

Approximate NLL small- x limit for the N³LO massive coefficient function

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In this document we will explain the construction for the approximate NLL small- x limit for the N³LO massive coefficient function, which is not currently known. The gluon coefficient function for F_2 can be written as (in Mellin space)

$$\begin{aligned} \tilde{\mathcal{C}}_2(N, M, \xi, \xi_\mu) = & \\ & \frac{\alpha_s}{3\pi} \frac{3}{2} \xi_\mu^{-M} \left(1 + \frac{\xi}{4}\right)^{-N} \frac{\Gamma(1-M)^3 \Gamma(1+M)}{(3-2M)(1+2M)\Gamma(2-2M)} \times \\ & \left[1 + M - \left(1 + M - \frac{\xi}{2}(2+3M-3M^2)\right) {}_2F_1\left(1-M, 1, \frac{3}{2}, -\frac{\xi}{4}\right)\right]. \end{aligned} \quad (1)$$

with

$$\xi = \frac{Q^2}{m^2}, \quad \xi_\mu = \frac{\mu^2}{m^2}. \quad (2)$$

See [arXiv:1708.07510] for the definition of $\tilde{\mathcal{C}}_2$. Now we can expand Eq. (1) for small M up to second order. What we find is

$$\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_2^{(0)} + M\tilde{\mathcal{C}}_2^{(1)} + M^2\tilde{\mathcal{C}}_2^{(2)} + \mathcal{O}(M^3). \quad (3)$$

Then we make the following substitution

$$\begin{aligned} M^0 &\rightarrow [M^0] = 1, \\ M^1 &\rightarrow [M^1] = M = a_s \gamma_0 + a_s^2 \gamma_1 + \mathcal{O}(a_s^3), \\ M^2 &\rightarrow [M^2] = M(M - a_s \beta_0) = a_s^2 (\gamma_0^2 - \gamma_0 \beta_0) + \mathcal{O}(a_s^3), \end{aligned} \quad (4)$$

where $a_s = \frac{\alpha_s}{4\pi}$ and $\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f$. In this way Eq. (3) becomes

$$\tilde{\mathcal{C}}_2 = \tilde{\mathcal{C}}_2^{(0)} + a_s \tilde{\mathcal{C}}_2^{(1)} \gamma_0 + a_s^2 \left(\tilde{\mathcal{C}}_2^{(1)} \gamma_1 + \tilde{\mathcal{C}}_2^{(2)} (\gamma_0^2 - \gamma_0 \beta_0) \right) + \mathcal{O}(a_s^4). \quad (5)$$

Observe that since $\tilde{\mathcal{C}}_2$ is $\mathcal{O}(a_s)$, then all the $\tilde{\mathcal{C}}_2^{(k)}$ are $\mathcal{O}(a_s)$. It means that the term proportional to a_s^2 in (5) is the $\mathcal{O}(a_s^3)$ expansion of the partonic structure function of the gluon in Mellin space. Moreover, observe that all of this must be taken in the limit $N \ll M$, and therefore the term $\left(1 + \frac{\xi}{4}\right)^{-N}$ in Eq. (1) can be replaced with 1. The value of γ_0 is known exactly, and is

$$\gamma_0^{\text{NLL}} = \frac{a_{11}}{N} + \frac{a_{10}}{N+1}, \quad (6)$$

while γ_1 is still not known. For this reason we used the approximate value

$$\gamma_1^{\text{NLL}} = \frac{a_{21}}{N} - \frac{2a_{21}}{N+1}. \quad (7)$$

In Eqs. (6)-(7) we defined

$$\begin{aligned} a_{11} &= C_A, \\ a_{10} &= -\frac{11C_A + 2n_f(1 - 2C_F/C_A)}{12}, \\ a_{21} &= n_f \frac{26C_F - 23C_A}{36}. \end{aligned} \quad (8)$$

Now we have to transform back from Mellin space to z -space. Using that the Mellin transform is defined as

$$\mathcal{M}[f(z)] \equiv f(N) = \int_0^1 dz z^{N-1} f(z), \quad (9)$$

one can show that

$$\begin{aligned}
\mathcal{M}[1] &= \frac{1}{N}, \\
\mathcal{M}[z] &= \frac{1}{N+1}, \\
\mathcal{M}[\log(z)] &= -\frac{1}{N^2}, \\
\mathcal{M}[z \log(z)] &= -\frac{1}{(N+1)^2},
\end{aligned} \tag{10}$$

