Approximate NLL small-x limit for the N^{3}LO massive coefficient function

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In this document I will explain the construction for the approximate NLL small-x limit for the N^{3}LO massive coefficient function, which is not currently known. Considering just the gluon contribution, in Mellin space the hadronic structure function F_2 can be written as

$$F_2(N,\xi) = \hat{F}_2(N,\xi) f_g(N,\mu^2), \tag{1}$$

with

$$\hat{F}_2(N,\xi) = K_2(N,\xi,\gamma) h(\gamma) \Big(\frac{m^2}{\mu^2}\Big)^{\gamma}, \tag{2}$$

where

$$h(\gamma) = \pi \alpha_{em} \alpha_s e_q^2 \bigg[\frac{7}{9} + \frac{41}{27}\gamma + \frac{244}{81}\gamma^2 + \mathcal{O}(\gamma^3)\bigg], \tag{3}$$

$$\begin{aligned} K_2(N,\xi,\gamma) = \Big(1+\frac{\xi}{4}\Big)^{-N} \frac{3}{(7-5\gamma)(1+2\gamma)} \bigg[\frac{2}{\xi}(1+\gamma) + \Big(1+\frac{\xi}{4}\Big)^{\gamma-1}\Big(2+3\gamma \\ - 3\gamma^2 - \frac{2}{\xi}(1+\gamma)\Big) {}_2F_1\Big(1-\gamma, \frac{\xi}{\xi+4}\Big)\bigg]. \end{aligned} \tag{4}$$

The function ${}_2F_1$ satisfies the expansion

$$\begin{aligned} {}_2F_1\big(1-\gamma, z^2\big) = \frac{1}{2z}\bigg\{ L(z) + \gamma\Big[H(-,+,z) + L(z)\log(1-z^2)\Big] - \frac{1}{2}\gamma^2\Big[H(-,+,-,z) \\ - H(-,+,z)\log(1-z^2) - L(z)\log^2(1-z^2)\Big] + \mathcal{O}(\gamma^3)\bigg\}, \end{aligned} \tag{5}$$

where $z = \sqrt{\xi/(\xi+4)}$ and

$$L(z) = \log\Big(\frac{1+z}{1-z}\Big), \tag{6}$$

$$H\Big(+,-,z\Big) = H_{1,1}(z) + H_{1,-1}(z) - H_{-1,1}(z) - H_{-1,-1}(z), \tag{7}$$

$$\begin{aligned} H\Big(-,+,-,z\Big) = H_{1,1,1}(z) - H_{1,1,-1}(z) + H_{1,-1,1}(z) - H_{1,-1,-1}(z) \\ - H_{-1,1,1}(z) + H_{-1,1,-1}(z) - H_{-1,-1,1}(z) + H_{-1,-1,-1}(z). \end{aligned} \tag{8}$$

Now we can expand Eq. (2) for small γ up to second order. What we find is

$$\hat{F}_2(N,\xi) = \hat{F}_2^{(0)}(N,\xi) + \gamma \hat{F}_2^{(1)}(N,\xi) + \gamma^2 \hat{F}_2^{(2)}(N,\xi) + \mathcal{O}(\gamma^3). \tag{9}$$

Then we make the following substitution

$$\gamma^0 \to \big[\gamma^0\big] = 1, \tag{10}$$

$$\gamma^1 \to \big[\gamma^1\big] = \gamma = \alpha_s\gamma_0 + \alpha_s^2\gamma_1 + \mathcal{O}(\alpha_s^3), \tag{11}$$

$$\gamma^2 \to \big[\gamma^2\big] = \gamma(\gamma - \alpha_s\beta_0) = \alpha_s^2(\gamma_0^2 - \gamma_0\beta_0) + \mathcal{O}(\alpha_s^3). \tag{12}$$

In this way Eq. (9) becomes

$$\hat{F}_2(N,\xi) = \hat{F}_2^{(0)}(N,\xi) + \alpha_s \hat{F}_2^{(1)}(N,\xi)\gamma_0 + \alpha_s^2\Big(\hat{F}_2^{(1)}(N,\xi)\gamma_1 + \hat{F}_2^{(2)}(N,\xi)\big(\gamma_0^2 - \gamma_0\beta_0\big)\Big). \tag{13}$$

