

“Experimental Data Processing”

Topic 5

"Model construction at state space under uncertainty"

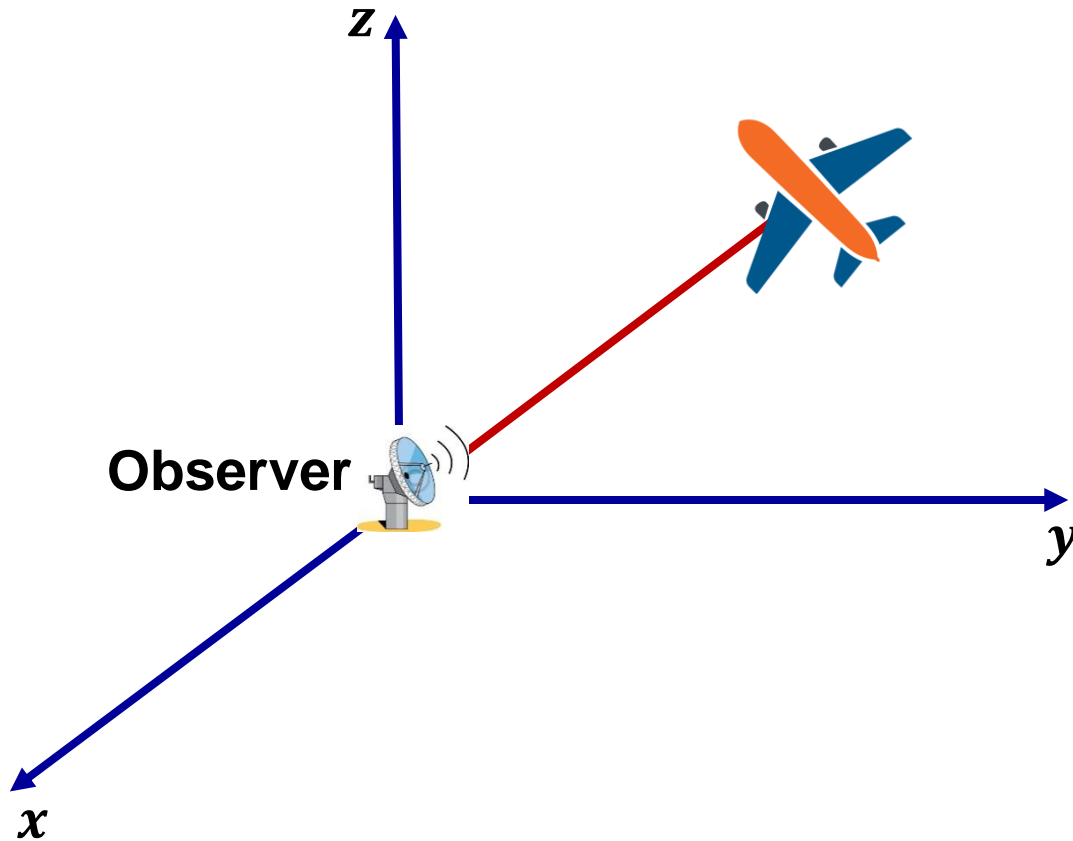
II. Extended Kalman filter for navigation and tracking

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Term 1B, October 2017

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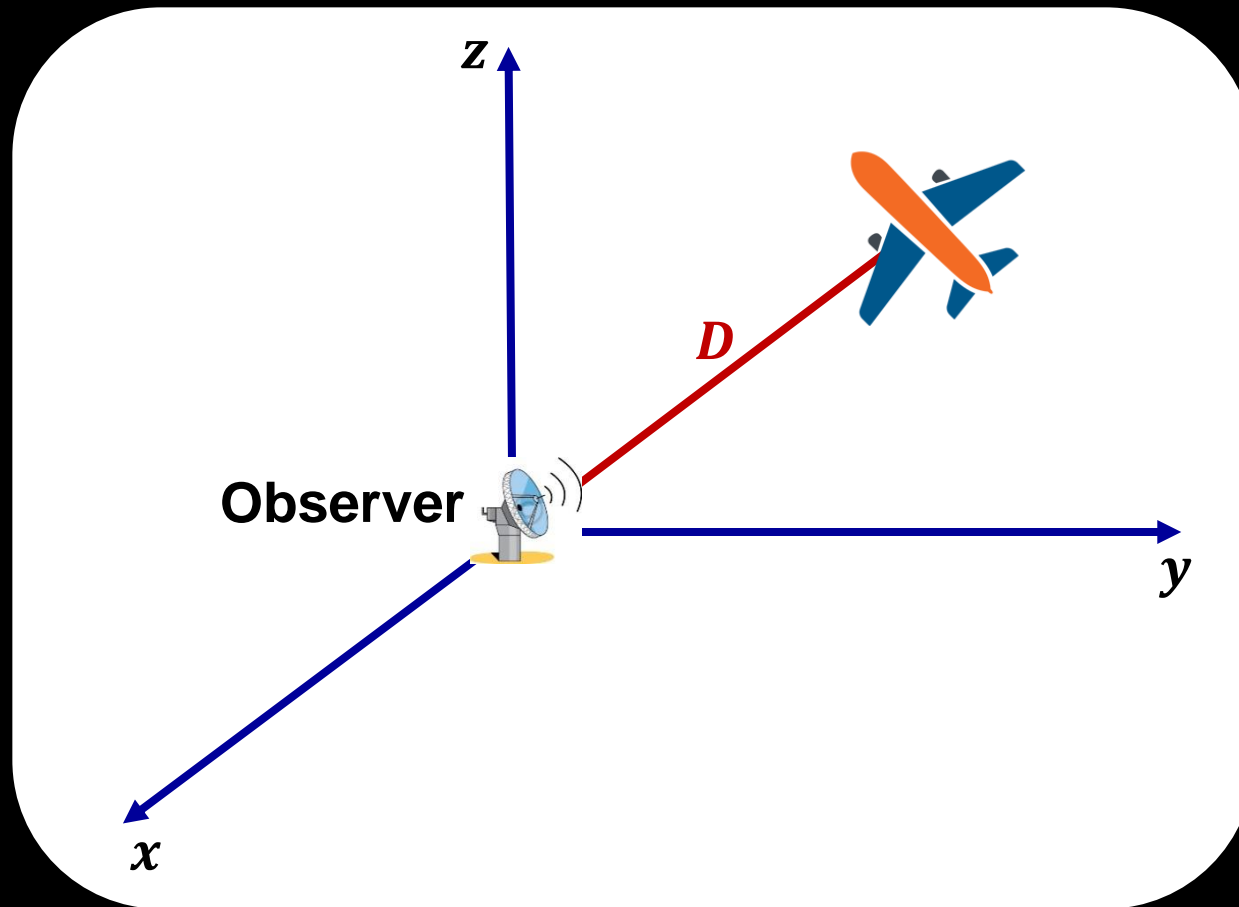
State of a moving object is characterized by state vector in Cartesian coordinate system



