AE 483 Homework 2 Solution

Question 1

Part a

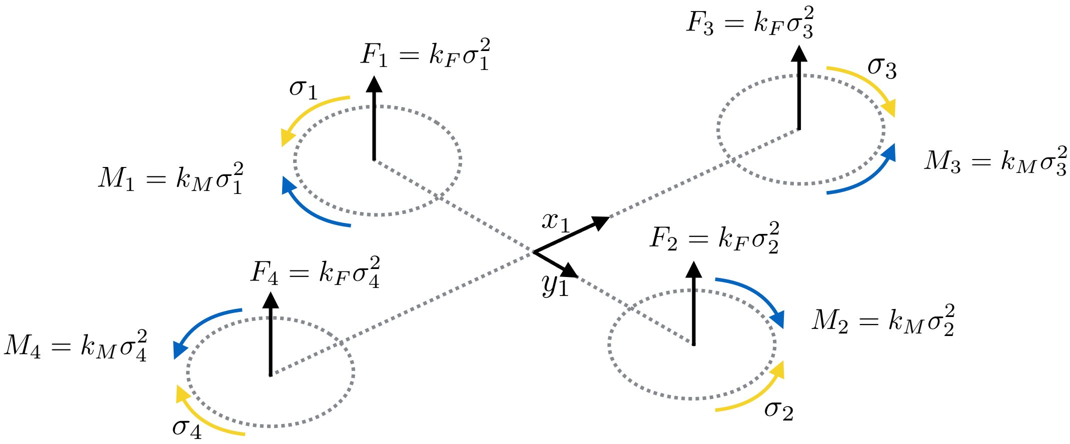

Figure 1: The free-body diagram that shows applied forces and torques

The total force and torque (due *only* to the rotors) are given below:

$$f^1_{\text{rotors}} = \begin{bmatrix} 0 \\ 0 \\ -k_F(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2) \end{bmatrix}, \quad \tau^1_{\text{rotors}} = \begin{bmatrix} k_F(\sigma_1^2-\sigma_2^2)l \\ k_F(\sigma_3^2-\sigma_4^2)l \\ k_M(\sigma_1^2+\sigma_2^2-\sigma_3^2-\sigma_4^2) \end{bmatrix}$$

Part b

The matrix $W \in \mathbb{R}^{4\times 4}$ should be clear from the following derivations:

$$u = \begin{bmatrix} k_F(\sigma_1^2-\sigma_2^2)l \\ k_F(\sigma_3^2-\sigma_4^2)l \\ k_M(\sigma_1^2+\sigma_2^2-\sigma_3^2-\sigma_4^2) \\ k_F(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2) \end{bmatrix} = \begin{bmatrix} k_Fl & -k_Fl & 0 & 0 \\ 0 & 0 & k_Fl & -k_Fl \\ k_M & k_M & -k_M & -k_M \\ k_F & k_F & k_F & k_F \end{bmatrix} \begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \\ \sigma_4^2 \end{bmatrix} = Ws$$

Part c

The determinant of the matrix W is

$$\det W = -8k_F^3k_Ml^2$$

It indicates that the matrix W is invertible if $k_F, k_M, l \neq 0$.

Part d

For each rotor, the maximum value $s_{\max} = 10^6$. Consider the following:

1. For the desired input $\bar{u} = \begin{bmatrix} 0 & 0 & 0 & 5 \end{bmatrix}^\top$, one may compute that

$$\bar{s} = \begin{bmatrix} 1.25 \times 10^5 \\ 1.25 \times 10^5 \\ 1.25 \times 10^5 \\ 1.25 \times 10^5 \end{bmatrix}, \ s = \begin{bmatrix} 1.25 \times 10^5 \\ 1.25 \times 10^5 \\ 1.25 \times 10^5 \\ 1.25 \times 10^5 \end{bmatrix}, \ u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

2. For the desired input $\bar{u} = \begin{bmatrix} 0.5 & -0.5 & 0.05 & 5 \end{bmatrix}^\top$, one may compute that

$$\bar{s} = \begin{bmatrix} 3 \times 10^5 \\ 2 \times 10^5 \\ -5 \times 10^4 \\ 5 \times 10^4 \end{bmatrix}, \ s = \begin{bmatrix} 3 \times 10^5 \\ 2 \times 10^5 \\ 0 \\ 5 \times 10^4 \end{bmatrix}, \ u = \begin{bmatrix} 0.5 \\ -0.25 \\ 0.045 \\ 5.5 \end{bmatrix}$$

