Karlsruher Institut für Technologie

Bachelor thesis

RIGHT-ANGLED COXETER GROUPS ARE RFRS

NICLAS RIST

May 16, 2024

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LOCATION:
Karlsruhe, Germany

TIME FRAME:
Novembre 11, 2023 - May 16, 2024

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2.1 COXETER GROUPS

First of all, we define the main object of interest we want to study.

Definition 2.1.1. *Let* S *be a set consisting of elements* s_i *indexed by an index set* I *with* $|I| = n < \infty$. *Let* $(m_{ij}) = M$ *be a symmetrical matrix in* $(\mathbb{N} \cup \{\infty\})^{n \times n}$, *where* $m_{ii} = 1$ *for all* i *and* $m_{ij} \geqslant 2$ *for* $i \neq j$. *Define a group* W *via the following presentation:*

$$W := \langle S \mid (s_i s_j)^{m_{ij}} = 1 \text{ for all } i, j \in I \rangle.$$

The pair (W, S) *is called a* Coxeter System *and* M *is called the corresponding* Coxeter Matrix. *A* Coxeter group *is a group isomorphic to a group* W, *corresponding to a Coxeter System* (W, S). *It is generated by the set* S.

In this work we will be particularly interested in a special class of Coxeter groups that we call right-angled. They are defined as follows, by imposing significant constraints on the entries of the Coxeter matrix.

Definition 2.1.2. *A Coxeter System* (W, S) *is* right-angled *if, for all distinct* $i, j \in I$, *the condition* $m_{ij} \in \{2, \infty\}$ *is satisfied. In this context, the group* W *is then called a* right-angled Coxeter group (RACG).

We give some important examples of Coxeter groups as well as right-angled Coxeter groups.

Example 2.1.3.
1. *Dihedral groups,* $D_{2m} \cong \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1 s_2)^m = 1 \rangle$ *are Coxeter groups for all* $m \in \mathbb{N}$.
2. *The triangle groups,* $\Delta(l, m, n) \cong \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^l = (st)^m = (tr)^n = 1 \rangle$ *with* l, m *and* n *integers greater or equal to 2 are Coxeter groups.*
3. *The infinite Dihedral group,* $D_\infty \cong \langle s, t \mid s^2 = t^2 = 1 \rangle$ *is a right-angled Coxeter group.*
4. *The free product,* $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \langle r, s, t \mid r^2 = s^2 = t^2 = 1 \rangle$ *is a right-angled Coxeter group.*

We can define a special type of subgroup, called a parabolic subgroup, within a Coxeter group W. These subgroups are constructed from a subset of the index set I. The definition is as follows:

Definition 2.1.4. *Let (W, S) be a Coxeter System as above with finite index set I, and J be a subset of the index set I. The group $W_J := \langle \{s_j \mid j \in J\} \rangle$, generated by the s_j in S is then called a* parabolic subgroup *of W. Moreover, we call any conjugate of W_J a parabolic subgroup as well.*

Once we have constructed the representation of Coxeter groups on a vector space as well as the Tits cone in the coming section, the parabolic subgroups will be a useful tool to form a deeper understanding of these objects. We will extensively use them in sections 2.4 and 2.5.

2.2 REPRESENTATION OF COXETER GROUPS

Given a Coxeter System (W, S), let V be a real vector space with basis $\{e_1, \dots, e_n\}$, where $n = |I| = |S|$. This provides a natural identification, $GL_n(V) \cong GL_n(\mathbb{R})$. We define a bilinear form B_W on V as follows:

$$B_W := \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right) & , \ m_{ij} < \infty \\ -1 & , \ m_{ij} = \infty \end{cases}.$$

By Definition 2.1.1, it is assured that $m_{ij} \geqslant 2$ for distinct i, j, ensuring the cosine term is non-positive. Consequently, we have $B_W(e_i, e_j) \leqslant 0$ for distinct i, j. Furthermore, from $m_{ii} = 1$, it follows that $B_W(e_i, e_i) = 1$. Using this bilinear form, we define hyperplanes with corresponding reflections for each basis element e_i as follows:

$$H_i := \{v \in V \mid B_W(e_i, v) = 0\}, \qquad \sigma_i : V \to V, \quad v \mapsto v - 2B_W(e_i, v)e_i.$$

Theorem 2.2.1. *The map given by:*

$$\rho : W \to GL_n(V) \cong GL_n(\mathbb{R}), \quad s_i \mapsto \sigma_i$$

is an injective homomorphism and therefore a faithful representation of W.

Before we prove the homomorphism property, we want to recall: A map $\varphi : S \to G$ from a set S to a group G extends to a homomorphism $\hat{\varphi} : \langle S \mid R \rangle \to G$, if and only if the induced homomorphism $\overline{\varphi} : F_S \to G$ from the free group over S satisfies $\overline{\varphi}(r) = e_G$ for every $r \in R$.

