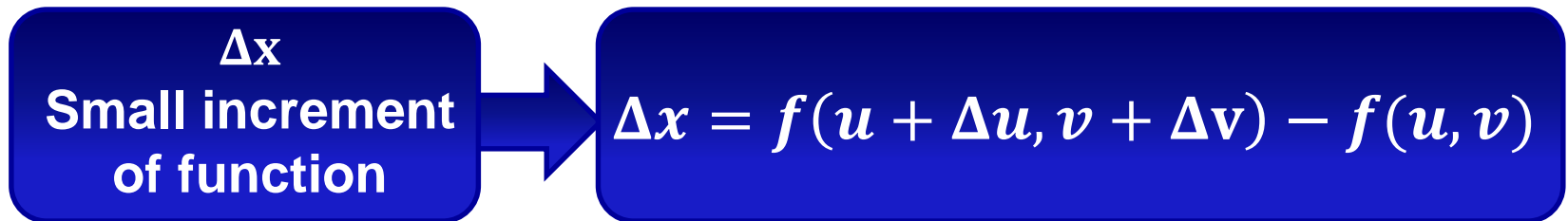
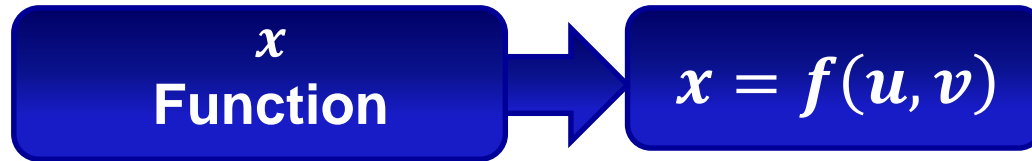
Taylor series

$$f(u + \Delta u) = f(u) + \frac{1}{1!} \frac{dx}{du} \Delta u + \frac{1}{2!} \frac{d^2x}{du^2} (\Delta u)^2 + \frac{1}{3!} \frac{d^3x}{du^3} (\Delta u)^3 + \dots +$$

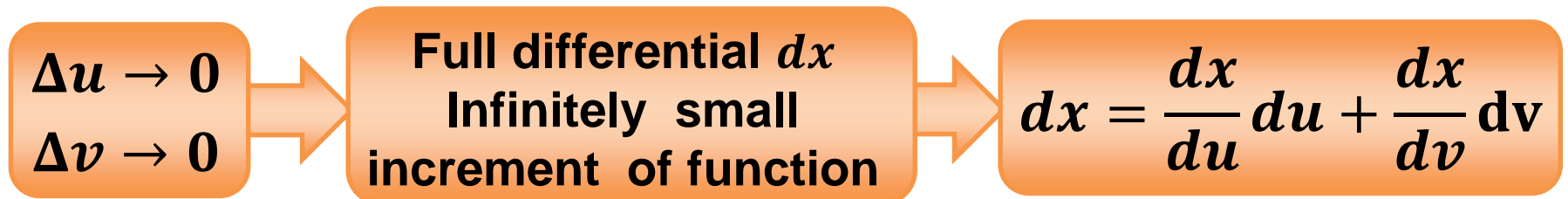
$$\Delta x = f(u + \Delta u) - f(u) \approx \frac{dx}{du} \Delta u$$

$\Delta u \rightarrow 0$ \rightarrow Differential dx
Infinitely small
increment of function $\rightarrow dx = \frac{dx}{du} du$

Full differential, example 2



$$\Delta x \approx \frac{dx}{du} \Delta u + \frac{dx}{dv} \Delta v$$



How to determine dependence of η_i^c on η_i^p ?

$$\eta_i^p = \begin{bmatrix} \eta_i^D \\ \eta_i^\beta \end{bmatrix}$$

Vector
of measurement errors
of range D_i^m and azimuth β

$$\eta_i^c = \begin{bmatrix} \eta_i^x \\ \eta_i^y \end{bmatrix}$$

Vector
of pseudo-measurement
errors of x_i^m and y_i^m

How to determine dependence of η_i^c on η_i^p ?