Part e

The total force and torque (both due to the rotors and due to gravity) are given below:

$$f^0 = R_1^0 f^1_{\text{rotors}} + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = R_1^0 \begin{bmatrix} 0 \\ 0 \\ -u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$$

$$\tau^1 = \tau^1_{\text{rotors}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Question 2

The equations of motion are:

$$
\begin{aligned}
\begin{bmatrix}\dot{o}_1\\ \dot{o}_2\\ \dot{o}_3\end{bmatrix} &= \begin{bmatrix}h_1\\ h_2\\ h_3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}\\
\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix} &= \begin{bmatrix}h_4\\ h_5\\ h_6\end{bmatrix} = \frac{1}{c_2}\begin{bmatrix}0&s_3&c_3\\0&c_2c_3&-c_2s_3\\c_2&s_2s_3&s_2c_3\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}\\
\begin{bmatrix}\dot{v}_1\\ \dot{v}_2\\ \dot{v}_3\end{bmatrix} &= \begin{bmatrix}h_7\\ h_8\\ h_9\end{bmatrix} = \frac{f^0}{m} = \frac{1}{m}R_1^0\begin{bmatrix}0\\0\\-u_4\end{bmatrix} + \begin{bmatrix}0\\0\\g\end{bmatrix}\\
&= -\frac{1}{m}\begin{bmatrix}c_1s_2c_3+s_1s_3\\ s_1s_2c_3-c_1s_3\\ c_2c_3\end{bmatrix}u_4 + \begin{bmatrix}0\\0\\g\end{bmatrix}\\
\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\\ \dot{\omega}_3\end{bmatrix} &= \begin{bmatrix}h_{10}\\ h_{11}\\ h_{12}\end{bmatrix} = J^{-1}(\tau - \widehat{\omega}J\omega)\\
&= J^{-1}\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} - \begin{bmatrix}0&-\omega_3&\omega_2\\ \omega_3&0&-\omega_1\\ -\omega_2&\omega_1&0\end{bmatrix}\begin{bmatrix}J_1&0&0\\0&J_2&0\\0&0&J_3\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}\right)\\
&= \begin{bmatrix}1/J_1&0&0\\0&1/J_2&0\\0&0&1/J_3\end{bmatrix}\begin{bmatrix}u_1+(J_2-J_3)\omega_2\omega_3\\ u_2+(J_3-J_1)\omega_1\omega_3\\ u_3+(J_1-J_2)\omega_1\omega_2\end{bmatrix}\\
&= \begin{bmatrix}[u_1+(J_2-J_3)\omega_2\omega_3]/J_1\\ [u_2+(J_3-J_1)\omega_1\omega_3]/J_2\\ [u_3+(J_1-J_2)\omega_1\omega_2]/J_3\end{bmatrix}
\end{aligned}
$$

Question 3

Part a

Given that $\theta = 0$, we should know $s_1 = s_2 = s_3 = 0, c_1 = c_2 = c_3 = 1$ for obvious reasons. From $\dot{v} = 0$, we could obtain:

$$
\dot{v} = \begin{bmatrix}0\\0\\-u_4/m\end{bmatrix} + \begin{bmatrix}0\\0\\g\end{bmatrix} = 0 \implies u_4 = mg
$$

From $w = \dot{w} = 0$, we could obtain

$$
\dot{w} = \begin{bmatrix}u_1/J_1\\u_2/J_2\\u_3/J_3\end{bmatrix} = 0 \implies u_1 = u_2 = u_3 = 0
$$

To summarize, the required conditions on inputs are:

$$u_1 = u_2 = u_3 = 0, u_4 = mg$$

Part b

Our goal is to linearize the differential equations derived in Question 2 around its hover conditions, and put them into the form of

$$\dot{x}_c = A_c x_c + B_c u_c$$

A superscript h is used to represent the value of each variable in hover conditions.

