Virtual Work Principle (Day 7)

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Preparatory reading:

- *Handout Day 7*
- *Chapter 4 in Cook et al.*
- *Read up on planar elasticity in tensor notation (c.f. in strength of materials II book by Tvergaard)*

Virtual Work Principle

$$\boxed{\int_V \{\delta\varepsilon\}^T\{\sigma\}\,dV = \int_S \{\delta\mathbf{u}\}^T\{F\}\,dS + \int_V \{\delta\mathbf{u}\}^T\{\Phi\}\,dV + \sum_i \{\delta\mathbf{u}\}_i^T\{p\}_i,}$$

which has to hold for any kinematically admissible displacement variations $\{\delta\mathbf{u}\}$, associated strain variations $\{\delta\varepsilon\}$ and nodal displacement variations $\{\delta D\}$.

Or

$$\boxed{\delta\Omega = \delta W}$$

where

- $\delta\Omega$ is variation in internal energy
- δW is variation in external work
- V is structural volume
- S is surface area
- $\{F\}$ is surface traction
- $\{\Phi\}$ is body force
- $\{\delta\mathbf{u}\}$ is virtual displacement variation
- $\{\sigma\}$ is the stress vector
- $\{\delta\varepsilon\}$ is the virtual strain variation vector
- $\{p\}_i$ is a concentrated (nodal) load vector

Definition:

$$\boxed{\delta\Omega \equiv \frac{\partial\Omega}{\partial D_i}\delta D_i}$$

Virtual Work Principle (in words)

Or in words:

> Arbitrary virtual displacement variations must lead to identical variations in internal and external work

Basically, it is and alternative way of expressing structural equilibrium (energy balance)

VWP can be derived in two (or more) ways

Part I

a) Stationarity principle

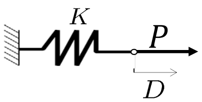
From minimum of Total Potential Energy (i.e reversible and non-dissipative)

Part II

b) Galerkin method

Derived from basic equilibrium equations, and holds for any system (incl. dissipative ones)

Minimum Total Potential Energy – 1dof problem



$$\delta\square \equiv \frac{\partial\square}{\partial D}\delta D$$

Elastic (strain) energy: $\Omega = \frac{1}{2}KD^2$

Potential work of external force: $W = PD$

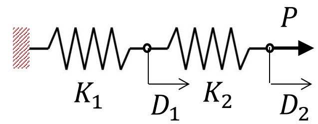
Total Potential Energy: $\Pi = \Omega - W = \frac{1}{2}KD^2 - PD$

Stationarity (minimum): $\frac{\partial\Pi}{\partial D} = \frac{\partial\Omega}{\partial D} - \frac{\partial W}{\partial D} = KD - P = 0$

Variational (VWP) form: $\delta\Pi = \delta\Omega - \delta W = \delta DKD - \delta DP = 0, \forall\delta D$

$$\Longrightarrow KD - P = 0 \Longrightarrow KD = P$$

Minimum Total Potential Energy – 2dof problem



Total Potential Energy:

$$\Pi = \frac{1}{2}K_1D_1^2 + \frac{1}{2}K_2(D_2 - D_1)^2 - PD_2$$

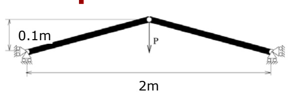
or: $$\Pi = \frac{1}{2}\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}^T \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} - \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}^T \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

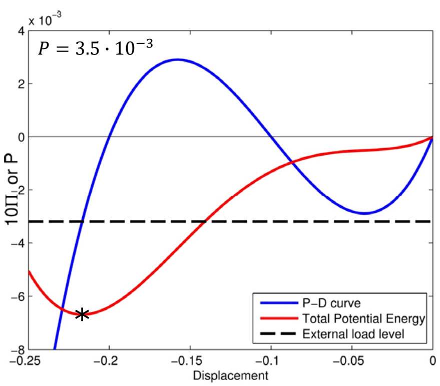
$$= \frac{1}{2}\{D\}^T[K]\{D\} - \{D\}^T\{P\}$$

