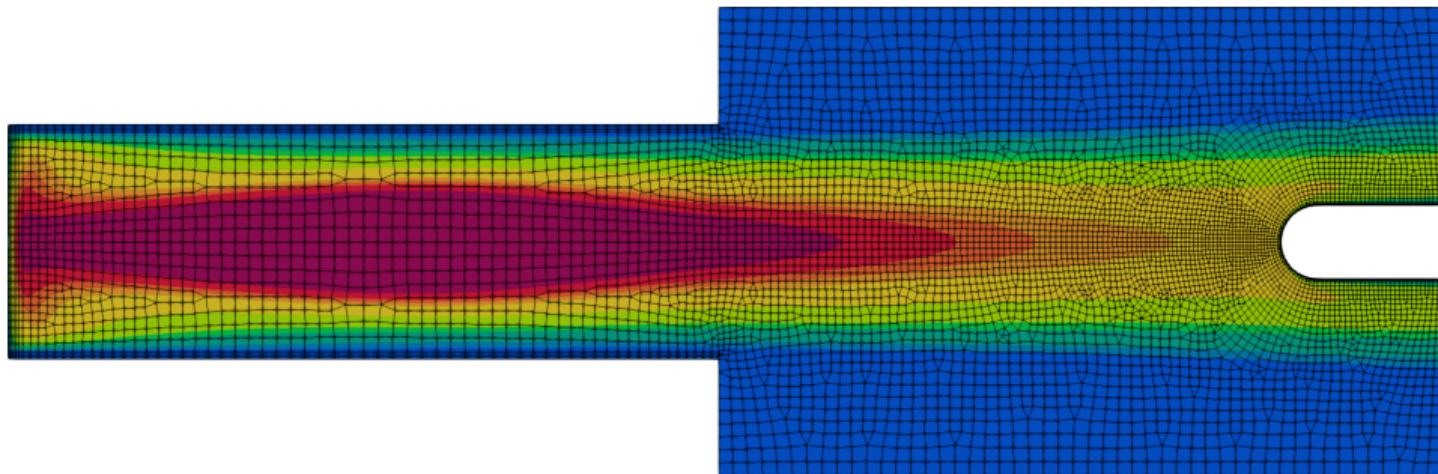


# A Hybridized Discontinuous Galerkin Solver for Inductively Coupled Plasma

Nicolas Corthouts



# Introduction

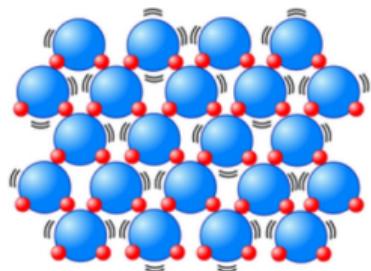
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## Etats de la matière: eau à pression atmosphérique

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# Etats de la matière: eau à pression atmosphérique

Solide

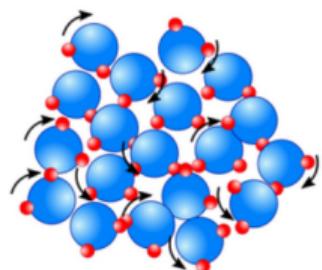
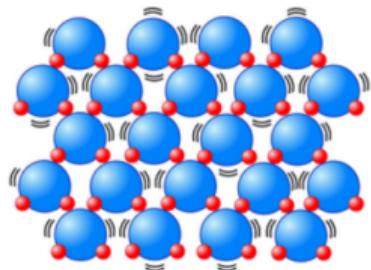


# Etats de la matière: eau à pression atmosphérique

Solide

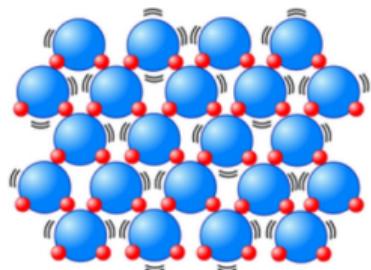


Liquide

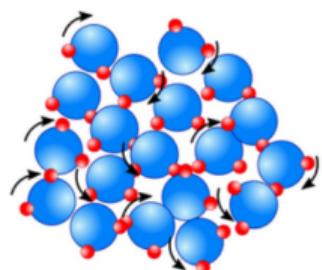


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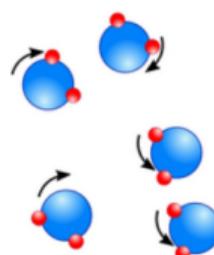
Solide



Liquide



Gaz



# Etats de la matière: eau à pression atmosphérique

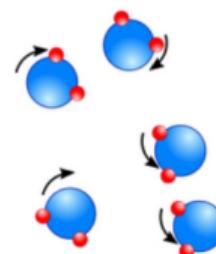
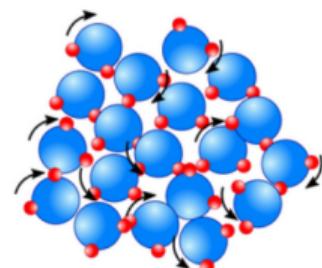
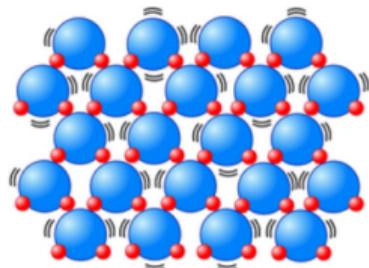
Solide



Liquide

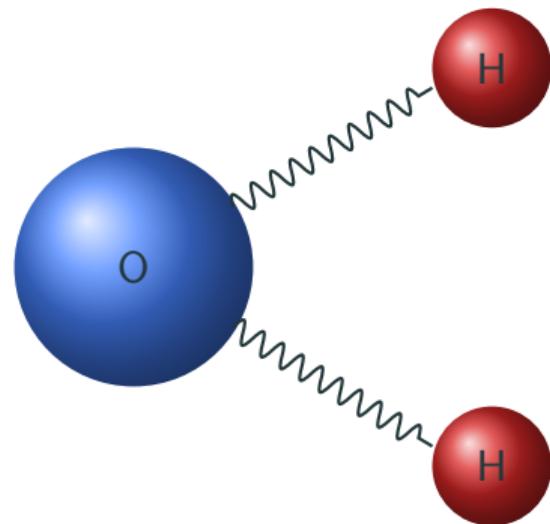


Gaz

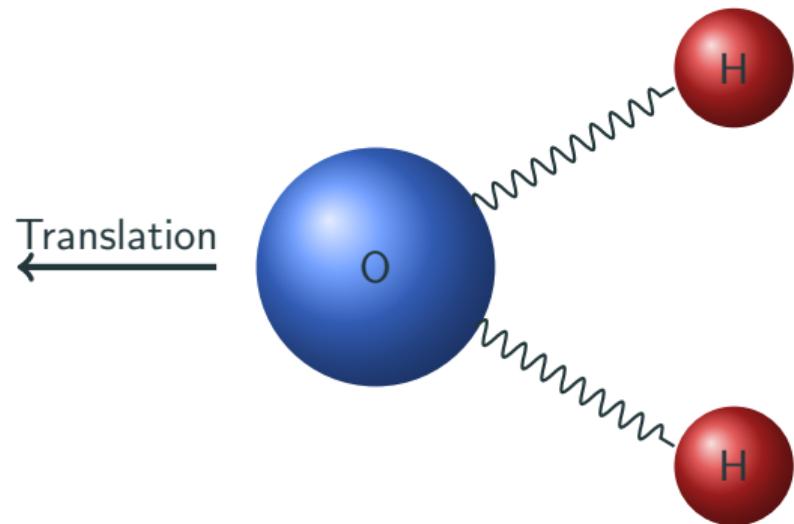


Que se passe-t-il si on chauffe encore plus?

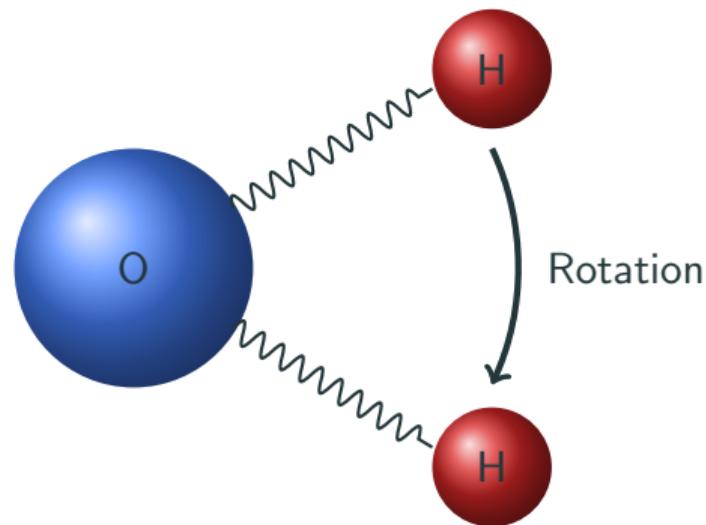
## Translation, rotation et vibration



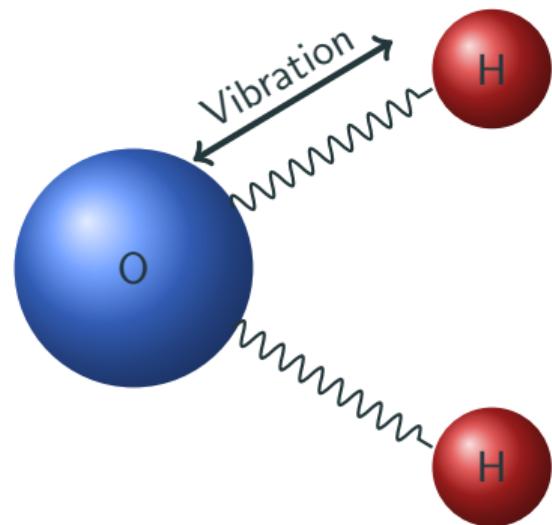
## Translation, rotation et vibration



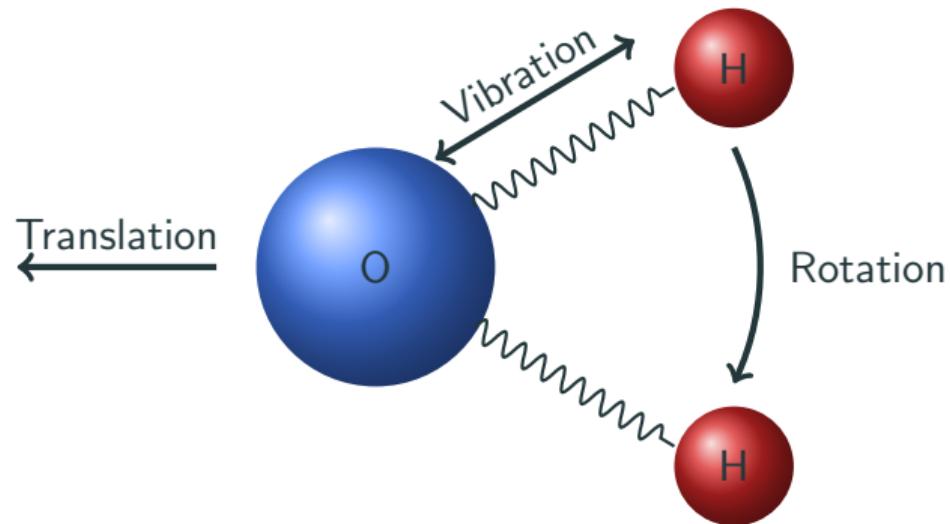
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## Translation, rotation et vibration



## Translation, rotation et vibration

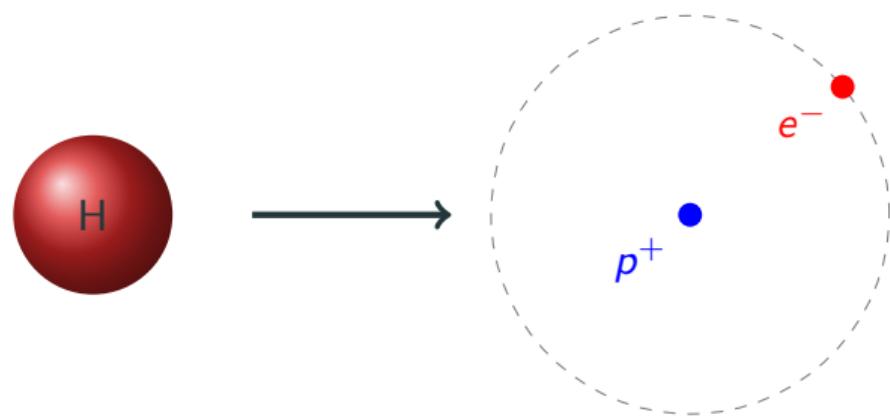


# Excitation et ionization

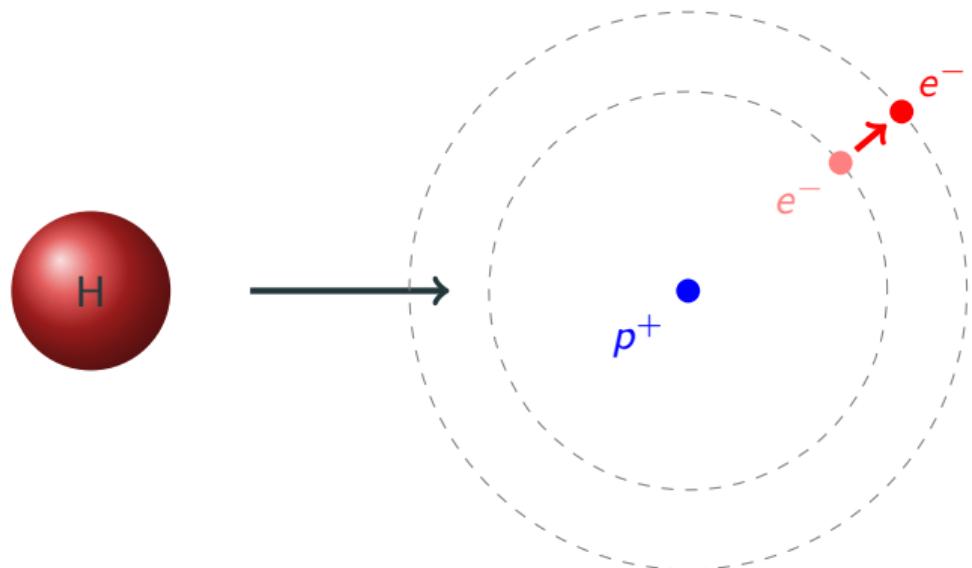
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## Excitation et ionization