$$\eta_i^p = \begin{bmatrix} \eta_i^D \\ \eta_i^\beta \end{bmatrix}$$

Vector
of measurement errors
of range D_i^m and azimuth β

$$\eta_i^c = \begin{bmatrix} \eta_i^x \\ \eta_i^y \end{bmatrix}$$

Vector
of pseudo-measurement
errors of x_i^m and y_i^m

1

$$x_i^m = D_i^m \sin \beta^m$$

$$y_i^m = D_i^m \cos \beta^m$$

How to determine dependence of η_i^c on η_i^p ?

$$\eta_i^p = \begin{bmatrix} \eta_i^D \\ \eta_i^\beta \end{bmatrix}$$

Vector
of measurement errors
of range D_i^m and azimuth β

$$\eta_i^c = \begin{bmatrix} \eta_i^x \\ \eta_i^y \end{bmatrix}$$

Vector
of pseudo-measurement
errors of x_i^m and y_i^m

1

$$\begin{aligned} x_i^m &= D_i^m \sin \beta^m \\ y_i^m &= D_i^m \cos \beta^m \end{aligned}$$

2

$$\begin{aligned} x_i^m &= x_i + \eta_i^x \\ y_i^m &= y_i + \eta_i^y \end{aligned}$$

How to determine dependence of η_i^c on η_i^p ?

$$\eta_i^p = \begin{bmatrix} \eta_i^D \\ \eta_i^\beta \end{bmatrix}$$

Vector
of measurement errors
of range D_i^m and azimuth β

$$\eta_i^c = \begin{bmatrix} \eta_i^x \\ \eta_i^y \end{bmatrix}$$

Vector
of pseudo-measurement
errors of x_i^m and y_i^m

1

$$\begin{aligned} x_i^m &= D_i^m \sin \beta^m \\ y_i^m &= D_i^m \cos \beta^m \end{aligned}$$

2

$$\begin{aligned} x_i^m &= x_i + \eta_i^x \\ y_i^m &= y_i + \eta_i^y \end{aligned}$$

3

$$\begin{aligned} D_i^m &= D_i + \eta_i^D \\ \beta_i^m &= \beta_i + \eta_i^\beta \end{aligned}$$

How to determine dependence of η_i^c on η_i^p ?

$$\eta_i^p = \begin{bmatrix} \eta_i^D \\ \eta_i^\beta \end{bmatrix}$$

Vector
of measurement errors
of range D_i^m and azimuth β

$$\eta_i^c = \begin{bmatrix} \eta_i^x \\ \eta_i^y \end{bmatrix}$$

Vector
of pseudo-measurement
errors of x_i^m and y_i^m

1

$$\begin{aligned} x_i^m &= D_i^m \sin \beta^m \\ y_i^m &= D_i^m \cos \beta^m \end{aligned}$$

2

$$\begin{aligned} x_i^m &= x_i + \eta_i^x \\ y_i^m &= y_i + \eta_i^y \end{aligned}$$

3

$$\begin{aligned} D_i^m &= D_i + \eta_i^D \\ \beta_i^m &= \beta_i + \eta_i^\beta \end{aligned}$$

Rewrite (1) using (2) and (3) for x_i

$$x_i + \eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta)$$

How to determine dependence of η_i^c on η_i^p ?

$$x_i + \eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta)$$

How to determine dependence of η_i^c on η_i^p ?

$$x_i + \eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta)$$



$$\eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta) - x_i$$

How to determine dependence of η_i^c on η_i^p ?

$$x_i + \eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta)$$



$$\eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta) - x_i$$



$$\eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta) - D_i \sin \beta_i$$

$$x = D \sin \beta$$

How to determine dependence of η_i^c on η_i^p ?

$$x_i + \eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta)$$



$$\eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta) - x_i$$



$$\eta_i^x = (D_i + \eta_i^D) \sin(\beta_i + \eta_i^\beta) - D_i \sin \beta_i$$

$$x = D \sin \beta$$

Or

$$\Delta x = (D + \Delta D) \sin(\beta + \Delta \beta) - D \sin \beta$$

How to determine dependence of η_i^c on η_i^p ?

$$\Delta x = (D + \Delta D) \sin(\beta + \Delta\beta) - D \sin\beta$$

$$x = D \sin\beta$$

$$\Delta y = (D + \Delta D) \cos(\beta + \Delta\beta) - D \cos\beta$$

$$y = D \cos\beta$$

How to determine dependence of η_i^c on η_i^p ?

