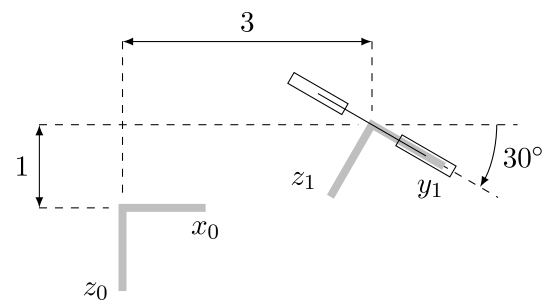
AE483 Homework #1: Quad-Rotor Kinematics

(due at the beginning of class on Monday, September 12)



1. Consider the quad-rotor shown above (side view). Assume the origin of frame 0 and the origin of frame 1 are both in the plane of the paper.

   (a) Express the position and orientation of frame 1 in the coordinates of frame 0.

   (b) A vector has coordinates

   $$v^1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

   in frame 1. Find the coordinates of this same vector in frame 0.

   (c) A point has coordinates

   $$p^1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

   in frame 1. Find the coordinates of this same point in frame 0.

   To solve this problem, you need to remember that

   - the position o_k^j of frame k describes the point o_k at the origin of frame k in the coordinates of frame j (it is a 3×1 matrix);
   - the orientation $R_k^j = \begin{bmatrix} x_k^j & y_k^j & z_k^j \end{bmatrix}$ of frame k describes the axes x_k, y_k, and z_k of frame k—three unit vectors—in the coordinates of frame j (it is a 3×3 matrix).

   You also need to remember the formulas for coordinate transformation:

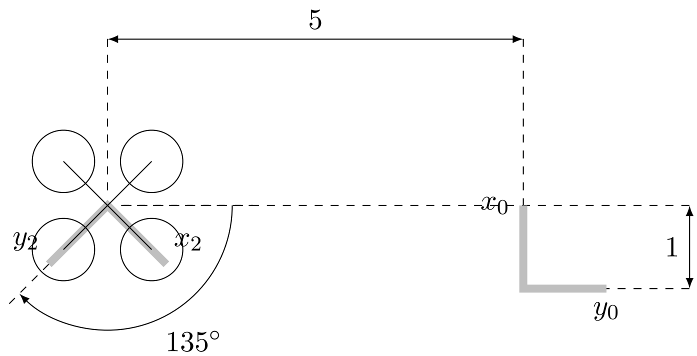
   - if a vector v has coordinates v^k in frame k, then it has coordinates

     $$v^j = R_k^j v^k$$

     in frame j;
   - if a point p has coordinates p^k in frame k, then it has coordinates

     $$p^j = o_k^j + R_k^j p^k$$

     in frame j.



2. Consider the quad-rotor shown above (top view). Assume the origin of frame 0 and the origin of frame 2 are both in the plane of the paper.

   (a) Express the position and orientation of frame 2 in the coordinates of frame 0.

   (b) A vector has coordinates
   $$w^0 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$
   in frame 0. Find the coordinates of this same vector in frame 2.

   (c) A point has coordinates
   $$q^0 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$
   in frame 0. Find the coordinates of this same point in frame 2.

   To solve this problem quickly, you need to remember the formula for inverse transformation:
   $$\left(o^k_j, R^k_j\right) = \left(-(R^j_k)^T o^j_k, (R^j_k)^T\right)$$
   Note in particular, as we saw in class, that
   $$R^k_j = (R^j_k)^{-1} = (R^j_k)^T,$$
   i.e., that the inverse of a rotation matrix is its transpose.

3. Consider again the quad-rotors from Problems 1 and 2. Assume that frame 0 was the same in both problems.

   (a) Express the position and orientation of frame 2 in the coordinates of frame 1.

   (b) Find the dot product $v \cdot w$.

   (c) Find the cross product $v \times w$.

   To solve this problem, you need to remember the formula for sequential transformation:
   $$(o^i_k, R^i_k) = \left(o^i_j + R^i_j o^j_k, R^i_j R^j_k\right).$$
   Note in particular, as we saw in class, that
   $$R^i_k = R^i_j R^j_k.$$

You also need to remember that the dot product $v \cdot w$ between two vectors v and w can be found by writing the coordinates of both v and w with respect to the same frame k, and then taking the matrix multiplication

$$v \cdot w = (v^k)^T w^k.$$

Similarly, the cross product $v \times w$ can be found by taking the matrix multiplication

$$v \times w = \widehat{(v^k)} w^k,$$

where $\widehat{a}$ is the "wedge" operation on a 3×1 matrix $a \in \mathbb{R}^3$ defined by

$$\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

In this expression, a_1, a_2, a_3 are the elements of a. It is easy to verify that these formulas give the same result as however you normally compute dot products and cross products.

