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Bachelor thesis

RIGHT-ANGLED COXETER GROUPS ARE RFRS

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2.1 COXETER GROUPS

First of all, we define the main object of interest we want to study.

Definition 2.1.1. *Let* S *be a set consisting of elements* s_i *indexed by an index set* I *with* $|I| = n < \infty$. *Let* $(m_{ij}) = M$ *be a symmetrical matrix in* $(\mathbb{N} \cup \{\infty\})^{n \times n}$, *where* $m_{ii} = 1$ *for all* i *and* $m_{ij} \geqslant 2$ *for* $i \neq j$. *Define a group* W *via the following presentation:*

$$W := \langle S \mid (s_i s_j)^{m_{ij}} = 1 \text{ for all } i, j \in I \rangle,$$

where the relations with $m_{ij} = \infty$ *are usually ommited, i.e. give trivial relations. The pair* (W, S) *is called a* Coxeter System *and* M *is called the corresponding* Coxeter Matrix. *A* Coxeter group *is a group isomorphic to a group* W, *corresponding to a Coxeter System* (W, S). *It is generated by the set* S.

In this work we will be particularly interested in a special class of Coxeter groups that we call right-angled. They are defined as follows, by imposing significant constraints on the entries of the Coxeter matrix.

Definition 2.1.2. *A Coxeter System* (W, S) *is* right-angled *if, for all distinct* $i, j \in I$, *the condition* $m_{ij} \in \{2, \infty\}$ *is satisfied. In this context, the group* W *is then called a* right-angled Coxeter group (RACG).

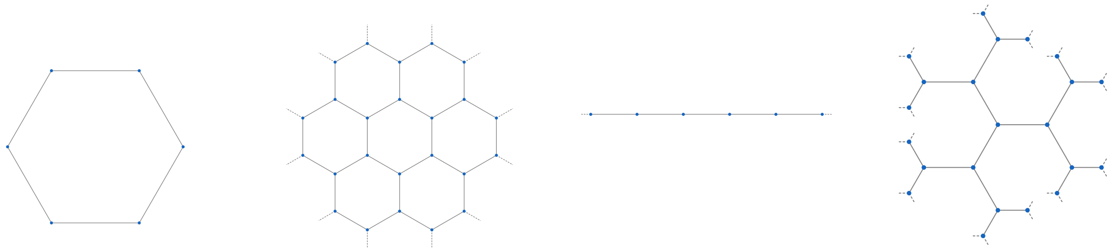
We give some important examples of Coxeter groups as well as right-angled Coxeter groups.

Example 2.1.3.

1. *Dihedral groups,* $D_{2m} \cong \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1 s_2)^m = 1 \rangle$ *are Coxeter groups for all* $m \in \mathbb{N}$.
2. *The triangle groups,* $\Delta(l, m, n) \cong \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^l = (st)^m = (tr)^n = 1 \rangle$ *with* l, m *and* n *integers greater or equal to 2 are Coxeter groups.*
3. *The infinite Dihedral group,* $D_\infty \cong \langle s, t \mid s^2 = t^2 = 1 \rangle$ *is a right-angled Coxeter group.*
4. *The free product,* $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \langle r, s, t \mid r^2 = s^2 = t^2 = 1 \rangle$ *is a right-angled Coxeter group.*

In Figure 2.1 below, a small portion of the combinatorial Cayley graphs of these examples are depicted.

Figure 2.1: The combinatorial Cayley graph of ...



(a) D_6 **(b)** $\Delta(3,3,3)$ **(c)** D_∞ **(d)** $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

We define a special type of subgroups, called parabolic subgroups within a Coxeter group W. These subgroups are constructed from a subset of the index set I. The definition is as follows:

Definition 2.1.4. *Let (W, S) be a Coxeter System as above with finite index set I, and J be a subset of the index set I. The group $W_J := \langle\{s_j \mid j \in J\}\rangle$, generated by the s_j in S with $j \in J$, is then called a* parabolic subgroup *of W. Moreover, we call any conjugate of W_J a parabolic subgroup as well.*

Once we have constructed the representation of Coxeter groups on a vector space as well as the Tits cone in the coming section, the parabolic subgroups will be a useful tool to form a deeper understanding of these objects. We will extensively use them in Sections 2.4 and 2.5.

2.2 REPRESENTATION OF COXETER GROUPS

Given a Coxeter System (W, S), let V be a real vector space with basis $\{e_1, \ldots, e_n\}$, where $n = |I| = |S|$. This provides a natural identification, $GL_n(V) \cong GL_n(\mathbb{R})$. We define a bilinear form B_W on V as follows:

$$B_W(e_i, e_j) := \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right) & , m_{ij} < \infty \\ -1 & , m_{ij} = \infty \end{cases}.$$

By Definition 2.1.1, it is assured that $m_{ij} \geqslant 2$ for distinct i, j, ensuring the cosine term is non-positive. Consequently, we have $B_W(e_i, e_j) \leqslant 0$ for distinct i, j. Furthermore, from $m_{ii} = 1$, it follows that $B_W(e_i, e_i) = 1$. Using this bilinear form, we define hyperplanes with corresponding reflections for each basis element e_i as follows:

$$H_i := \{v \in V \mid B_W(e_i, v) = 0\}, \qquad \sigma_i : V \to V, \quad v \mapsto v - 2B_W(e_i, v)e_i.$$

Theorem 2.2.1. *The map given by:*

$$\rho : W \to GL_n(V) \cong GL_n(\mathbb{R}), \quad s_i \mapsto \sigma_i$$

is an injective homomorphism and therefore a faithful representation of W.

Before we prove the homomorphism property, we recall: A map $\varphi : S \to G$ from a set S to a group G extends to a homomorphism $\hat{\varphi} : \langle S \mid R\rangle \to G$, if and only if the induced homomorphism $\overline{\varphi} : F_S \to G$ from the free group over S satisfies $\overline{\varphi}(r) = 1_G$ for every $r \in R$.

Proof. Observe that $\sigma_i^2 = \mathrm{id}$ in $GL_n(V)$ and thus to prove that ρ is a homomorphism, we apply the above to our situation to see that it suffices to show that the product $\sigma_i\sigma_j$ has order m_{ij} in $GL_n(V)$ for distinct $i, j \in I$. Also note that in the case of $m_{ij} = \infty$ there is nothing to prove as these relations are defined to be trivial in the presentation of W.

Thus, consider the two-dimensional subspace V_{ij} of V spanned by two basis vectors e_i, e_j and take a general element $v = \lambda \cdot e_i + \mu \cdot e_j$ in V_{ij} with $\lambda, \mu \in \mathbb{R}$ not simultaneously zero. As $m_{ij} < \infty$, the bilinear form B_W is positive definite by the following calculation

$$B_W(v, v) = \lambda^2 - 2\lambda\mu \cos\left(\frac{\pi}{m_{ij}}\right) + \mu^2 = \left(\lambda - \mu \cos\left(\frac{\pi}{m_{ij}}\right)\right)^2 + \mu^2 \sin^2\left(\frac{\pi}{m_{ij}}\right) > 0.$$

Identify $(V_{ij}, B_W|_{V_{ij}})$ with the euclidean plane $(\mathbb{R}^2, \langle\cdot,\cdot\rangle)$ up to a change of basis. Then σ_i, resp. σ_j act by orthogonal reflections in their corresponding hyperplanes H_i, resp. H_j intersected with V_{ij}. By having a look at the inner product of e_i and e_j

$$B_W(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right) = \cos\left(\pi - \frac{\pi}{m_{ij}}\right),$$

we observe that the angle between the two in V_{ij} is precisely $\pi - \frac{\pi}{m_{ij}}$. We conclude from this that the angle between their reflecting lines in $\frac{\pi}{m_{ij}}$ and the composition $\sigma_i\sigma_j$ turns out to be a rotation about $\frac{2\pi}{m_{ij}}$. As the composition $\sigma_i\sigma_j$ fixes the orthogonal complement $V_{ij}^{\perp}$ of V_{ij} by definition of the σ_i, we see that it has order m_{ij} on the whole space V.

This proves that ρ extends to a homomorphism. It remains to show the injectivity of ρ. This will be a consequence of a bigger result in section 2.4, see Corollary 2.4.4. $\square$

In contrast to the above proof, in the case of $m_{ij} = \infty$, we cannot expect our bilinear form to be positive definite. Let $v \in V_{ij}$ be as in the proof, then

$$B_W(v, v) = \lambda^2 - 2\lambda\mu + \mu^2 = (\lambda - \mu)^2 \geqslant 0.$$

This shows that we can at least expect it to be positive semidefinite. The direct calculation

$$(\sigma_i\sigma_j)(e_i) = \sigma_i(e_i + 2e_j) = e_i + 2(e_i + e_j),$$

together with an inductive argument, shows that $(\sigma_i\sigma_j)^n(e_i) = e_i + 2n(e_i + e_j)$. Therefore, the composition $\sigma_i\sigma_j$ has infinite order on V_{ij} and in particular on the whole of V. The following remark is a consequence of this, together with the observation in above proof.