Observe that since $h(\gamma)$ is $\mathcal{O}(\alpha_s)$, then all the $\hat{F}_{2,N}^{(k)}$ are $\mathcal{O}(\alpha_s)$. It means that the term proportional to α_s^2 in (13) is the $\mathcal{O}(\alpha_s^3)$ expansion of the partonic structure function of the gluon in Mellin space. Moreover, observe that all of this must be taken in the limit $N \ll \gamma$, and therefore the term $\left(1+\frac{\xi}{4}\right)^{-N}$ in Eq. (4) is expanded at zeroth order and gives 1. The value of γ_0 is known exactly, and is

$$\gamma_0^{\mathrm{NLL}} = \frac{a_{11}}{N} + \frac{a_{10}}{N+1}, \tag{14}$$

while γ_1 is still not known. For this reason we used the approximate value

$$\gamma_1^{\mathrm{NLL}} = \frac{a_{21}}{N} - \frac{2a_{21}}{N+1}, \tag{15}$$

where

$$a_{11} = \frac{C_A}{\pi}, \tag{16}$$

$$a_{10} = -\frac{11C_A + 2n_f(1-2C_F/C_A)}{12\pi}, \tag{17}$$

$$a_{21} = n_f\frac{26C_F - 23C_A}{36\pi^2}. \tag{18}$$

Now we have to transform back from Mellin space to z-space. Using that the Mellin transform is defined as

$$\mathcal{M}\big[f(z)\big] \equiv f(N) = \int_0^1 dz\, z^{N-1} f(z), \tag{19}$$

one can show that

$$\mathcal{M}\big[1\big] = \frac{1}{N}, \tag{20}$$

$$\mathcal{M}\big[z\big] = \frac{1}{N+1}, \tag{21}$$

$$\mathcal{M}\big[\log(z)\big] = -\frac{1}{N^2}, \tag{22}$$

$$\mathcal{M}\big[z\log(z)\big] = -\frac{1}{(N+1)^2}, \tag{23}$$

We find that, neglecting terms proportional to z and $z\log(z)$, and then dividing a factor z in order to find the coefficient function, our approximation of the next-to-leading logarithm expansion of the $\mathcal{O}(\alpha_s^3)$ gluon coefficient function for F_2 is

$$\begin{aligned}
zC_{2,g}^{[3](3)\mathrm{NLL}}\Big(z,\frac{m^2}{Q^2},\frac{m^2}{\mu^2}\Big) = {}&\\
a_{11}^2\Bigg[&-\frac{147}{27} - \frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{27}J(\xi)\left(\frac{92}{\xi}-71\right) + I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right) + \frac{8}{9}\left(\frac{10}{\xi}-13\right)\right)\\
&+\left(-\frac{160}{9} + \frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)L_\mu + \left(-\frac{16}{3} + \frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)L_\mu^2\Bigg]\log(z)\\
&+ a_{21}\left[\frac{16}{9} - \frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right) - \frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right]\\
&+ a_{10}a_{11}\left[\frac{2944}{27} + \frac{16}{3}K(\xi)\left(\frac{1}{\xi}-1\right) - \frac{16}{27}J(\xi)\left(\frac{92}{\xi}-71\right) + I(\xi)\left(-\frac{16}{3}L_\xi\left(\frac{1}{\xi}-1\right) - \frac{16}{9}\left(\frac{10}{\xi}-13\right)\right)\right]\\
&+ a_{11}\beta_0\left[-\frac{1472}{27} - \frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{27}J(\xi)\left(\frac{92}{\xi}-71\right) + I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right) + \frac{8}{9}\left(\frac{10}{\xi}-13\right)\right)\right]\\
&+\Bigg[a_{21}\left(\frac{32}{3} - \frac{16}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right) + a_{10}a_{11}\left(\frac{320}{9} - \frac{32}{3}I(\xi)\left(\frac{1}{\xi}-1\right) - \frac{16}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)\\
&\quad + a_{11}\beta_0\left(-\frac{160}{9} + \frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)\Bigg]L_\mu\\
&+\left[a_{10}a_{11}\left(\frac{32}{3} - \frac{16}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right) + a_{11}\beta_0\left(-\frac{16}{3} + \frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)\right]L_\mu^2
\end{aligned} \tag{24}$$