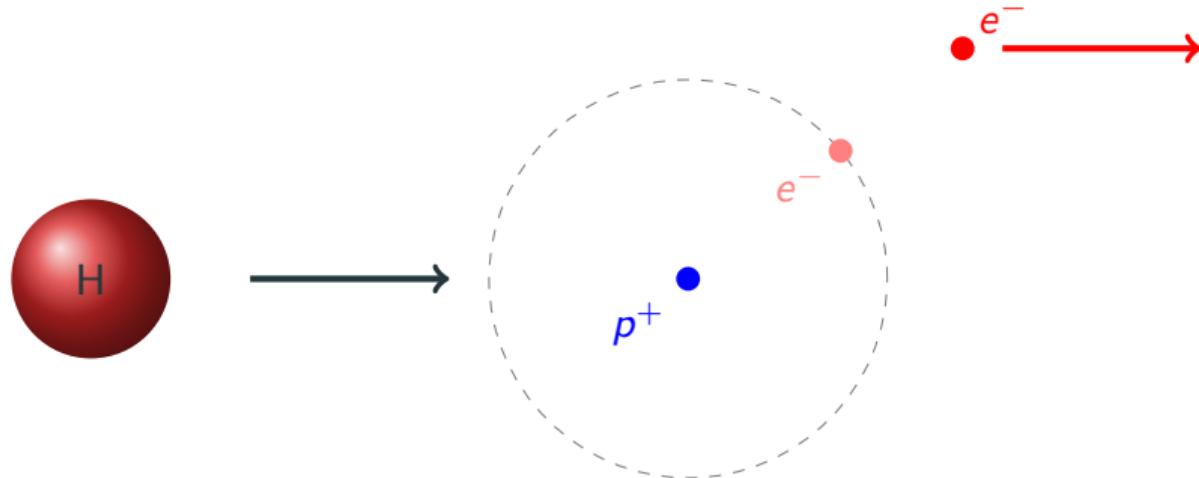


## Excitation et ionization



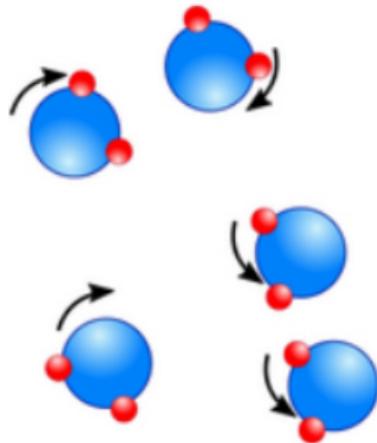
Si l'énergie reçue le permet, l'électron est dans un état **excité**. Il reviendra à son état initial en émettant de la lumière: c'est la **radiation**.

## Excitation et ionization



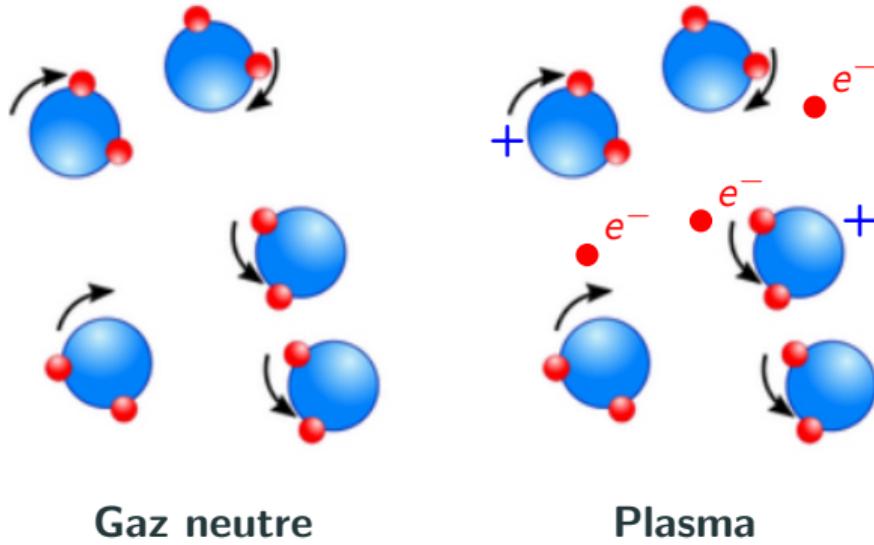
Si l'énergie reçue est trop grande, l'électron est arraché: il devient **libre**. L'atome d'hydrogène a été **ionisé**.

## Plasma: le quatrième état de la matière

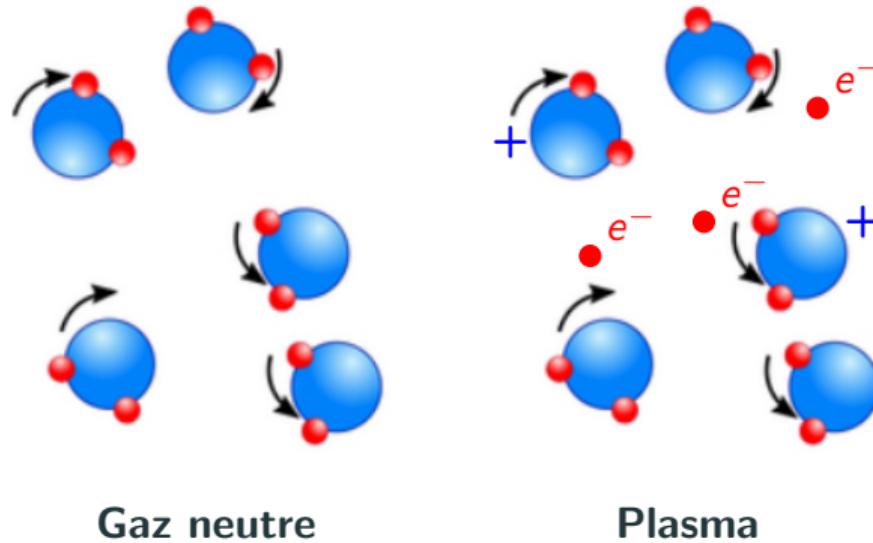


Gaz neutre

# Plasma: le quatrième état de la matière



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Un plasma est un gaz quasi-neutre composé de particules chargées (ions  $\bullet^+$  et électrons  $\bullet^{e^-}$ ) et neutres ( $\bullet^n$ ) démontrant un comportement collectif.<sup>1</sup>

<sup>1</sup> F. F. Chen. Introduction to Plasma Physics and Controlled Fusion. Ed. by Springer International Publisher. 2016

# Comportement collectif du plasma

**Collision = interaction entre particules**

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**Collisions de courte portée**



Contact direct (local & binaire)

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**Collisions de longue portée**



Force électrique (à distance, collectif)

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**Collision = interaction entre particules**

**Collisions de courte portée**



Contact direct (local & binaire)

**Collisions de longue portée**



Force électrique (à distance, collectif)

Si l'échelle est suffisamment grande ( $> 1 \mu\text{m}$  dans notre cas), le plasma est  
**quasi-neutre** grâce à la force électrique.

# Chimie dans les plasmas

**Les collisions entre les particules peuvent mener à des réactions chimiques.**

Si  $\tau_{chem}$  et  $\tau_{hydro}$  sont les temps caractéristiques de réaction chimique et d'écoulement:

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$$\tau_{chem} \simeq \tau_{hydro}$$

Hors équilibre chimique

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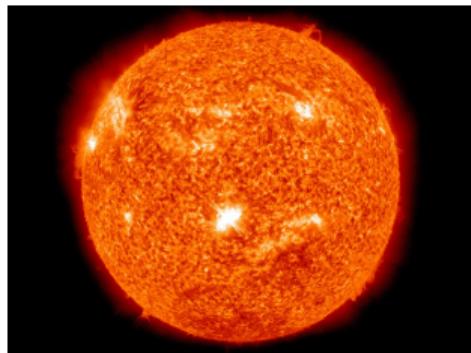
$$\tau_{chem} \ll \tau_{hydro}$$

Pas de réaction chimique

**Les plasmas sont soit en équilibre, soit hors équilibre.**

# Plasma dans la vie de tous les jours

**Les plasmas composent 90% de l'univers visible.**



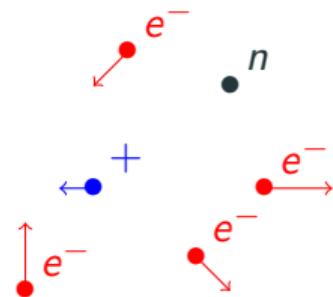
De plus en plus d'applications: fusion nucléaire, médecine, métallurgie, lasers, création de microprocesseurs, ...

## Plasma froids

Pour les plasma froids, l'énergie est d'abord emmagasinée par les électrons libres et cédée lors des collisions aux ions et neutres lourds.

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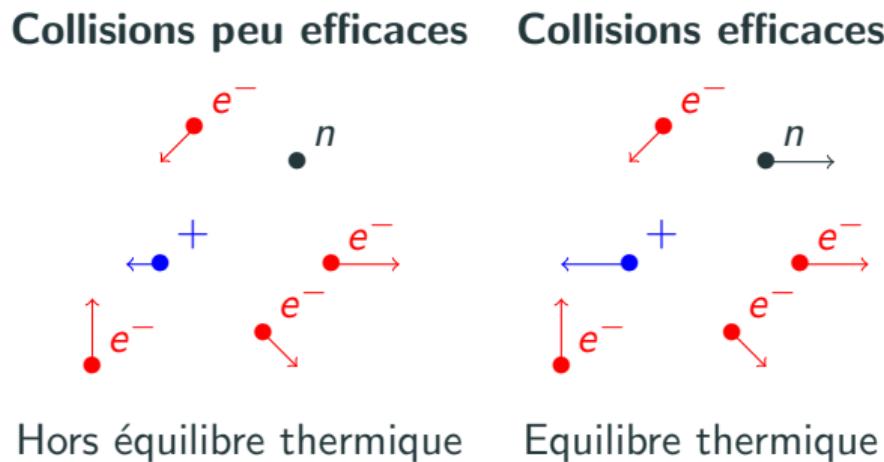
## Collisions peu efficaces



Hors équilibre thermique

# Plasma froids

Pour les plasma froids, l'énergie est d'abord emmagasinée par les électrons libres et cédée lors des collisions aux ions et neutres lourds.



## Plasma en réentrée atmosphérique

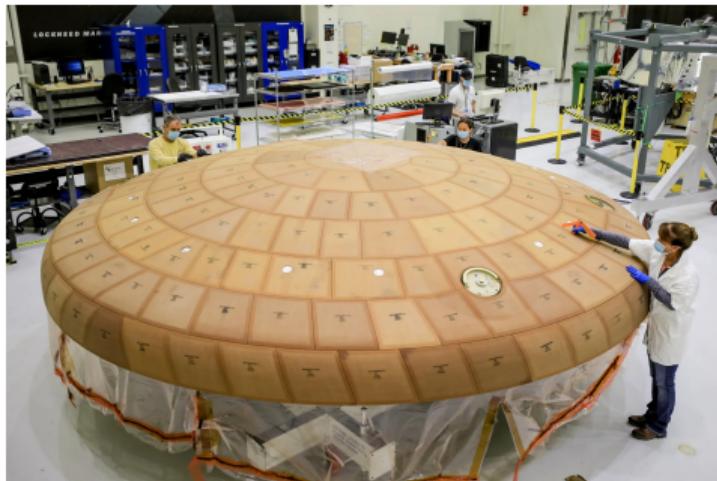


La grande vitesse de réentrée  $\simeq 10 \text{ km s}^{-1}$ , un choc suffisamment fort pour ioniser l'air  $\Rightarrow$  plasma.