Remark 2.2.2. *From last paragraph and the proof of Theorem 2.2.1, we conclude that two-generator subgroups of Coxeter groups are dihedral. Either of order $2m_{ij}$ or infinite order.*

We want to extend the action of W to the dual of the vector space V. This is achieved by acting on V^* via the dual representation of ρ, which we define by

$$\rho^* : W \to GL_n(V^*), \quad w \mapsto (\rho^*(w) : V^* \to V^*,\ \varphi \mapsto \rho^*(w)(\varphi)), \quad w \in W, \varphi \in V^*.$$

We can evaluate the functional $\rho^*(w)(\varphi)$ on some $v \in V$ via $(\rho^*(w)(\varphi))(v) := \varphi(\rho(w^{-1})(v))$. Notation wise, we will simply write $w(v)$, when $w \in W$ acts on $v \in V$ via $\rho(w)(v)$. Similarly, we write $w(\varphi)$ when we mean that $w \in W$ acts on some element $\varphi \in V^*$ of the dual space, via the dual representation $\rho^*(w)(\varphi)$. As in the case of the vector space V, we want to give a definition for the notion of a hyperplane with corresponding reflection in the dual space V^* as well. By dual hyperplane, we mean a subspace $H_i^* := \{\varphi \in V^* \mid \varphi(e_i) = 0\}$, and the corresponding dual reflections will be a map from V^* to V^*, given by:

$$\sigma_i^* : V^* \to V^*, \quad \varphi \mapsto \varphi \circ \sigma_i = \varphi - 2B_W(e_i, \cdot\,)\varphi(e_i).$$

To further explore Coxeter groups and their action via this representation, we need some more notation. In particular, we want a so-called *chamber*. This should be thought of as a cone over a polytope with finitely many faces such that the reflections in its codimension one faces correspond to the generators of W under the representation.

Definition 2.2.3. *The fundamental chamber* C *of the dual representation is the set, given by*

$$C := \{\varphi \in V^* \mid \varphi(e_i) \geqslant 0\ \forall i \in I\} \subset V^*.$$

Denote by $\{e_1^*, \dots, e_n^*\}$ the dual basis of V^* corresponding to the standard basis $\{e_1, \dots, e_n\}$ of V. Then we calculate, using the σ_i^* from above:

$$\sigma_i^*(e_j^*) = e_j^* - 2B_W(e_i,\cdot)e_j^*(e_i) = \begin{cases} e_j^* & \text{for } i \neq j \\ e_j^* - 2B_W(e_j,\cdot) & \text{for } i = j \end{cases},$$

which implies that each reflection σ_i^* fixes all the hyperplanes H_j^*, for distinct indices i and j. Moreover, note that the fundamental chamber can be written in the form

$$C = \bigcap_{i\in I}\{\varphi \in V^* \mid \varphi(e_i) \geqslant 0\} = \bigcap_{i\in I}(H_i^* \cup \{\varphi \in V^* \mid \varphi(e_i) > 0\}),$$

where we observe that the sets $H_i^* \cap C$ form the pairwise distinct codimension one faces of the chamber C. The open halfspaces $\{\varphi \in V^* \mid \varphi(e_i) > 0\}$ in the latter term will be called A_i^* and using these, we define the open fundamental chamber to be the intersection of the open halfspaces:

$$\operatorname{int}(C) = \mathring{C} = \bigcap_{i\in I} A_i^*.$$

As mentioned above, we want to study the action of our Coxeter group via the dual representation, acting by reflection in the faces H_i^*. However, in general the translates of the chamber under the group action won't cover the whole of V^*, which motivates the following definition:

Definition 2.2.4. *The Tits cone is the union of all W-translates of the chamber,* $WC := \bigcup_{w\in W} wC \subset V^*$.

As the name suggests, the fundamental chamber is a fundamental domain for the action of W on its Tits cone WC under the dual representation ρ^*. This will be proved in Theorem 2.5.7. While the formal defintion of the Tits cone provides a rigorous foundation, it is not very insightful from a geometric perspective. As one can think about the Tits cone quite geometrically, especially in low dimensions, we will take a closer look at an explicit example in the following section. Before doing so, we end this section with the following remark.

Remark 2.2.5. *One may ask why we transport everything to the dual space, instead of working in the standard representation* ρ. *For this, consider the infinite dihedral group* $D_\infty \cong \langle s, t \mid s^2 = t^2 = 1\rangle$. *We fix a basis* $\{e_1, e_2\}$ *of* V *and obtain that the bilinear form in this basis is given by the matrix*

$$B_W = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We observe that $H_1 = \mathit{span}\{e_1 + e_2\} = H_2$, *which implies that* σ_1 *and* σ_2 *fix the same hyperplane, despite having different* (-1)*-Eigenspaces (namely the span of* e_1, *resp.* e_2*). Therefore, in general, working in the standard representation won't result in a chamber, giving rise to the existence of the Tits cone. Now, passing to the dual space* V^* *by fixing the dual basis* $\{e_1^*, e_2^*\}$, *consider the dual reflections* σ_i^* *as discussed before. By the more general calculation earlier, we obtain*

$$\sigma_1^*(H_2^*) = H_2^* \quad \text{and} \quad \sigma_2^*(H_1^*) = H_1^*.$$

And furthermore, we note that for all $i, j \in \{1, 2\}$

$$\sigma_i^*(B_W(e_j, \cdot)) = B_W(e_j, \cdot) - 2B_W(e_i, \cdot)B_W(e_j, e_i) = -B_W(e_j, \cdot),$$

using that $B_W(e_j, \cdot) = -B_W(e_i, \cdot)$. *This shows that both dual reflections have the same* (-1)*-Eigenspace, but fix different hyperplanes (i. e., have different* $(+1)$*-Eigenspaces), resulting in a chamber as wished.*

2.3 THE TITS CONE – AN EXAMPLE

As an example we take a closer look at the free product $W \cong \langle r, s, t \mid r^2 = s^2 = t^2 = 1\rangle$, a right-angled Coxeter group from Example 2.1.3. We fix the basis $\{e_1, e_2, e_3\}$ and identify V with $\mathbb{R}^3$. In this basis, the bilinear form B_W is given by the matrix

$$B_W = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

By the spectral theorem we find a basis of orthonormal Eigenvectors, in which B_W is a diagonal matrix with its eigenvalues as entries. Using the Gram-Schmidt procedure, we get

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In other words, we have $V^T B_W V = D$ with $V \in O(n)$ and $D = \mathrm{diag}(2, 2, -1)$. Now, since we have a diagonal matrix, we can multiply the entries by squares, since the resulting matrix will be congruent to the given one:

Let $A = \mathrm{diag}(\mu_1, \ldots, \mu_n) \in \mathbb{R}^{n \times n}$,

$$S = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in (\mathbb{R} \setminus \{0\})^{n \times n} \implies S^T A S = \mathrm{diag}(\lambda_1^2 \mu_1, \ldots, \lambda_n^2 \mu_n).$$

To apply this and further transform our matrix D, define the invertible matrix T as follows

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which then implies $\widetilde{D} := T(V^T B_W V)T = \mathrm{diag}(1, 1, -1)$. The images of the basis vectors $\{e_1, e_2, e_3\}$ are given by the three columns of the matrix TV^T, namely:

$$\widetilde{e}_1 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \ \widetilde{e}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \text{ and } \widetilde{e}_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Note that we have $V \cong \mathbb{R}^3$, equipped with the inner product

$$\langle x, y \rangle_{2,1} := x^T \widetilde{D} y = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

Since $\langle \widetilde{e}_i, \widetilde{e}_i \rangle_{2,1} = 0$ for all $i \in \{1, 2, 3\}$, we have that these three vectors span an ideal triangle in the hyperboloid model of $\mathbb{H}^2$, given by $\langle x, y \rangle_{2,1} = -1$. One gets the ideal triangle by intersecting the hyperboloid with the hyperplanes spanned by each two of the $\widetilde{e}_i$ (they will only intersect in the surrounding cone of the hyperboloid).

Given these new coordinates under the transformation TV^T, the Tits cone will be given by $x_1^2 + x_2^2 - x_3^2 < 0$ union the images of $\widetilde{e}_1, \widetilde{e}_2$ and $\widetilde{e}_3$ under the reflection in the sides of the chamber, i.e. in the sides of the ideal triangle. Moreover, we get a subgroup of $O(2,1)_+$ generated by the following three matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{5}{3} & -\frac{4}{3} \\ 0 & \frac{4}{3} & \frac{5}{3} \end{pmatrix}, \begin{pmatrix} -1 & \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{\sqrt{3}} & -\frac{2}{3} & \frac{5}{3} \end{pmatrix} \text{ and } \begin{pmatrix} -1 & -\frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{3}} & -\frac{2}{3} & \frac{5}{3} \end{pmatrix}.$$

Each of the above matrices corresponds to one of the generators r, s and t of the right angled Coxeter group W under the transformation TV^T, constructed above.