State
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

D 1. Estimation of coordinates using measurements of distance D , azimuth β , and angle of elevation ε



State
vector

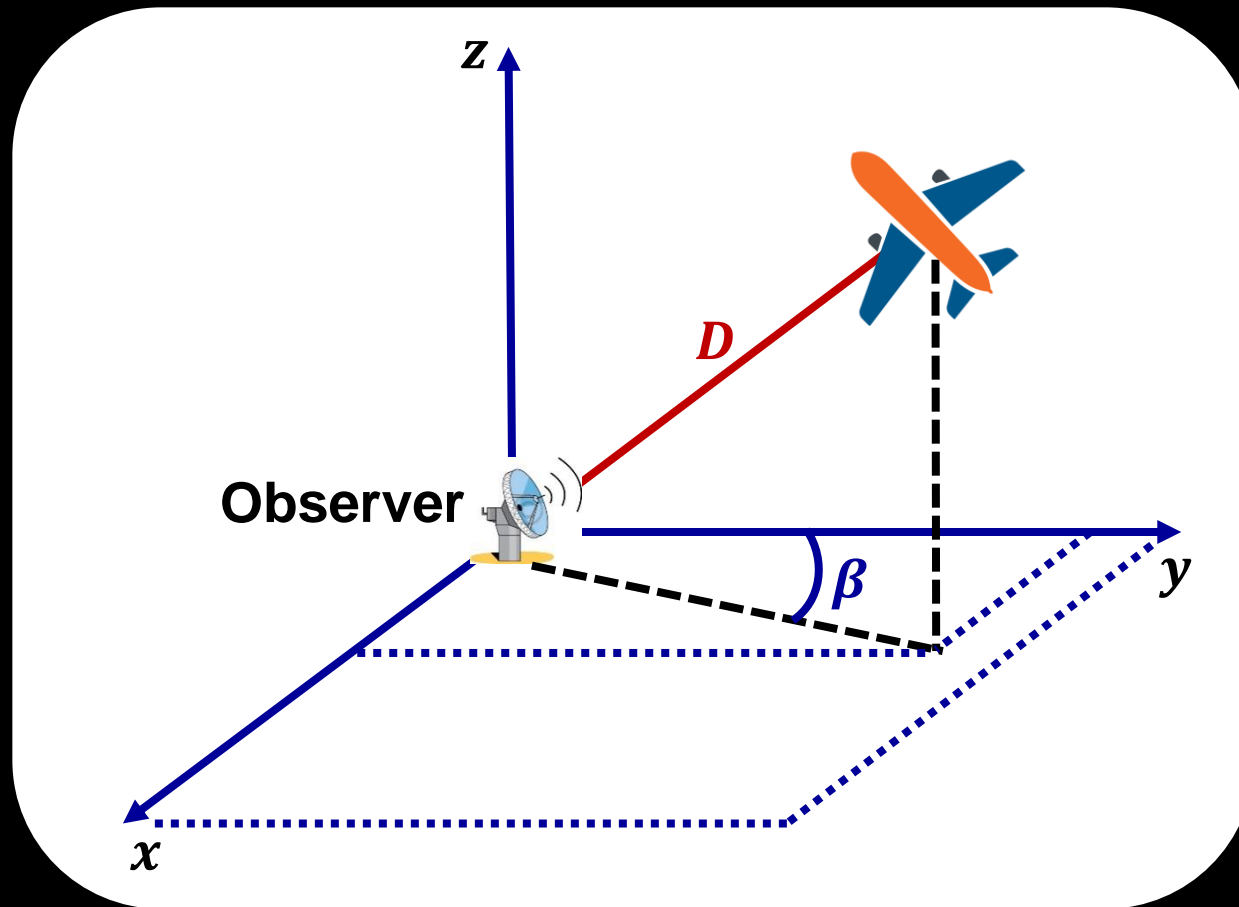
$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

Range
 D

Distance from
an observer to
a moving object

$$D_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$$

β 1. Estimation of coordinates using measurements of distance D , azimuth β , and angle of elevation ε



