Karlsruher Institut für Technologie

Bachelor thesis

RIGHT-ANGLED COXETER GROUPS ARE RFRS

NICLAS RIST

May 16, 2024

JProf. Dr. Claudio Llosa Isenrich

Department of Mathematics

Karlsruhe Institute of Technology

SUPERVISOR: JProf. Dr. Claudio Llosa Isenrich

LOCATION:
Karlsruhe, Germany

TIME FRAME:
Novembre 11, 2023 - May 16, 2024

CONTENTS

1 INTRODUCTION

As some sort of guidance for the reader, we want to outline the ideas of the next sections in the following. Clearly, we want to start by finding a candidate for the finite index subgroup W' of W that will satisfy the RFRS condition. This candidate will induce a manifold cover of the fundamental chamber C, corresponding to W and will be introduced in Section 3.2.

Continuing from there on, the general goal is to translate from a cofinal sequence of manifold covers of the fundamental chamber C, to a sequence of groups. To achieve this, there is no way around some orbifold theory, as the fundamental chamber C carries a natural orbifold structure. Thus, Section 3.3 will cover some aspects of the theory of orbifolds.

In Section 3.4 we will use the developed tools of the theory of orbifolds, to construct a cofinal sequence of orbifold covers, beginning in the fundamental chamber C. This sequence will in particular induce a cofinal sequence of subgroups, which we want to be a sequence of subgroups of the finite index subgroup W'.

This will be dealt with in Section 3.5, where the aforementioned sequence will somewhat be forced to be a sequence of subgroups in W'. We will see that this new cofinal sequence of groups induces a sequence of manifold covers, starting in the cover established in Section 3.2.

Up to then, we will have produced a cofinal sequence of subgroups living in W' and it remains to show that the rationally derived subgroup at step i, is a subgroup of the following subgroup at step $i+1$. This will be the content of the last Section 3.6, where we will watch loops bouncing off the faces in an orbifold.

Part I

PRELIMINARIES

2 PRELIMINARIES

2.1 COXETER GROUPS

First of all, we define the main object of interest we want to study.

Definition 2.1.1. *Let* S *be a set consisting of elements* s_i *indexed by an index set* I *with* $|I| = n < \infty$*. Let* $(m_{ij}) = M$ *be a symmetrical matrix in* $(\mathbb{N} \cup \{\infty\})^{n \times n}$*, where* $m_{ii} = 1$ *for all* i *and* $m_{ij} \geqslant 2$ *for* $i \neq j$*. Define a group* W *via the following presentation:*

$$W := \langle S \mid (s_i s_j)^{m_{ij}} = 1 \text{ for all } i, j \in I \rangle,$$

where the relations with $m_{ij} = \infty$ *are usually ommited, i.e. give trivial relations. The pair* (W, S) *is called a* Coxeter System *and* M *is called the corresponding* Coxeter Matrix*. A* Coxeter group *is a group isomorphic to a group* W*, corresponding to a Coxeter System* (W, S)*. It is generated by the set* S*.*

In this work we will be particularly interested in a special class of Coxeter groups that we call right-angled. They are defined as follows, by imposing significant constraints on the entries of the Coxeter matrix.

Definition 2.1.2. *A Coxeter System* (W, S) *is* right-angled *if, for all distinct* $i, j \in I$*, the condition* $m_{ij} \in \{2, \infty\}$ *is satisfied. In this context, the group* W *is then called a* right-angled Coxeter group (RACG)*.*

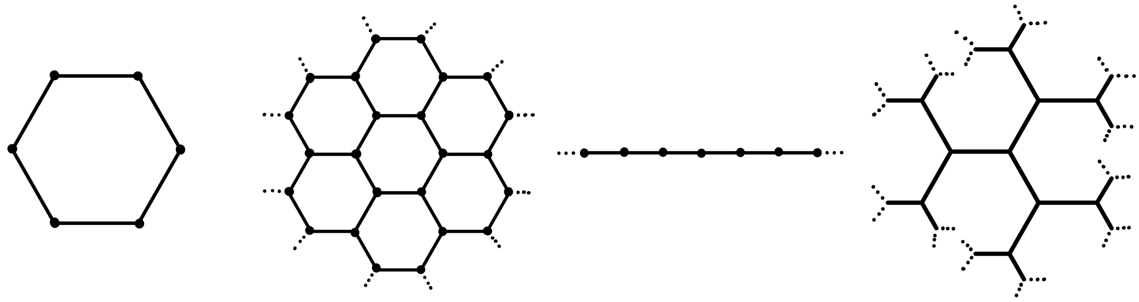
We give some important examples of Coxeter groups as well as right-angled Coxeter groups.

Example 2.1.3.

1. *Dihedral groups,* $D_{2m} \cong \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1 s_2)^m = 1 \rangle$ *are Coxeter groups for all* $m \in \mathbb{N}$*.*
2. *The triangle groups,* $\Delta(l, m, n) \cong \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^l = (st)^m = (tr)^n = 1 \rangle$ *with* l, m *and* n *integers greater or equal to 2 are Coxeter groups.*
3. *The infinite Dihedral group,* $D_\infty \cong \langle s, t \mid s^2 = t^2 = 1 \rangle$ *is a right-angled Coxeter group.*
4. *The free product,* $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \langle r, s, t \mid r^2 = s^2 = t^2 = 1 \rangle$ *is a right-angled Coxeter group.*

In Figure 2.1, small portions of the combinatorial Cayley graphs of these examples are depicted.

Figure 2.1: The combinatorial Cayley graph of ...



(a) D_6 (b) $\Delta(3,3,3)$ (c) D_∞ (d) $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

We define a special type of subgroups, called parabolic subgroups within a Coxeter group W. These subgroups are constructed from a subset of the index set I. The definition is as follows:

Definition 2.1.4. *Let (W,S) be a Coxeter System as above with finite index set I, and J be a subset of the index set I. The group $W_J := \langle\{s_j \mid j \in J\}\rangle$, generated by the s_j in S with $j \in J$, is then called a* parabolic subgroup *of W. Moreover, we call any conjugate of W_J a parabolic subgroup as well.*

Once we have constructed the representation of Coxeter groups on a vector space as well as the Tits cone in the coming section, the parabolic subgroups will be a useful tool to form a deeper understanding of these objects. We will extensively use them in Sections 2.4 and 2.5.

2.2 REPRESENTATION OF COXETER GROUPS

Given a Coxeter System (W,S), let V be a real vector space with basis $\{e_1, \dots, e_n\}$, where $n = |I| = |S|$. This provides a natural identification, $GL_n(V) \cong GL_n(\mathbb{R})$. We define a bilinear form B_W on V as follows:

$$B_W(e_i, e_j) := \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right) & , \ m_{ij} < \infty \\ -1 & , \ m_{ij} = \infty \end{cases}.$$

By Definition 2.1.1, it is assured that $m_{ij} \geqslant 2$ for distinct i,j, ensuring the cosine term is non-positive. Consequently, we have $B_W(e_i, e_j) \leqslant 0$ for distinct i,j. Furthermore, from $m_{ii} = 1$, it follows that $B_W(e_i, e_i) = 1$. Using this bilinear form, we define hyperplanes with corresponding reflections for each basis element e_i as follows:

$$H_i := \{v \in V \mid B_W(e_i, v) = 0\}, \qquad \sigma_i : V \to V, \quad v \mapsto v - 2B_W(e_i, v)e_i.$$

Theorem 2.2.1. *The map given by:*

$$\rho : W \to GL_n(V) \cong GL_n(\mathbb{R}), \quad s_i \mapsto \sigma_i$$

is an injective homomorphism and therefore a faithful representation of W.

Before we prove the homomorphism property, we recall: A map $\varphi : S \to G$ from a set S to a group G extends to a homomorphism $\hat{\varphi} : \langle S \mid R \rangle \to G$, if and only if the induced homomorphism $\overline{\varphi} : F_S \to G$ from the free group over S satisfies $\overline{\varphi}(r) = 1_G$ for every $r \in R$.

Proof. Observe that $\sigma_i^2 = \mathrm{id}$ in $GL_n(V)$ and thus to prove that ρ is a homomorphism, we apply the above to our situation to see that it suffices to show that the product $\sigma_i \sigma_j$ has order m_{ij} in $GL_n(V)$ for distinct $i, j \in I$. Also note that in the case of $m_{ij} = \infty$ there is nothing to prove as these relations are defined to be trivial in the presentation of W.

Thus, consider the two-dimensional subspace V_{ij} of V spanned by two basis vectors e_i, e_j and take a general element $v = \lambda \cdot e_i + \mu \cdot e_j$ in V_{ij} with $\lambda, \mu \in \mathbb{R}$ not simultaneously zero. As $m_{ij} < \infty$, the bilinear form B_W is positive definite by the following calculation

$$B_W(v,v) = \lambda^2 - 2\lambda\mu \cos\left(\frac{\pi}{m_{ij}}\right) + \mu^2 = \left(\lambda - \mu \cos\left(\frac{\pi}{m_{ij}}\right)\right)^2 + \mu^2 \sin^2\left(\frac{\pi}{m_{ij}}\right) > 0.$$

Identify $(V_{ij}, B_W|_{V_{ij}})$ with the euclidean plane $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$ up to a change of basis. Then σ_i, resp. σ_j act by orthogonal reflections in their corresponding hyperplanes H_i, resp. H_j intersected with V_{ij}. By having a look at the inner product of e_i and e_j

$$B_W(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right) = \cos\left(\pi - \frac{\pi}{m_{ij}}\right),$$

we observe that the angle between the two in V_{ij} is precisely $\pi - \frac{\pi}{m_{ij}}$. We conclude from this that the angle between their reflecting lines in $\frac{\pi}{m_{ij}}$ and the composition $\sigma_i \sigma_j$ turns out to be a rotation about $\frac{2\pi}{m_{ij}}$. As the composition $\sigma_i \sigma_j$ fixes the orthogonal complement $V_{ij}^{\perp}$ of V_{ij} by definition of the σ_i, we see that it has order m_{ij} on the whole space V.

This proves that ρ extends to a homomorphism. It remains to show the injectivity of ρ, which will be a consequence of a bigger result in section 2.4, see Corollary 2.4.4. □

In contrast to the above proof, in the case of $m_{ij} = \infty$, we cannot expect our bilinear form to be positive definite. Let $v \in V_{ij}$ be as in the proof, then

$$B_W(v,v) = \lambda^2 - 2\lambda\mu + \mu^2 = (\lambda - \mu)^2 \geqslant 0.$$

This shows that we can at least expect it to be positive semidefinite. The direct calculation

$$(\sigma_i \sigma_j)(e_i) = \sigma_i(e_i + 2e_j) = e_i + 2(e_i + e_j),$$

together with an inductive argument, shows that $(\sigma_i \sigma_j)^n(e_i) = e_i + 2n(e_i + e_j)$. Therefore, the composition $\sigma_i \sigma_j$ has infinite order on V_{ij} and in particular on the whole of V. The following remark is a consequence of this, together with the observation in above proof.

