Truss FEM

material nonlinearities

What does this mean to you?

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Materials Behavior

Linear materials:

$$\sigma = E\,\varepsilon$$

$$\frac{\partial \sigma}{\partial \varepsilon} = E \;= E_t$$

Nonlinear materials:

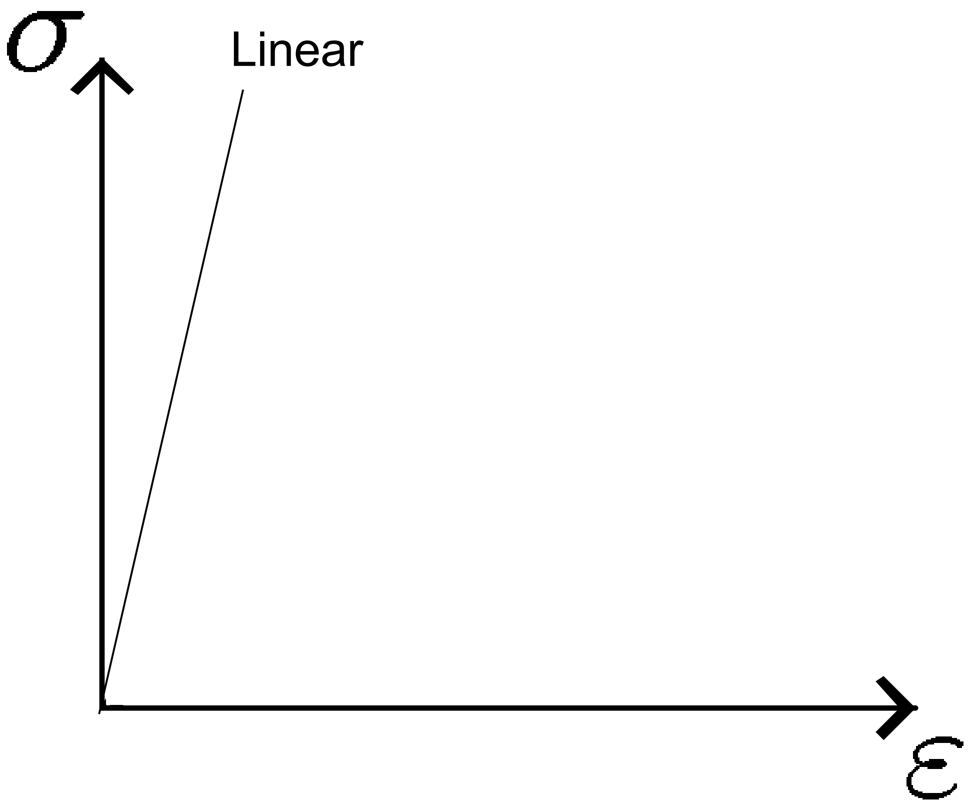
$$\frac{\partial \sigma}{\partial \varepsilon} = E_t$$

Slope of σ–ε curve

$$\Downarrow$$

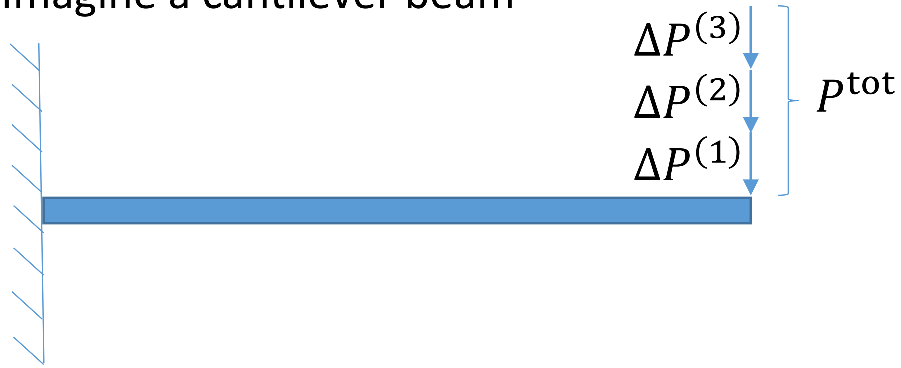
$$\partial\sigma = E_t\,\partial\varepsilon$$

(Please note: $\sigma = E\,\varepsilon$ no longer apply to non-linear materials!)

