Approximate NLL small-x limit for the N³LO massive coefficient function

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September 13, 2023

In this document we will explain the construction for the approximate NLL small-x limit for the N³LO massive coefficient function, which is not currently known. The gluon coefficient function for F_2 can be written as (in Mellin space)

$$
\tilde{\mathcal{C}}_2(N, M, \xi, \xi_\mu) =
$$
$$
\frac{\alpha_s}{3\pi} \frac{3}{2} \xi_\mu^{-M} \left(1 + \frac{\xi}{4}\right)^{-N} \frac{\Gamma(1-M)^3 \Gamma(1+M)}{(3-2M)(1+2M)\Gamma(2-2M)} \times
$$
$$
\left[1 + M - \left(1 + M - \frac{\xi}{2}(2 + 3M - 3M^2)\right) {}_2F_1\left(1 - M, 1, \frac{3}{2}, -\frac{\xi}{4}\right)\right] .
\tag{1}
$$

with

$$
\xi = \frac{Q^2}{m^2}, \quad \xi_\mu = \frac{\mu^2}{m^2} .
\tag{2}
$$

See [arXiv:1708.07510] for the definition of $\tilde{\mathcal{C}}_2$. Now we can expand Eq. (1) for small M up to second order. What we find is

$$
\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_2^{(0)} + M\tilde{\mathcal{C}}_2^{(1)} + M^2 \tilde{\mathcal{C}}_2^{(2)} + \mathcal{O}(M^3).
\tag{3}
$$

Then we make the following substitution

$$
\begin{aligned}
M^0 &\to \left[M^0\right] = 1, \\
M^1 &\to \left[M^1\right] = M = a_s \gamma_0 + a_s^2 \gamma_1 + \mathcal{O}(a_s^3), \\
M^2 &\to \left[M^2\right] = M(M - a_s \beta_0) = a_s^2(\gamma_0^2 - \gamma_0 \beta_0) + \mathcal{O}(a_s^3),
\end{aligned}
\tag{4}
$$

where $a_s = \frac{\alpha_s}{4\pi}$ and $\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f$. In this way Eq. (3) becomes

$$
\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_2^{(0)} + a_s \tilde{\mathcal{C}}_2^{(1)} \gamma_0 + a_s^2\left(\tilde{\mathcal{C}}_2^{(1)} \gamma_1 + \tilde{\mathcal{C}}_2^{(2)}\left(\gamma_0^2 - \gamma_0 \beta_0\right)\right) + \mathcal{O}(a_s^4).
\tag{5}
$$

Observe that since $\tilde{\mathcal{C}}_2$ is $\mathcal{O}(a_s)$, then all the $\tilde{\mathcal{C}}_2^{(k)}$ are $\mathcal{O}(a_s)$. It means that the term proportional to a_s^2 in (5) is the $\mathcal{O}(a_s^3)$ expansion of the partonic structure function of the gluon in Mellin space. Moreover, observe that all of this must be taken in the limit $N \ll M$, and therefore the term $\left(1 + \frac{\xi}{4}\right)^{-N}$ in Eq. (1) can be replaced with 1. The value of γ_0 is known exactly, and is

$$
\gamma_0^{\mathrm{NLL}} = \frac{a_{11}}{N} + \frac{a_{10}}{N+1},
\tag{6}
$$

while γ_1 is still not known. For this reason we used the approximate value

$$
\gamma_1^{\mathrm{NLL}} = \frac{a_{21}}{N} - \frac{2a_{21}}{N+1} .
\tag{7}
$$

In Eqs. (6)-(7) we defined

$$
\begin{aligned}
a_{11} &= C_A, \\
a_{10} &= -\frac{11C_A + 2n_f(1 - 2C_F/C_A)}{12}, \\
a_{21} &= n_f \frac{26C_F - 23C_A}{36} .
\end{aligned}
\tag{8}
$$

Now we have to transform back from Mellin space to z-space. Using that the Mellin transform is defined as

$$
\mathcal{M}\left[f(z)\right] \equiv f(N) = \int_0^1 dz\, z^{N-1} f(z),
\tag{9}
$$

one can show that

$$\mathcal{M}[1] = \frac{1}{N},$$
$$\mathcal{M}[z] = \frac{1}{N+1},$$
$$\mathcal{M}[\log(z)] = -\frac{1}{N^2},$$
$$\mathcal{M}[z\log(z)] = -\frac{1}{(N+1)^2},$$

(10)

