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A Monte Carlo Approach to the Pricing of Interest-Rate Derivatives

Candidato:
Nicolò Nuca

Relatore:
Prof. Guido Montagna

Correlatori:
Dott. Riccardo Bernini
Dott. Gianluca Molteni

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Contents

Introduction

Quite generally, it may be said that the theory of stochastic processes has spread into more and more disciplines. Concerning the field of mathematical finance, the essential idea of modeling an uncertain asset in terms of a Brownian motion appears simple but has quite numerous ramifications. A first significant step was achieved by Black and Scholes with their Nobel Prize model for the pricing of options. Since that time, the volume of exchange trading has been growing at an impressive rate, particularly for the case of derivative products. In this sense, the consequent increase in the available quantity of market data has attracted the interest of physicists whose methods and ideas developed in the field of complex systems became highly relevant to achieve reliable quantification and minimization of risk for the financial institutions involved. In fact, markets exhibit several of the properties that characterize an open system in which its components interact nonlinearly in presence of feedback, so that all information can be essentially described by an appropriate random hypothesis. The large number of market participants interacting with one other and to the external information imply a time series for the price of a given traded asset which is essentially indistinguishable from a stochastic process.

The thesis describes the research work done during the one-year internship at List SPA. The latter is an international company that provides advanced solutions for banking and finance, with particular reference to high-performance software and risk management frameworks. This thesis delves into this rapidly evolving area of finance, focusing specifically on the pricing models used for interest-rate derivatives. Following the cessation of Interbank Offered Rates (IBORs) settings at the end of 2021, industry focus is now on the transition to Risk Free Rates (RFRs) as the alternative benchmarks for many financial instruments, whose current total market exposure exceeds hundreds of trillions of dollars. The 2007-2008 financial crisis made clear the growing need to reform the process used to evaluate IBORs, which is predominantly based on banks' submissions, rather than actual market transaction. The interbank unsecured lending market that IBORs seek to represent has shrunk substantially, leading to concerns that they pose a systemic risk of manipulating their assessment process due to the lack of data. Consequently, national authorities have begun a replacement path which culminated in 2017 with the decision of the Financial Conduct Authority (FCA) of the United Kingdom to cease supervision over benchmark rates. This thesis is motivated by the need of ensuring the consistency of the adjusted pricing models and related calibration with respect to the market data on alternative reference rates that will be made progressively available.

The thesis is organized as follows. Before dealing with applications to finance, Chapter 1 discusses the historical approach to the basic concepts of stochastic processes and is meant to present the physics background to stochastic calculus. One example which will be cited later in the financial part is the treatment of the geometric Brownian motion, as it provides a viable model for the time evolution of some types of interest rates. For what concerns the more formal aspects of the theory of stochastic calculus, we avoid diving into the detailed proofs with the aim to provide an easy stepping stone onto the typical concepts and techniques. We have chosen to base the treatment on stochastic differential equation methods, although it is possible to derive an alternative approach which has its ground on the Fokker-Plank equation. Chapter 2 serves as an introduction to financial markets. We try to compile and explain the terminology and basic ideas around the concepts that will be highly used in the latter. The main argument shows that the value of a derivative in an arbitrage-free economy is the value of any replicating portfolio. We proceed following the popular and heavily exploited concept of numeraire which plays an important role in the theory of option pricing in the most recent literature and provides a powerful theoretical tool. In Chapter 3 we begin to consider the dynamics of interest rates, starting with the short-rate models. In this instance, we implicitly assume the whole yield curve to be completely determined by the evolution of its initial point, which is far from being reasonable in some scenario. In fact, without taking into account market volatility structures, the calibration often results poor. We introduce the general Heath-Jarrow-Morton (HJM) framework and discuss how the no arbitrage condition affects the evolution of the instantaneous forward rate. We turn to the market models, specifically the Libor Market Model, which is compatible with Black's cap formula as it imposes a lognormal dynamics to the forward rates in a particular probability measure. The main models are illustrated in different aspects ranging from theoretical formulation to an actual example of implementation in Python language. The calibration process, i.e. finding the best value for the parameters entering the equations such that the resulting values match the market quotes, in most cases involves the use of a minimization procedure. If not differently specified, it is done by mean of the Levenberg-Marquardt optimization algorithm. In Chapter 4 we present and discuss the original results of the internship work. We investigated the expected impact of the IBOR transition on the valuation of interest rate derivatives to assess the effectiveness of the new pricing models that have been developed to address these changes. In order to achieve this, we resumed the current literature and performed the simulations using different interest rate models. We follows the prompt by Lyashenko and Mercurio [LM19a] to construct an extension to the LMM which additionally is able to capture the dynamics of a backward looking rate. Based on the work by Moreni and Pallavicini [MP10] built on the IBOR rates, we adopt a single curve model which is suited to price derivatives on the new interest rate benchmark.

