1 Inverted Pendulum on a Cart: Equation Derivation

1.1 Initial Formulation

The typical equations for an inverted pendulum on a cart are derived using Lagrangian mechanics. The system has two degrees of freedom: the horizontal position of the cart (x) and the angle of the pendulum (θ).

1. Kinetic Energy (T):

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + L\dot{\theta}\cos\theta)^2 + (L\dot{\theta}\sin\theta)^2] \tag{1}$$

2. Potential Energy (V):

$$V = mgL(1 - \cos\theta) \tag{2}$$

3. Lagrangian ($\mathcal{L}$):

$$\mathcal{L} = T - V \tag{3}$$

4. Equations of Motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = F \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \tag{5}$$

1.2 Derivation to Implemented System

After applying the Lagrangian and simplifying, we get two coupled second-order differential equations:

1. For the cart:

$$(M + m)\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = F \tag{6}$$

2. For the pendulum:

$$L\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0 \tag{7}$$

To match the implemented system, we need to:

1. Add a damping term ($-d \cdot \dot{x}$) to the cart equation
2. Rearrange the equations to solve for $\ddot{x}$ and $\ddot{\theta}$

After rearrangement:

$$\ddot{x} = \frac{F - mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta - d\cdot\dot{x}}{M+m} \tag{8}$$

$$\ddot{\theta} = \frac{-F\cos\theta + (M+m)g\sin\theta + mL\dot{\theta}^2\sin\theta\cos\theta - d\cdot\dot{x}\cos\theta}{L(M+m\sin^2\theta)} \tag{9}$$

Now, let's define our state vector $\mathbf{x} = [x, \dot{x}, \theta, \dot{\theta}]$. The derivatives of this state vector will give us the system in the implemented 'pendulum_cart_ode' function:

$$\dot{x}_1 = x_2 \tag{10}$$

$$\dot{x}_2 = \frac{u - mL{x_4}^2\sin(x_3) + mLx_4\cos(x_3) - d\cdot x_2}{M+m} \tag{11}$$

$$\dot{x}_3 = x_4 \tag{12}$$

$$\dot{x}_4 = \frac{(M+m)g\sin(x_3) - \cos(x_3)(u - mL{x_4}^2\sin(x_3) + d\cdot x_2)}{L(M+m\sin^2(x_3))} \tag{13}$$

These equations are equivalent to those in the implemented 'pendulum_cart_ode' function, just written in a slightly different form. The main difference is that the function uses a common denominator $D = L(M + m\sin^2\theta)$ to simplify the expressions.