1 Eigenvalue method

1. We are interested here in the synthesis of linear phase FIR filters. We consider the particular case of a type I filter, of odd length N and symmetrical impulse response, whose transfer function is denoted $H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$. Let $M = \frac{N-1}{2}$. Verify that we can write

$$H\left(e^{i2\pi\nu}\right) = e^{-i2\pi\nu M}\, H_R\left(e^{i2\pi\nu}\right) \tag{1}$$

where $H_R\left(e^{i2\pi\nu}\right)$ is a real-valued function, called the amplitude response of filter H, defined by the equality $H_R\left(e^{i2\pi\nu}\right) = \boldsymbol{a}^T \boldsymbol{c}(\nu)$, where $\boldsymbol{c}(\nu) = [1, \cos(2\pi\nu), \ldots, \cos(2\pi M\nu)]^T$, and where the coefficients of vector $\boldsymbol{a} = [a_0, a_1, \ldots, a_M]^T$ are to be expressed in terms of $h(n)$.

We have

$$\begin{aligned} H(z) &= z^{-M}\left(\sum_{n=0}^{M-1} h(n)\, z^{-n+M} + h(M) + \sum_{n=M+1}^{N-1} h(n)\, z^{-n+M}\right) \\ &= z^{-M}\left(\sum_{n=0}^{M-1} h(n)\, z^{-n+M} + h(M) + \sum_{m=0}^{M-1} h(N-1-m)\, z^{m-M}\right) \\ &= z^{-M}\left(h(M) + \sum_{n=0}^{M-1} h(n)\,(z^{-n+M} + z^{n-M})\right). \end{aligned}$$

At $z = e^{i2\pi\nu}$, we get

$$H\left(e^{i2\pi\nu}\right) = e^{-i2\pi\nu M}\left(H(M) + 2\sum_{n=0}^{M-1} h(n)\cos(2\pi\nu(M-n))\right).$$

We thus retrieve (1) *with $a_0 = h(M)$ and $a_m = 2h(M-m)$ $\forall m \in [[1, M]]$.*

2. We wish to synthesize a low-pass filter with cutoff frequency $\nu_c \in \left]0, \frac{1}{2}\right[$ and whose stop-band starts at $\nu_a \in \left]\nu_c, \frac{1}{2}\right[$. The energy in the stop-band is $E_a = 2\int_{\nu_a}^{\frac{1}{2}} \left(H_R(e^{i2\pi\nu})\right)^2 \mathrm{d}\nu$. Show that we can write $E_a = \boldsymbol{a}^T \boldsymbol{P}\, \boldsymbol{a}$, where $\boldsymbol{P}$ is a positive semidefinite matrix, whose coefficients $\{\boldsymbol{P}_{(m,n)}\}_{(m,n)\in[[0,M]]^2}$ are to be determined in function of ν_a.

We have $E_a = 2\int_{\nu_a}^{\frac{1}{2}} \left(\boldsymbol{a}^T \boldsymbol{c}(\nu)\right)^2 \mathrm{d}\nu = \boldsymbol{a}^T \boldsymbol{P}\, \boldsymbol{a}$ with $\boldsymbol{P} = 2\int_{\nu_a}^{\frac{1}{2}} \boldsymbol{c}(\nu)\boldsymbol{c}(\nu)^T \mathrm{d}\nu$. Matrix $\boldsymbol{P}$ is positive semidefinite, as a sum of rank-1 positive semidefinite matrices. Moreover,

$$\begin{aligned} \boldsymbol{P}_{(m,n)} &= 2\int_{\nu_a}^{\frac{1}{2}} \cos(2\pi\nu m)\cos(2\pi\nu n)\mathrm{d}\nu \\ &= \int_{\nu_a}^{\frac{1}{2}} \cos(2\pi\nu(m+n)) + \cos(2\pi\nu(m-n))\mathrm{d}\nu \\ &= \left[\frac{\sin(2\pi\nu(m+n))}{2\pi(m+n)} + \frac{\sin(2\pi\nu(m-n))}{2\pi(m-n)}\right]_{\nu_a}^{\frac{1}{2}} \\ &= \frac{\delta(m+n)+\delta(m-n)}{2} - \nu_a\left(\operatorname{sinc}(2\pi\nu_a(m+n)) + \operatorname{sinc}(2\pi\nu_a(m-n))\right). \end{aligned}$$

3. Ideally, the amplitude response $H_R\left(e^{i2\pi\nu}\right)$ is equal to $H_R(1)$ in the bandwidth $[0, \nu_c]$. We therefore define the square error in the bandwidth as follows:

$$E_c = 2\int_0^{\nu_c} \left(H_R(e^{i2\pi\nu}) - H_R(1)\right)^2 \mathrm{d}\nu$$

Show that we can write $E_c = \boldsymbol{a}^T \boldsymbol{Q}\, \boldsymbol{a}$, where $\boldsymbol{Q}$ is a positive semidefinite matrix, whose coefficients $\left\{\boldsymbol{Q}_{(m,n)}\right\}_{(m,n)\in[[0,M]]^2}$ are to be determined in function of ν_c.