We find that, neglecting terms proportional to z and $z \log(z)$, our approximation of the next-to-leading logarithm expansion of the $\mathcal{O}(a_s^3)$ gluon coefficient function for F_2 is

$$\begin{aligned}
zC_{2,g}^{3\text{h.e.}}(z, \xi, \xi_\mu) = & \\
& a_{11}^2 \left[-\frac{147}{27} - \frac{8}{3} K(\xi) \left(\frac{1}{\xi} - 1 \right) + \frac{8}{27} J(\xi) \left(\frac{92}{\xi} - 71 \right) + I(\xi) \left(\frac{8}{3} L_\xi \left(\frac{1}{\xi} - 1 \right) + \frac{8}{9} \left(\frac{10}{\xi} - 13 \right) \right) \right. \\
& \quad \left. + \left(-\frac{160}{9} + \frac{16}{3} I(\xi) \left(\frac{1}{\xi} - 1 \right) + \frac{8}{9} J(\xi) \left(\frac{10}{\xi} - 13 \right) \right) L_\mu + \left(-\frac{16}{3} + \frac{8}{3} J(\xi) \left(\frac{1}{\xi} - 1 \right) \right) L_\mu^2 \right] \log(z) \\
& + a_{21} \left[\frac{16}{9} - \frac{16}{3} I(\xi) \left(\frac{1}{\xi} - 1 \right) - \frac{8}{9} J(\xi) \left(\frac{10}{\xi} - 13 \right) \right] \\
& + a_{10} a_{11} \left[\frac{2944}{27} + \frac{16}{3} K(\xi) \left(\frac{1}{\xi} - 1 \right) - \frac{16}{27} J(\xi) \left(\frac{92}{\xi} - 71 \right) + I(\xi) \left(-\frac{16}{3} L_\xi \left(\frac{1}{\xi} - 1 \right) - \frac{16}{9} \left(\frac{10}{\xi} - 13 \right) \right) \right] \\
& + a_{11} \beta_0 \left[-\frac{1472}{27} - \frac{8}{3} K(\xi) \left(\frac{1}{\xi} - 1 \right) + \frac{8}{27} J(\xi) \left(\frac{92}{\xi} - 71 \right) + I(\xi) \left(\frac{8}{3} L_\xi \left(\frac{1}{\xi} - 1 \right) + \frac{8}{9} \left(\frac{10}{\xi} - 13 \right) \right) \right] \\
& + \left[a_{21} \left(\frac{32}{3} - \frac{16}{3} J(\xi) \left(\frac{1}{\xi} - 1 \right) \right) + a_{10} a_{11} \left(\frac{320}{9} - \frac{32}{3} I(\xi) \left(\frac{1}{\xi} - 1 \right) - \frac{16}{9} J(\xi) \left(\frac{10}{\xi} - 13 \right) \right) \right. \\
& \quad \left. + a_{11} \beta_0 \left(-\frac{160}{9} + \frac{16}{3} I(\xi) \left(\frac{1}{\xi} - 1 \right) + \frac{8}{9} J(\xi) \left(\frac{10}{\xi} - 13 \right) \right) \right] L_\mu \\
& + \left[a_{10} a_{11} \left(\frac{32}{3} - \frac{16}{3} J(\xi) \left(\frac{1}{\xi} - 1 \right) \right) + a_{11} \beta_0 \left(-\frac{16}{3} + \frac{8}{3} J(\xi) \left(\frac{1}{\xi} - 1 \right) \right) \right] L_\mu^2,
\end{aligned} \tag{11}$$

where we defined $L_\mu = \log(1/\xi_\mu)$ and

$$\begin{aligned}
I(\xi) &= \frac{4}{\xi} \sqrt{\frac{\xi}{\xi+4}} H \left(+, -, \sqrt{\frac{\xi}{\xi+4}} \right), \\
J(\xi) &= \frac{4}{\xi} \sqrt{\frac{\xi}{\xi+4}} L \left(\sqrt{\frac{\xi}{\xi+4}} \right), \\
K(\xi) &= \frac{4}{\xi} \sqrt{\frac{\xi}{\xi+4}} H \left(-, +, -, \sqrt{\frac{\xi}{\xi+4}} \right),
\end{aligned} \tag{12}$$

