Approximate NLL small-x limit for the N^{3}LO massive coefficient function

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In this document we will explain the construction for the approximate NLL small-x limit for the N^{3}LO massive coefficient function, which is not currently known. The gluon coefficient function for F_2 can be written as (in Mellin space)

$$
\begin{aligned}
&\tilde{\mathcal{C}}_2(N, M, \xi, \xi_\mu) = \\
&\quad \frac{\alpha_s}{3\pi}\frac{3}{2}\xi_\mu^{-M}\left(1+\frac{\xi}{4}\right)^{-N}\frac{\Gamma(1-M)^3\Gamma(1+M)}{(3-2M)(1+2M)\Gamma(2-2M)}\times \\
&\quad \left[1+M-\left(1+M-\frac{\xi}{2}(2+3M-3M^2)\right){}_2F_1\left(1-M,1,\frac{3}{2},-\frac{\xi}{4}\right)\right].
\end{aligned} \tag{1}
$$

with

$$
\xi = \frac{Q^2}{m^2}\,, \quad \xi_\mu = \frac{\mu^2}{m^2}\,. \tag{2}
$$

See [arXiv:1708.07510] for the definition of $\tilde{\mathcal{C}}$. Now we can expand Eq. (1) for small M up to second order. What we find is

$$
\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_2^{(0)} + M\tilde{\mathcal{C}}_2^{(1)} + M^2\tilde{\mathcal{C}}_2^{(2)} + \mathcal{O}(M^3). \tag{3}
$$

Then we make the following substitution

$$
\begin{aligned}
&M^0 \to \left[M^0\right] = 1\,, \\
&M^1 \to \left[M^1\right] = M = a_s\gamma_0 + a_s^2\gamma_1 + \mathcal{O}(a_s^3)\,, \\
&M^2 \to \left[M^2\right] = M(M - a_s\beta_0) = a_s^2(\gamma_0^2 - \gamma_0\beta_0) + \mathcal{O}(a_s^3)\,,
\end{aligned} \tag{4}
$$

where $a_s = \frac{\alpha_s}{4\pi}$ and $\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f$. In this way Eq. (3) becomes

$$
\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_2^{(0)} + a_s\tilde{\mathcal{C}}_2^{(1)}\gamma_0 + a_s^2\Big(\tilde{\mathcal{C}}_2^{(1)}\gamma_1 + \tilde{\mathcal{C}}_2^{(2)}\big(\gamma_0^2 - \gamma_0\beta_0\big)\Big) + \mathcal{O}(a_s^3). \tag{5}
$$

Observe that since $\tilde{\mathcal{C}}_2$ is $\mathcal{O}(a_s)$, then all the $\tilde{\mathcal{C}}_2^{(k)}$ are $\mathcal{O}(a_s)$. It means that the term proportional to a_s^2 in (5) is the $\mathcal{O}(a_s^3)$ expansion of the partonic structure function of the gluon in Mellin space. Moreover, observe that all of this must be taken in the limit $N \ll M$, and therefore the term $\left(1+\frac{\xi}{4}\right)^{-N}$ in Eq. (1) can be replaced with 1. The value of M_0 is known exactly, and is

$$
\gamma_0^{\rm NLL} = \frac{a_{11}}{N} + \frac{a_{10}}{N+1}, \tag{6}
$$

while γ_1 is still not known. For this reason we used the approximate value

$$
\gamma_1^{\rm NLL} = \frac{a_{21}}{N} - \frac{2a_{21}}{N+1}\,. \tag{7}
$$

In Eqs. (6)-(7) we defined

$$
\begin{aligned}
&a_{11} = C_A, \\
&a_{10} = -\frac{11C_A + 2n_f(1 - 2C_F/C_A)}{12}, \\
&a_{21} = n_f\frac{26C_F - 23C_A}{36}.
\end{aligned} \tag{8}
$$