State
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

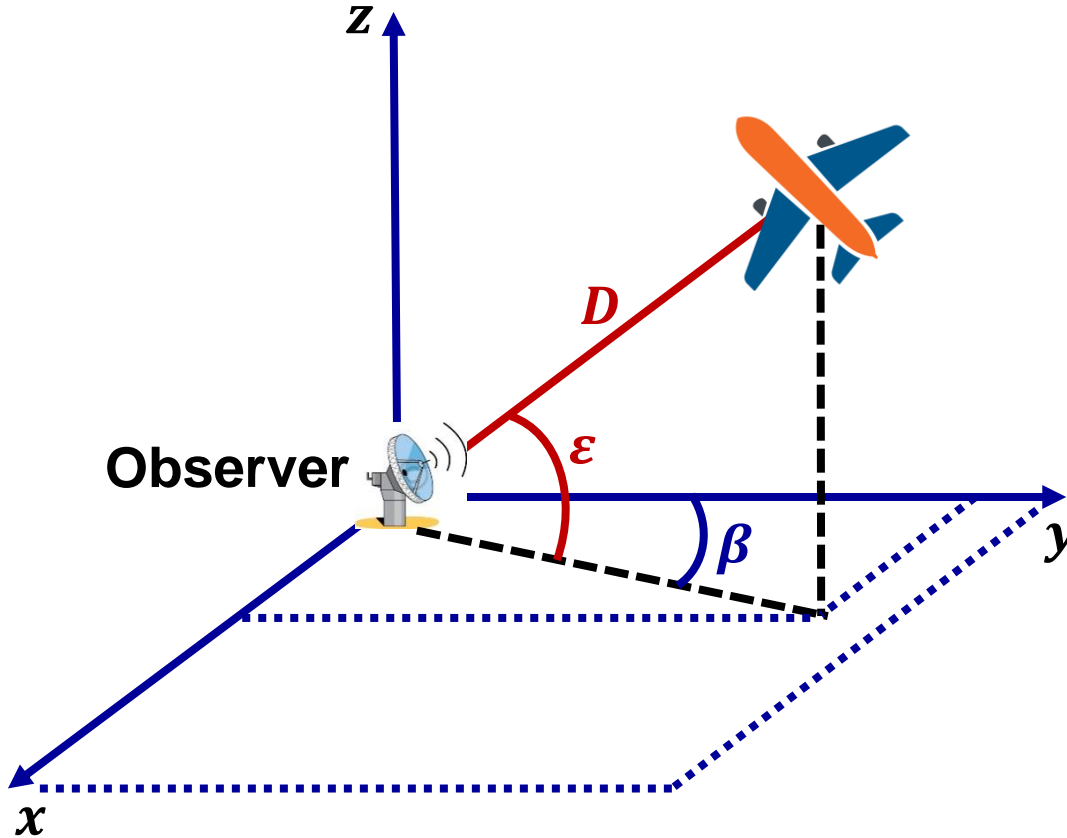
Azimuth
 β

Angle between direction
of North and projection
line in horizontal plane

$$\beta_i = \arctg \left(\frac{x_i}{y_i} \right)$$

ε

1. Estimation of coordinates using measurements of distance D , azimuth β , and angle of elevation ε



State
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

Angle of
elevation ε

Angle between the
horizontal plane and
direction of an object

$$\varepsilon_i = \arcsin \left(\frac{z_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}} \right)$$

1. Estimation of coordinates using measurements of distance D , azimuth β , and angle of elevation ε

Measurement equation

$$z_i = h(X_i) + \eta_i$$

Measurement
vector z_i

$$z_i = \begin{bmatrix} D_i^m \\ \beta_i^m \\ \varepsilon_i^m \end{bmatrix}$$

1. Estimation of coordinates using measurements of distance D , azimuth β , and angle of elevation ε

Measurement equation

$$z_i = h(X_i) + \eta_i$$

Measurement
vector z_i

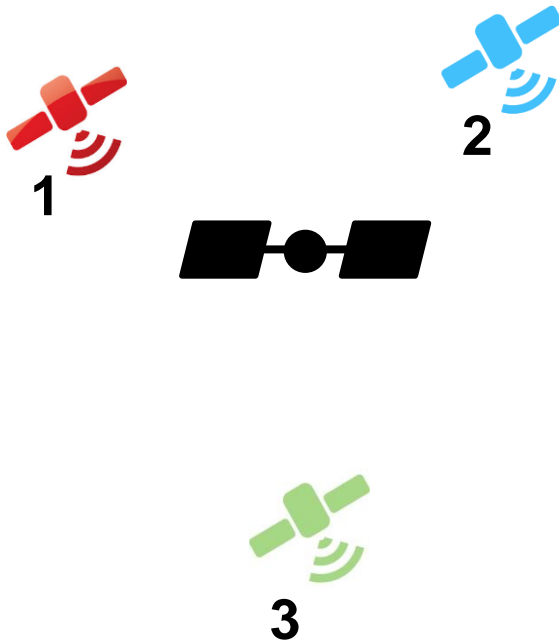
Nonlinear
function $h(X_i)$

$$z_i = \begin{bmatrix} D_i^m \\ \beta_i^m \\ \varepsilon_i^m \end{bmatrix}$$

$$h(X_i) = \begin{bmatrix} \sqrt{x_i^2 + y_i^2 + z_i^2} \\ \operatorname{arctg}\left(\frac{x_i}{y_i}\right) \\ \operatorname{arcsin}\left(\frac{z_i}{\sqrt{x_i^2 + y_i^2 + z_i^2}}\right) \end{bmatrix}$$