Remark 2.2.2. *From last paragraph and the proof of Theorem 2.2.1, we conclude that two-generator subgroups of Coxeter groups are dihedral. Either of order $2m_{ij}$ or infinite order.*

We want to extend the action of W to the dual of the vector space V. This is achieved by acting on V^* via the dual representation of ρ, which we define by

$$\rho^* : W \to GL_n(V^*), \quad w \mapsto (\rho^*(w) : V^* \to V^*,\ \varphi \mapsto \rho^*(w)(\varphi)), \quad w \in W, \varphi \in V^*.$$

We can evaluate the functional $\rho^*(w)(\varphi)$ on some $v \in V$ via $(\rho^*(w)(\varphi))(v) := \varphi(\rho(w^{-1})(v))$. Notation wise, we will simply write $w(v)$, when $w \in W$ acts on $v \in V$ via $\rho(w)(v)$. Similarly, we write $w(\varphi)$ when we mean that $w \in W$ acts on some element $\varphi \in V^*$ of the dual space, via the dual representation $\rho^*(w)(\varphi)$. As in the case of the vector space V, we want to give a definition for the notion of a hyperplane with corresponding reflection in the dual space V^* as well. By dual hyperplane, we mean a subspace $H_i^* := \{\varphi \in V^* \mid \varphi(e_i) = 0\}$, and the corresponding dual reflections will be a map from V^* to V^*, given by:

$$\sigma_i^* : V^* \to V^*, \quad \varphi \mapsto \varphi \circ \sigma_i = \varphi - 2B_W(e_i, \cdot\,)\varphi(e_i).$$

To further explore Coxeter groups and their action via this representation, we need some more notation. In particular, we want a so-called *chamber*. This should be thought of as a cone over a polytope with finitely many faces such that the reflections in its codimension one faces correspond to the generators of W under the representation.

Definition 2.2.3. *The fundamental chamber* C *of the dual representation is the set, given by*

$$C := \{\varphi \in V^* \mid \varphi(e_i) \geqslant 0\ \forall i \in I\} \subset V^*.$$

Denote by $\{e_1^*, \dots, e_n^*\}$ the dual basis of V^* corresponding to the standard basis $\{e_1, \dots, e_n\}$ of V. Then we calculate, using the σ_i^* from above:

$$\sigma_i^*(e_j^*) = e_j^* - 2B_W(e_i, \cdot)e_j^*(e_i) = \begin{cases} e_j^* & \text{for } i \neq j \\ e_j^* - 2B_W(e_j, \cdot) & \text{for } i = j \end{cases},$$

which implies that each reflection σ_i^* fixes all the hyperplanes H_j^*, for distinct indices i and j. Moreover, note that the fundamental chamber can be written in the form

$$C = \bigcap_{i \in I} \{\varphi \in V^* \mid \varphi(e_i) \geqslant 0\} = \bigcap_{i \in I} (H_i^* \cup \{\varphi \in V^* \mid \varphi(e_i) > 0\}),$$

where we observe that the sets $H_i^* \cap C$ form the pairwise distinct codimension one faces of the chamber C. The open halfspaces $\{\varphi \in V^* \mid \varphi(e_i) > 0\}$ in the latter term will be called A_i^* and using these, we define the open fundamental chamber to be the intersection of the open halfspaces:

$$\operatorname{int}(C) = \mathring{C} = \bigcap_{i \in I} A_i^*.$$

As mentioned above, we want to study the action of our Coxeter group via the dual representation, acting by reflection in the faces H_i^*. However, in general the translates of the chamber under the group action won't cover the whole of V^*, which motivates the following definition:

Definition 2.2.4. *The Tits cone is the union of all* W*-translates of the chamber,* $WC := \bigcup_{w \in W} wC \subset V^*$.

As the name suggests, the fundamental chamber is a fundamental domain for the action of W on its Tits cone WC under the dual representation ρ^*. This will be proved in Theorem 2.5.7. While the formal defintion of the Tits cone provides a rigorous foundation, it is not very insightful from a geometric perspective. As one can think about the Tits cone quite geometrically, especially in low dimensions, we will take a closer look at an explicit example in the following section. Before doing so, we end this section with the following remark.

Remark 2.2.5. *One may ask why we transport everything to the dual space, instead of working in the standard representation ρ. For this, consider the infinite dihedral group $D_\infty \cong \langle s, t \mid s^2 = t^2 = 1\rangle$. We fix a basis $\{e_1, e_2\}$ of V and obtain that the bilinear form in this basis is given by the matrix*

$$B_W = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We observe that $H_1 = \mathrm{span}\{e_1 + e_2\} = H_2$, which implies that σ_1 and σ_2 fix the same hyperplane, despite having different (-1)-Eigenspaces (namely the span of e_1, resp. e_2). Therefore, in general, working in the standard representation won't result in a chamber, giving rise to the existence of the Tits cone. Now, passing to the dual space V^* by fixing the dual basis $\{e_1^*, e_2^*\}$, consider the dual reflections σ_i^* as discussed before. By the more general calculation earlier, we obtain

$$\sigma_1^*(H_2^*) = H_2^* \quad \textit{and} \quad \sigma_2^*(H_1^*) = H_1^*.$$

And furthermore, we note that for all $i, j \in \{1, 2\}$

$$\sigma_i^*(B_W(e_j, \cdot)) = B_W(e_j, \cdot) - 2B_W(e_i, \cdot)B_W(e_j, e_i) = -B_W(e_j, \cdot),$$

using that $B_W(e_j, \cdot) = -B_W(e_i, \cdot)$. This shows that both dual reflections have the same (-1)-Eigenspace, but fix different hyperplanes (i. e., have different $(+1)$-Eigenspaces), resulting in a chamber as wished. We conclude with the following picture.

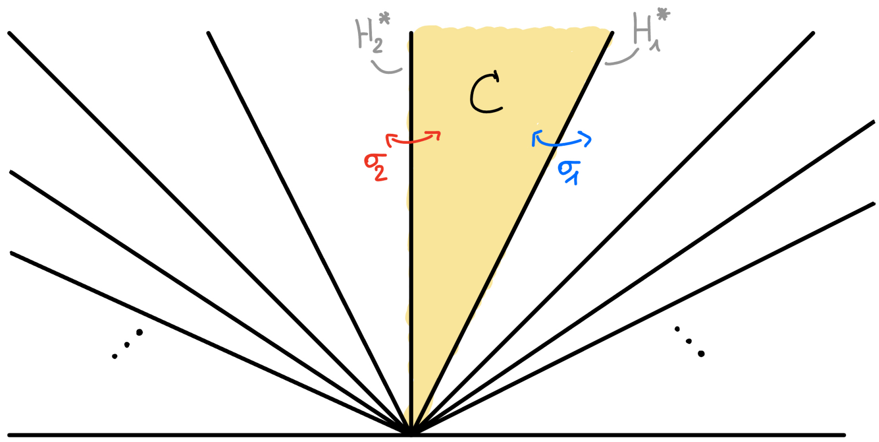


Figure 2.2: The chamber of D_∞

2.3 THE TITS CONE – AN EXAMPLE

As an example we take a closer look at the free product $W \cong \langle r, s, t \mid r^2 = s^2 = t^2 = 1\rangle$, a right-angled Coxeter group from Example 2.1.3. We fix the basis $\{e_1, e_2, e_3\}$ and identify V with $\mathbb{R}^3$. In this basis, the bilinear form B_W is given by the matrix

$$B_W = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

By the spectral theorem we find a basis of orthonormal Eigenvectors, in which B_W is a diagonal matrix with its eigenvalues as entries. Using the Gram-Schmidt procedure, we get

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In other words, we have $V^T B_W V = D$ with $V \in O(n)$ and $D = \operatorname{diag}(2, 2, -1)$. Now, since we have a diagonal matrix, we can multiply the entries by squares, since the resulting matrix will be congruent to the given one:

Let $A = \operatorname{diag}(\mu_1, \dots, \mu_n) \in \mathbb{R}^{n\times n}$,

$$S = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \in (\mathbb{R} \setminus \{0\})^{n\times n} \implies S^T A S = \operatorname{diag}(\lambda_1^2 \mu_1, \dots, \lambda_n^2 \mu_n).$$

To apply this and further transform our matrix D, define the invertible matrix T as follows

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which then implies $\widetilde{D} := T(V^T B_W V)T = \operatorname{diag}(1, 1, -1)$. The images of the basis vectors $\{e_1, e_2, e_3\}$ are given by the three columns of the matrix TV^T, namely:

$$\widetilde{e}_1 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \ \widetilde{e}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \text{ and } \widetilde{e}_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Note that we have $V \cong \mathbb{R}^3$, equipped with the inner product

$$\langle x, y\rangle_{2,1} := x^T \widetilde{D} y = x_1y_1 + x_2y_2 - x_3y_3.$$

Since $\langle \widetilde{e}_i, \widetilde{e}_i\rangle_{2,1} = 0$ for all $i \in \{1, 2, 3\}$, we have that these three vectors span an ideal triangle in the hyperboloid model of $\mathbb{H}^2$, given by $\langle x, y\rangle_{2,1} = -1$. One gets the ideal triangle by intersecting the hyperboloid with the hyperplanes spanned by each two of the $\widetilde{e}_i$ (they will only intersect in the surrounding cone of the hyperboloid).

Given these new coordinates under the transformation TV^T, the Tits cone will be given by $x_1^2 + x_2^2 - x_3^2 < 0$ union the images of $\tilde{e}_1, \tilde{e}_2$ and $\tilde{e}_3$ under the reflection in the sides of the chamber, i.e. in the sides of the ideal triangle. Moreover, we get a subgroup of $O(2,1)_+$ generated by the following three matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{5}{3} & -\frac{4}{3} \\ 0 & \frac{4}{3} & \frac{5}{3} \end{pmatrix}, \begin{pmatrix} -1 & \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{\sqrt{3}} & -\frac{2}{3} & \frac{5}{3} \end{pmatrix} \text{ and } \begin{pmatrix} -1 & -\frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{3}} & -\frac{2}{3} & \frac{5}{3} \end{pmatrix}.$$

Each of the above matrices corresponds to one of the generators r, s and t of the right angled Coxeter group W under the transformation TV^T, constructed above.

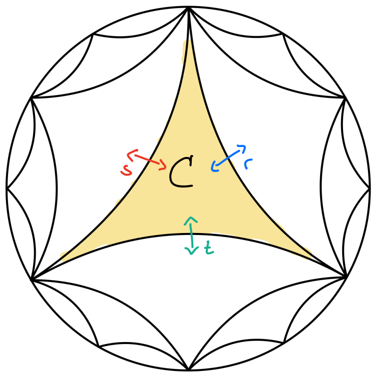


Figure 2.3: The fundamental chamber in the poincaré disc.

2.4 THE WORD METRIC AND THE FAITHFUL REPRESENTATION

Recall that for any finitely generated group $W = \langle S \rangle$, its Cayley graph induces a metric on W, relative to the generating set S. We call it the *word metric* of W relative to S and denote it by d_S. Also note that the word metric is left-invariant, meaning that for group elements $u, v, w \in W$, we have the equality $d_S(uv, uw) = d_S(v, w)$. Now, define the *word length* of an element $w \in W$ to be $\ell(w) := d_S(w, 1_W)$, the distance of an element to the neutral element of the group. Note that $\ell(w) = 0$ if and only if $w = 1_W$.

Lemma 2.4.1. *We collect some properties of the length function we will use later on.*

1. $\forall w \in W : \ \ell(w) = \ell(w^{-1})$
2. $\forall s \in S : \ \ell(s) = 1$ *and* $\ell(w) = 1 \iff w \in S^{\pm 1}$
3. $\forall v, w \in W : \ \ell(vw) \leqslant \ell(v) + \ell(w)$
4. $\forall v, w \in W : \ \ell(v) - \ell(w) \leqslant \ell(vw)$
5. $\forall w \in W, s \in S^{\pm 1} : \ \ell(w) - 1 \leqslant \ell(ws) \leqslant \ell(w) + 1$

Proof. All of the above statements follow from the fact that d_S is a left-invariant metric. $\square$

Coming back to Coxeter groups, by definition each generator $s \in S$ has order 2 in W. Therefore, we can write every non-trivial element in W as a sequence of generators in S. Note that in this sequence there might be redundencies, so that the following definition makes sense. We call an expression $w = s_{i_1} \cdots s_{i_r}$ for $i_1, \ldots, i_r \in I$ and $r \in \mathbb{N}$ *reduced*, if $\ell(w) = r$, i.e. w cannot be represented by a shorter word. These reduced expressions have the caveat of not being unique by any means.

