Truss FEM
$\boxed{\text{geometric nonlinearities}}$

What does this mean to you?

Kim Lau Nielsen

Department of Civil and Mechanical Engineering
Section of Solid Mechanics
Technical University of Denmark

Materials Behavior

(Repetition)

Common mistake ("copy/paste from Day 1"):

$$\{R\} = \{R_{int}\} - \{R_{ext}\} = \sum_e \{B_0\} N^e L_0^e - \{P\} = \{0\}.$$

with

?

$$N^e = A^e E \varepsilon = A^e E \{B_0\}^T \{d\}$$

Linear materials:

$$\sigma = E \varepsilon$$

$$\frac{\partial \sigma}{\partial \varepsilon} = E = E_t$$

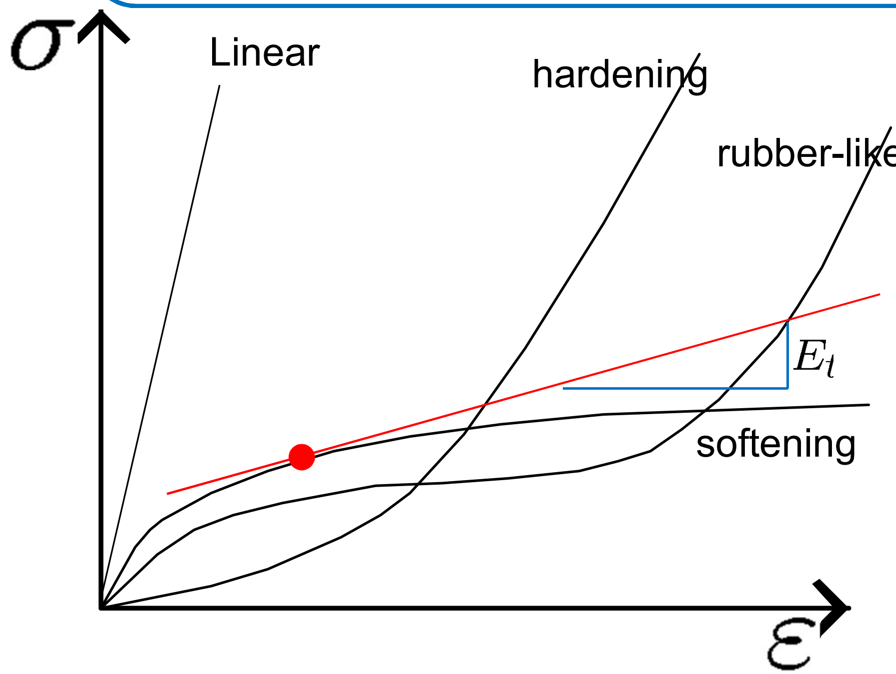
Nonlinear materials:

$$\frac{\partial \sigma}{\partial \varepsilon} = E_t$$

Slope of σ–ε curve

$$\Downarrow$$

$$\partial \sigma = E_t \, \partial \varepsilon$$



(Repetition)

Incremental Procedures

- Explicit methods (No equilibrium – Euler methods)
  - Pure Euler Method
  - Euler Method with one-step correction
- Implicit methods (equilibrium w. Newton-Raphson iterations)
  - Newton-Raphson Method (NR)
  - Modified NR Method

Pure Euler Method

(Repetition)

Recall the cantilever beam

Solution at increment n:

$$[K_t]^{n-1}\{\Delta D\} = \{\Delta P\}$$

Imagine you apply an external load increment

Calculate displacement increments

Remember BC's !

$$\Downarrow$$

$$\{D\}^n = \{D\}^{n-1} + \{\Delta D\}$$

Add displacement increments

Pseudo-code (Pure Euler Method)

(Repetition)

The most common programming error !

!!Remember to initialize arrays!!