We have $E_c = 2\int_0^{\nu_c} \left(\boldsymbol{a}^T(\boldsymbol{c}(\nu) - \boldsymbol{c}(0)\right)^2 \mathrm{d}\nu = \boldsymbol{a}^T \boldsymbol{Q}\, \boldsymbol{a}$ *with* $\boldsymbol{Q} = 2\int_0^{\nu_c} (\boldsymbol{c}(\nu) - \boldsymbol{c}(0))(\boldsymbol{c}(\nu) - \boldsymbol{c}(0))^T \mathrm{d}\nu$. *Matrix* $\boldsymbol{Q}$ *is positive semidefinite, as a sum of rank-1 positive semidefinite matrices. Moreover,*

$$\begin{aligned} \boldsymbol{Q}_{(m,n)} &= 2\int_0^{\nu_c} (\cos(2\pi\nu m) - 1)(\cos(2\pi\nu n) - 1)\mathrm{d}\nu \\ &= \int_0^{\nu_c} \cos(2\pi\nu(m+n)) + \cos(2\pi\nu(m-n)) - 2\cos(2\pi\nu m) - 2\cos(2\pi\nu n) + 2\,\mathrm{d}\nu. \end{aligned}$$

The end of the calculation is left to the reader.

4. The FIR filter synthesis method called *eigenvalue method* consists in minimizing with respect to $\boldsymbol{a}$ the cost function $E(\boldsymbol{a}) = \alpha E_c + (1-\alpha)E_a$, where $\alpha \in]0,1[$ is a trade-off parameter between pass-band and stop-band. We thus obtain $E(\boldsymbol{a}) = \boldsymbol{a}^T \boldsymbol{R}\, \boldsymbol{a}$, where $\boldsymbol{R} = \alpha\boldsymbol{Q} + (1-\alpha)\boldsymbol{P}$ is a positive semidefinite matrix. Show that vector $\boldsymbol{a}$ minimizes function E under unit norm constraint if and only if it is an eigenvector of $\boldsymbol{R}$, associated to the lowest eigenvalue (*Rayleigh's principle*).

   The Lagrangian of this optimization problem is

$$\mathcal{L}(\boldsymbol{a}, \lambda) = \boldsymbol{a}^T \boldsymbol{R}\, \boldsymbol{a} + \lambda(1 - \|\boldsymbol{a}\|^2).$$

   Its gradient w.r.t. $\boldsymbol{a}$ *is* $2\boldsymbol{R}\,\boldsymbol{a} - 2\lambda\boldsymbol{a}$. *This gradient is zero when* $\boldsymbol{R}\,\boldsymbol{a} = \lambda\boldsymbol{a}$ *(therefore* $\boldsymbol{a}$ *is an eigenvector of* $\boldsymbol{R}$*, associated to the eigenvalue* λ*), in which case* $E(\boldsymbol{a}) = \lambda$ *when* $\|\boldsymbol{a}\| = 1$*. Therefore vector* $\boldsymbol{a}$ *minimizes function* E *under unit norm constraint if and only if it is an eigenvector of* $\boldsymbol{R}$ *associated to the lowest eigenvalue* λ*.*

2 Synthesis of an integrator filter

We consider a digital signal $x(n)$, defined from an analog signal $x^a(t)$ sampled at sampling rate T: $x(n) = x^a(nT)$. This exercise aims at synthesizing a digital filter which allows to obtain, from the discrete signal $x(n)$, a sampled version of the integrated signal $y^a(t) = \int_{-\infty}^{t} x^a(u)\, du$.