Chapter 1

From Brownian motion to stochastic calculus

The aim of this first chapter is to present a certain number of basic notions in the probabilistic framework required to understand the financial models used in the latter. Stock market offers an ideal testing ground for the statistical ideas that arises from the level of randomness underpinning price fluctuations, so, it is not a surprise that most of the tools used in the theoretical approach to finance were inspired by statistical physics. The ubiquitous concept is the Brownian motion. In fact, its first theoretical description in the natural sciences was performed in 1905 by Einstein but the earliest formalization is due to Bachelier and it deals with the pricing of options in the stock market and derivative securities.

1.1 Einstein theory

The observation that small pollen grains when suspended in water are found to be in a very animated and irregular state of motion, was first investigated by R. Brown in 1827, and because of his pioneering work this phenomenon took his name. The riddle of Brownian motion did not have a satisfactory explanation until 1905, when Einstein published his famous paper. Simultaneously, the same explanation was independently discovered by Smoluchowski, who was also responsible for much of the experimental verification of the theory: the incessantly motion of liquid molecules in which the pollen grain is suspended causes exceedingly frequent impacts that generated the peculiar behavior.

This explanation is as elegant and rather simple as astonishing, considering that at the time it was formulated neither the existence of molecules was obvious nor the mathematical formalism to deal with such a intricate

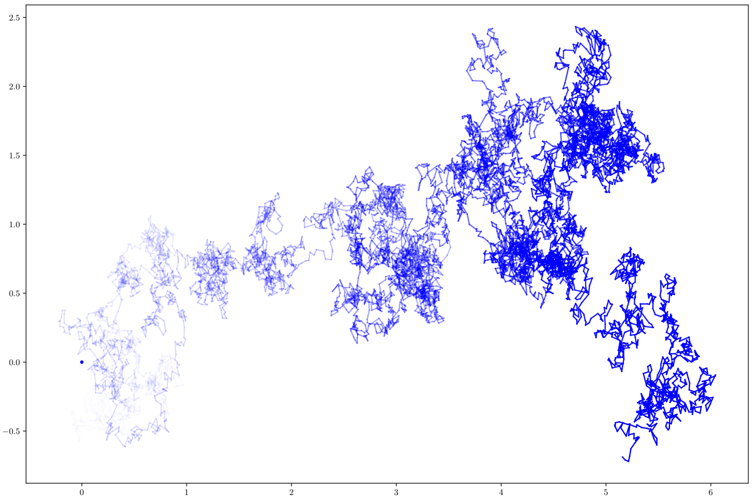

Figure 1.1: Simulation of two-dimensional Brownian motion with 5000 steps.

phenomenon. In fact, the motion of these molecules is so complicated that its effect can only be described in probabilistic terms of subsequent statistically independent impacts.

Indeed, for practical purposes, Einstein's explanation of the nature of Brownian motion must be regarded as the beginning of stochastic modeling of natural events. What follows now is a summary excerpt from Einstein's argument.
Suppose there be n suspended particles irregularly dispersed in a one-dimensional liquid. Assume that each individual particle execute a motion which is independent of the motions of all others and the movements of the same particle in different time intervals are independent processes, as long as the these time intervals are chosen not too small. Consider τ to be a small time interval compared to the observable time intervals, but nevertheless so large that the motion executed by the particle in two successive time interval τ can be thought of as events which are independent of each other. Let a time interval τ occur, and be Δ the increase along the x-axis different for each particle following a certain frequency law. A number dn of particles that experience a shift between Δ and $\Delta + d\Delta$ will follow an equation of the form

$$\mathrm{d}n = n\phi(\Delta)\,\mathrm{d}\Delta, \tag{1.1}$$

where

$$\int_{-\infty}^{\infty} \phi(\Delta)\, \mathrm{d}\Delta = 1, \tag{1.2}$$

and ϕ is peaked around small values of Δ and satisfies the condition

$$\phi(\Delta) = \phi(-\Delta). \tag{1.3}$$

Assuming that the number ν of particles per unit volume depends only on x and t, call $\nu = f\,(x, t)$. We compute the distribution of particles at time $t + \tau$ from the one at time t summing over all possible jumps leading to position x from position $x + \Delta$.

