From Discrete to Continuous Motion Planning

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Abstract In this paper, we demonstrate an equivalence between a large class of discrete motion planning problems, and piano mover's problems, which we refer to as "continuous motion planning problems". We first prove that under some assumptions, discrete motion planning in d dimensions can be transformed into continuous motion planning in $2d+1$ dimensions. Then we prove a more specific, similar equivalence for which the number of dimensions of the configuration space does not necessarily have to be increased. We study two simple cases where this theorem applies, and show that it can lead to original and efficient motion planning algorithms, which could probably be applied to a wide range of multi-contact planning problems. We apply this equivalence to a simulation of legged locomotion planning for a hexapod robot.

1 Introduction

After decades of research in motion planning, we now have plenty of tools to solve the quintessential piano mover's problem. Several sampling-based algorithms are known to be very efficient in practice, such as PRM [13], RRT [17], etc. Some methods have been improved in ordrer to get better convergence properties [12]. There exist also several libraries that contain state-of-the-art implementations of these algorithms and can be used in almost any configuration space, as long as the user defines the validity tests (i.e. collision checks) via the API. Examples include OMPL [18] and KineoWorks(TM). There are also several algorithms for path optimization ([7], [20]), and algorithms that take advantage of parallel architectures to reduce the computation costs [19].

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On the other hand, some classes of motion planning problems have been much less studied, and among them hybrid motion planning problems that have a continuous component but whose output must be a finite sequence of configurations (we call them "discrete motion planning problems"). A typical example is multi-contact planning [2] where the desired output is a finite sequence of contact configurations, two consecutive configurations differing by exactly one contact. In this paper we consider a general class of such discrete motion planning problems, and prove that they can be converted into continuous motion planning problems that are in essence nothing else but the piano mover's problem. More precisely, we first prove that discrete motion planning problems in d-dimensional configuration spaces can be converted into continuous motion planning problems in $(2d+1)$-dimensional configuration spaces. Then, we prove a more specific equivalence where an increase of dimensionality is not compulsory, and study two basic examples where it can be applied and leads to original motion planning algorithms. We then show that this equivalence can be advantageously used in real applications, and in particular apply it in simulation to plan the walking motion of a hexapod robot.

2 Notations and Definitions

We only consider metric configuration spaces, and denote by $dist()$ their distance functions. Let CS be a metric configuration space. Here are some important notations that we will extensively use:

- For $X \subset CS$ we denote by $\mathring{X}$ the interior of X, by $\bar{X}$ its closure, and by X^c its complement.
- For $X \subset CS$ and $s \in CS$ we pose $d(s,X) = \inf\{dist(s,s')|s' \in X\}$.
- While $\mathscr{P}(CS)$ denotes the set of subsets of CS, we denote by $\mathscr{P}_K(CS)$ the set of non-empty compact subsets of CS, and by $RegOp^*(CS)$ the set of non-empty bounded regular open subsets of CS, i.e. the non-empty bounded open subsets that are equal to the interior of their closure.
- We denote by $d_H()$ the Hausdorff distance on $\mathscr{P}(CS)$. It turns $\mathscr{P}_K(CS)$ into a metric space in its own right.
- For $s \in CS$ and $r > 0$, we denote by $\bar{\mathscr{S}}(s,r)$ the closed sphere of center s and radius r, and by $\mathring{\mathscr{S}}(s,r)$ the open sphere of center s and radius r.

We non-ambiguously use the same notations for any metric space other than CS.

Definition 1 (uniformly followable functions).
We say that a function $X : CS_1 \to RegOp^*(CS_2)$ is "uniformly Γ-followable" for $\Gamma > 0$ if it verifies the three following properties:

1. There exists $0 < \gamma < \Gamma$ such that $\forall s \in CS_1$, $\forall (s_a,s_b) \in X(s)^2$, $dist(s_a,s_b) < \gamma$.
2. The function $s \mapsto \bar{X}(s)$ is continuous on CS_1 w.r.t. the Hausdorff metric on $\mathscr{P}_K(CS_2)$, and the function $(s,s') \mapsto d(s,X(s')^c)$ is continuous on CS_1^2 (on CS_1^2 we use the distance $dist_\infty((s_\alpha,s'_\alpha),(s_\beta,s'_\beta)) = \max(dist(s_\alpha,s_\beta),dist(s'_\alpha,s'_\beta)))$.

3. For all $\eta > 0$, there exists $\lambda > 0$ such that $\forall (s,s') \in CS_1^2, d_H(\bar{X}(s),\bar{X}(s')) < \lambda$ implies that $\bar{X}(s) \cap \bar{X}(s')$ is non-empty and $d_H(\bar{X}(s),\bar{X}(s) \cap \bar{X}(s')) < \eta$.

Common functions that map elements of a metric configuration space to geometric shapes in another configuration space are often uniformly Γ-followable. This property can be seen as a form of uniform continuity.

3 Problem Definition and Related Work

We first define the class of continuous motion planning problems. As we previously mentioned, we are interested in problems that are in essence nothing but the piano mover's problem, i.e. problems for which classical sampling-based algorithms (e.g. PRM, RRT) readily apply. In particular, we do not let the possibility of adding non-holonomic constraints, or allowing only curvature-bounded paths, etc. So, let CS be a metric configuration space and let $dist()$ be its distance function. The property $C()$ defines the notion of collision-freeness, and the free space $FS = \{s \in CS | C(s)\}$ is assumed to be an open subset of CS.

Continuous motion planning problems are defined as follows:

Definition 2 (continuous motion planning problems).
INPUT: CS, $C()$, $s_i \in FS$ and $s_f \in FS$.
OBJECTIVE: find a continuous path $(s(t))_{t\in[0,1]}$ such that $\forall t \in [0,1]$, $C(s(t))$ (we call "valid" such a continuous path), and such that $s(0) = s_i$ and $s(1) = s_f$.

We will also consider slight variants of these problems where then initial and final configurations are not fixed but must simply belong to some sets.

