AE 483 Homework 3 Solution

Question 1

Part a

Given that $A > 0$, we should know by definition that

$$\begin{cases} \psi(x) > 0, & \text{if } x \neq 0 \\ \psi(x) = 0, & \text{if } x = 0 \end{cases}$$

Hence, there exists a unique minimizer $x^* = 0$, and the minimum value is $\psi(x^*) = 0$.

Part b

Similar to Part a, since $L > 0$, we should know by definition that

$$\begin{cases} \phi(x) > s, & \text{if } x \neq r \\ \phi(x) = s, & \text{if } x = r \end{cases}$$

Hence, there exists a unique minimizer $x^* = r$, and the minimum value is $\phi(x^*) = s$.

Part c

We shall start with the form as shown in Part b,

$$\begin{aligned} \gamma(x) &= \frac{1}{2}(x-r)^\top L(x-r) + s \\ &= \frac{1}{2}x^\top Lx - r^\top Lx + \frac{1}{2}r^\top Lr + s \\ &= \frac{1}{2}x^\top Fx + g^\top x + h \end{aligned}$$

Now, we could easily find the following:

$$L = F,\ r = -F^{-1}g,\ s = h - \frac{1}{2}g^\top F^{-1}g$$

According to Part b, we know that there exists a unique minimizer $x^* = -F^{-1}g$, and the minimum value is $\gamma(x^*) = h - \frac{1}{2}g^\top F^{-1}g$.

Part d

If h is changed, x^* will not be affected, but $\gamma(x^*)$ will be changed accordingly.

Part e

1. If $F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then F is positive definite. There exists a unique minimizer

$$x^* = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

and the minimum value is $\gamma(x^*) = 2.5$.

2. If $F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then F is only positive semidefinite. Consider the following:

$$\begin{aligned} \gamma(x) &= \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 5 \\ &= \frac{1}{2} x_1^2 + x_1 - 2x_2 + 5 \end{aligned}$$

and clearly $\gamma(x)$ can become arbitrarily small along x_2 direction. Hence, there does not exist a minimum value.

3. If $F = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$, then F is not even definite, and there does not exist a minimum value.

4. If $F = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, then F is positive definite. There exists a unique minimizer

$$x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the minimum value is $\gamma(x^*) = 4$.

Question 2

Some notations are changed here just for convenience, and they should be clear from the statement. We want to solve the following problem:

$$\begin{aligned}
\min_{u(0),\cdots,u(N-1)} \quad & \frac{1}{2}x(N)^\top Q_f x(N) + \frac{1}{2}\sum_{k=0}^{N-1} x(k)^\top Q x(k) + u(k)^\top R u(k) \\
\text{s.t.} \quad & x(k+1) = Ax(k) + Bu(k),\ \forall k = 0,\cdots,N-1 \\
& x(0) = x_0
\end{aligned}$$

where $Q_f > 0, Q > 0, R > 0$.

Part a

Define the value function $v : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned}
v(i,z) = \min_{u(i),\cdots,u(N-1)} \quad & \frac{1}{2}x(N)^\top Q_f x(N) + \frac{1}{2}\sum_{k=i}^{N-1} x(k)^\top Q x(k) + u(k)^\top R u(k) \\
\text{s.t.} \quad & x(k+1) = Ax(k) + Bu(k),\ \forall k = i,\cdots,N-1 \\
& x(i) = z
\end{aligned}$$

It gives the minimum cost-to-go value, starting from a state value z at the time step i.

Part b

Consider the following equation:

$$v(i,z) = \min_w \{\frac{1}{2}z^\top Qz + \frac{1}{2}w^\top Rw + v(i+1, Az+Bw)\}$$

First note that $\frac{1}{2}z^\top Qz + \frac{1}{2}w^\top Rw$ is the cost incurred at the time step i if $u(i) = w$. Then from the dynamics, $v(i+1, Az+Bw)$ is the minimum cost-to-go value, starting from where it lands at the time step $i+1$.

Part c

By definition of the value function in Part a, we have $v(N, x(N)) = \frac{1}{2}x(N)^\top Q_f x(N)$, and it implies that $P(N) = Q_f > 0$.