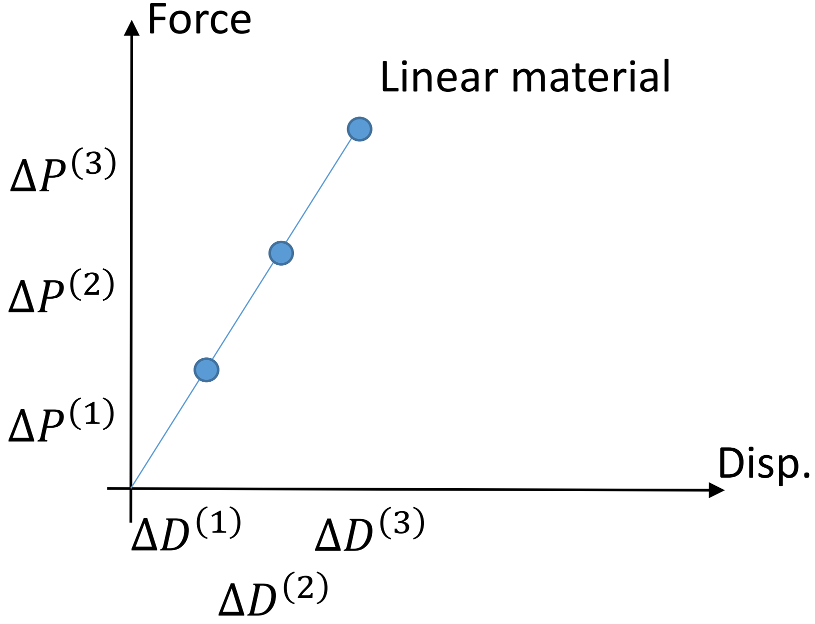


Incremental Method

Imagine a cantilever beam



- Apply full load in small increments
- Load at increment *n* : $P^n = P^{n-1} + \Delta P^n$
- Displacement at increment *n* : $D^n = D^{n-1} + \Delta D^n$
- Increment size : $\Delta P^n, \ \Delta D^n$



Incremental Procedures

- Explicit methods (No equilibrium – Euler methods)
  - Pure Euler Method
  - Euler Method with one-step correction

- Implicit methods (equilibrium w. Newton-Raphson iterations)
  - Newton-Raphson Method (NR)
  - Modified NR Method

Pure Euler Method

Recall the cantilever beam

Solution at increment *n*:

$$[K_t]^{n-1}\{\Delta D\} = \{\Delta P\}$$

Imagine you apply an external load increment

Calculate displacement increments

Remember BC's !

$$\Downarrow$$

$$\{D\}^n = \{D\}^{n-1} + \{\Delta D\}$$

Add displacement increments

Pseudo-code (Pure Euler Method)

$P^0 = 0$
$\Delta P^n = P^{final} / n_{incr}$
$D^0 = 0$

How do we get the tangent stiffness?

For load-increment $n = 1, 2, \ldots, n_{incr}$

$P^n = P^{n-1} + \Delta P^n$

$\boxed{\text{Calculate } K_t\left(D^{n-1}\right)}$

$\Delta D^n = \left(K_t\left(D^{n-1}\right)\right)^{-1} \Delta P^n$ (NB! Remember boundary conditions)