Now, for discrete motion planning, instead of continuous paths the outputs are finite sequences of configurations: the motion is abrupt between a configuration and the next one. Configurations must still be collision-free, but there is an additional relation $R()$ that defines a relationship between consecutive configurations. Such discrete motion planning problems arise in particular when simplified models are used to solve hybrid motion planning problems, i.e. problems that have both continuous and discrete aspects. For example, in footstep planning for humanoid robots, the motion of the robot is continuous, but the sequence of contacts with the ground is discrete. To make the problem easier we can use a simplified model in which the feasibility of a sequence of footsteps is not directly related to the actual continuous motion of the robot. In that case, $R()$ is be the relation that defines which next steps are feasible.

Here is our general definition of discrete motion planning problems:

Definition 3 (discrete motion planning problems).
INPUT: CS, $C()$, $R()$, $s_i \in FS$ and $s_f \in FS$.
OBJECTIVE: find a finite sequence of configurations $(s_1, s_2, \ldots, s_n)$ such that $\forall k \in \{1,\ldots,n-1\}$, we have $C(s_k)$, $C(s_{k+1})$, and $R(s_k, s_{k+1})$ (we call "valid" such a finite sequence), and such that $s_0 = s_i$ and $s_n = s_f$.

The purpose of this paper is to make a bridge between these two classes, from discrete to continuous motion planning problems.

There is a lot of related work, but the goal is often to transform a continuous motion planning problem into a discrete one. The simple fact that polygonal chains can approximate arbitrarily closely any continuous path can already be seen as a simple equivalence result that has for consequence the probabilistic completeness of most sampling-based algorithms for the piano mover's problem. A similar result for more complex problems is the small-time local controllability property [16] which allows to solve problems that require continuous solutions by looking for discrete sequences of small motions, which can for example belong to a finite collection of motion primitives [3]. Another bridge from continuous to discrete problems concerns collision checks: it has been shown that checking the validity of configurations along a continuous path can be done in a sound way without necessarily having to perform an infinite number of checks [6]. In any case, it is compulsory to make problems discrete if one wants to solve them with computers.

The objective of the present paper, i.e. transforming discrete problems into continuous ones, is much less common, but we can mention [1] where a reduction property shows that for some class of manipulation problems, the existence of a solution path with discrete "grasp" and "release" events is equivalent to the existence of a path where the grasp is continuously modified. This kind of equivalence can be especially useful when the hybrid nature of a problem makes it difficult to be solved. For example, in the problem of footstep planning for humanoid robots, the relationship between a footprint and the next one is continuous, but it is a discrete sequence of footprints that must be found. It is not easy to design an algorithm that would deal with both continuous and discrete aspects of the problem, and that is why the standard approach is to make the problem completely discrete by deciding in advance on a finite set of possible steps ([15], [4]). Some orthogonal approaches try to make the problem completely continuous. For example in [11] and [5], the robot first slides its feet on the ground. In both approaches it is shown that the continuous motions can always be transformed into finite sequences of steps, so the approaches are sound. Unfortunately, they are not complete: if the robot has no other choice but to step over some obstacles a solution can never be found. In [22], a sound and complete approach is proposed for a specific 2D walking robot whose discrete sequences of footsteps are produced from continuous paths. It is also used in [21] for more complex footstep planning. Basically, the result shown in [22] is a particular case of a more general theorem that we state in the present paper, and which can probably have many applications other than footstep planning.

Several algorithms have already been proposed to solve hybrid motion planning problems such as multi-modal or more specifically multi-contact motion planning, and it would be interesting to compare their efficiency to that of the algorithms based on the equivalence proposed in the present paper. In multi-modal motion planning, a finite or discrete number of *modes* correspond each to a submanifold of the configuration space, and the planner must choose a discrete sequence of modes as well as continuous single-mode paths through them. A general algorithm with good convergence properties has been introduced in [9]. It combines a graph search algorithm to find the sequence of modes, together with probabilistic roadmaps to plan the single-mode paths.

4 From Discrete Motion Planning in d Dimensions to Continuous Motion Planning in $2d+1$ Dimensions

In this section we demonstrate a quite general theorem that shows a strong relationship between discrete and continuous motion planning problems as we have defined them in the previous section. Our goal is to prove that for an arbitrary discrete motion planning problem (with just a few assumptions), it is possible to define an equivalent continuous motion planning problem. So, let us consider a discrete motion planning problem in an d-dimensional metric configuration space CS, defined by the collision-freeness property $C()$, the relation $R()$ and initial and final configurations s_i and s_f. We make only 3 not so restrictive but important assumptions related to the regularity of $R()$:

1. R is symmetric: $R(s_a, s_b) \Rightarrow R(s_b, s_a)$.
2. The sets $R(s) = \{s' \in CS | R(s,s')\}$ are path-connected: $\forall s \in CS$, $\forall (s_a, s_b) \in R(s)^2$, there exists a continuous path inside $R(s)$ from s_a to s_b.
3. $s \mapsto R(s)$ is a uniformly Γ-followable function from CS to $RegOp^*(CS)$, for some $\Gamma > 0$.

Under these assumptions, we prove that we can always define an equivalent continuous motion planning problem. To do so, the key is to define a new configuration space $\widetilde{CS}$ and a new notion of collision-freeness $\widetilde{C}()$. $\widetilde{CS}$ is simply $CS^2 \times (0, \Gamma)$, i.e. a metric space of dimension $2d+1$. The definition of $\widetilde{C}()$ is a bit more complex.

Definition 4 **($\widetilde{C}()$).**
$\widetilde{C}(s, s', \rho)$ is verified if and only if the two following properties are verified:

1. The set $A(s, s', \rho) = R(s) \cap \mathring{\mathscr{S}}(s', \rho)$ has a non-empty intersection with the free space: $\exists s_\alpha \in A(s, s', \rho) | C(s_\alpha)$. Note that the function $(s, s', \rho) \mapsto \bar{A}(s, s', \rho)$ is continuous.
2. The set $B(s, s', \rho) = \{s_b \in CS | R(s_b) \supset A(s, s', \rho)\}$ has also a non-empty intersection with the free space. Note that we always have $s \in B(s, s', \rho)$.