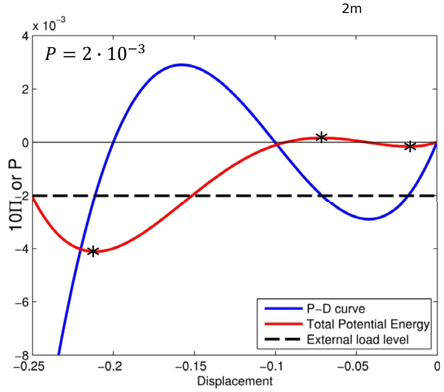
Stationarity: $\delta\Pi = \frac{\partial\Pi}{\partial\{D\}}\{\delta D\} = \{\delta D\}^T[K]\{D\} - \{\delta D\}^T\{P\} = 0, \forall\{\delta D\}$

$$\Longrightarrow [K]\{D\} = \{P\}$$

Minimum Total Potential Energy – non-linear problem







Minimum of Total Potential Energy – Warning!

Let the true (exact) solution be: $\{D_\infty\}$ i.e. $[K]\{D_\infty\} = \{P\}$

Total potential energy of the true solution: $\Pi_\infty = \frac{1}{2}\{D_\infty\}^T[K]\{D_\infty\} - \{D_\infty\}^T\{P\}$

Non-perfect (approximate) solution: $\{D\} = \{D_\infty\} + \epsilon\{\delta D\}$

Total potential energy of approximate solution:

$$\boxed{\Pi =} \frac{1}{2}\{D\}^T[K]\{D\} - \{D\}^T\{P\} = \frac{1}{2}(\{D_\infty\} + \epsilon\{\delta D\})^T[K](\{D_\infty\} + \epsilon\{\delta D\}) - (\{D_\infty\} + \epsilon\{\delta D\})^T\{P\}$$

$$= \frac{1}{2}\{D_\infty\}^T[K]\{D_\infty\} - \{D_\infty\}^T\{P\} + \epsilon(\{\delta D\}^T[K]\{D_\infty\} - \{\delta D\}^T\{P\}) + \frac{1}{2}\epsilon^2\{\delta D\}^T[K]\{\delta D\}$$

$$= \boxed{\Pi_\infty + \frac{1}{2}\epsilon^2\{\delta D\}^T[K]\{\delta D\}}$$

Hence: $\boxed{\Pi(\{D\}) \geq \Pi_\infty, \forall\{D\} \neq \{D_\infty\}}$

Minimum of Total Potential Energy – Warning!

The Total Potential Energy of an approximate solution is allways larger than for the true solution:

$$\boxed{\Pi(\{D\}) \geq \Pi_\infty \quad \text{for } \forall\{D\} \neq \{D_\infty\}}$$

This simplifies to the important observation for compliance:

$$-\frac{1}{2}\{D\}^T\{P\} \geq -\frac{1}{2}\{D_\infty\}^T\{P\} \quad \text{for } \forall\{D\} \neq \{D_\infty\}$$

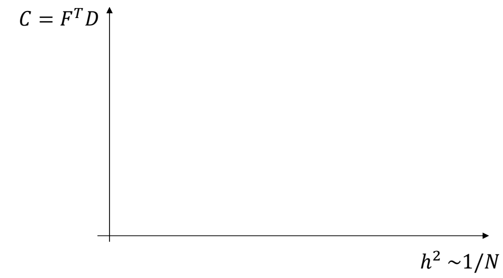
or:

$$\boxed{\{D\}^T\{P\} \leq \{D_\infty\}^T\{P\} \quad \text{for } \forall\{D\} \neq \{D_\infty\}}$$

i.e. the approximate FE solution is allways too "stiff" !!!

The error, however, gets smaller as we approach the exact solution.