2.4 THE WORD METRIC AND THE FAITHFUL REPRESENTATION

Recall that for any finitely generated group $W = \langle S \rangle$, its Cayley graph induces a metric on W, relative to the generating set S. We call it the *word metric* of W relative to S and denote it by d_S. Also note that the word metric is left-invariant, meaning that for group elements $u, v, w \in W$, we have the equality $d_S(uv, uw) = d_S(v, w)$. Now, define the *word length* of an element $w \in W$ to be $\ell(w) := d_S(w, 1_W)$, the distance of an element to the neutral element of the group. Note that $\ell(w) = 0$ if and only if $w = 1_W$.

Lemma 2.4.1. *We collect some properties of the length function we will use later on.*

1. $\forall w \in W : \ \ell(w) = \ell(w^{-1})$
2. $\forall s \in S : \ \ell(s) = 1$ *and* $\ell(w) = 1 \iff w \in S^{\pm 1}$
3. $\forall v, w \in W : \ \ell(vw) \leqslant \ell(v) + \ell(w)$
4. $\forall v, w \in W : \ \ell(v) - \ell(w) \leqslant \ell(vw)$
5. $\forall w \in W, s \in S^{\pm 1} : \ \ell(w) - 1 \leqslant \ell(ws) \leqslant \ell(w) + 1$

Proof. All of the above statements follow from the fact that d_S is a left-invariant metric. □

Coming back to Coxeter groups, by definition each generator $s \in S$ has order 2 in W. Therefore, we can write every non-trivial element in W as a sequence of generators in S. Note that in this sequence there might be redundencies, so that the following definition makes sense. We call an expression $w = s_{i_1} \cdots s_{i_r}$ for $i_1, \ldots, i_r \in I$ and $r \in \mathbb{N}$ *reduced*, if $\ell(w) = r$, i. e. w cannot be represented by a shorter word. These reduced expressions have the caveat of not being unique by any means.

Given a parabolic subgroup W_J of a Coxeter group W, it admits its own word metric with respect to $J \subset I$. Therefore each parabolic subgroup admits its own length function, which we denote by $\ell_J(w)$ for words w in W_J. In the following we will make use of the general fact that we have $\ell(w) \leqslant \ell_J(w)$ for all $w \in W_J$.

The length function turns out to be an important and powerful tool in studying Coxeter groups. Indeed, its role will be demonstrated in several of the forthcoming proofs, beginning with the following theorem which is a key step in proving faithfulness of our previously defined representation. But before doing so, we want to motivate the next theorem. We consider the set $\{w(e_i) \mid i \in I, w \in W\}$. Note that we fixed a basis in our representation vector space and therefore each of these elements can be written as a linear combination of the basis vectors. We will use this to define 'positive' and 'negative' elements. An element $w(e_i)$ is said to be

- positive (denoted $w(e_j) > 0$), if $w(e_j) = \sum_{i \in I} \lambda_i e_i$ with $\lambda_i \geqslant 0$ for all $i \in I$,
- negative (denoted $w(e_j) < 0$), if $w(e_j) = \sum_{i \in I} \lambda_i e_i$ with $\lambda_i \leqslant 0$ for all $i \in I$.

Theorem 2.4.2. *Let $w \in W$ and $s_i \in S$ for $i \in I$. Then $\ell(ws_i) > \ell(w)$ implies that $w(e_i) > 0$.*

Proof. The base case is trivial, as $\ell(w) = 0$ implies $w = 1_W$ and thus w fixes every basis element. Therefore, assume w is non-trivial and in reduced form. We state the induction hypothesis.

(IH) Let $v \in W$, so that $\ell(v) < \ell(w)$ and $\ell(vs_i) > \ell(v)$, then we have $v(e_i) > 0$.

Claim 1: There is a $j \in I$ such that $s_i \neq s_j$ and $\ell(ws_j) = \ell(w) - 1$.

Proof of Claim 1. Since w is in reduced form, write $w = s_{i_1} \cdots s_{i_k}$ and set $s_j = s_{i_k}$. This implies

$$\ell(ws_i) > \ell(w) > \ell(w) - 1 = \ell(ws_j),$$

and in particular $s_j \neq s_i$. This proves the first Claim. ■

Consider the parabolic subgroup $\langle s_i, s_j \rangle \leqslant W$, generated by these two distinct elements in S. Set $J := \{i, j\}$ in I and denote the subgroup by W_J. We use the decomposition $W = W/W_J \cdot W_J$ to decompose w into two parts, each living in one component. For this, consider a specific subset of the coset wW_J, given by

$$A := \{v \in W \mid vW_J = wW_J \text{ and } \ell(v) + \ell_J(v^{-1}w) = \ell(w)\}.$$

By definition, $w \in A$. With regard to the following, we choose $v \in A$ such that its length $\ell(v)$ is minimal. Now set $v_J = v^{-1}w$, which is equivalent to writing $w = vv_J$, giving us a decomposition of w as desired. Due to this decomposition, our analysis of w now boils down to studying the separate actions of v and v_J on V. First note that ws_j is contained in A as well. Clearly, $s_j^{-1} = s_j$ and thus we have that

$$s_j w^{-1} w = s_j \in W_J \text{ and } \ell(ws_j) + \ell_J(s_j) = \ell(w) - 1 + 1 = \ell(w).$$

By the choice of $\ell(v)$ to be minimal, we have $\ell(v) \leqslant \ell(ws_j) = \ell(w) - 1$, implying $\ell(v) < \ell(w)$. Hence, we are almost set up to apply the induction hypothesis to v and s_i. The last ingredient missing to do so is the following Claim.

Claim 2: For the lengths of vs_i and v, we have the relation: $\ell(vs_i) \geqslant \ell(v)$.

Proof of Claim 2. Assume towards contradiction: $\ell(vs_i) < \ell(v)$, equivalently $\ell(vs_i) = \ell(v) - 1$. Then:

$$\begin{aligned} \ell(w) = \ell(vv_J) = \ell(vs_is_iv^{-1}w) &\leqslant \ell(vs_i) + \ell(s_iv^{-1}w) \\ &\leqslant \ell(v) - 1 + \ell_J(v^{-1}w) + 1 = \ell(v) + \ell_J(v^{-1}w) = \ell(w). \end{aligned}$$

This means equality holds throughout and in particular $\ell(w) = \ell(vs_i) + \ell_J(s_iv^{-1}w)$. But this implies vs_i belongs to A, contradicting the minimality of $\ell(v)$. Thus, the claim holds. ■

Applying the induction hypothesis **(IH)** to v and s_i leaves us with $v(e_i) > 0$. The exact same argument applied to v and s_j shows $v(e_j) > 0$, so that we omit this here. Due to the decomposition and the fact that w lies in W_J, we are now reduced to the rank two situation. As we have seen that all two generator subgroups are dihedral, we work in the flat plane. But first, we have to observe the following claim.

Claim 3: For v_Js_i and v_J, we have the relation: $\ell_J(v_Js_i) \geqslant \ell(v)$.

Proof of Claim 3. Assume towards contradiction, that $\ell_J(v_J s_i) < \ell(v_J)$ holds. Then:

$$\ell(ws_i) = \ell(vv_J s_i) \leqslant \ell(v) + \ell(v_J s_i) \leqslant \ell(v) + \ell_J(v_J s_i) < \ell(v) + \ell_J(v_J) = \ell(w),$$

which contradicts the assumption of the theorem, that $\ell(ws_i) > \ell(w)$. ■

Moreover, this shows that every reduced expression of v_J in the parabolic subgroup W_J has to end in s_j. Otherwise, we would have $\ell(v_J) > \ell(v_J s_i)$, which contradicts *Claim 3*. To deduce the theorem, it suffices to show the following claim.

Claim 4: $(s_i s_j)(e_i) = \lambda_i e_i + \lambda_j e_j$ with $\lambda_i, \lambda_j \geqslant 0$, implying $(s_i s_j)(e_i) > 0$.

Proof of Claim 4. We already know that W_J is dihedral, either of order $2m_{ij}$ or infinite order. As pointed out above, every reduced expression of v_J - an alternating product of s_i and s_j - in W_J has to end in s_j. Using this, we distinguish the two cases.

1) $m_{ij} < \infty$: W_J is dihedral of order $2m_{ij}$. Note that the maximum length of $w \in W_J$ is precisely m_{ij} (as the Cayley graph is a cycle of length $2m_{ij}$) and the element of maximal length is represented by the reduced expressions $s_i s_j \cdots s_i s_j$ and $s_j s_i \cdots s_j s_i$. This implies that v_J has to have length strictly smaller m_{ij}, else it would have a reduced expression ending in s_i, contradicting the above. Then v_J is a product of less than $\lfloor \frac{m_{ij}}{2} \rfloor$ terms $s_i s_j$, and each of the products $s_i s_j$ is a rotation about $\frac{2\pi}{m_{ij}}$ towards e_j.
   Using that e_i and e_j are at angle $\pi - \frac{\pi}{m_{ij}}$, we see that v_J rotates e_i at most about $\pi - \frac{2\pi}{m_{ij}}$ towards e_j. Thus, $v_J(e_i)$ still lies inside the cone spanned by e_i and e_j. It could very likely be that we have another reflection s_j, but as the angle between e_i and the reflecting line is $\frac{\pi}{2} - \frac{\pi}{m_{ij}}$, the resulting vector still lies in the cone.