Given a parabolic subgroup W_J of a Coxeter group W, it admits its own word metric with respect to $J \subset I$. Therefore each parabolic subgroup admits its own length function, which we denote by $\ell_J(w)$ for words w in W_J. In the following we will make use of the general fact that we have $\ell(w) \leqslant \ell_J(w)$ for all $w \in W_J$.

The length function turns out to be an important and powerful tool in studying Coxeter groups. Indeed, its role will be demonstrated in several of the forthcoming proofs, beginning with the following theorem which is a key step in proving faithfulness of our previously defined representation. But before doing so, we want to motivate the next theorem. We consider the set $\{w(e_i) \mid i \in I, w \in W\}$. Note that we fixed a basis in our representation vector space and therefore each of these elements can be written as a linear combination of the basis vectors. We will use this to define 'positive' and 'negative' elements. An element $w(e_i)$ is said to be

- positive (denoted $w(e_j) > 0$), if $w(e_j) = \sum_{i \in I} \lambda_i e_i$ with $\lambda_i \geqslant 0$ for all $i \in I$,
- negative (denoted $w(e_j) < 0$), if $w(e_j) = \sum_{i \in I} \lambda_i e_i$ with $\lambda_i \leqslant 0$ for all $i \in I$.

Theorem 2.4.2. *Let $w \in W$ and $s_i \in S$ for $i \in I$. Then $\ell(ws_i) > \ell(w)$ implies that $w(e_i) > 0$.*

Proof. The base case is trivial, as $\ell(w) = 0$ implies $w = 1_W$ and thus w fixes every basis element. Therefore, assume w is non-trivial and in reduced form. We state the induction hypothesis.

(IH) Let $v \in W$, so that $\ell(v) < \ell(w)$ and $\ell(vs_i) > \ell(v)$, then we have $v(e_i) > 0$.

Claim 1: There is a $j \in I$ such that $s_i \neq s_j$ and $\ell(ws_j) = \ell(w) - 1$.

Proof of Claim 1. Since w is in reduced form, write $w = s_{i_1} \cdots s_{i_k}$ and set $s_j = s_{i_k}$. This implies

$$\ell(ws_i) > \ell(w) > \ell(w) - 1 = \ell(ws_j),$$

and in particular $s_j \neq s_i$. This proves the first Claim. ■

Consider the parabolic subgroup $\langle s_i, s_j \rangle \leqslant W$, generated by these two distinct elements in S. Set $J := \{i, j\}$ in I and denote the subgroup by W_J. We use the decomposition $W = W/W_J \cdot W_J$ to decompose w into two parts, each living in one component. For this, consider a specific subset of the coset wW_J, given by

$$A := \{v \in W \mid vW_J = wW_J \text{ and } \ell(v) + \ell_J(v^{-1}w) = \ell(w)\}.$$

By definition, $w \in A$. With regard to the following, we choose $v \in A$ such that its length $\ell(v)$ is minimal. Now set $v_J = v^{-1}w$, which is equivalent to writing $w = vv_J$, giving us a decomposition of w as desired. Due to this decomposition, our analysis of w now boils down to studying the separate actions of v and v_J on V. First note that ws_j is contained in A as well. Clearly, $s_j^{-1} = s_j$ and thus we have that

$$s_j w^{-1} w = s_j \in W_J \text{ and } \ell(ws_j) + \ell_J(s_j) = \ell(w) - 1 + 1 = \ell(w).$$

By the choice of $\ell(v)$ to be minimal, we have $\ell(v) \leqslant \ell(ws_j) = \ell(w) - 1$, implying $\ell(v) < \ell(w)$. Hence, we are almost set up to apply the induction hypothesis to v and s_i. The last ingredient missing to do so is the following Claim.

Claim 2: For the lengths of vs_i and v, we have the relation: $\ell(vs_i) \geqslant \ell(v)$.

Proof of Claim 2. Assume towards contradiction: $\ell(vs_i) < \ell(v)$, equivalently $\ell(vs_i) = \ell(v) - 1$. Then:

$$\begin{aligned} \ell(w) = \ell(vv_J) = \ell(vs_i s_i v^{-1} w) &\leqslant \ell(vs_i) + \ell(s_i v^{-1} w) \\ &\leqslant \ell(v) - 1 + \ell_J(v^{-1} w) + 1 = \ell(v) + \ell_J(v^{-1} w) = \ell(w). \end{aligned}$$

This means equality holds throughout and in particular $\ell(w) = \ell(vs_i) + \ell_J(s_i v^{-1} w)$. But this implies vs_i belongs to A, contradicting the minimality of $\ell(v)$. Thus, the claim holds. ∎

Applying the induction hypothesis **(IH)** to v and s_i leaves us with $v(e_i) > 0$. The exact same argument applied to v and s_j shows $v(e_j) > 0$, so that we omit this here. Due to the decomposition and the fact that w lies in W_J, we are now reduced to the rank two situation. As we have seen that all two generator subgroups are dihedral, we work in the flat plane. But first, we have to observe the following claim.

Claim 3: For $v_J s_i$ and v_J, we have the relation: $\ell_J(v_J s_i) \geqslant \ell(v)$.

Proof of Claim 3. Assume towards contradiction, that $\ell_J(v_J s_i) < \ell(v_J)$ holds. Then:

$$\ell(ws_i) = \ell(vv_J s_i) \leqslant \ell(v) + \ell(v_J s_i) \leqslant \ell(v) + \ell_J(v_J s_i) < \ell(v) + \ell_J(v_J) = \ell(w),$$

which contradicts the assumption of the theorem, that $\ell(ws_i) > \ell(w)$. ∎

Moreover, this shows that every reduced expression of v_J in the parabolic subgroup W_J has to end in s_j. Otherwise, we would have $\ell(v_J) > \ell(v_J s_i)$, which contradicts *Claim 3*. To deduce the theorem, it suffices to show the following claim.

Claim 4: $(s_i s_j)(e_i) = \lambda_i e_i + \lambda_j e_j$ with $\lambda_i, \lambda_j \geqslant 0$, implying $(s_i s_j)(e_i) > 0$.

Proof of Claim 4. We already know that W_J is dihedral, either of order $2m_{ij}$ or infinite order. As pointed out above, every reduced expression of v_J - an alternating product of s_i and s_j - in W_J has to end in s_j. Using this, we distinguish the two cases.

1) $m_{ij} < \infty$: W_J is dihedral of order $2m_{ij}$. Note that the maximum length of $w \in W_J$ is precisely m_{ij} (as the Cayley graph is a cycle of length $2m_{ij}$) and the element of maximal length is represented by the reduced expressions $s_i s_j \cdots s_i s_j$ and $s_j s_i \cdots s_j s_i$. This implies that v_J has to have length strictly smaller m_{ij}, else it would have a reduced expression ending in s_i, contradicting the above. Then v_J is a product of less than $\frac{m_{ij}}{2}$ terms $s_i s_j$, and each of the products $s_i s_j$ is a rotation about $\frac{2\pi}{m_{ij}}$ towards e_j.
Using that e_i and e_j are at angle $\pi - \frac{\pi}{m_{ij}}$, we see that v_J rotates e_i at most about $\pi - \frac{2\pi}{m_{ij}}$

towards e_j. Thus, $v_J(e_i)$ still lies inside the cone spanned by e_i and e_j. It could very likely be that we have another reflection s_j, but as the angle between e_i and the reflecting line is $\frac{\pi}{2} - \frac{\pi}{m_{ij}}$, the resulting vector still lies in the cone.

2) $m_{ij} = \infty$: In the case of the infinite dihedral group, we have previously seen that $(s_i s_j)^n(e_i) = e_i + 2n(e_i + e_j) = (2n+1)e_i + 2ne_j$. Thus, $\lambda_i, \lambda_j \geqslant 0$ with $|\lambda_i - \lambda_j| = 1$.

This proves the last Claim. ■

Now that we have established that $w = vv_J$ holds, for the element $w(e_i)$ we have

$$w(e_i) = (vv_J)(e_i) = v(\lambda_i e_i + \lambda_j e_j) = \lambda_i \underbrace{v(e_i)}_{>0} + \lambda_j \underbrace{v(e_j)}_{>0}.$$

Claim 4 implies $\lambda_i, \lambda_j \geqslant 0$, allowing us to conclude $w(e_i) > 0$. □

The result of the previous theorem readily extends to its converse, offering valuable insights as well. We shall formally record this implication as a corollary.

Corollary 2.4.3. *In the previous theorem, set $w = \widetilde{w}s_i$ for $i \in I$. Then $\ell(\widetilde{w}s_i) < \ell(w)$ implies $\ell(\widetilde{w}s_i s_i) > \ell(\widetilde{w}s_i)$. Thus we have $(ws_i)(e_i) = -w(e_i) > 0$, or equivalently $w(e_i) < 0$.*

With Corollary 2.4.3 and Theorem 2.4.2 in hand, we will finally be able to deduce the injectivity and thus faithfulness of our representation ρ.

Corollary 2.4.4. *The homomorphism of Theorem 2.2.1 is injective and thus a faithful representation.*

Proof. Assume $w \in \ker\{\rho\}$ non-trivial. Then there is an $i \in I$ with corresponding $s_i \in S$, such that $\ell(ws_i) < \ell(w)$. Now Corollary 2.4.3 implies $w(e_i) < 0$, but $w(e_i) = e_i > 0$ as $\rho(w) = \mathrm{id}_V$, which is a contradiction and thus, the statement holds. □

2.5 THE FUNDAMENTAL DOMAIN AND STABILIZERS

In this section, we will prove that the fundamental chamber $C \subset V^*$ is indeed a fundamental domain for the action of W on its Tits cone WC. Moreover, we will work out how stabilizers of points look like and then show that the Tits cone is really a convex cone in the dual space V^*.

We start by translating some results of the previous section to the language of chambers and halfspaces. The following two lemmas capture this essence.

Lemma 2.5.1. *Let $w \in W$ and $i \in I$. The relation $\ell(s_i w) > \ell(w)$ is equivalent to saying that w leaves the open chamber $\mathring{C}$ inside the open halfspace A_i^*, corresponding to the generator s_i. Put more formally, this states that the set $w(\mathring{C})$ is a subset of the open halfspace A_i^*.*

Proof. Observe that $\ell(s_i w) > \ell(w)$ is equivalent to $\ell(w^{-1}s_i) > \ell(w^{-1})$. Then by by Theorem 2.4.2, we have that $w^{-1}(e_i) > 0$. Take an arbitrary point $\varphi \in \mathring{C}$ in the open fundamental chamber, then we have that $w(\varphi)(e_i) = \varphi(w^{-1}(e_i))$. Since the chamber C lies in A_i^* for all $i \in I$ and $\varphi \in \mathring{C}$, we see that $w(\varphi) > 0$ is equivalent to $w^{-1}(e_i) > 0$. But this is true by assumption and since φ is arbitrary, we conclude that $w(\mathring{C}) \subset A_i^*$. □

Of course this lemma also has a corresponding converse result as in the previous section.