$$\Delta x = (D + \Delta D) \sin(\beta + \Delta\beta) - D \sin\beta$$

$$x = D \sin\beta$$

$$\Delta y = (D + \Delta D) \cos(\beta + \Delta\beta) - D \sin\beta$$

$$y = D \cos\beta$$

Taylor series

$$(D + \Delta D) \sin(\beta + \Delta\beta) \approx D \sin\beta + \frac{dx}{dD} \Delta D + \frac{dx}{d\beta} \Delta\beta$$

$$(D + \Delta D) \cos(\beta + \Delta\beta) \approx D \cos\beta + \frac{dy}{dD} \Delta D + \frac{dy}{d\beta} \Delta\beta$$

How to determine dependence of η_i^c on η_i^p ?

$$\Delta x = (D + \Delta D) \sin(\beta + \Delta\beta) - D \sin\beta$$

$$x = D \sin\beta$$

$$\Delta y = (D + \Delta D) \cos(\beta + \Delta\beta) - D \sin\beta$$

$$y = D \cos\beta$$

Taylor series

$$(D + \Delta D) \sin(\beta + \Delta\beta) \approx D \sin\beta + \frac{dx}{dD} \Delta D + \frac{dx}{d\beta} \Delta\beta$$

$$(D + \Delta D) \cos(\beta + \Delta\beta) \approx D \cos\beta + \frac{dy}{dD} \Delta D + \frac{dy}{d\beta} \Delta\beta$$

$$\Delta x = \frac{dx}{dD} \Delta D + \frac{dx}{d\beta} \Delta\beta$$

$$\Delta y = \frac{dy}{dD} \Delta D + \frac{dy}{d\beta} \Delta\beta$$

How to determine dependence of η_i^c on η_i^p ?