2. Estimation of coordinates using measurements only of distance D

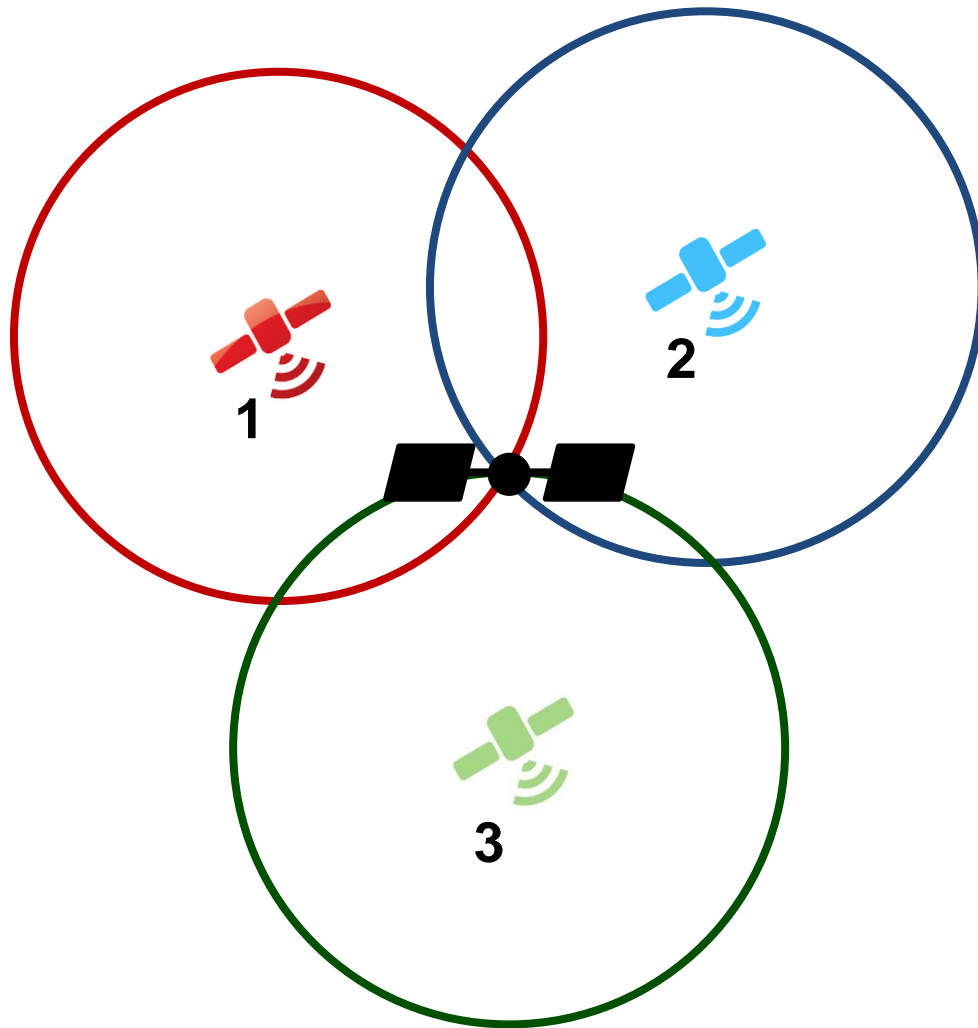
Three navigation stations
measure distance D
to a moving object



Moving object with
unknown coordinates x, y, z

Navigation stations with
known coordinates
 $(x_1, y_1, z_1); (x_2, y_2, z_2),$
 (x_3, y_3, z_3)

2. Estimation of coordinates using measurements only of distance D



The position of a satellite is at the intersecting points of circles



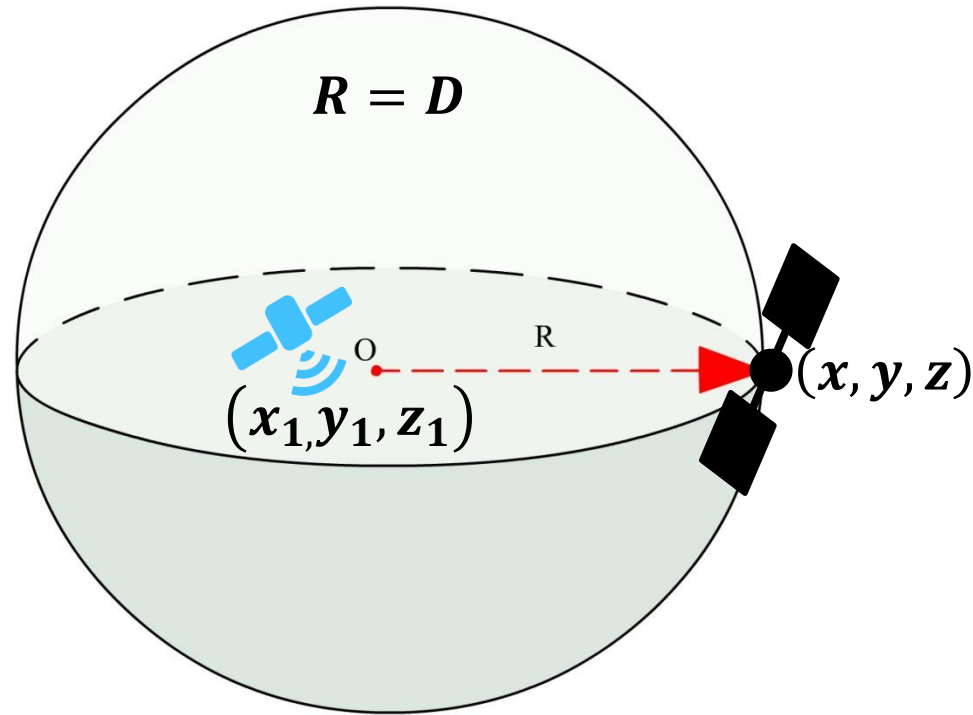
Moving object with unknown coordinates x, y, z



Navigation stations with known coordinates
 $(x_1, y_1, z_1); (x_2, y_2, z_2),$
 (x_3, y_3, z_3)

2. Estimation of coordinates using measurements only of distance D

Equation
of a 3-D sphere



Unknown coordinates
 x, y, z can be obtained
by solving system
of equations

$$\begin{aligned} D_1 &= \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2} \\ D_2 &= \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2} \\ D_3 &= \sqrt{(x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2} \end{aligned}$$

