# Probabilistic Graphical Models and Hidden Markov Models

Math189R: Mathematics of Big Data

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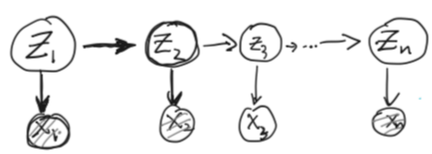
## 1 Hidden Markov Models (HMM)

### 1.1 Structure of Model

Figure 1:



**Attributes of Figure 1:**

$$
\begin{align}
z_i &= \text{“Hidden” variables}, \tag{1}\\
x_i &= \text{“Observed” data}, \tag{2}\\
z_1, z_2, ..., z_n &\in \{1, 2, ..., m\}, \tag{3}\\
x_1, x_2, ..., x_n &: X(\text{discrete}, \mathbb{R}, \mathbb{R}^n, ...). \tag{4}
\end{align}
$$

**Key:** The joint probability factorizes in the following way:

$$
\begin{align}
p(z_1, z_2, ..., z_n, x_1, x_2, ..., x_n) &= p(z_1)p(x_1 \mid z_1)\Pi_{k=2}^{n} p(z_k \mid z_{k-1})p(x_k \mid z_k), \tag{5}\\
p(z_1) &: \Pi(i), \tag{6}\\
p(x_1 \mid z_1) &: \mathbf{\Sigma}_{z_1}(x_1), \tag{7}\\
p(z_k \mid z_{k-1}) &: \Pi(z_{k-1}, z_k), \tag{8}\\
p(x_k \mid z_k) &: \mathbf{\Sigma}_{z_k}(x_k). \tag{9}
\end{align}
$$

**Model:**

$$p(z_1, z_2, ..., z_n) = \pi(i)\Sigma_{z_1}(x_1)\pi_{k=2}^{n} T(z_{k-1}, z_k)\boldsymbol{\Sigma}_{z_k}(x_k). \tag{10}$$

**Parameters of the Model:**

- Transition Probability: $T_{(i,j)} = p(z_{k+1} = j \mid z_k = i)$, where $i, j \in \{1, 2, ..., m\}$.
- Emission Probability:
  - Discrete Case (pmf): $\boldsymbol{\Sigma}_i(x) = p(x_k = x \mid z_k = i)$.
  - Continuous Case (pdf): $\boldsymbol{\Sigma}_i(x) = p(x \mid z_k = i)$.
- Initial Probability Distribution: $\Pi(i) = p(z_1 = i)$.

## 1.2 Forward and Backward Algorithm (F/B)

**Goal:** Compute $p(z_k \mid x)$, where $x = (x_1, ..., x_n)$. We assume the probabilities $p(x_k \mid z_k), p(z_k, z_{k-1}), p(z_1)$ are known.

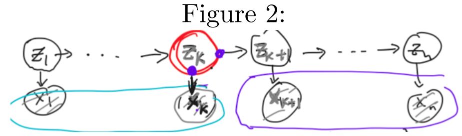
**Notation:**

$$x_{i:j} = (x_i, x_{i+1}, ..., x_j), \tag{11}$$
$$x_{k+1,n} = (x_{k+1}, ..., x_n). \tag{12}$$

**Method:**

- *Forward Algorithm:* Compute $p(z_k, x_{1:k})$ for all $k \in \{1, 2, ..., n\}$.
- *Backward Algorithm:* Compute $p(x_{k+1:n} \mid z_k)$ for all $k \in \{1, 2, ..., n-1\}$.