Question 1 Show that the integrated signal $y^a(t)$ can be written as the convolution product between the signal $x^a(t)$ and the analog filter $h^a(t) = 1$ if $t \geq 0$ and $h^a(t) = 0$ otherwise (Heaviside function). Is this filter causal? Is it stable? (reminder : the filter is stable if and only if $\int_{-\infty}^{+\infty} |h^a(t)|\, dt < +\infty$). Compute the transfer function $H^a(p) = \int_{-\infty}^{+\infty} h^a(t)\, e^{-pt}\, dt$ (Laplace transform of h^a, with $p \in \mathbb{C}$), and specify its domain of definition.

We have $(h^a * x^a)(t) = \int_{v\in\mathbb{R}} h^a(v)x^a(t-v)dv = \int_{v=0}^{+\infty} x^a(t-v)dv = \int_{u=-\infty}^{t} x^a(u)\, du$ *with the change of variable* $v = t - u$*. Filter* h^a *is causal by definition, and it is unstable because* $h^a \notin L^1(\mathbb{R})$*. We get* $H^a(p) = \int_0^{+\infty} e^{-pt}\, dt = \frac{1}{p}$*, which is defined for* $Re(p) > 0$*.*

2.1 Approximation by the rectangle method

We wish to approximate the integral of the signal $x^a(t)$ by the rectangle method (an example is given on Figure 1-(a)), which amounts to computing the integral of the interpolated signal

$$x_0^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f_0(t - mT)$$

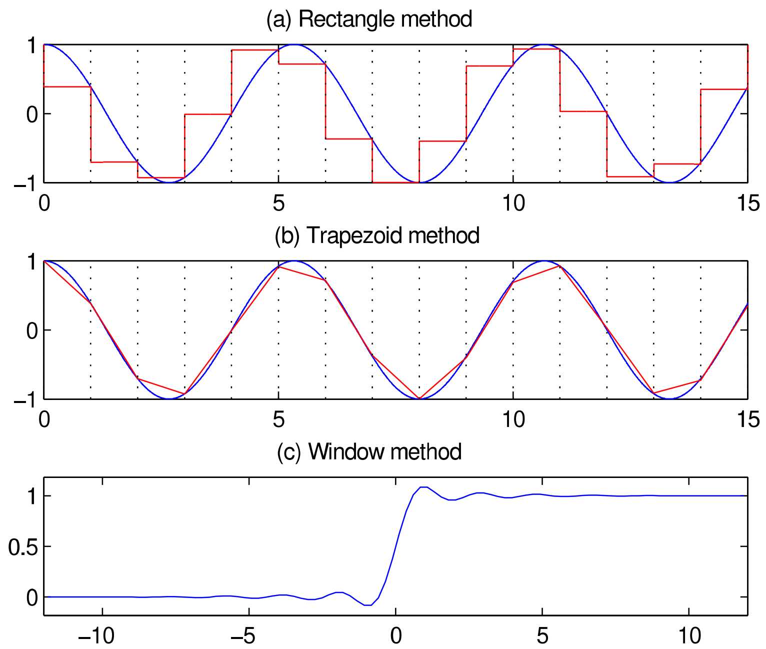

Figure 1: Three synthesis methods of an integrator filter

where $f_0(t) = 1$ if $t \in [-T, 0]$ and $f_0(t) = 0$ otherwise (rectangle function). We define the discrete-time integrated signal $y_0(n) = \int_{-\infty}^{nT} x_0^a(t)\,dt$.

Question 2 Show that $y_0(n)$ can be written as the convolution product between the signal $x(n)$ and a digital filter $h_0(n)$, and give the expression of its impulse response. Is this filter causal? Is it stable? Calculate the transfer function $H_0(z)$, and specify its domain of definition.

We have $y_0(n) = \sum\limits_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{nT} f_0(t-mT)\,dt = T \sum\limits_{m=-\infty}^{n} x(m) = \sum\limits_{m\in\mathbb{Z}} h_0(n-m)x(m)$ *with* $h_0(m) = T\, 1_{\mathbb{N}}(m)$. *Filter* h_0 *is causal by definition, and it is unstable because* $h_0 \notin l^1(\mathbb{Z})$*. We get* $H_0(z) = T \sum_{m\in\mathbb{N}} z^{-m} = \frac{T}{1-z^{-1}}$*, which is defined for* $|z| > 1$.

2.2 Approximation by the trapezoid method

We wish to approximate the integral of signal $x^a(t)$ by the trapezoid method (an example is given in Figure 1-(b)), which amounts to computing the integral of the interpolated signal

$$x_1^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f_1(t - mT)$$

where $f_1(t) = 1 - |t|/T$ if $t \in [-T, T]$ and $f_1(t) = 0$ elsewhere (triangle function). We define the discrete-time integrated signal $y_1(n) = \int_{-\infty}^{nT} x_1^a(t)\,dt$.