$$f(x, t+\tau)\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x+\Delta, t)\, \phi(\Delta)\, \mathrm{d}\Delta\, \mathrm{d}x. \tag{1.4}$$

Since τ is small, it holds

$$f(x, t+\tau) = f(x, t) + \tau \frac{\partial f(x, t)}{\partial t}. \tag{1.5}$$

Furthermore,

$$f(x+\Delta, t) = f(x, t) + \Delta \frac{\partial f(x, t)}{\partial x} + \frac{\Delta^2}{2!} \frac{\partial^2 f(x, t)}{\partial x^2}. \tag{1.6}$$

Since only small values of Δ have a relevant contribution, replacing these last results in Equation (1.4) leads to

$$\begin{aligned} f\,(x, t) + \tau \frac{\partial f(x, t)}{\partial t} &= f(x, t) \int_{-\infty}^{\infty} \phi(\Delta)\, \mathrm{d}\Delta \\ &+ \frac{\partial f(x, t)}{\partial x} \int_{-\infty}^{\infty} \Delta\, \phi(\Delta)\, \mathrm{d}\Delta \\ &+ \frac{\partial^2 f(x, t)}{\partial x^2} \int_{-\infty}^{\infty} \frac{\Delta^2}{2}\, \phi(\Delta)\, \mathrm{d}\Delta \end{aligned} \tag{1.7}$$

Due to Equation (1.3), the even terms on the right-hand side vanish. Introducing the *diffusion coefficient*

$$D = \frac{1}{\tau} \int_{-\infty}^{\infty} \frac{\Delta^2}{2}\, \phi\,(\Delta)\; \mathrm{d}\Delta \tag{1.8}$$

one obtains the differential equation of diffusion

$$\frac{\partial f\,(x, t)}{\partial t} = D \frac{\partial^2 f\,(x, t)}{\partial x^2} \tag{1.9}$$

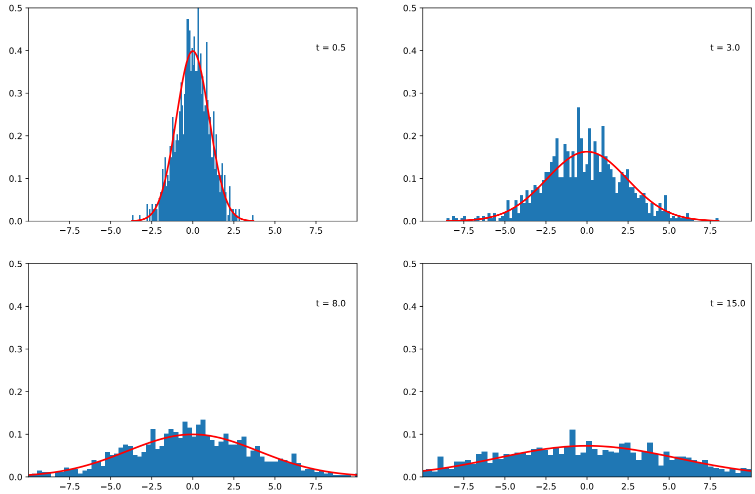

Figure 1.2: N=1000 samples from the solution of the diffusion equation at different times with D=1.