Obviously, verifying the property $\widetilde{C}()$ might be much more difficult than verifying $C()$, but we will discuss this later. We first demonstrate the following theorem:

Theorem 1. *The discrete motion planning problems in CS defined by $C()$ and $R()$ are equivalent to the continuous motion planning problems in $\widetilde{CS}$ defined by $\widetilde{C}()$: for any initial and final configurations $s_i \in CS$ and $s_f \in CS$, there exists a valid discrete sequence of configurations from s_i to s_f if and only if there exists a continuous path $(v(t))_{t \in [0,1]} \in \widetilde{CS}^{[0,1]}$ such that $\forall t \in [0,1], \widetilde{C}(v(t))$, and $s_i \in B(v(0))$, and $s_f \in A(v(1))$ or $s_f \in B(v(1))$.*

Section 4.1 and Section 4.2 are dedicated to the proof of each implication of this equivalence, but first we state the following lemma (the proof is straightforward and not given here):

Lemma 1. *Let $Y : [0,1] \to RegOp^*(CS)$ be a function such that $t \mapsto \bar{Y}(t)$ is continuous on $[0,1]$, and let $\kappa : CS \times [0,1] \to \mathbb{R}$ be a continuous function. Then $t \mapsto \sup\{\kappa(s,t) | s \in Y(t)\}$ is continuous on $[0,1]$.*

4.1 From a valid discrete sequence in CS to a valid continuous path in $\widetilde{CS}$

Theorem 2. *If there exists a valid sequence $(s_1 = s_i, s_2, \ldots, s_n = s_f)$ in CS, then there exists a valid continuous path $(v(t))_{t\in[0,1]}$ in $\widetilde{CS}$ such that $v(0)$ is of the form (s_1, s_2, ρ), and $v(1)$ of the form (s_{n-1}, s_n, ρ) or (s_n, s_{n-1}, ρ).*

Proof. We prove this implication by induction on n, the size of the valid sequence. For $n = 2$ the result is obvious since for any $0 < \rho < \Gamma$ we have $s_2 \in A(s_1, s_2, \rho)$, and, $A(s_1, s_2, \rho)$ being a subset of $R(s_1)$, we have also $s_1 \in B(s_1, s_2, \rho)$. Hence, the stationary path such that $\forall t \in [0,1], v(t) = (s_1, s_2, \rho)$, is valid.

Let us now assume that the result is true for any sequence of size n, and consider a valid sequence of size $n+1$: $(s_1 = s_i, s_1, \ldots, s_{n+1} = s_f)$. Let $(v(t))_{t\in[0,1]} \in \widetilde{CS}^{[0,1]}$ be a valid path such that $v(0) = (s_1, s_2, \rho_0)$, and $v(1) = (s_{n-1}, s_n, \rho_1)$ or $v(1) = (s_n, s_{n-1}, \rho_1)$.

First, we suppose that $v(1) = (s_{n-1}, s_n, \rho_1)$.

We have $s_{n-1} \in R(s_n)$ and $s_{n+1} \in R(s_n)$. Besides, $R(s_n)$ is path-connected so there exists a continuous path $\mu : [0,1] \to CS$ from s_{n-1} to s_{n+1} inside $R(s_n)$. For any $t \in [0,1]$, $R(\mu(t))$ is an open set that contains s_n. Since $s \mapsto R(s)$ is uniformly Γ-followable, $t \mapsto d(s_n, R(\mu(t)^c)$ is continuous on $[0,1]$. Besides, we always have $d(s_n, R(\mu(t)^c) > 0$. Thus, we deduce that there here exists $0 < \xi_{min} < \Gamma$ such that $\forall t \in [0,1]$, $R(\mu(t)) \supset \mathring{\mathscr{S}}(s_n, \xi_{min})$. First, we move continuously from (s_{n-1}, s_n, ρ_1) to $(s_{n-1}, s_n, \xi_{min})$ (this is a valid path). Then, we follow the path μ to move continuously from $(s_{n-1}, s_n, \xi_{min})$ to $(s_{n+1}, s_n, \xi_{min})$. Along this path any state $(\mu(t), s_n, \xi_{min})$ is such that $A(\mu(t), s_n, \xi_{min}) = \mathring{\mathscr{S}}(s_n, \xi_{min}) \subset R(s_{n-1}) \cap R(s_{n+1})$, and thus we have $s_n \in A(\mu(t), s_n, \xi_{min})$, $s_{n-1} \in B(\mu(t), s_n, \xi_{min})$, and $s_{n+1} \in B(\mu(t), s_n, \xi_{min})$. As a result, the path is valid. We deduce that appending these two paths to $(v(t))_{t\in[0,1]}$ gives us a valid continuous path from (s_1, s_2, ρ_0) to $(s_{n+1}, s_n, \xi_{min})$. This reasoning is illustrated in Fig. 1.

We now suppose that $v(1) = (s_n, s_{n-1}, \rho_1)$.

For $\xi_{max} > 0$ close enough to Γ, we have $\forall s \in R(s_n), A(s_n, s, \xi_{max}) = R(s_n)$. First, we move continuously from (s_n, s_{n-1}, ρ_1) to $(s_n, s_{n-1}, \xi_{max})$ (this is a valid path). Then, we follow a path inside $R(s_n)$ to go from $(s_n, s_{n-1}, \xi_{max})$ to $(s_n, s_{n+1}, \xi_{max})$. Along this path, any element (s_n, s, ξ_{max}) is such that $A(s_n, s, \xi_{max}) = R(s_n)$, which contains both s_{n-1} and s_{n+1}, and such that $s_n \in B(s_n, s, \xi_{max})$. Therefore the path is

