41525 FEM, Day 4

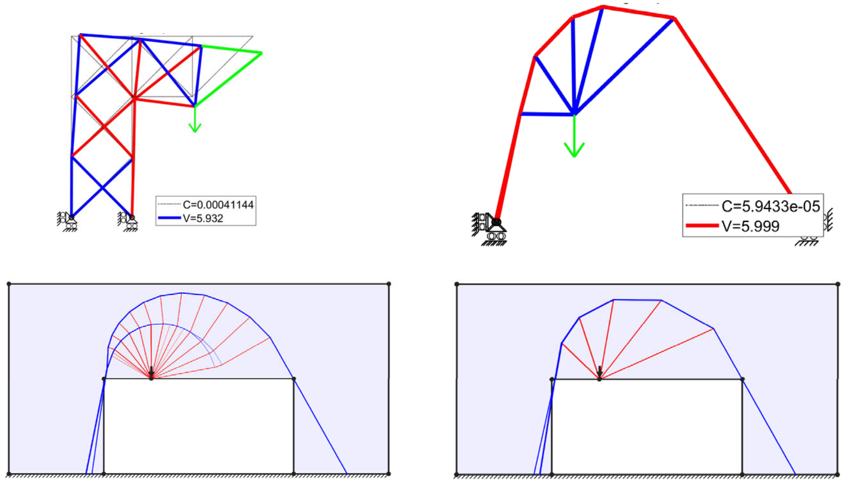
Topology Optimization for Truss Structures

