AE 483 Homework 1 Solution

Question 1

Part a

Let $\tilde{x}_1, \tilde{y}_1, \tilde{z}_1$ be three unit vectors on x_1, y_1, z_1 directions, respectively. In the coordinates of frame 0, they are

$$\tilde{x}_1^0 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \tilde{y}_1^0 = \begin{bmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \end{bmatrix}, \quad \tilde{z}_1^0 = \begin{bmatrix} -1/2 \\ 0 \\ \sqrt{3}/2 \end{bmatrix}$$

Then, the orientation of frame 1 in the coordinates of frame 0 is

$$R_1^0 = \begin{bmatrix} \tilde{x}_1^0 & \tilde{y}_1^0 & \tilde{z}_1^0 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3}/2 & -1/2 \\ -1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix}$$

The position of frame 1 in the coordinates of frame 0 is

$$o_1^0 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

Part b

$$v^0 = R_1^0 v^1 = \begin{bmatrix} 0 & \sqrt{3}/2 & -1/2 \\ -1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{3}/2 \\ -1 \\ 3/2 \end{bmatrix}$$

Part c

$$p^0 = o_1^0 + R_1^0 p^1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{3}/2 & -1/2 \\ -1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} + 7/2 \\ 0 \\ -\sqrt{3}/2 \end{bmatrix}$$

Question 2

Part a

The orientation of frame 2 in the coordinates of frame 0 is

$$R_2^0 = \begin{bmatrix} \tilde{x}_2^0 & \tilde{y}_2^0 & \tilde{z}_2^0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The position of frame 2 in the coordinates of frame 0 is

$$o_2^0 = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$$

Part b

$$w^2 = R_0^2 w^0 = (R_2^0)^\top w^0 = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 2 \end{bmatrix}$$

Part c

$$\begin{aligned} q^2 &= o_0^2 + R_0^2 q^0 = -(R_2^0)^\top o_2^0 + (R_2^0)^\top q^0 \\ &= \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2\sqrt{2} \\ -3\sqrt{2} \\ 1 \end{bmatrix} \end{aligned}$$

Question 3

Part a

The orientation of frame 2 in the coordinates of frame 1 is

$$\begin{aligned}
R_2^1 &= R_0^1 R_2^0 = (R_1^0)^\top R_2^0 \\
&= \begin{bmatrix} 0 & -1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{6}/4 & -\sqrt{6}/4 & 1/2 \\ \sqrt{2}/4 & \sqrt{2}/4 & \sqrt{3}/2 \end{bmatrix}
\end{aligned}$$

The position of frame 2 in the coordinates of frame 1 is

$$\begin{aligned}
o_2^1 &= o_0^1 + R_0^1 o_2^0 = -(R_1^0)^\top o_1^0 + (R_1^0)^\top o_2^0 \\
&= \begin{bmatrix} 0 & -1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix} \left(\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0 & -1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ -\sqrt{3} + 1/2 \\ \sqrt{3}/2 + 1 \end{bmatrix}
\end{aligned}$$

Part b

$$v \cdot w = (v^0)^\top w^0 = \begin{bmatrix} 3\sqrt{3}/2 & -1 & 3/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 2$$

Part c

$$v \times w = \widehat{(v^0)} w^0 = \begin{bmatrix} 0 & -3/2 & -1 \\ 3/2 & 0 & -3\sqrt{3}/2 \\ 1 & 3\sqrt{3}/2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7/2 \\ -3\sqrt{3} \\ 3\sqrt{3}/2 \end{bmatrix}$$

Question 4

Part a-c

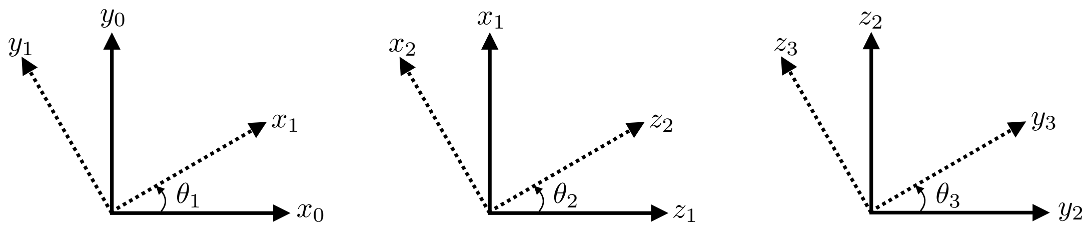

Figure 1: Sketch of the relationship between frames

Rotation matrices are listed below:

$$R_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ R_2^1 = \begin{bmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{bmatrix}, \ R_3^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_3 & -s_3 \\ 0 & s_3 & c_3 \end{bmatrix}$$