Now we have to transform back from Mellin space to z-space. Using that the Mellin transform is defined as

$$
\mathcal{M}\big[f(z)\big] \equiv f(N) = \int_0^1 dz\, z^{N-1} f(z), \tag{9}
$$

one can show that

$$
\begin{aligned}
\mathcal{M}\big[1\big] &= \frac{1}{N}, \\
\mathcal{M}\big[z\big] &= \frac{1}{N+1}, \\
\mathcal{M}\big[\log(z)\big] &= -\frac{1}{N^2}, \\
\mathcal{M}\big[z\log(z)\big] &= -\frac{1}{(N+1)^2},
\end{aligned}
\tag{10}
$$

We find that, neglecting terms proportional to z and $z\log(z)$, our approximation of the next-to-leading logarithm expansion of the $\mathcal{O}(a_s^3)$ gluon coefficient function for F_2 is

$$
\begin{aligned}
&zC_{2,g}^{[3](3)\mathrm{NLL}}\Big(z,\xi,\xi_\mu\Big) = \\
&\quad a_{11}^2\Bigg[-\frac{147}{27}-\frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right)+\frac{8}{27}J(\xi)\left(\frac{92}{\xi}-71\right)+I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right)+\frac{8}{9}\left(\frac{10}{\xi}-13\right)\right) \\
&\qquad +\left(-\frac{160}{9}+\frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right)+\frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)L_\mu+\left(-\frac{16}{3}+\frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)L_\mu^2\Bigg]\log(z) \\
&\quad + a_{21}\left[\frac{16}{9}-\frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right)-\frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right] \\
&\quad + a_{10}a_{11}\left[\frac{2944}{27}+\frac{16}{3}K(\xi)\left(\frac{1}{\xi}-1\right)-\frac{16}{27}J(\xi)\left(\frac{92}{\xi}-71\right)+I(\xi)\left(-\frac{16}{3}L_\xi\left(\frac{1}{\xi}-1\right)-\frac{16}{9}\left(\frac{10}{\xi}-13\right)\right)\right] \\
&\quad + a_{11}\beta_0\left[-\frac{1472}{27}-\frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right)+\frac{8}{27}J(\xi)\left(\frac{92}{\xi}-71\right)+I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right)+\frac{8}{9}\left(\frac{10}{\xi}-13\right)\right)\right] \\
&\quad + \Bigg[a_{21}\left(\frac{32}{3}-\frac{16}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)+a_{10}a_{11}\left(\frac{320}{9}-\frac{32}{3}I(\xi)\left(\frac{1}{\xi}-1\right)-\frac{16}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right) \\
&\qquad + a_{11}\beta_0\left(-\frac{160}{9}+\frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right)+\frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)\Bigg]L_\mu \\
&\quad + \left[a_{10}a_{11}\left(\frac{32}{3}-\frac{16}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)+a_{11}\beta_0\left(-\frac{16}{3}+\frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)\right]L_\mu^2,
\end{aligned}
\tag{11}
$$

where we defined $L_\mu = \log\left(1/\xi_\mu\right)$ and

$$
\begin{aligned}
I(\xi) =& \frac{4}{\xi}\sqrt{\frac{\xi}{\xi+4}}H\left(+,-,\sqrt{\frac{\xi}{\xi+4}}\right), \\
J(\xi) =& \frac{4}{\xi}\sqrt{\frac{\xi}{\xi+4}}L\left(\sqrt{\frac{\xi}{\xi+4}}\right), \\
K(\xi) =& \frac{4}{\xi}\sqrt{\frac{\xi}{\xi+4}}H\left(-,+,-,\sqrt{\frac{\xi}{\xi+4}}\right),
\end{aligned}
\tag{12}
$$

with

$$
\begin{aligned}
L(z) &= \log\!\left(\frac{1+z}{1-z}\right), \\
H\Big(+,-,z\Big) &= H_{1,1}(z)+H_{1,-1}(z)-H_{-1,1}(z)-H_{-1,-1}(z), \\
H\Big(-,+,-,z\Big) &= H_{1,1,1}(z)-H_{1,1,-1}(z)+H_{1,-1,1}(z)-H_{1,-1,-1}(z) \\
&\quad -H_{-1,1,1}(z)+H_{-1,1,-1}(z)-H_{-1,-1,1}(z)+H_{-1,-1,-1}(z).
\end{aligned}
\tag{13}
$$