Proof. Observe that $\sigma_i^2 = id$ in $GL_n(V)$ and thus, by applying the above to our situation we see that it suffices to show that the product $\sigma_i \sigma_j$ has order m_{ij} in $GL_n(V)$ for distinct $i, j \in I$. To do so, consider the two-dimensional subspace V_{ij} spanned by e_i and e_j in V. We take a general element $v = \lambda e_i + \mu e_j$, $\lambda, \mu \in \mathbb{R}$ in V_{ij} and distinguish the two cases in the definition of B_W:

1) $m_{ij} < \infty$: In this case B_W is positive definite, since for $v \neq 0$

$$B_W(v, v) = \lambda^2 - 2\lambda\mu\cos\left(\frac{\pi}{m_{ij}}\right) + \mu^2 = \left(\lambda - \mu\cos\left(\frac{\pi}{m_{ij}}\right)\right)^2 + \mu^2\sin^2\left(\frac{\pi}{m_{ij}}\right) > 0.$$

Up to a change of basis, we identify $(V_{ij}, B_W|_{V_{ij}})$ with the euclidean plane $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$. Now σ_i and σ_j act on V_{ij} by orthogonal reflections in the hyperplanes H_i, H_j intersected with V_{ij}. We take a look at the inner product of e_i, e_j

$$B_W(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right) = \cos\left(\pi - \frac{\pi}{m_{ij}}\right)$$

and obtain that the angle between them in V_{ij} is given by $\pi - \frac{\pi}{m_{ij}}$. Thus, the angle between the reflecting lines is $\frac{\pi}{m_{ij}}$ and the product $\sigma_i \sigma_j$ turns out to be a rotation by $\frac{2\pi}{m_{ij}}$, showing that $\sigma_i \sigma_j$ has order m_{ij} in the subspace V_{ij}. And since $\sigma_i \sigma_j$ fixes the orthogonal complement of V_{ij} by definition of the σ_i, it has order m_{ij} on the whole vector space V.

2) $m_{ij} = \infty$: By the following, we now have to deal with a non-positive definite form:

$$B_W(v, v) = \lambda^2 - 2\lambda\mu + \mu^2 = (\lambda - \mu)^2 \geqslant 0.$$

Indeed, we can only expect it to be positive semidefinite on V_{ij}. Using the calculation

$$(\sigma_i \sigma_j)(e_i) = \sigma_i(e_i + 2e_j) = e_i + 2(e_i + e_j),$$

together with an induction argument, we get that $(\sigma_i \sigma_j)^n(e_i) = e_i + 2n(e_i + e_j)$. Therefore, the concatenation has infinite order on V_{ij} and in particular on the whole of V.

This proves that ρ extends to a homomorphism. It remains to show the injectivity of ρ. This will be a consequence of a bigger result in section 2.4, see Corollary 2.4.4. □

Remark 2.2.2. *It is worth to note that in the above proof, we have seen that two-generator subgroups of Coxeter groups are dihedral. Either of order* $2m_{ij}$ *or infinite order.*

We want to extend the action of W to the dual of the vector space V. This is achieved by acting on V^* via the dual representation of ρ, which we define by

$$\rho^* : W \to GL_n(V^*), \quad w \mapsto (\rho^*(w)(\varphi))(v) = \varphi(\rho(w^{-1})(v)), \quad w \in W, \varphi \in V^*, v \in V.$$

Notation wise, we will simply write $w(v)$, when $w \in W$ acts on $v \in V$ via $\rho(w)(v)$. Similarly, we write $w(\varphi)$ when we mean that $w \in W$ acts on some element $\varphi \in V^*$ of the dual space, via the dual representation $\rho^*(w)(\varphi)$. As in the case of the vector space V, we want to give a definition for the notion of a hyperplane with corresponding reflection in the dual space V^* as well. By dual hyperplane, we mean a subspace $H_i^* := \{\varphi \in V^* \mid \varphi(e_i) = 0\}$, and the corresponding dual reflections will be a map from V^* to V^*, given by:

$$\sigma_i^* : V^* \to V^*, \quad \varphi \mapsto \varphi \circ \sigma_i = \varphi - 2B_W(e_i, \cdot\,)\varphi(e_i).$$

To further explore Coxeter groups and their action via this representation, we need some more notation. In particular, we want a so-called *chamber*. This should be thought of as a cone over a polytope with finitely many faces such that the reflections in its codimension one faces correspond to the generators of W under the representation.

Definition 2.2.3. *The fundamental chamber* C *of the dual representation is the set, given by*

$$C := \{\varphi \in V^* \mid \varphi(e_i) \geqslant 0 \ \forall i \in I\} \subset V^*.$$

Denote by $\{e_1^*, \dots, e_n^*\}$ the dual basis of V^* corresponding to the standard basis $\{e_1, \dots, e_n\}$ of V. Then we calculate, using the σ_i^* from above:

$$\sigma_i^*(e_j^*) = e_j^* - 2B_W(e_i, \cdot)e_j^*(e_i) = \begin{cases} e_j^* & \text{for } i \neq j \\ e_j^* - 2B_W(e_j, \cdot) & \text{for } i = j \end{cases},$$

which implies that each reflection σ_i^* fixes all the hyperplanes H_j^*, for distinct indices i and j. Moreover, note that the fundamental chamber can be written of the form

$$C = \bigcap_{i \in I} \{\varphi \in V^* \mid \varphi(e_i) \geqslant 0\} = \bigcap_{i \in I} (H_i^* \cup \{\varphi \in V^* \mid \varphi(e_i) > 0\}),$$

where we observe that the H_i^* form the pairwise distinct codimension one faces of the chamber C. The open halfspaces $\{\varphi \in V^* \mid \varphi(e_i) > 0\}$ in the latter term will be called A_i^* and using these, we define the open fundamental chamber to be the intersection of the open halfspaces:

$$\operatorname{int}(C) = \mathring{C} = \bigcap_{i \in I} A_i^*.$$

As mentioned above, we want to study the action of our Coxeter group via the dual representation, acting by reflection in the faces H_i^*. However, in general the translates of the chamber under the group action won't cover the whole of V^*, which motivates the following definition:

Definition 2.2.4. *The Tits cone is the union of all* W*-translates of the chamber,* $WC := \bigcup_{w \in W} wC \subset V^*$.

