Using Numerical Methods to Solve 2D Shallow Water Wave Equations

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Lab Summary

In this lab, we look more deeply into numerical methods to solve differential equations, generalizing previously studied methods to solve a coupled nonlinear set of partial differential equations known as the 2D shallow wave water equations. We begin by using mathematical equations to numerically map out a “sea/ocean floor” depth profile and initial wave condition. We then apply the Crank-Nicholson and second order central differences method in two dimensions to solve the shallow water wave equations. Finally, we will use real ocean floor depth profile data from the waters around Long Island, New York to simulate a shallow wave as it approaches the island. Due to the nature of the data used, we will also be exploring how to manipulate certain kinds of data files.

Physics Background

Shallow Water Wave Equations

Shallow water waves (SWWs) are characterized by their long wavelengths, typically hundreds of kilometers, relative to the depth of the water they reside in (i.e. an ocean or local body of water). Shallow waves are fast moving in deep water, with their wave speeds being roughly proportional to the depth of water[1] ($v \propto \sqrt{gb}$ where $g = 9.8m/s^2$ and b the depth of the water). As SWWs approach a shoreline, the depth of water and speed of the wave decrease. Due to this, the amplitude of the wave increases, in a process known as wave shoaling. The amplitude of a one-dimensional SWW is governed by Green's Law:[2] $h \propto b^{-0.25}$ where h is the amplitude of the wave. In the two-dimensional case, a decay term is present to account for the wave spreading out in the xy plane. The above relations break down once the amplitude of the wave is comparable to the depth of water.

Below are the SWW equations in two dimensional form, consisting of a set of nonlinear partial differential equations proportional to the height and spatial velocities of the wave[3]

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \tag{1}$$

$$\frac{\partial}{\partial t}(hu) + \frac{\partial}{\partial x}(Du^2 + \frac{1}{2}gh^2) + \frac{\partial}{\partial y}(Duv) = -gh\frac{\partial b}{\partial x} \tag{2}$$

$$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial y}(hD^2 + \frac{1}{2}gh^2) + \frac{\partial}{\partial x}(Duv) = -gh\frac{\partial b}{\partial y} \tag{3}$$

where u, v are the depth-averaged velocities in the x and y directions respectively, b is a function describing the depth profile of the body of water (i.e. ocean floor), and $D = b + h$. For ease of convenience later on, we define $X = hu$ and $Y = hv$. The above equations are subject to the initial conditions of $h(0, x, y)$ being a function that describes the initial wave. The boundary conditions are as follows: $X, Y = 0.0$, $h(t, 0, 0) = h(t, 1, 0)$, $h(t, N, 0) = h(t, N-1, 0)$, $h(t, 0, 0) = h(t, 0, 1)$, and $h(t, 0, N) = h(t, 0, N-1)$, where N denotes the boundary of a N by N domain. Equation 1 arises from conservation of mass in the water system and Equations 2 and 3 arise from conservation of momentum[4]. The above equations do not account for frictional, shear, or viscous effects.

Computational Background

Numerical Methods

The 2D SWW equations (1-3) are a set of partial differential equations that are both coupled and nonlinear. To solve them, we discretize each equation in the time and spatial domain separately. In the spatial domain, central differences of the following form

$$f_x(t, x, y) \approx \frac{f(t, x+h, y) - f(t, x-h, y)}{2\triangle x}, \quad f_y(t, x, y) \approx \frac{f(t, x, y+h) - f(t, x, y-h)}{2\triangle y}$$

are used to estimate the spatial derivatives. For the time domain, the time derivatives are estimated using both the Crank-Nicholson and central differences methods of the form[5]

$$\frac{f_{i,j}^{t+1} - f_{i,j}^{t}}{\triangle t} = \frac{1}{2}\left[F_{i,j}^{t+1}\left(t, x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) + F_{i,j}^{t+1}\left(t, x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right]$$

which are second order accurate[6]. The combination of these two methods allow Equations 1-3 to be written in their discretized form with index notation (Park 2007). The solution is implicit in nature, due to the fact that it draws on values from a future timestep:

$$h_{i,j}^{t+1} = h_{i,j}^{t} - \left[\frac{c_x}{4}(X_{i+1,,j}^{t+1} - X_{i-1,,j}^{t+1})\right] - \left[\frac{c_y}{4}(Y_{i,,j+1}^{t+1} - Y_{i,,j-1}^{t+1})\right] - \left[\frac{c_x}{4}(X_{i+1,,j}^{t} - X_{i-1,,j}^{t})\right] - \left[\frac{c_y}{4}(Y_{i,,j+1}^{t} - Y_{i,,j-1}^{t})\right] \tag{4}$$

$$\begin{aligned} X_{i,,j}^{t+1} \quad = \quad & X_{i,,j}^{t} - \frac{c_x}{2}\left(\left(\frac{X^2}{D}\right)_{i+1,j}^{t} - \left(\frac{X^2}{D}\right)_{i-1,j}^{t} + \frac{g}{2}\left(\left(h_{i+1,j}^{t}\right)^2 - \left(h_{i-1,j}^{t}\right)^2\right)\right) - \\ & \frac{c_y}{2}\left(\left(\frac{XY}{D}\right)_{i,j+1}^{k} - \left(\frac{XY}{D}\right)_{i,j-1}^{k}\right) - \frac{gc_x b_{i,j}}{2}(h_{i+1,j}^{t} - h_{i-1,j}^{t}) \end{aligned} \tag{5}$$

$$\begin{aligned} Y_{i,,j}^{t+1} \quad = \quad & Y_{i,,j}^{t} - \frac{c_y}{2}\left(\left(\frac{Y^2}{D}\right)_{i,j+1}^{t} - \left(\frac{Y^2}{D}\right)_{i,j-1}^{t} + \frac{g}{2}\left(\left(h_{i,j+1}^{t}\right)^2 - \left(h_{i,j-1}^{t}\right)^2\right)\right) - \\ & \frac{c_x}{2}\left(\left(\frac{XY}{D}\right)_{i+1,j}^{k} - \left(\frac{XY}{D}\right)_{i-1,j}^{k}\right) - \frac{gc_y b_{i,j}}{2}(h_{i,j+1}^{t} - h_{i,j-1}^{t}) \end{aligned} \tag{6}$$

where $c_x = \triangle t/\triangle x$ and $c_y = \triangle t/\triangle y$. To solve for the height of the SWW at the next time step (Equation 4), X and Y must first be solved at the next time step, for all $i, j \in [0, N]$, where N denotes the N by N size of the domain. $\triangle x$, $\triangle y$, $\triangle t$ denote the grid spacing in the x, y and time domains. A point to notice is that the above approximations break down when $D = 0$.

Lab Questions

The process mentioned prior is sub-divided into three questions, to aid in completion of it.

1. **Create depth profiles and initial waves:** In order for the SWW approximation to be valid, the depth profile (i.e. distance from the bottom of the ocean/sea floor to sea level), must be globally decreasing, analogous to the depth of water decreasing as you approach a shore. As noted above though, the sum of the depth profile and wave height at any point cannot equal zero. The shape of the depth profile and the initial condition of the wave both affect the propagation of the wave as it reaches the shoreline.

   (a) Create a couple of depth profile functions that describe the depth of a body of water as it approaches a shoreline. For efficiency in the next questions, ensure that the minimum value of the function is close to but not equal to 0.0 and the maximum is about 100.0, which occur farthest and closest away from the shoreline, respectively.

   (b) We will now create an initial wave in the body of water. Consider a two-dimensional Gaussian, as suggested by Park 2007, of the form

   $$h(t=0,x,y) = Aexp\left(-\frac{(x-x_0)^2}{\sigma_x} - \frac{(y-y_0)^2}{\sigma_y}\right)$$

   where h is height of the wave. Try out different values of A, x_0, y_0, σ_x and σ_y to see which produces the most "wave-like" initial condition. Plot the results on a 100 by 100 three dimensional grid.