$$
\begin{aligned}
&z\left(1+\frac{4}{\xi}\right)C_{L,g}^{[3](3)\mathrm{NLL}}\Big(z,\frac{m^2}{Q^2},\frac{m^2}{\mu^2}\Big)= \\
&\qquad a_{11}^2\Bigg[-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{128}{27}\left(17+\frac{120}{\xi}\right)+\frac{16}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad\quad +I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right) \\
&\qquad\quad +\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)+\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)L_\mu \\
&\qquad\quad +\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)L_\mu^2\Bigg]\log(z) \\
&\qquad +a_{21}\left[\frac{64}{9}\left(\frac{12}{\xi}-1\right)-\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right] \\
&\qquad +a_{10}a_{11}\Bigg[\frac{256}{27}\left(17+\frac{120}{\xi}\right)+\frac{64}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad\quad +I(\xi)\left(-\frac{64}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)-\frac{32}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\Bigg] \\
&\qquad +a_{11}\beta_0\Bigg[-\frac{128}{27}\left(17+\frac{120}{\xi}\right)-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)+\frac{16}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad\quad +I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\Bigg] \\
&\qquad +\Bigg[a_{21}\left(\frac{64}{3}\left(1+\frac{6}{\xi}\right)-\frac{64}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)\right) \\
&\qquad\quad +a_{10}a_{11}\left(\frac{128}{9}\left(\frac{12}{\xi}-1\right)-\frac{128}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right) \\
&\qquad\quad +a_{11}\beta_0\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)+\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\Bigg]L_\mu \\
&\qquad +\Bigg[a_{11}\beta_0\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)+a_{10}a_{11}\left(-\frac{64}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)+\frac{64}{3}\left(1+\frac{6}{\xi}\right)\right)\Bigg]L_\mu^2
\end{aligned}
\tag{25}
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\begin{aligned}
zC_{2,g}^{[3,0](3)\mathrm{NLL}}&\Big(z,\frac{m^2}{Q^2},\frac{m^2}{\mu^2}\Big) = \\
& a_{11}^2\Bigg[\frac{16}{27}\left(13\pi^2-92-72\zeta_3\right)-\frac{32}{27}\left(3\pi^2-71\right)L_Q-\frac{208}{9}L_Q^2+\frac{32}{9}L_Q^3 \\
&\qquad +L_\mu\left(\frac{32}{9}\left(\pi^2-5\right)+\frac{416}{9}L_Q-\frac{32}{3}L_Q^2\right)+L_\mu^2\left(-\frac{16}{3}+\frac{32}{3}L_Q\right)\Bigg]\log(x) \\
& -\frac{32}{27}a_{10}a_{11}\left(13\pi^2-92-72\zeta_3\right)+\frac{16}{27}a_{11}\beta_0\left(13\pi^2-92-72\zeta_3\right)-\frac{32}{9}a_{21}(\pi^2-5) \\
& +\left(-\frac{416}{9}a_{21}+\frac{64}{27}a_{10}a_{11}(3\pi^2-71)-\frac{32}{27}a_{11}\beta_0\left(3\pi^2-71\right)\right)L_Q \\
& +\left(\frac{416}{9}a_{10}a_{11}+\frac{32}{3}a_{21}-\frac{208}{9}a_{11}\beta_0\right)L_Q^2+\left(-\frac{64}{9}a_{10}a_{11}+\frac{32}{9}a_{11}\beta_0\right)L_Q^3 \\
& +L_\mu\Bigg(\frac{32}{3}a_{21}-\frac{64}{9}a_{10}a_{11}\left(\pi^2-5\right)+\frac{32}{9}a_{11}\beta_0\left(\pi^2-5\right)+\left(-\frac{832}{9}a_{10}a_{11}-\frac{64}{3}a_{21}+\frac{416}{9}a_{11}\beta_0\right)L_Q \\
&\qquad +\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_Q^2\Bigg) \\
& +L_\mu^2\left(\frac{32}{3}a_{10}a_{11}-\frac{16}{3}a_{11}\beta_0+\left(-\frac{64}{3}a_{10}a_{11}+\frac{32}{3}a_{11}\beta_0\right)L_Q\right)
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
zC_{L,g}^{[3,0](3)\mathrm{NLL}}&\Big(z,\frac{m^2}{Q^2},\frac{m^2}{\mu^2}\Big) = \\
& +a_{11}^2\left(\frac{32}{27}(3\pi^2-68)-\frac{64}{9}L_Q-\frac{32}{3}L_Q^2+L_\mu\left(\frac{64}{9}+\frac{64}{9}L_Q\right)-\frac{32}{3}L_\mu^2\right)\log(x) \\
& -\frac{64}{9}a_{21}-\frac{64}{27}a_{10}a_{11}(3\pi^2-68)+\frac{32}{27}a_{11}\beta_0(3\pi^2-68) \\
& +\left(\frac{128}{9}a_{10}a_{11}-\frac{64}{3}a_{21}-\frac{64}{9}a_{11}\beta_0\right)L_Q+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_Q^2 \\
& +L_\mu\left(-\frac{128}{9}a_{10}a_{11}+\frac{64}{3}a_{21}+\frac{64}{9}a_{11}\beta_0+\left(-\frac{128}{3}a_{10}a_{11}+\frac{64}{3}a_{11}\beta_0\right)L_Q\right) \\
& +\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_\mu^2
\end{aligned}
\tag{27}
$$

