1 CQF filter bank

A two-channel filter bank is defined by the diagram in Figure 1.

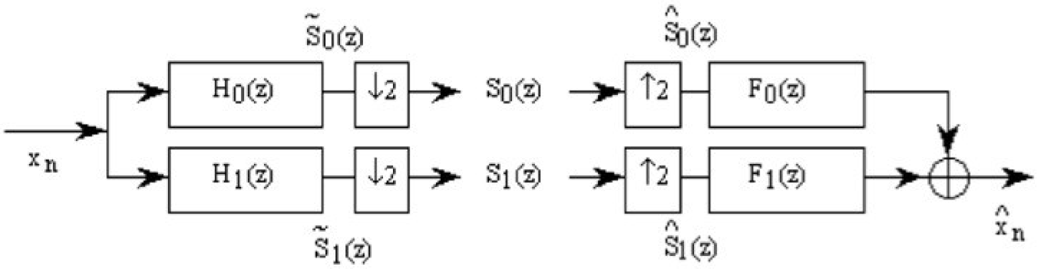

Figure 1: General diagram of a two-channel filter bank.

1. Express $\hat{X}(z)$ as a function of $X(z)$.

   The calculation is done in the course handout. We get

$$\widehat{X}(z) = T(z)X(z) + A(z)X(-z) \tag{1}$$

   where

$$T(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z)), \tag{2}$$

$$A(z) = \frac{1}{2}(H_0(-z)F_0(z) + H_1(-z)F_1(z)). \tag{3}$$

2. Deduce that the aliasing cancellation (AC) conditions of a 2-channel filter bank are $F_0(z) = H_1(-z)$ and $F_1(z) = -H_0(-z)$, and that its transfer function (TF) is $T(z) = \frac{1}{2}(H_0(z)F_0(z) + H_1(z)F_1(z))$.

   Indeed, if $F_0(z) = H_1(-z)$ and $F_1(z) = -H_0(-z)$, then $A(z) = 0$, therefore $\widehat{X}(z) = T(z)X(z)$, which proved that the transfer function is $T(z)$.

3. Now we assume that H_0 and H_1 are *conjugate quadrature filters* (CQF): $H_1(z) = -z^{-(N-1)}\widetilde{H}_0(-z)$ where $\tilde{H}_0(z) = H_0^*(\frac{1}{z})$ and N is even. Prove that equations (AC) and (CQF) imply that $\forall k \in \{0, 1\}$, $F_k(z) = z^{-(N-1)}\widetilde{H}_k(z)$: we say that the analysis and synthesis filters are *paraconjugate* (PC).

   The AC condition yields $F_0(z) = H_1(-z)$, then the CQF condition yields $H_1(-z) = z^{-(N-1)}\widetilde{H}_0(z)$ because N is even, therefore $F_0(z) = z^{-(N-1)}\widetilde{H}_0(z)$. In the same way, the AC condition yields $F_1(z) = -H_0(-z)$, then the CQF condition yields $H_0(-z) = -z^{-(N-1)}\widetilde{H}_1(z)$ because N is even, therefore $F_1(z) = z^{-(N-1)}\widetilde{H}_1(z)$.

4. Finally we assume that $H_0(z)$ is a *symmetric power* (SP) filter: $\widetilde{H}_0(z)H_0(z) + \widetilde{H}_0(-z)H_0(-z) = 2c$. Prove that equations (TF), (CQF), (PC) et (SP) imply that $T(z) = cz^{-(N-1)}$: the CQF filter bank guarantees perfect reconstruction.

   Substituting equations PC into TF yields $T(z) = \frac{z^{-(N-1)}}{2}(H_0(z)\widetilde{H}_0(z) + H_1(z)\widetilde{H}_1(z))$. Then substituting equation CQF into this last equation yields $T(z) = \frac{z^{-(N-1)}}{2}(H_0(z)\widetilde{H}_0(z)+\widetilde{H}_0(-z)H_0(-z))$. Finally, substituting equation SP into this last equation yields $T(z) = cz^{-(N-1)}$, which proves the perfect reconstruction of the CQF filter bank.

