

In this document we compute the double convolution between a regular coefficient function $C(x)$ and a splitting function $P(x)$ containing a regular, a singular and a local part, i.e.

$$P(x) = P^R(x) + P^S(x) + P^L\delta(x-1) \quad (1)$$

where P^R is regular in $x \rightarrow 1$, while P^S is singular in $x \rightarrow 1$.

The convolution between $C(x)$ and $P(x)$ is

$$\begin{aligned} (C \otimes P)(x) &= \int_x^{x_{\max}} \frac{dz_2}{z_2} C(z_2) P^R\left(\frac{x}{z_2}\right) + \int_{\frac{x}{x_m}}^1 dz_2 P^S(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) - C(x) \int_0^{\frac{x}{x_m}} dz_2 P^S(z_2) + C(x) P^L \\ &= \int_x^{x_{\max}} \frac{dz_2}{z_2} C(z_2) P^R\left(\frac{x}{z_2}\right) + \int_{\frac{x}{x_m}}^1 dz_2 P^S(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) + C(x) \left(P^L - \mathcal{P}^S\left(\frac{x}{x_m}\right) \right) \end{aligned} \quad (2)$$

where we defined

$$\mathcal{P}^S(x) = \int_0^x dz P^S(z)$$

and we used that $C(x)$ is defined for $0 < x < x_m$ and zero outside.

The double convolution is defined as

$$\begin{aligned} (C \otimes P \otimes P)(x) &= \int_x^{x_{\max}} \frac{dz_1}{z_1} (C \otimes P)(z_1) P^R\left(\frac{x}{z_1}\right) \\ &\quad + \int_{\frac{x}{x_m}}^1 dz_1 P^S(z_1) \left(\frac{(C \otimes P)\left(\frac{x}{z_1}\right)}{z_1} - (C \otimes P)(x) \right) \\ &\quad + (C \otimes P)(x) \left(P^L - \mathcal{P}^S\left(\frac{x}{x_m}\right) \right) \end{aligned} \quad (3)$$

Let's consider each piece separately.

1 Regular term

$$\begin{aligned} \int_x^{x_{\max}} \frac{dz_1}{z_1} (C \otimes P)(z_1) P^R\left(\frac{x}{z_1}\right) &= \\ \int_x^{x_m} \frac{dz_1}{z_1} P^R\left(\frac{x}{z_1}\right) \left[\int_{z_1}^{x_m} \frac{dz_2}{z_2} C(z_2) P^R\left(\frac{z_1}{z_2}\right) + \int_{\frac{z_1}{x_m}}^1 P^S(z_2) \left(\frac{C\left(\frac{z_1}{z_2}\right)}{z_2} - C(z_1) \right) + C(z_1) \left(P^L - \mathcal{P}^S\left(\frac{z_1}{x_m}\right) \right) \right] \\ &= \int_x^{x_m} dz_1 \int_{z_1}^{x_m} dz_2 \frac{1}{z_1 z_2} P^R\left(\frac{x}{z_1}\right) C(z_2) P^R\left(\frac{z_1}{z_2}\right) + \int_x^{x_m} dz_1 \int_{\frac{z_1}{x_m}}^1 dz_2 \frac{P^R\left(\frac{x}{z_1}\right) P^S(z_2)}{z_1} \left(\frac{C\left(\frac{z_1}{z_2}\right)}{z_2} - C(z_1) \right) \\ &\quad - \int_x^{x_m} dz_1 \int_0^{\frac{z_1}{x_m}} dz_2 \frac{P^R\left(\frac{x}{z_1}\right) C(z_1) P^S(z_2)}{z_1} + P^L \int_x^{x_m} \frac{dz_1}{z_1} P^R\left(\frac{x}{z_1}\right) C(z_1) \end{aligned}$$

In the second integral of the last line we can perform the change of variable $z_2 \rightarrow \frac{z_1}{z_2}$, so that we get

$$\begin{aligned} &= \int_x^{x_m} dz_1 \int_{z_1}^{x_m} dz_2 \frac{1}{z_1 z_2} P^R\left(\frac{x}{z_1}\right) C(z_2) P^R\left(\frac{z_1}{z_2}\right) + \int_x^{x_m} dz_1 \int_{z_1}^{x_m} \frac{dz_2}{z_2} P^R\left(\frac{x}{z_1}\right) P^S\left(\frac{z_1}{z_2}\right) \left(\frac{C(z_2)}{z_1} - \frac{C(z_1)}{z_2} \right) \\ &\quad - \int_x^{x_m} dz_1 \frac{P^R\left(\frac{x}{z_1}\right) C(z_1) \mathcal{P}^S\left(\frac{z_1}{x_m}\right)}{z_1} + P^L \int_x^{x_m} \frac{dz_1}{z_1} P^R\left(\frac{x}{z_1}\right) C(z_1) \end{aligned}$$

Now the double integrals can be trasformed in integrals over rectangles, using the theta functions:

$$\begin{aligned} &= \int \int_x^{x_m} dz_1 dz_2 \theta(z_2 - z_1) \frac{1}{z_1 z_2} P^R\left(\frac{x}{z_1}\right) C(z_2) P^R\left(\frac{z_1}{z_2}\right) \\ &\quad + \int \int_x^{x_m} dz_1 dz_2 \theta(z_2 - z_1) \frac{1}{z_2} P^R\left(\frac{x}{z_1}\right) P^S\left(\frac{z_1}{z_2}\right) \left(\frac{C(z_2)}{z_1} - \frac{C(z_1)}{z_2} \right) \\ &\quad - \int_x^{x_m} dz_1 \frac{P^R\left(\frac{x}{z_1}\right) C(z_1) \mathcal{P}^S\left(\frac{z_1}{x_m}\right)}{z_1} + P^L \int_x^{x_m} \frac{dz_1}{z_1} P^R\left(\frac{x}{z_1}\right) C(z_1) \end{aligned} \quad (4)$$

The last equation is what has been implemented in the code with Monte Carlo integrals.