## Système de protection thermique et destruction des déchets spatiaux

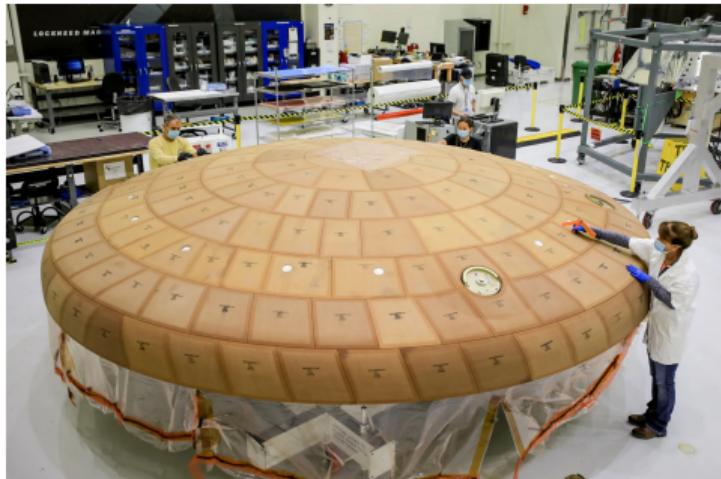
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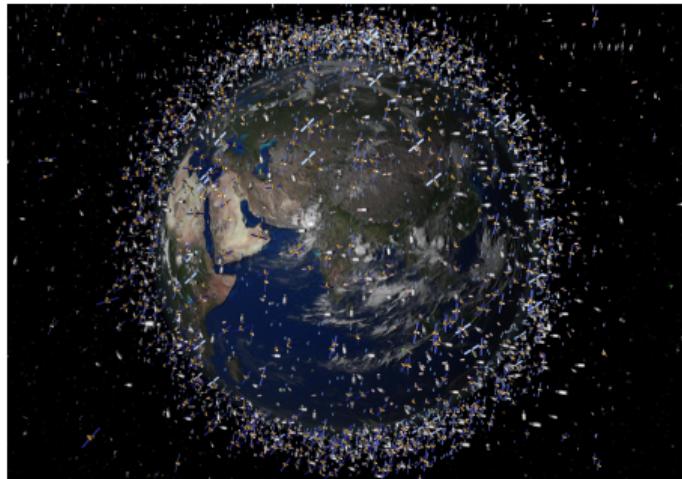


TPS

# Système de protection thermique et destruction des déchets spatiaux

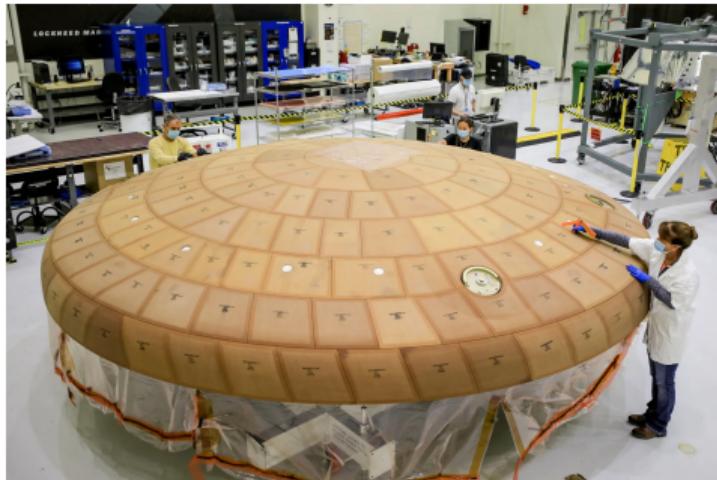


TPS



Déchets spatiaux

# Système de protection thermique et destruction des déchets spatiaux



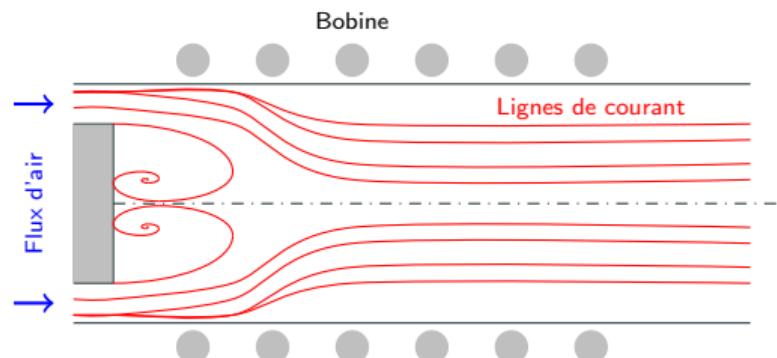
TPS



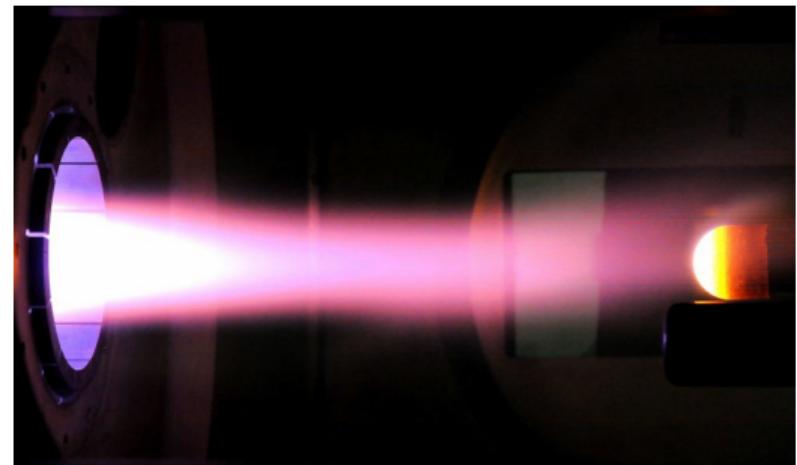
Déchets spatiaux

Nécessité de développer des machines expérimentales reproduisant les plasmas de réentrée atmosphérique pour étudier ces applications.

# Plasma à induction



Torche

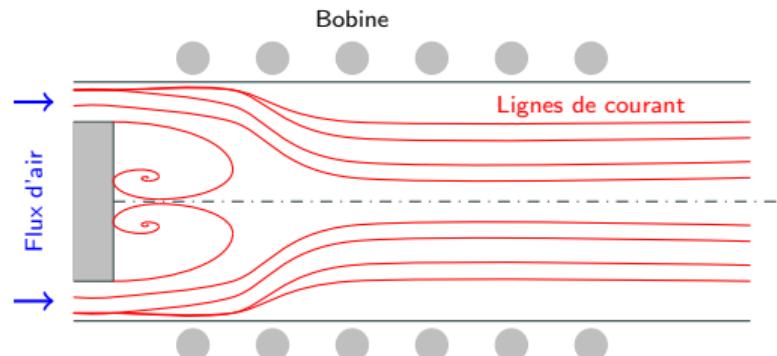


Chambre

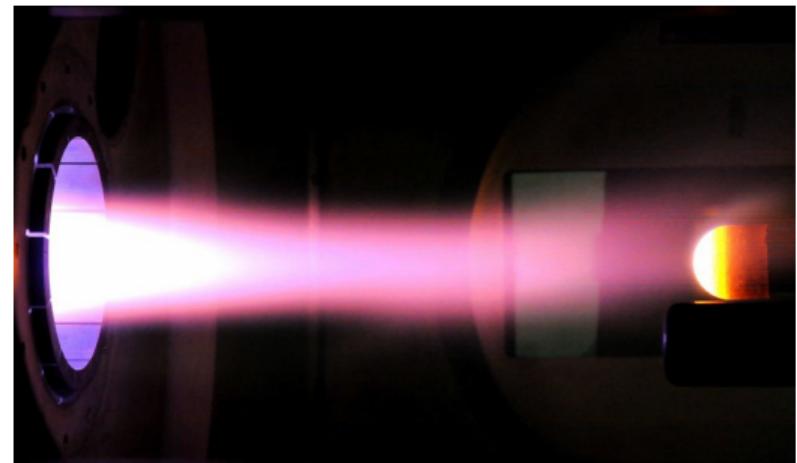
Examples: Plasmatron (VKI), Plasmatron X (Illinois), IPG (Russie), ...

Basé sur le principe de **transfer de chaleur local**.

# Plasma à induction



Torche

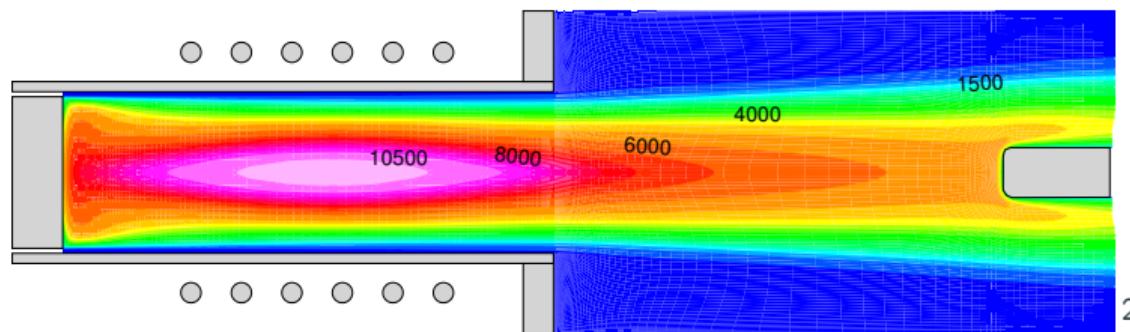


Chambre

Résoudre les équations de Maxwell, de Navier-Stokes + modèles physico-chimiques + modèle de radiation.

# Plasma à induction: besoin de plus

Des solvers numériques (volumes finis) ont été développés afin de préparer au mieux les expériences dans les torches à induction.



<sup>2</sup>Thierry Magin.

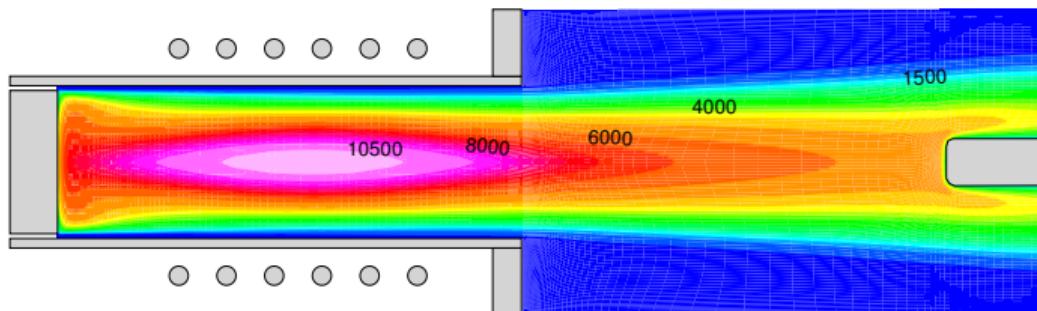
## Motivations de la thèse

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**La plupart des solvers actuels**

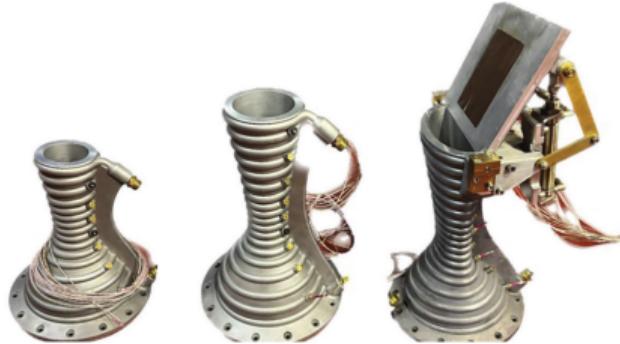
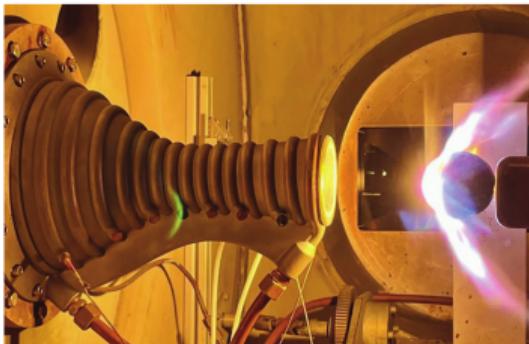
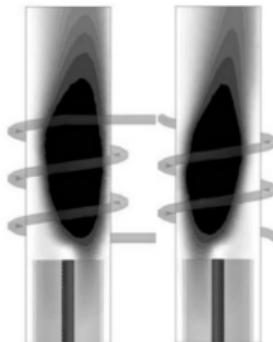
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- Ne représentent que des géométries axisymétriques et simples.



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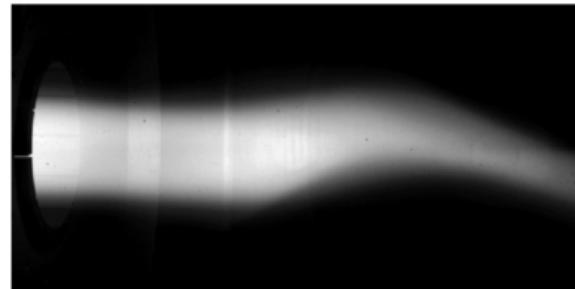
*Besoin d'un solver 3D.*

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- Utilisent un modèle à l'équilibre chimique et thermique sans radiation.

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*Besoin d'un solver 3D.*

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*Besoin d'un solver instationnaire.*

- Utilisent un modèle à l'équilibre chimique et thermique sans radiation.

*C'est faux! Car la déséquilibre chimique et thermique ont été observés dans les plasma à induction! La radiation joue aussi un rôle important dans le transfer de chaleur.*

## La plupart des solvers actuels

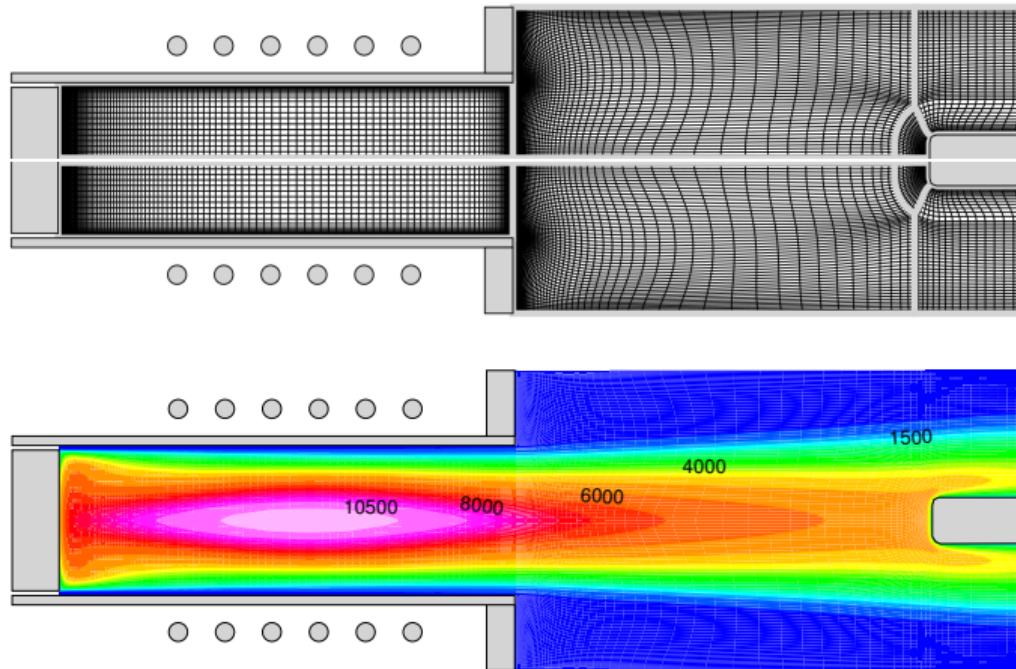
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*Besoin d'un solver avec une chimie plus détaillée et de la radiation.*

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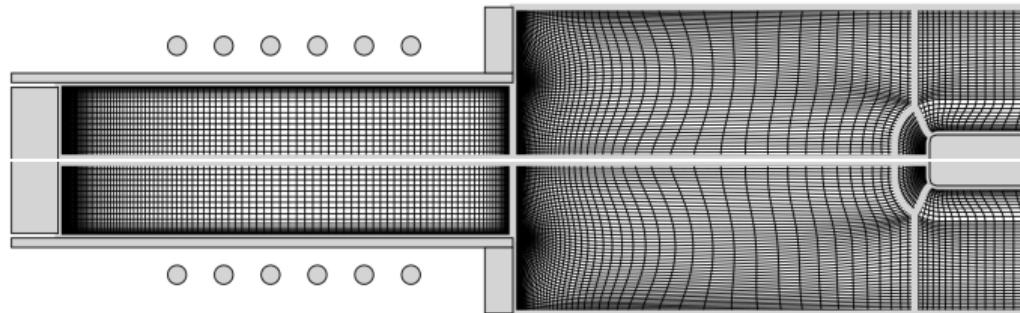
**Peut-on améliorer les solvers volume finis actuels afin de pouvoir prendre en compte toute la physique?**

## Le problème des solvers volume finis actuels.



Pour les volumes finis, la solution est constante sur chaque éléments.