2. **Solving the 2D SWW equations using Central Differences and Crank-Nicholson Methods:** Write a code that solves the two dimensional shallow water wave equations in discretized form (Equations 4-6) for the height of the wave, h, at an arbitrary time. The general idea is to use values at one time step to solve for values at the next time step. Notice that at the boundaries of the grid, a forward or backwards difference must be used instead of a central difference, in order to correctly calculate the height. Plot the results with an overplot of the depth profile at various times to show the propagation of the wave. Use one of the depth functions and initial waves found in 1a) and 1b) and a grid size of 100 by 100. Use values of $\triangle x = \triangle y = 1.0$ and $\triangle t = 1/600$ sec. As mentioned above, start by determining M and N at a specific t, then use this to calculate h. You must also calculate D after calculating h.

   From the plots, describe the motion of the wave. Comment on the stability of the approximation. How sensitive is the simulation to the size of your time steps, with respect to stability? Is there a relationship between the number of time steps and time step value ($\triangle t$)?

3. **Real World Application (Long Island shoreline):** Using the model developed above, a simulation of a shallow wave propagating towards Long Island will be performed, using real water depth data. The data to be used is part of the NOS Hydrographic Survey, a digital acquisition survey run in partnership with NOAA. The data has been collected since 1965 and can be found here

   https://pubs.usgs.gov/of/1998/of98-502/chapt6/constopo.htm

   (a) The data itself covers an area of about 1 degree by 3 degrees latitude/longitude around Long Island and consists of 1,665,317 data points, each one containing a depth value at a specific latitude and longitude. The data is not discretized into evenly spaced grid points and is in .gz format. The program 7-Zip can be used to convert the file to a .dat file. The .dat file can then be opened with a data program, such as Notepad, and saved as a text file. Write a code that extracts the data points, fills a 100 by 100 matrix with them and plot the resulting profile as in 1a). This matrix will then be used as the depth profile, b, as mentioned prior. *Hint:* you need not use every point from the survey, one out of every ten will suffice.

   (b) Plot a shallow water wave propagating as in 2) over the Long Island depth profile found in 3a). Make note of any differences between this wave and the wave generated in 2)

Lab Solutions

Question 1

a) Below are three plots of depth profiles. The first plot uses a hyperbolic tangent function to simulate the varying slope of the sea bed, the second a linear relation, and the third a square root relation. All three plots follow the conditions outlined in the question.