Different approximation bla bla bla...

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\begin{aligned}
&zC^{[3](3)\mathrm{NLL}}_{2,g}\Big(z, \frac{m^2}{Q^2}, \frac{m^2}{\mu^2}\Big) = \\
&\quad a_{11}^2\Bigg[-\frac{147}{27} - \frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{27}J(\xi)\left(\frac{92}{\xi}-71\right) + I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right) + \frac{8}{9}\left(\frac{10}{\xi}-13\right)\right) \\
&\qquad + \left(-\frac{160}{9} + \frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right) + \frac{8}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right)L_\mu + \left(-\frac{16}{3} + \frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)L_\mu^2\Bigg]\log(z) \\
&\quad + a_{10}a_{11}\left[\frac{2944}{27} + \frac{16}{3}K(\xi)\left(\frac{1}{\xi}-1\right) - \frac{16}{27}J(\xi)\left(\frac{92}{\xi}-71\right) + I(\xi)\left(-\frac{16}{3}L_\xi\left(\frac{1}{\xi}-1\right) - \frac{16}{9}\left(\frac{10}{\xi}-13\right)\right)\right] \\
&\quad + a_{11}\beta_0\Bigg[-\frac{1472}{27} - \frac{8}{3}K(\xi)\left(\frac{1}{\xi}-1\right) - \frac{640}{9}\log 2 + \frac{140}{3}\zeta_3 \\
&\qquad + I(\xi)\left(\frac{8}{3}L_\xi\left(\frac{1}{\xi}-1\right) + \frac{8}{9}\left(\frac{10}{\xi}-13\right) + \frac{64}{3}\left(\frac{1}{\xi}-1\right)\log 2 - 14\left(\frac{1}{\xi}-1\right)\zeta_3\right) \\
&\qquad + J(\xi)\left(\frac{8}{27}\left(\frac{92}{\xi}-71\right) + \frac{32}{9}\left(\frac{10}{\xi}-13\right)\log 2 - \frac{7}{3}\left(\frac{10}{\xi}-13\right)\zeta_3\right)\Bigg] \\
&\quad + L_\mu\Bigg[a_{10}a_{11}\left(\frac{320}{9} - \frac{32}{3}I(\xi)\left(\frac{1}{\xi}-1\right) - \frac{16}{9}J(\xi)\left(\frac{10}{\xi}-13\right)\right) \\
&\qquad + a_{11}\beta_0\Bigg(-\frac{160}{9} + \frac{16}{3}I(\xi)\left(\frac{1}{\xi}-1\right) - \frac{28}{3}\log 2 + 28\zeta_3 \\
&\qquad\quad + J(\xi)\left(\frac{8}{9}\left(\frac{10}{\xi}-13\right) + \frac{64}{3}\left(\frac{1}{\xi}-1\right)\log 2 - 14\left(\frac{1}{\xi}-1\right)\zeta_3\right)\Bigg)\Bigg] \\
&\quad + \left[a_{10}a_{11}\left(\frac{32}{3} - \frac{16}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right) + a_{11}\beta_0\left(-\frac{16}{3} + \frac{8}{3}J(\xi)\left(\frac{1}{\xi}-1\right)\right)\right]L_\mu^2
\end{aligned}
\tag{28}
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\begin{aligned}
&z\left(1+\frac{4}{\xi}\right)C_{L,g}^{[3](3)\mathrm{NLL}}\Big(z,\frac{m^2}{Q^2},\frac{m^2}{\mu^2}\Big) = \\