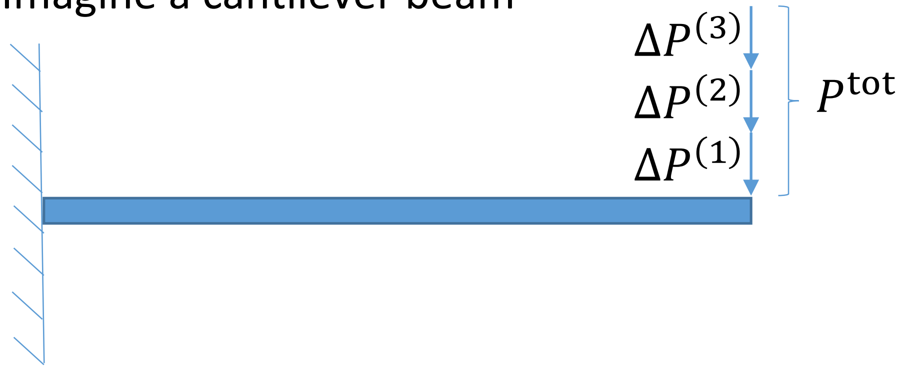
$D^n = D^{n-1} + \Delta D^n$

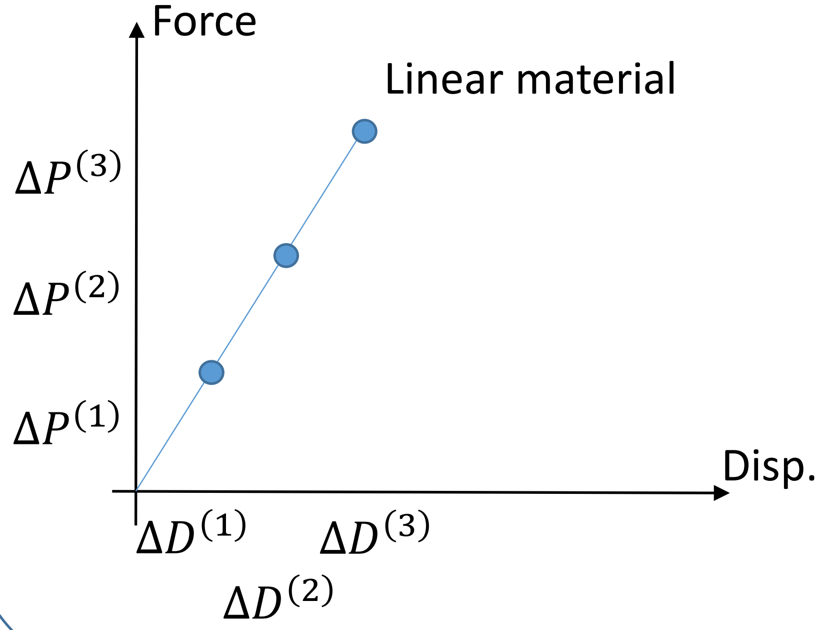
End load-increments

where *nincr* is the number of load increments.

A test case

Imagine a cantilever beam





Pseudo-code (Pure Euler Method)

$P^0 = 0$
$\Delta P^n = P^{final}/n_{incr}$
$D^0 = 0$

How do we get the tangent stiffness?

For load-increment $n = 1, 2, \ldots, n_{incr}$

$P^n = P^{n-1} + \Delta P^n$

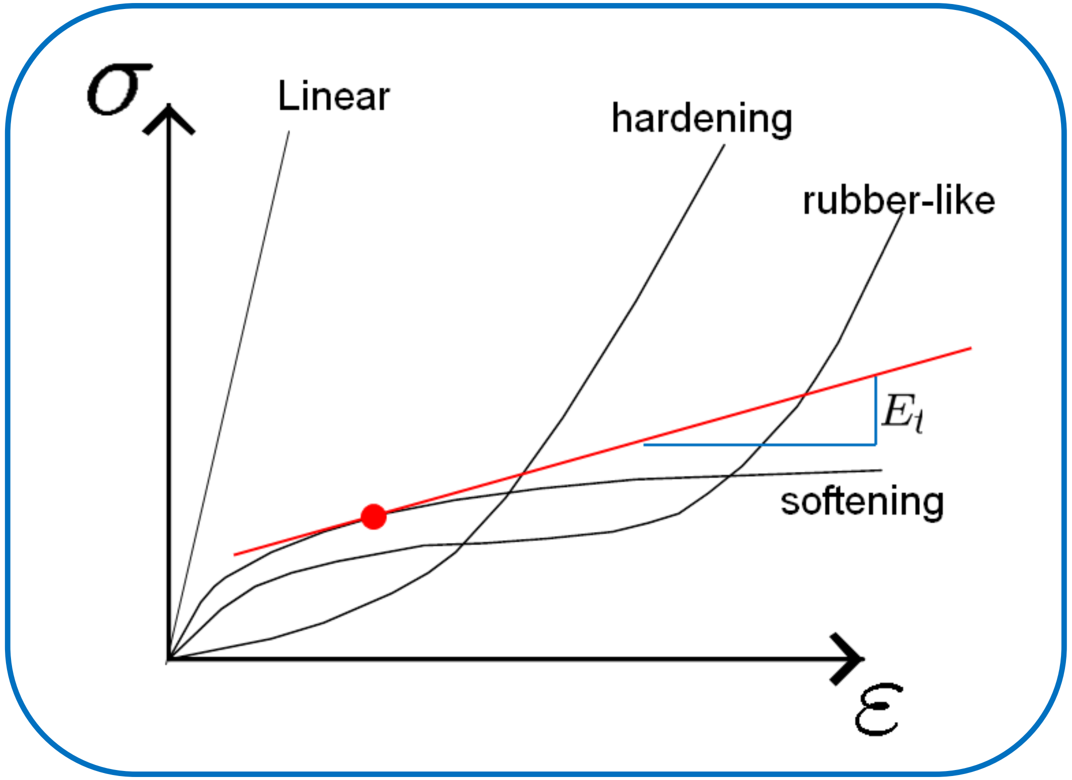
$\boxed{\text{Calculate } K_t(D^{n-1})}$