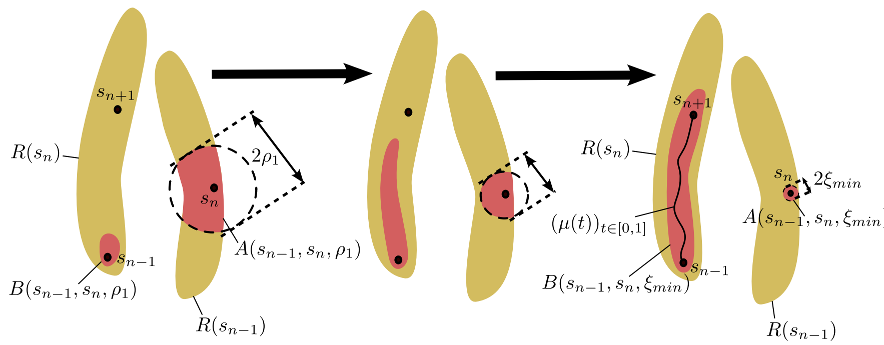


Fig. 1 We illustrate how to go continuously from $v(1) = (s_{n-1}, s_n, \rho_1)$ to $(s_{n+1}, s_n, \xi_{min})$ with a valid path. When decreasing ρ, the size of $B(s_{n-1}, s_n, \rho)$ increases, and when ρ tends to zero, $B(s_{n-1}, s_n, \rho)$ "converges" towards $R(s_n)$. For some value ξ_{min}, we not only have $s_{n+1} \in B(s_{n-1}, s_n, \xi_{min})$, we have $s_{n+1} \in B(\mu(t), s_n, \xi_{min})$ for all configurations $\mu(t)$ along the path from s_{n-1} to s_{n+1}. As a result, the continuous path $(v'(t))$ from $v(1)$ to $(s_{n+1}, s_n, \xi_{min})$ is such that at all time, $A(v'(t))$ contains s_n, which is collision-free, and $B(v'(t))$ contains either s_{n-1} or s_{n+1}, which are both collision-free, and thus $\widetilde{C}(v'(t))$ is verified. As a consequence, the path is valid.

valid (cf. Fig. 2), and appending this path and the previous one to $(v(t))_{t\in[0,1]}$ gives us a valid continuous path from (s_1, s_2, ρ_0) to $(s_n, s_{n+1}, \xi_{min})$. That concludes the demonstration of Theorem 2. $\square$

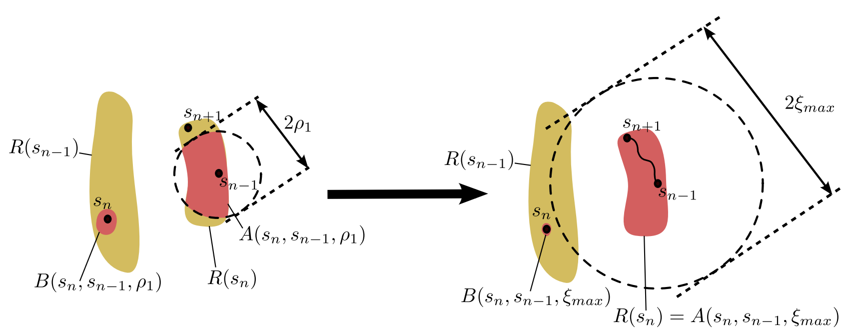


Fig. 2 We illustrate how to go continuously from $v(1) = (s_n, s_{n-1}, \rho_1)$ to $(s_n, s_{n+1}, \xi_{max})$ with a valid path. When increasing ρ, the size of $A(s_n, s_{n-1}, \rho)$ increases, and at some point it becomes equal to $R(s_n)$. For $\rho = \xi_{max}$, we even have: $\forall s \in R(s_n)$, $A(s_n, s, \rho) = R(s_n)$. It follows that we can first go from $v(1)$ to $(s_n, s_{n-1}, \xi_{max})$ with a valid path, and then from $(s_n, s_{n-1}, \xi_{max})$ to $(s_n, s_{n+1}, \xi_{max})$, again with a valid path.

4.2 From a valid continuous path in $\widetilde{CS}$ to a valid discrete sequence in CS

Theorem 3. *If there exists a continuous path $(v(t))_{t\in[0,1]} \in \widetilde{CS}^{[0,1]}$ such that $\forall t \in [0,1]$, $\widetilde{C}(v(t))$, and $s_i \in B(v(0))$, and $s_f \in A(v(1))$ or $s_f \in B(v(1))$, then there exists in CS a valid finite sequence of configurations from s_i to s_f.*

Proof. Let us write: $v(t) = (s_\alpha(t), s_\beta(t), \rho(t))$.

For any collision-free configuration $s \in CS$ and $t \in [0,1]$ we define:

$$d_{obs}(s,t) = \min(\min\{dist(s,s')|s' \in FS^c\}, \min\{dist(s,s')|s' \in \mathring{\mathscr{S}}(s_\beta(t),\rho(t))^c\})$$

It can be verified that $(s,t) \mapsto d_{obs}(s,t)$ is a continuous function on $CS \times [0,1]$. We know that $t \mapsto \bar{R}(s_\alpha(t))$ is continuous on $[0,1]$. Using Lemma 1, it follows that $t \mapsto \delta_{obs}(t) = \sup\{d_{obs}(s,t)|s \in \bar{R}(s_\alpha(t))\}$ is also continuous on $[0,1]$. Since $\forall t \in [0,1]$, $\widetilde{C}(v(t))$, we can deduce that $\forall t \in [0,1]$, $\delta_{obs}(t) > 0$. As a result there exists $\Delta_{obs} > 0$ such that $\forall t \in [0,1]$, $\delta_{obs}(t) > \Delta_{obs}$.

Since $t \mapsto v(t)$ is uniformly continuous on $[0,1]$, and $s \mapsto R(s)$ is uniformly Γ-followable, there exists $\eta_1 > 0$ such that:

$$\forall (t,t') \in [0,1], |t-t'| < \eta_1 \Rightarrow d_H(\bar{R}(s_\alpha(t)), \bar{R}(s_\alpha(t)) \cap \bar{R}(s_\alpha(t'))) < \tfrac{1}{8}\Delta_{obs},$$

with $\bar{R}(s_\alpha(t)) \cap \bar{R}(s_\alpha(t'))$ non-empty.

