L06: Rigid Body Transformations and Sensors

* Summary of Lecture 05

$\dot{x} = f(x, u)$ Transition equation: continuous-time model

Constraints in $f(x,u)$ → wheels, joints, etc.

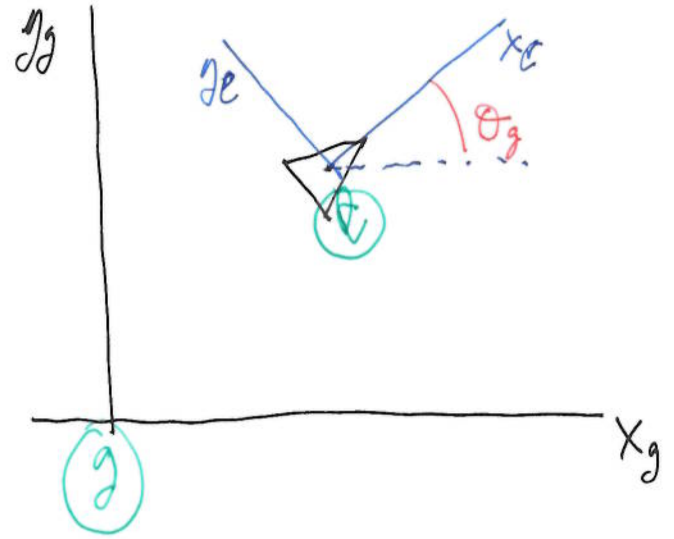
$x_t = g(x_{t-1}, u_t)$ Transition function: discrete-time model obtained by integrating $f(x,u)$. Can be non-linear.

Probabilistic model:
- Add noise to the action/state space
- linearize $g(\cdot)$
- Covariance projection $x_t \sim \mathcal{N}(g(\mu_{t-1}, u_t), G_t \Sigma G_t^T + R)$

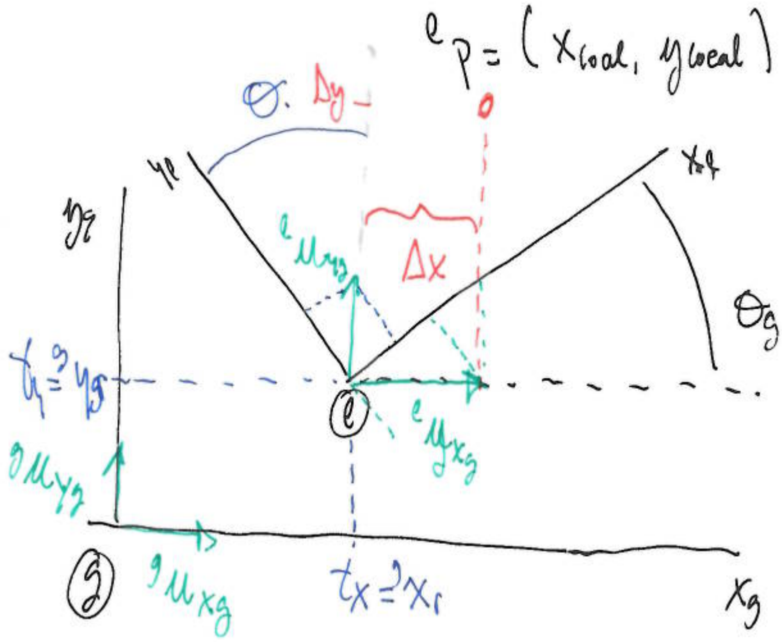
* 2D-Rigid Body Transformations

global frame → $\;{}^g x_r = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$ ← robot

This 2D pose can be interpreted as a transformation between coordinate frames. ⇒ XYT parametrization



Robot pose might be the origin of a coordinate frame e (local) with respect to a global coordin. g.



1) $^{g}u_{x_g}$, $^{g}u_{y_g}$ unitari vectors in the global frame.