Part d

First note that the cost $v(N, x(N))$ has the desired quadratic form. To evaluate at $N-1$, consider the following:

$$\begin{aligned}
v(N-1, z) &= \min_w \{\frac{1}{2} z^\top Q z + \frac{1}{2} w^\top R w + v(N, Az + Bw)\} \\
&= \min_w \{\frac{1}{2} z^\top Q z + \frac{1}{2} w^\top R w + \frac{1}{2}(Az + Bw)^\top P(N)(Az + Bw)\} \\
&= \min_w \{\frac{1}{2} w^\top \underbrace{(R + B^\top P(N) B)}_{F} w + \underbrace{z^\top A^\top P(N) B}_{g^\top} w + \underbrace{\frac{1}{2} z^\top (Q + A^\top P(N) A) z\}}_{h}
\end{aligned}$$

Given $R > 0, P(N) > 0$, it is easy to verify that $F > 0$. If we slightly change the notation here, according to Question 1(c), the unique minimizer and the minimum value are:

$$\begin{aligned}
u(N-1) &= w^* = -F^{-1} g \\
&= -\underbrace{(R + B^\top P(N) B)^{-1} B^\top P(N) A}_{K(N-1)} x(N-1)
\end{aligned}$$

$$\begin{aligned}
v(N-1, x(N-1)) &= h - \frac{1}{2} g^\top F^{-1} g \\
&= \frac{1}{2} x(N-1)^\top \underbrace{[Q + A^\top P(N) A - A^\top P(N) B (R + B^\top P(N) B)^{-1} B^\top P(N) A]}_{P(N-1)} x(N-1)
\end{aligned}$$

Analogously, we could express $P(i)$ and $K(i)$ by:

$$\begin{aligned}
P(i) &= Q + A^\top P(i+1) A - A^\top P(i+1) B (R + B^\top P(i+1) B)^{-1} B^\top P(i+1) A \\
K(i) &= (R + B^\top P(i+1) B)^{-1} B^\top P(i+1) A
\end{aligned}$$

In addition, if $Q, R, P(i+1) > 0$, we could verify that $P(i) > 0$ as well.

Question 3

Part a-b

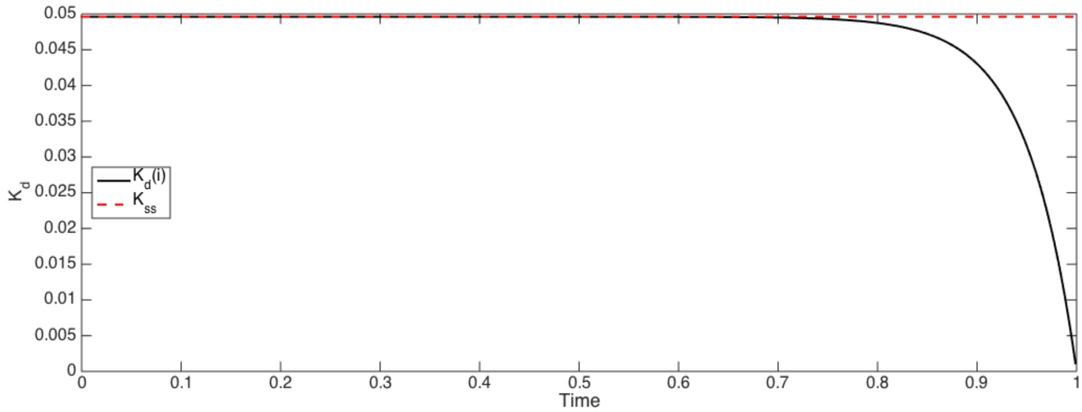

Figure 1: Comparison of $K_d(i)$ and K_{ss}

As i is getting small, $K_d(i)$ converges to 4.962×10^{-2}.

Part c

If we use the function dlqr in MATLAB, the control gain is 4.962×10^{-2} as well.

Part d

Use the expression derived in Question 2(d), and let $P(i+1) = P(i) = P_{ss}$, we can write

$$P_{ss} = Q + A^\top P_{ss} A - A^\top P_{ss} B (R + B^\top P_{ss} B)^{-1} B^\top P_{ss} A$$

To solve the above equation for P_{ss}, we could find that

$$P_{ss} = -19754.1 \text{ or } 50.1263$$

Using the positive value of P_{ss}, we could find the corresponding value of K_d, and it is

$$K_{ss} = (R + B^\top P_{ss} B)^{-1} B^\top P_{ss} A = 4.962 \times 10^{-2}$$

which is the same as what we found in Part b and c.

Part e

If we used the control law $u_d(i) = -K_{ss}x_d(i)$ at each time step $i = 0, \cdots, n-1$, then trajectories of x_d and u_d are plotted below, and the state x_d converges to the origin.