2) $m_{ij} = \infty$: In the case of the infinite dihedral group, we already have seen in the proof of Theorem 2.2.1 that $(s_i s_j)^n(e_i) = e_i + 2n(e_i + e_j)$. Thus, $\lambda_i, \lambda_j \geqslant 0$ with $|\lambda_i - \lambda_j| = 1$.

This proves the last Claim. ■

Now that we have established that $w = vv_J$ holds, for the element $w(e_i)$ we have

$$w(e_i) = (vv_J)(e_i) = v(\lambda_i e_i + \lambda_j e_j) = \lambda_i \underbrace{v(e_i)}_{>0} + \lambda_j \underbrace{v(e_j)}_{>0}.$$

Claim 4 implies $\lambda_i, \lambda_j \geqslant 0$, allowing us to conclude $w(e_i) > 0$. □

The result of the previous theorem readily extends to its converse, offering valuable insights as well. We shall formally record this implication as a corollary.

Corollary 2.4.3. *In the previous theorem, set $w = \widetilde{w}s_i$ for $i \in I$. Then $\ell(\widetilde{w}s_i) < \ell(w)$ implies $\ell(\widetilde{w}s_i s_i) > \ell(\widetilde{w}s_i)$. Thus we have $(ws_i)(e_i) = -w(e_i) > 0$, or equivalently $w(e_i) < 0$.*

With Corollary 2.4.3 and Theorem 2.4.2 in hand, we will finally be able to deduce the injectivity and thus faithfulness of our representation ρ.

Corollary 2.4.4. *The homomorphism of Theorem 2.2.1 is injective and thus a faithful representation.*

Proof. Assume $w \in \ker\{\rho\}$ non-trivial. Then there is an $i \in I$ with corresponding $s_i \in S$, such that $\ell(ws_i) < \ell(w)$. Now Corollary 2.4.3 implies $w(e_i) < 0$, but $w(e_i) = e_i > 0$ as $\rho(w) = \mathrm{id}_V$, which is a contradiction and thus, the statement holds. □

2.5 THE FUNDAMENTAL DOMAIN AND STABILIZERS

In this section, we will prove that the fundamental chamber $C \subset V^*$ is indeed a fundamental domain for the action of W on its Tits cone WC. Moreover, we will work out how stabilizers of points look like and then show that the Tits cone is really a convex cone in the dual space V^*.

We start by translating the results of the previous section to the language of chambers and halfspaces. The following two lemmas capture this essence.

Lemma 2.5.1. *Let $w \in W$ and $i \in I$. The relation $\ell(s_i w) > \ell(w)$ is equivalent to saying that w leaves the open chamber $\mathring{C}$ inside the open halfspace A_i^*, corresponding to the generator s_i. Put more formally, this states that the set $w(\mathring{C})$ is a subset of the open halfspace A_i^*.*

Proof. Observe that $\ell(s_i w) > \ell(w)$ is equivalent to $\ell(w^{-1}s_i) > \ell(w^{-1})$. Then by by Theorem 2.4.2, we have that $w^{-1}(e_i) > 0$. Take an arbitrary point $\varphi \in \mathring{C}$ in the open fundamental chamber, then we have that $w(\varphi)(e_i) = \varphi(w^{-1}(e_i))$. Since the chamber C lies in A_i^* for all $i \in I$ and $\varphi \in \mathring{C}$, we see that $w(\varphi) > 0$ is equivalent to $w^{-1}(e_i) > 0$. But this is true by assumption and since φ is arbitrary, we conclude that $w(\mathring{C}) \subset A_i^*$. □

Of course this lemma also has a corresponding converse result as in the previous section.

Lemma 2.5.2. *Let $w \in W$ and $i \in I$. The relation $\ell(s_i w) < \ell(w)$ is equivalent to saying that w now moves the open chamber $\mathring{C}$ inside the open halfspace, complementary to A_i^*. More formally, this states that the set $w(\mathring{C})$ is a subset of the open halfspace $s_i(A_i^*)$, as the s_i permute A_i^* and its corresponding complementary halfspace.*

Proof. Take $w' = s_i w$ for some $i \in I$ and $w \in W$ so that $\ell(s_i w) > \ell(w)$. Then we have that the length of $s_i w' = w$ is strictly smaller then then length of w'. By applying above lemma we get that $w(\mathring{C}) \subset A_i^*$, which is equivalent to $s_i w'(\mathring{C}) \subset A_i^*$. Apply s_i to both sides leaves us with $w'(\mathring{C}) \subset s_i(A_i^*)$, which proves the lemma. □

Building onto this insight, we now want to study the action of parabolic subgroups on the Tits cone to get an understanding of the stabilizers of points. To do so, decompose the fundamental chamber into subsets corresponding to the parabolic subgroups of W as follows.

Definition 2.5.3. *Given a parabolic subgroup W_J corresponding to $J \subset I$, set*

$$C_J := \bigcap_{j \in J} H_j^* \cap \bigcap_{k \notin J} A_k^*.$$

We call these the corresponding parabolic subsets (of the fundamental chamber).

Example 2.5.4. 1. *When the set* J *is empty, the corresponding parabolic subset* $C_\emptyset$ *coincides with the entire chamber* C*. Conversely, when* J *contains all indices,* C_J *reduces to the singleton* $\{0\}$*.*

2. *If* J *is a proper subset of* I *with cardinality one, then the corresponding subset* C_J *coincides with a codimension-one face of the chamber* C*.*

Theorem 2.5.5. *Let* $w \in W$ *and* $J, K \subset I$ *be subsets. Then* $w(C_J) \cap C_K \neq \emptyset$ *implies* $J = K$*,* $w \in W_J$ *and thus* $w(C_J) = C_J$*. In particular, the isotropy groups of the sets* C_J *are the parabolic subgroups* W_J*.*

Proof. Let $w \in W$ and $J, K \subset I$ be subsets, such that $w(C_J) \cap C_K \neq \emptyset$. The proof is by induction on the length of w. The base case $\ell(w) = 0$ is trivial, as then w is trivial.

Assume that $\ell(w) > 0$ and choose $i \in I$, such that $\ell(s_i w) < \ell(w)$. Writing $w = s_i(s_i w)$, by Lemma 2.5.2 we know that w moves the open chamber $\mathring{C}$ into the open halfspace $s_i(A_i^*)$, i.e. $w(\mathring{C}) \subset s_i(A_i^*)$. Now using the continuity of the group action, we note that $w(C) \subset \overline{s_i(A_i^*)}$. Recall that by definition, the fundamental chamber C lies in the halfspaces $\overline{A_i^*}$ for all $i \in I$. Thus, we record that $w(C) \cap C \subset H_i^*$ and since s_i fixes the corresponding H_i^* by definition, it fixes every point in the intersection of C and its translate $w(C)$. Note that the sets C_J and C_K are subsets of the fundamental chamber C and therefore, s_i fixes every point in the non-empty set $w(C_J) \cap C_K$. But if s_i fixes some point φ in C_K, we calculate

$$\varphi(e_i) = s_i(\varphi)(e_i) = \varphi(s_i(e_i)) = -\varphi(e_i) \implies \varphi(e_i) = 0 \iff \varphi \in H_i^*.$$

We deduce $i \in K$, respectively $s_i \in W_K$. Using this together with the assumption, we get that $s_i w(C_J) \cap C_K = s_i(w(C_J) \cap C_K)$ is non-empty. We apply the induction hypothesis to the element $s_i w$, to see that $J = K$ and $s_i w \in W_J$. Finally, since $s_i \in W_J = W_K$, we have that $s_i w \in W_J$ implies $w \in W_J$, proving the theorem. □

Before proceeding with proving that the fundamental chamber lives up to its name, we want to clearly state what is meant by a fundamental domain.

Definition 2.5.6. *Let* G *be a group, acting on a topological space* X*. We call a closed subset* $F \subset X$ *a* (strict) fundamental domain*, if for each* $x \in F$ *its orbit* $\mathrm{Orb}(x)$ *intersects* F *in exactly one point.*

Note that, by definition of the Tits cone WC, every W-orbit of a point $\varphi \in C$ meets the fundamental chamber C in at least one point, namely φ. Thus, it suffices to proof that each W-orbit meets C in at most one point, to prove the following theorem.