$\Delta D^n = (K_t(D^{n-1}))^{-1} \Delta P^n$ (NB! Remember boundary conditions)

$D^n = D^{n-1} + \Delta D^n$

End load-increments

where *nincr* is the number of load increments.



Tangent Stiffness Matrix

Definition:

$$[K_t] = \frac{\partial\{R\}}{\partial\{D\}} = \frac{\partial\{R_{\text{int}}\}}{\partial\{D\}}$$

(Recall that; $\{R\} = \{R_{int}\} - \{R_{ext}\}$

$= \sum_e \{B_0\}\, N^e\, L_0^e - \{P\}$ with: $N^e = A^e\, \sigma(\varepsilon)$)

Bar element & nonlinear material:

$$[K_t] = \frac{\partial\{R_{\text{int}}\}}{\partial\{D\}} = \frac{\partial}{\partial\{D\}}\Big(\sum_e \{B_0\} N^e L_0^e\Big)$$

(Recall that; $\{B_0\} = \frac{1}{L_0^2}\{-\Delta x,\ -\Delta y,\ \Delta x,\ \Delta y\}^T$)

with $\{B_0\}$ evaluated at element level.

$$= \sum_e \{B_0\}\frac{\partial N^e}{\partial\{d\}} L_0^e = \sum_e \{B_0\}\frac{\partial \sigma}{\partial\{d\}} A^e L_0^e$$

$$= \sum_e \{B_0\}\frac{\partial \sigma}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial\{d\}} A^e L_0^e = \sum_e \{B_0\}\frac{\partial \sigma}{\partial \varepsilon}\{B_0\}^T A^e L_0^e$$

Tangent Stiffness Matrix

Local tangent stiffness matrix:

$$[k_t] = \frac{\partial \sigma}{\partial \varepsilon} A^e L_0^e \{B_0\}\{B_0\}^T$$

Local stiffness matrix (for comparison):

$$[k] = E^e A^e L_0^e \{B_0\}\{B_0\}^T$$

Linear material:

$$\frac{\partial \sigma}{\partial \varepsilon} = E \qquad \Rightarrow$$ Tangent stiffness matrix = stiffness matrix

Euler Method with One-Step Correction

Solution at increment *n*:

Imagine you apply an external load increment

$$[K_t]^{n-1}\{\Delta D\} = \{\Delta P\} - \{R\}^{n-1}$$

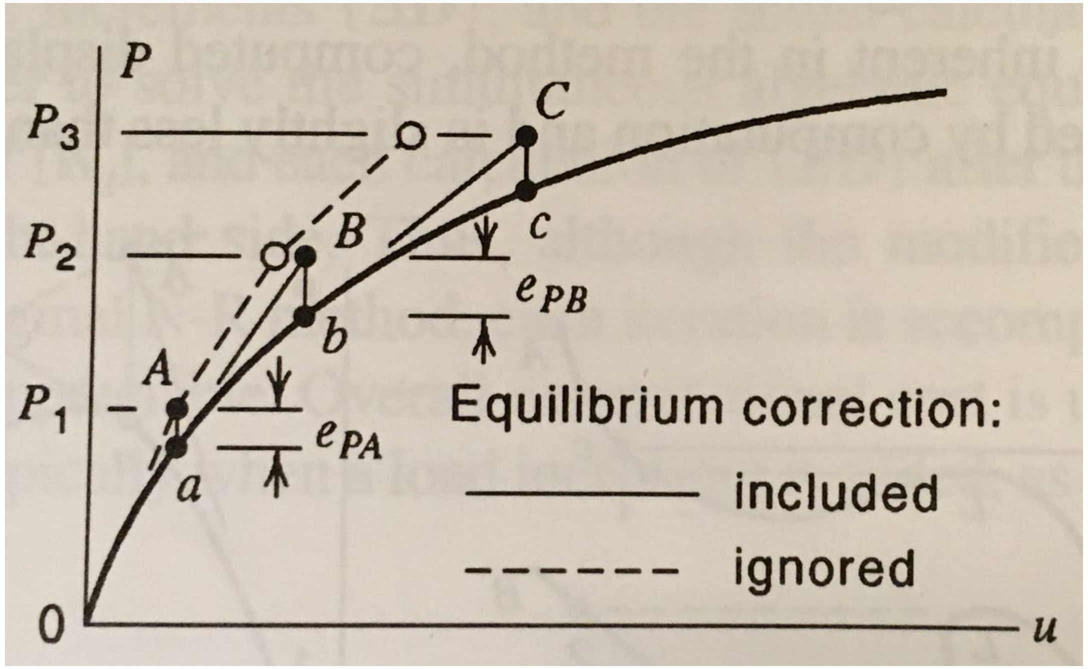
Calculate displacement increments

Subtract residual from previous step

$$\Downarrow$$

$$\{D\}^n = \{D\}^{n-1} + \{\Delta D\}$$

Add displacement increments



Cook, Fig. 17.2-5

Pseudo-code (Modified Euler Method)

$P^0 = 0$
$\Delta P^n = P^{final} / n_{incr}$
$D^0 = 0$
$R^0 = 0$

For load-increment $n = 1, 2, \ldots, n_{incr}$

$P^n = P^{n-1} + \Delta P^n$

Calculate $K_t(D^{n-1})$

$\Delta D^n = (K_t(D^{n-1}))^{-1}(\Delta P^n - R^{n-1})$ (NB! Remember boundary conditions)

$D^n = D^{n-1} + \Delta D^n$

Calculate $R^n = R^n_{int}(D^n) - P^n$

End load-increments

Newton Raphson Method

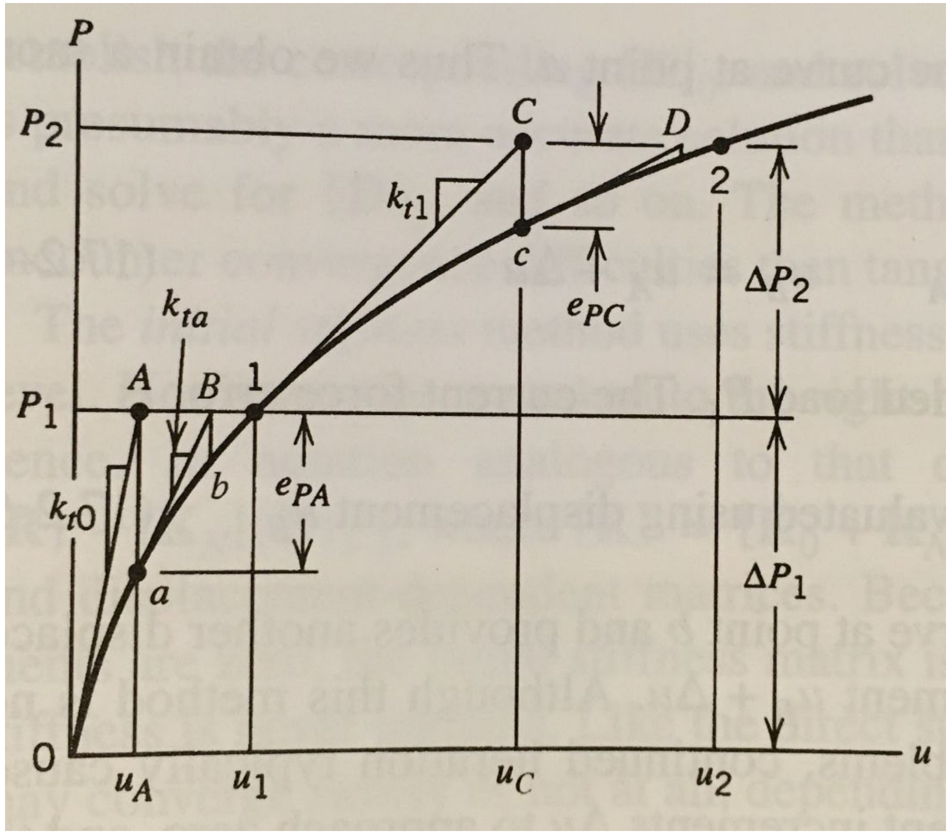
Solution at increment *n*:

$$[K_t]^{n-1}\{\Delta D\} = \{\Delta P\}$$
$$\Downarrow$$
$$\{D\}_1^n = \{D\}^{n-1} + \{\Delta D\}$$

Remember BC's!