2) Transform $^{g}u \rightarrow {}^{e}u$ in the new local frame.

$$^{l}u_{x_g} = [\cos\theta, -\sin\theta]^T$$

$$^{l}u_{y_g} = [\sin\theta, \cos\theta]^T$$

3) Project ^{l}p in local frame to the unitari vectors, expressed in local frame.

$$\Delta x = {}^{l}p^{T} \cdot {}^{l}u_{x_g} = [x_{local}, y_{local}] \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} = x_{loc}\cos\theta - y_{loc}\sin\theta$$

Scalar: magnitude of the vector, in the new axis.

$$\Delta y = {}^{l}p^{T} \cdot {}^{l}u_{y_g} = [x_{local}, y_{local}] \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} = x_{loc}\sin\theta + y_{loc}\cos\theta$$

4) Translate the vector $(\Delta x, \Delta y)$

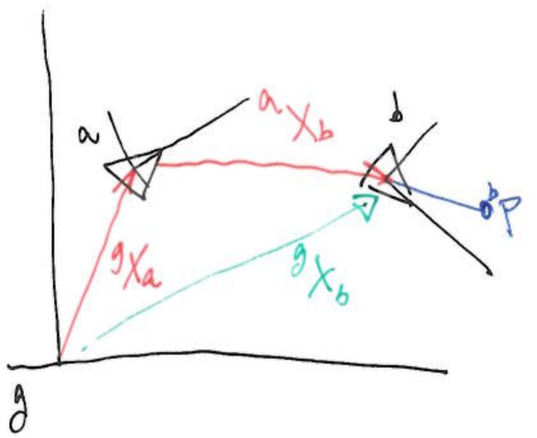
$$^{g}p = \begin{bmatrix} ^{g}x_p \\ ^{g}y_p \end{bmatrix} = \begin{bmatrix} x_{local}\cos\theta_g - y_{local}\sin\theta_g + t_x \\ x_{local}\cdot\sin\theta_g + y_{local}\cos\theta_g + t_y \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta_g & -\sin\theta_g & t_x \\ \sin\theta_g & \cos\theta_g & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{local} \\ y_{local} \\ 1 \end{bmatrix}$$

(homogeneous coordinates.)

$$ {}^{g}p = \begin{bmatrix} {}^{g}R_{\ell}(\theta_r) & t \\ 0_{1\times2} & 1 \end{bmatrix} \cdot {}^{\ell}p = {}^{g}H_{\ell} \cdot {}^{\ell}p = {}^{g}p $$

* RBT composition



$$ {}^{g}p = {}^{g}H_{b} \cdot {}^{b}p $$

Noncommutative
The order matters

$$ {}^{a}p = {}^{a}H_{b} \cdot {}^{b}p $$

$$ {}^{g}p = {}^{g}H_{a}\, {}^{a}p = {}^{g}H_{a}\, {}^{a}H_{b} \cdot {}^{b}p $$

* Smith, Self and Cheeseman notation (SSC)

Compact way to represent RBT + covariance projection.

$$ {}^{g}H_{b} = {}^{g}H_{a}\, {}^{a}H_{b} \longrightarrow X_{gb} = X_{ga} \oplus X_{ab} $$

and covariances

(cross-cov.)

$$ \Sigma_{gb} \quad \text{from} \quad \Sigma_{ga},\ \Sigma_{ab},\ \Sigma_{ga,ab} $$

Ex: from LOG :

$$ X_t = X_{t-1} \oplus u_t $$

Read SSC paper (on reading @ canvas) to expand this concept.

* Group of rotations, an introduction

from 2D-RBT $H = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$ from XYT pose.

if translation is zero, $t = [0, 0]^T$, only Rotation.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ 1 parameter.