Lemma 2.5.2. *Let $w \in W$ and $i \in I$. The relation $\ell(s_i w) < \ell(w)$ is equivalent to saying that w now moves the open chamber $\mathring{C}$ inside the open halfspace, complementary to A_i^*. More formally, this states that the set $w(\mathring{C})$ is a subset of the open halfspace $s_i(A_i^*)$, as the s_i permute A_i^* and its corresponding complementary halfspace.*

Proof. Take $w' = s_i w$ for some $i \in I$ and $w \in W$ so that $\ell(s_i w) > \ell(w)$. Then we have that the length of $s_i w' = w$ is strictly smaller then then length of w'. By applying above lemma we get that $w(\mathring{C}) \subset A_i^*$, which is equivalent to $s_i w'(\mathring{C}) \subset A_i^*$. Apply s_i to both sides leaves us with $w'(\mathring{C}) \subset s_i(A_i^*)$, which proves the lemma. □

Building onto this insight, we now want to study the action of parabolic subgroups on the Tits cone to get an understanding of the stabilizers of points. To do so, decompose the fundamental chamber into subsets corresponding to the parabolic subgroups of W as follows.

Definition 2.5.3. *Given a parabolic subgroup W_J corresponding to $J \subset I$, set*

$$C_J := \bigcap_{j \in J} H_j^* \cap \bigcap_{k \notin J} A_k^*.$$

We call these the corresponding parabolic subsets *(of the fundamental chamber).*

Example 2.5.4.

1. *When the set J is empty, the corresponding parabolic subset $C_\emptyset$ coincides with the entire chamber C. Conversely, when J contains all indices, C_J reduces to the singleton $\{0\}$.*
2. *If J is a proper subset of I with cardinality one, then the corresponding subset C_J coincides with a codimension-one face of the chamber C.*

Theorem 2.5.5. *Let $w \in W$ and $J, K \subset I$ be subsets. Then $w(C_J) \cap C_K \neq \emptyset$ implies $J = K$, $w \in W_J$ and thus $w(C_J) = C_J$. In particular, the isotropy groups of the sets C_J are the parabolic subgroups W_J.*

Proof. Let $w \in W$ and $J, K \subset I$ be subsets, such that $w(C_J) \cap C_K \neq \emptyset$. The proof is by induction on the length of w. The base case $\ell(w) = 0$ is trivial, as then w is trivial.

Assume that $\ell(w) > 0$ and choose $i \in I$, such that $\ell(s_i w) < \ell(w)$. Writing $w = s_i(s_i w)$, by Lemma 2.5.2 we know that w moves the open chamber $\mathring{C}$ into the open halfspace $s_i(A_i^*)$, i.e. $w(\mathring{C}) \subset s_i(A_i^*)$. Now using the continuity of the group action, we note that $w(C) \subset \overline{s_i(A_i^*)}$. Recall that by definition, the fundamental chamber C lies in the halfspaces $\overline{A_i^*}$ for all $i \in I$. Thus, we record that $w(C) \cap C \subset H_i^*$ and since s_i fixes the corresponding H_i^* by definition, it fixes every point in the intersection of C and its translate $w(C)$. Note that the sets C_J and C_K are subsets of the fundamental chamber C and therefore, s_i fixes every point in the non-empty set $w(C_J) \cap C_K$. But if s_i fixes some point φ in C_K, we calculate

$$\varphi(e_i) = s_i(\varphi)(e_i) = \varphi(s_i(e_i)) = -\varphi(e_i) \implies \varphi(e_i) = 0 \iff \varphi \in H_i^*.$$

From what we can deduce that $i \in K$, respectively $s_i \in W_K$. Using this together with the assumption, we get that $s_i w(C_J) \cap C_K = s_i(w(C_J) \cap C_K)$ is non-empty. We apply the induction hypothesis to the element $s_i w$, to see that $J = K$ and $s_i w \in W_J$. Finally, since $s_i \in W_J = W_K$, we have that $s_i w \in W_J$ implies $w \in W_J$, proving the theorem. □

Before proceeding with proving that the fundamental chamber lives up to its name, we want to clearly state what is meant by a fundamental domain.

Definition 2.5.6. *Let G be a group, acting on a topological space X. We call a closed subset $F \subset X$ a* fundamental domain, *if for each $x \in F$ its orbit $\mathrm{Orb}(x)$ intersects F in exactly one point.*

Note that, by definition of the Tits cone WC, every W-orbit of a point $\varphi \in C$ meets the fundamental chamber C in at least one point, namely φ. Thus, it suffices to proof that each W-orbit meets C in at most one point, to prove the following theorem.

Theorem 2.5.7. *The fundamental chamber is a fundamental domain for the action of the Coxeter group W on its Tits cone WC, justifying its name.*

Proof. Assume that $\varphi, \psi \in C$ lie in the same W-orbit, but in different parabolic subsets C_J, respectively C_K of the fundamental chamber. Since they lie in the same orbit, there is a $w \in W$ with $\varphi = w(\psi)$. Thus, the intersection $w(C_J) \cap C_K$ is non-empty and Theorem 2.5.5 implies equality of J and K, as well as $w \in W_J$. We deduce $\varphi = w(\psi) = \psi$. Thus, every W-orbit of a point $\varphi \in C$ meets the fundamental chamber C at most in φ, proving the theorem. □

Define a set $\mathcal{C}$ as the union of all translates of possible parabolic subsets C_J i. e., define $\mathcal{C}$ by

$$\mathcal{C} := \bigcup_{J \subset I} \bigcup_{w \in W/W_J} w(C_J).$$

We want to emphasize here that by Theorem 2.5.5 the sets of the form $w(C_J)$ in the Tits cone WC are all disjoint for different $J \subset I$ and w ranging over the coset W/W_J. Thus, the sets of $\mathcal{C}$ form a partition of the Tits cone. This decomposition (altough not into chambers) is a key component in the following theorem.

Theorem 2.5.8. *The Tits cone WC is a convex cone, and every closed line segment in the Tits cone meets only finitely many of the sets in $\mathcal{C}$.*

Proof. First note that the fundamental chamber is a convex cone as the intersection of the finitely many closed halfspaces $\overline{A_i^*}$. This implies that the Tits cone is a cone as well. We will prove the convexity by showing that every closed segment between any two points in the Tits cone is contained in it. Furthermore, we will prove that these segments can be covered by finitely many of the sets in the above defined union $\mathcal{C}$, implying the latter statement.

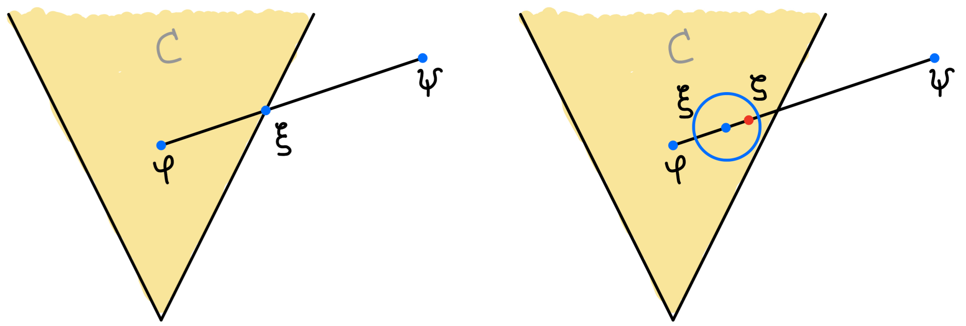
Consider the closed segment $[\varphi, \psi]$ with $\varphi, \psi \in WC$ and without loss of generality, assume the endpoints lie in different chambers C and $w(C)$ for $w \in W$. Proceed by induction on the word length $\ell(w)$. The base case $\ell(w) = 0$ reduces to $\varphi, \psi \in C$. Since C is convex and can trivially be covered by finitely many of the C_J, this case has been dealt with.

Therefore, we now assume $\ell(w) > 0$. Intersect the segment $[\varphi, \psi]$ with the fundamental chamber C, to receive two new segments $[\varphi, \xi]$ and $[\xi, \psi]$ (cf. Figure 2.4a). The first segment can be covered by finitely many of the sets in $\mathcal{C}$, as it lies inside the fundamental chamber C. Thus, we need to show that we can cover the second segment $[\xi, \psi]$ by finitely many of these sets. Assume further, we have a $J \subset I$, such that $\psi \in s_i(A_i^*)$ for an $i \in J$ and $\psi \in \overline{A_i^*}$ for all $i \notin J$. Then we have that $\psi \notin C$.

Claim: ξ has to lie in one of the codimension-one faces H_i^*, $i \in J$.

Proof of Claim. Assume that ξ lies in the open halfspace A_i^* for some $i \in J$. Then every point ζ in the intersection of a neighborhood of ξ, contained in A_i^*, with the segment $[\xi, \psi]$ has to also lie in A_i^*. Clearly, $\zeta \in \overline{A_i^*}$ for $i \notin J$ holds as well, implying that $\zeta \in C$. But this is a contradiction to the decomposition of the initial segment $[\varphi, \psi]$. Thus, ξ has to lie in one of the H_i^*. ■

Using the assumptions $\psi \in s_i(A_i^*)$ and $\psi \in w(C)$, we deduce $w(C) \subset \overline{s_i(A_i^*)}$, hence by continuity of the action $w(\mathring{C}) \subset s_i(A_i^*)$. By Lemma 2.5.2 this is equivalent to $\ell(s_i w) < \ell(w)$ and we are set up to apply the induction hypothesis to ξ and $s_i(\psi) \in s_i w(C)$. This produces a cover of $[\xi, s_i(\psi)]$ by finitely many sets in $\mathcal{C}$. But since we established that ξ has to lie in H_i^*, translation by s_i gives $[s_i(\xi), s_i^2(\psi)] = [\xi, \psi]$, and thus we can cover this segment by finitely many sets as well. The result follows. □



(a) A path through the Tits cone. (b) A neighborhood of ξ.

Figure 2.4: A schematic picture to the proof.

To summarize, the essence of this section is that first of all the fundamental chamber is indeed a fundamental domain of the action of a Coxeter group on its Tits cone. This was shown in Theorem 2.5.7. Furthermore, we want to emphasize that each point in the interior of the fundamental chamber has trivial stabilizer, which is a direct consequence of Theorem 2.5.5. Moreover, every point in a parabolic subset C_J of C, for J a subset of I, is stabilized by the corresponding parabolic subgroup W_J.

2.6 COVERING ACTIONS – A TOOLBOX

In this section we will assemble a set of tools, mostly coming from algebraic topology, connecting group actions to the notion of coverings. They will be somewhat essential in the proof of the main theorem in the next chapter. We start by saying what is meant by a covering.

Definition 2.6.1. *Let* X, Y *be topological spaces. A continuous map* $p : Y \to X$ *is a* covering map *and* Y *a* covering space *for* X*, if every point* x *in* X *has an open neighborhood* U*, such that the preimage of* U *under* p *is a disjoint union of open sets* U_i *in* Y *for an index set* I*. Furthermore, the map* p *has to be a local homeomorphism, meaning the restriction of* p *to the* U_i *is a homeomorphism onto its image* $p(U_i)$*. Then,* U *is called* evenly covered *and* $|I|$ *the degree of the covering, while the open sets* U_i *are called the* sheets *over* U *and the preimage of an* x *in* X *is called a* fiber *of* x*.*

It turns out that restricting the action of a group on a topological space in the right way will give coverings by quotienting out the action. We make this precise in the following.

Definition 2.6.2. *Let* G *be a group, acting on a space* X*. The action is said to be* properly discontinuous*, if every point in* X *has a neighborhood* U *such that the set* $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ *is finite.*

Definition 2.6.3. *Let* G *be a group, acting on a space* X*. The action is said to be a* covering (space) action *if every point has a neighborhood* U *such that the set* $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ *consists only of the neutral element.*

Obviously, this last condition is even more restrictive. Yet, at least for Hausdorff spaces, there is a connection between the two. In addition to acting properly discontinuously, we have to demand the action to be free.