Part d

$$R_3^0 = R_1^0 R_2^1 R_3^2 = \begin{bmatrix} c_1c_2 & c_1s_2s_3 - s_1c_3 & s_1s_3 + c_1s_2c_3 \\ s_1c_2 & s_1s_2s_3 + c_1c_3 & s_1s_2c_3 - c_1s_3 \\ -s_2 & c_2s_3 & c_2c_3 \end{bmatrix}$$

Part e

Let $\theta_2 = 0, \theta_1 = \alpha, \theta_3 = \beta$, then the rotation matrix is

$$R_3^0 \xlongequal{\theta_2=0} \begin{bmatrix} c_1 & -s_1c_3 & s_1s_3 \\ s_1 & c_1c_3 & -c_1s_3 \\ 0 & s_3 & c_3 \end{bmatrix} \xlongequal{\theta_1=\alpha,\theta_3=\beta} \begin{bmatrix} \cos\alpha & -\sin\alpha\cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & \cos\alpha\cos\beta & -\cos\alpha\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

Now let $\theta_2 = \pi, \theta_1 = \alpha - \pi, \theta_3 = \beta - \pi$, then the rotation matrix is

$$R_3^0 \xlongequal{\theta_2=\pi} \begin{bmatrix} -c_1 & -s_1c_3 & s_1s_3 \\ -s_1 & c_1c_3 & -c_1s_3 \\ 0 & -s_3 & -c_3 \end{bmatrix} \xlongequal{\theta_1=\alpha-\pi,\theta_3=\beta-\pi} \begin{bmatrix} \cos\alpha & -\sin\alpha\cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & \cos\alpha\cos\beta & -\cos\alpha\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}$$

Apparently, we have infinite choices of θ_1, θ_2 and θ_3 that result in the same orientation.

Part f

Let $\theta_2 = \pi/2$, then the rotation matrix is

$$R_3^0 \xlongequal{\theta_2=\pi/2} \begin{bmatrix} 0 & c_1s_3 - s_1c_3 & s_1s_3 + c_1c_3 \\ 0 & s_1s_3 + c_1c_3 & s_1c_3 - c_1s_3 \\ -1 & 0 & 0 \end{bmatrix} \xlongequal{\gamma=\theta_1-\theta_3} \begin{bmatrix} 0 & -\sin\gamma & \cos\gamma \\ 0 & \cos\gamma & \sin\gamma \\ -1 & 0 & 0 \end{bmatrix}$$

If I always choose $\theta_1 = \theta_3$ (in which case $\gamma = 0$), then it is the same orientation and there are infinite choices of θ_1, θ_2 and θ_3.

Question 5

Part a-c

$$w_{0,1}^1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \quad w_{1,2}^2 = \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix}, \quad w_{2,3}^3 = \begin{bmatrix} \dot{\theta}_3 \\ 0 \\ 0 \end{bmatrix}$$

Part d

$$\begin{aligned} w_{0,1}^3 &= R_1^3 w_{0,1}^1 = R_2^3 R_1^2 w_{0,1}^1 = (R_3^2)^\top (R_2^1)^\top w_{0,1}^1 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{bmatrix} \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \\ &= \begin{bmatrix} c_2 & 0 & -s_2 \\ s_2 s_3 & c_3 & c_2 s_3 \\ s_2 c_3 & -s_3 & c_2 c_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} -s_2 \\ c_2 s_3 \\ c_2 c_3 \end{bmatrix} \dot{\theta}_1 \end{aligned}$$

$$\begin{aligned} w_{1,2}^3 &= R_2^3 w_{1,2}^2 = (R_3^2)^\top w_{1,2}^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ c_3 \\ -s_3 \end{bmatrix} \dot{\theta}_2 \end{aligned}$$

Part e

The matrix A should be clear from the following derivations:

$$\begin{aligned} w_{0,3}^3 &= w_{0,1}^3 + w_{1,2}^3 + w_{2,3}^3 \\ &= \begin{bmatrix} -s_2 \\ c_2 s_3 \\ c_2 c_3 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} 0 \\ c_3 \\ -s_3 \end{bmatrix} \dot{\theta}_2 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_3 = \begin{bmatrix} -s_2 & 0 & 1 \\ c_2 s_3 & c_3 & 0 \\ c_2 c_3 & -s_3 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \end{aligned}$$

Part f

$$B = A^{-1} = \frac{1}{c_2} \begin{bmatrix} 0 & s_3 & c_3 \\ 0 & c_2 c_3 & -c_2 s_3 \\ c_2 & s_2 s_3 & s_2 c_3 \end{bmatrix}$$

Part g

Consider Part f again,

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{c_2} \begin{bmatrix} 0 & s_3 & c_3 \\ 0 & c_2 c_3 & -c_2 s_3 \\ c_2 & s_2 s_3 & s_2 c_3 \end{bmatrix} w_{0,3}^3$$

Fix the angular velocity $w_{0,3}^3$ and make it very small. As $\theta_2 \to \pi/2$, we know that $c_2 \to 0$, and it implies that $\dot{\theta}_1, \dot{\theta}_3 \to \infty$.