- Properties:
  + R is an orthonormal matrix: $R \cdot R^T = I$
  + $\det R = 1$

More generally, all possible matrices R conform a group with r.t. $\boxed{\text{multiplication}}$ operation

↳ Special Orthogonal Group : $SO(n)$, $n = 2, 3$ dims

def: $SO(n) = \{ R = \mathbb{R}^{n \times n} \mid RR^T = I \wedge \det R = 1 \}$

$\therefore R_1, R_2 \in SO(n) \qquad R_1 \cdot R_2 = R_3 \in SO(n)$

- Representation of $SO(3)$ ($SO(2)$ no problem)
  - Euler angles (ϕ, θ, ψ). Problems: conventions and Gimbal lock.
  - Quaternions $q = (q_w, q_x, q_y, q_z)$ s.t. $\|q\| = 1$
  - Lie algebra $w \in \mathbb{R}^3$. Great for optimization.

* Group of RBT, an introduction

$$H = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

. translation $\Rightarrow$ homogeneous coordinates $p = [x, y, 1]^T$

We chained transformations.

Special Euclidean group $SE(n)$ (n dims.)

def: $SE(n) = \{ H = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \setminus R \in SO(n) \wedge t \in \mathbb{R}^n \}$

$\therefore T_1, T_2 \in SE(n) \qquad T_1 \cdot T_2 = T_3 \in SE(n)$

$T_2 \cdot T_1 \neq T_1 \cdot T_2$ non commutitive!

- Representation of $SE(3)$ ($SE(2) \rightarrow XYT$)
  - Decouple translation and rotation (Euler, quat)
  - Lie algebra: $\xi \in \mathbb{R}^6$. Great for optimization!

(Suggested reading: Introduction to SE(3). Blanco (@ canvas))

Sensor Models

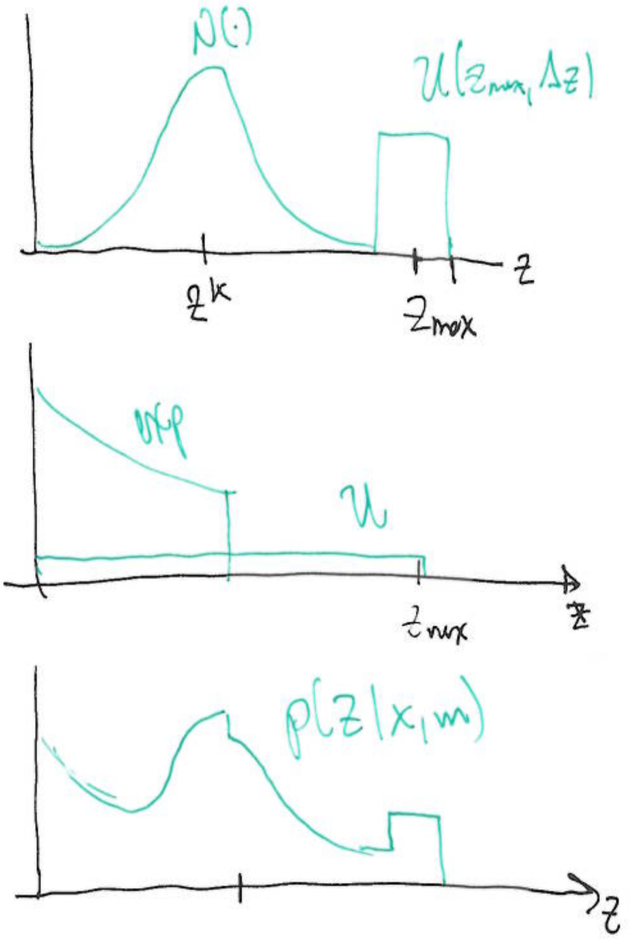
$p(z_t | x_t)$ Observation distribution (LOH)

$z_t = h(x_t, \delta_t)$ Observation function

Ex sensors: Acc, gyros, sonar, radar, laser-range-finder, camera, ...

we'll focus here.

* Location-based sensor: 1 beam.