Lemma 2.6.4. *Let* G *be a group that acts freely and properly discontinuously on a Hausdorff space* X*. Then the action of* G *is a covering action in the above sense.*

Proof. Let X be a Hausdorff space and G be a group acting freely, properly discontinuously on X. Then for an open neighborhood U of $x \in X$, the set $M := \{g \in G \mid gU \cap U \neq \emptyset\}$ is finite. For these $g \in M$ we pick pairwise disjoint Neighborhoods V_g of gx, which is possible since X is Hausdorff and G acts freely (thus $gx \neq x$). Finally, set $V = \left(\bigcap_{g \in M} g^{-1}V_g\right) \cap U$, which is open as finite intersection of open sets and by definition a neighborhood of x. □

To finally construct coverings from group actions and connect them in some sense, we need a last condition on the space the group acts on.

Definition 2.6.5. *A path-connected topological space* X *is called* simply connected*, if* $\pi_1(X) \cong \{1\}$*.*

Given a group G acting on a space X, we will denote by $\mathrm{Orb}_G(x)$ the orbit of a point $x \in X$ under the action of the group G, i.e. $\mathrm{Orb}_G(x) := \{gx \mid g \in G\} \subset X$.

Lemma 2.6.6. *Let* G *be a group, acting by a covering action on a simply connected topological space* X. *Then the quotient map* $p_G : (X, x_0) \to (G\backslash X, \mathrm{Orb}(x_0))$ *is a covering map and* $\pi_1(G\backslash X, \mathrm{Orb}(x_0)) \cong G$.

Proof. Let U be an open neighborhood of $x \in X$, such that $\{g \in G \mid gU \cap U \neq \emptyset\} = \{1\}$.

Claim 1: The map p_G restricted to U is a continuous bijection onto its image $p_G(U)$.

Proof of Claim 1.

- Continuity: $V \subset G\backslash X$ is open if and only if $p_G^{-1}(V)$ is open and thus p_G is continuous.
- Surjectivity: Since orbits of points are non-empty, p_G is surjective.
- Injectivity: Assume $x, y \in X$ distinct with $p_G(x) = p_G(y)$. But then there is a non-trivial element $g \in G$, $g \neq 1_G$ with $gx = y$, implying $g(U) \cap U \neq \emptyset$. This is a contradiction to G acting by a covering action. ■

Claim 2: Every point $\mathrm{Orb}(x) \in G\backslash X$ has a neighborhood $V \subset G\backslash X$ with $p_G^{-1}(V) = \bigsqcup_{g\in G} gU$.

Proof of Claim 2. By assumption the sets gU are all disjoint neighborhoods of $gx \in X$ for all $g \in G$. Moreover, for $V := p_G(U) \subset G\backslash X$, we have $p_G^{-1}(V) = \bigsqcup_{g\in G} gU$, proving the Claim. ■

Claim 3: The map $p_G|_{gU} \to V = p_G(U)$ is a homeomorphism for all $g \in G$.

Proof of Claim 3. Note that the action of G on X is by homeomorphisms and $U \subset X$ is open. Then the union $\bigsqcup_{g\in G} gU$ is open as well and since $p_G^{-1}(V) = \bigsqcup_{g\in G} gU$, the set V is open in the quotient topology of $G\backslash X$. Therefore, the map p_G restricted to gU for $g \in G$ is an open map and hence, by *Claim 1* a homeomorphism. ■

Now all conditions in Definition 2.6.1 are satisfied, so p_G is a covering map. For the last part of the statement, note that X is simply connected, implying that it is the universal cover of $G\backslash X$. By definition, G is the group of Deck transformations, thus $\pi_1(G\backslash X, \mathrm{Orb}(x_0)) \cong G$. □

3 THE MAIN THEOREM

Let us state and recall the main theorem we want to prove.

Theorem. *A finitely generated right-angled Coxeter group W has a finite index subgroup W', such that W' is residually finite and rationally solvable (RFRS).*

The goal of this whole chapter is to explain Agol's proof of this theorem, using the preliminaries and tools we have developed. We will break the proof down into smaller pieces and will discuss in each section a specific aspect of the proof.

3.1 THE RFRS PROPERTY

We should state clearly what we have to show. This is essentially the construction of a finite index subgroup, that is RFRS. But what does it mean for a group G to be RFRS? We shall start by giving the definition of the RFRS condition to shed some light onto our goal.

Definition 3.1.1. *Let G be a group. Then, G satisfies the RFRS condition, if there is a sequence of subgroups $G = G_0 > G_1 > \ldots$, satisfying the following conditions.*

1. *for each i, $G_i \lhd G$ is a normal subgroup of G,*
2. *for each i, the index of G_i in G, $[G : G_i]$ is finite,*
3. *the intersection $\bigcap_i G_i = \{1\}$ is the trivial group, meaning the sequence is cofinal, and*
4. *for each i, the group $(G_i)_r^{(1)} := \ker\{G_i \to \mathbb{Q} \otimes_{\mathbb{Z}} (G_i)_{ab}\}$ is a subgroup of G_{i+1}.*

Observe that in practice, it suffices to show that conditions 2 to 4 hold. To justify this, let $G = G_0 > G_1 > \cdots$ be a cofinal sequence of finite index subgroups $G_i < G$ with $(G_i)_r^{(1)} \leqslant G_{i+1}$. Then, for each i, we pass to the core of G_i in G, which is defined by $\mathrm{Core}(G_i) := \bigcap_{g \in G} gG_ig^{-1}$. By construction, the core of a subgroup G_i is normal in G and furthermore, we claim:

Claim: If G_i has finite index in G, its core will also have finite index in G.

Proof of Claim. Let G_i be a subgroup in G with $[G : G_i] < \infty$. Then, G acts (by left multiplication) on the left cosets of G_i in G and the kernel of this action consists of the elements $h \in G$ with

$$hgG_i = gG_i \iff g^{-1}hgG_i = G_i \iff g^{-1}hg \in G_i \iff h \in gG_ig^{-1} \quad \forall g \in G.$$

We note that the kernel is exactly the core of G_i. Moreover, the quotient ${}^{G}\!/\!{}_{\mathrm{Core}(G_i)}$ embeds into the symmetric group $\mathrm{Sym}({}^{G}\!/\!{}_{G_i})$, as the left action on the cosets permutes them. As the order of this symmetric group is $[G : G_i]!$, we see that if G_i has finite index in G, the core has finite index as well. ∎

Thus, to see that this new sequence satisfies the RFRS condition, it remains to check that the rational derived subgroup $(\mathrm{Core}(G_i))_r^{(1)}$ is a subgroup of the group $\mathrm{Core}(G_{i+1})$.

Clearly, for $g \in G$ we have the equality $g(G_i)_r^{(1)}g^{-1} = (gG_ig^{-1})_r^{(1)}$, from which we obtain

$$(\mathrm{Core}(G_i))_r^{(1)} \leqslant \bigcap_{g\in G} (gG_ig^{-1})_r^{(1)} = \bigcap_{g\in G} g(G_i)_r^{(1)}g^{-1} \leqslant \bigcap_{g\in G} gG_{i+1}g^{-1} = \mathrm{Core}(G_{i+1}).$$

To summarize, by accepting to pass to the sequence of core subgroups in G, we can drop the first condition on the sequence of subgroups in the RFRS condition.

Remark 3.1.2. *Note that an abelian group A has a natural structure as a $\mathbb{Z}$-module. Thus, the tensor product with the rationals is well-defined and tensoring A with $\mathbb{Q}$ kills the torsion part of the abelian group A. Whence, we have that $\mathbb{Q} \otimes_{\mathbb{Z}} A \cong \mathbb{Q} \otimes_{\mathbb{Z}} {}^{A}\!/\!{}_{Torsion}$. To see this, take an elementary tensor $r \otimes a$ in $\mathbb{Q} \otimes_{\mathbb{Z}} A$ so that a is a torsion-element of order n. Then we get the following equalities,*

$$r \otimes a = r \cdot \frac{n}{n} \otimes a = r \cdot \frac{1}{n} \otimes n \cdot a = 0.$$

In particular, for the abelianization G_{ab}, the group $\mathbb{Q} \otimes_{\mathbb{Z}} G_{ab}$ is isomorphic to $\mathbb{Q} \otimes_{\mathbb{Z}} {}^{G_{ab}}\!/\!{}_{Torsion}$. Moreover, the torsion-free abelianization embeds into $\mathbb{Q} \otimes_{\mathbb{Z}} G_{ab}$ and the so called rational derived subgroup $(G)_r^{(1)}$ is the kernel of the homomorphism from G to its torsion-free abelianization tensored with the rationals.

3.2 CONSTRUCTION OF THE MANIFOLD COVER

Consider the abelianization W_{ab} of W, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Now, the abelianization yields a homomorphism $\alpha : W \to W_{\mathrm{ab}}$. In the following we will focus on its kernel $\ker\alpha$. Note that by the first homomorphism theorem, $\ker\alpha$ has finite index in W, since

$$\left|{}^{W}\!/\!{}_{\ker\alpha}\right| = \left|{}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}}\right|^n = 2^n < \infty.$$

Here we used the fact that W is finitely generated, whence $n = |I| = |S| < \infty$.

Furthermore, note that for each $J \subset I$ with W_J finite, we have an isomorphism between W_J and $({}^{\mathbb{Z}}\!/\!{}_{2\mathbb{Z}})^{|J|}$. Thus, the restriction of α to each such subgroup $\alpha|_{W_J}$ is an injective homomorphism. We now use that in the right-angled case, the isotropy subgroups of codimension-k faces are all of this form. This can be seen, as the isotropy subgroup of a codimension-k face F is generated by the reflections in the k codimension-1 faces, whose intersection forms F. As all these codimension-1 faces meet at right-angles, the generators commute pairwise. Therefore, all isotropy subgroups inject into the abelianization of W. This implies that the intersection of an isotropy subgroup with the kernel of α is trivial and consequently no isotropy group is contained in the kernel $\ker \alpha$. Since finite subgroups are contained in isotropy subgroups, the kernel $\ker \alpha$ acts freely on the Tits cone WC corresponding to W.

In particular, by Theorem 2.5.5 the action of W on the interior of its Tits cone $\mathrm{int}(WC)$ is properly discontinuous. By Theorem 2.5.8 the Tits cone is also a convex cone, implying that it has trivial fundamental group, whence is simply-connected. Having all this information, we are able to apply Lemma 2.6.6 to obtain that the following map is a covering

$$\mathrm{int}(WC) \longrightarrow \mathrm{int}(WC)/_{\ker \alpha} \qquad x \mapsto \mathrm{Orb}_{\ker \alpha}(x).$$

We conclude the results of this section in the following proposition.

Proposition 3.2.1. *Let W be a right-angled Coxeter group with corresponding Tits cone WC. Then, there is a finite index subgroup $W' \leqslant W$, acting by covering action on the interior of the Tits cone, implying that the quotient $\mathrm{int}(WC)/_{W'}$ is a manifold. It is given by $W' = \ker\{W \to W_{ab}\} = \ker \alpha$.*

3.3 SOME ORBIFOLD THEORY

In this section, we want to elaborate more on the natural orbifold structure of the fundamental chamber C. We start by giving a formal definition of an orbifold. We break the definition down into smaller pieces, starting with local models, sometimes called (orbifold) charts.

Definition 3.3.1. *A* local model *is a pair $(\widetilde{U}, \Gamma)$, where $\widetilde{U} \subset \mathbb{R}^n$ is open and Γ is a finite subgroup of the group of diffeomorphisms of $\widetilde{U}$, denoted* $\mathrm{diffeo}(\widetilde{U})$*, acting on $\widetilde{U}$. By abusing notation, we will sometimes say that the quotient $U = \widetilde{U}/_{\Gamma}$ is the local model.*

Now that we have defined the local structure of an orbifold, we want to translate between these local models. This is being made precise by orbifold maps.