## Le problème des solvers volume finis actuels.



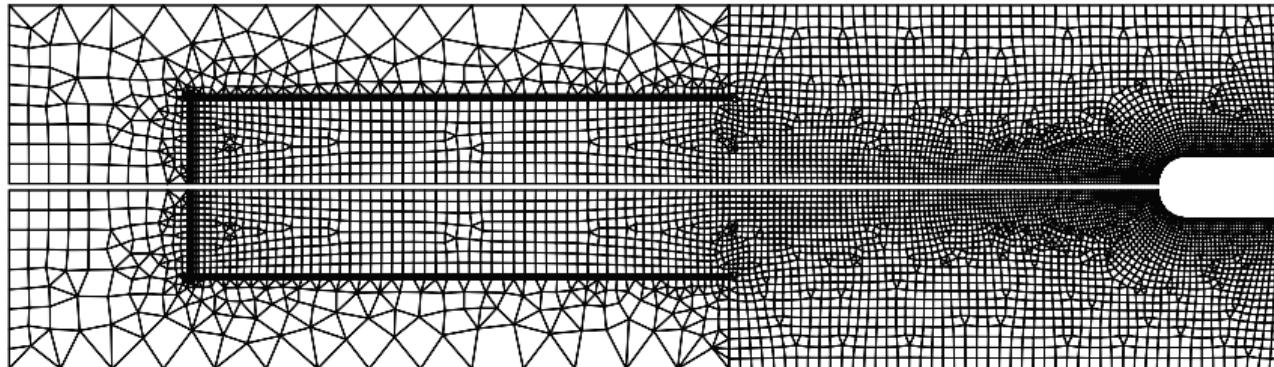
**Les volumes finis nécessitent un maillage de bonne qualité, et donc très fin dans les régions de grande variation ou de géométrie complexe ⇒**

- Demande beaucoup d'attention au maillage.
- Quasi impossible pour les géométries complexes.
- Demande un maillage trop fin partout pour une physique plus complexe.

## But de la thèse

Développer un solver faisant partie des méthodes de Galerkin discontinues pour les plasma à induction.

- La solution n'est plus constante sur chaque élément.
- Beaucoup plus grande flexibilité de maillage.



## Research questions

With the objective of producing a new high order solver for inductively coupled plasma, we will try to answer the following questions:

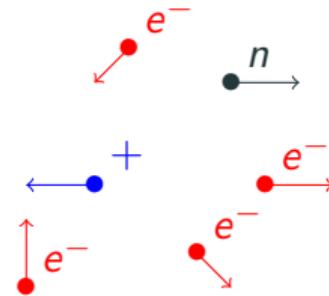
- Q1 In addition of being precise, can a high-order solver be robust for inductively coupled plasma applications?**
- Q2 Is the developed solver user-friendly?**
- Q3 Can the new solver be easily adapted to the new experiments performed in ICP facilities?**

## **Model for inductively coupled plasma**

---

## Hypothesis on the plasma

Collision-dominated and thermal plasma



$$T_e \simeq T_h$$

## Hypothesis on the plasma

---

**Collision-dominated and thermal plasma**

**Quasi-neutrality**

Characteristic length greater than the Debye length, *i.e.* the distance at which

$$E_{\text{thermal}} \simeq E_{\text{elec}}$$

so

$$q \simeq 0$$

## Hypothesis on the plasma

---

**Collision-dominated and thermal plasma**

**Quasi-neutrality**

**No displacement current**

ELM waves perturb electrons, but they are assumed to return fast to equilibrium.

ELM wavelength are much greater than the plasma length scale.

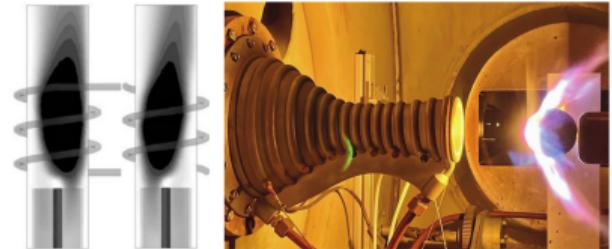
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# Hypothesis on the plasma

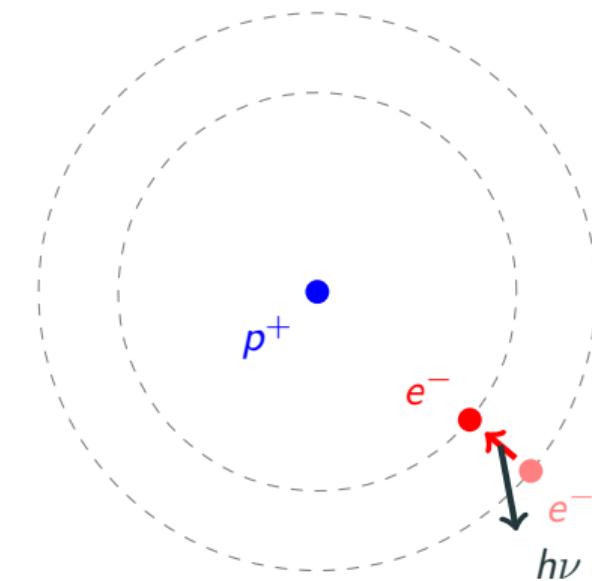
**Collision-dominated and thermal plasma**

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**Axisymmetry**

**Non-radiative plasma**



## Hypothesis on the plasma

---

**Collision-dominated and thermal plasma**

**Quasi-neutrality**

**No displacement current**

**Axisymmetry**

**Non-radiative plasma**

**Local thermodynamic equilibrium**

Chemical reaction return very fast to equilibrium.

In reality, non equilibrium can be observed in the torch.

Thermal + chemical equilibrium = local thermodynamic equilibrium.

## Hypothesis on the plasma

---

**Collision-dominated and thermal plasma**

**Quasi-neutrality**

**No displacement current**

**Axisymmetry**

**Non-radiative plasma**

**Local thermodynamic equilibrium**

**No elemental de-mixing**

Elemental de-mixing accounts for the diffusion of elements. In the case of LTE, gives results closer to non-equilibrium models.

Cheaper to solve as the elements have no production rates.

# Hypothesis on the plasma

Collision-dominated and thermal plasma

Quasi-neutrality

No displacement current

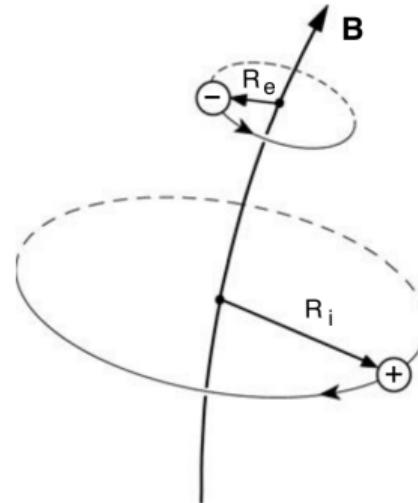
Axisymmetry

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Unmagnetized plasma



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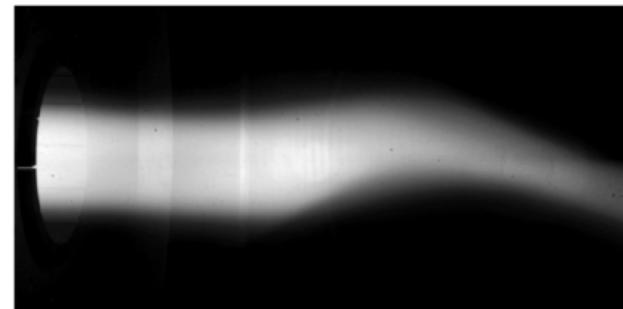
**Non-radiative plasma**

**Local thermodynamic equilibrium**

**No elemental de-mixing**

**Unmagnetized plasma**

**Steady-state**



We average the equations over one period of the induction current.

# Hypothesis on the plasma

**Collision-dominated and thermal plasma**

**Quasi-neutrality**

**No displacement current**

**Axisymmetry**

**Non-radiative plasma**

**Local thermodynamic equilibrium**

**No elemental de-mixing**

**Unmagnetized plasma**

**Steady-state**

The electric field:

- It is ambipolar ( $j_z = j_r = 0$ ).
- The coils are thin parallel wires surrounding the facility.
- Axisymmetric
- $E_{tot} = E_C + E_P$
- We use phasor notation.

## Equations governing the plasma

### Navier-Stokes equations

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{F}_L$$

$$\partial_t \left( \rho e + \frac{1}{2} \rho \|\mathbf{v}\|^2 \right) + \nabla \cdot \left( \rho e \mathbf{v} + \frac{1}{2} \rho \|\mathbf{v}\|^2 \mathbf{v} + p \mathbf{v} \right) = \nabla \cdot (\boldsymbol{\tau} \mathbf{v}) + \nabla \cdot \mathbf{q} + P_J$$

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$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{F}_L$$

$$\partial_t \left( \rho e + \frac{1}{2} \rho \|\mathbf{v}\|^2 \right) + \nabla \cdot \left( \rho e \mathbf{v} + \frac{1}{2} \rho \|\mathbf{v}\|^2 \mathbf{v} + p \mathbf{v} \right) = \nabla \cdot (\boldsymbol{\tau} \mathbf{v}) + \nabla \cdot \mathbf{q} + P_J$$

$$\boldsymbol{\tau} = \eta \left( \nabla \mathbf{v} + \nabla \mathbf{v}^T \right) - \frac{2}{3} \eta \nabla \cdot \mathbf{v}$$

$$\mathbf{q} = -k \nabla T$$

$$F_z^L = \frac{\sigma_e}{4\pi f} \left[ E_I^{Im} \partial_z E_I^{Re} - E_I^{Re} \partial_z E_I^{Im} \right]$$

$$F_r^L = \frac{\sigma_e}{4\pi f} \left[ E_I^{Im} \frac{1}{r} \partial_r (r E_I^{Re}) - E_I^{Re} \frac{1}{r} \partial_r (r E_I^{Im}) \right]$$

$$P_J = \frac{\sigma_e}{2} \left[ (E_I^{Im})^2 + (E_I^{Re})^2 \right]$$

## Electric field equation

$$\partial_{zz}^2 E_P + \frac{1}{r} \partial_r (r \partial_r E_P) - \frac{E_P}{r^2} = i2\pi f \mu_0 \sigma_e (E_C + E_P)$$

## Electric field equation

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$\mu_0$  : magnetic permeability

$\sigma_e$  : electron electric conductivity

$f$  : induction frequency

## Electric field equation

$$\partial_{zz}^2 E_P + \frac{1}{r} \partial_r (r \partial_r E_P) - \frac{E_P}{r^2} = i 2\pi f \mu_0 \sigma_e (E_C + E_P)$$

$$E_C = \sum_{I=1}^{N_{coil}} i f \mu_0 I_C \sqrt{\frac{r_0}{r}} \left[ 2 \frac{E_2(k_I)}{k_I} - E_1(k_I) \left( \frac{2}{k_I} - k_I \right) \right]$$

$$E_1(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}} \quad E_2(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2(\theta)} d\theta$$

The transport properties are computed using Mutation++

$$\eta, k, \sigma_e$$



---

Multicomponent Thermodynamic And Transport properties for IONized gases in C++

## Coupling of the systems

---

### Navier-Stokes

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{F}_L$$

$$\partial_t \left( \rho e + \frac{1}{2} \rho \|\mathbf{v}\|^2 \right) + \nabla \cdot \left( \rho e \mathbf{v} + \frac{1}{2} \rho \|\mathbf{v}\|^2 \mathbf{v} + p \mathbf{v} \right) = \nabla \cdot (\boldsymbol{\tau} \mathbf{v}) + \nabla \cdot \mathbf{q} + P_J$$

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---

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### Maxwell

$$\partial_{zz}^2 E_P + \frac{1}{r} \partial_r (r \partial_r E_P) - \frac{E_P}{r^2} = i 2 \pi f \mu_0 \sigma_e (E_C + E_P)$$