$P^0 = 0$
$\Delta P^n = P^{final} / n_{incr}$
$D^0 = 0$

For load-increment $n = 1, 2, \ldots, n_{incr}$

$P^n = P^{n-1} + \Delta P^n$

$\boxed{\text{Calculate } K_t(D^{n-1})}$

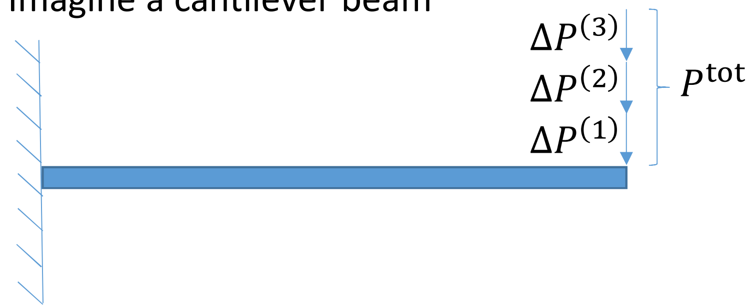
$\Delta D^n = (K_t(D^{n-1}))^{-1} \Delta P^n$ (NB! Remember boundary conditions)

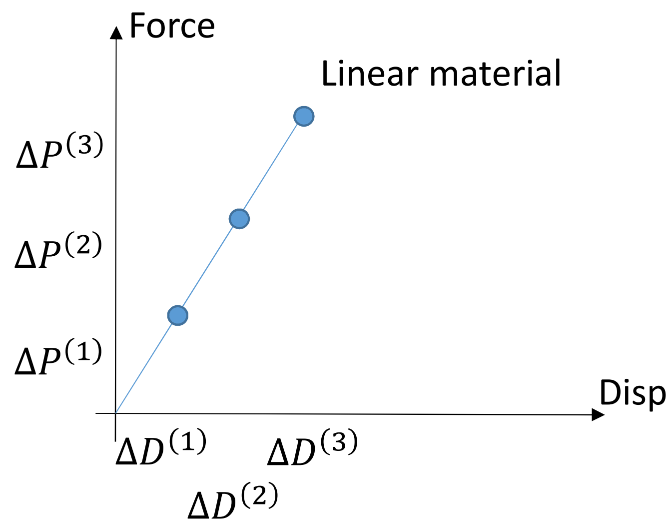
$D^n = D^{n-1} + \Delta D^n$

End load-increments

where *nincr* is the number of load increments.

Imagine a cantilever beam





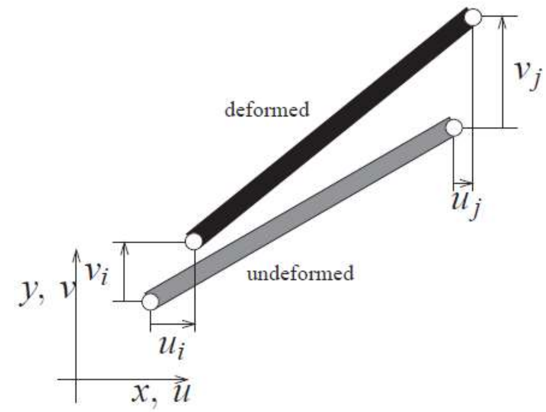
Linear strain

(Recall from day 1-2)

- Cauchy strain (engineering strain)

$$\varepsilon_L = \frac{L_1 - L_0}{L_0}$$

Bar element:



$$L_1 = \sqrt{(\Delta x + \Delta u)^2 + (\Delta y + \Delta v)^2}$$

$$\approx L_0 + \frac{\Delta x \Delta u + \Delta y \Delta v}{L_0}$$

$$\Rightarrow \quad \varepsilon_L = \frac{\Delta x \Delta u + \Delta y \Delta v}{L_0^2}$$

Small displacement assumption

FEM notation:

$$\varepsilon_L = \frac{1}{L_0^2}\{-\Delta x \ \ -\Delta y \ \ \Delta x \ \ \Delta y\}\{d\} = \{B_0\}^T\{d\}$$