There also exists $\eta_2 > 0$ such that $\forall (t,t') \in [0,1], |t-t'| < \eta_2 \Rightarrow dist(s_\beta(t), s_\beta(t')) < \frac{1}{8}\Delta_{obs}$, and $\forall (t,t') \in [0,1], |t-t'| < \eta_2 \Rightarrow |\rho(t) - \rho(t')| < \frac{1}{8}\Delta_{obs}$. Let N be a positive integer such that $1/N < \min(\eta_1, \eta_2)$.

We now consider $v(0/N), v(1/N), \ldots, v(N/N)$ and try to construct a valid sequence $(s_1, s_2, \ldots, s_{2N+1})$ such that $s_1 \in B(v(0))$ and $\forall i \in \{1,\ldots,N\}$, $s_{2i} \in A(v(i/N))$ and $s_{2i+1} \in B(v(i/N))$. We pose $s_1 = s_i \in B(v(0))$. Now, let us assume that we have been able to construct such a sequence up to s_{2k+1} with $0 \leq k < N$. We try to construct s_{2k+2} and s_{2k+3}. Let us write: $A\left(v\left(\frac{k}{N}\right)\right) = A(s_\alpha, s_\beta, \rho)$ and $A\left(v\left(\frac{k+1}{N}\right)\right) = A(s'_\alpha, s'_\beta, \rho')$. We have:

$$dist(s_\beta, s'_\beta) < \tfrac{1}{8}\Delta_{obs} \text{ and } |\rho - \rho'| < \tfrac{1}{8}\Delta_{obs} \text{ and } d_H(\bar{R}(s_\alpha), \bar{R}(s_\alpha) \cap \bar{R}(s'_\alpha)) < \tfrac{1}{8}\Delta_{obs}$$

There exists s collision-free in $R(s_\alpha)$ such that $d_{obs}(s, k/N) > \frac{1}{2}\Delta_{obs} > 0$. s belongs to $R(s_\alpha) \cap \mathring{\mathscr{S}}(s_\beta, \rho) = A(v(k/N))$. We have $dist(s, s_\beta) < \rho - \frac{1}{2}\Delta_{obs}$ and:

$$dist(s, s'_\beta) < \rho - \tfrac{1}{2}\Delta_{obs} + dist(s_\beta, s'_\beta) < \rho - \tfrac{1}{2}\Delta_{obs} + \tfrac{1}{8}\Delta_{obs} < \rho - |\rho - \rho'| - \tfrac{1}{4}\Delta_{obs}$$
$$dist(s, s'_\beta) < \rho' - \tfrac{1}{4}\Delta_{obs}$$

Besides, $d_H(\bar{R}(s_\alpha), \bar{R}(s_\alpha) \cap \bar{R}(s'_\alpha)) < \frac{1}{8}\Delta_{obs}$. Therefore, there exists $s_\cap \in \bar{R}(s_\alpha) \cap \bar{R}(s'_\alpha)$ such that $dist(s_\cap, s) < \frac{1}{8}\Delta_{obs}$. Since $\frac{1}{8}\Delta_{obs} < \frac{1}{2}\Delta_{obs}$, $s_\cap$ is necessarily collision-free. We also have $dist(s_\cap, s'_\beta) < \rho' - \frac{1}{8}\Delta_{obs}$ and $dist(s_\cap, s_\beta) < \rho - \frac{3}{8}\Delta_{obs}$. We deduce that $s_\cap$ is a collision-free configuration that belongs to $A(v(k/N)) \cap A(v((k+1)/N))$. Since we have $s_{2k+1} \in B(v(k/N))$, it follows that $s_\cap \in R(s_{2k+1})$. Therefore, we can set $s_{2k+2} = s_{\cap}$. For s_{2k+3}, we can take any collision-free configuration inside $B(v((k+1)/N))$.

By iteration, we can obtain a valid sequence $(s_1 = s_i, s_2, \ldots, s_{2N+1})$. However, at this point s_{2N+1} is not necessarily equal to s_f. But $s_{2N+1} \in B(v(1))$ and $s_{2N} \in A(v(1))$. If $s_f \in B(v(1))$, we have $R(s_{2N}, s_f)$ and thus we can set $s_{2N+1} = s_f$, while if $s_f \in A(v(1))$ we have $R(s_{2N+1}, s_f)$, and thus we can add a new configuration $s_{2N+2} = s_f$. The sequence constructed is valid (we illustrate its construction in Fig. 3), and this concludes the demonstration of Theorem 3 as well as the demonstration of Theorem 1. □

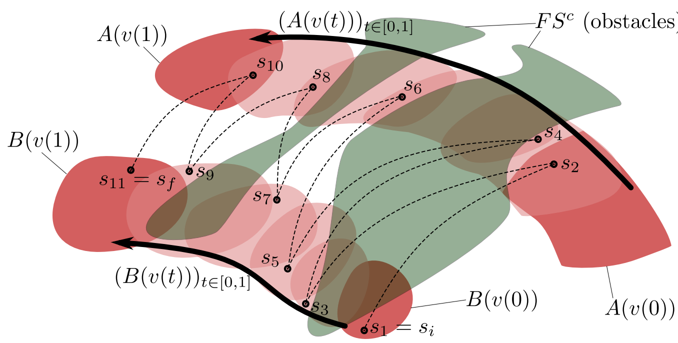


Fig. 3 From a valid continuous path to a valid sequence.

What shows Theorem 1 is that a large range of d-dimensional discrete motion planning problems can be converted into equivalent $(2d+1)$-dimensional continuous motion planning problems. This result is interesting in itself, and might have a wide scope of potential applications, but because of the curse of dimensionality which particularly affects sampling-based motion planning algorithms, using such a conversion to actually solve discrete motion planning problems might be convenient but not be very efficient. In the next section, we show that with different assumptions on the relation $R()$, it is possible to obtain a similar equivalence without necessarily having to increase the dimensionality of the configuration space.