Ole Sigmund

Reading material: Chapter 4 in Course Notes

Competition results

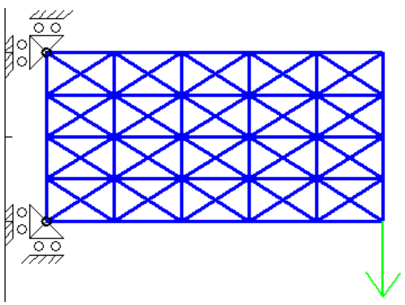
Worst **Best**



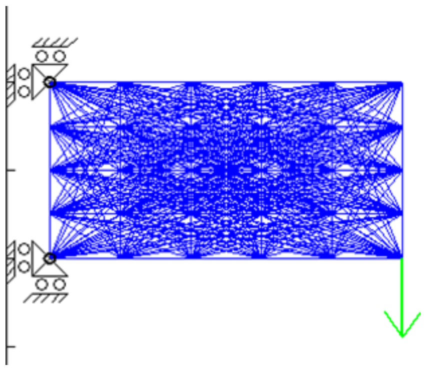
2% higher C

https://www.layopt.com/truss/

Ground structures

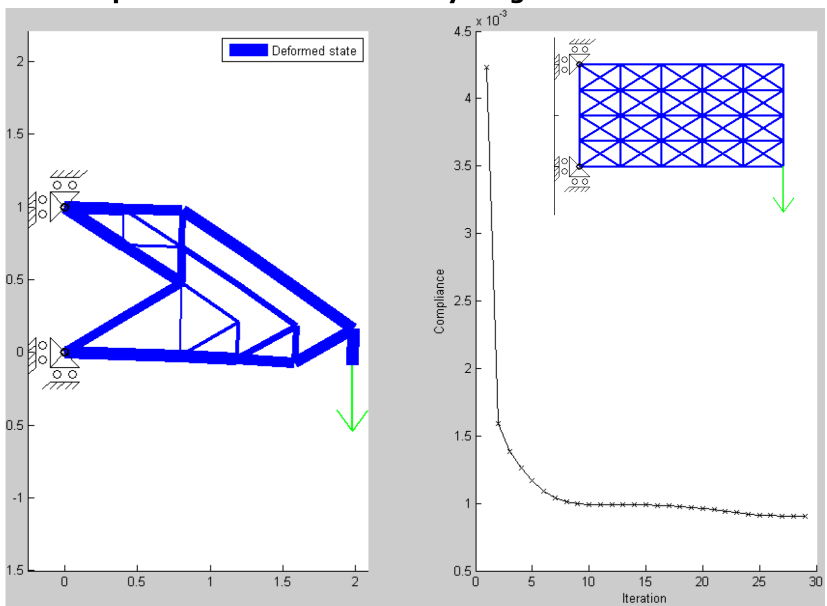


Only neighbor connections

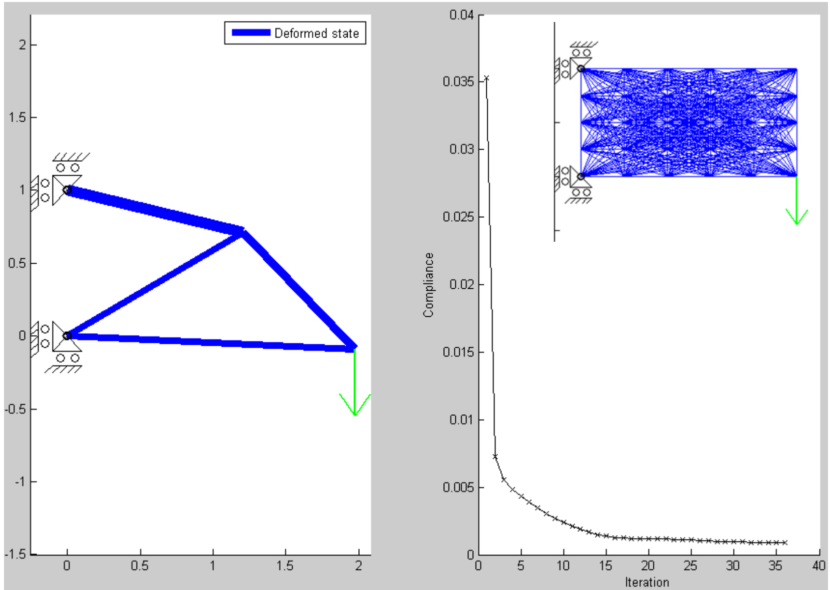


All nodes connected to all

Optimized structure for only neighbor connections



Optimized structure for full connectivity



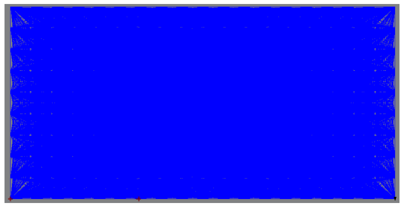
0/1 Integer problem

Combinations:

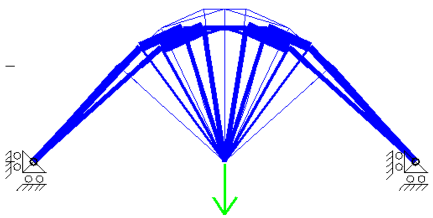
$$\boxed{\frac{N!}{(N-M)!M!}}$$

N=10, M=5 => 252
N=20, M=10 => 185.000
N=40, M=20 => $1.4 \cdot 10^{9}$
N=100, M=50 => 10^{29}
N=435, M=200 => 10^{152}

Topology Optimization

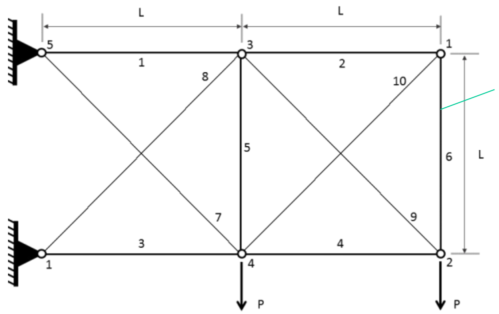


FIExtract: 19x11 nodes, 21736 elements

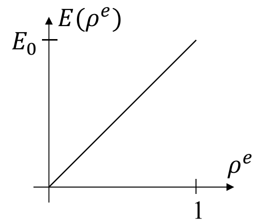


Vol=8.00, C=$8.5 \cdot 10^{-5}$

Design variables

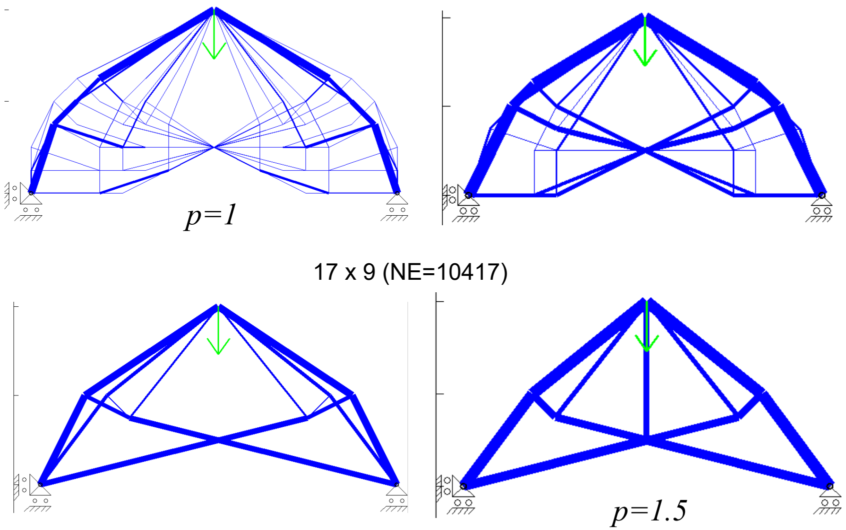


$$E^e = E(\rho^e) = (\rho^e)\ E_0$$



$$\boxed{0 < \rho_{min} \leq \rho^e \leq 1} \qquad \boxed{[k^e] = (\rho^e)\ [k_0^e]}$$

Result of topology optimization (increasing penalization)



$p \in [1, 1.5]$

Formulation of optimization problem

Objective function : Maximize stiffness

Minimize displacement

Minimize compliance

$C = \{D\}^T\{P\} = \{D\}^T[K]\{D\} = \sum_e \{d^e\}^T[k^e]\{d^e\}$

Volume Constraint : $V = \sum_e \rho^e v^e = \{\rho\}^T\{v\} \le V^*$

Variables $\{\rho\}$: $\{0\} < \{\rho_{min}\} \le \{\rho\} \le \{1\}$

Stiffness – ρ relation : $E(\rho^e) = (\rho^e)^p E_0$ or $[k^e] = (\rho^e)^p [k_0^e]$