As the name suggests, the fundamental chamber is a fundamental domain for the action of W on the Tits cone WC under the dual representation ρ^*. This will be proved in Theorem 2.5.7. While the formal defintion of the Tits cone provides a rigorous foundation, it is not very insightful from a geometric perspective. As one can think about the Tits cone quite geometrically, especially in low dimensions, we will take a closer look at an explicit example in the following section. Before doing so, we end this section with the following remark.

Remark 2.2.5. *One may ask why we transport everything to the dual space, instead of working in the standard representation* ρ. *For this, consider the infinite dihedral group* $D_\infty \cong \langle s, t \mid s^2 = t^2 = 1\rangle$. *We fix a basis* $\{e_1, e_2\}$ *of* V *and obtain that the bilinear form in this basis is given by the matrix*

$$B_W = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

We observe that $H_1 = \mathrm{span}\{e_1 + e_2\} = H_2$, *which implies that* σ_1 *and* σ_2 *fix the same hyperplane, despite having different* (-1)*-Eigenspaces (namely the span of* e_1, *resp.* e_2*). Therefore, in general, working in the standard representation won't result in a chamber, giving rise to the existence of the Tits* *cone. Now, passing to the dual space* V^* *by fixing the dual basis* $\{e_1^*, e_2^*\}$*, consider the dual reflections* σ_i^* *as discussed before. By the more general calculation earlier, we obtain*

$$\sigma_1^*(H_2^*) = H_2^* \quad \textit{and} \quad \sigma_2^*(H_1^*) = H_1^*.$$

And furthermore, we note that for all $i, j \in \{1, 2\}$

$$\sigma_i^*(B_W(e_j, \cdot)) = B_W(e_j, \cdot) - 2B_W(e_i, \cdot)B_W(e_j, e_i) = -B_W(e_j, \cdot),$$

using that $B_W(e_j, \cdot) = -B_W(e_i, \cdot)$*. This shows that both dual reflections have the same* (-1)*-Eigenspace, but fix different hyperplanes (i. e., have different* $(+1)$*-Eigenspaces), resulting in a chamber as wished.*

2.3 THE TITS CONE – AN EXAMPLE

As an example we take a closer look at the free product $W \cong \langle r, s, t \mid r^2 = s^2 = t^2 = 1 \rangle$ from Example 2.1.3. We fix the basis $\{e_1, e_2, e_3\}$ and identify V with $\mathbb{R}^3$. In this basis, the bilinear form B_W is given by the matrix

$$B_W = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

By the spectral theorem we find a basis of orthonormal Eigenvectors, in which B_W is a diagonal matrix with its eigenvalues as entries. Using the Gram-Schmidt procedure, we get

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In other words, we have $V^T B_W V = D$ with $V \in O(n)$ and $D = \mathrm{diag}(2, 2, -1)$. Now, since we have a diagonal matrix, we can multiply the entries by squares, since the resulting matrix will be congruent to the given one:

$$\text{Let } A = \mathrm{diag}(\mu_1, \dots, \mu_n) \in \mathbb{R}^{n \times n},$$

$$S = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in (\mathbb{R} \setminus \{0\})^{n \times n} \implies S^T A S = \mathrm{diag}(\lambda_1^2 \mu_1, \dots, \lambda_n^2 \mu_n).$$

To apply this and further transform our matrix D, define the invertible matrix T as follows

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which then implies $\tilde{D} := T(V^T B_W V)T = \mathrm{diag}(1, 1, -1)$. The images of the basis vectors $\{e_1, e_2, e_3\}$ are given by the three columns of the matrix TV^T, namely:

$$\tilde{e}_1 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \quad \tilde{e}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \text{ and } \tilde{e}_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{6} \\ \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Note that we have $V \cong \mathbb{R}^3$, equipped with the inner product

$$\langle x, y \rangle_{2,1} := x^T \tilde{D} y = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

Since $\langle \tilde{e}_i, \tilde{e}_i \rangle_{2,1} = 0$ for all $i \in \{1, 2, 3\}$, we have that these three vectors span an ideal triangle in the hyperboloid model of $\mathbb{H}^2$, given by $\langle x, y \rangle_{2,1} = -1$. One gets the ideal triangle by intersecting the hyperboloid with the hyperplanes spanned by each two of the $\tilde{e}_i$ (they will only intersect in the surrounding cone of the hyperboloid).

Given these new coordinates under the transformation TV^T, the Tits cone will be given by $x_1^2 + x_2^2 - x_3^2 < 0$ union the images of $\tilde{e}_1, \tilde{e}_2$ and $\tilde{e}_3$ under the reflection in the sides of the chamber i. e., in the sides of the ideal triangle. Moreover, we get a subgroup of $O(2, 1)_+$ generated by the three matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{5}{3} & -\frac{4}{3} \\ 0 & \frac{4}{3} & \frac{5}{3} \end{pmatrix}, \quad \begin{pmatrix} -1 & \frac{2}{\sqrt{3}} & -\frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{\sqrt{3}} & -\frac{2}{3} & \frac{5}{3} \end{pmatrix} \text{ and } \begin{pmatrix} -1 & -\frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{3}} & -\frac{2}{3} & \frac{5}{3} \end{pmatrix}.$$

Each of the above matrices corresponds to one of the generators r, s and t of the right angled Coxeter group W under the transformation TV^T, constructed above.