with

$$\begin{aligned}
L(z) &= \log \left(\frac{1+z}{1-z} \right), \\
H(+, -, z) &= H_{1,1}(z) + H_{1,-1}(z) - H_{-1,1}(z) - H_{-1,-1}(z), \\
H(-, +, -, z) &= H_{1,1,1}(z) - H_{1,1,-1}(z) + H_{1,-1,1}(z) - H_{1,-1,-1}(z) \\
&\quad - H_{-1,1,1}(z) + H_{-1,1,-1}(z) - H_{-1,-1,1}(z) + H_{-1,-1,-1}(z).
\end{aligned} \tag{13}$$

We can apply the same procedure for F_L : all the steps we did to get Eq. (11) are unchanged and the only difference will

be in the from of the function in Eq. (1). In this case we have to use

$$\begin{aligned}\tilde{C}_L(N, M, \xi, \xi_\mu) = & \frac{\alpha_s}{3\pi} \frac{3}{2} \left(1 + \frac{\xi}{4}\right)^{-N} \xi_\mu^{-M} \frac{\Gamma(1-M)^3 \Gamma(1+M)}{(3-2M)(1+2M)\Gamma(2-2M)} \frac{4}{4+\xi} \times \\ & \left[3 + \frac{\xi}{2}(1-M) - \left(3 + \xi(1-M) \left(1 - \frac{\xi}{4}M\right)\right) {}_2F_1\left(1-M, 1, \frac{3}{2}, -\frac{\xi}{4}\right) \right].\end{aligned}\quad (14)$$

We get that

$$\begin{aligned}z \left(1 + \frac{4}{\xi}\right) C_{L,g}^{3\text{h.e.}}(z, \xi, \xi_\mu) = & a_{11}^2 \left[-\frac{32}{3\xi} K(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{128}{27} \left(17 + \frac{120}{\xi}\right) + \frac{16}{27} J(\xi) \left(3 + \frac{136}{\xi} + \frac{480}{\xi^2}\right) \right. \\ & + I(\xi) \left(\frac{32}{3\xi} L_\xi \left(1 + \frac{3}{\xi}\right) + \frac{16}{9} \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) \\ & + \left(\frac{64}{3\xi} I(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{64}{9} \left(\frac{12}{\xi} - 1\right) + \frac{16}{9} J(\xi) \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) L_\mu \\ & \left. + \left(\frac{32}{3\xi} J(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{32}{3} \left(1 + \frac{6}{\xi}\right) \right) L_\mu^2 \right] \log(z) \\ & + a_{21} \left[\frac{64}{9} \left(\frac{12}{\xi} - 1\right) - \frac{64}{3\xi} I(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{16}{9} J(\xi) \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right] \\ & + a_{10} a_{11} \left[\frac{256}{27} \left(17 + \frac{120}{\xi}\right) + \frac{64}{3\xi} K(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{32}{27} J(\xi) \left(3 + \frac{136}{\xi} + \frac{480}{\xi^2}\right) \right. \\ & \left. + I(\xi) \left(-\frac{64}{3\xi} L_\xi \left(1 + \frac{3}{\xi}\right) - \frac{32}{9} \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) \right] \\ & + a_{11} \beta_0 \left[-\frac{128}{27} \left(17 + \frac{120}{\xi}\right) - \frac{32}{3\xi} K(\xi) \left(1 + \frac{3}{\xi}\right) + \frac{16}{27} J(\xi) \left(3 + \frac{136}{\xi} + \frac{480}{\xi^2}\right) \right. \\ & \left. + I(\xi) \left(\frac{32}{3\xi} L_\xi \left(1 + \frac{3}{\xi}\right) + \frac{16}{9} \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) \right] \\ & + \left[a_{21} \left(\frac{64}{3} \left(1 + \frac{6}{\xi}\right) - \frac{64}{3\xi} J(\xi) \left(1 + \frac{3}{\xi}\right) \right) \right. \\ & + a_{10} a_{11} \left(\frac{128}{9} \left(\frac{12}{\xi} - 1\right) - \frac{128}{3\xi} I(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{32}{9} J(\xi) \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) \\ & \left. + a_{11} \beta_0 \left(\frac{64}{3\xi} I(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{64}{9} \left(\frac{12}{\xi} - 1\right) + \frac{16}{9} J(\xi) \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) \right] L_\mu \\ & + \left[a_{11} \beta_0 \left(\frac{32}{3\xi} J(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{32}{3} \left(1 + \frac{6}{\xi}\right) \right) + a_{10} a_{11} \left(-\frac{64}{3\xi} J(\xi) \left(1 + \frac{3}{\xi}\right) + \frac{64}{3} \left(1 + \frac{6}{\xi}\right) \right) \right] L_\mu^2. \quad (15)\end{aligned}$$