We find that, neglecting terms proportional to z and $z\log(z)$, our approximation of the next-to-leading logarithm expansion of the $\mathcal{O}(a_s^3)$ gluon coefficient function for F_2 is

$$
\begin{aligned}
zC_{2,g}^{[3](3)\text{h.e.}}\left(z,\xi,\xi_\mu\right) = \\
a_{11}^2 & \left[-\frac{147}{27} - \frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{27}J(\xi)\left(\frac{92}{\xi}-71\right) + I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right) + \frac{8}{9}\left(\frac{10}{\xi}-13\right)\right) \right. \\
& \left. + \left(-\frac{160}{9} + \frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)L_\mu + \left(-\frac{16}{3} + \frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)L_\mu^2 \right]\log(z) \\
& + a_{21}\left[\frac{16}{9} - \frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right) - \frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right] \\
& + a_{10}a_{11}\left[\frac{2944}{27} + \frac{16}{3}K(\xi)\left(\frac{1}{\xi}-1\right) - \frac{16}{27}J(\xi)\left(\frac{92}{\xi}-71\right) + I(\xi)\left(-\frac{16}{3}L_\xi\left(\frac{1}{\xi}-1\right) - \frac{16}{9}\left(\frac{10}{\xi}-13\right)\right)\right] \\
& + a_{11}\beta_0\left[-\frac{1472}{27} - \frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{27}J(\xi)\left(\frac{92}{\xi}-71\right) + I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right) + \frac{8}{9}\left(\frac{10}{\xi}-13\right)\right)\right] \\
& + \left[a_{21}\left(\frac{32}{3} - \frac{16}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right) + a_{10}a_{11}\left(\frac{320}{9} - \frac{32}{3}I(\xi)\left(\frac{1}{\xi}-1\right) - \frac{16}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)\right. \\
& \left. + a_{11}\beta_0\left(-\frac{160}{9} + \frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)\right]L_\mu \\
& + \left[a_{10}a_{11}\left(\frac{32}{3} - \frac{16}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right) + a_{11}\beta_0\left(-\frac{16}{3} + \frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)\right]L_\mu^2,
\end{aligned}
$$

(11)

where we defined $L_\mu = \log(1/\xi_\mu)$, $L_\xi = \log(1+\xi/4)$ and

$$I(\xi) = \frac{4}{\xi}\sqrt{\frac{\xi}{\xi+4}}H\left(+,-,\sqrt{\frac{\xi}{\xi+4}}\right),$$
$$J(\xi) = \frac{4}{\xi}\sqrt{\frac{\xi}{\xi+4}}L\left(\sqrt{\frac{\xi}{\xi+4}}\right),$$
$$K(\xi) = \frac{4}{\xi}\sqrt{\frac{\xi}{\xi+4}}H\left(-,+,-,\sqrt{\frac{\xi}{\xi+4}}\right),$$

(12)

with

$$L(z) = \log\left(\frac{1+z}{1-z}\right),$$
$$H\left(+,-,z\right) = H_{1,1}(z) + H_{1,-1}(z) - H_{-1,1}(z) - H_{-1,-1}(z),$$
$$H\left(-,+,-,z\right) = H_{1,1,1}(z) - H_{1,1,-1}(z) + H_{1,-1,1}(z) - H_{1,-1,-1}(z)$$
$$- H_{-1,1,1}(z) + H_{-1,1,-1}(z) - H_{-1,-1,1}(z) + H_{-1,-1,-1}(z).$$

(13)

We can apply the same procedure for F_L: all the steps we did to get Eq. (11) are unchanged and the only difference will

be in the from of the function in Eq. (1). In this case we have to use

$$\tilde{\mathcal{C}}_L(N, M, \xi, \xi_\mu) =$$
$$\frac{\alpha_s}{3\pi}\frac{3}{2}\left(1+\frac{\xi}{4}\right)^{-N}\xi_\mu^{-M}\frac{\Gamma(1-M)^3\Gamma(1+M)}{(3-2M)(1+2M)\Gamma(2-2M)}\frac{4}{4+\xi}\times$$
$$\left[3+\frac{\xi}{2}(1-M)-\left(3+\xi(1-M)\left(1-\frac{\xi}{4}M\right)\right){}_2F_1\left(1-M,1,\frac{3}{2},-\frac{\xi}{4}\right)\right]. \tag{14}$$