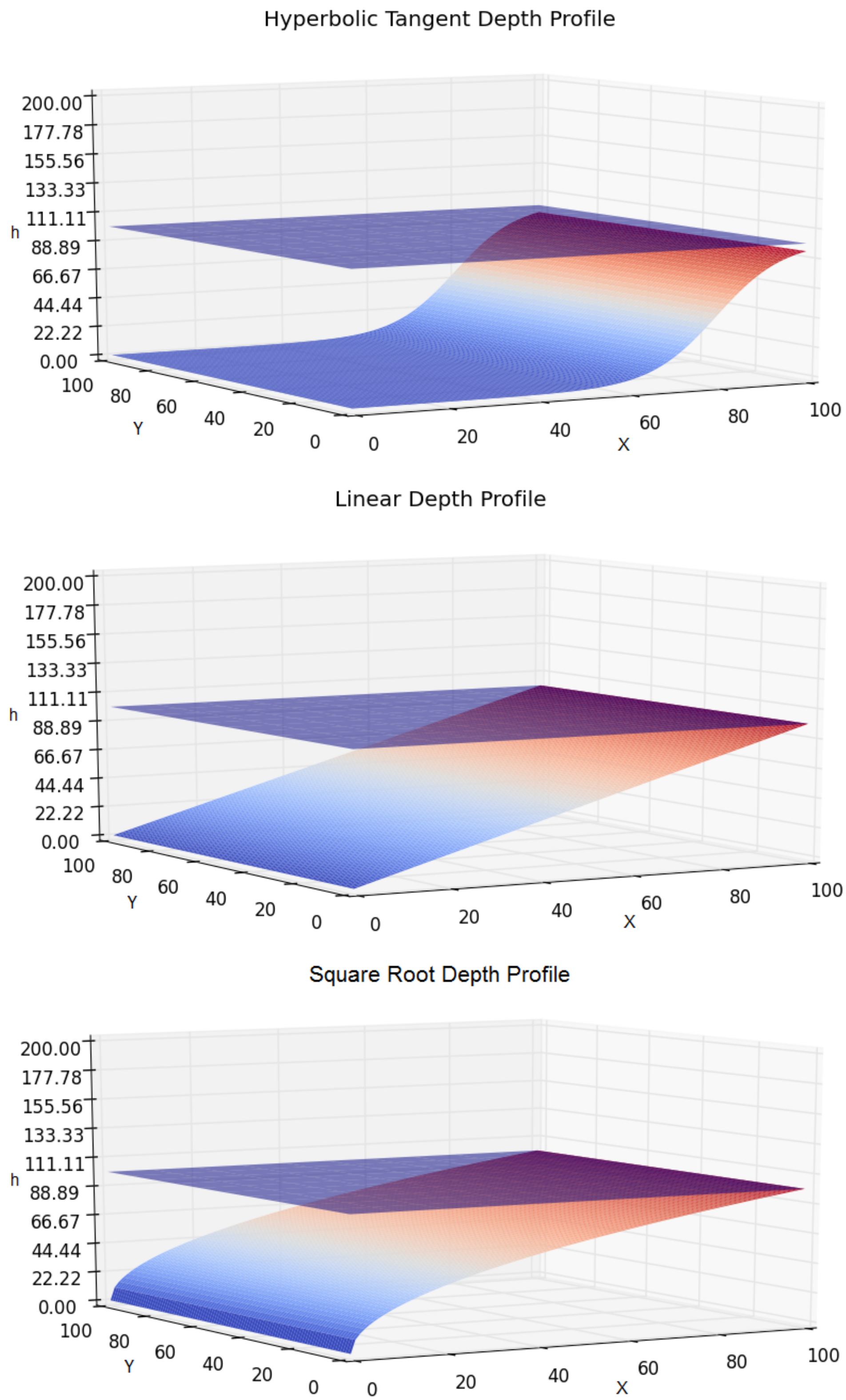
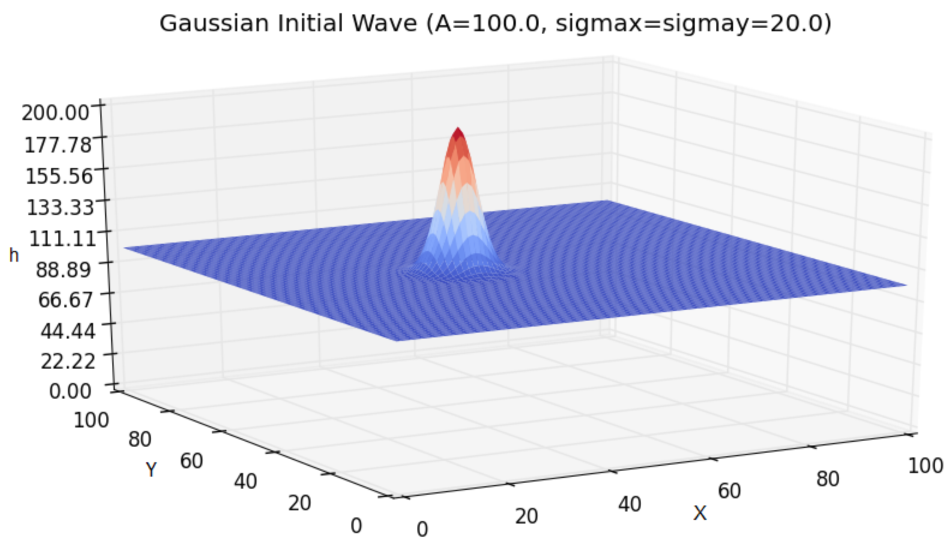


Figure 1: **Various depth profiles describing a sea/ocean bed as it approaches a shoreline. Sea level is taken to be h=100, denoted by the flat plane in all plots.**

b) Below are plots of two initial wave sources. Notice that the Gaussian equation does not describe the wave itself. As the Gaussian "falls" into the plane (i.e. propagated over time), it creates a wave, which then approaches the shore. Given this, the Gaussian equation can be seen as a wave generator of sorts.