2.4 THE WORD METRIC AND THE FAITHFUL REPRESENTATION

Recall that for any finitely generated group $W = \langle S \rangle$, its Cayley graph induces a metric on W, relative to the generating set S. We call it the *word metric* of W relative to S and denote it by d_S. Also note that the word metric is left-invariant, meaning that for group elements $u, v, w \in W$, we have the equality $d_S(uv, uw) = d_S(v, w)$. Now, define the *word length* of an element $w \in W$ to be $\ell(w) := d_S(w, 1_W)$, the distance of an element to the neutral element of the group. Note that $\ell(w) = 0$ if and only if $w = 1_W$.

Lemma 2.4.1. *We collect some properties of the length function we will use later on.*

1. $\forall w \in W: \ \ell(w) = \ell(w^{-1})$
2. $\forall s \in S: \ \ell(s) = 1$ *and* $\ell(w) = 1 \iff w \in S^{\pm 1}$
3. $\forall v, w \in W: \ \ell(vw) \leqslant \ell(v) + \ell(w)$
4. $\forall v, w \in W: \ \ell(v) - \ell(w) \leqslant \ell(vw)$

5. $\forall w \in W, s \in S^{\pm 1}: \ \ell(w) - 1 \leqslant \ell(ws) \leqslant \ell(w) + 1$

Proof. All of the above statements follow from the fact that d_S is a left-invariant metric. □

Coming back to Coxeter groups, by definition each generator $s \in S$ has order 2 in W. Therefore, we can write every non-trivial element in W as a sequence of generators in S. Note that in this sequence there might be redundencies, so that the following definition makes sense. We call an expression $w = s_{i_1} \cdots s_{i_r}$ for $i_1, \ldots, i_r \in I$ and $r \in \mathbb{N}$ *reduced*, if $\ell(w) = r$, i. e. w cannot be represented by a shorter word. These reduced expressions have the caveat of not being unique by any means.

Given a parabolic subgroup W_J of a Coxeter group W, it admits its own word metric with respect to $J \subset I$. Therefore each parabolic subgroup admits its own length function, which we denote by $\ell_J(w)$ for words w in W_J. In the following we will make use of the general fact that we have $\ell(w) \leqslant \ell_J(w)$ for all $w \in W_J$.

The length function turns out to be an important and powerful tool in studying Coxeter groups. Indeed, its role will be demonstrated in several of the forthcoming proofs, beginning with the following theorem which is a key step in proving faithfulness of our previously defined representation.

Theorem 2.4.2. *Let $w \in W$ and $s_i \in S$ for $i \in I$. Then $\ell(ws_i) > \ell(w)$ implies that $w(e_i) > 0$.*

We prove the theorem by induction on $\ell(w)$ and by decomposing w, we will see how w acts on V. Note that by writing $w(e_i)$ for some $w \in W$ and $i \in I$, we mean that $e_j^*(w(e_i)) \geqslant 0$ for all $j \in I$. Put in other words, this means that $w(e_i)$ is contained in every $\overline{A_j^*} = A_j^* \cup H_j^*$ for $j \in I$.

Proof. The base case is trivial, as $\ell(w) = 0$ implies $w = 1_W$ and thus w fixes every basis element. Therefore, assume w is non-trivial and in reduced form. We state the induction hypothesis.

(IH) $\ell(vs_i) > \ell(v)$ for some $v \in W$ and $s_i \in S$ implies that $v(e_i) > 0$.

Claim 1: There is a $j \in I$ such that $s_i \neq s_j$ and $\ell(ws_j) = \ell(w) - 1$.

Proof of Claim 1. Since w is in reduced form, write $w = s_{i_1} \cdots s_{i_k}$ and set $s_j = s_{i_k}$. This implies

$$\ell(ws_i) > \ell(w) > \ell(w) - 1 = \ell(ws_j),$$

and in particular $s_j \neq s_i$. This proves the first Claim. ■

Consider the parabolic subgroup $\langle s_i, s_j \rangle \leqslant W$ generated by these two distinct elements in S. Denote this subgroup by W_J, for $J = \{i, j\} \subset I$. Our goal now is to decompose w into two parts, one living in the subgroup W_J and the other in its complement. For this, consider a specific subset of the coset wW_J, given by

$$A := \{v \in W \mid vW_J = wW_J \ \text{ and } \ \ell(v) + \ell_J(v^{-1}w) = \ell(w)\}.$$

By definition, $w \in A$. With regard to the following, we choose $v \in A$ such that its length $\ell(v)$ is minimal. Now set $v_J = v^{-1}w$, which is equivalent to writing $w = vv_J$, giving us a decomposition of w as desired. Due to this decomposition, our analysis of w now boils down to studying the separate actions of v and v_J on V. First note that ws_j is contained in A as well, since $s_j^{-1} = s_j$ and thus

$$s_j w^{-1} w = s_j \in W_J \text{ and } \ell(ws_j) + \ell_J(s_j) = \ell(w) - 1 + 1 = \ell(w).$$

By our choice of $\ell(v)$ to be minimal, we see $\ell(v) \leqslant \ell(ws_j) = \ell(w) - 1$. Thus, we are almost set up to apply the induction hypothesis to v and s_i. The last ingredient for this is the following

Claim 2: For the lengths of vs_i and v, we have the relation: $\ell(vs_i) \geqslant \ell(v)$.