Minimum of Total Potential Energy – convergence



* Holds for certain continuity conditions

Minimum of TPE – continuum version

Elastic strain energy

$$\Omega = \frac{1}{2}\int_V \{\varepsilon\}^T\{\sigma\}dV = \frac{1}{2}\int_V \{\varepsilon\}^T[C]\{\varepsilon\}dV$$

Potential work of external forces

$$W = \int_{S_F} \{\mathbf{u}\}^T\{F\}dS + \int_V \{\mathbf{u}\}^T\{\Phi\}dV.$$

Stationarity

$$\delta\Pi = \delta\Omega - \delta W = \int_V \{\delta\varepsilon\}^T\{\sigma\}dV - \int_{S_F} \{\delta\mathbf{u}\}^T\{F\}dS - \int_V \{\delta\mathbf{u}\}^T\{\Phi\}dV = 0$$

Hence, VWP can be derived from minimum of potential energy principle

PS! Derivations like this was used for above derivations: $\delta\Omega \equiv \frac{\partial\Omega}{\partial\{D\}}\{\delta D\} = \frac{\partial\Omega}{\partial\{\varepsilon\}}\frac{\partial\{\varepsilon\}}{\partial\{D\}}\{\delta D\} = \frac{\partial\Omega}{\partial\{\varepsilon\}}\{\delta\varepsilon\}$, $\quad \delta\square \equiv \frac{\partial\square}{\partial D}\delta D$

Integration by parts (Green's theorem)

Green's theorem may be seen as an application of "Integration by Parts"

$$\boxed{\int_a^b f\frac{\partial g}{\partial x}dx = -\int_a^b \frac{\partial f}{\partial x}g dx + [f\, g]_a^b} \quad \text{1d}$$

Consider the volume integral

$$\int_V \phi\frac{\partial\psi}{\partial x}dV.$$

This may also be written as

$$\int_{y_L}^{y_U}\int_{x_L}^{x_R} \phi\frac{\partial\psi}{\partial x}\, dx\, dy.$$

By partial integration this is equal to

$$-\int_V \frac{\partial\phi}{\partial x}\psi dV + \int_{y_L}^{y_U} [\phi\psi]_{x_L}^{x_R} dy = -\int_V \frac{\partial\phi}{\partial x}\psi dV + \int_{y_L}^{y_U} [(\phi\psi)_{x=x_R} - (\phi\psi)_{x=x_L}]\, dy$$

On the right and left hand sides of the boundary we have that

$$dy = n_x dS \ \text{ and } \ dy = -n_x dS,$$

Therefore the last integral of (1) may be rewritten as a surface integral

$$\int_S \phi\psi n_x dS.$$

which results in

$$\boxed{\int_V \phi\frac{\partial\psi}{\partial x}dV = -\int_V \frac{\partial\phi}{\partial x}\psi dV + \int_S \phi\psi n_x dS} \quad \text{2d/3d}$$

Galerkin method

Write up PDE (equilibrium) and boundary conditions

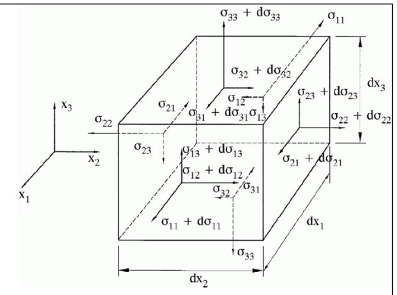
1. Convert PDE to weak form (introduce test fields)

2. Apply Green's theorem (integration by parts)

3. Discretetize system (c.f. Days 1 or 5)

Mechanical equilibrium equations

$$\sigma_{ji,j} \equiv \frac{\partial \sigma_{ji}}{\partial x_j}$$

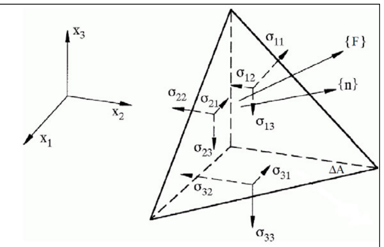


Volume force equilibrium

$$\sigma_{ji,j} + \Phi_i = 0$$

Moment equilibrium

$$\sigma_{ij} = \sigma_{ji}$$



Surface traction equilibrium

$$\sigma_{ij} n_i = F_j$$

Galerkin method (1)