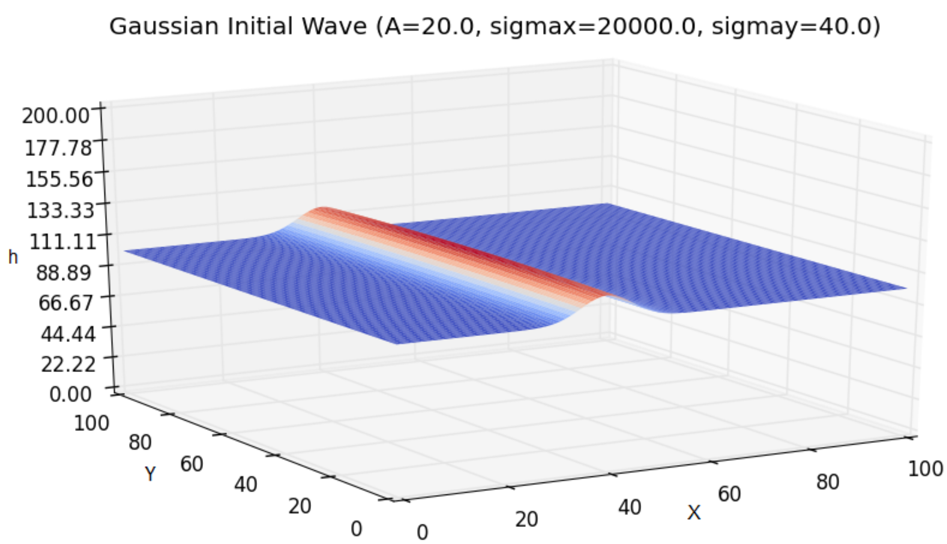


Figure 2: **Two Gaussian wave generators with differing parameters. Sea level is again taken to be h=100. Notice also that the parameters can be manipulated such that the Gaussian looks like a wave (second image).**

Question 2

Below is the shallow water wave propagation of the first initial Gaussian wave over the hyperbolic depth profile at various times.