5 More Specific Reductions With Less Increase of Dimensionality

In this section, we relax the previous assumptions on $R()$. However, we assume that $R()$ is reflexive, and that there exists another metric configuration space Ω and a function $f : \Omega \to RegOp^*(CS)$ such that the four following properties are verified:

1. f is uniformly Γ-followable for some $\Gamma > 0$.
2. $\forall \varphi \in \Omega$, $f(\varphi)$ is such that $\forall (s,s') \in f(\varphi)^2$, $R(s,s')$.
3. $\forall (s,s') \in CS^2$ such that $R(s,s')$, there exists $\varphi \in \Omega$ such that $s \in f(\varphi)$ and $s' \in f(\varphi)$.
4. $\forall (s,s',s'') \in CS^3$ such that $R(s,s')$ and $R(s',s'')$, and $\forall \varphi_0 \in \Omega$ such that $s \in f(\varphi_0)$ and $s' \in f(\varphi_0)$, there exists a continuous path from φ_0 to a configuration φ_1 verifying $s'' \in f(\varphi_1)$, such that for any configuration φ along this path, we have $s' \in f(\varphi)$.

We define yet another notion of collision-freeness $C_\Omega()$:

Definition 5 **($C_\Omega()$).** $\varphi \in \Omega$ verifies $C_\Omega(\varphi)$ if and only if the intersection between $f(\varphi)$ and the free space is non-empty, i.e. $\exists s \in f(\varphi)$ such that $C(s)$.

We have the following equivalence:

Theorem 4. *There exists a valid finite sequence from s_i to s_f in CS if and only if there exists a continuous path $(\chi(t))_{t\in[0,1]} \in \Omega^{[0,1]}$ such that $s_i \in f(\chi(0))$, $s_f \in f(\chi(1))$, and $\forall t \in [0,1]$, $C_\Omega(\chi(t))$.*

The next two sections are dedicated to the proof of each implication of this equivalence, while in sections 5.3 and 5.4, we study two examples of discrete motion planning problems where Theorem 4 applies.

5.1 From a valid discrete sequence in CS to a valid continuous path in Ω

Theorem 5. *If there exists a valid sequence $(s_1 = s_i, s_2, \ldots, s_n = s_f)$, then there exists a valid continuous path $(\chi(t))_{t\in[0,1]}$ such that $s_1 \in f(\chi(0))$, $s_2 \in f(\chi(0))$, $s_{n-1} \in f(\chi(1))$ and $s_n \in f(\chi(1))$.*

Proof. We prove this implication by induction on n, the size of the valid sequence. For $n = 2$, we have $R(s_i, s_f)$, and there exists $\varphi \in \Omega$ such that $s_i \in f(\varphi)$ and $s_f \in f(\varphi)$. The stationary path such that $\forall t \in [0,1], \chi(t) = \varphi$, is valid.

Let us now assume that the result is true for any sequence of size $n \geq 2$, and consider a valid sequence of size $n+1$: $(s_1 = s_i, s_2, \ldots, s_{n+1} = s_f)$. Let $(\chi(t))_{t\in[0,1]} \in \Omega^{[0,1]}$ be a valid path such that $s_0 \in f(\chi(0))$, $s_1 \in f(\chi(0))$, $s_{n-1} \in f(\chi(1))$ and $s_n \in f(\chi(1))$. We have $R(s_{n-1}, s_n)$ and $R(s_n, s_{n+1})$, so there exists a continuous path $(\vartheta(t))_{t\in[0,1]} \in \Omega^{[0,1]}$ from $\chi(1)$ to a configuration $\varphi_f \in \Omega$ verifying $s_{n+1} \in f(\varphi_f)$, such that for any configuration ϑ along this path, we have $s_n \in f(\vartheta)$, and thus $C_\Omega(\vartheta)$. Appending this path to the path $(\chi(t))_{t\in[0,1]}$ gives us a valid continuous path, and this conludes the demonstration of Theorem 5. $\square$

5.2 From a valid continuous path in Ω to a valid discrete sequence in CS

Theorem 6. *If there exists a valid continuous path* $(\chi(t))_{t\in[0,1]} \in \Omega^{[0,1]}$ *with* $s_i \in \chi(0)$ *and* $s_f \in \chi(1)$*, then there exists a valid sequence* $(s_1 = s_i, s_2, \dots, s_n = s_f)$*.*

Proof. The demonstration is a bit similar to the one of Theorem 3, and we give here only a sketch of the proof. Using the fact that f is uniformly Γ-followable, we can show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}^*$ such that the sequence $(\chi(0/N), \chi(1/N), \dots, \chi(N/N))$ verifies the following property: $\forall k \in \{0, \dots, N-1\}$, for any configuration $s \in f(\chi(k/N))$, the sphere $\mathring{\mathscr{S}}(s, \varepsilon)$ intersects the non-empty set $f(\chi(k/N)) \cap f(\chi((k+1)/N))$. For ε small enough, we can show that $\forall t \in [0,1]$, there exists $s_t \in f(\chi(t))$ collision-free and such that the sphere $\mathring{\mathscr{S}}(s_t, \varepsilon)$ is entirely inside FS. We thus deduce that all the sets $f(\chi(k/N)) \cap f(\chi((k+1)/N))$ have a non-empty intersection with FS. Using the property that two elements s, s' of the same set $f(\chi(t))$ are always such that $R(s, s')$, we can construct a valid sequence $(s_1, s_2, \dots, s_{N+2})$ such that $s_1 = s_i$, $s_{N+2} = s_f$, and $\forall k \in \{2, \dots, N+1\}$, we have $s_k \in f(\chi((k-2)/N)) \cap f(\chi((k-1)/N)$. Such a construction is illustrated in Fig. 4, and it concludes the demonstration of Theorem 6. $\square$

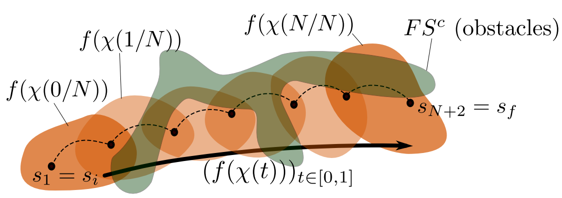


Fig. 4 From a valid continuous path $(\chi(t))_{t\in[0,1]} \in \Omega^{[0,1]}$ to a valid finite sequence of configurations in CS.