Proof of Claim 2. Assume towards contradiction: $\ell(vs_i) < \ell(v)$, equivalently $\ell(vs_i) = \ell(v) - 1$. Then:

$$\begin{aligned}\ell(w) = \ell(vv_J) = \ell(vs_is_iv^{-1}w) &\leqslant \ell(vs_i) + \ell(s_iv^{-1}w) \\ &\leqslant \ell(v) - 1 + \ell_J(v^{-1}w) + 1 = \ell(v) + \ell_J(v^{-1}w) = \ell(w).\end{aligned}$$

This means equality holds throughout and in particular $\ell(w) = \ell(vs_i) + \ell_J(s_iv^{-1}w)$. But this implies vs_i belongs to A, contradicting the minimality of $\ell(v)$. ∎

Applying the induction hypothesis **(IH)** to v and s_i leaves us with $v(e_i) > 0$. The exact same argument applied to v and s_j shows $v(e_j) > 0$, so that we omit this here. Next, our goal is to study how v_J acts on V. More precisely, how it acts on the basis element e_i.

Claim 3: For v_Js_i and v_J we have the relation: $\ell_J(v_Js_i) \geqslant \ell(v)$.

Proof of Claim 3. Assume towards contradiction, that $\ell_J(v_Js_i) < \ell(v_J)$ holds. Then:

$$\ell(ws_i) = \ell(vv_Js_i) \leqslant \ell(v) + \ell(v_Js_i) \leqslant \ell(v) + \ell_J(v_Js_i) < \ell(v) + \ell_J(v_J) = \ell(w),$$

which contradicts the assumption of the theorem, that $\ell(ws_i) > \ell(w)$. ∎

Moreover, this shows that all reduced expressions of v_J in the parabolic subgroup W_J have to end in s_j. Else we would have $\ell(v_J) > \ell(v_Js_i)$, contradicting *Claim 3*. It remains to show

Claim 4: s_is_j maps e_i to a non-negative linear combination of e_i and e_j.

Proof of Claim 4. Note that W_J is dihedral, either of order $2m_{ij}$ or infinite order. Then, as v_J lies in W_J, any reduced expression of v_J in W_J is an alternating product of s_i and s_j ending in s_j. Now, distinguish by the order of the group W_J:

1) $m_{ij} < \infty$: W_J is dihedral of order $2m_{ij}$. Thus, any element is of length $< m_{ij}$ and has a unique word corresponding to the edge labels in the Cayley graph. Note that the maximum length of $w \in W_J$ is precisely m_{ij} (as the Cayley graph is a cycle of length $2m_{ij}$). The element of length m_{ij} is represented by the reduced expressions $s_is_j \cdots s_j$ and $s_js_i \cdots s_i$. This implies that v_J has to have length smaller m_{ij}, else it would have a reduced expression ending in s_i, contradicting the above. Then v_J is a product of less than $\frac{m_{ij}}{2}$ terms s_is_j, and each of the products s_is_j is a counterclockwise rotation about $\frac{2\pi}{m_{ij}}$. So v_J rotates e_i at maximum $\pi - \frac{2\pi}{m_{ij}}$, which still lies inside the cone spanned by e_i and e_j. In particular the resulting vector is a non-negative linear combination.

2) $m_{ij} = \infty$: In the case of the infinite dihedral group, we already have seen in the proof of Theorem 2.2.1 that $(s_i s_j)^n(e_i) = e_i + 2n(e_i + e_j)$.

This proves the last Claim. ■

Since v_J and v both map e_i to a non-negative linear combination of e_i and e_j, so does w due to its decomposition. Therefore, the proof is now complete. □

The result of the previous theorem readily extends to its converse, offering valuable insights as well. We shall formally record this implication as a corollary.

Corollary 2.4.3. *In the previous theorem, set $w = \tilde{w}s_i$ for $i \in I$. Then $\ell(\tilde{w}s_i) < \ell(w)$ implies $\ell(\tilde{w}s_i s_i) > \ell(\tilde{w}s_i)$. Thus we have $(ws_i)(e_i) = -w(e_i) > 0$, or equivalently $w(e_i) < 0$.*

With Corollary 2.4.3 and Theorem 2.4.2 in hand, we will finally be able to deduce the injectivity and thus faithfulness of our representation ρ.

Corollary 2.4.4. *The homomorphism of Theorem 2.2.1 is injective and thus a faithful representation.*

Proof. Assume $w \in \ker\{\rho\}$ non-trivial. Then there is an $i \in I$ with corresponding $s_i \in S$, such that $\ell(ws_i) < \ell(w)$. Now Corollary 2.4.3 implies $w(e_i) < 0$, but $w(e_i) = e_i > 0$ as $\rho(w) = \mathrm{id}_V$, which is a contradiction and thus, the statement holds. □

2.5 THE FUNDAMENTAL DOMAIN AND STABILIZERS

In this section, we will prove that the fundamental chamber $C \subset V^*$ is indeed a fundamental domain for the action of W on its Tits cone WC. Moreover, we will work out how stabilizers of points look like and then show that the Tits cone is really a convex cone in the dual space V^*.

We start by translating the results of the previous section to the language of chambers and halfspaces. The following two lemmas capture this essence.