2. Estimation of coordinates using measurements only of distance D

Measurement equation

$$z_i = h(X_i) + \eta_i$$

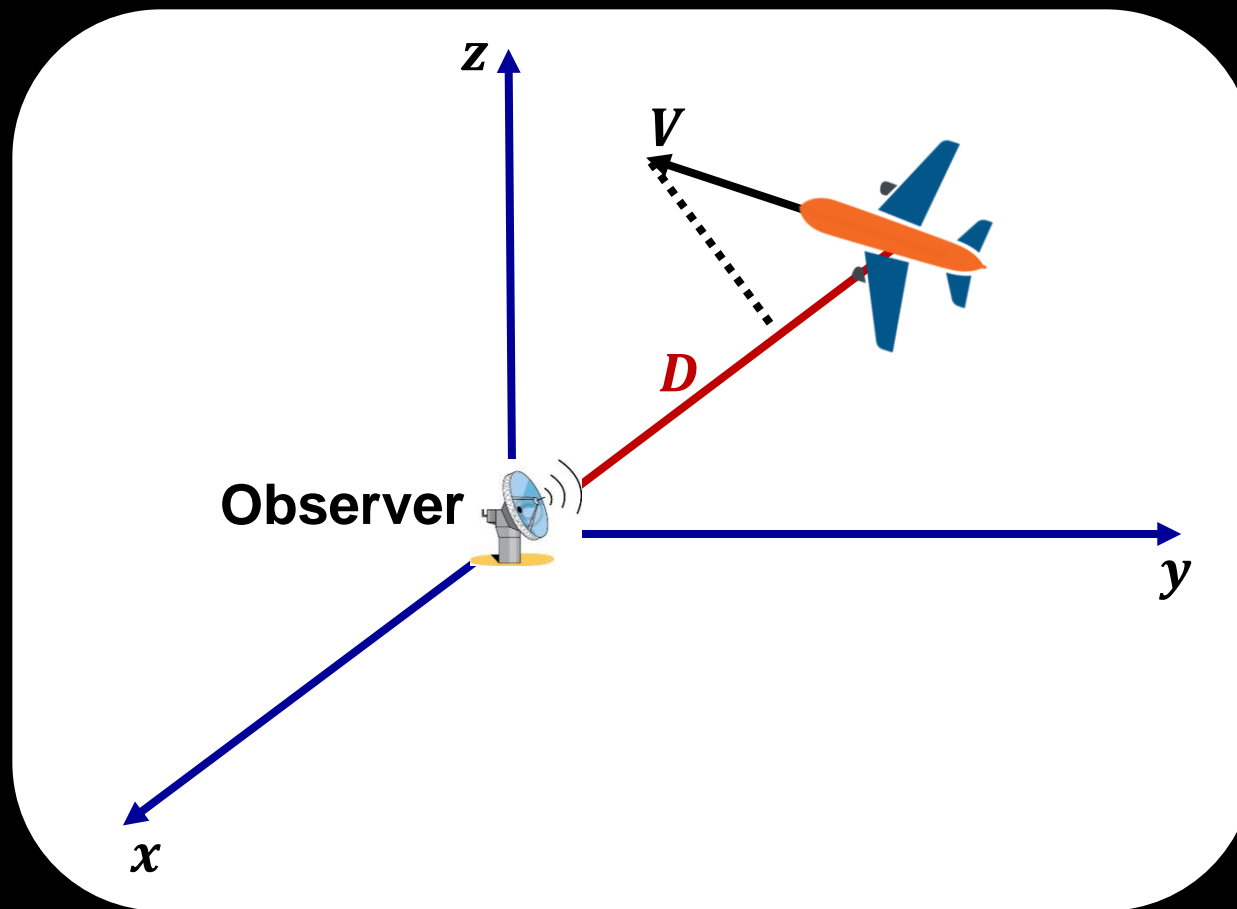
Measurement
vector z_i

$$z_i = \begin{bmatrix} D_1^m \\ D_2^m \\ D_3^m \end{bmatrix}$$

Nonlinear
function $h(X_i)$

$$h(X_i) = \begin{bmatrix} \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2} \\ \sqrt{(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2} \\ \sqrt{(x_3 - x)^2 + (y_3 - y)^2 + (z_3 - z)^2} \end{bmatrix}$$

3. Estimation of velocity using measurements of Doppler velocity



State
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \\ z_i \\ V_i^z \end{bmatrix}$$

Doppler
velocity V_d

Projection of V on
vector D - radial
component of V

$$V_d = \frac{xV^x + yV^y + zV^z}{D}$$

3. Estimation of velocity using measurements of Doppler velocity

Measurement equation

$$z_i = h(X_i) + \eta_i$$

Measurement
vector z_i

$$z_i = |V_d^m|$$

Nonlinear
function $h(X_i)$

$$h(X_i) = \frac{xV^x + yV^y + zV^z}{\sqrt{x^2 + y^2 + z^2}}$$

4. Nonlinear model of a geostationary satellite orbit

System of differential equations in Celestial Reference System (CRS)

$$\begin{cases} \dot{\mathbf{r}} = \mathbf{V} \\ \dot{\mathbf{V}} = -GM_o \frac{\mathbf{r}}{|\mathbf{r}|^3} + \mathbf{F}(\mathbf{r}, \mathbf{V}, t) \end{cases}$$

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{V} = \begin{bmatrix} V^x \\ V^y \\ V^z \end{bmatrix}$$