It is now useful to compute the large Q limits of Eqs. (11),(15). What we get is

$$\begin{aligned}z C_{2,g}^{[3,0](3)\text{h.e.}}(z, \xi, \xi_\mu) = & a_{11}^2 \left[\frac{16}{27} (13\pi^2 - 92 - 72\zeta_3) - \frac{32}{27} (3\pi^2 - 71) L_Q - \frac{208}{9} L_Q^2 + \frac{32}{9} L_Q^3 \right. \\ & \left. + L_\mu \left(\frac{32}{9} (\pi^2 - 5) + \frac{416}{9} L_Q - \frac{32}{3} L_Q^2 \right) + L_\mu^2 \left(-\frac{16}{3} + \frac{32}{3} L_Q \right) \right] \log(x) \\ & - \frac{32}{27} a_{10} a_{11} (13\pi^2 - 92 - 72\zeta_3) + \frac{16}{27} a_{11} \beta_0 (13\pi^2 - 92 - 72\zeta_3) - \frac{32}{9} a_{21} (\pi^2 - 5) \\ & + \left(-\frac{416}{9} a_{21} + \frac{64}{27} a_{10} a_{11} (3\pi^2 - 71) - \frac{32}{27} a_{11} \beta_0 (3\pi^2 - 71) \right) L_Q\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{416}{9} a_{10} a_{11} + \frac{32}{3} a_{21} - \frac{208}{9} a_{11} \beta_0 \right) L_Q^2 + \left(-\frac{64}{9} a_{10} a_{11} + \frac{32}{9} a_{11} \beta_0 \right) L_Q^3 \\
& + L_\mu \left(\frac{32}{3} a_{21} - \frac{64}{9} a_{10} a_{11} (\pi^2 - 5) + \frac{32}{9} a_{11} \beta_0 (\pi^2 - 5) + \left(-\frac{832}{9} a_{10} a_{11} - \frac{64}{3} a_{21} + \frac{416}{9} a_{11} \beta_0 \right) L_Q \right. \\
& \quad \left. + \left(\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right) L_Q^2 \right) \\
& + L_\mu^2 \left(\frac{32}{3} a_{10} a_{11} - \frac{16}{3} a_{11} \beta_0 + \left(-\frac{64}{3} a_{10} a_{11} + \frac{32}{3} a_{11} \beta_0 \right) L_Q \right), \tag{16}
\end{aligned}$$

$$\begin{aligned}
z C_{L,g}^{[3,0](3)\text{h.e.}}(z, \xi, \xi_\mu) = & \\
& + a_{11}^2 \left(\frac{32}{27} (3\pi^2 - 68) - \frac{64}{9} L_Q - \frac{32}{3} L_Q^2 + L_\mu \left(\frac{64}{9} + \frac{64}{9} L_Q \right) - \frac{32}{3} L_\mu^2 \right) \log(x) \\
& - \frac{64}{9} a_{21} - \frac{64}{27} a_{10} a_{11} (3\pi^2 - 68) + \frac{32}{27} a_{11} \beta_0 (3\pi^2 - 68) \\
& + \left(\frac{128}{9} a_{10} a_{11} - \frac{64}{3} a_{21} - \frac{64}{9} a_{11} \beta_0 \right) L_Q + \left(\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right) L_Q^2 \\
& + L_\mu \left(-\frac{128}{9} a_{10} a_{11} + \frac{64}{3} a_{21} + \frac{64}{9} a_{11} \beta_0 + \left(-\frac{128}{3} a_{10} a_{11} + \frac{64}{3} a_{11} \beta_0 \right) L_Q \right) \\
& + \left(\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right) L_\mu^2. \tag{17}
\end{aligned}$$

where we used that $L_Q = \log(1/\xi)$.