Consider the following:

$$\delta \dot{o}_1 = \frac{\partial h_1}{\partial v_1}\bigg|_{v^h} \delta v_1 = \delta v_1$$

$$\delta \dot{o}_2 = \frac{\partial h_2}{\partial v_2}\bigg|_{v^h} \delta v_2 = \delta v_2$$

$$\delta \dot{o}_3 = \frac{\partial h_3}{\partial v_3}\bigg|_{v^h} \delta v_3 = \delta v_3$$

$$\delta \dot{\theta}_1 = \sum_{z=\{\theta_2,\theta_3,\omega_2,\omega_3\}} \frac{\partial h_4}{\partial z}\bigg|_{\theta^h,\omega^h} \delta z = \delta \omega_3$$

$$\delta \dot{\theta}_2 = \sum_{z=\{\theta_3,\omega_2,\omega_3\}} \frac{\partial h_5}{\partial z}\bigg|_{\theta^h,\omega^h} \delta z = \delta \omega_2$$

$$\delta \dot{\theta}_3 = \sum_{z=\{\theta_1,\theta_2,\theta_3,\omega_2,\omega_3\}} \frac{\partial h_6}{\partial z}\bigg|_{\theta^h,\omega^h} \delta z = \delta \omega_1$$

$$\delta \dot{v}_1 = \sum_{z=\{\theta_1,\theta_2,\theta_3,u_4\}} \frac{\partial h_7}{\partial z}\bigg|_{\theta^h,u^h} \delta z = -g\delta \theta_2$$

$$\delta \dot{v}_2 = \sum_{z=\{\theta_1,\theta_2,\theta_3,u_4\}} \frac{\partial h_8}{\partial z}\bigg|_{\theta^h,u^h} \delta z = g\delta \theta_3$$

$$\delta \dot{v}_3 = \sum_{z=\{\theta_2,\theta_3,u_4\}} \frac{\partial h_9}{\partial z}\bigg|_{\theta^h,u^h} \delta z = -\frac{1}{m}\delta u_4$$

$$\delta \dot{\omega}_1 = \sum_{z=\{\omega_2,\omega_3,u_1\}} \frac{\partial h_{10}}{\partial z}\bigg|_{\omega^h,u^h} \delta z = \frac{1}{J_1}\delta u_1$$

$$\delta \dot{\omega}_2 = \sum_{z=\{\omega_1,\omega_3,u_2\}} \frac{\partial h_{11}}{\partial z}\bigg|_{\omega^h,u^h} \delta z = \frac{1}{J_2}\delta u_2$$

$$\delta \dot{\omega}_3 = \sum_{z=\{\omega_1,\omega_2,u_3\}} \frac{\partial h_{12}}{\partial z}\bigg|_{\omega^h,u^h} \delta z = \frac{1}{J_3}\delta u_3$$

Then the matrices A_c, B_c are given by:

$$A_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{m} \\ \frac{1}{J_1} & 0 & 0 & 0 \\ 0 & \frac{1}{J_2} & 0 & 0 \\ 0 & 0 & \frac{1}{J_3} & 0 \end{bmatrix}$$

Part c

By the continuous time state space equations, we could write:

$$\dot{x}_c(i\Delta t) = A_c x_c(i\Delta t) + B_c u_c(i\Delta t)$$

To substitute the above equation into the linear approximation form, we have:

$$\begin{aligned} x_c((i+1)\Delta t) &\approx x_c(i\Delta t) + \dot{x}_c(i\Delta t)\Delta t \\ &= x_c(i\Delta t) + [A_c x_c(i\Delta t) + B_c u_c(i\Delta t)]\Delta t \\ &= (I + A_c\Delta t)x_c(i\Delta t) + B_c\Delta t u_c(i\Delta t) \end{aligned}$$

Since $x_d(i) = x_c(i\Delta t), u_d(i) = u_c(i\Delta t)$, the above equation can be translated into:

$$x_d(i+1) = (I + A_c\Delta t)x_d(i) + B_c\Delta t u_d(i)$$

which is in discrete time state space form. And matrices A_d, B_d are given by

$$A_d = I + A_c\Delta t,\ B_d = B_c\Delta t$$