We get that

$$z\left(1+\frac{4}{\xi}\right)C_{L,g}^{[3](3)\text{h.e.}}\left(z,\xi,\xi_\mu\right) =$$

$$a_{11}^2\left[-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{128}{27}\left(17+\frac{120}{\xi}\right)+\frac{16}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right)\right.$$

$$+I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)$$

$$+\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)+\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)L_\mu$$

$$\left.+\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)L_\mu^2\right]\log(z)$$

$$+a_{21}\left[\frac{64}{9}\left(\frac{12}{\xi}-1\right)-\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right]$$

$$+a_{10}a_{11}\left[\frac{256}{27}\left(17+\frac{120}{\xi}\right)+\frac{64}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right)\right.$$

$$\left.+I(\xi)\left(-\frac{64}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)-\frac{32}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\right]$$

$$+a_{11}\beta_0\left[-\frac{128}{27}\left(17+\frac{120}{\xi}\right)-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)+\frac{16}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right)\right.$$

$$\left.+I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\right]$$

$$+\left[a_{21}\left(\frac{64}{3}\left(1+\frac{6}{\xi}\right)-\frac{64}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)\right)\right.$$

$$+a_{10}a_{11}\left(\frac{128}{9}\left(\frac{12}{\xi}-1\right)-\frac{128}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)$$

$$\left.+a_{11}\beta_0\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)+\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\right]L_\mu$$

$$+\left[a_{11}\beta_0\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)+a_{10}a_{11}\left(-\frac{64}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)+\frac{64}{3}\left(1+\frac{6}{\xi}\right)\right)\right]L_\mu^2. \tag{15}$$

It is now useful to compute the large Q limits of Eqs. (11),(15). What we get is

$$zC_{2,g}^{[3,0](3)\text{h.e.}}\left(z,\xi,\xi_\mu\right) =$$

$$a_{11}^2\left[\frac{16}{27}\left(13\pi^2-92-72\zeta_3\right)-\frac{32}{27}\left(3\pi^2-71\right)L_Q-\frac{208}{9}L_Q^2+\frac{32}{9}L_Q^3\right.$$

$$\left.+L_\mu\left(\frac{32}{9}\left(\pi^2-5\right)+\frac{416}{9}L_Q-\frac{32}{3}L_Q^2\right)+L_\mu^2\left(-\frac{16}{3}+\frac{32}{3}L_Q\right)\right]\log(x)$$

$$-\frac{32}{27}a_{10}a_{11}\left(13\pi^2-92-72\zeta_3\right)+\frac{16}{27}a_{11}\beta_0\left(13\pi^2-92-72\zeta_3\right)-\frac{32}{9}a_{21}(\pi^2-5)$$

$$+\left(-\frac{416}{9}a_{21}+\frac{64}{27}a_{10}a_{11}(3\pi^2-71)-\frac{32}{27}a_{11}\beta_0\left(3\pi^2-71\right)\right)L_Q$$

$$+\left(\frac{416}{9}a_{10}a_{11}+\frac{32}{3}a_{21}-\frac{208}{9}a_{11}\beta_0\right)L_Q^2+\left(-\frac{64}{9}a_{10}a_{11}+\frac{32}{9}a_{11}\beta_0\right)L_Q^3$$

$$+L_\mu\left(\frac{32}{3}a_{21}-\frac{64}{9}a_{10}a_{11}\left(\pi^2-5\right)+\frac{32}{9}a_{11}\beta_0\left(\pi^2-5\right)+\left(-\frac{832}{9}a_{10}a_{11}-\frac{64}{3}a_{21}+\frac{416}{9}a_{11}\beta_0\right)L_Q\right.$$

$$\left.+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_Q^2\right)$$

$$+L_\mu^2\left(\frac{32}{3}a_{10}a_{11}-\frac{16}{3}a_{11}\beta_0+\left(-\frac{64}{3}a_{10}a_{11}+\frac{32}{3}a_{11}\beta_0\right)L_Q\right)\,,\tag{16}$$

$$zC_{L,g}^{[3,0](3)\text{h.e.}}\left(z,\xi,\xi_\mu\right)=$$

$$+a_{11}^2\left(\frac{32}{27}(3\pi^2-68)-\frac{64}{9}L_Q-\frac{32}{3}L_Q^2+L_\mu\left(\frac{64}{9}+\frac{64}{9}L_Q\right)-\frac{32}{3}L_\mu^2\right)\log(x)$$

$$-\frac{64}{9}a_{21}-\frac{64}{27}a_{10}a_{11}(3\pi^2-68)+\frac{32}{27}a_{11}\beta_0(3\pi^2-68)$$

$$+\left(\frac{128}{9}a_{10}a_{11}-\frac{64}{3}a_{21}-\frac{64}{9}a_{11}\beta_0\right)L_Q+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_Q^2$$

$$+L_\mu\left(-\frac{128}{9}a_{10}a_{11}+\frac{64}{3}a_{21}+\frac{64}{9}a_{11}\beta_0+\left(-\frac{128}{3}a_{10}a_{11}+\frac{64}{3}a_{11}\beta_0\right)L_Q\right)$$

$$+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_\mu^2\,.\tag{17}$$

where we used that $L_Q=\log\left(1/\xi\right)$.