Now we want to construct an uncertainty band for our approximation. It will be done using a different expression for γ_1^{NLL} , i.e.

$$\gamma_1^{\text{NLL}'} = \beta_0 a_{11} \left(\frac{21}{8} \zeta_3 - 4 \log 2 \right) \left(\frac{1}{N} - \frac{4N}{(N+1)^2} \right). \tag{18}$$

Applying the exact same procedure we find

$$\begin{aligned}
z C_{2,g}'^{3\text{h.e.}}(z, \xi, \xi_\mu) = & \\
& a_{11}^2 \left[-\frac{147}{27} - \frac{8}{3} K(\xi) \left(\frac{1}{\xi} - 1 \right) + \frac{8}{27} J(\xi) \left(\frac{92}{\xi} - 71 \right) + I(\xi) \left(\frac{8}{3} L_\xi \left(\frac{1}{\xi} - 1 \right) + \frac{8}{9} \left(\frac{10}{\xi} - 13 \right) \right) \right. \\
& \quad \left. + \left(-\frac{160}{9} + \frac{16}{3} I(\xi) \left(\frac{1}{\xi} - 1 \right) + \frac{8}{9} J(\xi) \left(\frac{10}{\xi} - 13 \right) \right) L_\mu + \left(-\frac{16}{3} + \frac{8}{3} J(\xi) \left(\frac{1}{\xi} - 1 \right) \right) L_\mu^2 \right] \log(z) \\
& + a_{10} a_{11} \left[\frac{2944}{27} + \frac{16}{3} K(\xi) \left(\frac{1}{\xi} - 1 \right) - \frac{16}{27} J(\xi) \left(\frac{92}{\xi} - 71 \right) + I(\xi) \left(-\frac{16}{3} L_\xi \left(\frac{1}{\xi} - 1 \right) - \frac{16}{9} \left(\frac{10}{\xi} - 13 \right) \right) \right] \\
& + a_{11} \beta_0 \left[-\frac{1472}{27} - \frac{8}{3} K(\xi) \left(\frac{1}{\xi} - 1 \right) - \frac{640}{9} \log 2 + \frac{140}{3} \zeta_3 \right. \\
& \quad + I(\xi) \left(\frac{8}{3} L_\xi \left(\frac{1}{\xi} - 1 \right) + \frac{8}{9} \left(\frac{10}{\xi} - 13 \right) + \frac{64}{3} \left(\frac{1}{\xi} - 1 \right) \log 2 - 14 \left(\frac{1}{\xi} - 1 \right) \zeta_3 \right) \\
& \quad \left. + J(\xi) \left(\frac{8}{27} \left(\frac{92}{\xi} - 71 \right) + \frac{32}{9} \left(\frac{10}{\xi} - 13 \right) \log 2 - \frac{7}{3} \left(\frac{10}{\xi} - 13 \right) \zeta_3 \right) \right] \\
& + L_\mu \left[a_{10} a_{11} \left(\frac{320}{9} - \frac{32}{3} I(\xi) \left(\frac{1}{\xi} - 1 \right) - \frac{16}{9} J(\xi) \left(\frac{10}{\xi} - 13 \right) \right) \right. \\
& \quad + a_{11} \beta_0 \left(-\frac{160}{9} + \frac{16}{3} I(\xi) \left(\frac{1}{\xi} - 1 \right) - \frac{28}{3} \log 2 + 28 \zeta_3 \right. \\
& \quad \left. \left. + J(\xi) \left(\frac{8}{9} \left(\frac{10}{\xi} - 13 \right) + \frac{64}{3} \left(\frac{1}{\xi} - 1 \right) \log 2 - 14 \left(\frac{1}{\xi} - 1 \right) \zeta_3 \right) \right) \right] \\
& + \left[a_{10} a_{11} \left(\frac{32}{3} - \frac{16}{3} J(\xi) \left(\frac{1}{\xi} - 1 \right) \right) + a_{11} \beta_0 \left(-\frac{16}{3} + \frac{8}{3} J(\xi) \left(\frac{1}{\xi} - 1 \right) \right) \right] L_\mu^2, \tag{19}
\end{aligned}$$