Lemma 2.5.1. *Let $w \in W$ and $i \in I$. The relation $\ell(s_i w) > \ell(w)$ is equivalent to saying that w leaves the open chamber $\mathring{C}$ inside the open halfspace A_i^*, corresponding to the generator s_i. Put more formally, this states that the set $w(\mathring{C})$ is a subset of the open halfspace A_i^*.*

Proof. Observe that $\ell(s_i w) > \ell(w)$ is equivalent to $\ell(w^{-1}s_i) > \ell(w^{-1})$. Then by by Theorem 2.4.2, we have that $w^{-1}(e_i) > 0$. Take an arbitrary point $\varphi \in \mathring{C}$ in the open fundamental chamber, then we have that $w(\varphi)(e_i) = \varphi(w^{-1}(e_i))$. Since the chamber C lies in A_i^* for all $i \in I$ and $\varphi \in \mathring{C}$, we see that $w(\varphi) > 0$ is equivalent to $w^{-1}(e_i) > 0$. But this is true by assumption and since φ is arbitrary, we conclude that $w(\mathring{C}) \subset A_i^*$. □

Of course this lemma also has a corresponding converse result as in the previous section.

Lemma 2.5.2. *Let $w \in W$ and $i \in I$. The relation $\ell(s_i w) < \ell(w)$ is equivalent to saying that w now moves the open chamber $\mathring{C}$ inside the open halfspace, complementary to A_i^*. More formally, this states that the set $w(\mathring{C})$ is a subset of the open halfspace $s_i(A_i^*)$, as the s_i permute A_i^* and its corresponding complementary halfspace.*

Proof. Take $w' = s_i w$ for some $i \in I$ and $w \in W$ so that $\ell(s_i w) > \ell(w)$. Then we have that the length of $s_i w' = w$ is strictly smaller then then length of w'. By applying above lemma we get that $w(\mathring{C}) \subset A_i^*$, which is equivalent to $s_i w'(\mathring{C}) \subset A_i^*$. Apply s_i to both sides leaves us with $w'(\mathring{C}) \subset s_i(A_i^*)$, which proves the lemma. □

Building onto this insight, we now want to study the action of parabolic subgroups on the Tits cone to get an understanding of the stabilizers of points. To do so, decompose the fundamental chamber into subsets corresponding to the parabolic subgroups of W as follows.

Definition 2.5.3. *Given a parabolic subgroup W_J corresponding to $J \subset I$, set*

$$C_J := \bigcap_{j \in J} H_j^* \cap \bigcap_{k \notin J} A_k^*.$$

We call these the corresponding parabolic subsets (of the fundamental chamber).

Example 2.5.4.

1. *When the set J is empty, the corresponding parabolic subset $C_\emptyset$ coincides with the entire chamber C. Conversely, when J contains all indices, C_J reduces to the singleton $\{0\}$.*
2. *If J is a proper subset of I with cardinality one, then the corresponding subset C_J coincides with a codimension-one face of the chamber C.*

Theorem 2.5.5. *Let $w \in W$ and $J, K \subset I$ be subsets. Then $w(C_J) \cap C_K \neq \emptyset$ implies $J = K$, $w \in W_J$ and thus $w(C_J) = C_J$. In particular, the isotropy groups of the sets C_J are the parabolic subgroups W_J.*

Proof. Let $w \in W$ and $J, K \subset I$ be subsets, such that $w(C_J) \cap C_K \neq \emptyset$. The proof is by induction on the length of w. The base case $\ell(w) = 0$ is trivial, as then w is trivial.

Assume that $\ell(w) > 0$ and choose $i \in I$, such that $\ell(s_i w) < \ell(w)$. Writing $w = s_i(s_i w)$, by Lemma 2.5.2 we know that w moves the open chamber $\mathring{C}$ into the open halfspace $s_i(A_i^*)$, i.e. $w(\mathring{C}) \subset s_i(A_i^*)$. Now using the continuity of the group action, we note that $w(C) \subset \overline{s_i(A_i^*)}$. Recall that by definition, the fundamental chamber C lies in the halfspaces $\overline{A_i^*}$ for all $i \in I$. Thus, we record that $w(C) \cap C \subset H_i^*$ and since s_i fixes the corresponding H_i^* by definition, it fixes every point in the intersection of C and its translate $w(C)$. Note that the sets C_J and C_K are subsets of the fundamental chamber C and therefore, s_i fixes every point in the non-empty set $w(C_J) \cap C_K$. But if s_i fixes some point φ in C_K, we calculate

$$\varphi(e_i) = s_i(\varphi)(e_i) = \varphi(s_i(e_i)) = -\varphi(e_i) \implies \varphi(e_i) = 0 \iff \varphi \in H_i^*.$$

We deduce $i \in K$, respectively $s_i \in W_K$. Using this together with the assumption, we get that $s_i w(C_J) \cap C_K = s_i(w(C_J) \cap C_K)$ is non-empty. We apply the induction hypothesis to the element $s_i w$, to see that $J = K$ and $s_i w \in W_J$. Finally, since $s_i \in W_J = W_K$, we have that $s_i w \in W_J$ implies $w \in W_J$, proving the theorem. □

Before proceeding with proving that the fundamental chamber lives up to its name, we want to clearly state what is meant by a fundamental domain.