Now we want to construct an uncertainty band for our approximation. It will be done using a different expression for γ_1^{NLL}, i.e.

$$\gamma_1^{\text{NLL}'}=\beta_0a_{11}\left(\frac{21}{8}\zeta_3-4\log 2\right)\left(\frac{1}{N}-\frac{4N}{(N+1)^2}\right)\,.\tag{18}$$

Applying the exact same procedure we find

$$zC_{2,g}^{'[3](3)\text{h.e.}}\left(z,\xi,\xi_\mu\right)=$$

$$a_{11}^2\left[-\frac{147}{27}-\frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right)+\frac{8}{27}J(\xi)\left(\frac{92}{\xi}-71\right)+I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right)+\frac{8}{9}\left(\frac{10}{\xi}-13\right)\right)\right.$$

$$\left.+\left(-\frac{160}{9}+\frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right)+\frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)L_\mu+\left(-\frac{16}{3}+\frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)L_\mu^2\right]\log(z)$$

$$+a_{10}a_{11}\left[\frac{2944}{27}+\frac{16}{3}K(\xi)\left(\frac{1}{\xi}-1\right)-\frac{16}{27}J(\xi)\left(\frac{92}{\xi}-71\right)+I(\xi)\left(-\frac{16}{3}L_\xi\left(\frac{1}{\xi}-1\right)-\frac{16}{9}\left(\frac{10}{\xi}-13\right)\right)\right]$$

$$+a_{11}\beta_0\left[-\frac{1472}{27}-\frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right)-\frac{640}{9}\log 2+\frac{140}{3}\zeta_3\right.$$

$$+I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right)+\frac{8}{9}\left(\frac{10}{\xi}-13\right)+\frac{64}{3}\left(\frac{1}{\xi}-1\right)\log 2-14\left(\frac{1}{\xi}-1\right)\zeta_3\right)$$

$$\left.+J(\xi)\left(\frac{8}{27}\left(\frac{92}{\xi}-71\right)+\frac{32}{9}\left(\frac{10}{\xi}-13\right)\log 2-\frac{7}{3}\left(\frac{10}{\xi}-13\right)\zeta_3\right)\right]$$

$$+L_\mu\left[a_{10}a_{11}\left(\frac{320}{9}-\frac{32}{3}I(\xi)\left(\frac{1}{\xi}-1\right)-\frac{16}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)\right.$$

$$+a_{11}\beta_0\left(-\frac{160}{9}+\frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right)-\frac{28}{3}\log 2+28\zeta_3\right.$$

$$\left.\left.+J(\xi)\left(\frac{8}{9}\left(\frac{10}{\xi}-13\right)+\frac{64}{3}\left(\frac{1}{\xi}-1\right)\log 2-14\left(\frac{1}{\xi}-1\right)\zeta_3\right)\right)\right]$$

$$+\left[a_{10}a_{11}\left(\frac{32}{3}-\frac{16}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)+a_{11}\beta_0\left(-\frac{16}{3}+\frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)\right]L_\mu^2\,,\tag{19}$$

$$z\left(1+\frac{4}{\xi}\right)C'^{[3](3)\text{h.e.}}_{L,g}\left(z,\xi,\xi_\mu\right)=$$

$$a_{11}^2\left[-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{128}{27}\left(17+\frac{120}{\xi}\right)+\frac{16}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right)\right.$$

$$+I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)$$

$$+\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)+\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)L_\mu$$

$$\left.+\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)L_\mu^2\right]\log(z)$$

$$+a_{10}a_{11}\left[\frac{64}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)+\frac{256}{27}\left(17+\frac{120}{\xi}\right)-\frac{32}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right)\right.$$

$$\left.+I(\xi)\left(-\frac{64}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)-\frac{32}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\right]$$

$$a_{11}\beta_0\left[-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{128}{27}\left(17+\frac{120}{\xi}\right)-\frac{256}{9}\left(\frac{12}{\xi}-1\right)\log 2+\frac{56}{3}\left(\frac{12}{\xi}-1\right)\zeta_3\right.$$

$$+I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)+\frac{256}{3\xi}\left(1+\frac{3}{\xi}\right)\log 2-\frac{56}{\xi}\left(1+\frac{3}{\xi}\right)\zeta_3\right)$$

$$\left.+J(\xi)\left(\frac{16}{27}\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right)+\frac{64}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\log 2-\frac{14}{3}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\zeta_3\right)\right]$$