Theorem 2.5.7. *The fundamental chamber is a fundamental domain for the action of the Coxeter group* W *on its Tits cone* WC*, justifying its name.*

Proof. Assume that $\varphi, \psi \in C$ lie in the same W-orbit, but in different parabolic subsets C_J, respectively C_K of the fundamental chamber. Since they lie in the same orbit, there is a $w \in W$ with $\varphi = w(\psi)$. Thus, the intersection $w(C_J) \cap C_K$ is non-empty and Theorem 2.5.5 implies equality of J and K, as well as $w \in W_J$. We deduce $\varphi = w(\psi) = \psi$. Thus, every W-orbit of a point $\varphi \in C$ meets the fundamental chamber C at most in φ, proving the theorem. □

Define a set $\mathcal{C}$ as the union of all translates of possible parabolic subsets C_J i. e., define $\mathcal{C}$ by

$$\mathcal{C} := \bigcup_{J \subset I} \bigcup_{w \in W/W_J} w(C_J).$$

We want to emphasize here that by Theorem 2.5.5 the sets of the form $w(C_J)$ in the Tits cone WC are all disjoint for different $J \subset I$ and w ranging over the coset W/W_J. Thus, the sets of $\mathcal{C}$ form a partition of the Tits cone. This decomposition (altough not into chambers) is a key component in the following theorem.

Theorem 2.5.8. *The Tits cone WC is a convex cone, and every closed line segment in the Tits cone meets only finitely many of the sets in $\mathcal{C}$.*

Proof. First note that the fundamental chamber is a convex cone as the intersection of the finitely many closed halfspaces $\overline{A_i^*}$. This implies that the Tits cone is a cone as well. We will prove the convexity by showing that every closed segment between any two points in the Tits cone is contained in it. Furthermore, we will prove that these segments can be covered by finitely many of the sets in the above defined union $\mathcal{C}$, implying latter statement.

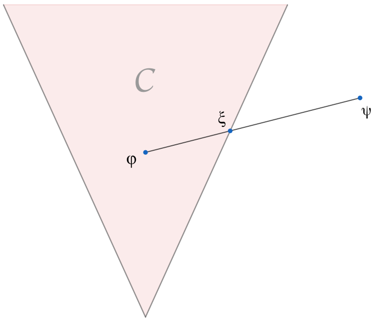
Consider the closed segment $[\varphi, \psi]$ with $\varphi, \psi \in WC$ and assume the endpoints lie in different chambers. Proceed by induction on the word length $\ell(w)$. The base case $\ell(w) = 0$ reduces to $\varphi, \psi \in C$. Since C is convex and can trivially be covered by finitely many of the C_J, this case has been dealt with.

Therefore, we now assume $\ell(w) > 0$. Intersect the segment $[\varphi, \psi]$ with the fundamental chamber C, to receive two new segments $[\varphi, \xi]$ and $[\xi, \psi]$ (cf. Figure 2.2a). The first segment can be covered by finitely many of the sets in $\mathcal{C}$, as it lies inside the fundamental chamber C. Thus, we need to show that we can cover the second segment $[\xi, \psi]$ by finitely many of these sets. Assume further, we have a $J \subset I$, such that $\psi \in s_i(A_i^*)$ for an $i \in J$ and $\psi \in \overline{A_i^*}$ for all $i \notin J$. Then we have that $\psi \notin C$.

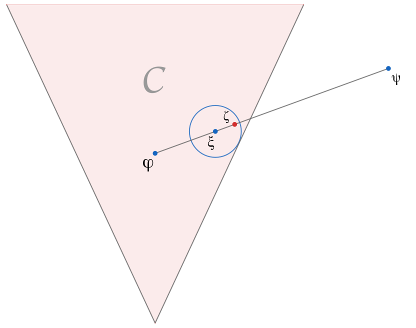
Claim 1: ξ has to lie in one of the codimension-one faces H_i^*, $i \in J$.

Proof of Claim 1. Assume that ξ lies in the open halfspace A_i^* for some $i \in J$. Then every point ζ in the intersection of a neighborhood of ξ, contained in A_i^*, with the segment $[\xi, \psi]$ has to also lie in A_i^*. Clearly, $\zeta \in \overline{A_i^*}$ for $i \notin J$ holds as well, implying that $\zeta \in C$. But this is a contradiction to the decomposition of the initial segment $[\varphi, \psi]$. Thus, ξ has to lie in one of the H_i^*. ■

Using the assumptions $\psi \in s_i(A_i^*)$ and $\psi \in w(C)$, we deduce $w(C) \subset \overline{s_i(A_i^*)}$, hence by continuity of the action $w(\mathring{C}) \subset s_i(A_i^*)$. By Lemma 2.5.2 this is equivalent to $\ell(s_i w) < \ell(w)$ and we are set up to apply the induction hypothesis to ξ and $s_i(\psi) \in s_i w(C)$. This produces a cover of $[\xi, s_i(\psi)]$ by finitely many sets in $\mathcal{C}$. But since we established that ξ has to lie in H_i^*, translation by s_i gives $[s_i(\xi), s_i^2(\psi)] = [\xi, \psi]$, and thus we can cover this segment by finitely many sets as well. The result follows. □



(a) A path through the Tits cone.



(b) A neighborhood of ξ.

To summarize, the essence of this section is that first of all the fundamental chamber is indeed a fundamental domain of the action of a Coxeter group on its Tits cone. This was shown in Theorem 2.5.7. Furthermore, we want to emphasize that each point in the interior of the fundamental chamber has trivial stabilizer, which is a direct consequence of Theorem 2.5.5. Moreover, every point in a parabolic subset C_J of C, for J a subset of I, is stabilized by the corresponding parabolic subgroup W_J.

2.6 COVERING ACTIONS – A TOOLBOX

In this section we will assemble a set of tools, mostly coming from algebraic topology, connecting group actions to the notion of coverings. They will be somewhat essential in the proof of the main theorem in the next chapter. We start by saying what is meant by a covering.

Definition 2.6.1. *Let X, Y be topological spaces. A continuous map $p : Y \to X$ is a* covering map *and Y a* covering space *for X, if every point x in X has an open neighborhood U, such that the preimage of U under p is a disjoint union of open sets U_i in Y for an index set I. Furthermore, the map p has to be a local homeomorphism, meaning the restriction of p to the U_i is a homeomorphism onto its image $p(U_i)$. Then, U is called* evenly covered *and $|I|$ the degree of the covering, while the open sets U_i are called the* sheets *over U and the preimage of an x in X is called a* fiber *of x.*

It turns out that restricting the action of a group on a topological space in the right way will give coverings by quotienting out the action. We make this precise in the following.

Definition 2.6.2. *Let G be a group, acting on a space X. The action is said to be* properly discontinuous, *if every point in X has a neighborhood U such that the set $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is finite.*

Definition 2.6.3. *Let G be a group, acting on a space X. The action is said to be a* covering (space) action *if every point has a neighborhood U such that the set $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ consists only of the neutral element.*

Obviously, this last condition is even more restrictive. Yet, at least for Hausdorff spaces, there is a connection between the two. In addition to acting properly discontinuously, we have to demand the action to be free.

Lemma 2.6.4. *Let* G *be a group that acts freely and properly discontinuously on a Hausdorff space* X. *Then the action of* G *is a covering action in the above sense.*

Proof. Let X be a Hausdorff space and G be a group acting freely, properly discontinuously on X. Then for an open neighborhood U of $x \in X$, the set $M := \{g \in G \mid gU \cap U \neq \emptyset\}$ is finite. For these $g \in M$ we pick pairwise disjoint Neighborhoods V_g of gx, which is possible since X is Hausdorff and G acts freely (thus $gx \neq x$). Finally, set $V = \left(\bigcap_{g \in M} g^{-1}V_g\right) \cap U$, which is open as finite intersection of open sets and by definition a neighborhood of x. □

To finally construct coverings from group actions and connect them in some sense, we need a last condition on the space the group acts on.

Definition 2.6.5. *A path-connected topological space* X *is called* simply connected, *if* $\pi_1(X) \cong \{1\}$.

Lemma 2.6.6. *Let* G *be a group, acting by a covering action on a simply connected topological space* X. *Then the quotient map* $p_G : (X, x_0) \to (G\backslash X, \mathrm{Orb}(x_0))$ *is a covering map and* $\pi_1(G\backslash X, \mathrm{Orb}(x_0)) \cong G$.

Proof. Let U be an open neighborhood of $x \in X$, such that $\{g \in G \mid gU \cap U \neq \emptyset\} = \{1\}$.

Claim 1: The map p_G restricted to U is a continuous bijection onto its image $p_G(U)$.

Proof of Claim 1.

- Continuity: $V \subset G\backslash X$ is open if and only if $p_G^{-1}(V)$ is open and thus p_G is continuous.
- Surjectivity: Since orbits of points are non-empty, p_G is surjective.
- Injectivity: Assume $x, y \in X$ distinct with $p_G(x) = p_G(y)$. But then there is a non-trivial element $g \in G$, $g \neq 1_G$ with $gx = y$, implying $g(U) \cap U \neq \emptyset$. This is a contradiction to G acting by a covering action. ■

Claim 2: Every point $\mathrm{Orb}(x) \in G\backslash X$ has a neighborhood $V \subset G\backslash X$ with $p_G^{-1}(V) = \bigsqcup_{g \in G} gU$.