Stiffness matrix for solid bar

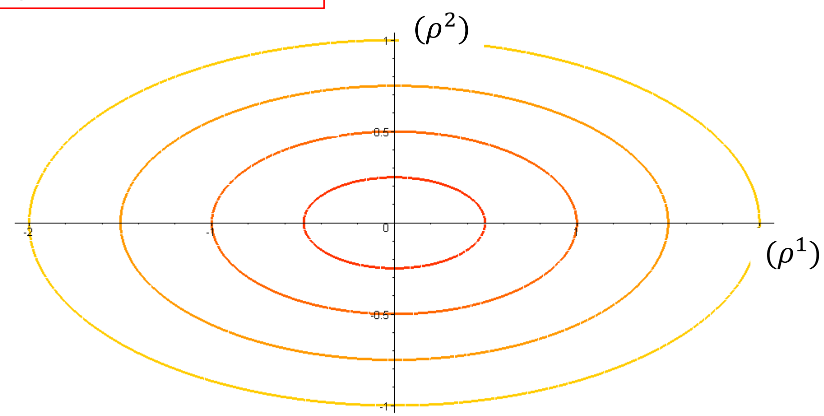
$$\boxed{\begin{aligned} \min_{\{\rho\}} :\ & C = \{D\}^T[K]\{D\} \\ s.t. :\ & g = \{\rho\}^T\{v\} - V^* \le 0 \\ :\ & \{0\} < \{\rho_{min}\} \le \{\rho\} \le \{1\} \\ :\ & [K(\{\rho\})]\{D\} = \{P\} \end{aligned}}$$

$$[K] = \sum_e [k^e] = \sum_e (\rho^e)^p [k_0^e]$$

Optimization crash course

Unconstrained optimization problem

$$\boxed{\min_{\rho^1, \rho^2} :\ f = (\rho^1)^2 + 4(\rho^2)^2}$$



Optimality criterion: unconstrained optimization problem

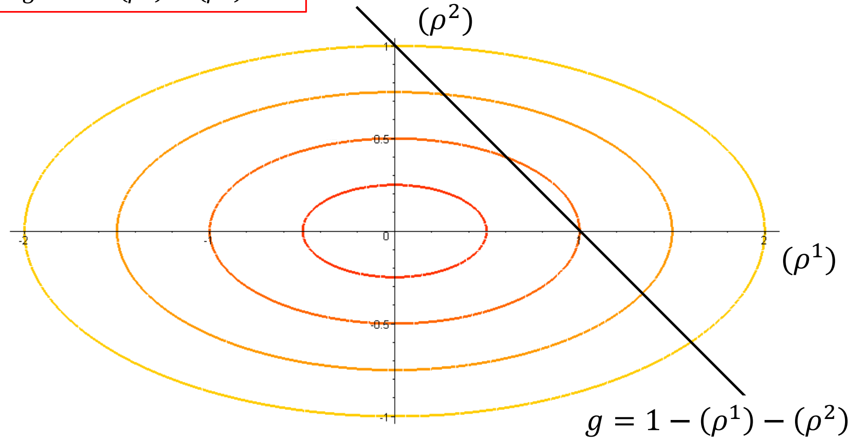
$$\boxed{\nabla f = \{0\}}$$

where: f Objective function

Definitions: $$\nabla f \equiv \begin{Bmatrix} \frac{\partial f}{\partial \rho^1} \\ \frac{\partial f}{\partial \rho^2} \\ \vdots \\ \frac{\partial f}{\partial \rho^{NE}} \end{Bmatrix}, \qquad f' \equiv \frac{\partial f}{\partial \rho^e}$$

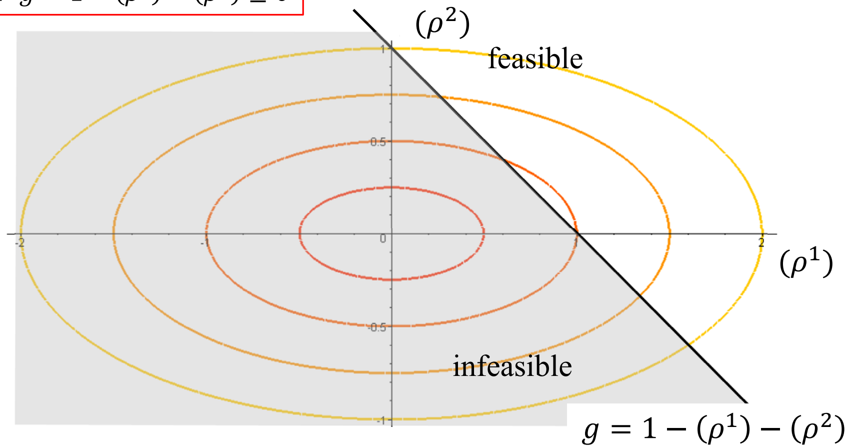
Constrained optimization problem

$$\boxed{\begin{array}{ll} \min\limits_{\rho^1,\rho^2}: & f = (\rho^1)^2 + 4(\rho^2)^2 \\ s.t.: & g = 1 - (\rho^1) - (\rho^2) \leq 0 \end{array}}$$