Iteration until convergence (R=0):

$$[K_t]_i^{n-1}\{\Delta D\}_i = -\{R\}_i^n$$
$$\Downarrow$$
$$\{D\}_i^n = \{D\}_{i-1}^n + \{\Delta D\}_i$$



Cook, Fig. 17.2-2a

Pseudo-code (Newton-Raphson Method)

For load-increment $n = 1, 2, ..., n_{incr}$

$P^n = P^{n-1} + \Delta P^n$

$D_0^n = D^{n-1}$

For equilibrium iterations $i = 0, 1, ..., i_{max}$

Calculate $R_i^n = R_{int}(D_i^n) - P^n$

Stop iterations when $||R_i^n|| \leq \varepsilon_{stop}||P^n||$

Calculate $K_t(D_i^n)$ (and factorize)

Solve equilibrium equation $\Delta D_i^n = -(K_t(D_i^n))^{-1} R_i^n$ (NB! Remember BC)

Update displacements $D_{i+1}^n = D_i^n + \Delta D_i^n$

End equilibrium iterations

$D^n = D_i^n$

End load-increments

Modified NR-Method

Solution at increment *n*:

$$[K_t]^{n-1}\{\Delta D\} = \{\Delta P\}$$

$$\Downarrow$$

$$\{D\}_1^n = \{D\}^{n-1} + \{\Delta D\}$$

Iteration until convergence (R=0):

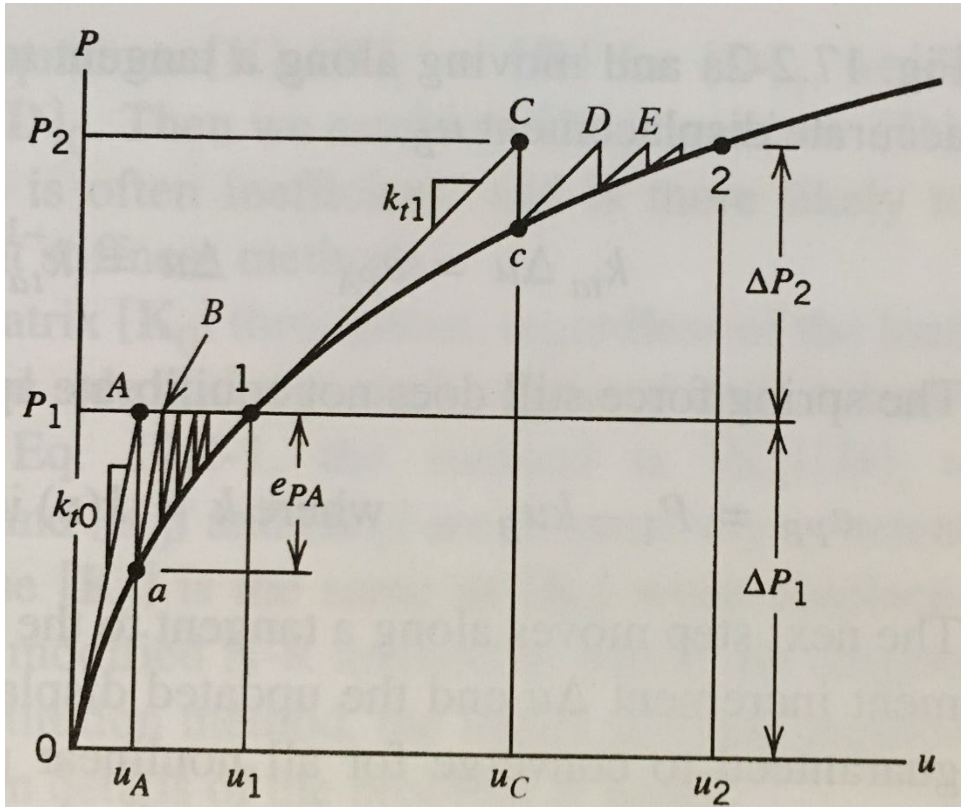
Here lies the difference?

$$\boxed{[K_t]^{n-1}}\{\Delta D\}_i = -\{R\}_i^n$$

$$\Downarrow$$

$$\{D\}_i^n = \{D\}_{i-1}^n + \{\Delta D\}_i$$

The Modified NR-Method do not update the tangent stiffness.