Proof of Claim 2. By assumption the sets gU are all disjoint neighborhoods of $gx \in X$ for all $g \in G$. Moreover, for $V := p_G(U) \subset G\backslash X$, we have $p_G^{-1}(V) = \bigsqcup_{g \in G} gU$, proving the Claim. ■

Claim 3: The map $p_G|_{gU} \to V = p_G(U)$ is a homeomorphism for all $g \in G$.

Proof of Claim 3. Note that the action of G on X is by homeomorphisms and $U \subset X$ is open. Then the union $\bigsqcup_{g \in G} gU$ is open as well and since $p_G^{-1}(V) = \bigsqcup_{g \in G} gU$, the set V is open in the quotient topology of $G\backslash X$. Therefore, the map p_G restricted to gU for $g \in G$ is an open map and hence, by *Claim 1* a homeomorphism. ■

Now all conditions in Definition 2.6.1 are satisfied, so p_G is a covering map. For the last part of the statement, note that X is simply connected, implying that it is the universal cover of $G\backslash X$. By definition, G is the group of Deck transformations, thus $\pi_1(G\backslash X, \mathrm{Orb}(x_0)) \cong G$. □

3 THE MAIN THEOREM

Let us quickly recall the main theorem we want to prove.

Theorem. *A finitely generated right-angled Coxeter group* W *has a finite index subgroup* W'*, such that* W' *is residually finite and rationally solvable (RFRS).*

Before proceeding with the proof, we should make clear what we have to show. This is essentially the existence of a cofinal sequence of subgroups, satisfying the RFRS condition. But what does it mean to be RFRS? We shall give the definition, to shed some light onto our goal.

Definition 3.0.1. *Let* G *be a group. Then,* G *satisfies the RFRS condition, if there is a sequence of subgroups* $G = G_0 > G_1 > \dots$*, satisfying the following conditions.*

1. *for each* i*,* $G_i \triangleleft G$ *is a normal subgroup of* G*,*
2. *for each* i*, the index* $[G : G_i]$ *is finite,*
3. *the intersection* $\bigcap_i G_i$ *is the trivial group and*
4. *for each* i*,* $(G_i)_r^{(1)} := \ker\{G_i \to \mathbb{Q} \otimes_{\mathbb{Z}} (G_i)_{ab}\}$ *is a subgroup of* G_{i+1}*.*

Observe that in praxis, it suffices to show that conditions 2 to 4 hold. To see this, let $G = G_0 > G_1 > \dots$ be a cofinal sequence of finite index subgroups $G > G_i$ with $(G_i)_r^{(1)} \leqslant G_{i+1}$. Then, for each i we pass to the core of G_i, given by $\mathrm{Core}(G_i) := \bigcap_{g \in G} gG_ig^{-1}$. By construction the core of a subgroup G_i is normal in G and furthermore, we claim the following.

Claim: If G_i has finite index in G, its core will also have finite index in G.

Proof of Claim. Let G_i be a subgroup in G with $[G : G_i] < \infty$. Then, G acts (by left multiplication) on the left cosets of G_i in G and the kernel of this action consists of the $h \in G$ with

$$hgG_i = gG_i \iff g^{-1}hgG_i = G_i \iff g^{-1}hg \in G_i \iff h \in gG_ig^{-1} \quad \forall g \in G.$$

We note that the kernel is exactly the core of G_i. Moreover, the quotient ${}^{G}\!/_{\mathrm{Core}(G_i)}$ embeds into the symmetric group $\mathrm{Sym}({}^{G}\!/_{G_i})$, whose order is $[G : G_i]!$. Thus, we see that if G_i has finite index in G, the core has finite index as well. ∎

So, it remains to check that $(\mathrm{Core}(G_i))_r^{(1)}$ is a subgroup of $\mathrm{Core}(G_{i+1})$. Clearly, for $g \in G$ we have the equality $g((G_i)_r^{(1)})g^{-1} = (gG_ig^{-1})_r^{(1)}$, from which we obtain

$$(\mathrm{Core}(G_i))_r^{(1)} \leqslant \bigcap_{g \in G} (gG_ig^{-1})_r^{(1)} = \bigcap_{g \in G} g(G_i)_r^{(1)}g^{-1} \leqslant \bigcap_{g \in G} gG_{i+1}g^{-1} = \mathrm{Core}(G_{i+1}).$$

We see that if we are willing to pass to the sequence of core subgroups, we can drop the first condition in the definition of the RFRS condition.

Remark 3.0.2. *Note that tensoring an abelian group* A *(as* $\mathbb{Z}$*-module) with the rationals, 'kills' the torsion part of* A*, whence* $\mathbb{Q} \otimes_{\mathbb{Z}} A \cong {}^{A}\!/_{Torsion}$*. In particular, for the abelianization* G_{ab}*, the group* $\mathbb{Q} \otimes_{\mathbb{Z}} G_{ab}$ *is isomorphic to the torsion free abelianization* ${}^{G_{ab}}\!/_{Torsion}$*.*

3.1 CONSTRUCTION OF THE MANIFOLD COVER

Consider the abelianization W_{ab} of W, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Now, the abelianization yields a homomorphism $\alpha : W \to W_{\text{ab}}$ to whose kernel $\ker \alpha$, we will turn our attention to in the following. Note that by the first homomorphism theorem, $\ker \alpha$ has finite index in W, since

$$\left| {}^{W}\!/_{\ker \alpha} \right| = \left| {}^{\mathbb{Z}}\!/_{2\mathbb{Z}} \right|^n = 2^n < \infty.$$

Here we used the fact that W is finitely generated, whence $n = |I| = |S| < \infty$.

Furthermore, note that for each $J \subset I$ with W_J finite, we have an isomorphism between W_J and $({}^{\mathbb{Z}}\!/_{2\mathbb{Z}})^{|J|}$. Thus, the restriction of α to each such subgroup $\alpha|_{W_J}$ is an injective homomorphism. We now use that in the right-angled case, the isotropy subgroups of codimension-k faces are all of this form. This can be seen, as the isotropy subgroup of a codimension-k face F is generated by the reflections in the k codimension-1 faces, whose intersection forms F. As all these codimension-1 faces meet at angle $\frac{\pi}{2}$, the generators commute pairwise. Therefore, all isotropy subgroups inject into the abelianization of W. This implies that the intersection of an isotropy subgroup with the kernel of α is trivial and consequently no isotropy group is contained in the kernel $\ker \alpha$. Since finite subgroups are contained in isotropy subgroups, the kernel $\ker \alpha$ acts freely on the Tits cone WC corresponding to W.

In particular, by Theorem 2.5.5 the action of W on the interior of its Tits cone $\mathrm{int}(WC)$ is properly discontinuously. By Theorem 2.5.8 the Tits cone is also a convex cone, implying that it has trivial fundamental group, whence is simply-connected. Having all this information, we are able to apply Lemma 2.6.6 to obtain the covering

$$\mathrm{int}(WC) \longrightarrow {}^{\mathrm{int}(WC)}\!/_{\ker \alpha} \qquad x \mapsto \mathrm{Orb}_{\ker \alpha}(x).$$

We conclude the results of this section in the following proposition.

Proposition 3.1.1. *Let* W *be a right-angled Coxeter group with corresponding Tits cone* WC*. Then, there is a finite index subgroup* $W' \leqslant W$*, acting by covering action on the interior of the Tits cone, implying that the quotient* ${}^{\mathrm{int}(WC)}\!/_{W'}$ *is a manifold. In particular,* $W' = \ker\{W \to W_{ab}\} = \ker \alpha$*.*

3.2 SOME ORBIFOLD THEORY

In this section, we want to elaborate more on the natural orbifold structure of the fundamental chamber C. We start by giving a formal definition of an orbifold. We break the definition down into smaller pieces, starting with local models, sometimes called (orbifold) charts.

Definition 3.2.1. *A* local model *is a pair* $(\widetilde{U}, \Gamma)$*, where* $\widetilde{U} \subset \mathbb{R}^n$ *is open and* Γ *is a finite subgroup of the group of diffeomorphisms of* $\widetilde{U}$*, denoted* $\text{diffeo}(\widetilde{U})$*, acting on* $\widetilde{U}$*. By abusing notation, we will sometimes say that the quotient* $U = {}^{\widetilde{U}}\!/\!_{\Gamma}$ *is the local model.*

Now that we have defined the local structure of an orbifold, we want to translate between these local models. This is being made precise by orbifold maps.