2 Transmultiplexer

We implement the transmultiplexer represented in Figure 2 by means of the filters defined in Exercise 1.

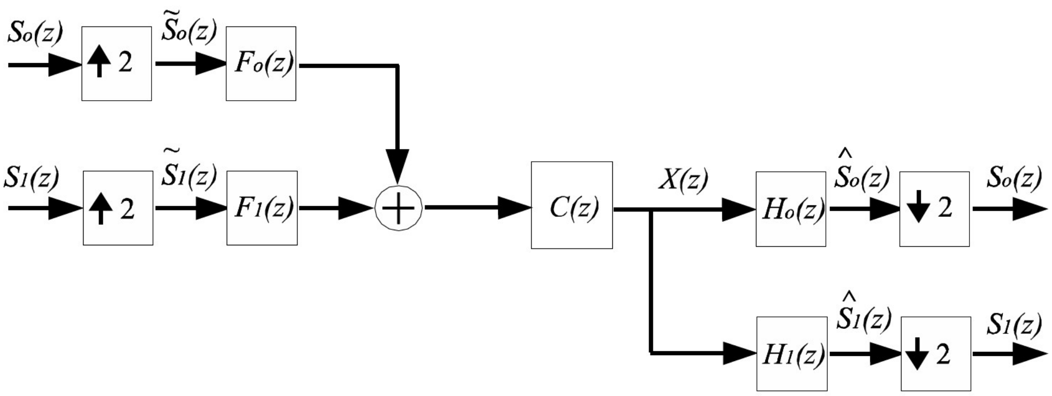

Figure 2: Transmultiplexer.

1. Prove that the transmultiplexer guarantees perfect reconstruction at the output when $C(z) = z^{-1}$.

   We already know that the diagram in Figure 1 is equivalent to the transfer function $cz^{-(N-1)}$ where N is even. This proves that the left block in Figure 1, preceded by $cz^{-(N-1)}$, defines a linear transform which is exactly the inverse of the linear transform defined by the right block in Figure 1. Besides, we notice that Figure 2 is obtained by switching the two blocks in Figure 1 and inserting filter $C(z)$ between them. Consequently, if we choose $C(z) = cz^{-(N-1)}$, then the transmultiplixer is equivalent to the identity transform, which is a particular case of perfect reconstruction. Equivalently, another perfect reconstruction solution is obtained by simply choosing $C(z) = z^{-1}$, because N is even, and because the transmultiplexer involves decimations and insertions of zeros of order 2.

2. From now on, filter $C(z)$ will represent the transfer function of a transmission channel between the encoder and the decoder. It is uniformly equal to 1 if the channel is transparent, but in general, the transmission channel is imperfect, and its transfer function $C(z)$ is not constant. In order to simplify, let us assume that $C(z)$ is of the form $1 - \alpha z^{-1}$. In order to keep the perfect reconstruction property at the output of the transmultiplexer, we will have to introduce just after $C(z)$ a causal filter $D(z)$ such that $C(z)D(z) = dz^{-n_0}$, where n_0 is an odd number.

   (a) How to choose $D(z)$ if $\alpha = 0.9$?
   If $\alpha = 0.9$, then we can simply set $D(z) = \frac{dz^{-n_0}}{1-\alpha z^{-1}}$, because the causal implementation of this filter is stable.

   (b) What problem do we encounter if $\alpha = 1.2$? Propose an approximate solution.
   If $\alpha = 1.2$, then the causal implementation of the filter $D(z) = \frac{dz^{-n_0}}{1-\alpha z^{-1}}$ is unstable. The stable implementation cannot be used either, because it is IIR and anti-causal. A solution would consist in approximating this filter by a causal FIR filter. This can be achieved by truncating the anti-causal, infinite impulse response of the stable implementation of $D(z)$ to a given finite length, and translating it over time, so as to make it causal. The quality of this approximation can be improved by increasing the length of the causal FIR filter obtained in this way.

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