$$+L_\mu\left[a_{10}a_{11}\left(-\frac{128}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)+\frac{128}{9}\left(\frac{12}{\xi}-1\right)-\frac{32}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\right.$$

$$+a_{11}\beta_0\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)-\frac{256}{3}\left(1+\frac{6}{\xi}\right)\log 2+56\left(1+\frac{6}{\xi}\right)\zeta_3\right.$$

$$\left.\left.+J(\xi)\left(\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)+\frac{256}{3\xi}\left(1+\frac{3}{\xi}\right)\log 2-\frac{56}{\xi}\left(1+\frac{3}{\xi}\right)\zeta_3\right)\right)\right]$$

$$+\left[a_{11}\beta_0\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)+a_{10}a_{11}\left(-\frac{64}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)+\frac{64}{3}\left(1+\frac{6}{\xi}\right)\right)\right]L_\mu^2,\quad(20)$$

whose large Q limits are

$$zC'^{[3,0](3)\text{h.e.}}_{2,g}\left(z,\xi,\xi_\mu\right)=$$

$$a_{11}^2\left[\frac{16}{27}\left(13\pi^2-92-72\zeta_3\right)-\frac{32}{27}\left(3\pi^2-71\right)L_Q-\frac{208}{9}L_Q^2+\frac{32}{9}L_Q^3\right.$$

$$\left.+L_\mu\left(\frac{32}{9}\left(\pi^2-5\right)+\frac{416}{9}L_Q-\frac{32}{3}L_Q^2\right)+L_\mu^2\left(-\frac{16}{3}+\frac{32}{3}L_Q\right)\right]\log(x)$$

$$+\frac{4}{27}a_{11}\beta_0\left(-368+52\pi^2-480\log 2+96\pi^2\log 2+27\zeta_3-63\pi^2\zeta_3\right)-\frac{32}{27}a_{10}a_{11}\left(13\pi^2-92-72\zeta_3\right)$$

$$+L_Q\left(\frac{64}{27}a_{10}a_{11}(3\pi^2-71)-\frac{4}{27}a_{11}\beta_0(-568+24\pi^2-1248\log 2+819\zeta_3)\right)$$

$$+L_Q^2\left(\frac{416}{9}a_{10}a_{11}-\frac{4}{9}a_{11}\beta_0\left(52+96\log 2-63\zeta_3\right)\right)+\left(-\frac{64}{9}a_{10}a_{11}+\frac{32}{9}a_{11}\beta_0\right)L_Q^3$$

$$L_\mu\left[-\frac{64}{9}a_{10}a_{11}\left(\pi^2-5\right)+\frac{4}{9}a_{11}\beta_0(-40+8\pi^2-96\log 2+63\zeta_3)\right.$$

$$\left.+L_Q\left(-\frac{832}{9}a_{10}a_{11}+\frac{8}{9}a_{11}\beta_0(52+96\log 2-63\zeta_3)\right)+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_Q^2\right]$$

$$+ L_\mu^2 \left[\frac{32}{3} a_{10} a_{11} - \frac{16}{3} a_{11} \beta_0 + \left(-\frac{64}{3} a_{10} a_{11} + \frac{32}{3} a_{11} \beta_0 \right) L_Q \right], \tag{21}$$

$$zC_{L,g}'^{[3,0](3)\text{h.e.}} \left(z, \xi, \xi_\mu \right) =$$

$$+ a_{11}^2 \left(\frac{32}{27} (3\pi^2 - 68) - \frac{64}{9} L_Q - \frac{32}{3} L_Q^2 + L_\mu \left(\frac{64}{9} + \frac{64}{9} L_Q \right) - \frac{32}{3} L_\mu^2 \right) \log(x)$$

$$- \frac{64}{27} a_{10} a_{11} \left(3\pi^2 - 68 \right) + \frac{8}{27} a_{11} \beta_0 (-272 + 12\pi^2 + 96 \log 2 - 63\zeta_3)$$

$$+ L_Q \left(\frac{128}{9} a_{10} a_{11} + \frac{8}{9} a_{11} \beta_0 (-8 + 96 \log 2 - 63\zeta_3) \right) + \left[\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right] L_Q^2$$

$$+ L_\mu \left[-\frac{128}{9} a_{10} a_{11} - \frac{8}{9} a_{11} \beta_0 \left(-8 + 96 \log 2 - 63\zeta_3 \right) + \left(-\frac{128}{3} a_{10} a_{11} + \frac{64}{3} a_{11} \beta_0 \right) L_Q \right]$$

$$+ \left[\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right] L_\mu^2 . \tag{22}$$