Definition 3.2.2. *An* orbifold map *between local models* $(\widetilde{U}_i, \Gamma_i), (\widetilde{U}_j, \Gamma_j)$ *is a pair of maps* $(\widetilde{\psi}, \varphi)$*, consisting of a smooth map* $\widetilde{\psi} : \widetilde{U}_i \to \widetilde{U}_j$ *and a homomorphism of groups* $\varphi : \Gamma_i \to \Gamma_j$*. We enforce the map* $\widetilde{\psi}$ *to be* φ*-equivariant, meaning that for all* $g \in \Gamma_i$ *and all* $\widetilde{x} \in \widetilde{U}_i$*,* $\widetilde{\psi}(g\widetilde{x}) = \varphi(g)\widetilde{\psi}(\widetilde{x})$ *holds. Then* $\widetilde{\psi}$ *induces a map* $\psi : {}^{\widetilde{U}_i}\!/\!_{\Gamma_i} \to {}^{\widetilde{U}_j}\!/\!_{\Gamma_j}$*, between the local models. When all three of these maps are injective, we call* ψ *a local isomorphism.*

Now that we have these local definitions, we 'glue' them together, to obtain an orbifold. Before we do so, we recall some notions from topology.

Suppose, we are given an open cover $\{U_i\}$ of a topological space X. It is said to be *locally finite*, if every $x \in X$ admits a neighborhood N such that $N \cap U_i$ is empty for all but finitely many i. The open cover $\{U_i\}$ is a *refinement* of an open cover $\{V_j\}$ of X, if for every V_j there is a U_i with $U_i \subseteq V_j$. Now, the topological space X is said to be *paracompact*, if every open cover of X admits such a locally finite refinement.

Definition 3.2.3. *An* n*-dimensional* (smooth) orbifold Q *is a pair* $(X_Q, \mathcal{A})$*. The space* X_Q *is a paracompact Hausdorff space, called the* underlying space*. The set* $\mathcal{A}$ *is called an* orbifold atlas*, consisting of charts* (U_i, ϕ_i)*, indexed by some set* I *and satisfying the following conditions:*

- *the* U_i *form an open cover of the underlying space* X_Q*,*
- *for each* U_i *there exists a local model* ${}^{\widetilde{U}_i}\!/\!_{\Gamma_i}$ *with a homeomorphism* $\phi_i : U_i \to {}^{\widetilde{U}_i}\!/\!_{\Gamma_i}$ *and*
- *charts have to be compatible, meaning that for* $U_i \subset U_j$ *the inclusion is a local isomorphism.*

To sketch the connection between manifolds and orbifolds, let us mention one more thing.

Definition 3.2.4. *The* local group $\text{loc}(x)$ *of some* x *in a local model* ${}^{\widetilde{U}}\!/\!_{\Gamma}$ *is the isotropy group of any* $\widetilde{x}$ *in* $\widetilde{U}$*, getting projected onto* x*. The* singular locus $\Sigma(Q)$ *of an orbifold* Q *consists of all points in the underlying space* X_Q *with non-trivial local group, i.e.* $\Sigma(Q) = \{x \in X_Q \mid \text{loc}(x) \neq \{1\}\}$.

By this definition, we see that an orbifold with empty singular locus is just a manifold. Furthermore, when thinking about an orbifold, we can just think about the underlying space and label each element in the singular locus by its local group.

Many basic examples arise by taking the quotient relative to a properly discontinuous group action on $\mathbb{R}^n$. We want to mention at least some examples.

Example 3.2.5.

1. *Consider the flat plane $\mathbb{R}^2$ with the action of a cyclic group $\mathbb{Z}/n\mathbb{Z}$ by rotation about the origin. The arising orbifold is a cone with singular point the origin and cone angle $\frac{2\pi}{n}$.*
2. *Consider the sphere S^2 with the action of a cyclic group $\mathbb{Z}/2\mathbb{Z}$ by rotation about the north pole N. The arising orbifold now is called a* teardrop *with singular point, the north pole N.*

By starting with a topological space X, this is actually a more general way to construct orbifolds. Given a properly discontinuously action of a group G on X, simply pass to the quotient space X/G to obtain an orbifold. In contrast, to obtain a manifold, one has to demand the action to be free as well. This is not the only analogy between manifolds and orbifolds. The concept of orbifold coverings translates almost one to one from the ordinary case.

Definition 3.2.6. *An orbifold covering $p : Q' \to Q$ is a continuous map on the underlying spaces $X_{Q'} \to X_Q$, such that for each point $x \in X_Q$ there is a local model $U = \widetilde{U}/\Gamma$ around x and each component V_i of $p^{-1}(U)$ is homeomorphic to $\widetilde{U}/\Gamma_i$ for a subgroup Γ_i of Γ. Furthermore, the restriction $p|_{V_i} : V_i \to U$ corresponds to the natural projection $\widetilde{U}/\Gamma_i \to \widetilde{U}/\Gamma$.*

Remark 3.2.7. *We want to mention here, that the universal cover of an orbifold is exactly what we think of, i.e. the initial object in the category of orbifold coverings. Moreover, there is a notion of orbifold fundamental groups as well and they behave to orbifold covers the same as fundamental groups to coverings in the ordinary case. In particular, subgroups (of index n) of the orbifold fundamental group correspond to (n-sheeted) orbifold coverings.*

Having all this in mind, we have two ways of approaching the fundamental chamber C. One way, it obtains its natural orbifold structure as the quotient $\mathrm{int}(WC)/W$. As the Tits cone is simply connected, it is the universal cover of C and we observe that the orbifold fundamental group of C is precisely W. On the other hand, the fundamental chamber is the quotient of the manifold $\mathrm{int}(WC)/W'$ by a finite group action.

Since W' is a finite index subgroup of W, Remark 3.2.7 implies that we get a covering map p so that the diagram on the right commutes. Thus, both of the approaches to the fundamental chamber C agree in the sense that the projection map $\mathrm{int}(WC) \to \mathrm{int}(WC)/W$ factors through the manifold cover, constructed in Section 3.1.

$$\begin{tikzcd}
\mathrm{int}(WC) \arrow[r] \arrow[d] & \mathrm{int}(WC)/W \\
\mathrm{int}(WC)/W' \arrow[ur, dashed, "p"'] &
\end{tikzcd}$$

3.3 THE COFINAL COVER

In this section, we want to at least *sketch* the proof of the existence of a cofinal sequence of two-fold covers, arising by iteratively reflecting in faces. Some more results from the theory of orbifolds are needed for the whole proof. We will state and use them without proof.

In the following, doubling along a face means the construction of a sequence of polytopes $P_k = P_{k-1} \cup g_{k-1}(P_{k-1})$, where g_{k-1} is the reflection in one of the faces of the previous polytope P_{k-1}. The following proposition will tell us, that in the case of a right-angled Coxeter group, we are able to double the fundamental chamber C in one of its faces and are still able to cover the whole Tits cone WC by reflecting in the faces of the resulting chamber.

Proposition 3.3.1. *Let $P_k \subset \mathbb{R}^n$ be a polytope with orbifold structure, orbifold fundamental group $\pi_1^{orb}(P_k)$ and universal cover $\widetilde{P}$. Then, the fundamental group $\pi_1^{orb}(P_{k+1})$ is isomorphic to the group generated by reflections in the faces of the double P_{k+1}. Furthermore, the fundamental group $\pi_1^{orb}(P_{k+1})$ of the double is an index-2 subgroup of $\pi_1^{orb}(P_k)$ and the following two conditions are satisfied.*

1. $\forall k \in \mathbb{N} : \forall g \in \pi_1^{orb}(P_k) : g(\mathring{P}_k) \cap \mathring{P}_k \neq \emptyset \implies g = 1$, *and*
2. $\forall k \in \mathbb{N} : \bigcup_{g \in \pi_1^{orb}(P_k)} g(P_k) = \widetilde{P}$.

Proof. We won't proof this here. □

To state the other helping lemma we will need, we first want to show that each chamber in the Tits cone can uniquely be labeled by elements of its corresponding group.

Lemma 3.3.2. *Let W be a Coxeter group and WC its Tits cone. The chambers of the form $w(C)$ in the Tits cone can be labeled uniquely by elements in W.*

Proof. Let $v(C), w(C)$ be chambers in WC for $v, w \in W$, such that $v \neq w$ and $w(C) = v(C)$. But then, by Theorem 2.5.5 we have that

$$w(\mathring{C}) \cap v(\mathring{C}) = w(\mathring{C} \cap w^{-1}v(\mathring{C})) \neq \emptyset \implies w^{-1}v = 1 \iff w = v.$$

Thus, the chambers can be labeled in a unique way. □

We proceed with another helpful result, whose proof relies on the aforementioned results.