Different strain measures

Cauchy (engineering strain):

$$\varepsilon = \frac{L_1 - L_0}{L_0}$$

Swaiger:

$$\varepsilon = \frac{L_1 - L_0}{L_1}$$

Green-Lagrange strain:

$$\varepsilon_G = \frac{L_1^2 - L_0^2}{2L_0^2}$$

True strain:

$$d\varepsilon = \frac{dL}{L} \quad \Rightarrow \quad \int_0^{\varepsilon} d\varepsilon = \int_{L_0}^{L_1} \frac{dL}{L} \quad \Rightarrow \quad \varepsilon = \ln(\frac{L_1}{L_0})$$

Two formulations

We choose this one

- Total Lagrangian approach
  - All quantities are referred back to the initial material configuration (the reference frame), where also the integration is carried out.
  - "Non-physical" stress and strain measures are employed.
- Updated Lagrangian approach
  - All quantities refers to the current configuration (or deformed shape), where also the integration is carried out.
  - Physical stress and strain measures are employed.

Principle of Virtual Work

$$\int_V \{\delta\varepsilon\}^T \{\sigma\}\, dV = \int_S \{\delta\mathbf{u}\}^T \{F\}\, dS$$

Notice; the energy will always be a physical quantity. ← **IMPORTANT**

Alternative strain measures

- Exact Cauchy strain for bar element

$$\varepsilon = \frac{\sqrt{(\Delta x + \Delta u)^2 + (\Delta y + \Delta v)^2} - L_0}{L_0}$$

This is, however, not practical For FEM computations.

- The Green-Lagrange strain

$$\varepsilon_G = \frac{L_1^2 - L_0^2}{2L_0^2}$$

(For small elongation/compression)

Expanding terms: $L_1^2 - L_0^2 = (\Delta x + \Delta u)^2 + (\Delta y + \Delta v)^2 - (\Delta y^2 + \Delta x^2)$

$$\Rightarrow \qquad \varepsilon_G = \frac{\Delta x \Delta u + \Delta y \Delta v}{L_0^2} + \frac{\Delta u^2 + \Delta v^2}{2L_0^2}$$

$$= \varepsilon_L + \frac{\Delta u^2 + \Delta v^2}{2L_0^2}$$

Green-Lagrange strain

$$\varepsilon_G = \frac{\Delta x \Delta u + \Delta y \Delta v}{L_0^2} + \frac{\Delta u^2 + \Delta v^2}{2L_0^2}$$

FEM notation:

$$\varepsilon_G = \{B_0\}^T\{d\} + \tfrac{1}{2}\{d\}^T \tfrac{1}{L_0^2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \{d\}$$

$$= \{B_0\}^T\{d\} + \tfrac{1}{2}\{B_d\}^T\{d\}$$

where

$$\{B_d\} = \tfrac{1}{L_0^2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \{d\} = \tfrac{1}{L_0^2}\{-\Delta u \ -\Delta v \ \Delta u \ \Delta v\}^T$$

Green-Lagrange strain

FEM notation:

$$\varepsilon_G = \{B_0\}^T\{d\} + \tfrac{1}{2}\{B_d\}^T\{d\}$$

Nonlinear strain displacement vector:

$$\frac{\partial \varepsilon_G}{\partial \{d\}} = \{B_0\}^T + \{d\}^T \frac{1}{L_0^2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= \{B_0\}^T + \{B_d\}^T = \{\bar{B}\}^T$$

Nonlinear FEM

- Internal forces (for bar elements):

$$\delta\varepsilon_G = \{\bar{B}\}^T\{\delta d\}$$

$$\Rightarrow \quad \{R_{\text{int}}\} = \sum_e \{\bar{B}\} N_G^e L_0^e$$

Recall from linear analysis

$$\{R_{\text{int}}\} = \sum_e \{B_0\}\, N^e\, L_0^e$$

- Bar forces:

$$N_G^e = A^e \sigma_G^e$$

$$\sigma_G^e = E\varepsilon_G = E(\{B_0\}^T + \tfrac{1}{2}\{B_d\}^T)\{d\}$$

Notice; the stress is not physical, but it is the 2nd Piola-Kirchhoff stress measure "work conjugate" to the Green-Lagrange strain.