5.3 Flea motion planning

First, let us consider the simple example of flea motion planning, which has been introduced in [21] to illustrate the method used in [22] and [21] to convert footstep planning into classical continuous motion planning. Compared to the version presented in [21], we make some small modifications in the definition of the problem so as to fit the premises of our theorems (which could also be slightly modified in order to directly apply to the version of [21]).

The flea is a point in a 2D environment, and thus we use $CS = \mathbb{R}^2$ as the configuration space. There are obstacles in this 2D environment such that the free space $FS = \{s \in \mathbb{R}^2 | C(s)\}$ is an open set. The flea can make jumps in any direction and of any length strictly less than $l_{max} > 0$. The goal is to find a sequence of jumps from a location (x_A, y_A) to a location (x_B, y_B) while avoiding the obstacles. This problem clearly fits our definition of discrete motion planning problem, with, for

$(s,s') \in (\mathbb{R}^2)^2$, $R(s,s') \Leftrightarrow dist(s,s') < l_{max}$. This relation $R()$ is reflexive. Let us consider the function $f : \mathbb{R}^2 \to RegOp^*(\mathbb{R}^2)$ such that $f(s) = \mathring{\mathscr{S}}(s, \frac{l_{max}}{2})$. It can be verified that f is uniformly $2l_{max}$-followable. Besides, the other properties required by Theorem 4 are also verified:

- $\forall s \in \mathbb{R}^2$, any two points s', s'' in $f(s)$ are such that $dist(s',s'') < l_{max}$, and therefore we have $R(s',s'')$.
- For any two points s, s' verifying $R(s,s')$, there exists an open disk of radius $\frac{l_{max}}{2}$ containing both points, and thus there exists $s'' \in \mathbb{R}^2$ such that $s \in f(s'')$ and $s' \in f(s'')$.
- As explained in Fig. 5 the fourth property required for Theorem 4 to apply is also verified.

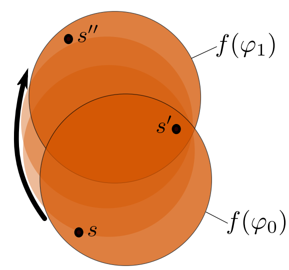


Fig. 5 For any three points s, s', s'' verifying $R(s,s')$ and $R(s',s'')$, and any $\varphi_0 \in \mathbb{R}^2$ such that $s \in f(\varphi_0)$ and $s' \in f(\varphi_0)$, there exists a continuous path from φ_0 to a configuration φ_1 verifying $s'' \in f(\varphi_1)$, and such that any configuration φ along this path verifies $s' \in f(\varphi)$.

As a consequence of these properties, we can apply Theorem 4. This means that trying to solve the flea motion planning problem is equivalent to trying to find a valid continuous path for the disk of radius $\frac{l_{max}}{2}$. It turns out that this equivalence gives a very efficient algorithm to solve the flea motion planning problem. Indeed, the new notion of collision-freeness for the disk is the following one: a configuration of the disk is collision-free if and only if there exists a point inside the disk that is outside the obstacles. This new notion of collision-freeness is called “weakly collision-freeness” in [21]. To check this property, we can apply to the obstacles a morphological operation of erosion by an open sphere of radius $\frac{l_{max}}{2}$ (see [23]), and the collision-freeness of the disk becomes equivalent to the classical collision-freeness of the center of the disk in the environment with the eroded obstacles. So, once the eroded obstacles are obtained, we can use any classical sampling-based algorithm to find a short continuous collision-free path for the disk. To actually convert this continuous path to a finite sequence of jumps, we can apply the greedy approach already used in [21] which consists in repeatedly trying to jump from the current disk $f(\varphi(t))$ to a disk $f(\varphi(t'))$ with t' as large as possible and obtained by dichotomy.

Let us add a few comments on the problem of flea motion planning. Firstly, the flea motion planning problem can be extended to $\mathbb{R}^d$ for $d > 2$ (the $R(s)$ sets become d-dimensional spheres), and Theorem 4 still applies. Secondly, the conditions of Theorem 1 are also verified. Therefore, we could have used Theorem 1 to convert flea motion planning into a continuous motion planning problem in a configuration space with $2 \times 2 + 1 = 5$ dimensions. But, because with flea motion planning the relation $R()$ is geometrically simple, it was possible to use a similar equivalence to convert it into a 2-dimensional continuous motion planning problem. A question of prime importance is raised by this remark: for a given discrete motion planning problem, how can we simplify or approximate the relation R so as to obtain an equivalent continuous motion planning problem in a configuration space with as few dimensions as possible? We show an example of such simplification in the next section.

5.4 A variant of the flea motion planning problem

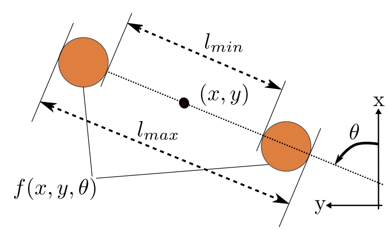
Let us first consider a variant of the flea motion planning problem where the flea cannot make jumps smaller than some fixed length. The relation $R()$ becomes: $R(s,s') \Leftrightarrow l_{min} < dist(s,s') < l_{max}$. With this variant, $R()$ is not reflexive and Theorem 4 cannot be applied. In order not to have to apply Theorem 1 and do the continuous motion planning in a configuration space of dimension 5, we modify a bit the relation $R()$. We pose:

$$R(s,s') \Leftrightarrow l_{min} < dist(s,s') < l_{max} \vee dist(s,s') < l_{max} - l_{min}$$

It still seems difficult to apply Theorem 4 with a new configuration space Ω of dimension 2 like we just did for flea motion planning, and we leave it as an open question. However, we show that we can do it with a space of dimension 3.