## Coupling of the systems

---

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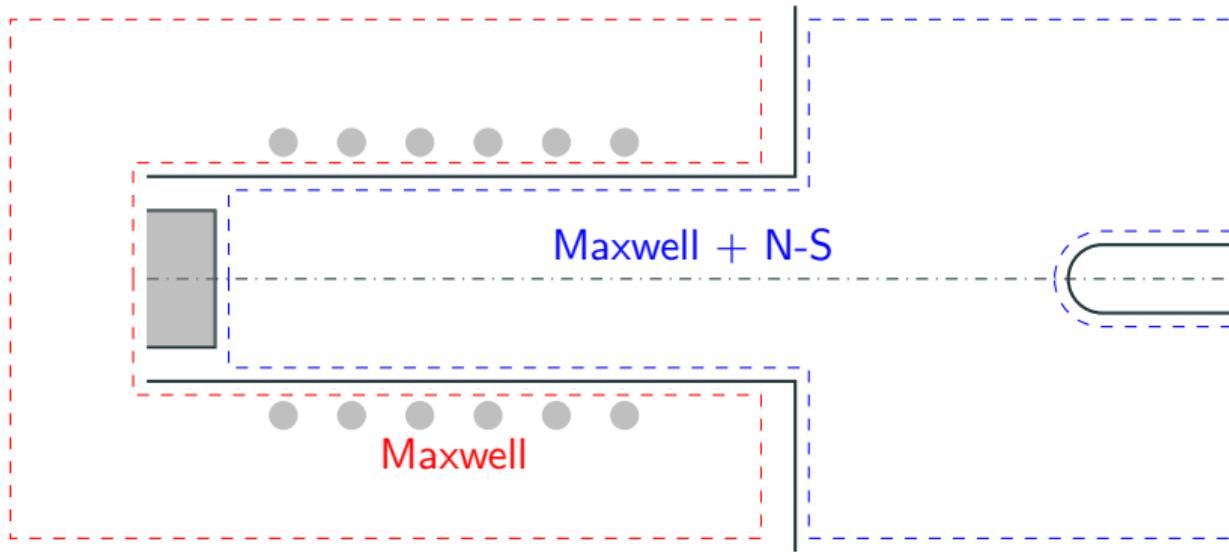
$$\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{F}_L$$

$$\partial_t \left( \rho e + \frac{1}{2} \rho \|\mathbf{v}\|^2 \right) + \nabla \cdot \left( \rho e \mathbf{v} + \frac{1}{2} \rho \|\mathbf{v}\|^2 \mathbf{v} + p \mathbf{v} \right) = \nabla \cdot (\boldsymbol{\tau} \mathbf{v}) + \nabla \cdot \mathbf{q} + \mathbf{P}_J$$

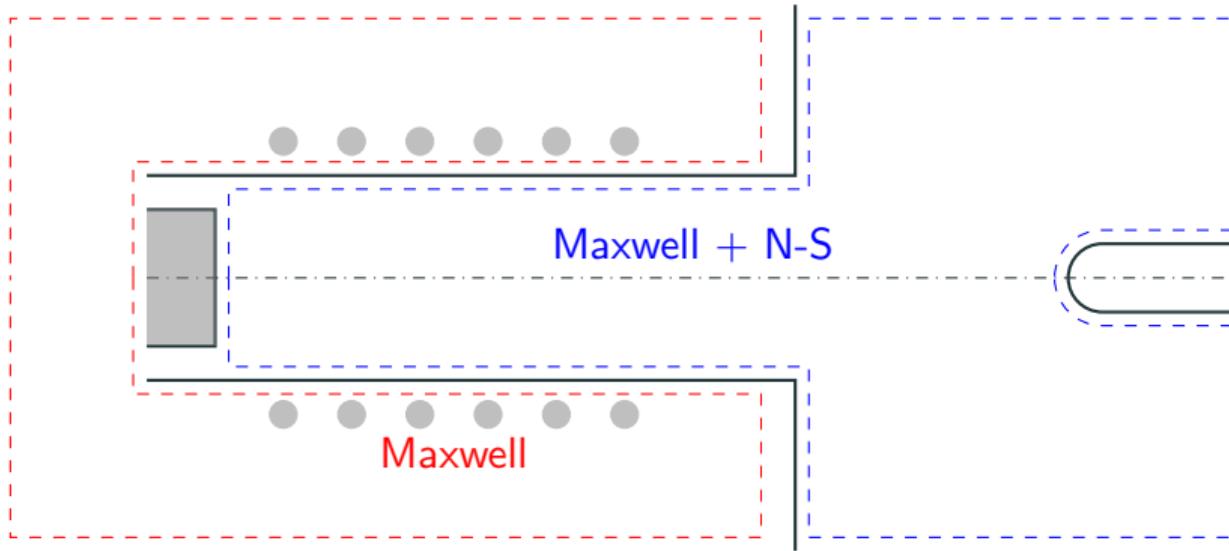
### Maxwell

$$\partial_{zz}^2 E_P + \frac{1}{r} \partial_r (r \partial_r E_P) - \frac{E_P}{r^2} = i 2 \pi f \mu_0 \sigma_e (E_C + E_P)$$

## Computational domain

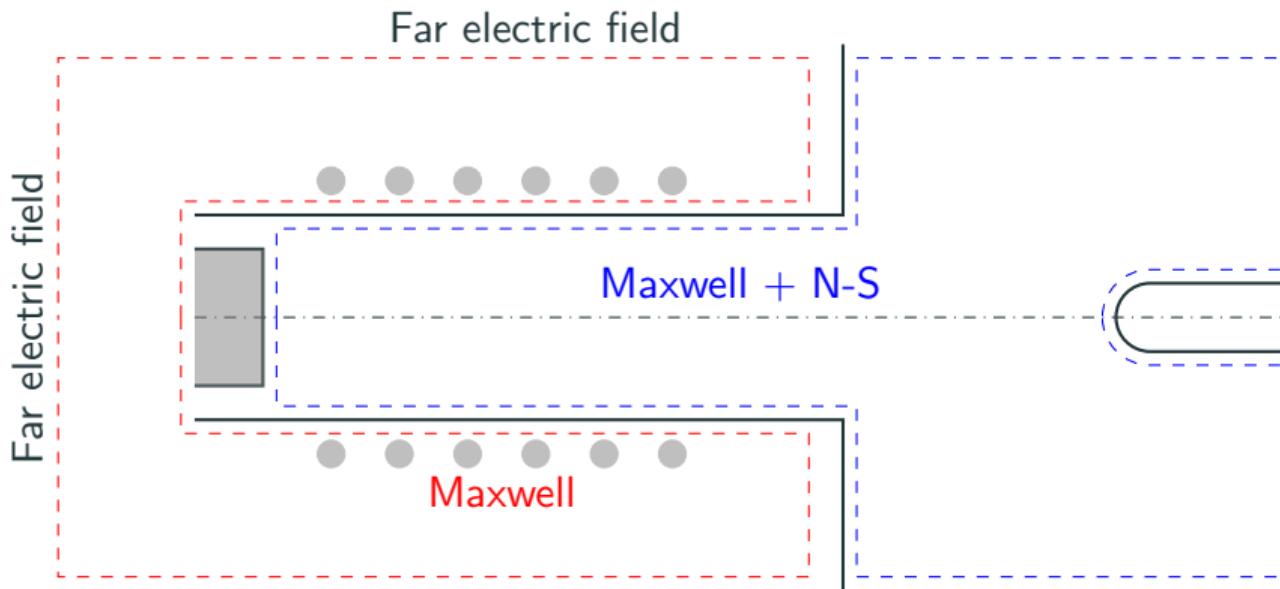


## Computational domain



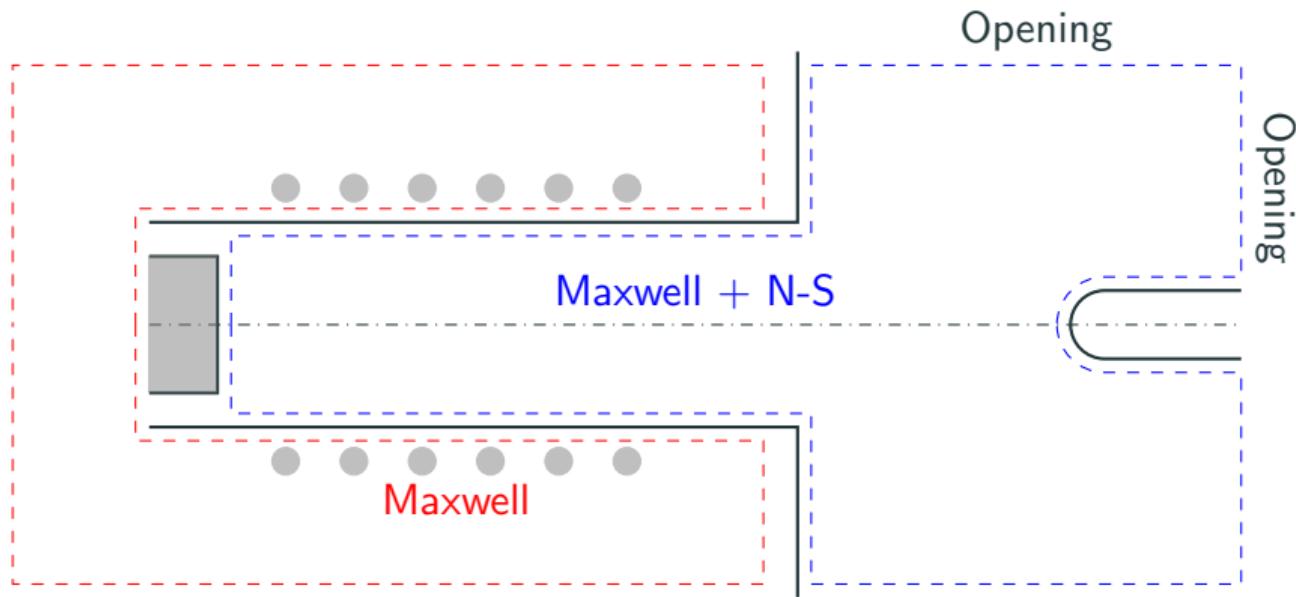
**Symmetry axis**  $\partial_r p = 0$ ,  $\partial_r u_z = 0$ ,  $u_r = 0$ ,  $\partial_r T = 0$  and  $E_P = 0$ .

## Computational domain



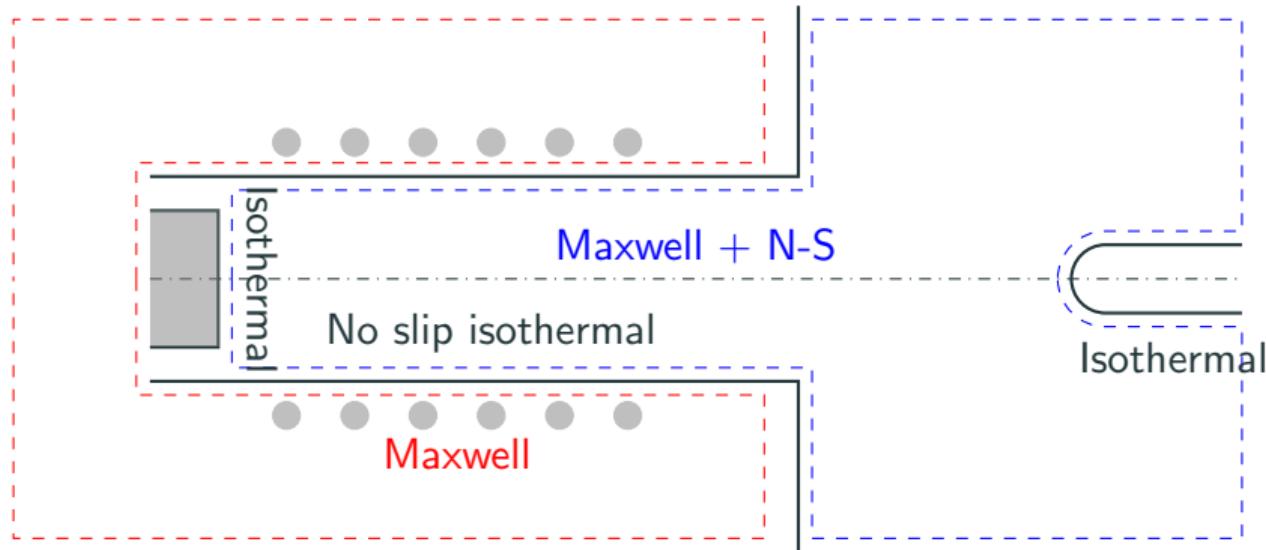
**Vanishing far electric field**  $E_P = 0$ .

## Computational domain



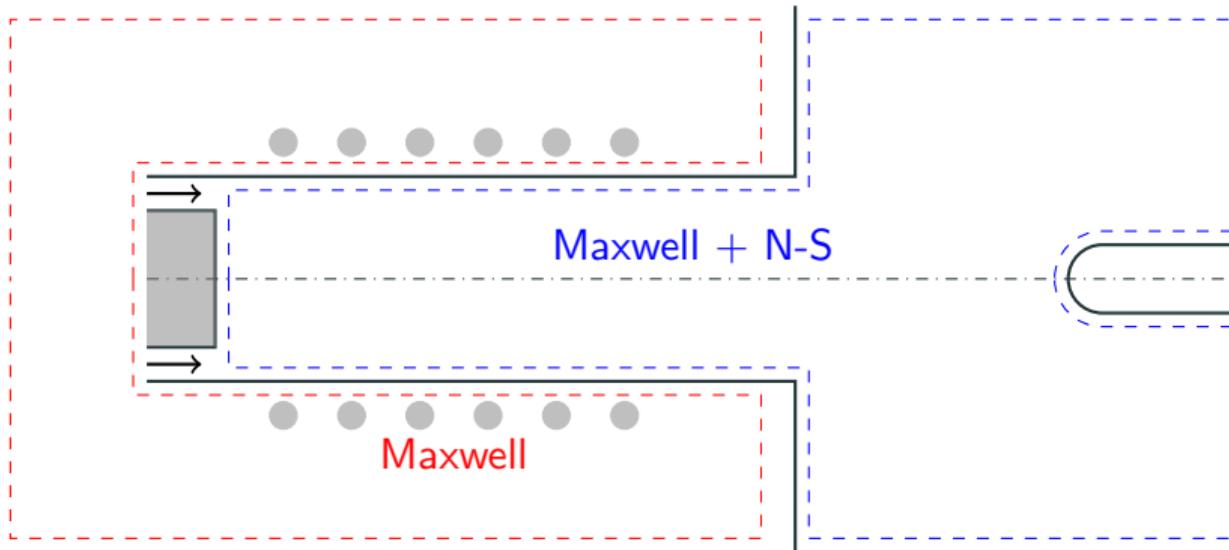
**Openings**  $p = p_0$

## Computational domain



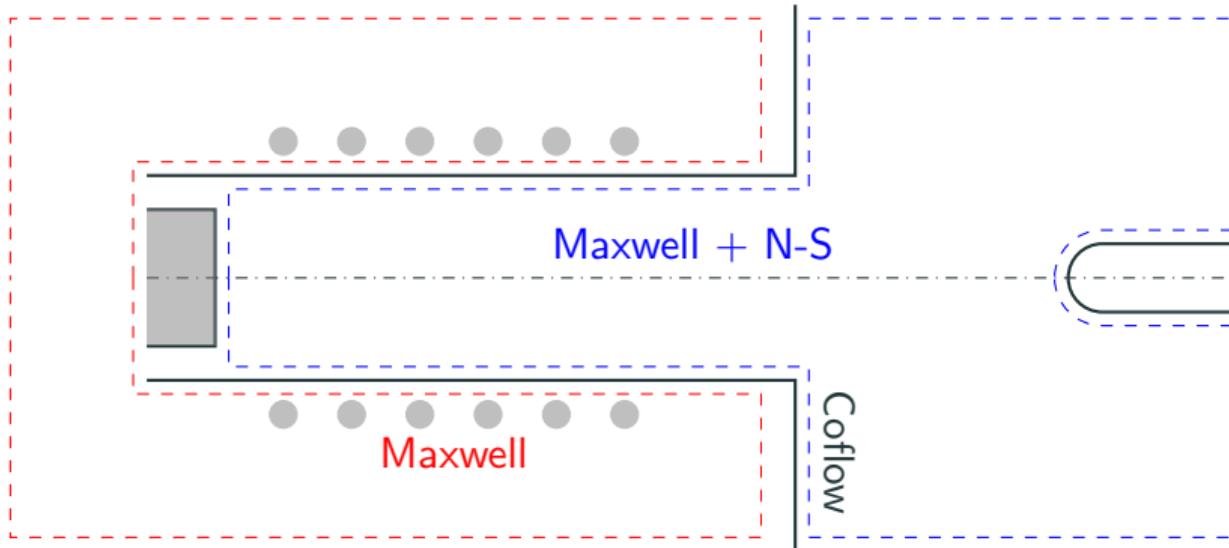
No-slip wall isothermal  $T = T_{wall}$  and  $\mathbf{u} = \mathbf{0}$ .