Definition 2.5.6. *Let* G *be a group, acting on a topological space* X. *We call a closed subset* $F \subset X$ *a* (strict) fundamental domain, *if for each* $x \in F$ *its orbit* $\mathrm{Orb}(x)$ *intersects* F *in exactly one point.*

Note that, by definition of the Tits cone WC, every W-orbit of a point $\varphi \in C$ meets the fundamental chamber C in at least one point, namely φ. Thus, it suffices to proof that each W-orbit meets C in at most one point, to prove the following theorem.

Theorem 2.5.7. *The fundamental chamber is a fundamental domain for the action of the Coxeter group* W *on its Tits cone* WC *and thus, justifies its name.*

Proof. Assume that $\varphi, \psi \in C$ lie in the same W-orbit, but in different parabolic subsets C_J, respectively C_K of the fundamental chamber. Since they lie in the same orbit, there is a $w \in W$ with $\varphi = w(\psi)$. Thus, the intersection $w(C_J) \cap C_K$ is non-empty and Theorem 2.5.5 implies equality of J and K, as well as $w \in W_J$. We deduce $\varphi = w(\psi) = \psi$. Thus, every W-orbit of a point $\varphi \in C$ meets the fundamental chamber C at most in φ, proving the theorem. $\square$

Define a set $\mathcal{C}$ as the union of all translates of possible parabolic subsets W_J i. e., define $\mathcal{C}$ by

$$\mathcal{C} := \bigcup_{J \subset I} \bigcup_{w \in W/W_J} w(C_J).$$

We want to emphasize here that by Theorem 2.5.5 the sets of the form $w(C_J)$ in the fundamental chamber C are all disjoint for different $J \subset I$ and w ranging over the coset W/W_J. Thus, the sets of $\mathcal{C}$ form a decomposition of the Tits cone. This decomposition (altough not into chambers) is a key component in the following theorem.

Theorem 2.5.8. *The Tits cone* WC *is a convex cone, and every closed line segment in the Tits cone meets only finitely many of the sets in* $\mathcal{C}$.

Proof. First note that the fundamental chamber is a convex cone as the intersection of the finitely many closed halfspaces $\overline{A_i^*}$. This implies that the Tits cone is a cone as well. We will prove the convexity by showing that every closed segment between any two points in the Tits cone is contained in it. Furthermore, we will prove that these segments can be covered by finitely many of the sets in the above defined union $\mathcal{C}$, implying latter statement.

Consider the closed segment $[\varphi, \psi]$ with $\varphi, \psi \in WC$ and assume the endpoints lie in different chambers. Proceed by induction on the word length $\ell(w)$. The base case $\ell(w) = 0$ reduces to $\varphi, \psi \in C$. Since C is convex and can trivially be covered by finitely many of the C_J, this case has been dealt with.

Therefore, we now assume $\ell(w) > 0$. Intersect the segment $[\varphi, \psi]$ with the fundamental chamber C, to receive two new segments $[\varphi, \xi]$ and $[\xi, \psi]$ (cf. Figure 2.1). The first segment can be covered by finitely many of the sets in $\mathcal{C}$, as it lies inside the fundamental chamber C. Thus, we need to show that we can cover the second segment $[\xi, \psi]$ by finitely many of these sets. Assume further, that $\psi \in s_i(A_i^*)$ for an $i \in J$ and $\psi \in \overline{A_i^*}$ for $i \notin J$ for some $J \subset I$, so that $\psi \notin C$.

Claim 1: ξ has to lie in one of the codimension-one faces H_i^* for $i \in J$.

Proof of Claim 1. Assume that ξ lies in the open halfspace A_i^* for some $i \in J$. Then every point ζ in the intersection of a neighborhood of ξ contained in A_i^* and the segment $[\xi, \psi]$ has to also lie in A_i^* for $i \in J$. Clearly, $\zeta \in \overline{A_i^*}$ for $i \notin J$ holds as well, implying that $\zeta \in C$. But this is a contradiction to the decomposition of the initial segment $[\varphi, \psi]$. Thus, ξ has to lie in one of the H_i^*, for $i \in J$. ■

Using the assumptions $\psi \in s_i(A_i^*)$ and $\psi \in w(C)$, we deduce $w(C) \subset \overline{s_i(A_i^*)}$, hence by continuity of the action $w(\mathring{C}) \subset s_i(A_i^*)$. By Lemma 2.5.2 this is equivalent to $\ell(s_i w) < \ell(w)$ and we are set up to apply the induction hypothesis to ξ and $s_i(\psi) \in s_i w(C)$. This produces a covering of $[\xi, s_i(\psi)]$ by finitely many sets in $\mathcal{C}$. But since we established that ξ has to lie in H_i^*, translation by s_i gives $[s_i(\xi), s_i^2(\psi)] = [\xi, \psi]$, and thus we can cover this segment by finitely many sets as well. The result follows. □

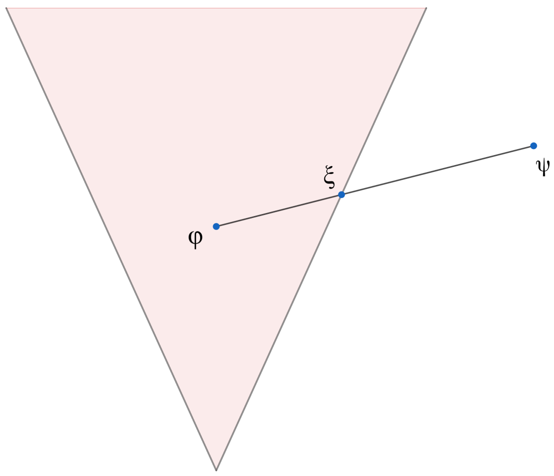

Figure 2.1: A path through the Tits cone.