Cook, Fig. 17.2-2b

Pseudo-code (Modified NR-Method)

For load-increment $n = 1, 2, \ldots, n_{incr}$

$P^n = P^{n-1} + \Delta P^n$

$D_0^n = D^{n-1}$

$\boxed{\text{Calculate } K_t(D_0^n) \text{ (and factorize)}}$

Use LU-factorization in Matlab (see p. 10 in the lecture notes).

For equilibrium iterations $i = 0, 1, \ldots, i_{max}$

Calculate $R_i^n = R_{int}(D_i^n) - P^n$

Stop iterations when $||R_i^n|| \leq \varepsilon_{stop}||P^n||$

Solve equilibrium equation $\Delta D_i^n = -(K_t(D_0^n))^{-1} R_i^n$ (NB! Remember BC)

Update displacements $D_{i+1}^n = D_i^n + \Delta D_i^n$

End equilibrium iterations

$D^n = D_i^n$

End load-increments

Elastic-Plastic Analysis

Typical nonlinear material is metal plasticity

Do you think the presented methods represent metal behavior accurately?

Elastic unloading requires a model;

$\sigma_{max} = \sigma_Y$
if $|\sigma| < \sigma_{max}$ then
$\quad E_{ep} = E$ (Elastic)
else
$\quad E_{ep} = E_t$ (Plastic)
$\quad \sigma_{max} = |\sigma|$
end
(isotropic hardening)

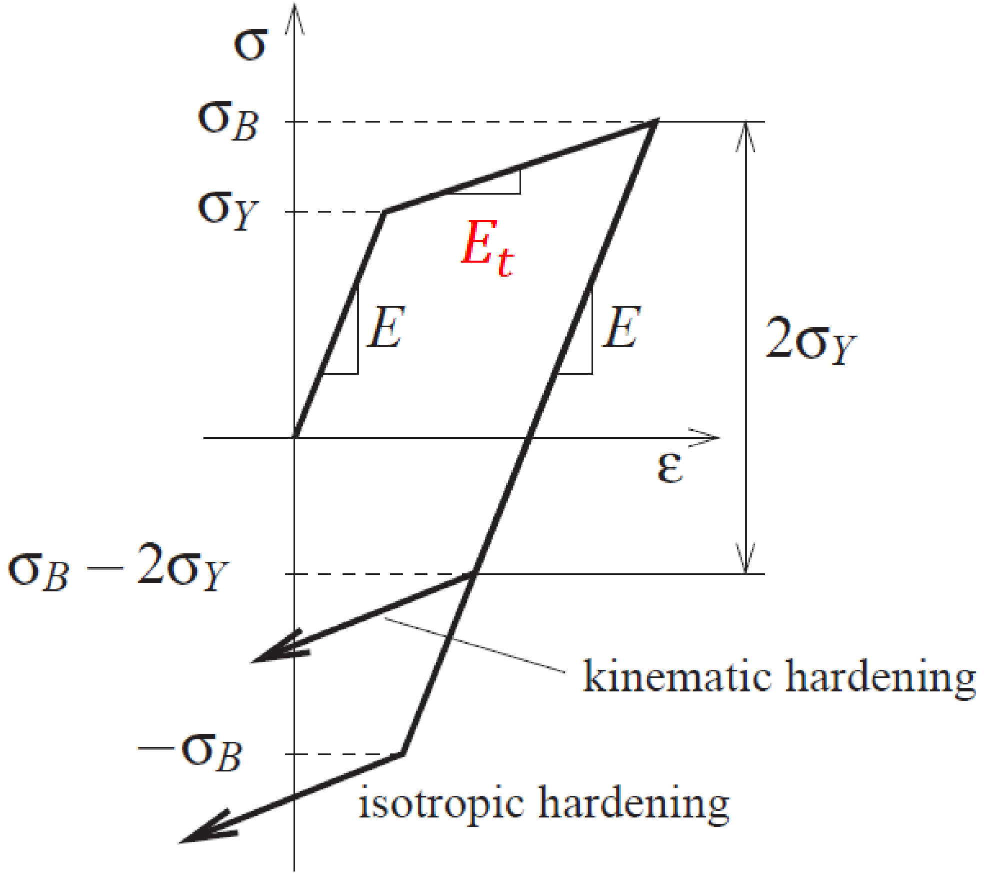


Figure 2.1: Isotropic and kinematic hardening.