Coordinates and components of velocity of a geostationary satellite in CRS

$$GM_o$$

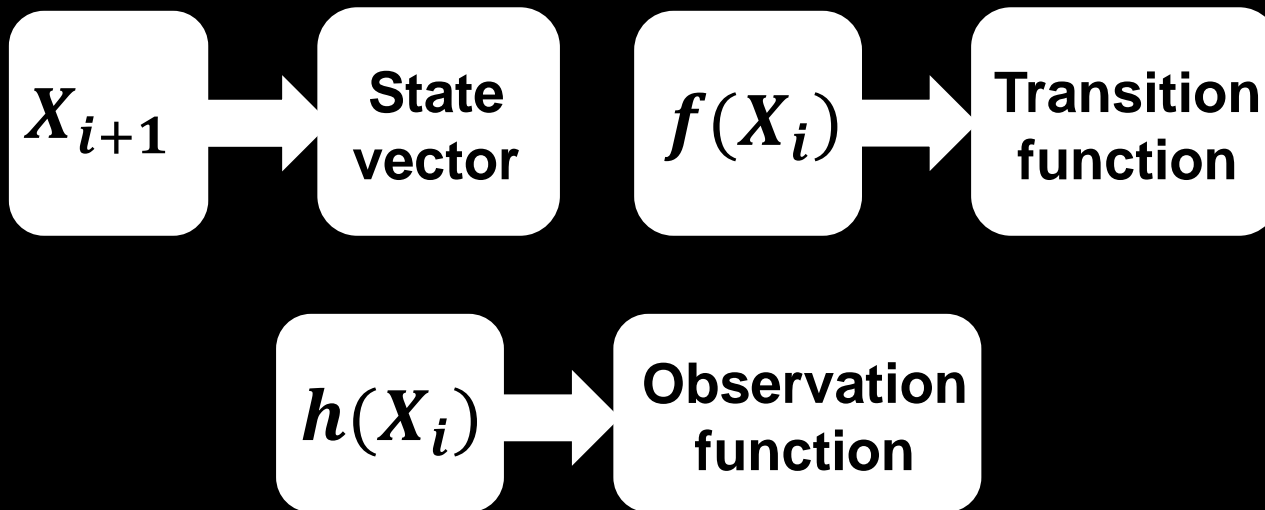
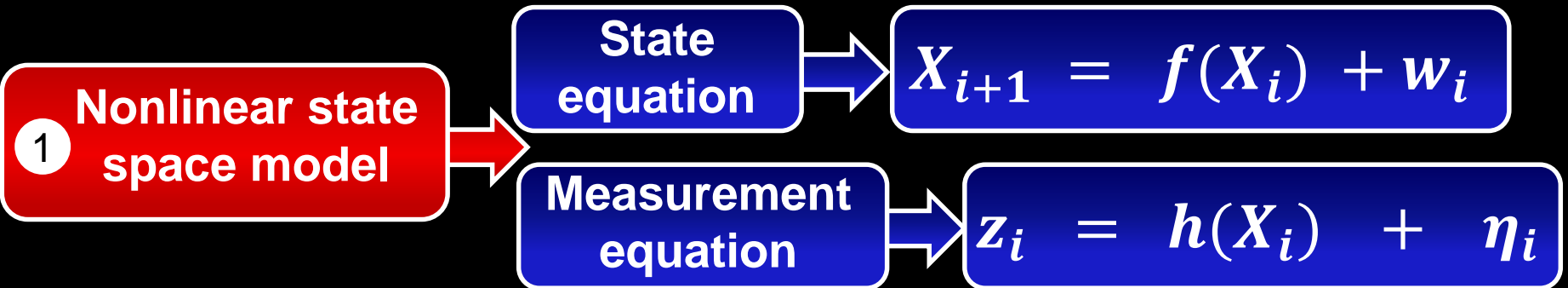
Gravitational constant multiplied by the Earth's mass

$$\mathbf{F}(\mathbf{r}, \mathbf{V}, t)$$

Disturbing acceleration related with:

- nonspherical gravity field of the Earth;
- gravitational disturbances of Moon and Sun;
- light pressure from Sun

Extended Kalman filter



Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$



Filtered and predicted estimates at time i

Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$



Filtered and predicted estimates at time i

Let's produce Taylor series for
 $f(X_i)$ and $h(X_i)$ around estimates $\hat{X}_{i,i}$ and $\hat{X}_{i+1,i}$

Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$

Filtered and predicted estimates at time i

Let's produce Taylor series for $f(X_i)$ and $h(X_i)$ around estimates $\hat{X}_{i,i}$ and $\hat{X}_{i+1,i}$

State
equation

$$f(X_i) \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i} (X_i - \hat{X}_{i,i})$$

Measurement
equation

$$h(X_{i+1}) \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1,i})}{dX_i} (X_{i+1} - \hat{X}_{i+1,i})$$

Extended Kalman filter

$\hat{X}_{i,i}, \hat{X}_{i+1,i}$

Filtered and predicted estimates at time i

Let's produce Taylor series for $f(X_i)$ and $h(X_i)$ around estimates $\hat{X}_{i,i}$ and $\hat{X}_{i+1,i}$

State
equation

$$f(X_i) \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i} (X_i - \hat{X}_{i,i})$$

Measurement
equation

$$h(X_{i+1}) \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1,i})}{dX_i} (X_{i+1} - \hat{X}_{i+1,i})$$

Let's substitute these expressions for $f(X_i)$ and $h(X_{i+1})$ in state space model (1)

Extended Kalman filter

State
equation

$$X_{i+1} \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i} (X_i - \hat{X}_{i,i}) + w_i$$

Measurement
equation

$$z_{i+1} \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} (X_{i+1} - \hat{X}_{i+1,i}) + \eta_i$$

Extended Kalman filter

State
equation

$$X_{i+1} \approx f(\hat{X}_{i,i}) + \frac{df(\hat{X}_{i,i})}{dX_i} (X_i - \hat{X}_{i,i}) + w_i$$

Measurement
equation

$$z_{i+1} \approx h(\hat{X}_{i+1,i}) + \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} (X_{i+1} - \hat{X}_{i+1,i}) + \eta_i$$