The PDE (equilibrium) and boundary conditions

Volume force equilibrium:	$\sigma_{ji,j} + \Phi_i = 0$	in V
Moment equilibrium:	$\sigma_{ij} = \sigma_{ji}$	in V
Surface traction equilibrium:	$\sigma_{ij} n_i = F_j$	on S_F
Support BC's:	$u_i = u_i^*$	on S_u.

Also referred to as the "strong form" … and not suitable for computations

Integration by parts (Green's theorem)

Green's theorem may be seen as an application of "Integration by Parts"

$$\boxed{\int_a^b f \frac{\partial g}{\partial x} dx = -\int_a^b \frac{\partial f}{\partial x} g dx + [f\ g]_a^b} \quad \text{1d}$$

Consider the volume integral

$$\int_V \phi \frac{\partial \psi}{\partial x} dV.$$

This may also be written as

$$\int_{y_L}^{y_U} \int_{x_L}^{x_R} \phi \frac{\partial \psi}{\partial x}\, dx\, dy.$$

By partial integration this is equal to

$$-\int_V \frac{\partial \phi}{\partial x} \psi dV + \int_{y_L}^{y_U} [\phi\psi]_{x_L}^{x_R} dy = -\int_V \frac{\partial \phi}{\partial x} \psi dV + \int_{y_L}^{y_U} \left[(\phi\psi)_{x=x_R} - (\phi\psi)_{x=x_L}\right] dy$$

On the right and left hand sides of the boundary we have that

$$dy = n_x dS \text{ and } dy = -n_x dS,$$

Therefore the last integral of (1) may be rewritten as a surface integral

$$\int_S \phi\psi n_x dS.$$

which results in

$$\boxed{\int_V \phi \frac{\partial \psi}{\partial x} dV = -\int_V \frac{\partial \phi}{\partial x} \psi dV + \int_S \phi\psi n_x dS} \quad \text{2d/3d}$$

Galerkin method (1+2)

"test function", "weight function", "virtual displacement" etc.

$$\sigma_{ji,j}+\Phi_i=0 \text{ in } V$$
$$\sigma_{ij}=\sigma_{ji} \text{ in } V$$
$$\sigma_{ij}n_i=F_j \text{ on } S_F$$
$$u_i=u_i^* \text{ on } S_u.$$

Step 1 – Multiply by test field and integrate

$$\int_V \delta u_i\,(\sigma_{ji,j}+\Phi_i)\,dV=0$$,with δu_i being kinematically admissible

Step 2 – Employ Green's Theorem (or integration by part)

… and rewrite the first term (using Green's Theorem);

$$\int_V \delta u_i\sigma_{ji,j}\,dV=-\int_V \underbrace{\delta u_{i,j}\sigma_{ji}}\,dV+\int_S \underbrace{\delta u_i\sigma_{ji}n_j}\,dS$$

$\delta u_i=0$ on S_u

$\delta u_{i,j}\sigma_{ji}=\delta\varepsilon_{ij}\sigma_{ij}$

$\sigma_{ji}n_j=F_i$ on S_F

Galerkin method (2+3)

Note 1: this time we made no assumption on the constitutive behaviour.

Note 2: The Principle of Virtual Work Principle is generally applicable – and always used!