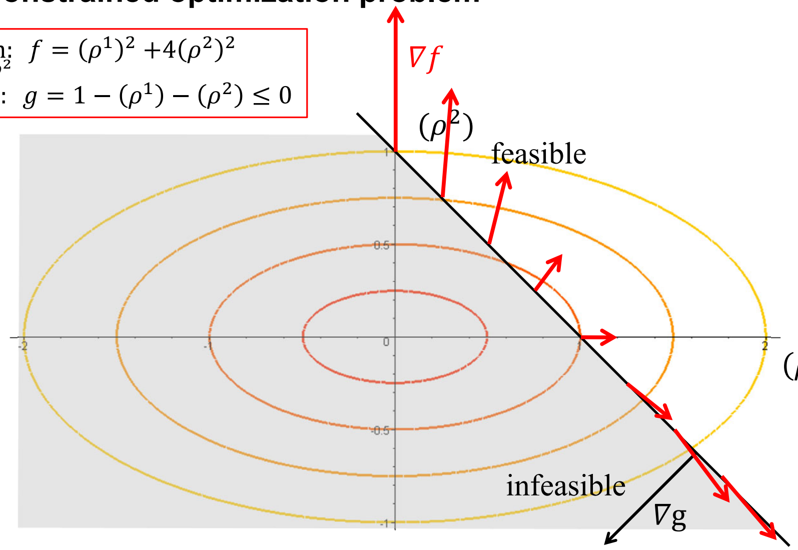
Constrained optimization problem

$$\boxed{\begin{array}{ll} \min\limits_{\rho^1,\rho^2}: & f = (\rho^1)^2 + 4(\rho^2)^2 \\ s.t.: & g = 1 - (\rho^1) - (\rho^2) \leq 0 \end{array}}$$



Constrained optimization problem

$$\boxed{\begin{array}{ll} \min\limits_{\rho^1,\rho^2}: & f = (\rho^1)^2 + 4(\rho^2)^2 \\ s.t.: & g = 1 - (\rho^1) - (\rho^2) \leq 0 \end{array}}$$



Optimality criterion: constrained optimization problem

$$\boxed{\nabla f + \lambda \nabla g = \{0\}}$$

where:

f Objective function

g Constraint function ($g \leq 0$)

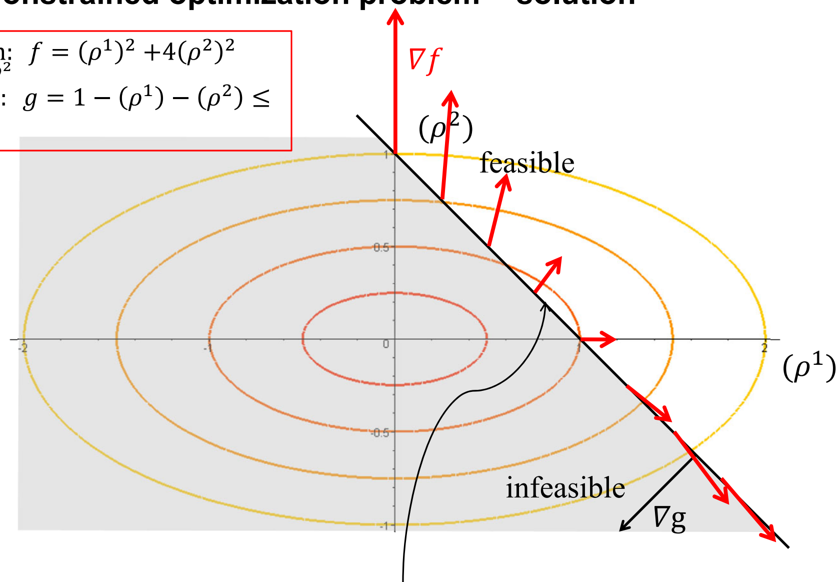
λ Lagrange multiplier

Definitions: $\nabla f \equiv \left\{ \begin{array}{c} \frac{\partial f}{\partial \rho^1} \\ \frac{\partial f}{\partial \rho^2} \\ \vdots \\ \frac{\partial f}{\partial \rho^{NE}} \end{array} \right\}, \qquad f' \equiv \frac{\partial f}{\partial \rho^e}$

Constrained optimization problem - solution

$\min_{\rho^1,\rho^2}:\ f = (\rho^1)^2 + 4(\rho^2)^2$

$s.t.:\ g = 1 - (\rho^1) - (\rho^2) \leq 0$



Optimum: $(\rho^1, \rho^2) = (0.8, 0.2)$

Sensitivity analysis (constraint function)

Constraint function:

$$g = \{\rho\}^T\{v\} - V^* = \sum_e \rho^e v^e - V^* \leq 0$$

Gradient:

$$g' \equiv \frac{\partial g}{\partial \rho^e} = v^e \quad \Rightarrow \quad \nabla g = \{v\}$$

Sensitivity analysis (objective function) I

Objective function:

$$f = \{D\}^T[K]\{D\} = \sum_e \{d^e\}^T[k^e]\{d^e\} = \sum_e (\rho^e)^p \{d^e\}^T[k_0^e]\{d^e\}$$