## Computational domain



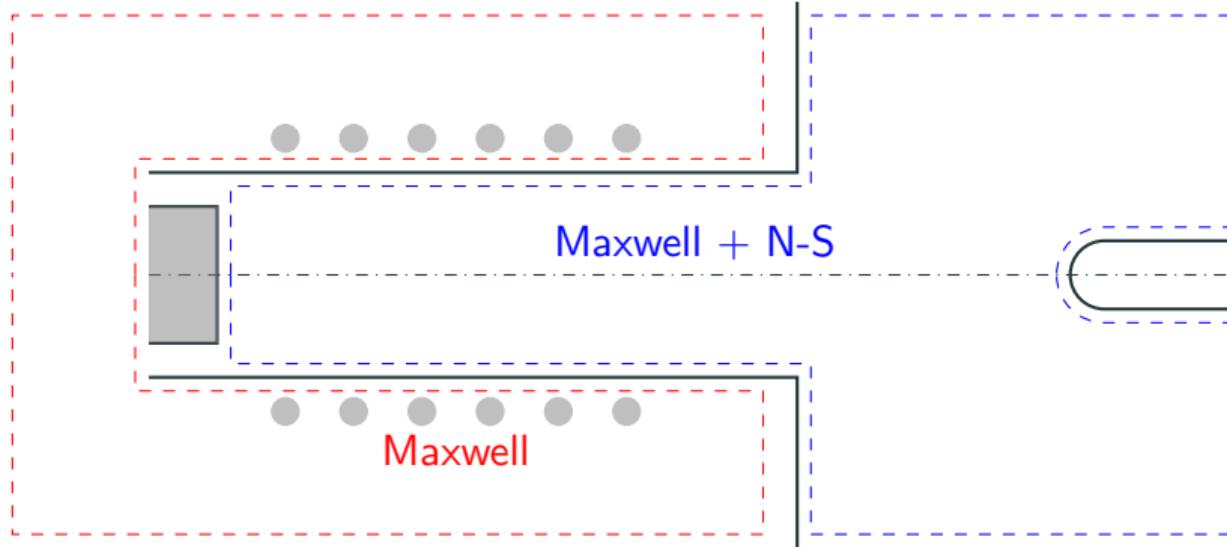
Inflow  $u_z = U_{in}$ ,  $u_r = 0$  and  $T = T_{inlet}$ .

## Computational domain



**Stabilizing co-flow**  $T = T_{wall}$ ,  $\mathbf{u} = u_{coflow}\mathbf{e}_z$

## Computational domain

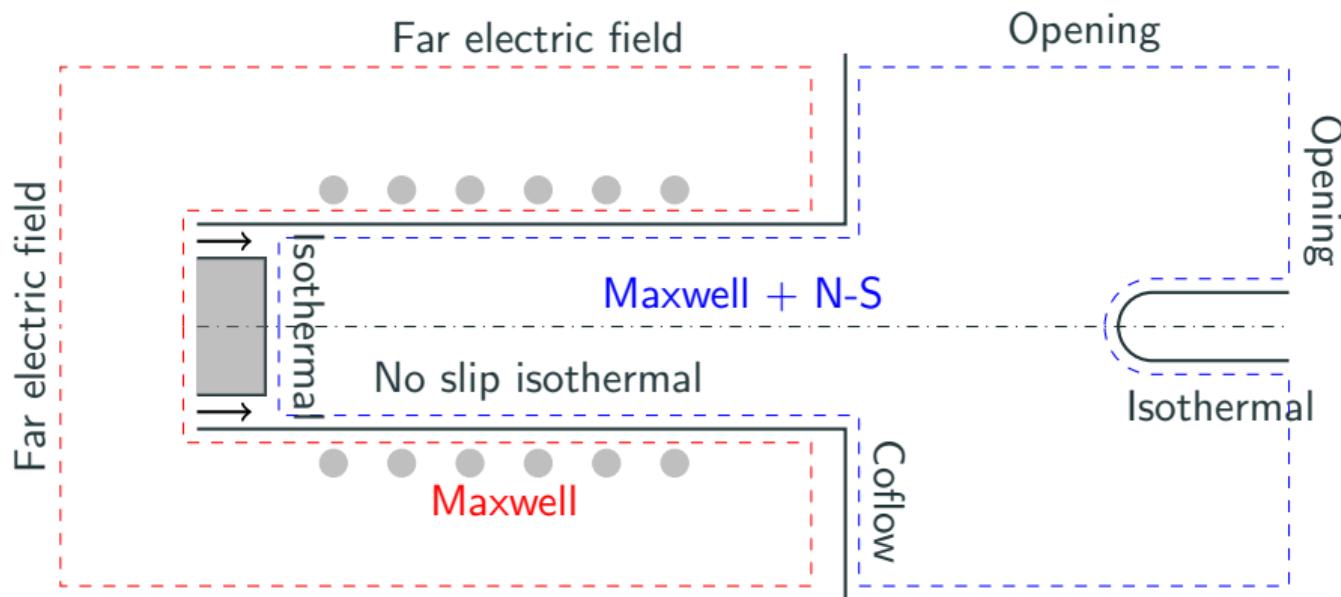


## ICP and Maxwell interface

$$\nabla E_p^{Re,NS} \cdot \mathbf{n}^{NS} = \nabla E_p^{Re,MAX} \cdot \mathbf{n}^{MAX}$$

$$\nabla E_p^{Im,NS} \cdot \mathbf{n}^{NS} = \nabla E_p^{Im,MAX} \cdot \mathbf{n}^{MAX}$$

## Computational domain



## **Hybridized discontinuous Galerkin method**

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## Conservative form of the equations

---

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + p \mathbb{I}) - \nabla \cdot \boldsymbol{\tau} = \mathbf{F}_L$$

$$\partial_t (\rho E) + \nabla \cdot (\rho E \mathbf{v} + p \mathbf{v}) - \nabla \cdot (\boldsymbol{\tau} \mathbf{v} + \mathbf{q}) = P_J$$

$$-\nabla \cdot (\nabla E_P) = -\frac{E_P}{r^2} - i2\pi f \mu_0 \sigma_e (E_C + E_P)$$

## Conservative form of the equations

---

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + p \mathbb{I}) - \nabla \cdot \boldsymbol{\tau} = \mathbf{F}_L$$

$$\begin{aligned} \partial_t (\rho E) + \nabla \cdot (\rho E \mathbf{v} + p \mathbf{v}) - \nabla \cdot (\boldsymbol{\tau} \mathbf{v} + \mathbf{q}) &= P_J \\ &\quad - \nabla \cdot (\nabla E_P) = -\frac{E_P}{r^2} - i2\pi f \mu_0 \sigma_e (E_C + E_P) \end{aligned}$$

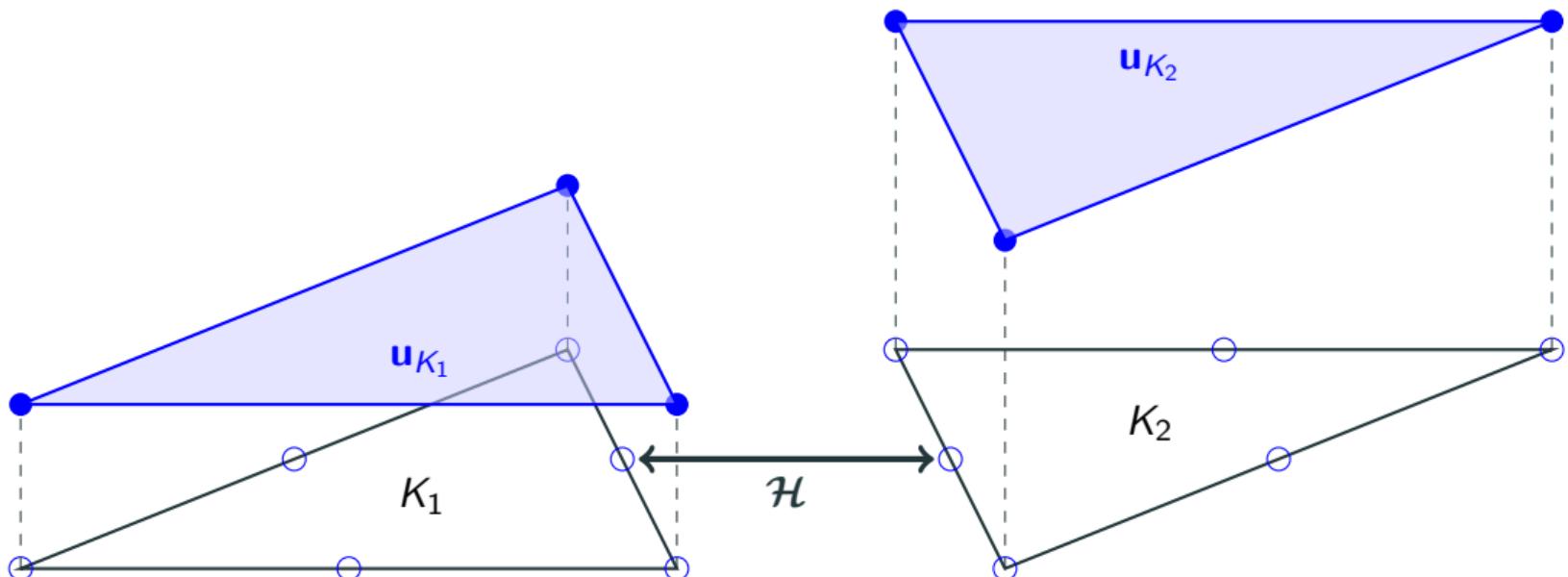
$$\partial_t \mathbf{w}(\mathbf{u}) + \nabla \cdot \mathbf{F}_c(\mathbf{u}) - \nabla \cdot \mathbf{F}_v(\mathbf{u}, \nabla \mathbf{u}) = \mathbf{S}(\mathbf{u}, \nabla \mathbf{u})$$

## Conservative form of the equations

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + p \mathbb{I}) - \nabla \cdot \boldsymbol{\tau} &= \mathbf{F}_L \\ \partial_t (\rho E) + \nabla \cdot (\rho E \mathbf{v} + p \mathbf{v}) - \nabla \cdot (\boldsymbol{\tau} \mathbf{v} + \mathbf{q}) &= P_J \\ &\quad - \nabla \cdot (\nabla E_P) = -\frac{E_P}{r^2} - i2\pi f \mu_0 \sigma_e (E_C + E_P)\end{aligned}$$

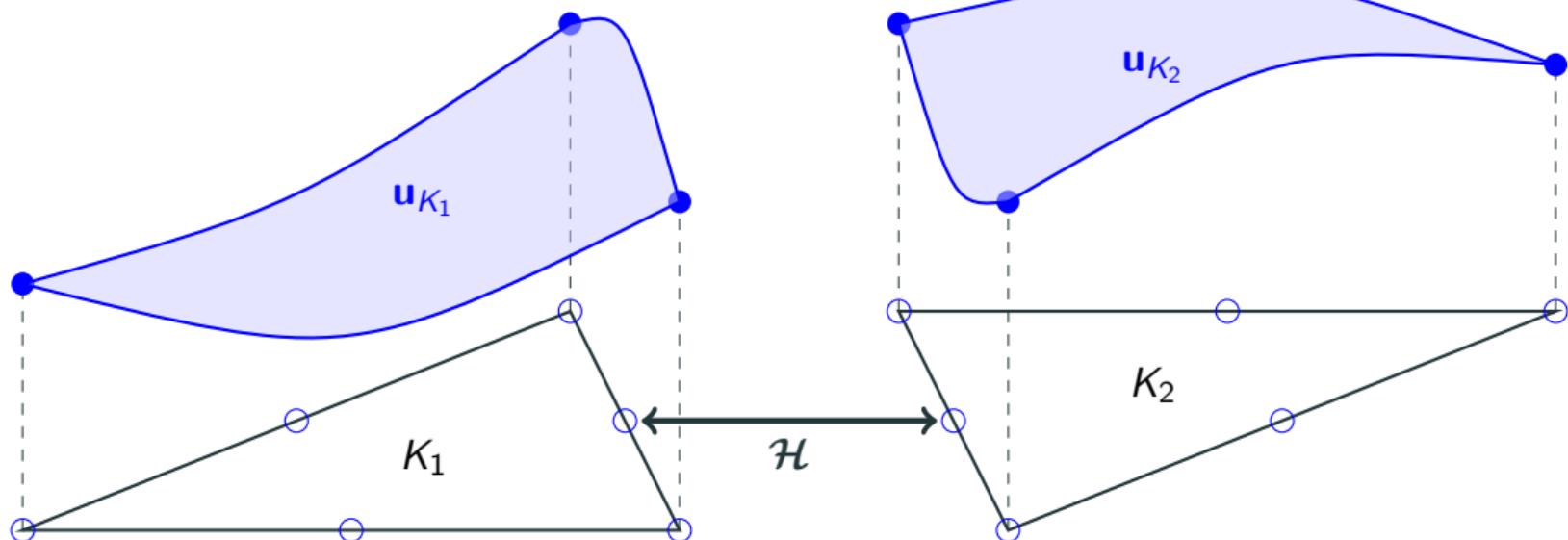
$$\partial_t \mathbf{w}(\mathbf{u}) + \nabla \cdot \mathbf{F}_c(\mathbf{u}) - \nabla \cdot \mathbf{F}_v(\mathbf{u}, \nabla \mathbf{u}) = \mathbf{S}(\mathbf{u}, \nabla \mathbf{u})$$