Question 3 Show that $y_1(n)$ can be written as the convolution product between the signal $x(n)$ and a digital filter $h_1(n)$, and give the expression of its impulse response. Is this filter causal? Is it stable?

We have $y_1(n) = \sum\limits_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{nT} f_1(t-mT)\,dt = T(\frac{1}{2}x(n) + \sum\limits_{m=-\infty}^{n-1} x(m)) = \sum\limits_{m\in\mathbb{Z}} h_1(n-m)x(m)$ *with* $h_1(m) = T\,(\frac{1}{2}\delta_0(m) + 1_{\mathbb{N}^*}(m))$*. Filter* h_1 *is causal by definition, and it is unstable because* $h_1 \notin l^1(\mathbb{Z})$*.*

Question 4 Show that this method is equivalent to determining the digital filter from the analog filter of Question 1 by using the bilinear transformation (hint: we can identify the two transfer functions).
We get $H_1(z) = \frac{1}{2} + \sum_{m\in\mathbb{N}^*} z^{-m} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$*, which is defined for* $|z| > 1$*. With the bilinear transform, we retrieve* $H_1(z) = H^a(p) = \frac{1}{p}$ *with* $p = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$*.*

2.3 Synthesis by the window method

We now wish to determine the integral of the signal $x^a(t)$ exactly. To do this, we assume that $x^a(t)$ satisfies the assumptions of the Shannon-Nyquist's theorem. It can then be reconstructed exactly from its samples:

$$x^a(t) = \sum_{m=-\infty}^{+\infty} x^a(mT)\, f(t-mT)$$

where $f(t) = \text{sinc}\left(\frac{t}{T}\right)$ avec $\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$. We define the discrete time integrated signal $y(n) = \int_{-\infty}^{nT} x^a(t)\,dt$.

Question 5 Show that $y(n)$ can be written as the convolution product between the signal $x(n)$ and the digital filter $h(n) = T\int_{-\infty}^{n} \text{sinc}(u)\,du$ (hint: we will assume that $x^a(t)$ satisfies strong enough assumptions to be able to switch $\int$ and $\sum$).
We have $y(n) = \sum\limits_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{nT} f(t-mT)\,dt = \sum\limits_{m=-\infty}^{+\infty} x(m) \int_{-\infty}^{(n-m)T} f(t)\,dt = \sum\limits_{m\in\mathbb{Z}} h(n-m)x(m)$ *with* $h(m) = \int_{-\infty}^{mT} f(t)\,dt = \int_{-\infty}^{mT} \text{sinc}\left(\frac{t}{T}\right) dt = T\int_{-\infty}^{m} \text{sinc}(u)\,du$ *with the change of variable* $t = uT$*.*

Question 6 The impulse response of filter h is represented in Figure 1-(c) (for $T = 1$). What phenomenon can be observed compared to the impulse responses calculated previously? Is this filter causal? Is it stable ? (hint: $h(n) \underset{n\to+\infty}{\longrightarrow} T$)
In Figure 1-(c), we observe a Gibbs phenomenon in the time domain. This filter is non longer causal, nor stable because $h \notin l^1(\mathbb{Z})$*.*
Since $h(n) \underset{n\to+\infty}{\longrightarrow} T$, it does not seem reasonable to synthesize filter h by directly applying the window method, which consists in truncating the impulse response. Instead, we define filter $G(z) = (1-z^{-1})\,H(z)$, whose impulse response decreases towards 0 at infinity. This filter $G(z)$ can be synthesized by the window method. We can then deduce an integrating filter $H(z) = \frac{G(z)}{1-z^{-1}}$.

Question 7 Show that the impulse response of the filter $g(n)$ is symmetrical with respect to $\frac{1}{2}$, and is upper bounded in absolute value by $O\left(\frac{1}{n}\right)$ (the proof is simple but it may be useful to make a drawing).
We have $G(z) = (1-z^{-1})H(z)$*, therefore* $g(n) = h(n) - h(n-1) = T\int_{n-1}^{n} \text{sinc}(u)\,du = O(\frac{1}{n})$*. Moreover,* $g(1-n) = T\int_{-n}^{1-n} \text{sinc}(u)\,du = g(n)$*, thus the impulse response* $g(n)$ *is symmetrical with respect to* $\frac{1}{2}$*.*