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1) Finite Difference (FD) approach:

$$\frac{\partial f}{\partial \rho^e} \approx \frac{\Delta f}{\Delta \rho^e} = \frac{f(\{\rho\} + \epsilon\{e\}_e) - f(\{\rho\})}{\epsilon}$$

where

$$\{e\}_e = \{0, 0, \ldots, \underset{e}{1}, \ldots, 0, 0\}^T$$

NB! FD approach requires *NE* extra FE analyses ☹

Sensitivity analysis (objective function) II

Objective function:

$$f = \{D\}^T[K]\{D\} = \sum_e \{d^e\}^T[k^e]\{d^e\} = \sum_e (\rho^e)^p \{d^e\}^T[k_0^e]\{d^e\}$$

---

2) Adjoint approach (using compact form $f = D^T K D$)

$$f' = D'^T K D + D^T K' D + D^T K D' = D^T K' D + 2 D^T K D' \quad (*)$$

"Trick" - differentiate FE equation ($KD = P$):

$$K'D + KD' = 0 \ \Rightarrow KD' = -K'D$$ (assuming *P* independent on design) $(**)$

Insert (**) in (*)

$$\boxed{f' =} D^T K' D - 2 D^T K' D = \boxed{-D^T K' D}$$

Sensitivity analysis (objective function) III

Objective function:

$$f = \{D\}^T[K]\{D\} = \sum_e \{d^e\}^T[k^e]\{d^e\} = \sum_e (\rho^e)^p \{d^e\}^T[k_0^e]\{d^e\}$$

---

2) Adjoint approach (contd')

$$\boxed{f'} \equiv \frac{\partial f}{\partial \rho^e} = -D^T K' D = -D^T \frac{\partial K}{\partial \rho^e} D = -\{d^e\}^T \frac{\partial [k^e]}{\partial \rho^e}\{d^e\}$$

$$= \boxed{-p(\rho^e)^{p-1}\{d^e\}^T[k_0^e]\{d^e\}}$$

Adjoint approach requires *no* extra FE analysis ☺

Optimality criterion

Constraint gradient: $$\frac{\partial g}{\partial \rho^e} = v^e \quad (*)$$

Objective gradient: $$\frac{\partial f}{\partial \rho^e} = -p(\rho^e)^{p-1}\{d^e\}^T[k_0^e]\{d^e\} \quad (**)$$

Optimality Criterion: $$\nabla f + \lambda \nabla g = \{0\} \ \Leftrightarrow$$

$$\frac{\partial f}{\partial \rho^e} + \lambda \frac{\partial g}{\partial \rho^e} = 0, \ e = 1, \ldots, NE \Leftrightarrow$$

$$\frac{-\frac{\partial f}{\partial \rho^e}}{\lambda \frac{\partial g}{\partial \rho^e}} = 1, \ e = 1, \ldots, NE \quad (***)$$

$$\Updownarrow$$

Insert (*) and (**) in (***): $$\boxed{\frac{p\{d^e\}^T[k^e]\{d^e\}}{\lambda v^e \rho^e} = 1, \qquad e = 1, \ldots, NE}$$

Physical meaning of optimality condition

Optimality condition:
(holds for "active elements") $$\boxed{\frac{p}{\lambda}\frac{\{d^e\}^T[k^e]\{d^e\}}{v^e\rho^e} = 1, \qquad e = 1, \ldots, NE}$$

i.e. strain energy pr. material volume must be constant

or

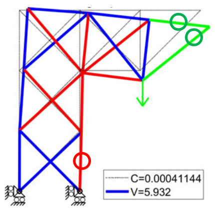
Strain Energy Density (SED) must be constant in all elements not meeting the lower or upper density bounds (i.e. for the active elements)!

Competition results

$\sigma\varepsilon = 1.0\text{e-}03$ *

{0.0670
0.0066
0.1396
0.0066
0.0066
0.0670
0.0066
0.0066
0.0670
0.1340
0.4758
0.3307
0.0439
0.0538
0.0232
0.0269
0.0463
0.0269
0.0000
0 }

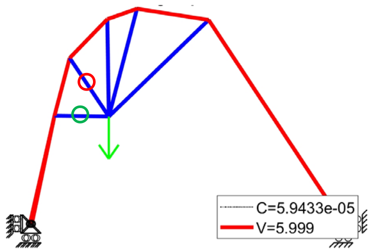
Worst



$\sigma\varepsilon = 1.0\text{e-}03$ *

{0.0147
0.0148
0.0001
0.0150
0.0150
0.0152
0.0078
0.0045
0.0076
0.0088
0.0088
0.0048
0.0048
0.0135
0.0090 }