# Finite volume and (hybridized) discontinuous Galerkin



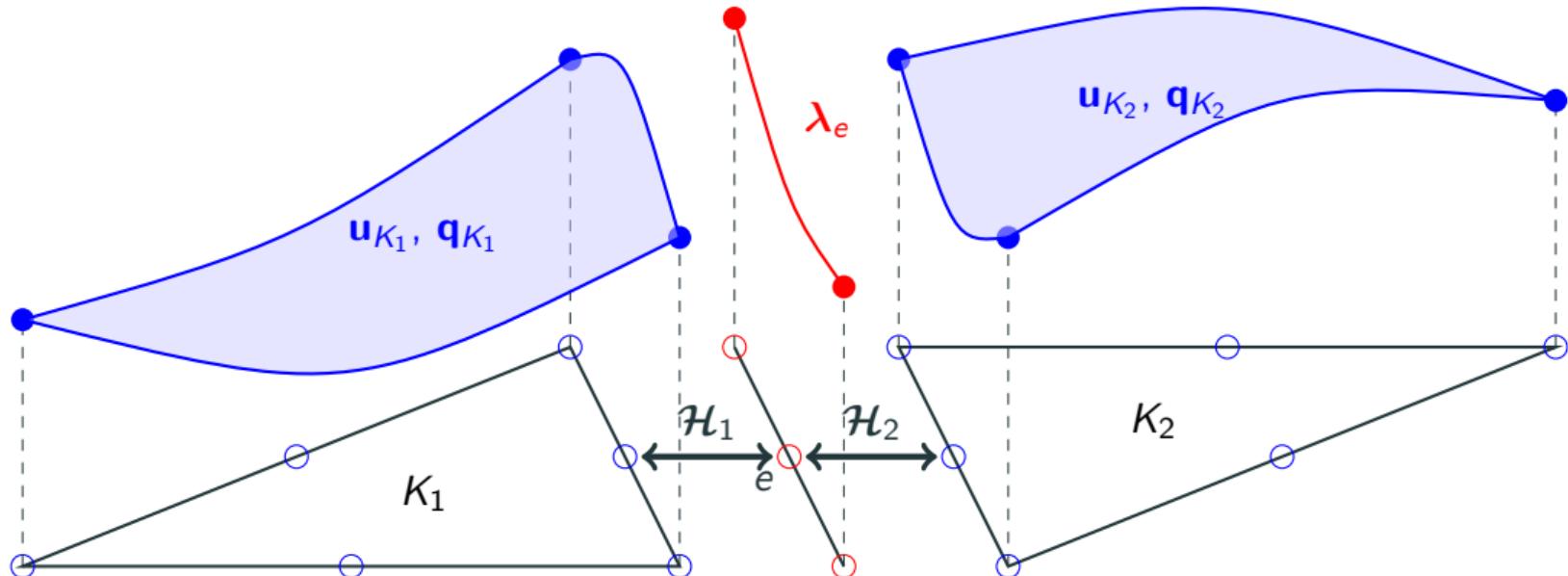
Finite volumes

# Finite volume and (hybridized) discontinuous Galerkin



Discontinuous Galerkin

# Finite volume and (hybridized) discontinuous Galerkin



Hybridized discontinuous Galerkin

## Model problem

$$\begin{aligned}\partial_t \mathbf{w}(\mathbf{u}) + \nabla \cdot (\mathbf{F}_c(\mathbf{u}) - \mathbf{F}_v(\mathbf{u}, \nabla \mathbf{u})) &= \mathbf{S}(\mathbf{u}, \nabla \mathbf{u}) && \text{on } \Omega \\ \mathbf{u} = \mathbf{u}_{bc} & && \text{on } \partial\Omega_D \\ \mathbf{F}_v \cdot \mathbf{n} = \mathbf{F}_{v,n,bc} & && \text{on } \partial\Omega_N \\ \mathbf{u}(t = 0) = \mathbf{U} & && \text{on } \Omega\end{aligned}$$

$\mathbf{F}_{c,v}$  : convective and diffusive fluxes.

$\Omega$  : domain of the problem.

$\mathbf{S}$  : source terms.

$\partial\Omega_{D,N}$  : domain boundary for Dirichlet and

$w$  : conservative variables.

Neumann BC.

$\mathbf{u}, \mathbf{u}_{bc}$  : solution and boundary condition.  $\mathbf{U}$  : initial data.

## Model problem

$$\begin{aligned}\partial_t \mathbf{w}(\mathbf{u}) + \nabla \cdot (\mathbf{F}_c(\mathbf{u}) - \mathbf{F}_v(\mathbf{u}, \nabla \mathbf{u})) &= \mathbf{S}(\mathbf{u}, \nabla \mathbf{u}) && \text{on } \Omega \\ \mathbf{u} &= \mathbf{u}_{bc} && \text{on } \partial\Omega_D \\ \mathbf{F}_v \cdot \mathbf{n} &= \mathbf{F}_{v,n,bc} && \text{on } \partial\Omega_N \\ \mathbf{u}(t = 0) &= \mathbf{U} && \text{on } \Omega\end{aligned}$$

Before going further, some notation:

$$\begin{aligned}(a, b)_K &= \int_K ab \, dV, \\ \langle a, b \rangle_{\partial K} &= \int_{\partial K} ab \, dS.\end{aligned}$$

## From weak form to discrete DG problem

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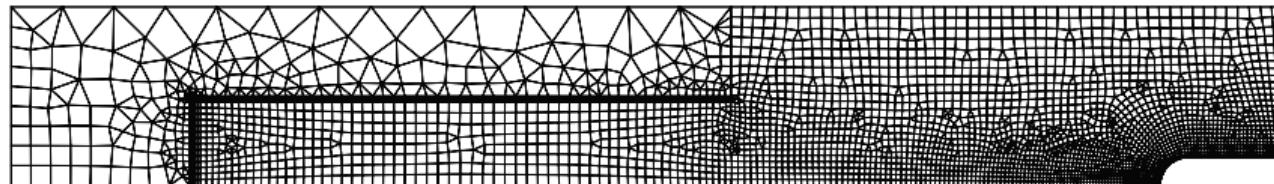
$$(\partial_t \mathbf{w} - \mathbf{S}, w)_\Omega - (\mathbf{F}_c - \mathbf{F}_v, \nabla w)_\Omega + \langle (\mathbf{F}_c - \mathbf{F}_v) \cdot \mathbf{n}, w \rangle_{\partial\Omega} = 0, \quad \forall w \in L_2(\Omega)$$

Multiplying by a function  $w \in L_2(\Omega)$  and integrating over the domain gives the **weak form**.

## From weak form to discrete DG problem

$$(\partial_t \mathbf{w} - \mathbf{S}, w)_{\mathcal{T}} - (\mathbf{F}_c - \mathbf{F}_v, \nabla w)_{\mathcal{T}} + \sum_{K \in \mathcal{T}} \langle (\mathbf{F}_c - \mathbf{F}_v) \cdot \mathbf{n}, w \rangle_{\partial K} = 0, \quad \forall w \in L_2(\Omega)$$

The DG discretization consists in first dividing the domain  $\Omega$  in a collection of non-overlapping elements  $\mathcal{T}$ .



## From weak form to discrete DG problem

---

$$(\partial_t \mathbf{w} - \mathbf{S}, w)_{\mathcal{T}} - (\mathbf{F}_c - \mathbf{F}_v, \nabla w)_{\mathcal{T}} + \sum_{K \in \mathcal{T}} \langle (\mathbf{F}_c - \mathbf{F}_v) \cdot \mathbf{n}, w \rangle_{\partial K} = 0, \quad \forall w \in W_h$$

We restrict the functions to a subset of  $L_2(\Omega)$  which is finite dimensional. A common choice is

$$W_h = \{ w \in L^2(\Omega) : w|_K \in \mathcal{P}^p(K), \forall K \in \mathcal{T} \} \subset L_2(\Omega)$$

## From weak form to discrete DG problem

---

$$(\partial_t \mathbf{w} - \mathbf{S}, w)_T - (\mathbf{F}_c - \mathbf{F}_v, \nabla w)_T + \sum_{K \in \mathcal{T}} \langle \mathcal{H}, w \rangle_{\partial K} = 0, \quad \forall w \in W_h$$

For stabilizing the method, the convective and diffusive fluxes on the interface are approximated

$$(\mathbf{F}_c - \mathbf{F}_v) \cdot \mathbf{n}_K \simeq \mathcal{H}$$

## From weak form to discrete DG problem

---

$$(\partial_t \mathbf{w}_h - \mathbf{S}_h, w)_T - (\mathbf{F}_{c,h} - \mathbf{F}_{v,h}, \nabla w)_T + \sum_{K \in \mathcal{T}} \langle \mathcal{H}, w \rangle_{\partial K} = 0, \quad \forall w \in W_h$$

The solution is also approximated by

$$\mathbf{u} \simeq \mathbf{u}_h \in W_h$$

## From weak form to discrete DG problem

---

$$(\partial_t \mathbf{w}_h - \mathbf{S}_h, \varphi_{K,i})_K - (\mathbf{F}_{c,h} - \mathbf{F}_{v,h}, \nabla \varphi_{K,i})_K + \langle \mathcal{H}, \varphi_{K,i} \rangle_{\partial K} = 0, \quad \forall \varphi_{K,i} \in W_h$$

Finally, if the  $\varphi_{K,i}$  form a local basis of  $W_h$  on  $K$ ,  $i \in \{1, 2, \dots, p\}$ , then it suffices to verify the equation for each  $\varphi_{K,i}$  on each element, and

$$\mathbf{u} \simeq \mathbf{u}_h = \sum_{K \in \mathcal{T}} \sum_{i=1}^p \mathbf{u}_{K,i} \varphi_{K,i}$$

The  $\mathbf{u}_i$  are the degrees of freedom.

## From weak form to discrete DG problem

$$(\partial_t \mathbf{w}_h - \mathbf{S}_h, \varphi_{K,i})_K - (\mathbf{F}_{c,h} - \mathbf{F}_{v,h}, \nabla \varphi_{K,i})_K + \langle \mathcal{H}, \varphi_{K,i} \rangle_{\partial K} = 0, \quad \forall \varphi_{K,i} \in W_h$$

- Because the  $\varphi_{K,i}$  are in general not continuous across elements, discontinuous solutions can be represented.
- If  $\varphi_K = 1$  everywhere, the finite volume method is retrieved.
- The art of designing DG is contained in the numerical fluxes (*cfr. later*).

## From DG to HDG

$$(\partial_t \mathbf{w}_h - \mathbf{S}_h, \varphi_{K,i})_K - (\mathbf{F}_{c,h} - \mathbf{F}_{v,h}, \nabla \varphi_{K,i})_K + \langle \mathcal{H}, \varphi_{K,i} \rangle_{\partial K} = 0, \quad \forall \varphi_{K,i} \in W_h$$

$$(\mathbf{q}_h, \boldsymbol{\tau}_{K,i})_K - (\mathbf{u}_h, \nabla \boldsymbol{\tau}_{K,i})_K + \langle \mathbf{u}_h, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_0} + \langle \mathbf{u}_{bc}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_{bc}} = 0, \quad \forall \boldsymbol{\tau}_{K,i} \in V_h$$

We solve now for the solution gradient

$$\mathbf{q} = \nabla \mathbf{u}$$

with the subset

$$V_h = \left\{ \mathbf{v} \in L^2(\Omega) : \mathbf{v}|_K \in (\mathcal{P}^p(K))^D, \forall K \in \mathcal{T} \right\}$$

and local basis  $\boldsymbol{\tau}_{K,i}$

$$\mathbf{q}_h = \sum_{K \in \mathcal{T}} \sum_{i=1}^p \mathbf{q}_{K,i} \boldsymbol{\tau}_{K,i}$$

## From DG to HDG

$$\begin{aligned}(\partial_t \mathbf{w}_h - \mathbf{S}_h, \varphi_{K,i})_K - (\mathbf{F}_{c,h} - \mathbf{F}_{v,h}, \nabla \varphi_{K,i})_K + \langle \mathcal{H}(\mathbf{u}_h, \mathbf{q}_h, \lambda_h), \varphi_{K,i} \rangle_{\partial K} &= 0, \quad \forall \varphi_{K,i} \in W_h \\(\mathbf{q}_h, \boldsymbol{\tau}_{K,i})_K - (\mathbf{u}_h, \nabla \boldsymbol{\tau}_{K,i})_K + \langle \lambda, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_0} + \langle \mathbf{u}_{bc}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_{bc}} &= 0, \quad \forall \boldsymbol{\tau}_{K,i} \in V_h\end{aligned}$$

We first introduce  $\Gamma$ , the set of traces between elements

$$\Gamma = \{e : e = K_i \cap K_j; \forall K_i, K_j \in \mathcal{T}, K_i \neq K_j\}.$$

We introduce the hybrid unknown on the element trace  $\lambda$  with the function subset

$$M_h = \{\mu \in L^2(\Gamma) : \mu|_e \in \mathcal{P}^p(e), \forall e \in \Gamma\}$$

and local basis  $\mu_{e,i}$

$$\lambda_h = \sum_{e \in \Gamma} \sum_{i=1}^p \lambda_{e,i} \mu_{e,i}$$

## From DG to HDG

---

$$(\partial_t \mathbf{w}_h - \mathbf{S}_h, \varphi_{K,i})_K - (\mathbf{F}_{c,h} - \mathbf{F}_{v,h}, \nabla \varphi_{K,i})_K + \langle \mathcal{H}(\mathbf{u}_h, \mathbf{q}_h, \boldsymbol{\lambda}_h), \varphi_{K,i} \rangle_{\partial K} = 0, \quad \forall \varphi_{K,i} \in W_h$$

$$(\mathbf{q}_h, \boldsymbol{\tau}_{K,i})_K - (\mathbf{u}_h, \nabla \boldsymbol{\tau}_{K,i})_K + \langle \boldsymbol{\lambda}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_0} + \langle \mathbf{u}_{bc}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_{bc}} = 0, \quad \forall \boldsymbol{\tau}_{K,i} \in V_h$$

$$\langle [[\mathcal{H}]], \mu_{e,i} \rangle_e = 0, \quad \forall \mu_{e,i} \in M_h$$

Equation for  $\boldsymbol{\lambda}$ : continuity of the normal numerical flux across the interfaces.