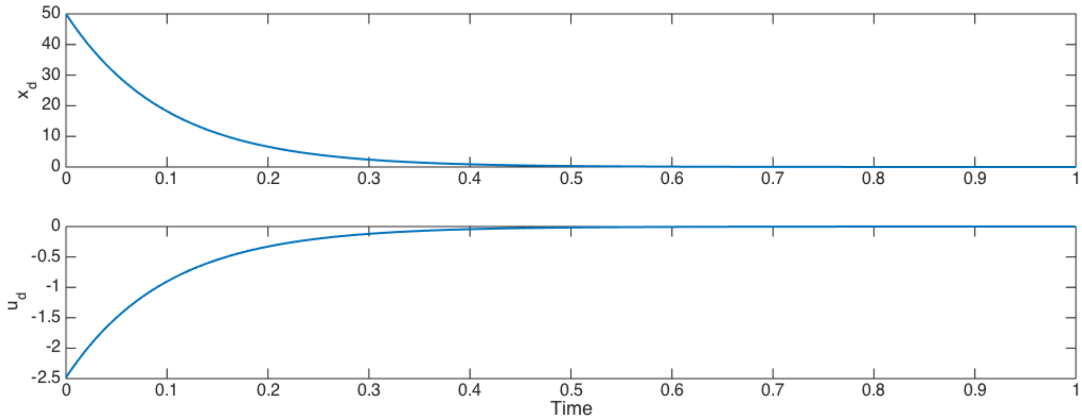

Figure 2: Trajectories of x_d and u_d, with $Q_d = 1, R_d = 1$

Part f

In this part, we will change the values of Q_d and R_d, compute K_d using the function dlqr in MATLAB, and compare trajectories of x_d and u_d to those shown in Figure 2 (corresponding to $Q_d = R_d = 1$).

1. If we increase (or decrease) Q_d and R_d, but keeping their ration the same, then trajectories of x_d and u_d would be the same as well.

2. If we use $Q_d = 100, R_d = 1$, then trajectories are shown in Figure 3 below:

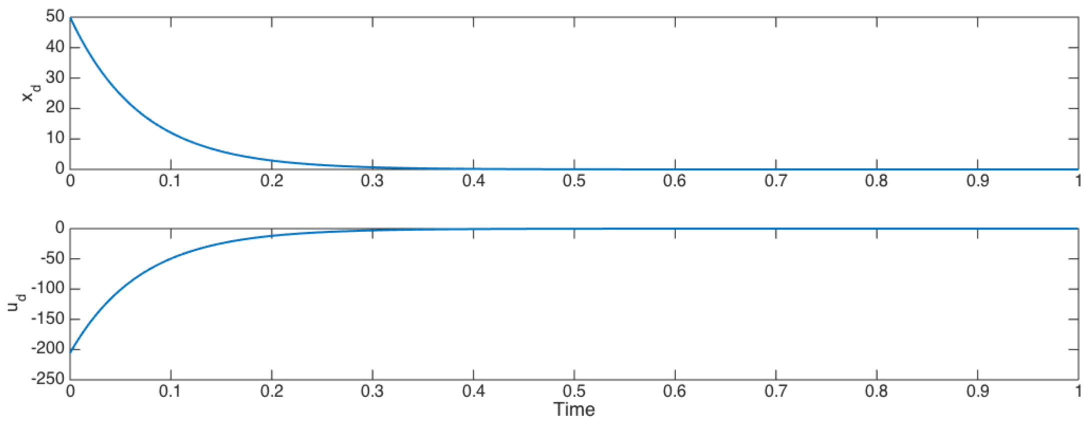

Figure 3: Trajectories of x_d and u_d, with $Q_d = 100, R_d = 1$

We could find that x_d converges faster, at the cost of using a much larger u_d, especially at the beginning of the simulation.

3. If we use $Q_d = 1, R_d = 100$, then trajectories are shown in Figure 4 below:

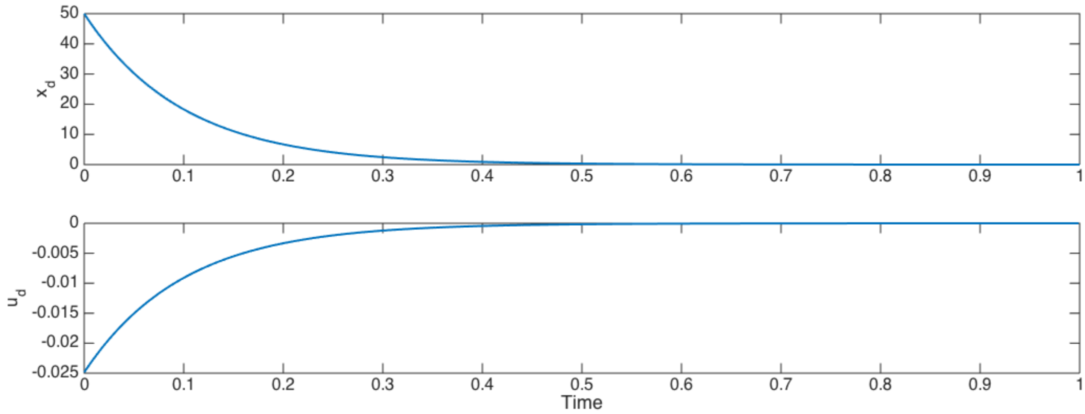

Figure 4: Trajectories of x_d and u_d, with $Q_d = 1, R_d = 100$

It describes that we could use a very small u_d to stabilize the system state x_d, with almost the same convergence rate (only slightly slower).