Step 2 – Employ Green's Theorem (continued)

Rearranging terms yields;

$$\int_V \delta\varepsilon_{ij}\,\sigma_{ij}\,dV=\int_{S_F}\delta u_i\,F_i\,dS+\int_V \delta u_i\,\Phi_i\,dV,\ \forall\,\delta u_i \text{ kin. adm.}$$

$$\sigma_{ji,j}+\Phi_i=0 \text{ in } V$$
$$\sigma_{ij}=\sigma_{ji} \text{ in } V$$
$$\sigma_{ij}n_i=F_j \text{ on } S_F$$
$$u_i=u_i^* \text{ on } S_u.$$

or in matrix notation;

$$\int_V \{\delta\varepsilon\}^T\{\sigma\}\,dV=\int_{S_F}\{\delta\mathbf{u}\}^T\{F\}\,dS+\int_V\{\delta\mathbf{u}\}^T\{\Phi\}\,dV$$

Step 3 – Discretize the system (as in Days 1 or 5)

$$\{\mathbf{u}\}=\begin{Bmatrix}u\\v\end{Bmatrix}=\begin{bmatrix}N_1&0&N_2&\dots&0\\0&N_1&0&\dots&N_n\end{bmatrix}\begin{Bmatrix}u_1\\v_1\\u_2\\\vdots\\v_n\end{Bmatrix}=[N]\{d\}$$

etc.

Imposing displacement boundary conditions (3 ways)

"0's and 1's" Day 1

Penalty approach Cook 13.3

In the alternative approach we add a penalty term to the potential energy Π

$$\tilde{\Pi}=\Pi+\alpha\int_{S_u}(u_i-u_i^*)^2\,dS.$$

Taking the variation, we get

$$\delta\tilde{\Pi}=\delta\Pi+\alpha\int_{S_u}2\delta u_i\,(u_i-u_i^*)\,dS.$$

This results in the penalty method for imposing displacement const[…] in DAY 1 of the course.

Lagrangian multipliers Cook 13.2

Here we augment the potential energy Π with an extra term

$$\tilde{\Pi}=\Pi+\int_{S_u}\lambda\,(u_i-u_i^*)\,dS.$$

Taking the variation, we get

$$\delta\tilde{\Pi}=\delta\Pi+\int_{S_u}\delta\lambda\,(u_i-u_i^*)\,dS+\int_{S_u}\delta u_i\,\lambda\,dS.$$

This results in an extra set of equations and unknowns that have to be included in the global system equations before solving

$$\begin{bmatrix}\boldsymbol{K}&\boldsymbol{C}^T\\\boldsymbol{C}&0\end{bmatrix}\begin{Bmatrix}\boldsymbol{D}\\\boldsymbol{\lambda}\end{Bmatrix}=\begin{Bmatrix}\boldsymbol{R}\\\boldsymbol{Q}\end{Bmatrix}$$

Today

Exercise 7.1

Poisson's equation represents a well-established model for steady-state heat conduction in a solid material. The strong form of the equation posed on a domain V with a partitioned boundary S ($S=S_u\cup S_h$ and $S_u\cap S_h=0$) takes the form

$$-\nabla\cdot(k\nabla u)=f \quad \text{in } V \tag{7.39a}$$
$$(k\nabla u)\cdot\mathbf{n}=h \quad \text{on } S_h \tag{7.39b}$$
$$u=0 \quad \text{on } S_u \tag{7.39c}$$

where $u=u(x,y)$ is the scalar temperature field, $k>0$ is the conductivity constant of the material and f is the volumetric heating. $\mathbf{n}$ denotes the outward unit normal and h is a prescribed heat flux. To simplify the problem only $u=0$ is prescribed on S_u.

For the PDE (7.39) derive

- *the corresponding variational problem, (i.e. the Virtual Work Principle) for the heat conduction problem.*
- *the discrete finite element equation based on the variational problem*
- *element stiffness matrices, surface load vectors, etc., equivalent to the expressions derived for stiffness analysis in chapter 5. Please also use the notation of chapter 5.*

+finish band implementation and stress plots from Day 6

Proceeding to iso-parametric formulation on Day 8!

Second Assignment

First 2 (out of 4) sub-assignments for 2. Assignment available today

5pm on Learn

NB! For first exercise download band minimization programs bandfem/renum from Inside

Mid-term evaluation

Please answer questionnaire on Learn!

Available from noon today until noon 25 October (Monday after the fall break)