Figure 2:



We have the following probability relationships:

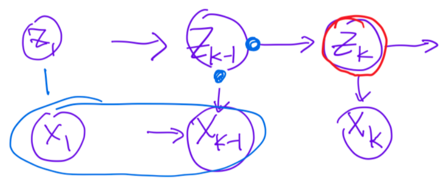
$$\begin{aligned} p(z_k \mid x) \propto p(z_K, x) &= p(z_k, x_{1:k}, x_{k+1:n}), & (13)\\ &= p(x_{k+1,n}, z_k, x_{1:k}), & (14)\\ &= p(x_{k+1,n} \mid z_k)p(z_k, x_{1:k}). & (15) \end{aligned}$$

**Equation:**

$$\begin{aligned} p(z_k \mid x) &\propto p(x_{k+1:n} \mid z_k)p(z_k, x_{1:k}), \text{ where} & (16)\\ p(x_{k+1:n} \mid z_k) &: \text{Backward Part} & (17)\\ p(z_k, x_{1:k}) &: \text{Forward Part}. & (18) \end{aligned}$$

**Forward Algorithm: Trick for Hidden Markov Model** We will employ a trick to yield the final equation for the forward algorithm.

Figure 3:



We use the following relationships:

$$\begin{aligned}
\alpha_k(z_k) &= p(z_k, x_{1:k}) && (19)\\
&= \sum_{z_{k-1}=1}^{m} p(z_k, z_{k-1}, x_{1:k}) && (20)\\
&= \sum_{z_{k-1}}^{m} p(x_k \mid z_k) p(z_k, z_{k-1}, x_{1:k-1}), \text{ where} && (21)\\
p(z_k, z_{k-1}) &= p(x_k, z_k, z_k - 1, x_{1,k-1}), && (22)\\
p(z_k, z_{k-1}, x_{1:k-1}) &= p(z_k \mid z_{k-1}) p(z_{k-1}, x_{1:k-1}), \text{ and} && (23)\\
p(z_{k-1}, x_{1:k-1}) &= \alpha_{k-1}(z_{k-1}). && (24)
\end{aligned}$$

Finally, we have:

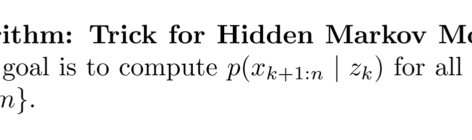
$$\alpha_k(z_k) = \sum_{z_{k-1}}^{m} p(x_k \mid z_k) p(z_k \mid z_{k-1}) \alpha_{k-1}(z_{k-1}) \quad (25)$$

so we can compute $\alpha_1, \alpha_2, ..., \alpha_n$ by:

$$\alpha_k(z_k) = p(z_k, x_{1:k}) \text{ for all } k. \quad (26)$$

**Backward Algorithm: Trick for Hidden Markov Model** We still have $x_1, x_2, ..., x_n$. Our goal is to compute $p(x_{k+1:n} \mid z_k)$ for all $k \in \{1, 2, ..., m-1\}$ and $z_k \in \{1, 2, ..., m\}$.

Figure 4:



We use the following relationships:

$$\beta_k(z_k) = p(x_{k+1,n} \mid z_k) \tag{27}$$

$$= \sum_{z_{k+1}=1}^{m} p(x_{k+1,n}, z_{k+1} \mid z_k) \tag{28}$$

$$= \sum_{z_{k+1}=1}^{m} p(x_{k+1:n}, z_{k+1}) p(x_{k+1} \mid z_{k+1}) p(z_{k+1} \mid z_k), \text{ where} \tag{29}$$

$$p(x_{k+1:n}, z_{k+1}) = \beta_{k+1}(z_{k+1}), \tag{30}$$

$$p(x_{k+1} \mid z_{k+1}) : \text{ Emission Probability (known)}, \tag{31}$$

$$p(z_{k+1} \mid z_k) : \text{ Transition Probability (known)}. \tag{32}$$

We arrive at the recursive formula:

$$\beta_n(z_n) = 1 \tag{33}$$

$$\beta_k(z_k) = \sum_{z_{k+1}=1}^{m} \beta_{k+1}(z_{k+1}) p(x_{k+1} \mid z_{k+1}) p(z_{k+1} \mid z_k) \tag{34}$$

$$\text{for all } k \in \{1, 2, ..., n-1\}. \tag{35}$$