The jump operator is given by

$$[[\mathbf{u}]] = \mathbf{u}_+ \cdot \mathbf{n}_+ + \mathbf{u}_- \cdot \mathbf{n}_-$$

## From DG to HDG

$$\begin{aligned} & (\partial_t \mathbf{w}_h - \mathbf{S}_h, \varphi_{K,i})_K - (\mathbf{F}_{c,h} - \mathbf{F}_{v,h}, \nabla \varphi_{K,i})_K + \langle \mathcal{H}(\mathbf{u}_h, \mathbf{q}_h, \boldsymbol{\lambda}_h), \varphi_{K,i} \rangle_{\partial K} = 0, \quad \forall \varphi_{K,i} \in W_h \\ & (\mathbf{q}_h, \boldsymbol{\tau}_{K,i})_K - (\mathbf{u}_h, \nabla \boldsymbol{\tau}_{K,i})_K + \langle \boldsymbol{\lambda}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_0} + \langle \mathbf{u}_{bc}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_{bc}} = 0, \quad \forall \boldsymbol{\tau}_{K,i} \in V_h \\ & \langle [[\mathcal{H}]], \mu_{e,i} \rangle_e = 0, \quad \forall \mu_{e,i} \in M_h \end{aligned}$$

But why HDG over DG?

## From DG to HDG

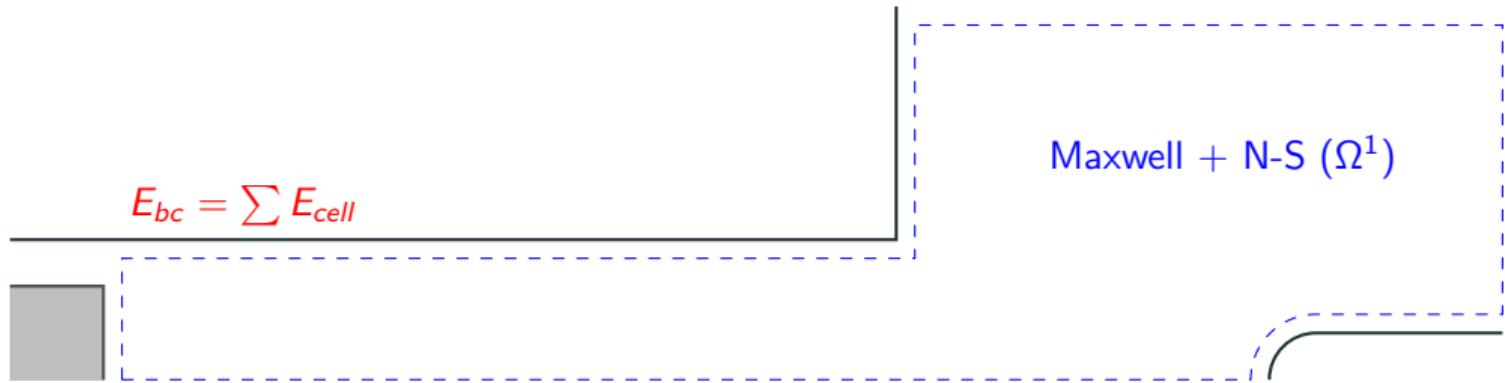
$$\begin{aligned}(\partial_t \mathbf{w}_h - \mathbf{S}_h, \varphi_{K,i})_K - (\mathbf{F}_{c,h} - \mathbf{F}_{v,h}, \nabla \varphi_{K,i})_K + \langle \mathcal{H}(\mathbf{u}_h, \mathbf{q}_h, \boldsymbol{\lambda}_h), \varphi_{K,i} \rangle_{\partial K} &= 0, \quad \forall \varphi_{K,i} \in W_h \\(\mathbf{q}_h, \boldsymbol{\tau}_{K,i})_K - (\mathbf{u}_h, \nabla \boldsymbol{\tau}_{K,i})_K + \langle \boldsymbol{\lambda}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_0} + \langle \mathbf{u}_{bc}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_{bc}} &= 0, \quad \forall \boldsymbol{\tau}_{K,i} \in V_h \\\langle [[\mathcal{H}]], \mu_{e,i} \rangle_e &= 0, \quad \forall \mu_{e,i} \in M_h\end{aligned}$$

**But why HDG over DG?**

As will be seen later, HDG allows for static condensation, effectively reducing the number of DOFs when using a Newton solver.

# ICP as a multi-domain problem

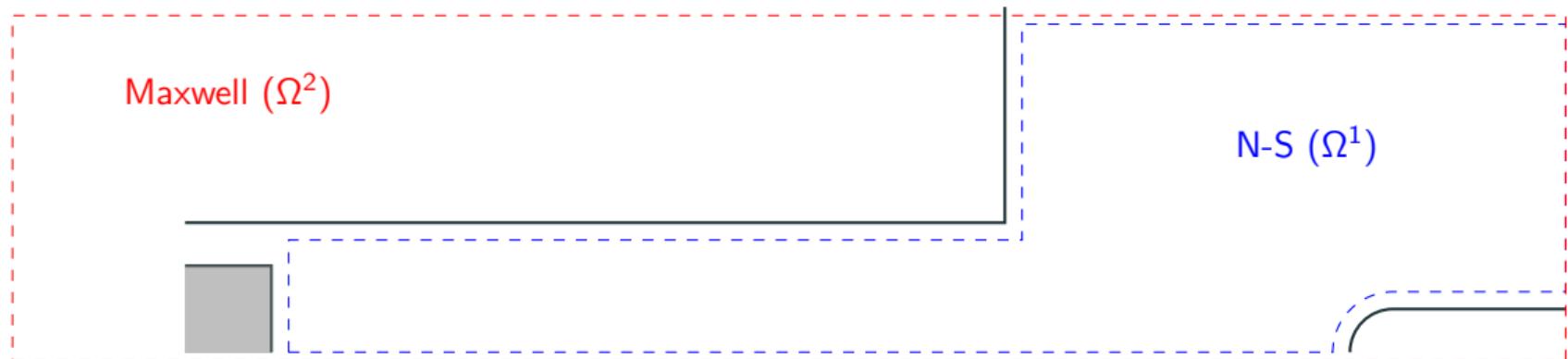
Single domain + integral boundary value



Compact but fills the jacobian matrix, making it incompatible with HDG.

# ICP as a multi-domain problem

## Two overlapping domain



Most widely used. Solve N-S while Maxwell frozen, and *vice versa*. Slow convergence.

# ICP as a multi-domain problem

## Multi-domain



Solves the system in a fully coupled manner, with two domains. The method we employ here.

## Modification of the model problem

$$\partial_t \mathbf{w}^I(\mathbf{u}) + \nabla \cdot (\mathbf{F}_c^I(\mathbf{u}) - \mathbf{F}_v^I(\mathbf{u}, \nabla \mathbf{u})) = \mathbf{S}^I(\mathbf{u}, \nabla \mathbf{u}), \quad \text{on } \Omega^I$$

$$\mathbf{u}^I = \mathbf{u}_{bc}^I, \quad \mathbf{x} \in \partial\Omega_d^I$$

$$\mathbf{F}_v^I \cdot \mathbf{n} = \mathbf{F}_{v,n,bc}^I, \quad \text{on } \partial\Omega_n^I$$

$$\mathcal{F}^{I,I'}(\mathbf{u}^I, \nabla \mathbf{u}^I, \mathbf{u}^{I'}, \nabla \mathbf{u}^{I'}) = 0, \quad \text{on } \gamma^{I,I'}, I \neq I'$$

$$\mathbf{u}^I(t=0) = \mathbf{U}^I, \quad \text{on } \Omega^I$$

## Modification of the model problem

$$\partial_t \mathbf{w}^I(\mathbf{u}) + \nabla \cdot (\mathbf{F}_c^I(\mathbf{u}) - \mathbf{F}_v^I(\mathbf{u}, \nabla \mathbf{u})) = \mathbf{S}^I(\mathbf{u}, \nabla \mathbf{u}), \quad \text{on } \Omega^I$$

$$\mathbf{u}^I = \mathbf{u}_{bc}^I, \quad \mathbf{x} \in \partial\Omega_d^I$$

$$\mathbf{F}_v^I \cdot \mathbf{n} = \mathbf{F}_{v,n,bc}^I, \quad \text{on } \partial\Omega_n^I$$

$$\mathcal{F}^{I,I'}(\mathbf{u}^I, \nabla \mathbf{u}^I, \mathbf{u}^{I'}, \nabla \mathbf{u}^{I'}) = 0, \quad \text{on } \gamma^{I,I'}, \quad I \neq I'$$

$$\mathbf{u}^I(t=0) = \mathbf{U}^I, \quad \text{on } \Omega^I$$

## Modification of the model problem

$$\begin{aligned}\partial_t \mathbf{w}^I(\mathbf{u}) + \nabla \cdot (\mathbf{F}_c^I(\mathbf{u}) - \mathbf{F}_v^I(\mathbf{u}, \nabla \mathbf{u})) &= \mathbf{S}^I(\mathbf{u}, \nabla \mathbf{u}), \quad \text{on } \Omega^I \\ \mathbf{u}^I = \mathbf{u}_{bc}^I, & \quad \mathbf{x} \in \partial\Omega_d^I \\ \mathbf{F}_v^I \cdot \mathbf{n} = \mathbf{F}_{v,n,bc}^I, & \quad \text{on } \partial\Omega_n^I \\ \mathcal{F}^{I,I'}(\mathbf{u}^I, \nabla \mathbf{u}^I, \mathbf{u}^{I'}, \nabla \mathbf{u}^{I'}) &= 0, \quad \text{on } \gamma^{I,I'}, I \neq I' \\ \mathbf{u}^I(t=0) = \mathbf{U}^I, & \quad \text{on } \Omega^I\end{aligned}$$

For ICP,

$$\mathcal{F} = \begin{pmatrix} \nabla E_p^{Re,NS} \cdot \mathbf{n}^{NS} - \nabla E_p^{Re,MAX} \cdot \mathbf{n}^{MAX} \\ \nabla E_p^{Im,NS} \cdot \mathbf{n}^{NS} - \nabla E_p^{Im,MAX} \cdot \mathbf{n}^{MAX} \end{pmatrix}$$

## Modification of the discrete formulation

$$\begin{aligned} & \left( \partial_t \mathbf{w}_h - \mathbf{S}_h^K, \varphi_{K,i} \right)_K - \left( \mathbf{F}_{h,c}^K - \mathbf{F}_{h,v}^K, \nabla \varphi_{K,i} \right)_K + \left\langle \boldsymbol{\mathcal{H}}^K, \varphi_{K,i} \right\rangle_{\partial K} = 0 \\ & (\mathbf{q}_K, \boldsymbol{\tau}_{K,i})_K - (\mathbf{u}_K, \nabla \boldsymbol{\tau}_{K',i})_K + \langle \boldsymbol{\lambda}_{\partial K}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_0} + \langle \mathbf{u}_{bc}, \boldsymbol{\tau}_{K,i} \cdot \mathbf{n} \rangle_{\partial K_{bc}} = 0 \\ & \langle [[\boldsymbol{\mathcal{H}}]], \mu_{e,i} \rangle_{e \setminus \bar{\Gamma}} + \langle \tilde{\mathcal{F}}^e, \mu_{e,i} \rangle_{e \cap \bar{\Gamma}} = 0 \end{aligned}$$

with  $\bar{\Gamma}$  the set of interfaces between subdomains.

## Numerical fluxes : convection

$$\mathcal{H} = \mathcal{H}_c - \mathcal{H}_v$$

$$\mathcal{H}_c = R\mathcal{H}_{c,n}(\mathbf{u}, \lambda) = R(\dot{m} + \dot{m}_p)\Psi_\lambda - R|\dot{m}|(\Psi_\lambda - \Psi_U) + \mathbf{P}$$

with

$$\Psi = \begin{pmatrix} 1 & v^\perp & v^\parallel & v_\theta & e + \frac{1}{2}\|\mathbf{v}\|^2 + \frac{p}{\rho} \end{pmatrix}^T$$

$$\mathbf{P} = \begin{pmatrix} 0 & p_\lambda & 0 & 0 \end{pmatrix}^T$$

## Numerical fluxes : diffusion

$$\mathcal{H} = \mathcal{H}_c - \mathcal{H}_v$$

$$\mathcal{H}_v(\mathbf{u}, \lambda, \mathbf{q}) = \mathbf{F}_v(\lambda, \mathbf{q}) \cdot \mathbf{n} + \gamma(h)(\lambda - \mathbf{u})$$

with

$$\gamma = C \max_{K \ni f} \left( \frac{1}{2\mathcal{V}} \sum_{f \in K} \mathcal{A} \right)$$

## Solution strategy

## Iterative procedure for power

## Results

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## **Conclusions and future work**

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