Rigorous characterization of

$p(z|x,m)$ Likelihood built

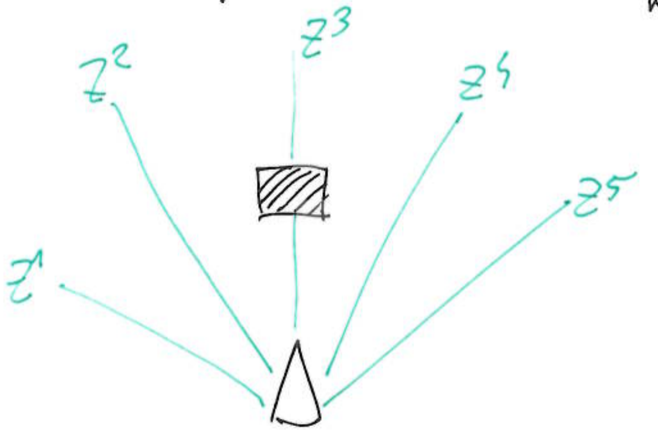
assumes a map of objects

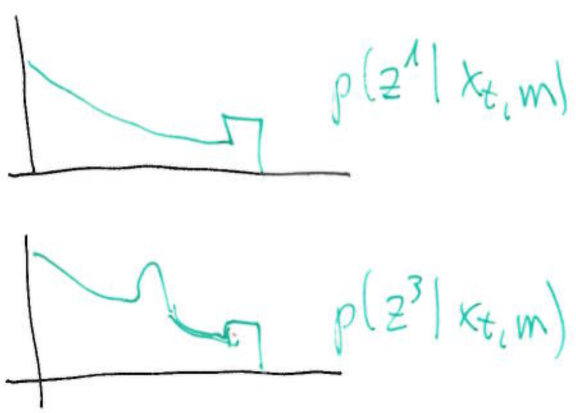
1) Normal centered at m
2) Max distance (no object)
3) Unexpected obstacles (people) before hitting m.
4) Uniform random.

Result: weighted sum.

N-beams

$$p(z^k|x_t,m) = \prod_k^K p(z^k|x_t,m)$$, independence on z^k





Problems: Non-smooth result (hitting/not hitting object)

Expensive (raycast) and unprecise.

we won't use this.

* Feature-based measurements model

extract features f from observations $f(z_t) = \{f_1, f_2, \dots f_n\}$

Ex: lines, corners, point descriptors, objects, etc...

$\underbrace{\text{lines, corners}}_{\text{laser}}$ $\underbrace{\qquad\qquad\qquad\qquad}_{\text{C. Vision.}}$

∘ Landmark : feature which corresponds to physical objects

$$(2D) \quad f_i = \begin{bmatrix} \text{'range'} \\ \text{'bearing'} \\ \text{'signature'} \end{bmatrix} = \begin{bmatrix} r \\ \phi \\ s \end{bmatrix}$$

optional to alleviate data association problem.
(color, size, $\mathbb{R}^{60}$ embedding, etc...)

→ each feature corresponds to a location $[m_{i,x}, m_{i,y}]^T$

$$\begin{bmatrix} r_t^j \\ \phi_t^j \\ s_t^j \end{bmatrix} = \begin{bmatrix} \sqrt{(m_{i,x} - x)^2 + (m_{i,y} - y)^2} \\ \operatorname{atan2}(m_{i,y} - y, m_{i,x} - x) - \theta \\ s_i \end{bmatrix} + \begin{bmatrix} \delta_{\sigma_r^2} \\ \delta_{\sigma_\phi^2} \\ \delta_{\sigma_s^2} \end{bmatrix}, \quad \delta \sim N(0, \Sigma_\delta)$$

data association i^{th} map location m_i corresponds to the j^{th} feature.

$$f^j = h(x_t, m_i) + \delta_t$$

$$f \sim N(f(z_t); h(x_t, m_i), \Sigma_\delta) \quad \text{Probabilistic model}$$

Sampling x_t from m_i: ProbRob 180

(Next class. ProbRob, Ch 3)