State
equation

$$X_{i+1} \approx \frac{df(\hat{X}_{i,i})}{dX_i} X_i + w_i + f(\hat{X}_{i,i}) - \frac{df(\hat{X}_{i,i})}{dX_i} \hat{X}_{i,i}$$

Measurement
equation

$$z_{i+1} \approx \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} X_{i+1} + \eta_i + h(\hat{X}_{i+1,i}) - \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \hat{X}_{i+1,i}$$

Unknown terms

Known terms

Extended Kalman filter

State
equation

$$X_{i+1} \approx \frac{df(\hat{X}_{i,i})}{dX_i} X_i + w_i + u_i$$

Measurement
equation

$$z_{i+1} \approx \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} X_{i+1} + \eta_{i+1} + y_{i+1}$$

Known values

$$u_i = f(\hat{X}_{i,i}) - \frac{df(\hat{X}_{i,i})}{dX_i} \hat{X}_{i,i}$$

$$y_{i+1} = h(\hat{X}_{i+1,i}) - \frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \hat{X}_{i+1,i}$$

Recurrent algorithm of Kalman filter

① Prediction (extrapolation)

Prediction of state vector at time i

$$\hat{X}_{i+1,i} = \frac{df(\hat{X}_{i,i})}{dX_i} \hat{X}_{i,i} + u_i$$

Prediction error covariance matrix

$$P_{i+1,i} = \frac{df(\hat{X}_{i,i})}{dX_i} P_{i,i} \left(\frac{df(\hat{X}_{i,i})}{dX_i} \right)^T + Q_i$$

Recurrent algorithm of Kalman filter

① Prediction (extrapolation)

Prediction of state vector at time i

$$\hat{X}_{i+1,i} = \frac{df(\hat{X}_{i,i})}{dX_i} \hat{X}_{i,i} + u_i$$

Prediction error covariance matrix

$$P_{i+1,i} = \frac{df(\hat{X}_{i,i})}{dX_i} P_{i,i} \left(\frac{df(\hat{X}_{i,i})}{dX_i} \right)^T + Q_i$$



More accurate prediction from state equation

$$\hat{X}_{i+1,i} = f(\hat{X}_{i,i})$$

Recurrent algorithm of Kalman filter

② Filtration

Adjustment of predicted estimate

Improved estimate by incorporating a new measurement

$$\hat{X}_{i+1,i+1} = \hat{X}_{i+1,i} + K_{i+1}(z_{i+1} - h(\hat{X}_{i+1,i}))$$

Residual

Filter gain, weight of residual

$$K_{i+1} = P_{i+1,i} \left(\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right)^T \left[\left(\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right) P_{i+1,i} \left(\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right)^T + R_i \right]^{-1}$$

Filtration error covariance matrix

$$P_{i+1,i+1} = \left[I - K_{i+1} \left(\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right) \right] P_{i+1,i}$$

Laboratory work 12

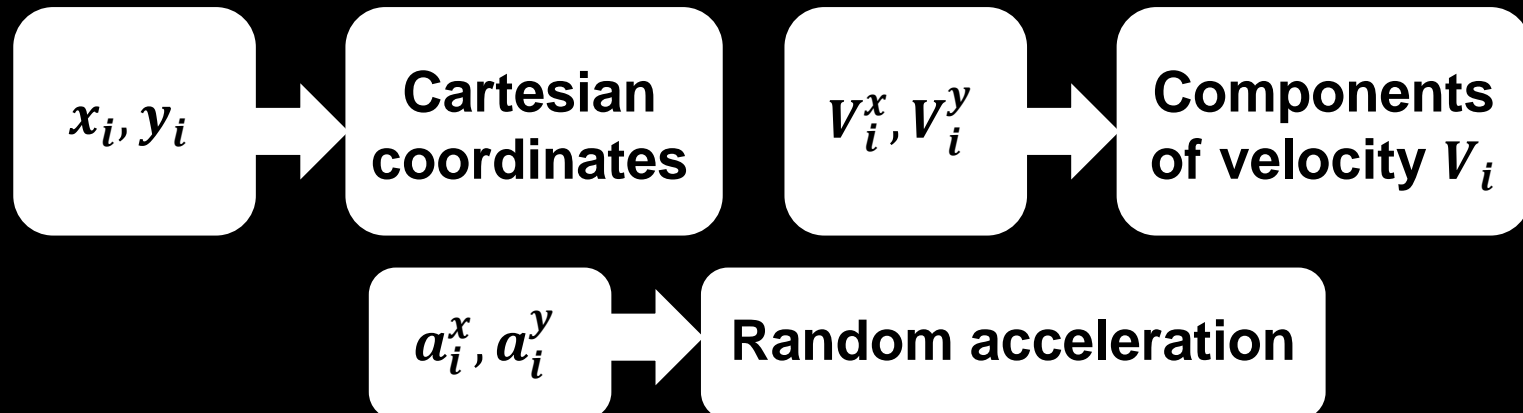
Motion model is in Cartesian coordinate system