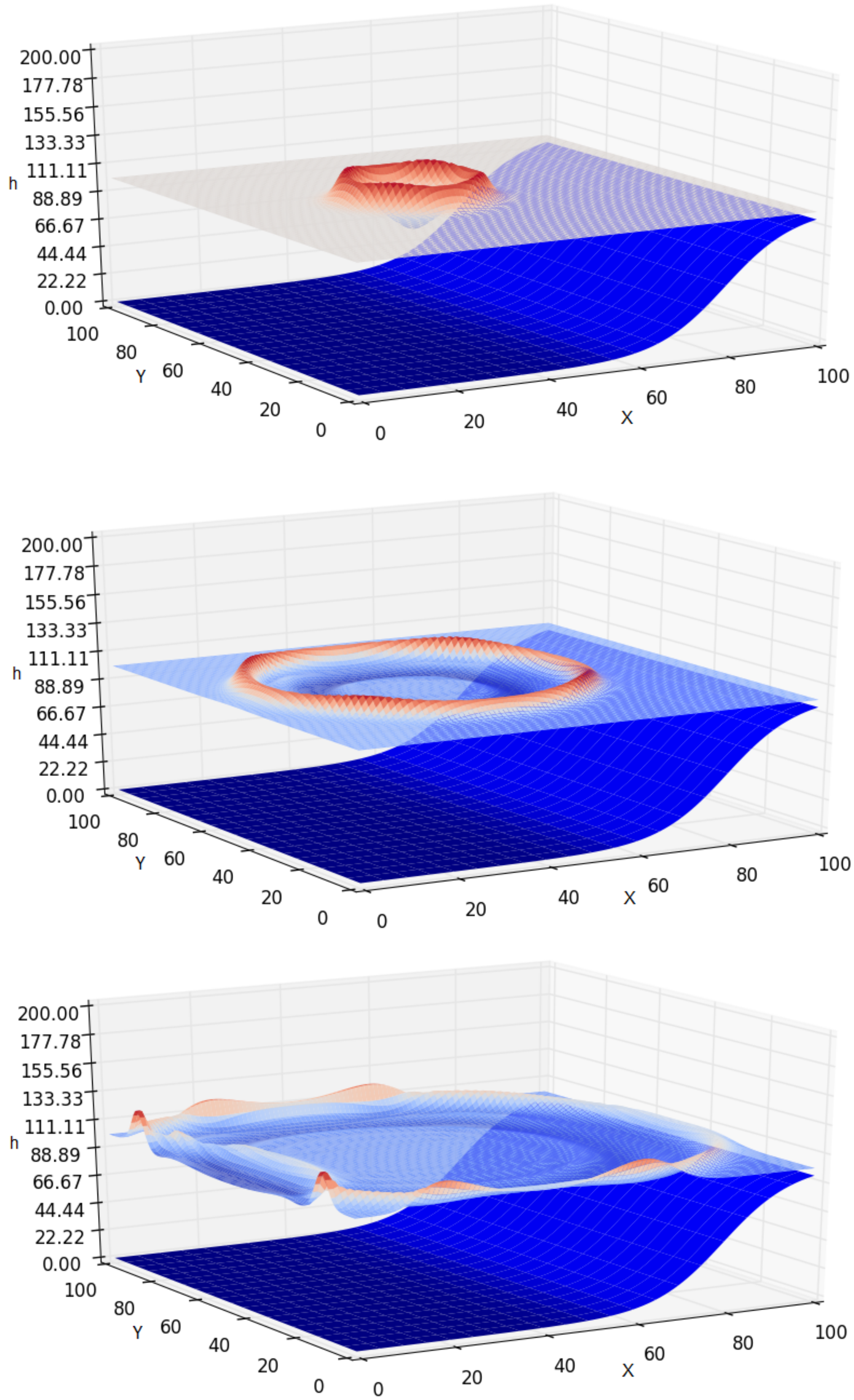


Figure 3: **Propagation of a shallow water wave at t=0.3, 0.88, and 1.53 seconds over a hyperbolic tangent depth profile. The first Gaussian in 1b) was used as the wave generator.**

As the Gaussian "falls", a radial wave is created and spreads. As the wave approaches the shore, it appears to increase in height slightly, in accordance with predictions. There does appear to be a relationship between stability and time step. At certain time steps, the above central difference/Crank-Nicolson apporximations become unstable. Through various trials, it was found that stability issues occur when the number of time steps roughly equals twice the inverse of the time step value ($N_t \sim 2/\triangle t$). The following plot shows instability in a wave simulation at t=2.23 seconds with a time step of $\triangle t = 1/600$

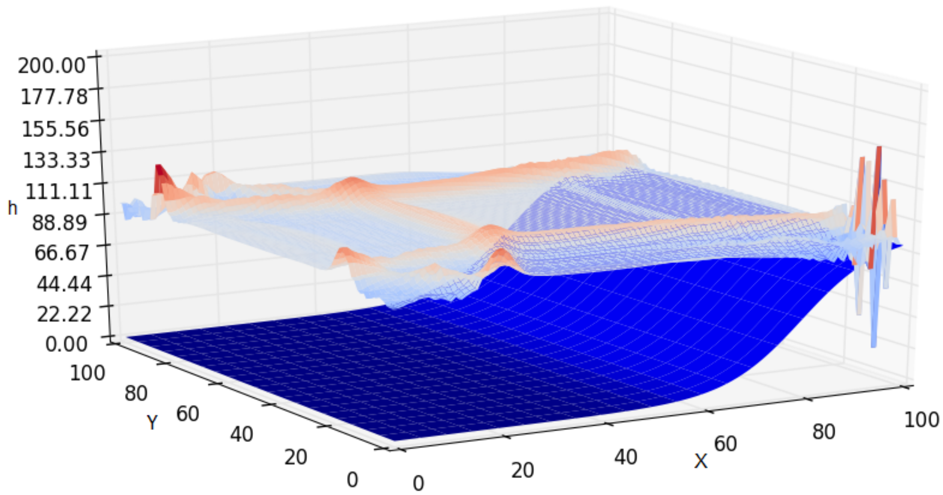


Figure 4: **Wave propagation under the conditions of 2) at a later time (t=2.23). The simulation is in the process of becoming unstable, indicated by the spike forming in the corner of the grid.**