Definition 3.3.2. *An* orbifold map *between local models $(\widetilde{U}_i, \Gamma_i), (\widetilde{U}_j, \Gamma_j)$ is a pair of maps $(\widetilde{\psi}, \varphi)$, consisting of a smooth map $\widetilde{\psi} : \widetilde{U}_i \to \widetilde{U}_j$ and a homomorphism of groups $\varphi : \Gamma_i \to \Gamma_j$. We enforce the map $\widetilde{\psi}$ to be φ-equivariant, meaning that for all $g \in \Gamma_i$ and all $\widetilde{x} \in \widetilde{U}_i$, $\widetilde{\psi}(g\widetilde{x}) = \varphi(g)\widetilde{\psi}(\widetilde{x})$ holds. Then $\widetilde{\psi}$ induces a map $\psi : \widetilde{U}_i/_{\Gamma_i} \to \widetilde{U}_j/_{\Gamma_j}$, between the local models. When all three of these maps are injective, we call ψ a local isomorphism.*

Now that we have these local definitions, we 'glue' them together, to obtain an orbifold. Before we do so, we recall some notions from topology.

Suppose, we are given an open cover $\{U_i\}$ of a topological space X. It is said to be *locally finite*, if every $x \in X$ admits a neighborhood N such that $N \cap U_i$ is empty for all but finitely many of the indices i. The open cover $\{U_i\}$ is a *refinement* of an open cover $\{V_j\}$ of X, if for every V_j there is a U_i with $U_i \subseteq V_j$. Now, the topological space X is said to be *paracompact*, if every open cover of X admits such a locally finite refinement.

Definition 3.3.3. *An n-dimensional* (smooth) orbifold Q *is a pair* $(X_Q, \mathcal{A})$. *The space* X_Q *is a paracompact Hausdorff space, called the* underlying space. *The set* $\mathcal{A}$ *is called an* orbifold atlas, *consisting of charts* (U_i, ϕ_i), *indexed by some set* I *and satisfying the following conditions:*

- *the* U_i *form an open cover of the underlying space* X_Q,
- *for each* U_i *there exists a local model* $\widetilde{U}_i/\Gamma_i$ *with a homeomorphism* $\phi_i : U_i \to \widetilde{U}_i/\Gamma_i$ *and*
- *charts have to be compatible, meaning that for* $U_i \subset U_j$ *the inclusion is a local isomorphism.*

To sketch the connection between manifolds and orbifolds, let us mention one more thing.

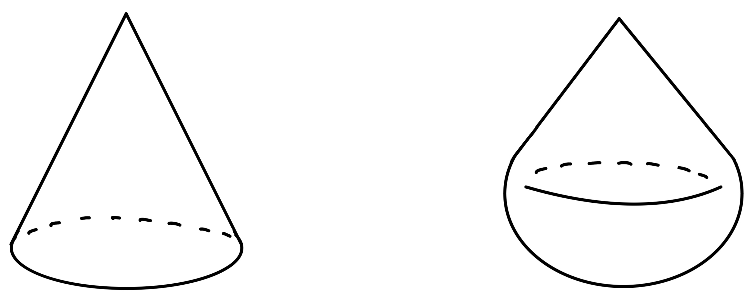
Definition 3.3.4. *The* local group $\mathrm{loc}(x)$ *of some* x *in a local model* $\widetilde{U}/\Gamma$ *is the isotropy group of any* $\widetilde{x}$ *living in* $\widetilde{U}$, *getting projected onto* x. *The* singular locus $\Sigma(Q)$ *of an orbifold* Q *consists of all points in the underlying space* X_Q *with non-trivial local group, i.e.* $\Sigma(Q) = \{x \in X_Q \mid \mathrm{loc}(x) \neq \{1\}\}$.

By this definition, we see that an orbifold with empty singular locus is just a manifold. Furthermore, when thinking about an orbifold, we can just think about the underlying space and label each element in the singular locus by its local group.

Many basic examples arise by taking the quotient relative to a properly discontinuous group action on $\mathbb{R}^n$. We want to mention at least some examples.

Example 3.3.5.

1. *Consider the flat plane* $\mathbb{R}^2$ *with the action of a cyclic group* $\mathbb{Z}/n\mathbb{Z}$ *by rotation about the origin. The arising orbifold is a cone with singular point the origin and cone angle* $\frac{2\pi}{n}$.
2. *Consider the sphere* S^2 *with the action of a cyclic group* $\mathbb{Z}/n\mathbb{Z}$ *by rotation about the north pole* N. *The arising orbifold now is called a* teardrop *with singular point, the north pole* N.



(a) The cone orbifold. **(b)** The teardrop orbifold.

The concept of orbifold coverings translates almost one to one from the ordinary case.

Definition 3.3.6. *An orbifold covering* $p : Q' \to Q$ *is a continuous map on the underlying spaces* $X_{Q'} \to X_Q$, *such that for each point* $x \in X_Q$ *there is a local model* $U = \widetilde{U}/\Gamma$ *around* x *and each component* V_i *of* $p^{-1}(U)$ *is homeomorphic to* $\widetilde{U}/\Gamma_i$ *for a subgroup* Γ_i *of* Γ. *Furthermore, the restriction* $p|_{V_i} : V_i \to U$ *corresponds to the natural projection* $\widetilde{U}/\Gamma_i \to \widetilde{U}/\Gamma$.

Remark 3.3.7. *We want to mention here, that the universal cover of an orbifold is exactly what we think of, i.e. the initial object in the category of orbifold coverings. Moreover, there is a notion of orbifold fundamental groups, which we won't define here. For us it is only important to know that they behave to orbifold covers the same as fundamental groups to coverings in the ordinary case. In particular, subgroups (of index n) of the orbifold fundamental group correspond to (n-sheeted) orbifold coverings.*

Having all this in mind, we have two ways of approaching the fundamental chamber C. One way, it obtains its natural orbifold structure as the quotient $\mathrm{int}(WC)/W$. As the Tits cone is simply connected, it is the universal cover of C and we observe that the orbifold fundamental group of C is precisely W. On the other hand, the fundamental chamber is the quotient of the manifold $\mathrm{int}(WC)/W'$ by a finite group action.

Since W' is a finite index subgroup of W, Remark 3.3.7 implies that we get a covering map p so that the diagram on the right commutes. Thus, both of the approaches to the fundamental chamber C agree in the sense that the projection map $\mathrm{int}(WC) \to \mathrm{int}(WC)/W$ factors through the manifold cover, constructed in Section 3.1.

$$\begin{tikzcd} \mathrm{int}(WC) \arrow[r] \arrow[d] & \mathrm{int}(WC)/W \\ \mathrm{int}(WC)/W' \arrow[ur, dashed, "p"'] & \end{tikzcd}$$

3.4 THE COFINAL COVER

In this section, we *sketch* the proof of the existence of a cofinal sequence of two-fold covers, arising by iteratively reflecting in faces. Some more results from the theory of orbifolds would be needed for the whole proof. Since they go beyond the scope of this work, we will state and use them without proof.

In the following, doubling along faces means the construction of a sequence of right-angled polytopes $P_k = P_{k-1} \cup g_{k-1}(P_{k-1})$, where g_{k-1} is the reflection in one of the faces of the previous polytope P_{k-1}, denoted by F_{k-1}. The following proposition will tell us, that in the case of a right-angled Coxeter group, we are able to double the fundamental chamber C in one of its faces and are still able to cover the whole Tits cone WC by reflecting in the faces of the resulting polytope.

Proposition 3.4.1. *Let $P_k \subset \mathbb{R}^n$ be a right-angled polytope with orbifold structure, coming from a doubling sequence starting in $P_0 = C$. Denote its orbifold fundamental group by $\pi_1^{orb}(P_k)$ and its universal cover by $\widetilde{P}$. Then, the fundamental group $\pi_1^{orb}(P_{k+1})$ is isomorphic to the group generated by reflections in the faces of the double P_{k+1}. Furthermore, the fundamental group $\pi_1^{orb}(P_{k+1})$ of the double is an index-2 subgroup of $\pi_1^{orb}(P_k)$ and the following two conditions are satisfied.*

1. $\forall k \in \mathbb{N} : \forall g \in \pi_1^{orb}(P_k) : g(\mathring{P}_k) \cap \mathring{P}_k \neq \emptyset \implies g = 1$, *and*

2. $\forall k \in \mathbb{N} : \bigcup_{g \in \pi_1^{orb}(P_k)} g(P_k) = \widetilde{P}$.

Proof. We will omit this proof here, as it goes beyond the scope of this work. $\square$

To state the other auxiliary lemma we will need, we first want to show that each chamber in the Tits cone can uniquely be labeled by elements of its corresponding group.

Lemma 3.4.2. *Let W be a Coxeter group and WC its Tits cone. The chambers of the form $w(C)$ in the Tits cone can be labeled uniquely by elements in W.*

Proof. Let $v(C), w(C)$ be chambers in WC for $v, w \in W$, such that $v \neq w$ and $w(C) = v(C)$. But then, by Theorem 2.5.5 we have that

$$w(\mathring{C}) \cap v(\mathring{C}) = w(\mathring{C} \cap w^{-1}v(\mathring{C})) \neq \emptyset \implies w^{-1}v = 1 \iff w = v.$$

Thus, the chambers can be labeled in a unique way. □

Remark 3.4.3. *We remark that each of the right-angled polytopes P_k can be decomposed into a union of translates of the fundamental chamber C. This means for each polytope P_k there is a $n_k \in \mathbb{N}$ and elements $w_{i,k} \in W$, so that $P_k = \bigcup_{i=1}^{n_k} w_{i,k}(C)$.*

We proceed with another helpful result, whose proof relies on the aforementioned results.

Lemma 3.4.4. *Let $C \subset \mathbb{R}^n$ be the fundamental chamber of a Coxeter group W. The non-trivial labels $w_{i,k} \neq 1_W$ of the chambers, defining the polytope $P_k = \bigcup_{i=1}^{n_k} w_{i,k}(C)$, are* not *contained in the reflection group, generated by reflecting in its faces.*

Proof. Note that by Lemma 3.4.2, the chambers of the form $w(C)$ are uniquely labeled by the elements $w \in W$. Thus, let $P_k := \bigcup_{i=1}^{n_k} w_{i,k}(C)$ be as in the lemma and $w_{i,k}$ one of its defining labels. Assume, $w_{i,k} \neq 1_W$ is contained in the reflection group generated by reflecting in the faces of P_k. But then we have that $w_{i,k}(\mathring{P}_k) \cap \mathring{P}_k \neq \emptyset$, contradicting Proposition 3.4.1. □

Using this insight, we define a graph G with vertices $V(G) := \{w(C) \mid w \in W\}$ and edges $E(G) := W \times S$. The endpoint map is given by $\delta : W \times S \to 2^{V(G)},\ (w, s) \mapsto (w(C), ws(C))$. Thus, two vertices $w(C)$ and $v(C)$ have an edge, if there is an $s \in S$, such that $ws = v$. Then, clearly we have $wsw^{-1}w(C) = ws(C) = v(C)$. Note that the element wsw^{-1} is the only element, that flips the edge $\{w(C), ws(C)\}$. Thus, the map $W \times S \to W,\ (w, s) \mapsto wsw^{-1}$ defines a labeling of the edges by elements in W. We will call the combinatorial graph of G (meaning that every double edge gets collapsed) the *chamber graph* $\mathrm{Cham}(W, S)$ of W.

This graph is canonically isomorphic to the combinatorial Cayley graph of the group W and we thus see that the Cayley graph 'embeds' into the Tits cone WC.

Theorem 3.4.5. *There exists a cofinal doubling sequence P_k, coming from the fundamental chamber $P_0 = C$. This means that we have $WC = \bigcup_{n \in \mathbb{N}} P_n$.*

Proof. The proof is by induction on the word length $\ell(w)$. For $\ell(w) = 0$, the corresponding chamber is the fundamental chamber C, which is covered by itself.