Let us consider the function $f : SE(2) \rightarrow RegOp^{*}(\mathbb{R}^2)$ as described in Fig. 6. With this function, the properties required by Theorem 4 are verified (we do not

Fig. 6 $f(x,y,\theta)$ is the union of two open disk of radius $\frac{l_{max}-l_{min}}{2}$, the first one of center $(x,y) + \frac{l_{max}-l_{min}}{4}(\cos\theta, \sin\theta)$, and the second one of center $(x,y) - \frac{l_{max}-l_{min}}{4}(\cos\theta, \sin\theta)$.



demonstrate it here), and so the theorem applies. This means that trying to find a sequence of jumps for this new variant of the flea motion planning problem is equivalent to trying to find a continuous collision-free path in $SE(2)$. The same greedy method as for flea motion planning can be applied. So, we have just seen that a slight modification of $R()$ enabled us to use Theorem 4 and obtain an equivalent continuous motion planning problem in a configuration space with 3 dimensions rather than 5 with Theorem 1. This can be interesting, but depending on the problem, such a simplification might not make sense, especially if like here it increases the motion capabilities. In general, it is better to look for simplifications that reduce the motion capabilities, leading to conservative approaches. Another remark of importance is that short paths in the continuous space are not necessarily converted into short sequences of jumps, even with the greedy approach. In practice, it seems more likely to be true with the standard flea motion planning problem. It would be interesting to investigate under which circumstances it is always possible to convert short continuous paths into short finite sequences, or more precisely to try to transfer bounds of sub-optimality from the continuous paths to the resulting finite sequences, but this is out of the scope of this paper.

6 Applications

The technique used in [22] and [21] for footstep planning for humanoid robots can be seen as an application of Theorem 4. In the present paper, we show that it is easy to extend the method in order to plan the walking motion of a hexapod robot (cf. Fig. 7). Our objective is to make the hexapod walk on uneven terrain with non-gaited locomotion planning (which is typically computationally costly). The uneven terrain is described by a heightmap, i.e. a function $z = F(x,y)$. The heightmap can be used to set the height of the contact positions, and we ignore the contact orientations, so we use $(\mathbb{R}^2)^6$ as the configuration space (it is easy to define a heuristic that sets a unique whole-body configuration from the 6 contact positions; in particular, we require the robot main body to remain horizontal). Here is how we define the relation $C()$: from the heightmap we infer what locations are allowed for individual contacts, and for a configuration in $(\mathbb{R}^2)^6$, we require our heuristic to lead to a valid whole-body configuration that does not collide with the heightmap.

We try to apply Theorem 4 with the configuration space $\Omega = SE(2)$. To do so, we simplify the walking capabilities of the hexapod. For a configuration (x,y,θ), we define $f(x,y,\theta)$ as the set of configurations in $(\mathbb{R}^2)^6$ such that each contact belongs to an open disk, as shown in Fig. 7 (it is important that the disks are disjoints). With this assumption, we first replace $C()$ by heuristic checks: for a configuration $s \in f(x,y,\theta)$, $C(s)$ is verified if the contacts are safe (i.e. the heightmap is almost flat around their locations), if the maximum height difference between two contacts with the ground is less than some threshold, and if the maximum height of the heightmap in the "robot zone" (i.e. the convex hull of the contacts) is not much higher than the height of the contacts. Here is how we simplify the walking capabilities of the hexapod: we consider transitions where all the 6 legs are moved at the same time, and we say that a transition from $s \in (\mathbb{R}^2)^6$ to $s' \in (\mathbb{R}^2)^6$ is allowed if and only if there exists $(x,y,\theta) \in SE(2)$ such that $s \in f(x,y,\theta)$ and $s' \in f(x,y,\theta)$. With this restriction, we can verify that the conditions of Theorem 4 apply (only the fourth property is difficult to verify), and thus we can use the equivalence to convert our problem of locomotion planning into a continuous motion planning in $SE(2)$ (we use the library OMPL and the algorithm RRT-Connect [14] to perform the motion planning). Once the conversion of a continuous path is done, we obtain a finite sequence of transitions for which the 6 foot locations are changed at each transition. It is not difficult to convert it into a sequence of feasible transitions where at most 3 feet are moved at the same time (but sometimes 2, or just 1).

This original technique for legged locomotion planning is convenient and fast: in the example described in Fig. 7 where the hexapod must go across an uneven and challenging terrain, the whole planning (continuous planning *and* two-stage conversion into a discrete sequence of steps) was done in 57ms on an Intel(R) Core(TM) i7 1.60GHz CPU. We cannot readily use this method to solve planning problems as complex as the ones considered in [8], but it is a good compromise between gaited methods and more complex approaches such as [8]. It would be interesting to try to make an advantageous use of our method in advanced software architecturesf or multi-contact motion planning, such as the ones presented in [10] or [24].

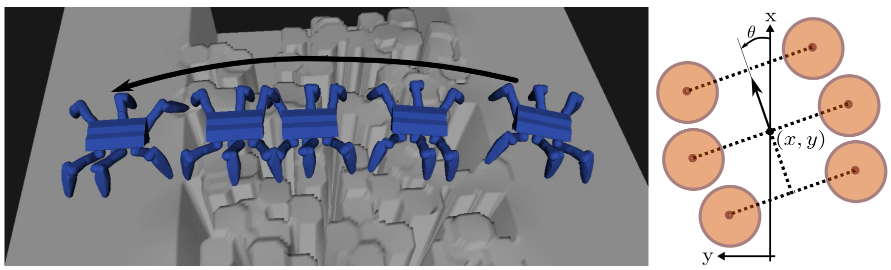


Fig. 7 *On the left*: the motion of the hexapod accross this challenging terrain was planned in 57ms. *On the right*: the six open disks that constrain the configurations and steps of the hexapod (each leg must have its contact with the ground within its assigned disk).

7 Conclusion

In this paper we have proved two new equivalence results between discrete and continuous motion planning. They can be used to convert discrete problems into continuous ones that are similar to the piano mover's problem, and thus enable the application of standard motion planning algorithms to a new class of problems. We have shown that it leads to original and efficient techniques for legged locomotion planning, and expect various other types of applications such as for example regrasp planning. In future work, it would be interesting to study more precisely the complexity of the algorithms made possible by our approach, and in particular the complexity of the new collision checks, an issue that we did not address in the present paper. Another question of importance is whether a similar equivalence could be obtained with motion planning problems with kinodynamic constraints, which is not obvious at all at this stage.

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