$$\Delta \mathbf{x} = \frac{dx}{dD} \Delta D + \frac{dx}{d\beta} \Delta \beta$$



$$\eta^x = \frac{dx}{dD} \eta^D + \frac{dx}{d\beta} \eta^\beta$$

$$\Delta \mathbf{y} = \frac{dy}{dD} \Delta D + \frac{dy}{d\beta} \Delta \beta$$



$$\eta^y = \frac{dy}{dD} \eta^D + \frac{dy}{d\beta} \eta^\beta$$

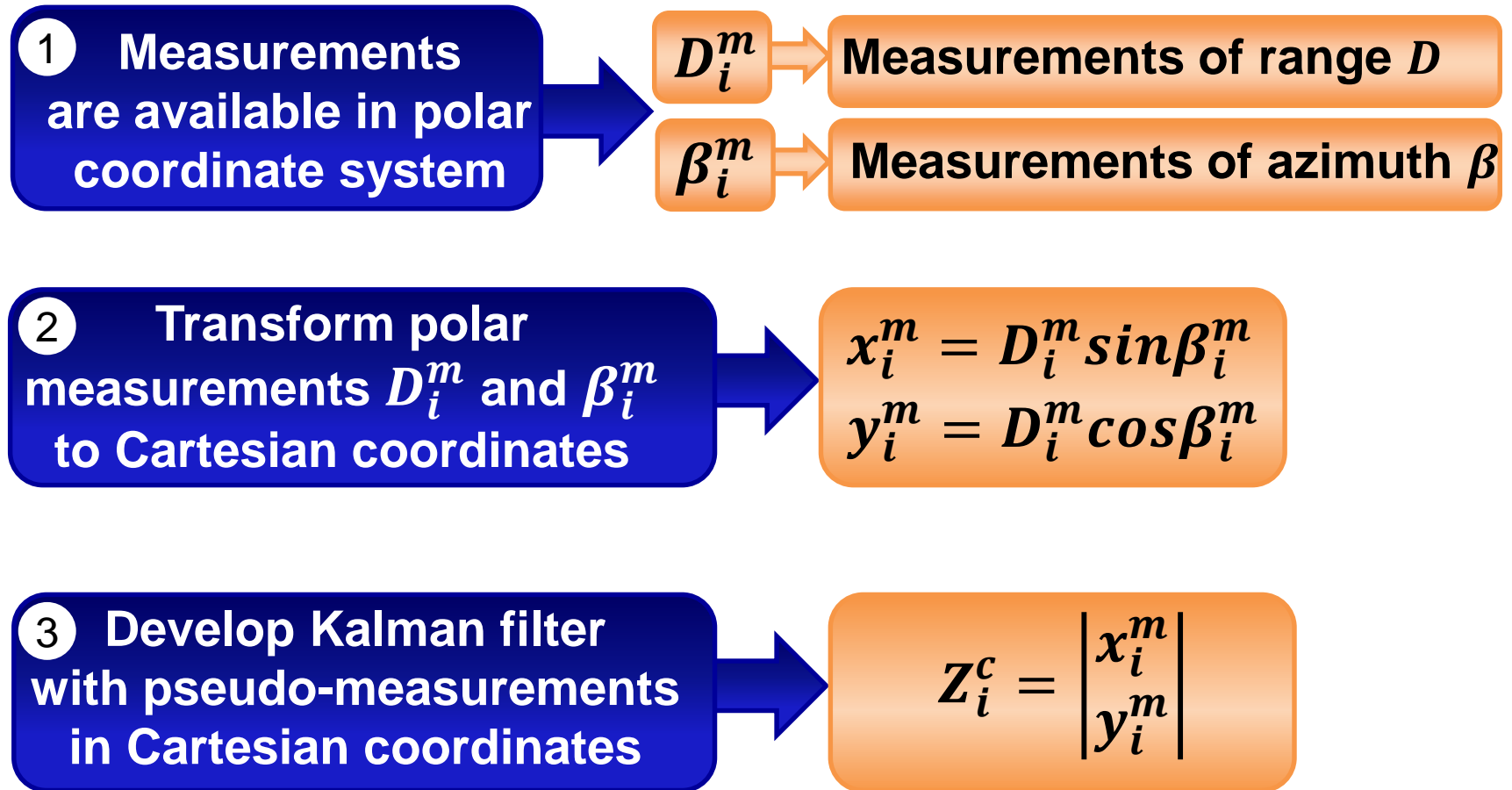
$$\frac{dx}{dD} = \sin \beta$$

$$\frac{dx}{d\beta} = D \cos \beta$$

$$\frac{dy}{dD} = \cos \beta$$

$$\frac{dy}{d\beta} = -D \sin \beta$$

Summary, scheme of estimation algorithm



Summary, state-space model

State equation

$$X_i = \Phi_{i,i-1} X_{i-1}$$

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \end{bmatrix}$$

$$\Phi_{i,i-1} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Measurement equation

$$Z_i^c = H X_i + \eta_i^c$$

$$Z_i^c = \begin{bmatrix} x_i^m \\ y_i^m \end{bmatrix}$$

$$\begin{aligned} x_i^m &= D_i^m \sin \beta_i^m \\ y_i^m &= D_i^m \cos \beta_i^m \end{aligned}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\eta_i^c = \begin{bmatrix} \eta_i^x \\ \eta_i^y \end{bmatrix} = \begin{bmatrix} \eta_i^D \sin \beta_i^m + \eta_i^\beta D_i^m \cos \beta_i^m \\ \eta_i^D \cos \beta_i^m - \eta_i^\beta D_i^m \sin \beta_i^m \end{bmatrix}$$

Noises η_i^D , η_i^β - errors of D_i^m and β_i^m
Are characterized by variances σ_D^2 , σ_β^2

Covariance matrix of measurement error R

Measurement
noise



$$\eta_i^c = \begin{bmatrix} \eta_i^x \\ \eta_i^y \end{bmatrix} = \begin{bmatrix} \eta_i^D \sin \beta_i^m + \eta_i^\beta D_i^m \cos \beta_i^m \\ \eta_i^D \cos \beta_i^m - \eta_i^\beta D_i^m \sin \beta_i^m \end{bmatrix}$$