The corresponding physical (engineering) quantities are;

$$\sigma^e = \sigma_G^e L_1/L_0 \quad \text{and} \quad N^e = N_G^e L_1/L_0$$

Tangent Stiffness Matrix

- Definition (just like Day 2):

$$[K_t] = \frac{\partial\{R\}}{\partial\{D\}} = \frac{\partial\{R_{\text{int}}\}}{\partial\{D\}} = \frac{\partial}{\partial\{D\}}\sum_e \{\bar{B}\} N_G^e L_0^e$$

(Recall that; $\{R\} = \{R_{int}\} - \{R_{ext}\}$)

$$= \sum_e \left(\frac{\partial\{\bar{B}\}}{\partial\{d\}} N_G^e L_0^e + \{\bar{B}\} L_0^e \frac{\partial N_G^e}{\partial\{d\}} \right)$$

- Computing terms:

$$\frac{\partial\{\bar{B}\}}{\partial\{d\}} = \frac{\partial\{B_d\}}{\partial\{d\}} = \frac{1}{L_0^2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial N_G^e}{\partial\{d\}} = A^e E \frac{\partial \varepsilon_G}{\partial\{d\}} = A^e E \{\bar{B}\}^T$$

Tangent Stiffness Matrix

- Collecting terms:

$$[K_t] = \sum_e \left(\frac{\partial\{B_d\}}{\partial\{d\}} N_G^e L_0^e + A^e E L_0^e \{\bar{B}\}\{\bar{B}\}^T \right)$$

(Recall that; $\{\bar{B}\}^T = \{B_0\}^T + \{B_d\}^T$)

$$= \sum_e \left([k_\sigma^e] + [k_0^e] + [k_d^e] \right)$$

- Contributions:

$$[k_\sigma^e] = \frac{\partial\{B_d\}}{\partial\{d\}} N_G^e L_0^e = \frac{1}{L_0^2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} N_G^e L_0^e$$ (Stress stiffness)

$$[k_0^e] = A^e E L_0^e \{B_0\}\{B_0\}^T$$ (Initial linear stiffness)

$$[k_d^e] = A^e E L_0^e \{B_0\}\{B_d\}^T + A^e E L_0^e \{B_d\}\{B_0\}^T + A^e E L_0^e \{B_d\}\{B_d\}^T$$ (Displacement stiffness)

Exercises

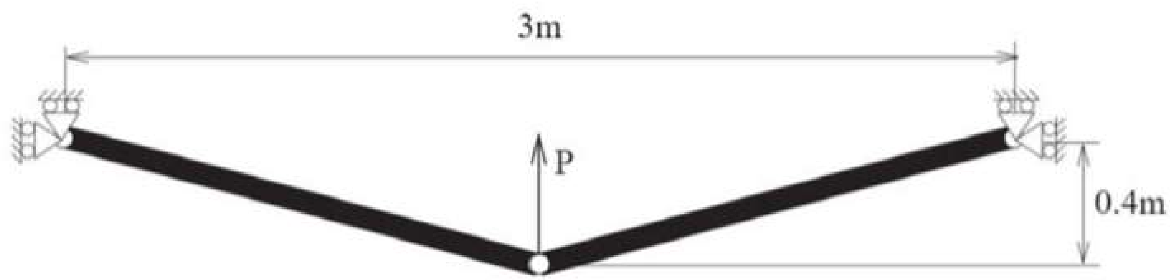


Figure 3.1: The 2-bar VonMises truss.

Exercise 3.1

Implement a (non-modified) NR geometrically non-linear truss analysis algorithm based on an incremental scheme.

Use the program to solve the Von Mises truss problem illustrated in Figure 3.1 (use e.g. $P = 0.03$ and 20 increments). Compare the result with the analytical solution (Krenk 1993)

$$P = 2EA\left(\frac{a}{L_0}\right)^3\left[\frac{D}{a} - \frac{3}{2}\left(\frac{D}{a}\right)^2 + \frac{1}{2}\left(\frac{D}{a}\right)^3\right], \tag{3.19}$$

where D is the vertical displacement of the center node and a is the undeformed height and L_0 is the undeformed bar length. Remark that this solution assumes $a \ll L_0$.