Question 3:

a) Refer to code for description of how to convert the Long Island data into a grid format. The resulting depth profile from the Long Island data is as follows

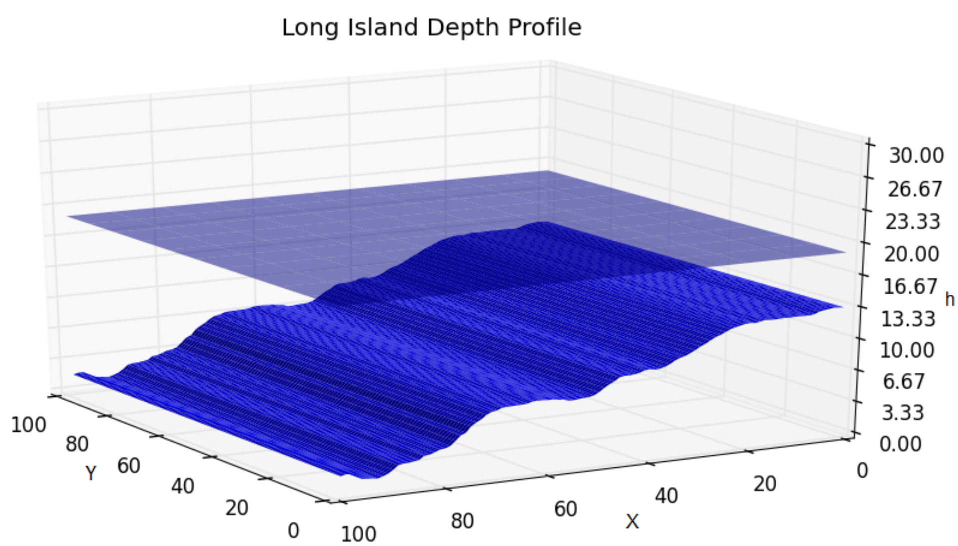


Figure 5: **The depth profile calculated using the Long Island data. Notice here that sea level is at h=18.9.**

b) The process of 2) was repeated using the Long Island data, with the following results.