We can apply the same procedure for F_L: all the steps we did to get Eq. (11) are unchanged and the only difference will be in the from of the function in Eq. (1). In this case we have to use

$$
\begin{aligned}
&\tilde{\mathcal{C}}_L(N,M,\xi,\xi_\mu) = \\
&\quad \frac{\alpha_s}{3\pi}\frac{3}{2}\left(1+\frac{\xi}{4}\right)^{-N}\xi_\mu^{-M}\frac{\Gamma(1-M)^3\Gamma(1+M)}{(3-2M)(1+2M)\Gamma(2-2M)}\frac{4}{4+\xi}\times \\
&\quad \left[3+\frac{\xi}{2}(1-M)-\left(3+\xi(1-M)\left(1-\frac{\xi}{4}M\right)\right){}_2F_1\left(1-M,1,\frac{3}{2},-\frac{\xi}{4}\right)\right].
\end{aligned}
\tag{14}
$$

We get that

$$
\begin{aligned}
&z\left(1+\frac{4}{\xi}\right)C_{L,g}^{[3](3)\mathrm{NLL}}\left(z,\xi,\xi_\mu\right) = \\
&\quad a_{11}^2\Bigg[-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{128}{27}\left(17+\frac{120}{\xi}\right)+\frac{16}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad +I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right) \\
&\qquad +\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)+\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)L_\mu \\
&\qquad +\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)L_\mu^2\Bigg]\log(z) \\
&\quad +a_{21}\left[\frac{64}{9}\left(\frac{12}{\xi}-1\right)-\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right] \\
&\quad +a_{10}a_{11}\Bigg[\frac{256}{27}\left(17+\frac{120}{\xi}\right)+\frac{64}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad +I(\xi)\left(-\frac{64}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)-\frac{32}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\Bigg] \\
&\quad +a_{11}\beta_0\Bigg[-\frac{128}{27}\left(17+\frac{120}{\xi}\right)-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)+\frac{16}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad +I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\Bigg] \\
&\quad +\Bigg[a_{21}\left(\frac{64}{3}\left(1+\frac{6}{\xi}\right)-\frac{64}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)\right) \\
&\qquad +a_{10}a_{11}\left(\frac{128}{9}\left(\frac{12}{\xi}-1\right)-\frac{128}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right) \\
&\qquad +a_{11}\beta_0\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)+\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\Bigg]L_\mu \\
&\quad +\left[a_{11}\beta_0\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)+a_{10}a_{11}\left(-\frac{64}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)+\frac{64}{3}\left(1+\frac{6}{\xi}\right)\right)\right]L_\mu^2.
\end{aligned}
\tag{15}
$$