(IH) The chambers with labels v of length $\ell(v) = k$ are covered by a polytope $P_k = \bigcup_{i=1}^{n_k} w_{i,k}(C)$ in the doubling sequence $\{P_k\}_{k \in \mathbb{N}}$.

We need to show that the chambers $w(C)$ with labels $w \in W$ of length $\ell(w) = k+1$ can be covered by doubling P_k finitely many times in faces. First observe that there is a label v, whose chamber $v(C)$ is contained in P_k and that is connected to w by an edge in the Cayles graph. Since the Cayley graph sits inside the Tits cone and is isomorphic to the chamber graph, the chambers $w(C)$ and $v(C)$ have to be adjacent, sharing a face that is contained in a face of P_k. Then, we can distinguish two cases. Either, $w(C)$ is already contained in P_k and we are done, or we double P_k in the face shared by $v(C)$ and $w(C)$, so that $w(C)$ is contained in P_{k+1}. Since W is finitely generated, there are finitely many more labels $w' \in W$ of length $\ell(w') = k+1$ and we repeat the above procedure for each of them. Thus, every chamber in the Tits cone can be covered by doubling finitely many times in faces of the fundamental chamber C. $\square$

Remark 3.4.6. *Note that by Proposition 3.4.1 the elements corresponding to labels of chambers inside a right-angled polytope P_k are not contained in $\pi_1^{orb}(P_k)$ and $\pi_1^{orb}(P_{k+1})$ is a subgroup in $\pi_1^{orb}(P_k)$ of index two. By Theorem 3.4.5 we have a doubling sequence $\{P_k\}_{k \in \mathbb{N}}$ with $\bigcup_{n \in \mathbb{N}} P_n = WC$. As the fundamental group of WC is trivial (it is a convex cone), the intersection of the fundamental groups of the P_k is trivial as well. Thus, the cofinality of the doubling sequence translates to the group side.*

Example 3.4.7. *As an example, transporting the Tits cone of the free product from Example 2.1.3 to the poincaré disc, a portion of the chamber graph is depicted below.*

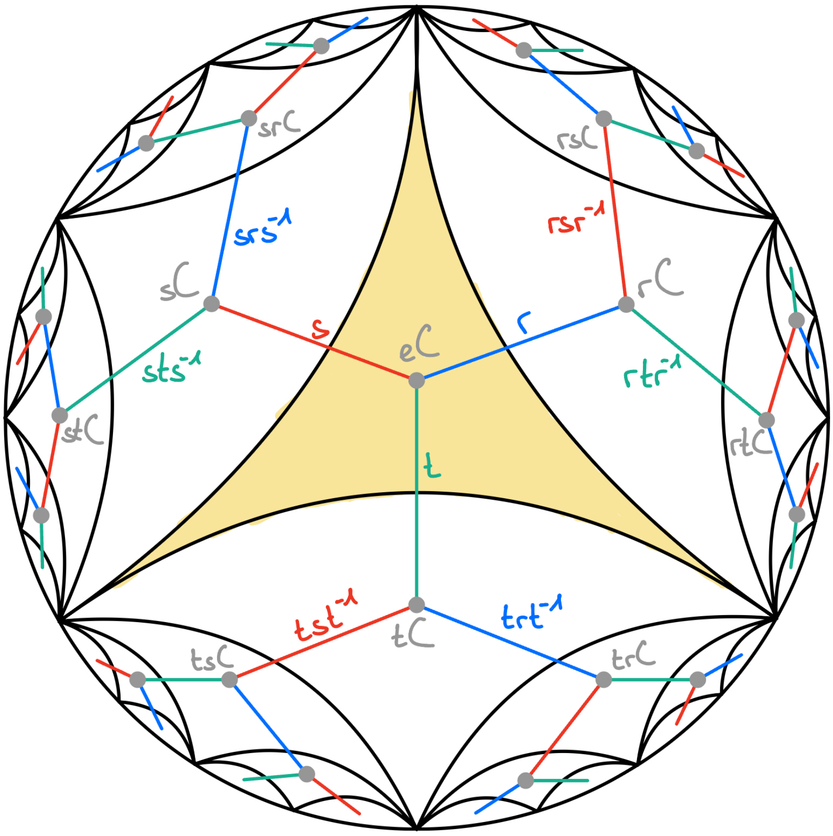


Figure 3.1: The chamber graph of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

3.5 THE COFINAL MANIFOLD SEQUENCE

We finally have developed enough tools, to construct the desired sequence of subgroups in W'. Following Theorem 3.4.5, we fix a cofinal doubling sequence P_k, starting in the fundamental chamber $P_0 = C$ corresponding to the dual representation of W, i.e. we fix a cofinal sequence of two-fold orbifold covers $C \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$. Then, we define the groups $W_i := \pi_1^{orb}(P_i) \cap W'$ for $i \in \{0\} \cup \mathbb{N}$, each inducing a manifold via the quotient ${}^{\mathrm{int}(WC)}\!/\!_{W_i}$. Clearly, each W_i is a subgroup of W' as well as it is a subgroup of the orbifold fundamental group of the polytope (with orbifold structure) P_i. Using the theory of orbifold coverings, we observe that the construction of these manifolds implies that they are coverings of the polytopes P_i for each i. By Remark 3.4.6, the cofinal two-fold tower induces a descending series of fundamental groups of the polytopes P_i. Intersecting them with the finite index subgroup W' will not and thus, the descending series of subgroups formed by the groups W_i is again cofinal.

To keep track of our progress, we note that to show that W' is RFRS, it remains to show the following two conditions:

1. each $W_i := \pi_1^{orb}(P_i) \cap W'$ has finite index in the group W' and
2. $(W_i)_r^{(1)} = \ker\{W_i \to \mathbb{Q} \otimes_{\mathbb{Z}} (W_i)_{ab}\}$ is a subgroup of W_{i+1}.

The first condition is essentially a consequence of the fact that the sequence of orbifold fundamental groups has index precisely two in each step. To conclude, we prove the following lemma.

Lemma 3.5.1. *The homomorphism* $\phi : {}^{W_i}\!/\!_{W_{i+1}} \to {}^{\pi_1^{orb}(P_i)}\!/\!_{\pi_1^{orb}(P_{i+1})}$ *is injective.*

Proof. Suppose, $g, h \in {}^{W_i}\!/\!_{W_{i+1}}$ with $\phi(g) = \phi(h)$, which is equivalent to

$$g \cdot \pi_1^{orb}(P_{i+1}) = h \cdot \pi_1^{orb}(P_{k+1}).$$

Since g and h lie in ${}^{W_i}\!/\!_{W_{i+1}}$, we have that $gW_{i+1} = hW_{i+1}$, equivalently $g^{-1}h \in W_{i+1}$ from what we conclude that $g = h$ in the quotient ${}^{W_i}\!/\!_{W_{i+1}}$ and the result follows. □

An immediate consequence of this is the following result, we record here.

Corollary 3.5.2. *The index of* W_{i+1} *in* W_i *is less than or equal to 2.*

Based on this observation, we can conclude that the first of the above conditions holds for the constructed sequence of groups. We state this here as a corollary.

Corollary 3.5.3. *The index of* W_i *in* W' *is finite.*

Proof. By the above corollary, we have that the index of W_{i+1} in W_i is less than or equal to 2. Thus, for the index of W_i in W' we have the upper bound $[W' : W_i] \leqslant 2^i < \infty$ for $i \in \mathbb{N}$. □

3.6 LOOPS BOUNCING OFF FACES

As discussed in the previous section, we need to show that the rational derived group $(W_i)_r^{(1)}$ is a subgroup of W_{i+1}. First, recall the following result, addressing abelian quotients.

Lemma 3.6.1. *Let* G *be a group with normal subgroup* $N \triangleleft G$*. The quotient group* ${}^{G}\!/\!_{N}$ *is abelian if and only if the derived subgroup* $G^{(1)} = [G, G]$ *of* G *is contained in the normal subgroup* N*.*

Proof. '$\Rightarrow$' As N is normal in G and the quotient is abelian, for any $g, h \in G$ we have that

$$ghN = gN \cdot hN = hN \cdot gN = hgN \iff g^{-1}h^{-1}gh \in N.$$

This is equivalent to the commutator $[g, h]$ being in N for all $g, h \in G$, therefore $[G, G] \subset N$.

'$\Leftarrow$' As now, $[G, G] \subset N$ for any $g \in G$ and $n \in N$, we have

$$gng^{-1} = gng^{-1}n^{-1}n = [g, n]n.$$

Since $[G, G] \subset N$, this element is again in N so that N is normal in G. To see that the quotient by N is abelian, we take $g, h \in {}^{G}\!/\!_{N}$ and calculate

$$gN \cdot hN = ghN = hgg^{-1}h^{-1}ghN = hg[g^{-1}, h^{-1}]N = hgN = hN \cdot gN.$$

The result follows directly. □

Lemma 3.6.2. *Let* G *be a group and* $N \leqslant G$ *a subgroup of index two. Then,* N *is normal in* G*.*

Proof. Let g be an element in G. As N has index two in G, we have that either $g \in N$ or $g \in G \backslash N$. The first case is trivial, as $gN = N = Ng$ and in the second case we have that $gN = G \backslash N = Ng$, since cosets partition the group G. This implies that N is normal in G. □

Using these two auxiliary lemmas, we can prove the following intermediate result.

Proposition 3.6.3. *The quotient group* ${}^{W_i}\!/\!_{W_{i+1}}$ *is either isomorphic to* ${}^{\mathbb{Z}}\!/\!_{2\mathbb{Z}}$ *or the trivial group. Furthermore, the quotient map* $W_i \to {}^{W_i}\!/\!_{W_{i+1}}$ *factors through the abelianization and the derived subgroup* $(W_i)^{(1)}$ *of* W_i *is a subgroup of* W_{i+1}*.*

Proof. The first part is a direct consequence of Corollary 3.5.2 and Lemma 3.6.2. Thus, the quotient ${}^{W_i}\!/\!_{W_{i+1}}$ is in particular abelian and we are able to apply Lemma 3.6.1. This implies that the derived subgroup $(W_i)^{(1)} = [W_i, W_i]$ is a subgroup of W_{i+1}. Stated differently, we have a factorization of $W_i \to {}^{W_i}\!/\!_{W_{i+1}}$ through the abelianization $(W_i)_{ab}$. □

Remark 3.6.4. *Recall that by Remark 3.1.2 torsion elements will not survive in the rational derived subgroup* $(W_i)_r^{(1)}$ *of* W_i*, and by Proposition 3.6.3 we already have that* $(W_i)^{(1)}$ *is a subgroup of* W_{i+1}*. We claim that it suffices to show that every element* $w \in W_i$ *with non-trivial image in the quotient group* W_i/W_{i+1} *is not torsion in the abelianization* $(W_i)_{ab}$*. This is justified, as by Proposition 3.6.3, we have a factorization through the abelianization of* W_i*. Then, since* w *is not torsion in* $(W_i)_{ab}$*, it will admit a factorization through the torsion-free abelianization.*

$$\begin{tikzcd} W_i \arrow[rr] \arrow[dr] & & W_i/W_{i+1} \cong \mathbb{Z}/2\mathbb{Z} \\ & (W_i)_{ab}/\text{Torsion} \arrow[ur, dashed] & \end{tikzcd}$$

In particular, the group W_i/W_{i+1} *is a quotient of the torsion-free abelianization* $(W_i)_{ab}/\text{Torsion}$*.*

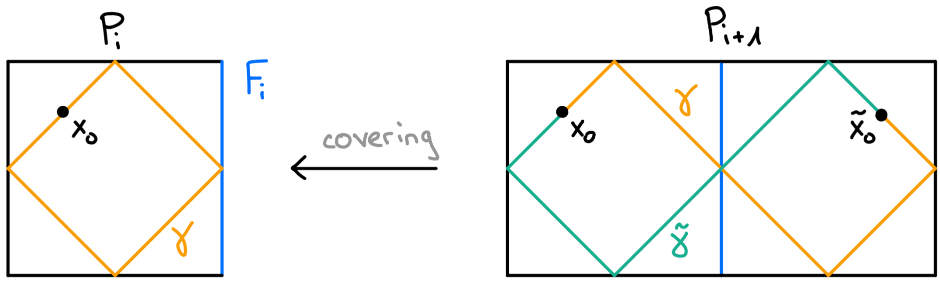
In the following we will denote the manifolds $\text{int}(WC)/W_i$ by M_i for $i \in \{0\} \cup \mathbb{N}$ and the projections to the orbifolds P_i by $p_i : M_i \to P_i$. Furthermore, we fix an element w that is contained in W_i but not in W_{i+1}, together with a loop γ, which represents the element w in the manifold M_i.