Best



Heuristic update rule

SED high => increase density
SED low => decrease density

$$\rho^e = \rho^e_{old}\left(\frac{-\frac{\partial f}{\partial \rho^e}}{\lambda\frac{\partial g}{\partial \rho^e}}\right)^\eta = \rho^e_{old}\,(B^e)^\eta \quad \text{, where } B^e = \frac{-\frac{\partial f}{\partial \rho^e}}{\lambda\frac{\partial g}{\partial \rho^e}} = \frac{p\{d^e\}^T[k^e]\{d^e\}}{\lambda v^e \rho^e}$$

$$\rho^e = \begin{cases} \rho_{min} & \text{if } \rho^e_{old}(B^e)^\eta \le \rho_{min} \\ \rho^e_{old}(B^e)^\eta & \text{otherwise "active elements"} \\ 1 & \text{if } \rho^e_{old}(B^e)^\eta \ge 1 \end{cases}$$

Satisfying the volume constraint

$$\rho^e = \rho^e_{old}\left(\frac{-\frac{\partial f}{\partial \rho^e}}{\lambda\frac{\partial g}{\partial \rho^e}}\right)^\eta$$

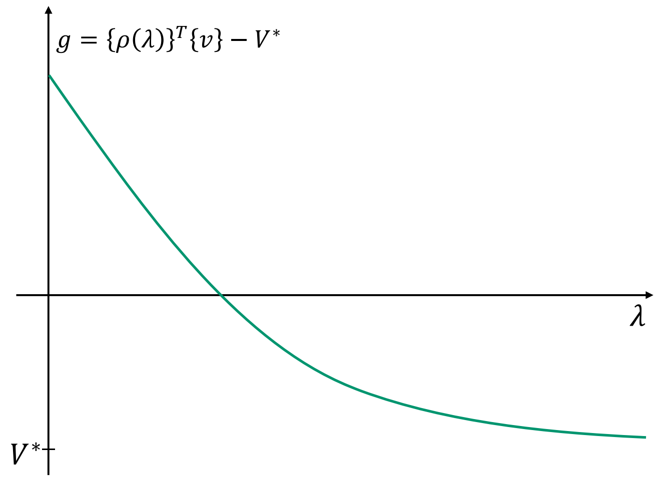
$$g = g(\lambda) = g(\{\rho(\lambda)\} = \{\rho(\lambda)\}^T\{v\} - V^* = 0$$

Bi-section scheme

Guess extremal values of λ (e.g. $\lambda_1 = 10^{-10}$ and $\lambda_2 = 10^{10}$)
While $(\lambda_2 - \lambda_1)/(\lambda_1 + \lambda_2) > \varepsilon$ (e.g. $\varepsilon = 10^{-5}$)
  $\lambda_{mid} = (\lambda_1 + \lambda_2)/2$
  if $g(\lambda_{mid}) > 0$
    $\lambda_1 = \lambda_{mid}$
  else
    $\lambda_2 = \lambda_{mid}$
  end
end while

Guess extremal values of λ (e.g. $\lambda_1 = 10^{-10}$ and $\lambda_2 = 10^{10}$)
While $(\lambda_2 - \lambda_1)/(\lambda_1 + \lambda_2) > \varepsilon$ (e.g. $\varepsilon = 10^{-5}$)
  $\lambda_{mid} = (\lambda_1 + \lambda_2)/2$
  if $g(\lambda_{mid}) > 0$
    $\lambda_1 = \lambda_{mid}$
  else
    $\lambda_2 = \lambda_{mid}$
  end
end while

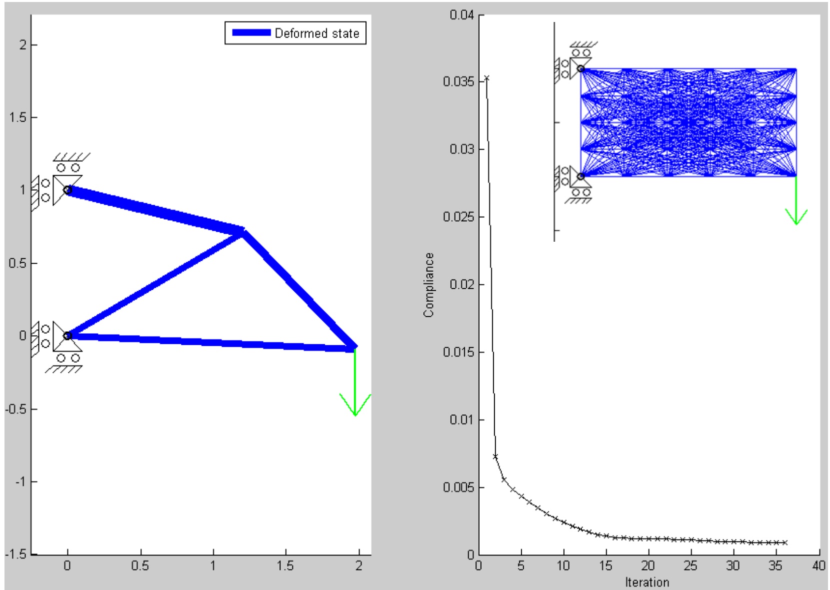
$$\rho^e = \rho^e_{old}\left(\frac{-\frac{\partial f}{\partial \rho^e}}{\lambda\frac{\partial g}{\partial \rho^e}}\right)^\eta$$



Topology optimization algorithm

Initialize $\{\rho\}$ and V^* (choose initial $\{\rho\}$ to satisfy V^*)
for iopt = 1, 2, ... max_iopt
    Set $\{\rho_{old}\} = \{\rho\}$
    Solve $[K]\{D\} = \{P\}$
    Calculate $\frac{df}{d\rho^e}$ and $\frac{dg}{d\rho^e}$
    Find $\{\rho\}$ by bi-section method (separated function)
    If $||\{\rho_{old}\} - \{\rho\}|| < \varepsilon\, ||\{\rho\}||$ break
    Plot convergence (f(iopt))
    Plot structure (e.g. with bar thicknesses proportional to relative element densities)
end

Optimized structure for full connectivity



Exercise 4.1

Implement the Optimality Criteria approach for topology optimization of truss structures modelled by small displacements and strains.