Exercises

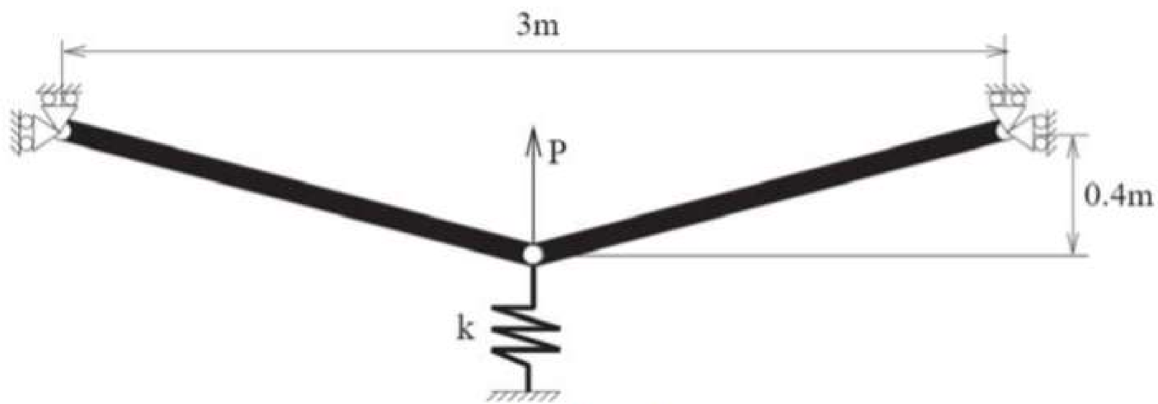


Figure 3.2: A modified VonMises truss problem.

Exercise 3.2

Solve the problem shown in Figure 3.2 for varying spring stiffnesses k and *plot the solution in a force-displacement diagram.*

Here the analytical solution is

$$P = 2EA\left(\frac{a}{L_0}\right)^3\left[\frac{D}{a} - \frac{3}{2}\left(\frac{D}{a}\right)^2 + \frac{1}{2}\left(\frac{D}{a}\right)^3\right] + kD. \tag{3.20}$$

NB! Remember to include the spring stiffness in both the stiffness matrix and the residual.

Exercises

Exercise 3.3

Solve the bigger problem sketched in Figure 3.3. *Can you relate the response to the analytical expression for limit loads of slender elastic columns?*

$$P_{crit} = \frac{\pi^2}{4}\frac{EI}{L^2} \tag{3.21}$$

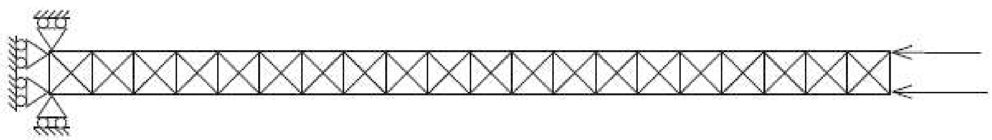


Figure 3.3: A slender truss beam. (NB! Crossing beams are not connected).

Hint: Find the effective product of Young's modulus and the moment of inertia EI by subjecting the beam to a small vertical load at the end point (as seen in Figure 3.4) and use your knowledge of simple beam theory.

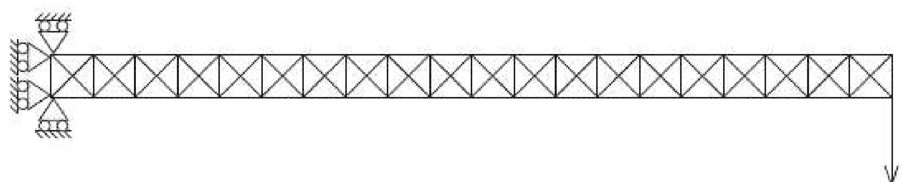


Figure 3.4: A slender truss beam in bending.

Exercises

- On the hand-in exercises …
  - Ole will introduce the last two sub-exercises next Wednesday
  - Four assignments (one related to each day)
  - Hand-in a report of 4 pages no later than Tuesday (on 27/9) at 10pm
  - Solution to *exercises can be added to the hand-in by adding <u>one</u> extra page.