4. Consider the ZYX body-axis Euler Angle sequence:

   - frame 1 is generated by a rotation θ_1 about z_0;
   - frame 2 is generated by a rotation θ_2 about y_1;
   - frame 3 is generated by a rotation θ_3 about x_2.

   Please do the following:

   (a) Carefully draw a picture of the relationship between frames 0 and 1, and find R_1^0.

   (b) Carefully draw a picture of the relationship between frames 1 and 2, and find R_2^1.

   (c) Carefully draw a picture of the relationship between frames 2 and 3, and find R_3^2.

   (d) Write the matrix multiplication that could be done to compute R_3^0.

   (e) Suppose $\theta_2 \neq \pm\pi/2$. Give an example of two different choices of $\theta_1, \theta_2, \theta_3 \in (-\pi, \pi]$ that result in the same orientation (any orientation you want), and describe this orientation by computing R_3^0. Are there other choices of θ_1, θ_2, and θ_3, subject to the same constraints, that also result in the same orientation? If so, how many?

   (f) Suppose $\theta_2 = \pm\pi/2$ (you choose the sign). Give an example of two different choices of $\theta_1, \theta_3 \in (-\pi, \pi]$ that result in the same orientation (any orientation you want), and describe this orientation by computing R_3^0. Are there other choices of θ_1 and θ_3, subject to the same constraints, that also result in the same orientation? If so, how many?

   Please use the notation $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. (Note that the frames in this and subsequent problems are different from the frames in the first three problems.)

5. Consider again the ZYX body-axis Euler Angle sequence. Suppose that θ_1, θ_2, and θ_3 are functions of time.

   (a) Write $w_{0,1}^1$ in terms of $\dot{\theta}_1$.

   (b) Write $w_{1,2}^2$ in terms of $\dot{\theta}_2$.

   (c) Write $w_{2,3}^3$ in terms of $\dot{\theta}_3$.

   (d) Express $w_{0,1}$ and $w_{1,2}$ in the coordinates of frame 3.

(e) Find the angular velocity $w_{0,3}^3$ in terms of the angular rates $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$. Write your answer as

$$w_{0,3}^3 = A \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

for some 3×3 matrix $A \in \mathbb{R}^{3\times 3}$ that depends only on the Euler Angles, not on their angular rates or on components of the angular velocity.

(f) Find the angular rates $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in terms of the angular velocity $w_{0,3}^3$. Write your answer as

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = B w_{0,3}^3$$

for some 3×3 matrix $B \in \mathbb{R}^{3\times 3}$ that depends only on the Euler Angles, not on their angular rates or on components of the angular velocity.

(g) Is it possible for the angular velocity to be very small while the angular rates are very large, perhaps even undefined? Explain, in light of your answer to Problems 4(e)-4(f).

To solve this problem, you need to remember that $w_{j,k}^k$ is the angular velocity of frame k with respect to frame j, written in the coordinates of frame k. By "angular velocity," we mean the unique vector $w_{j,k}$ for which

$$\dot{R}_k^j = R_k^j \widehat{\left(w_{j,k}^k\right)},$$

where $\dot{R}_k^j$ is obtained by taking the time derivative of each element of R_k^j. You also need to remember that angular velocities add. Finally, you need to remember how to invert a 3×3 matrix (look it up or use software that does symbolic math).

6. This semester, you will build a quad-rotor simulator in MATLAB. As you know, simulation is an important part of the design cycle. Indeed, you will be using a simulator in lab. If you can build your own version of this simulator, then you can easily change it to model other vehicles as well (a hex-rotor, for example), a useful thing both for doing UAV research in graduate school and for building your own business around UAVs. Here, you will implement the kinematics part of this simulator. Your code will:

   - Create a movie of the quadrotor moving with linear and angular velocity

     $$v_{0,1}^0 = e^{-t/4} \begin{bmatrix} \sin(t) \\ \sin(2t) \\ \sin(3t) \end{bmatrix} \qquad w_{0,1}^1 = 10e^{-t} \begin{bmatrix} \sin(t) \\ \sin(2t) \\ \sin(3t) \end{bmatrix}$$

     having started from a position and orientation in which frames 0 and 1 are aligned. Your movie should show both the quadrotor and axes representing frame 1.
   - Plot the position of the quadrotor as a function of time (i.e., the components of o_1^0).
   - Plot the orientation of the quadrotor as a function of time (i.e., the ZYX Euler Angles).

You will do so by modifying `hw1.m`, available on the course website. You will also need to download `quadmodel.mat`, which contains a geometric model of the quadrotor that you will draw. The parts of `hw1.m` that need to be changed are clearly marked (search for "`MODIFY`"). Submission details will be forthcoming. You may work either alone or in pairs on this problem (the rest of the assignment should be submitted individually). Please start early.