Hint 1: Call a function "bisect", that uses the bi-section method to return the value of the design variable vector $\{\rho\}$ as a function of $\{\rho_{old}\}$, V^*, $\{\frac{df}{d\rho}\}$ and $\{\frac{dg}{d\rho}\}$, i.e.

$$[\{\rho\}] = \text{bisect}(\{\rho_{old}\}, V^*, \{\tfrac{df}{d\rho}\}, \{\tfrac{dg}{d\rho}\})$$

Program this function in a separate file named 'bisect.m' for example. In Matlab, functions begin with the line

function [outdata, ..] = bisect(indata, ..)

Test the program on the example from Exercise 1.1 (Fig. 1.3 but add the two missing diagonal bars) with $\rho_{min} = 10^{-6}$, $V^* = 10$ and $P = 0.01$. *Plot the compliance as a function of the iteration number. Check that the strain energy density (e.g. $\sigma\varepsilon/\rho$) is constant for each active (i.e. not governed by the lower or upper density bound) element. (NB! remark that the element stress now should be computed as $\sigma = E\varepsilon = (\rho^e)^p E_0 \varepsilon$).*

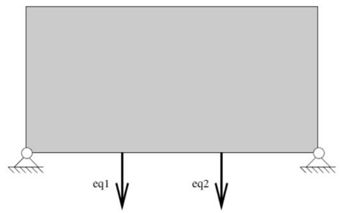
i.e. not ρ_{min} or 1 !!

Exercise 4.2

Study the effect of varying the damping parameter η? Which value should be chosen for fastest convergence? Use a bigger example than the simple 11-bar truss for this study.

***Exercise 4.3**

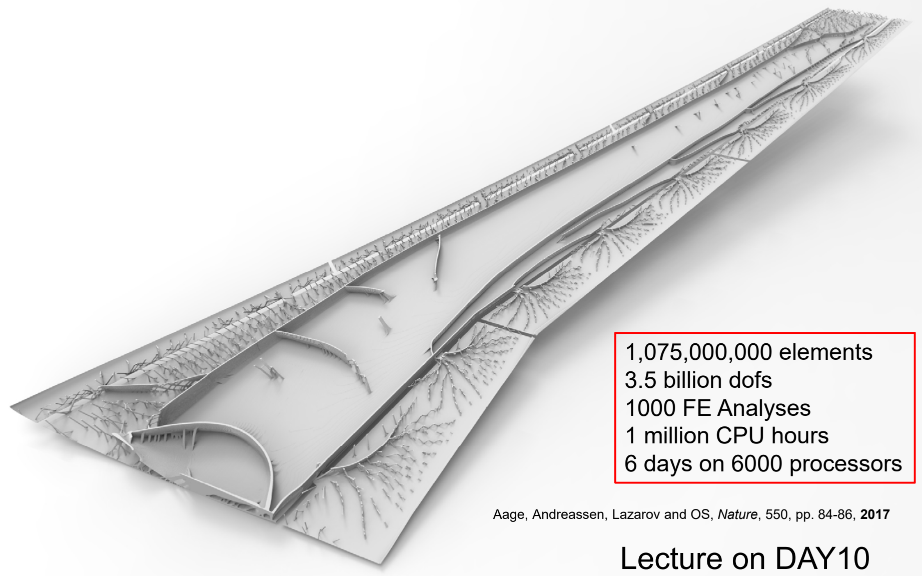
Implement and solve multi-loadcase problems (objective is weighted sum of compliances for different load cases). What is the difference in applying the loads simultaneously or independently? Hint: minimize either $f_1 = \{D_{1+2}\}^T(\{P_1\} + \{P_2\})$ where $[K]\{D_{1+2}\} = \{P_1\} + \{P_2\}$ or $f_2 = \{D_1\}^T\{P_1\} + \{D_2\}^T\{P_2\}$ where $[K]\{D_1\} = \{P_1\}$ and $[K]\{D_2\} = \{P_2\}$. In words: in the first case the two loads are applied simultaneously, in the second case the two loads are applied in two different load cases.



***Exercise 4.4**

Implement everything in three dimensions and demonstrate by an example.