&\qquad a_{11}^2\Bigg[-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{128}{27}\left(17+\frac{120}{\xi}\right)+\frac{16}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad\quad +I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right) \\
&\qquad\quad +\left(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)+\frac{16}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)L_\mu \\
&\qquad\quad +\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)L_\mu^2\Bigg]\log(z) \\
&\qquad +a_{10}a_{11}\Bigg[\frac{64}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)+\frac{256}{27}\left(17+\frac{120}{\xi}\right)-\frac{32}{27}J(\xi)\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right) \\
&\qquad\quad +I(\xi)\left(-\frac{64}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)-\frac{32}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right)\Bigg] \\
&\qquad a_{11}\beta_0\Bigg[-\frac{32}{3\xi}K(\xi)\left(1+\frac{3}{\xi}\right)-\frac{128}{27}\left(17+\frac{120}{\xi}\right)-\frac{256}{9}\left(\frac{12}{\xi}-1\right)\log 2+\frac{56}{3}\left(\frac{12}{\xi}-1\right)\zeta_3 \\
&\qquad\quad +I(\xi)\left(\frac{32}{3\xi}L_\xi\left(1+\frac{3}{\xi}\right)+\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)+\frac{256}{3\xi}\left(1+\frac{3}{\xi}\right)\log 2-\frac{56}{\xi}\left(1+\frac{3}{\xi}\right)\zeta_3\right) \\
&\qquad\quad +J(\xi)\left(\frac{16}{27}\left(3+\frac{136}{\xi}+\frac{480}{\xi^2}\right)+\frac{64}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\log 2-\frac{14}{3}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\zeta_3\right)\Bigg] \\
&\qquad +L_\mu\Bigg[a_{10}a_{11}\left(-\frac{128}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)+\frac{128}{9}\left(\frac{12}{\xi}-1\right)-\frac{32}{9}J(\xi)\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)\right) \\
&\qquad\quad +a_{11}\beta_0\Bigg(\frac{64}{3\xi}I(\xi)\left(1+\frac{3}{\xi}\right)-\frac{64}{9}\left(\frac{12}{\xi}-1\right)-\frac{256}{3}\left(1+\frac{6}{\xi}\right)\log 2+56\left(1+\frac{6}{\xi}\right)\zeta_3 \\
&\qquad\qquad +J(\xi)\left(\frac{16}{9}\left(-3-\frac{4}{\xi}+\frac{24}{\xi^2}\right)+\frac{256}{3\xi}\left(1+\frac{3}{\xi}\right)\log 2-\frac{56}{\xi}\left(1+\frac{3}{\xi}\right)\zeta_3\right)\Bigg)\Bigg] \\
&\qquad +\Bigg[a_{11}\beta_0\left(\frac{32}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)-\frac{32}{3}\left(1+\frac{6}{\xi}\right)\right)+a_{10}a_{11}\left(-\frac{64}{3\xi}J(\xi)\left(1+\frac{3}{\xi}\right)+\frac{64}{3}\left(1+\frac{6}{\xi}\right)\right)\Bigg]L_\mu^2