It is now useful to compute the large Q limits of Eqs. (11),(15). What we get is

$$
\begin{aligned}
&zC_{2,g}^{[3,0](3)\mathrm{NLL}}\left(z,\xi,\xi_\mu\right) = \\
&\quad a_{11}^2\Bigg[\frac{16}{27}\left(13\pi^2-92-72\zeta_3\right)-\frac{32}{27}\left(3\pi^2-71\right)L_Q-\frac{208}{9}L_Q^2+\frac{32}{9}L_Q^3 \\
&\qquad +L_\mu\left(\frac{32}{9}\left(\pi^2-5\right)+\frac{416}{9}L_Q-\frac{32}{3}L_Q^2\right)+L_\mu^2\left(-\frac{16}{3}+\frac{32}{3}L_Q\right)\Bigg]\log(x) \\
&\quad -\frac{32}{27}a_{10}a_{11}\left(13\pi^2-92-72\zeta_3\right)+\frac{16}{27}a_{11}\beta_0\left(13\pi^2-92-72\zeta_3\right)-\frac{32}{9}a_{21}(\pi^2-5) \\
&\quad +\left(-\frac{416}{9}a_{21}+\frac{64}{27}a_{10}a_{11}(3\pi^2-71)-\frac{32}{27}a_{11}\beta_0\left(3\pi^2-71\right)\right)L_Q
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{416}{9}a_{10}a_{11}+\frac{32}{3}a_{21}-\frac{208}{9}a_{11}\beta_0\right)L_Q^2+\left(-\frac{64}{9}a_{10}a_{11}+\frac{32}{9}a_{11}\beta_0\right)L_Q^3\\
&+L_\mu\Bigg(\frac{32}{3}a_{21}-\frac{64}{9}a_{10}a_{11}\left(\pi^2-5\right)+\frac{32}{9}a_{11}\beta_0\left(\pi^2-5\right)+\left(-\frac{832}{9}a_{10}a_{11}-\frac{64}{3}a_{21}+\frac{416}{9}a_{11}\beta_0\right)L_Q\\
&\qquad+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_Q^2\Bigg)\\
&+L_\mu^2\left(\frac{32}{3}a_{10}a_{11}-\frac{16}{3}a_{11}\beta_0+\left(-\frac{64}{3}a_{10}a_{11}+\frac{32}{3}a_{11}\beta_0\right)L_Q\right)\,,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&zC_{L,g}^{[3,0](3)\mathrm{NLL}}\Big(z,\xi,\xi_\mu\Big)=\\
&\qquad+a_{11}^2\left(\frac{32}{27}(3\pi^2-68)-\frac{64}{9}L_Q-\frac{32}{3}L_Q^2+L_\mu\left(\frac{64}{9}+\frac{64}{9}L_Q\right)-\frac{32}{3}L_\mu^2\right)\log(x)\\
&\qquad-\frac{64}{9}a_{21}-\frac{64}{27}a_{10}a_{11}(3\pi^2-68)+\frac{32}{27}a_{11}\beta_0(3\pi^2-68)\\
&\qquad+\left(\frac{128}{9}a_{10}a_{11}-\frac{64}{3}a_{21}-\frac{64}{9}a_{11}\beta_0\right)L_Q+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_Q^2\\
&\qquad+L_\mu\left(-\frac{128}{9}a_{10}a_{11}+\frac{64}{3}a_{21}+\frac{64}{9}a_{11}\beta_0+\left(-\frac{128}{3}a_{10}a_{11}+\frac{64}{3}a_{11}\beta_0\right)L_Q\right)\\
&\qquad+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_\mu^2\,.
\end{aligned}
\tag{17}
$$

where we used that $L_Q=\log(1/\xi)$.

Now we want to construct an uncertainty band for our approximation. It will be done using a different expression for γ_1^{NLL}, i.e.

$$
\gamma_1^{\mathrm{NLL}'}=\beta_0a_{11}\left(\frac{21}{8}\zeta_3-4\log 2\right)\left(\frac{1}{N}-\frac{4N}{(N+1)^2}\right)\,.
\tag{18}
$$