Lemma 3.3.3. *Let $C \subset \mathbb{R}^n$ be the fundamental chamber of a Coxeter group W. The non-trivial labels $w_i \neq 1_W$ of the chambers, defining the polytope $P = \bigcup_{i=1}^n w_i(C)$, are* not *contained in the reflection group, generated by reflecting in its faces.*

Proof. Note that by Lemma 3.3.2, the chambers of the form $w(C)$ are uniquely labeled by $w \in W$. Thus, let $P := \bigcup_{i=1}^n w_i(C)$ be as in the lemma and w_i one of its defining labels. Assume, $w_i \neq 1_W$ is contained in the reflection group generated by reflecting in the faces of P. But then we have that $w_i(\mathring{P}) \cap \mathring{P} \neq \emptyset$, contradicting Proposition 3.3.1. □

Using this insight, we define a graph G with vertices $V(G) := \{w(C) \mid w \in W\}$ and edges $E(G) := W \times S$. The endpoint map is given by $\delta : W \times S \to 2^{V(G)}$, $(w, s) \mapsto (w(C), ws(C))$. Thus, two vertices $w(C)$ and $v(C)$ have an edge, if there is an $s \in S$, such that $ws = v$. Then, clearly we have $wsw^{-1}w(C) = ws(C) = v(C)$. Note that the element wsw^{-1} is the only element, that flips the edge $\{w(C), ws(C)\}$. Thus, the map $W \times S \to W$, $(w, s) \mapsto wsw^{-1}$ defines a labeling of the edges by elements in W. We will call the combinatorial graph of G (meaning that every double edge gets collapsed) the *chamber graph* $\mathrm{Cham}(W, S)$ of W.

It is obvious, that this graph is isomorphic to the combinatorial Cayley graph of the group W and we see that the Cayley graph 'embeds' into the Tits cone WC. This will be useful to prove the main theorem of this section next.

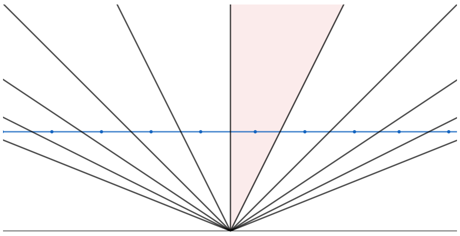
Theorem 3.3.4. *The doubling sequence P_k, starting in the fundamental chamber $P_0 = C$, is cofinal. This means, the chamber P_n for some n, covers the whole Tits cone.*

Proof. The proof is by induction on the word length $\ell(w)$. For $\ell(w) = 0$, the corresponding chamber is the fundamental chamber C, which is covered by itself.

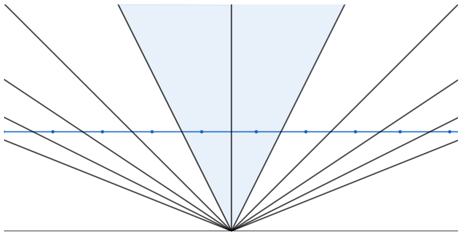
(IH) Chambers with labels v of length $\ell(v) = k - 1$ are covered by a polytope $P = \bigcup_{i=1}^{n} w_i(C)$.

We need to show that chambers of the form $w(C)$ with $\ell(w) = k$ can be covered by reflecting in a face of the polytope P. Using the chamber graph $\mathrm{Cham}(W, S)$, constructed above, we deduce that for each such w, there is a $v \in W$ of length $\ell(v) = k - 1$, connected to w by an edge in $\mathrm{Cham}(W, S)$. This is due to the fact that the chamber graph is isomorphic to the Cayley graph of W and the length function of W is induced by the Cayley graph. But then, the chamber $w(C)$ is adjacent to the chamber $v(C)$ and the latter is contained in P by **(IH)**. Thus, either $w(C)$ itself is already contained in P or we reflect in the face shared by $w(C)$ and $v(C)$ to cover $w(C)$ with the double P' of P. Now, continuing with the next element w of length k, we eventually double P' and repeat this for all such w. This iterative doubling process is valid by Proposition 3.3.1 and by induction every chamber $g(C)$ for $g \in W$ can be covered. □

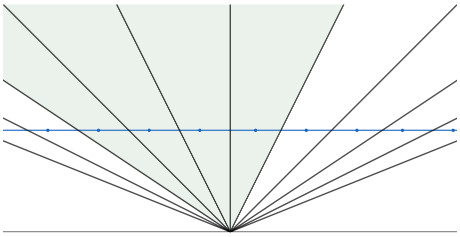
Remark 3.3.5. *Using Proposition 3.3.1, we see that the cofinality of this sequence translates to the group side. More precisely, this sequence of two-fold covers induces a sequence of index-2 subgroups via the fundamental groups of the covering spaces. As we will at some point cover the whole Tits cone, whose fundamental group is trivial, the intersection of all these groups is trivial.*



(a) The fundamnetal chamber C.



(b) The double polytope $P = C \cup s_1(C)$



(c) The double polytope $P' = P \cup s_1 s_2 s_1(P)$

Figure 3.1: An example doubling sequence in D_∞

3.4 THE MANIFOLD SEQUENCE

We finally have developed enough tools, to construct the desired sequence of subgroups in W'. Following Theorem 3.3.4, we fix a cofinal doubling sequence P_k, starting in the fundamental chamber $P_0 = C$, i.e. a cofinal sequence of two-fold orbifold covers $C \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$. Then, we define the groups $W_i := \pi_1^{\mathrm{orb}}(P_i) \cap W'$ for $i \in \{0\} \cup \mathbb{N}$, each inducing a manifold ${}^{WC}/_{W_i}$. Using the theory of orbifold coverings, we mention that these manifolds cover the orbifolds ${}^{WC}/_{\pi_1^{\mathrm{orb}}(P_i)}$. By Remark 3.3.5, the groups W_i form a cofinal descending series of subgroups.

Thus, to show that W' is RFRS, it remains to show the following two conditions:

1. each W_i has finite index in W' and
2. $(W_i)_r^{(1)} := \ker\{W_i \to \mathbb{Q} \otimes_{\mathbb{Z}} (W_i)_{ab}\}$ is a subgroup of W_{i+1}.

We will start with the first condition.

Lemma 3.4.1. *The homomorphism* $\phi : {}^{W_i}/_{W_{i+1}} \to {}^{\pi_1^{orb}(P_i)}/_{\pi_1^{orb}(P_{i+1})}$ *is injective.*

Proof. □

Corollary 3.4.2. *The index of W_{i+1} in W_i is less than or equal to 2.*

Proof. □

Corollary 3.4.3. *The index of W_i in W' is finite.*

Proof. □

3.5 LOOPS BOUNCING OFF FACES

We still have to show that the rational derived group $(G_i)_r^{(1)}$, given by $\ker\left\{G_i \to \mathbb{Q} \otimes_{\mathbb{Z}} {}^{G_i}\!/\!_{G_{i+1}}\right\}$ is a subgroup of the group G_{i+1}. Recall that a quotient group ${}^{H}\!/\!_{N}$ is abelian, if and only if the derived subgroup $H^{(1)} := [H, H]$ is contained in the normal subgroup N. As we are in the situation that we have either ${}^{G_i}\!/\!_{G_{i+1}} \cong {}^{\mathbb{Z}}\!/\!_{2\mathbb{Z}}$ or ${}^{G_i}\!/\!_{G_{i+1}} \cong \{1\}$, we conclude that the derived subgroup $G_i^{(1)}$ is a subgroup of G_{i+1}. Thus, to show that $(G_i)_r^{(1)}$ is a subgroup of G_{i+1}, it suffices to show that each element $g \in G_i$ with non-trivial image in the quotient ${}^{G_i}\!/\!_{G_{i+1}}$ is not torsion. These elements will then admit a factorization through the torsion-free abelianization.

$$\begin{tikzcd} G_i \arrow[rr] \arrow[dr] & & {}^{G_i}\!/\!_{G_{i+1}} \cong {}^{\mathbb{Z}}\!/\!_{2\mathbb{Z}} \\ & {}^{(G_i)_{ab}}\!/\!_{\text{Torsion}} \arrow[ur, dashed] & \end{tikzcd}$$

Now, take an element g in G_i that is not contained in G_{i+1} and a representative loop γ of g in the manifold D_i'. As discussed above, we have a projection map from D_i' to P_i. Watching the projection of γ bouncing off the faces inside the polytope P_i, we observe it hitting the face F_i an odd number of times. If it would hit F_i an even number of times, γ would lift to P_{i+1} and g would be contained in G_{i+1}, a contradiction. Thus, in the manifold cover D_i' of P_i, we see γ intersecting the face F_i an odd number of times.

To see that the loop γ represents an element of infinite order, we will construct a morphism of fundamental groups. Namely, from the fundamental group $\pi_1(D_i')$ of the manifold D_i' to the fundamental group of the circle $\pi_1(S^1)$. If γ maps non-trivially to $\pi_1(S^1)$, it has to represent an element of infinite order, as $\pi_1(S^1) \cong \mathbb{Z}$. Before proceeding with the morphism, we point out the two-sidedness of the face F_i in D_i'.

Definition 3.5.1. *Let* M *be a manifold and* $F \subset M$ *be a submanifold. Then,* F *is called* two-sided *in* M, *if locally it looks like the product with an intervall. More precisely, there is a neighborhood* N *with* $N \cong F \times (-\varepsilon, \varepsilon)$ *for suitable* $\varepsilon > 0$, *where* F *is identified with* $F \times \{0\}$.

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