\end{aligned}
\tag{29}
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\begin{aligned}
&zC_{2,g}^{[3,0](3)\mathrm{NLL}}\Big(z,\frac{m^2}{Q^2},\frac{m^2}{\mu^2}\Big) = \\
&\quad a_{11}^2\Bigg[\frac{16}{27}\left(13\pi^2-92-72\zeta_3\right)-\frac{32}{27}\left(3\pi^2-71\right)L_Q-\frac{208}{9}L_Q^2+\frac{32}{9}L_Q^3 \\
&\qquad +L_\mu\left(\frac{32}{9}\left(\pi^2-5\right)+\frac{416}{9}L_Q-\frac{32}{3}L_Q^2\right)+L_\mu^2\left(-\frac{16}{3}+\frac{32}{3}L_Q\right)\Bigg]\log(x) \\
&\quad +\frac{4}{27}a_{11}\beta_0\left(-368+52\pi^2-480\log 2+96\pi^2\log 2+27\zeta_3-63\pi^2\zeta_3\right)-\frac{32}{27}a_{10}a_{11}\left(13\pi^2-92-72\zeta_3\right) \\
&\quad +L_Q\left(\frac{64}{27}a_{10}a_{11}(3\pi^2-71)-\frac{4}{27}a_{11}\beta_0(-568+24\pi^2-1248\log 2+819\zeta_3)\right) \\
&\quad +L_Q^2\left(\frac{416}{9}a_{10}a_{11}-\frac{4}{9}a_{11}\beta_0\left(52+96\log 2-63\zeta_3\right)\right)+\left(-\frac{64}{9}a_{10}a_{11}+\frac{32}{9}a_{11}\beta_0\right)L_Q^3 \\
&\quad L_\mu\Bigg[-\frac{64}{9}a_{10}a_{11}\left(\pi^2-5\right)+\frac{4}{9}a_{11}\beta_0(-40+8\pi^2-96\log 2+63\zeta_3) \\
&\qquad +L_Q\left(-\frac{832}{9}a_{10}a_{11}+\frac{8}{9}a_{11}\beta_0(52+96\log 2-63\zeta_3)\right)+\left(\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right)L_Q^2\Bigg] \\
&\quad +L_\mu^2\left[\frac{32}{3}a_{10}a_{11}-\frac{16}{3}a_{11}\beta_0+\left(-\frac{64}{3}a_{10}a_{11}+\frac{32}{3}a_{11}\beta_0\right)L_Q\right]
\end{aligned}
\tag{30}
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$$
\begin{aligned}
&zC_{L,g}^{[3,0](3)\mathrm{NLL}}\Big(z,\frac{m^2}{Q^2},\frac{m^2}{\mu^2}\Big) = \\
&\quad +a_{11}^2\left(\frac{32}{27}(3\pi^2-68)-\frac{64}{9}L_Q-\frac{32}{3}L_Q^2+L_\mu\left(\frac{64}{9}+\frac{64}{9}L_Q\right)-\frac{32}{3}L_\mu^2\right)\log(x) \\
&\quad -\frac{64}{27}a_{10}a_{11}\left(3\pi^2-68\right)+\frac{8}{27}a_{11}\beta_0(-272+12\pi^2+96\log 2-63\zeta_3) \\
&\quad +L_Q\left(\frac{128}{9}a_{10}a_{11}+\frac{8}{9}a_{11}\beta_0(-8+96\log 2-63\zeta_3)\right)+\left[\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right]L_Q^2 \\
&\quad +L_\mu\left[-\frac{128}{9}a_{10}a_{11}-\frac{8}{9}a_{11}\beta_0\left(-8+96\log 2-63\zeta_3\right)+\left(-\frac{128}{3}a_{10}a_{11}+\frac{64}{3}a_{11}\beta_0\right)L_Q\right] \\
&\quad +\left[\frac{64}{3}a_{10}a_{11}-\frac{32}{3}a_{11}\beta_0\right]L_\mu^2
\end{aligned}
\tag{31}
$$