Applying the exact same procedure we find

$$
\begin{aligned}
&zC_{2,g}^{[3](3)\mathrm{NLL}}\Big(z,\xi,\xi_\mu\Big)=\\
&a_{11}^2\Bigg[-\frac{147}{27}-\frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right)+\frac{8}{27}J(\xi)\left(\frac{92}{\xi}-71\right)+I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right)+\frac{8}{9}\left(\frac{10}{\xi}-13\right)\right)\\
&\quad+\left(-\frac{160}{9}+\frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right)+\frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)L_\mu+\left(-\frac{16}{3}+\frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)L_\mu^2\Bigg]\log(z)\\
&+a_{10}a_{11}\left[\frac{2944}{27}+\frac{16}{3}K(\xi)\left(\frac{1}{\xi}-1\right)-\frac{16}{27}J(\xi)\left(\frac{92}{\xi}-71\right)+I(\xi)\left(-\frac{16}{3}L_\xi\left(\frac{1}{\xi}-1\right)-\frac{16}{9}\left(\frac{10}{\xi}-13\right)\right)\right]\\
&+a_{11}\beta_0\Bigg[-\frac{1472}{27}-\frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right)-\frac{640}{9}\log 2+\frac{140}{3}\zeta_3\\
&\quad+I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right)+\frac{8}{9}\left(\frac{10}{\xi}-13\right)+\frac{64}{3}\left(\frac{1}{\xi}-1\right)\log 2-14\left(\frac{1}{\xi}-1\right)\zeta_3\right)\\
&\quad+J(\xi)\left(\frac{8}{27}\left(\frac{92}{\xi}-71\right)+\frac{32}{9}\left(\frac{10}{\xi}-13\right)\log 2-\frac{7}{3}\left(\frac{10}{\xi}-13\right)\zeta_3\right)\Bigg]\\
&+L_\mu\Bigg[a_{10}a_{11}\left(\frac{320}{9}-\frac{32}{3}I(\xi)\left(\frac{1}{\xi}-1\right)-\frac{16}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)\\
&\quad+a_{11}\beta_0\Bigg(-\frac{160}{9}+\frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right)-\frac{28}{3}\log 2+28\zeta_3\\
&\qquad+J(\xi)\left(\frac{8}{9}\left(\frac{10}{\xi}-13\right)+\frac{64}{3}\left(\frac{1}{\xi}-1\right)\log 2-14\left(\frac{1}{\xi}-1\right)\zeta_3\right)\Bigg)\Bigg]\\
&+\left[a_{10}a_{11}\left(\frac{32}{3}-\frac{16}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)+a_{11}\beta_0\left(-\frac{16}{3}+\frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)\right]L_\mu^2\,,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&z\left(1+\frac{4}{\xi}\right)C_{L,g}^{[3](3)\mathrm{NLL}}\left(z,\xi,\xi_\mu\right)= \\
&\quad a_{11}^2\Bigg[-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{128}{27}\left(17+\frac{120}{\xi}\right)+\frac{16}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad +I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right) \\
&\qquad +\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)+\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)L_\mu \\
&\qquad +\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)L_\mu^2\Bigg]\log(z) \\
&\quad +a_{10}a_{11}\Bigg[\frac{64}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)+\frac{256}{27}\left(17+\frac{120}{\xi}\right)-\frac{32}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad +I(\xi)\left(-\frac{64}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)-\frac{32}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\Bigg] \\
&\quad a_{11}\beta_0\Bigg[-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{128}{27}\left(17+\frac{120}{\xi}\right)-\frac{256}{9}\left(\frac{12}{\xi}-1\right)\log 2+\frac{56}{3}\left(\frac{12}{\xi}-1\right)\zeta_3 \\
&\qquad +I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)+\frac{256}{3\xi}\left(1+\frac{3}{\xi}\right)\log 2-\frac{56}{\xi}\left(1+\frac{3}{\xi}\right)\zeta_3\right) \\
&\qquad +J(\xi)\left(\frac{16}{27}\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right)+\frac{64}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\log 2-\frac{14}{3}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\zeta_3\right)\Bigg] \\
&\quad +L_\mu\Bigg[a_{10}a_{11}\left(-\frac{128}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)+\frac{128}{9}\left(\frac{12}{\xi}-1\right)-\frac{32}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right) \\
&\qquad +a_{11}\beta_0\Bigg(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)-\frac{256}{3}\left(1+\frac{6}{\xi}\right)\log 2+56\left(1+\frac{6}{\xi}\right)\zeta_3 \\
&\qquad\quad +J(\xi)\left(\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)+\frac{256}{3\xi}\left(1+\frac{3}{\xi}\right)\log 2-\frac{56}{\xi}\left(1+\frac{3}{\xi}\right)\zeta_3\right)\Bigg)\Bigg] \\
&\quad +\left[a_{11}\beta_0\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)+a_{10}a_{11}\left(-\frac{64}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)+\frac{64}{3}\left(1+\frac{6}{\xi}\right)\right)\right]L_\mu^2\,,
\end{aligned}
\tag{20}
$$