To summarize, the essence of this section is that first of all the fundamental chamber is indeed a fundamental domain of the action of a Coxeter group on its Tits cone. This was shown in Theorem 2.5.7. Furthermore, we want to emphasize that each point in the fundamental chamber has trivial stabilizer, which is a consequence of Theorem 2.5.5. Moreover, every point in a set C_J, for J a subset of I, is stabilized by the corresponding parabolic subgroup W_J.

2.6 COVERING ACTIONS – A TOOLBOX

Definition 2.6.1. *Let X, Y be topological spaces. A continuous map $p : Y \to X$ is a* covering map *and Y a* covering space *for X, if every point x in X has an open neighborhood U, such that the preimage of U under p is a disjoint union of open sets U_i in Y for an index set I. Furthermore, the map p has to be a local homeomorphism, meaning the restriction of p to the U_i is a homeomorphism. Then, U is called* evenly covered *and* $|I|$ *the degree of the covering, while the open sets* U_i *are called the* sheets *over* U *and the preimage of an* x *in* X *is called a* fiber *of* x.

Definition 2.6.2. *Let* G *be a group, acting on a space* X*. If the following is satisfied:*

$$\forall x \in X : \exists U \in \mathcal{U}_x^X : \ |\{g \in G \mid g(U) \cap U \neq \emptyset\}| < \infty$$

we say that the action is properly discontinuous *on* X.

Definition 2.6.3. *Let* G *be a group, acting on a space* X*. If the more restrictive condition*

$$\forall x \in X : \exists U \in \mathcal{U}_x^X : \ \{g \in G \mid g(U) \cap U \neq \emptyset\} = \{1\}$$

is satisfied, we call the action a covering (space) action*.*

Lemma 2.6.4. *Let* G *be a group, that acts freely and properly discontinuous on a Hausdorff space* X*. Then the action is a covering action in the above sense.*

Proof. Let X be a Hausdorff space and G be a group acting freely, properly discontinuously on X. Then for a open Neighborhood U of $x \in X$ the set $M := \{g \in G \mid gU \cap U \neq \emptyset\}$ is finite. For these $g \in M$ we pick pairwise disjoint Neighborhoods V_g of gx, which is possible since G acts freely, thus $gx \neq x$ and X is Hausdorff. Finally, set $V = \bigcap_{g \in M} g^{-1} V_g \cap U$, which is open as finite intersection of open sets and by definition a Neighborhood of x. □

Definition 2.6.5. *A path-connected topological space* X *is called* simply connected, *if* $\pi_1(X) \cong \{1\}$.

Lemma 2.6.6. *Let* G *be a group, acting by covering action on a simply connected topological space* X*. Then the quotient map* $p_G : (X, x_0) \to (G\backslash X, G \cdot x_0)$ *is a covering map and* $\pi_1(G\backslash X, G \cdot x_0) \cong G$.

Proof. Let U be an open neighborhood of $x \in X$, such that $\{g \in G \mid gU \cap U \neq \emptyset\} = \{1\}$.

Claim 1: The map p_G restricted to U is a continuous bijection.

Proof of Claim 1. Continuity: $U \subset G\backslash X$ is open if and only if $p^{-1}(U)$ is open and thus p is continuous Surjectivity: Since orbits are not empty, p is surjective Injectivity: Assume $x, y \in X$ with $p(x) = p(y)$, but then there is $g \in G\backslash\{1\}$ with $gx = y$ implying $gU \cap U \neq \emptyset$. This is a contradiction to G acting by covering action. ■

Claim 2: Every point $Gx \in G\backslash X$ has a neighborhood $V \subset G\backslash X$ with $p_G^{-1}(V) = \bigsqcup_{i \in I} U_i$.

Proof of Claim 2. By assumption the sets gU are all disjoint neighborhoods of $gx \in X$ for all $g \in G$. Moreover, for $V := p_G(U) \subset G\backslash X$, we have $p_G^{-1}(V) = \bigsqcup_{g \in G} gU$, proving the Claim. ■

Claim 3: The map $p_G|_{gU} \to V = p_G(U)$ is an homeomorphism for all $g \in G$.

Proof of Claim 3. Note that the action of G on X is by homeomorphisms and $U \subset X$ is open. Then the union $\bigsqcup_{g \in G} gU$ is open as well and since $p_G^{-1}(V) = \bigsqcup_{g \in G} gU$, the set V is open in the quotient topology of $G\backslash X$. Therefore, the map p_G restricted to gU for $g \in G$ is an open map and hence, by *Claim 1* a local homeomorphism. ■

Now all conditions in Definition 2.6.1 are satisfied, so p_G is a covering map. For the last part of the statement, take a homotopy class of loops $[\gamma] \in \pi_1(G\backslash X, Gx_0)$ with representative γ. Define a map $\varphi : \pi_1(G, x_0) \to G$ on homotopy classes $[\gamma]$, such that $\varphi([\gamma])$ takes $\tilde{\gamma}(0) = x_0$ to $\tilde{\gamma}(1)$, where $\tilde{\gamma} : [0, 1] \to X$ is a lift of γ. Now if γ' is another representative of $[\gamma]$ then, $\widetilde{\overline{\gamma}' \cdot \gamma}$ is the lift of a contractible curve in $G\backslash X$. □

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