Exercises

2.8 Exercises

Edof = [2*IX(e,1)-1 , 2*IX(e,1) , 2*IX(e,2)-1 , 2*IX(e,2)]

Hints:

- Before you start today's exercises, try to simplify and clean up your code from last time (for example, use the "edof=[...]" command as discussed in the lecture). This will make todays extensions easier.

- Make a new directory called "DAY2" (under the "FEM" directory) and copy example files and truss program to this directory.

- Keep backup copies of old files that work!

Rubber is a typical example of a non-linear material. The Signorini stress-strain relation that mimics rubber response can in 1D be written as

$$\sigma(\varepsilon) = c_1 \left(\lambda - \lambda^{-2}\right) + c_2 \left(1 - \lambda^{-3}\right) + c_3 \left(1 - 3\lambda + \lambda^3 - 2\lambda^{-3} + 3\lambda^{-2}\right), \tag{2.21}$$

where $\lambda = 1 + c_4\varepsilon$ is the stretch.

The tangent stiffness modulus for this material can be found by differentiation

$$E_t(\varepsilon) = \frac{d\sigma}{d\varepsilon} = c_4 \left[c_1 \left(1 + 2\lambda^{-3}\right) + 3\, c_2\, \lambda^{-4} + 3\, c_3 \left(-1 + \lambda^2 - 2\lambda^{-3} + 2\lambda^{-4}\right)\right]. \tag{2.22}$$

Note that this semi-artificial material model only works for $\lambda > 0$, i.e. $\varepsilon > -1/c_4$. If the strain gets below this value in one of your examples, you should decrease the load.

Exercise 2.1

For later check of your FE-code, *start by writing a small Matlab script that plots the force as a function of displacement for a bar of length 3 and area 2 made of the rubber-like material*. Use the material constants $c_1 = 1$, $c_2 = 50$, $c_3 = 0.1$, $c_4 = 100$ and a maximum force of 200 to get started.

Implement the Euler method for truss structures with the rubber like material law (2.21) and (2.22). *Plot the force-displacement curve of a straight truss structure (Figure 2.2) for different increment sizes and compare them with the analytical curve.* To get started, you may use the values, $A = 2$, $P^{final} = 200$, 20 load increments and therefore $\Delta P = 10$. The number of load increments (n_{incr}) and the final load (P^{final}) should be given in the input file.

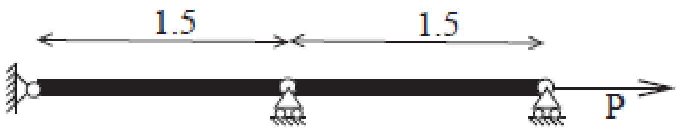


Figure 2.2: Simple 2-bar test specimen.

Exercise 2.2

Implement the "incremental with one-step equilibrium-correction" method for the 2-bar truss structure with the same non-linear material law as above. *Plot the force displacement curve for the simple 2-bar truss for different increment sizes and compare them with the analytical curve and the curves for the pure Euler method.*

Exercise 2.3

Implement an incremental scheme with NR-method equilibrium iterations and *test it on the 2-bar problem like in the previous exercises*. The maximum (NR) equilibrium iterations (e.g. i_max=100) should be given in the input file.

Exercise 2.4

Implement the modified NR-method and *test it on the 2-bar problem like in the previous exercises*. Remember that you can factorize the stiffness matrix once pr. load step and then save time (for larger problems) by doing simple forward-backward-substitution for each iteration.

***Exercise 2.5**

Implement the (pure) incremental method for elastic-plastic analysis (with loading and unloading) and *compare results for isotropic and kinematic hardening models. For illustrative purposes make the elastic elements blue and the plastic ones red.*