whose large Q limits are

$$
\begin{aligned}
&zC_{2,g}^{[3,0](3)\mathrm{NLL}}\left(z,\xi,\xi_\mu\right)= \\
&\quad a_{11}^2\Bigg[\frac{16}{27}\left(13\pi^2-92-72\zeta_3\right)-\frac{32}{27}\left(3\pi^2-71\right)L_Q-\frac{208}{9}L_Q^2+\frac{32}{9}L_Q^3 \\
&\qquad +L_\mu\left(\frac{32}{9}\left(\pi^2-5\right)+\frac{416}{9}L_Q-\frac{32}{3}L_Q^2\right)+L_\mu^2\left(-\frac{16}{3}+\frac{32}{3}L_Q\right)\Bigg]\log(x) \\
&\quad +\frac{4}{27}a_{11}\beta_0\left(-368+52\pi^2-480\log 2+96\pi^2\log 2+27\zeta_3-63\pi^2\zeta_3\right)-\frac{32}{27}a_{10}a_{11}\left(13\pi^2-92-72\zeta_3\right) \\
&\quad +L_Q\left(\frac{64}{27}a_{10}a_{11}(3\pi^2-71)-\frac{4}{27}a_{11}\beta_0(-568+24\pi^2-1248\log 2+819\zeta_3)\right) \\
&\quad +L_Q^2\left(\frac{416}{9}a_{10}a_{11}-\frac{4}{9}a_{11}\beta_0\left(52+96\log 2-63\zeta_3\right)\right)+\left(-\frac{64}{9}a_{10}a_{11}+\frac{32}{9}a_{11}\beta_0\right)L_Q^3 \\
&\quad L_\mu\Bigg[-\frac{64}{9}a_{10}a_{11}\left(\pi^2-5\right)+\frac{4}{9}a_{11}\beta_0(-40+8\pi^2-96\log 2+63\zeta_3) \\
&\qquad +L_Q\left(-\frac{832}{9}a_{10}a_{11}+\frac{8}{9}a_{11}\beta_0(52+96\log 2-63\zeta_3)\right)+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_Q^2\Bigg]
\end{aligned}
$$

$$+ L_\mu^2 \left[\frac{32}{3} a_{10} a_{11} - \frac{16}{3} a_{11} \beta_0 + \left(-\frac{64}{3} a_{10} a_{11} + \frac{32}{3} a_{11} \beta_0 \right) L_Q \right], \tag{21}$$

$$\begin{aligned}
zC_{L,g}^{[3,0](3)\mathrm{NLL}}\Big(z,\xi,\xi_\mu\Big) = \\
&+ a_{11}^2 \left(\frac{32}{27}(3\pi^2 - 68) - \frac{64}{9} L_Q - \frac{32}{3} L_Q^2 + L_\mu \left(\frac{64}{9} + \frac{64}{9} L_Q \right) - \frac{32}{3} L_\mu^2 \right) \log(x) \\
&- \frac{64}{27} a_{10} a_{11} \left(3\pi^2 - 68\right) + \frac{8}{27} a_{11} \beta_0 (-272 + 12\pi^2 + 96 \log 2 - 63\zeta_3) \\
&+ L_Q \left(\frac{128}{9} a_{10} a_{11} + \frac{8}{9} a_{11} \beta_0 (-8 + 96 \log 2 - 63\zeta_3) \right) + \left[\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right] L_Q^2 \\
&+ L_\mu \left[-\frac{128}{9} a_{10} a_{11} - \frac{8}{9} a_{11} \beta_0 \left(-8 + 96 \log 2 - 63\zeta_3\right) + \left(-\frac{128}{3} a_{10} a_{11} + \frac{64}{3} a_{11} \beta_0 \right) L_Q \right] \\
&+ \left[\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right] L_\mu^2 .
\end{aligned} \tag{22}$$