$$x_i = x_{i-1} + V_{i-1}^x T + \frac{a_{i-1}^x T^2}{2}$$

$$V_i^x = V_{i-1}^x + a_{i-1}^x T$$

$$y_i = y_{i-1} + V_{i-1}^y T + \frac{a_{i-1}^y T^2}{2}$$

$$V_i^y = V_{i-1}^y + a_{i-1}^y T$$



State-space model, state equation

State
equation

$$X_i = \Phi_{i,i-1} X_{i-1} + G a_{i-1}$$

State
vector

$$X_i = \begin{bmatrix} x_i \\ V_i^x \\ y_i \\ V_i^y \end{bmatrix}$$

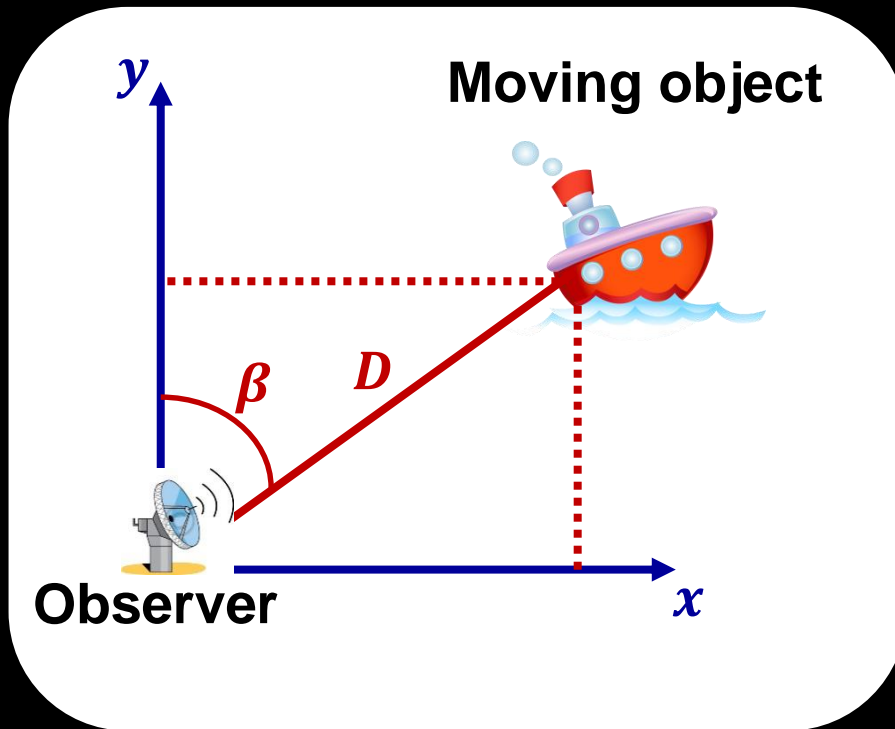
Transition
matrix

$$\Phi_{i,i-1} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Input
matrix

$$G = \begin{bmatrix} \frac{T^2}{2} & 0 \\ T & 0 \\ 0 & \frac{T^2}{2} \\ 0 & T \end{bmatrix}$$

State-space model, measurement equation



$$D = \sqrt{x^2 + y^2}$$

$$\beta = \arctg\left(\frac{x}{y}\right)$$

$$x = D \sin \beta$$
$$y = D \cos \beta$$

Measurement
vector

$$z_i = \begin{bmatrix} D_i^m \\ \beta_i^m \end{bmatrix}$$

D_i^m

Measurements of range D

β_i^m

Measurements of azimuth β

State-space model, measurement equation

Measurement equation

$$z_i = h(X_i) + \eta_i$$

$$\eta_i = \begin{bmatrix} \eta_i^D \\ \eta_i^\beta \end{bmatrix}$$

Measurement
vector z_i

$$z_i = \begin{bmatrix} D_i^m \\ \beta_i^m \end{bmatrix}$$

Nonlinear
function $h(X_i)$

$$h(X_i) = \begin{bmatrix} \sqrt{x_i^2 + y_i^2} \\ \arctg\left(\frac{x_i}{y_i}\right) \end{bmatrix}$$

Recurrent algorithm of Kalman filter

① Prediction (extrapolation)

Prediction of state vector at time $i + 1$ using i measurements

$$\hat{X}_{i+1,i} = \Phi_{i+1,i} \hat{X}_{i,i}$$

Prediction error covariance matrix

$$P_{i+1,i} = \Phi_{i+1,i} P_{i,i} \Phi_{i+1,i}^T + Q_i$$

$$P_{i+1,i} = E[(X_{i+1} - X_{i+1,i})(X_{i+1} - X_{i+1,i})^T]$$

$X_{i+1,i}$

First subscript $i + 1$
denotes time on which
the prediction is made

Second subscript i
represents the number of
measurements to get $X_{i+1,i}$

Recurrent algorithm of Kalman filter

② Filtration

Adjustment of predicted estimate

Improved estimate by incorporating a new measurement

$$\hat{X}_{i+1,i+1} = \hat{X}_{i+1,i} + K_{i+1}(z_{i+1} - h(\hat{X}_{i+1,i}))$$

Residual

Filter gain, weight of residual

$$K_{i+1} = P_{i+1,i} \left(\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right)^T \left[\left(\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right) P_{i+1,i} \left(\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right)^T + R_i \right]^{-1}$$

Filtration error covariance matrix

$$P_{i+1,i+1} = \left[I - K_{i+1} \left(\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} \right) \right] P_{i+1,i}$$

Recurrent algorithm of Kalman filter

Nonlinear
function $h(X_i)$

$$h(X_i) = \begin{bmatrix} \sqrt{x_i^2 + y_i^2} \\ \operatorname{arctg}\left(\frac{x}{y}\right) \end{bmatrix}$$

Derivative with respect
to X_{i+1} at point $\hat{X}_{i+1,i}$

$$\frac{dh(\hat{X}_{i+1,i})}{dX_{i+1}} = \begin{bmatrix} \frac{x_{i+1,i}}{\sqrt{x_{i+1,i}^2 + y_{i+1,i}^2}} & 0 & \frac{y_{i+1,i}}{\sqrt{x_{i+1,i}^2 + y_{i+1,i}^2}} & 0 \\ \frac{y_{i+1,i}}{x_{i+1,i}^2 + y_{i+1,i}^2} & 0 & -\frac{x_{i+1,i}}{x_{i+1,i}^2 + y_{i+1,i}^2} & 0 \end{bmatrix}$$