Lemma 3.6.5. *The loop* γ *intersects the preimage* $p_i^{-1}(F_i)$ *of the face* F_i *in which* P_i *is doubled to obtain the polytope* P_{i+1}*, an odd number of times.*

Proof. Project γ down to the polytope P_i generically, using the projection p_i. As the polytopes P_i have cyclic local group of order two everywhere on their codimension-one faces, the projection of γ gets reflected in the faces of the polytopes P_i. Now, consider the face F_i in which P_i gets doubled. Then, $p_i(\gamma)$ hits this face an odd number of times, else $p_i(\gamma)$ would lift to a loop in the double P_{i+1}, and therefore, the element w represented by γ would lift to the group W_{i+1}. This contradicts our choice of w, not being contained in W_{i+1}. In particular, observing the loop γ directly up in the manifold M_i, we see it intersecting the preimage $p_i^{-1}(F_i)$ of the face F_i an odd number of times, which proves the lemma. ☐

One might think of the orbifolds P_i as polytopes that are formed by mirrors, and the loop γ as a closed path formed by a ray of light, being reflected in these mirrors. This interpretation also motivated the name of this section. Before proceeding, we want to provide a more explicit picture of the above argument. We record this in the following remark.

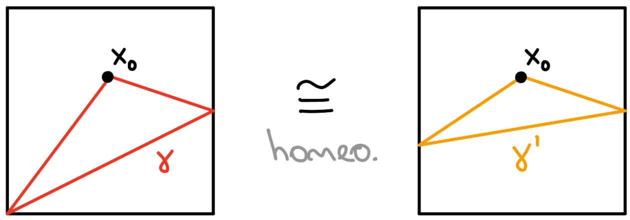
Remark 3.6.6. *We want to explain what is happening here in a more explicit way. To do so, we look at the following picture, which one should think of in a schematic way. We see a loop* γ *(which, strictly speaking, is the projection of a loop in the manifold* M_i*) in the polytope* P_i*, being reflected in the codimension-one faces of* P_i*. The face of interest* F_i *is colored in blue and we assume* γ *to be the orange path, walked around exactly once, starting in the basepoint* x_0*.*



By doubling the polytope P_i along the face F_i, we land on the right side of this picture. Observe that the basepoint x_0 now has a corresponding double, denoted by $\tilde{x}_0$. The more interesting part is, that now the loop γ gets unwound to a path going from x_0 to $\tilde{x}_0$. If, on the other hand, we consider the loop γ^2, i.e. the loop γ gets run through twice, consequently hitting the face F_i twice, by our reasoning above, we should see a loop coming from γ^2 in the two-fold cover P_{i+1}. Indeed, the loop γ^2 lifts to a loop P_{i+1} and is pictured by the concatenated loop $\tilde{\gamma} \cdot \gamma$ in P_{i+1} on the right.

Now that we have a more intuitive view on our reasoning, we have to deal with one more minor issue. We will mention it in the following remark, but will omit the details.

Remark 3.6.7. *Note that we could also have a loop γ like in the left of the following image.*



The difficulty with such loops is that the codimension-2 faces of a polytope also consist of singular points. Their local group may very well be different from the local groups of the points in codimension-1 faces. Nevertheless, we fortunately do not have to deal with them. In fact, we can always find a loop γ', homotopy equivalent to γ and that is only hitting codimension-1 faces.

In the following, our goal will be to obtain a homomorphism of fundamental groups, from the fundamental group of the manifold $\pi_1(M_i)$ to the fundamental group of the circle $\pi_1(S^1)$. This morphism will be induced by a continuous quotient map. To make this precise, we have to introduce one more concept, we have not yet mentioned. This is the fact that the preimage of a face F_i in the manifold M_i is two-sided (moreover it is oriented and embedded). This is a technical condition on the preimage $p_i^{-1}(F_i) \subset M_i$, but it will play an important role in the following. We give the definition of what it means to be two-sided.

Definition 3.6.8. *Let M be a manifold. A submanifold $F \subset M$ is called* two-sided *in M, if it locally looks like the product with an interval. More precisely, there is a neighborhood N_F with $N_F \cong F \times (-\varepsilon, \varepsilon)$ for suitable $\varepsilon > 0$, where the submanifold F is identified with the fiber $F \times \{0\}$.*

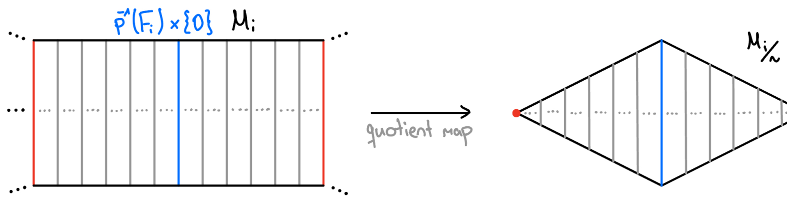
Denote by $N_{F_i} \subset M_i$ the neighborhood coming from the two-sidedness of the preimage $p_i^{-1}(F_i) \subset M_i$. Using this neighborhood, we can construct the morphism in the next lemma.

Lemma 3.6.9. *There exists a continuous quotient map* $q : M_i \to S^1$ *that induces a homomorphism of groups* $q^* : \pi_1(M_i) \to \pi_1(S^1)$*, between the fundamental groups.*

Proof. We will give an constructive proof. To do so, introduce an equivalence relation on the points in the manifold M_i as follows:

$$x \sim y \qquad :\Longleftrightarrow \qquad x, y \in M_i \backslash N_{F_i} \quad \text{or} \quad x = y.$$

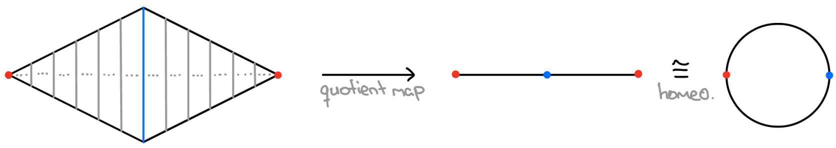
Equip the quotient ${}^{M_i}\!/\!_{\sim}$ with the quotient topology. Then, by passing to this quotient, we see a double cone with identified cone points.



Continuing from this space on, we have to pass to another quotient. Namely, we have to collapse each of the fibers $p_i^{-1}(F_i) \times \{z\}$ for every $z \in (-\varepsilon, \varepsilon)$ to a single point. This can be done by introducing another equivalence relation on the quotient space ${}^{M_i}\!/\!_{\sim}$ by

$$x \sim_1 y \qquad :\Longleftrightarrow \qquad \exists\, z \in (-\varepsilon, \varepsilon) : \; x, y \in p_i^{-1}(F_i) \times \{z\}.$$

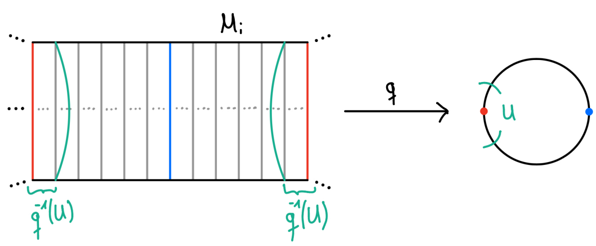
Equip this quotient again with the quotient topology and by passing to this further quotient, we are left with the interval $(-\varepsilon, \varepsilon)$, whose endpoints are identified.



As all the quotient spaces carry the quotient topology that is defined just right to make such constructions work, we only have to think about the red point. Consider the composed quotient map $q : M_i \to S^1$ and a neighborhood U around the red point. Then, U is of the form $[-\varepsilon, -\varepsilon + \delta) \cup (\varepsilon - \delta, \varepsilon]$ for some $\delta \in (0, \varepsilon)$ and it is open in the quotient if and only if its preimage $p_i^{-1}(U)$ is open in M_i. The preimage of U in M_i is of the form

$$q^{-1}(U) = M_i \backslash N_{F_i} \cup p_i^{-1}(F_i) \times [-\varepsilon, -\varepsilon + \delta) \cup p_i^{-1}(F_i) \times (\varepsilon - \delta, \varepsilon],$$

which is open, as its complement $M_i \backslash q^{-1}(U) = p_i^{-1}(F_i) \times [-\varepsilon + \delta, \varepsilon - \delta]$ is closed in M_i.



Thus, the quotient map $q : M_i \to S^1$ is continuous and therefore induces the desired homomorphism $q^* : \pi_1(M_i) \to \pi_1(S^1)$. $\square$

Recalling the discussion above, we know that the loop γ, intersects the preimage of F_i an odd number of times in M_i. Projecting it down to S^1, we see that the projection has to wind around the circle at least once and for that reason has to represent a non-trivial element in $\pi_1(S^1)$. This implies that the element w, represented by the loop γ, has to have infinite order in $(W_i)_{ab}$ and therefore is not contained in the rationally derived subgroup $(W_i)_r^{(1)}$. Thus, the loop 2γ is non-trivial in the group $\mathbb{Q} \otimes_{\mathbb{Z}} {W_i}/{(W_i)_r^{(1)}}$. As γ is a representative of w, the loop 2γ is a representative of w^2 and it immediately follows that $w^2 \notin (W_i)_r^{(1)}$, as it cannot be torsion by preceding argument.

But by the fact that we have $[W_i : W_{i+1}] \leqslant 2$, the element w^2 has to be contained in W_{i+1} and we see that the group ${W_i}/{W_{i+1}} \cong {\mathbb{Z}}/{2\mathbb{Z}}$ is a quotient of the torsion-free abelianization, ${(W_i)_{ab}}/{\text{Torsion}} \cong \mathbb{Q} \otimes_{\mathbb{Z}} (W_i)_{ab}$ by the before mentioned factorization. Thus, we have finally shown that every element mapping non-trivially to the quotient ${W_i}/{W_{i+1}}$ is not torsion, allowing us to deduce that $(W_i)_r^{(1)}$ is a subgroup of W_{i+1}, which completes the proof.

BIBLIOGRAPHY

[1] Ian Agol. "Criteria for virtual fibering." In: *Journal of Topology* (2008).

[2] Michael W. Davis. *The Geometry and Topology of Coxeter Groups. (LMS-32)* -. Kassel: Princeton University Press, 2012. ISBN: 978-1-400-84594-1.

[3] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2002. ISBN: 0-521-79540-0.

[4] James E. Humphreys. *Reflection Groups and Coxeter Groups* -. Cambridge: Cambridge University Press, 1992. ISBN: 978-0-521-43613-7.

[5] Anne Thomas. *Geometric and Topological Aspects of Coxeter Groups and Buildings* -. European Mathematical Society Publishing House, 2018. ISBN: 978-3-037-19189-7.