$$\begin{aligned}
& z \left(1 + \frac{4}{\xi}\right) C'_{L,g}{}^{3\text{h.e.}}(z, \xi, \xi_\mu) = \\
& a_{11}^2 \left[-\frac{32}{3\xi} K(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{128}{27} \left(17 + \frac{120}{\xi}\right) + \frac{16}{27} J(\xi) \left(3 + \frac{136}{\xi} + \frac{480}{\xi^2}\right) \right. \\
& \quad + I(\xi) \left(\frac{32}{3\xi} L_\xi \left(1 + \frac{3}{\xi}\right) + \frac{16}{9} \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) \\
& \quad + \left(\frac{64}{3\xi} I(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{64}{9} \left(\frac{12}{\xi} - 1\right) + \frac{16}{9} J(\xi) \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) L_\mu \\
& \quad \left. + \left(\frac{32}{3\xi} J(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{32}{3} \left(1 + \frac{6}{\xi}\right) \right) L_\mu^2 \right] \log(z) \\
& + a_{10} a_{11} \left[\frac{64}{3\xi} K(\xi) \left(1 + \frac{3}{\xi}\right) + \frac{256}{27} \left(17 + \frac{120}{\xi}\right) - \frac{32}{27} J(\xi) \left(3 + \frac{136}{\xi} + \frac{480}{\xi^2}\right) \right. \\
& \quad \left. + I(\xi) \left(-\frac{64}{3\xi} L_\xi \left(1 + \frac{3}{\xi}\right) - \frac{32}{9} \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) \right] \\
& a_{11} \beta_0 \left[-\frac{32}{3\xi} K(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{128}{27} \left(17 + \frac{120}{\xi}\right) - \frac{256}{9} \left(\frac{12}{\xi} - 1\right) \log 2 + \frac{56}{3} \left(\frac{12}{\xi} - 1\right) \zeta_3 \right. \\
& \quad + I(\xi) \left(\frac{32}{3\xi} L_\xi \left(1 + \frac{3}{\xi}\right) + \frac{16}{9} \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) + \frac{256}{3\xi} \left(1 + \frac{3}{\xi}\right) \log 2 - \frac{56}{\xi} \left(1 + \frac{3}{\xi}\right) \zeta_3 \right) \\
& \quad + J(\xi) \left(\frac{16}{27} \left(3 + \frac{136}{\xi} + \frac{480}{\xi^2}\right) + \frac{64}{9} \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \log 2 - \frac{14}{3} \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \zeta_3 \right) \left. \right] \\
& + L_\mu \left[a_{10} a_{11} \left(-\frac{128}{3\xi} I(\xi) \left(1 + \frac{3}{\xi}\right) + \frac{128}{9} \left(\frac{12}{\xi} - 1\right) - \frac{32}{9} J(\xi) \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) \right) \right. \\
& \quad + a_{11} \beta_0 \left(\frac{64}{3\xi} I(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{64}{9} \left(\frac{12}{\xi} - 1\right) - \frac{256}{3} \left(1 + \frac{6}{\xi}\right) \log 2 + 56 \left(1 + \frac{6}{\xi}\right) \zeta_3 \right. \\
& \quad \left. \left. + J(\xi) \left(\frac{16}{9} \left(-3 - \frac{4}{\xi} + \frac{24}{\xi^2}\right) + \frac{256}{3\xi} \left(1 + \frac{3}{\xi}\right) \log 2 - \frac{56}{\xi} \left(1 + \frac{3}{\xi}\right) \zeta_3 \right) \right) \right] \\
& + \left[a_{11} \beta_0 \left(\frac{32}{3\xi} J(\xi) \left(1 + \frac{3}{\xi}\right) - \frac{32}{3} \left(1 + \frac{6}{\xi}\right) \right) + a_{10} a_{11} \left(-\frac{64}{3\xi} J(\xi) \left(1 + \frac{3}{\xi}\right) + \frac{64}{3} \left(1 + \frac{6}{\xi}\right) \right) \right] L_\mu^2, \quad (20)
\end{aligned}$$