R
Covariance
matrix



$$R = E[\eta_i^c \cdot (\eta_i^c)^T]$$

$$R = \begin{bmatrix} \sin^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \cos^2 \beta_i^m \sigma_\beta^2 & \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] \\ \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] & \cos^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \sin^2 \beta_i^m \sigma_\beta^2 \end{bmatrix}$$

$$\sigma_D^2$$

→ Variance of measurement error of range D

$$\sigma_\beta^2$$

→ Variance of measurement error of azimuth β

Goals of the laboratory work

1

Analyze instability zone of a filter

**When object is
close to observer**



**Navigation
system may
become blind**



**Related with
ill-conditioned
matrix R**

Condition number of measurement error covariance matrix R

$$R = \begin{vmatrix} \sin^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \cos^2 \beta_i^m \sigma_\beta^2 & \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] \\ \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] & \cos^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \sin^2 \beta_i^m \sigma_\beta^2 \end{vmatrix}$$

$$\sigma_D^2$$

→ Variance of measurement error of range D

$$\sigma_\beta^2$$

→ Variance of measurement error of azimuth β

Eigen values
of matrix R

$$\Rightarrow \det(R - \lambda I) = 0$$

$$\begin{aligned} \lambda_1 &= \sigma_D^2 \\ \lambda_2 &= (D_i^m)^2 \sigma_\beta^2 \end{aligned}$$

Condition number of measurement error covariance matrix R

$$R = \begin{bmatrix} \sin^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \cos^2 \beta_i^m \sigma_\beta^2 & \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] \\ \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] & \cos^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \sin^2 \beta_i^m \sigma_\beta^2 \end{bmatrix}$$

$$\sigma_D^2$$

→ Variance of measurement error of range D

$$\sigma_\beta^2$$

→ Variance of measurement error of azimuth β

Eigen values
of matrix R

$$\Rightarrow \det(R - \lambda I) = 0$$

$$\begin{aligned} \lambda_1 &= \sigma_D^2 \\ \lambda_2 &= (D_i^m)^2 \sigma_\beta^2 \end{aligned}$$

Condition
number

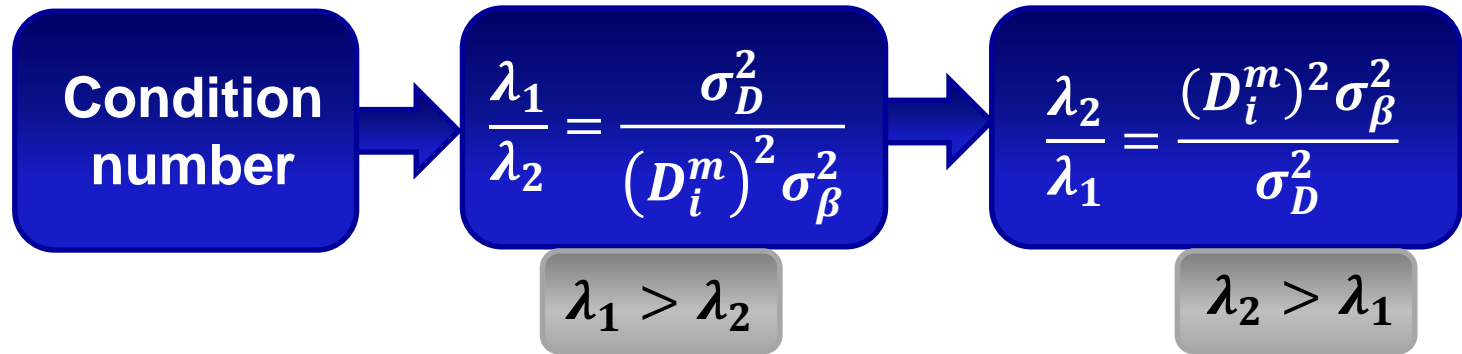
$$\frac{\lambda_1}{\lambda_2} = \frac{\sigma_D^2}{(D_i^m)^2 \sigma_\beta^2}$$