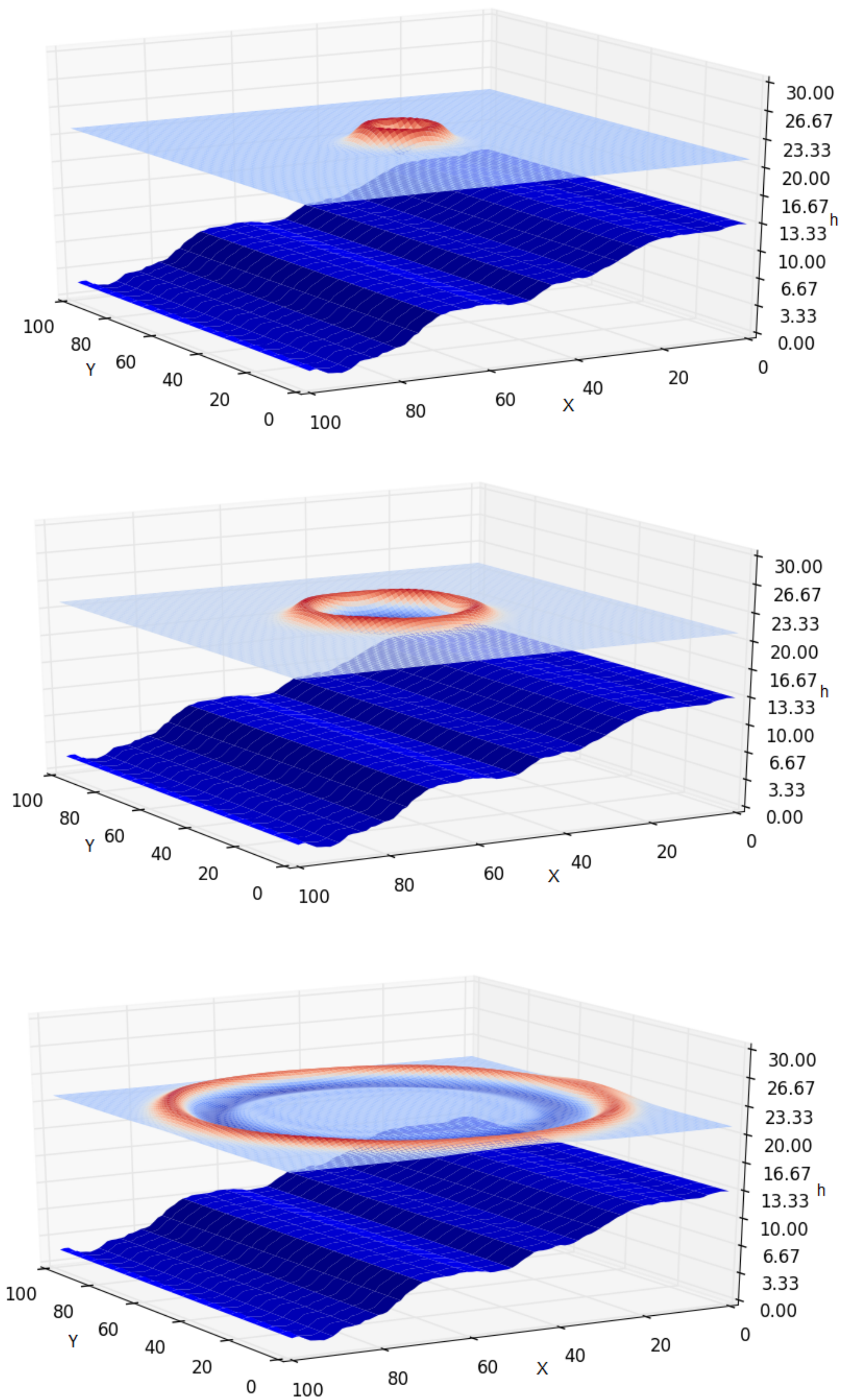


Figure 6: **Propagation of a shallow water wave in the waters surrounding Long Island at t=0.3, 0.88, and 2.48 seconds. Recall that the initial amplitude of the Gaussian wave varies from that in Figure 3 and that the same time step of 1/600 sec was used.**

As seen in 2), the wave spreads out and increases in height slightly as it approaches the shore. Note that the first two plots are at the same times as the plots in 2) however a smaller amplitude Gaussian wave was used, due to the fact that the Long Island depth profile only varied in height by about 16 meters, whereas our original simulation varied by about 100 meters. Compared to the wave propagation in 2), the wave over the Long Island depth profile appears to be moving generally slower; it just reaches the edge of the grid at $t = 2.48$ seconds, in contrast with the original simulation. Stability is also affected, as there is no indication of unstableness in the plot at $t = 2.48$ seconds, in contrast to the plot in Figure 4.

It appears that further investigation is required to accurately determine how stability of the above approximation is dependent on time step, depth profile, and perhaps initial wave amplitude.

References

[1] Tao, Terry. (2011, 13 March). *The shallow water wave equation and tsunami propagation.* Retrieved from terrytao.wordpress.com

[2] Synolakis C.E. 1991. *Green's law and the evolution of solitary waves*, Phys. Fluids, Ser. A, 3, 490–492.

[3] Kühbacher, C. 2009. *Shallow Water Derivation and Applications.* (Presentation)

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[5] Summers, P. 2012. *2D Heat Equation Modeled by Crank-Nicolson Method.*

[6] Kaus, B.J.P., *Numerische Methoden 1.*

[7] Park, J. 2007. *Numerical Simulation of Wave Propagation Using the Shallow Water Equations.*