whose large Q limits are

$$\begin{aligned}
& z C'_{2,g}{}^{[3,0](3)\text{h.e.}}(z, \xi, \xi_\mu) = \\
& a_{11}^2 \left[\frac{16}{27} (13\pi^2 - 92 - 72\zeta_3) - \frac{32}{27} (3\pi^2 - 71) L_Q - \frac{208}{9} L_Q^2 + \frac{32}{9} L_Q^3 \right. \\
& \quad \left. + L_\mu \left(\frac{32}{9} (\pi^2 - 5) + \frac{416}{9} L_Q - \frac{32}{3} L_Q^2 \right) + L_\mu^2 \left(-\frac{16}{3} + \frac{32}{3} L_Q \right) \right] \log(x) \\
& + \frac{4}{27} a_{11} \beta_0 (-368 + 52\pi^2 - 480 \log 2 + 96\pi^2 \log 2 + 27\zeta_3 - 63\pi^2 \zeta_3) - \frac{32}{27} a_{10} a_{11} (13\pi^2 - 92 - 72\zeta_3) \\
& + L_Q \left(\frac{64}{27} a_{10} a_{11} (3\pi^2 - 71) - \frac{4}{27} a_{11} \beta_0 (-568 + 24\pi^2 - 1248 \log 2 + 819\zeta_3) \right) \\
& + L_Q^2 \left(\frac{416}{9} a_{10} a_{11} - \frac{4}{9} a_{11} \beta_0 (52 + 96 \log 2 - 63\zeta_3) \right) + \left(-\frac{64}{9} a_{10} a_{11} + \frac{32}{9} a_{11} \beta_0 \right) L_Q^3 \\
& L_\mu \left[-\frac{64}{9} a_{10} a_{11} (\pi^2 - 5) + \frac{4}{9} a_{11} \beta_0 (-40 + 8\pi^2 - 96 \log 2 + 63\zeta_3) \right. \\
& \quad \left. + L_Q \left(-\frac{832}{9} a_{10} a_{11} + \frac{8}{9} a_{11} \beta_0 (52 + 96 \log 2 - 63\zeta_3) \right) + \left(\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right) L_Q^2 \right]
\end{aligned}$$

$$+ L_\mu^2 \left[\frac{32}{3} a_{10} a_{11} - \frac{16}{3} a_{11} \beta_0 + \left(-\frac{64}{3} a_{10} a_{11} + \frac{32}{3} a_{11} \beta_0 \right) L_Q \right], \quad (21)$$

$$\begin{aligned} zC'_{L,g}{}^{[3,0](3)\text{h.e.}}(z, \xi, \xi_\mu) = & \\ & + a_{11}^2 \left(\frac{32}{27} (3\pi^2 - 68) - \frac{64}{9} L_Q - \frac{32}{3} L_Q^2 + L_\mu \left(\frac{64}{9} + \frac{64}{9} L_Q \right) - \frac{32}{3} L_\mu^2 \right) \log(x) \\ & - \frac{64}{27} a_{10} a_{11} (3\pi^2 - 68) + \frac{8}{27} a_{11} \beta_0 (-272 + 12\pi^2 + 96 \log 2 - 63\zeta_3) \\ & + L_Q \left(\frac{128}{9} a_{10} a_{11} + \frac{8}{9} a_{11} \beta_0 (-8 + 96 \log 2 - 63\zeta_3) \right) + \left[\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right] L_Q^2 \\ & + L_\mu \left[-\frac{128}{9} a_{10} a_{11} - \frac{8}{9} a_{11} \beta_0 (-8 + 96 \log 2 - 63\zeta_3) + \left(-\frac{128}{3} a_{10} a_{11} + \frac{64}{3} a_{11} \beta_0 \right) L_Q \right] \\ & + \left[\frac{64}{3} a_{10} a_{11} - \frac{32}{3} a_{11} \beta_0 \right] L_\mu^2. \end{aligned} \quad (22)$$