Advanced Topology Optimization



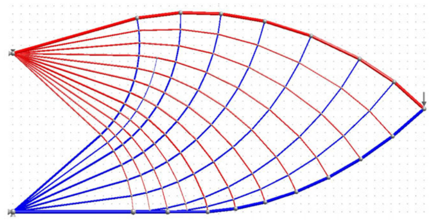
1,075,000,000 elements
3.5 billion dofs
1000 FE Analyses
1 million CPU hours
6 days on 6000 processors

Aage, Andreassen, Lazarov and OS, *Nature*, 550, pp. 84-86, **2017**

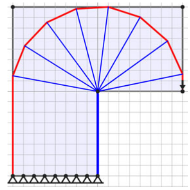
Lecture on DAY10

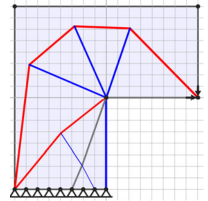
http://www.dr.dk/DR2/Danskernes+akademi/IT_teknik/Optimale_huller.htm

Alternative truss optimization codes



http://www.upct.es/~deyc/software/tto.php





https://www.layopt.com/truss/

First assignment

Formal rules

- This first assignment consists of four subassignments in total (one pr. course day).
- Each subassignment must be answered on one separate page including figures and discussions.
- Apart from the 4 pages discussed above, a cover page must include the following:
  - Group number and code.
  - Student's names and student's numbers.
  - A statement on who is responsible for what parts of the assignment and codes[1].
- One (in total) additional and separate page may be added containing answers to *-exercises related to the subassignments. Recall that one can still achieve top grades without solving any *-exercises!
- Use minimum 12 pt, single line spacing, 2cm margin left and right, 3cm margin top and bottom on your A4 paper. (NB! This document is formatted according to these rules)
- The report must be uploaded to DTU Learn as one separate pdf-file before the deadline (Tuesday September 27th at 10pm) and handed to the teacher in hard copy with signatures before the following lecture at the latest.
- One separate pdf-file containing source codes for one representative Matlab code pr. subassignment must be uploaded to DTU Learn as well.

Recommendations:

- Use the feedback sheet (see the sample outline next page) as work guidance and check-list.
- The one page answers to each subassignment may contain plots, graphs, equations and text.
- All results must be accompanied by short and precise explanations, discussions and critical interpretations. Note that these discussions and interpretations also count in the evaluation.
- Create simple, clear and readable illustrations and graphs.

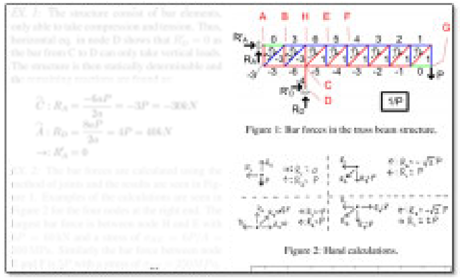
[1]From the DTU rules: *"It must be clearly specified for which sections each student has the (main) responsibility. A group project is not deemed to be individualized if the students merely state that they have contributed equally to all sections of the report or the like. If a group project does not comply with these requirements for individualization or other formal requirements, the report may be rejected and no assessment given."*

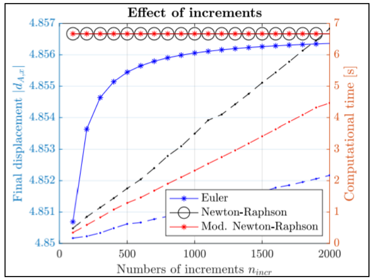
Tentative evaluation scheme for Assignment 1

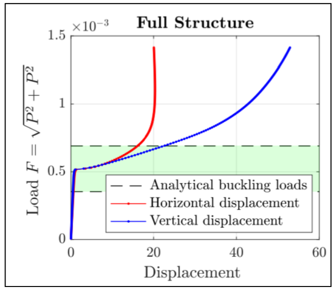
Group code:

	ok	NA	Unacceptable	Just acceptable	Average	Good	Excellent	Remarks
Report specific issues								
Layout and format								
Matlab codes								
Cover page								
Page limit								
Competition 1								
Clarity and preciseness								
Language								
Readability of graphs								
Code organization								
Subassignment 1								
Ex. 1								
Ex. 2								
Ex. 3								
Ex. 4								
Ex. 5								
Ex. 6								
Subassignment 2								
Ex. 1								
Ex. 2								
Ex. 3								
Ex. 4								
Ex. 5								
Subassignment 3								
Ex. 1								
Ex. 2								
Ex. 3								
Ex. 4								
Subassignment 4								
Ex. 1								
Ex. 2								
Ex. 3								

Examples from prior reports







Office hours before first assignment

(Check latest updates on Learn)

Person	Thursday morning (22/9)	Thursday afternoon (22/9)	Friday morning (23/9)	Friday afternoon (23/9)	Monday morning (26/9)	Monday afternoon (26/9)	Tuesday morning (27/9)	Tuesday afternoon (27/9)
OS		13-14 (404/136)				13-14 (404/136)		
Kim	10-11 (404/118)				9-10 (404/118)			
Christoffer		15-16 (404/131)					10-11 (404/131)	
Peter	9-10 (404/131)		9-10 (404/131)					
Yafeng				13-14 (404/126)		15-16 (404/126)		
Federico			11-12 (404/111)		11-12 (404/111)			

NB! Please come well preparred!

Deadline Tuesday, September 27th, 10pm!!