$$\lambda_1 > \lambda_2$$

$$\frac{\lambda_2}{\lambda_1} = \frac{(D_i^m)^2 \sigma_\beta^2}{\sigma_D^2}$$

$$\lambda_2 > \lambda_1$$

Condition number of measurement error covariance matrix R



Condition number of measurement error covariance matrix R

Condition
number

$$\frac{\lambda_1}{\lambda_2} = \frac{\sigma_D^2}{(D_i^m)^2 \sigma_\beta^2}$$

$$\frac{\lambda_2}{\lambda_1} = \frac{(D_i^m)^2 \sigma_\beta^2}{\sigma_D^2}$$

$$\lambda_1 > \lambda_2$$

$$\lambda_2 > \lambda_1$$

If condition number
of matrix $R \approx 1$

R – well-conditioned
matrix

$$\sigma_D^2 \approx (D_i^m)^2 \sigma_\beta^2$$

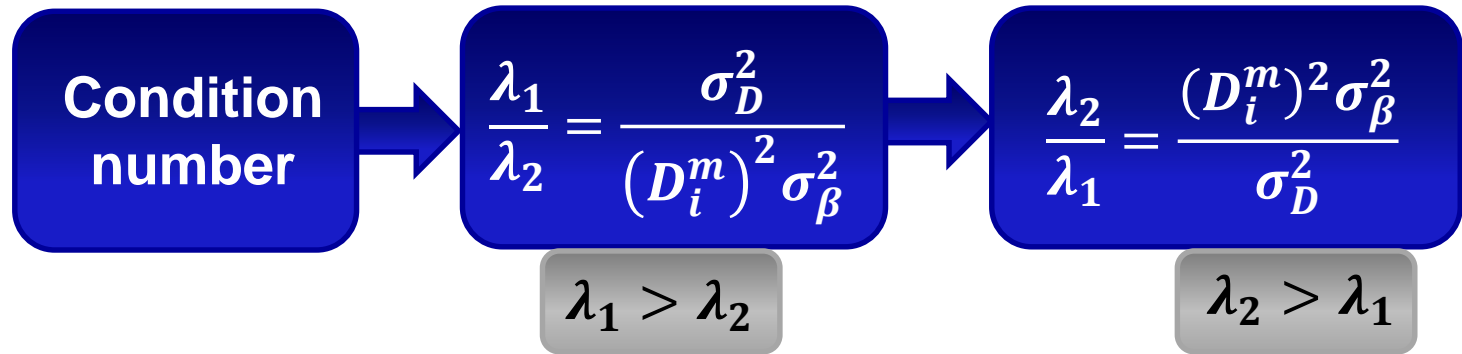
$$R = \begin{vmatrix} \sin^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \cos^2 \beta_i^m \sigma_\beta^2 & \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] \\ \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] & \cos^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \sin^2 \beta_i^m \sigma_\beta^2 \end{vmatrix}$$



$$R = \begin{vmatrix} \sin^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \cos^2 \beta_i^m \sigma_\beta^2 & 0 \\ 0 & \cos^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \sin^2 \beta_i^m \sigma_\beta^2 \end{vmatrix}$$

Independent assimilation for x, y

Condition number of measurement error covariance matrix R



If condition number
of matrix $R > 1000$

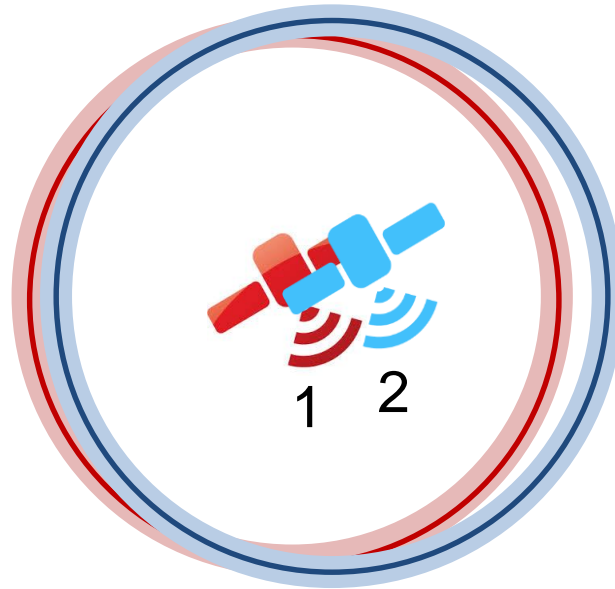
R – ill-conditioned
matrix

$$R = \begin{vmatrix} \sin^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \cos^2 \beta_i^m \sigma_\beta^2 & \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] \\ \sin \beta_i^m \cos \beta_i^m [\sigma_D^2 - (D_i^m)^2 \sigma_\beta^2] & \cos^2 \beta_i^m \sigma_D^2 + (D_i^m)^2 \sin^2 \beta_i^m \sigma_\beta^2 \end{vmatrix}$$

Estimation accuracy is decreased
Filter may diverge

III-conditioned problem

**Validity
of applying
a technique**



-?



Man-made satellite



Navigation satellite

II-conditioned problem

Satellite position is undefined!