2 Singular term

$$\begin{aligned}
& \int_{\frac{x}{x_m}}^1 dz_1 P^S(z_1) \left\{ \frac{(C \otimes P)\left(\frac{x}{z_1}\right)}{z_1} - (C \otimes P)(x) \right\} = \\
& \int_{\frac{x}{x_m}}^1 dz_1 P^S(z_1) \left\{ \frac{1}{z_1} \left[\int_{\frac{x}{z_1}}^{x_m} \frac{dz_2}{z_2} C(z_2) P^R\left(\frac{x}{z_1 z_2}\right) + \int_{\frac{x}{z_1 x_m}}^1 dz_2 P^S(z_2) \left(\frac{C\left(\frac{x}{z_1 z_2}\right)}{z_2} - C\left(\frac{x}{z_1}\right) \right) + C\left(\frac{x}{z_1}\right) \left(P^L - \mathcal{P}^S\left(\frac{x}{z_1 x_m}\right) \right) \right] \right. \\
& \left. - \left[\int_x^{x_m} \frac{dz_2}{z_2} C(z_2) P^R\left(\frac{x}{z_2}\right) + \int_{\frac{x}{x_m}}^1 dz_2 P^S(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) + C(x) \left(P^L - \mathcal{P}^S\left(\frac{x}{x_m}\right) \right) \right] \right\} \\
& = \int_{\frac{x}{x_m}}^1 dz_1 \int_{\frac{x}{z_1}}^{x_m} dz_2 \frac{1}{z_1 z_2} P^S(z_1) C(z_2) P^R\left(\frac{x}{z_1 z_2}\right) + \int_{\frac{x}{x_m}}^1 dz_1 \int_{\frac{x}{z_1 x_m}}^1 dz_2 \frac{P^S(z_1) P^S(z_2)}{z_1} \left(\frac{C\left(\frac{x}{z_1 z_2}\right)}{z_2} - C\left(\frac{x}{z_1}\right) \right) \\
& - \int_{\frac{x}{x_m}}^1 dz_1 \frac{P^S(z_1) \mathcal{P}^S\left(\frac{x}{z_1 x_m}\right)}{z_1} C\left(\frac{x}{z_1}\right) + P^L \int_{\frac{x}{x_m}}^1 \frac{dz_1}{z_1} P^S(z_1) C\left(\frac{x}{z_1}\right) \\
& - \int_{\frac{x}{x_m}}^1 dz_1 \int_x^{x_m} \frac{dz_2}{z_2} P^S(z_1) C(z_2) P^R\left(\frac{x}{z_2}\right) - \int_{\frac{x}{x_m}}^1 dz_1 \int_{\frac{x}{x_m}}^1 dz_2 P^S(z_1) P^S(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) \\
& + \int_{\frac{x}{x_m}}^1 dz_1 P^S(z_1) C(x) \mathcal{P}^S\left(\frac{x}{x_m}\right) - P^L \int_{\frac{x}{x_m}}^1 dz_1 P^S(z_1) C(x)
\end{aligned}$$

Collecting the terms with same integration boundaries, and using theta functions, we get

$$\begin{aligned}
& = \int_{\frac{x}{x_m}}^1 dz_1 \int_x^{x_m} dz_2 P^S(z_1) \left(\frac{P^R\left(\frac{x}{z_1 z_2}\right) \theta\left(z_2 - \frac{x}{z_1}\right)}{z_1} - P^R\left(\frac{x}{z_2}\right) \right) \frac{C(z_2)}{z_2} \\
& + \int_{\frac{x}{x_m}}^1 dz_1 \int_{\frac{x}{x_m}}^1 dz_2 P^S(z_1) \left[\frac{P^S(z_2)}{z_1} \left(\frac{C\left(\frac{x}{z_1 z_2}\right)}{z_2} - C\left(\frac{x}{z_1}\right) \right) \theta\left(z_2 - \frac{x}{z_1 x_m}\right) - P^S(z_2) \left(\frac{C\left(\frac{x}{z_2}\right)}{z_2} - C(x) \right) \right] \\
& - \int_{\frac{x}{x_m}}^1 dz_1 P^S(z_1) \left(\frac{C\left(\frac{x}{z_1}\right) \mathcal{P}^S\left(\frac{x}{z_1 x_m}\right)}{z_1} - C(x) \mathcal{P}^S\left(\frac{x}{x_m}\right) \right) \\
& + P^L \int_{\frac{x}{x_m}}^1 dz_1 P^S(z_1) \left(\frac{C\left(\frac{x}{z_1}\right)}{z_1} - C(x) \right)
\end{aligned}$$

Observe that all these terms are finite since the singularity of P^S for $z \rightarrow 1$ is integrable since it always appears together with a term of the form $f(x/z)/z - f(x)$ so that the integral

$$\int_{x/x_m}^1 dz P^S(z) \left(\frac{f\left(\frac{x}{z}\right)}{z} - f(x) \right)$$

is finite.