If we assume the initial condition $f(x,0) = n\delta(x)$, meaning that all particles are initially at the origin of our reference, the solution is completely determined

$$f(x,t) = \frac{n}{\sqrt{4\pi Dt}} \exp\left(-x^2/4Dt\right) \tag{1.10}$$

To show the evolution of the system as the time passes, one can draw multiple values from such a distribution evaluated at consecutive time and re-create the behavior of the initial data. It is now possible to derive the displacement λ in the direction of the x-axis that a particle experiences on average, or, to be precise, the square root of the expected value of the square of the displacement.

$$\lambda = \sqrt{\langle x^2 \rangle} = \sqrt{2Dt} \tag{1.11}$$

It is worth to point out, as shown in figure 1.3, that the jumps' effect on the particle in the early stages of motion could lead to a further resulting position compared to a regular linear drift.
Recall that Equation (1.10) describes a *probability density function*, i.e. when integrated on a volume $[x, x + \mathrm{d}x]$ its value is the probability of finding there a fraction $\mathrm{d}n$ of particles. To give a meaning to this argument while considering an individual particle, the integral of x weighted by $f(x,t)$ could be thought as representing the average position of multiple copies of the same particle

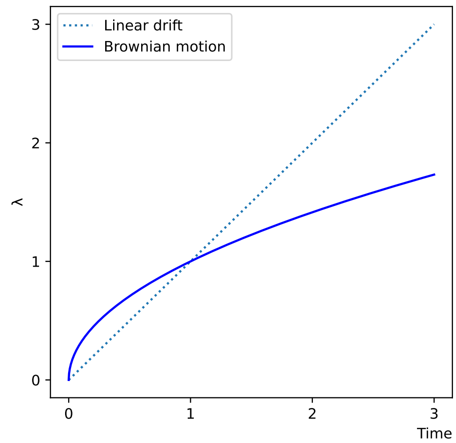

Figure 1.3: Typical displacement of Brownian motion in comparison to a deterministic linear drift.

over a large number of realizations of its random history. Figure 1.1 shown a simulation of the typical situation which occur in nature. This has been done by dividing a time interval of one unit into a thousand equal intervals. The value of the step multiplied by a number extracted from a normal random variable provides a good approximation of the Brownian increment. In the following, every figures of this kind is generated using this procedure.

1.2 Langevin interpretation

While the movement of a particle performing Brownian motion appears to be quite random and chaotic, it must nevertheless be describable by the same equation as is any other dynamical system.
Newton's equation of motion for the particle of radius a and mass m in a fluid medium with viscosity η is

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t} = F(t)\,\mathrm{d}t \tag{1.12}$$

where $F(t)$ is the total instantaneous force acting at time t due to interaction with the surrounding medium. Here the irregular fluctuation in time of the fluid concentration corresponds to the irregular kicks to the dust particle: in principle if the positions of the surrounding molecules are known as a function of time, the force is a known variable. In this sense it is not a random force at all. Still, it is not practical to look for an exact expression for F and it is desirable to include the total contribution from the density variation in a random term $\xi(t)$ that takes into account these numerous deterministic sources. It need also to be taken into account a friction force $-\gamma v(t)$, proportional to the velocity of the Brownian particle. The friction coefficient γ is given by Stokes' law $\gamma = 6\pi\eta a$. Then the equations of motion are:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v(t), \tag{1.13}$$

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = -\frac{\gamma}{m}v(t) + \frac{\xi(t)}{m}. \tag{1.14}$$

The effect of the occasional impacts is summarized by giving the first and second moments of the fluctuating force

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t_1)\, \xi(t_2) \rangle = \delta(t_1 - t_2). \tag{1.15}$$

The delta function in time indicate that there is no correlation between impacts in any distinct times, as a consequence of the separation of time scales discussed in the previous section: any memory between forces at different times are lost due to frequent collisions. It's convenient to assume that the fluctuating force has a Gaussian distribution determined by the moments in Equation (1.15). Given these mathematical assumption, it is not obvious that the Equation (1.14) has a unique solution, or even that $\dot{v}$ exists. There is a standard existence theorem for differential equation which guarantee the existence of a local solution if $\xi(t)$ is continuous. Still, it may not be unique, unless some stronger conditions are imposed: a further analysis of this arguments is made in the next section.

Now in order to obtain some results, it's convenient to rearrange the equation in a more useful way. Multiplying Equation (1.14) by x and using Leibniz rule one finds, dropping the obvious dependencies in the notation

$$\frac{m}{2}\frac{\mathrm{d}^2 x^2}{\mathrm{d}t^2} - mv^2 = -3\pi\eta a \frac{\mathrm{d}x^2}{\mathrm{d}t} + x\,\xi \tag{1.16}$$

From statistical mechanics, it is known that the mean kinetic energy of the Brownian particle should, in equilibrium, reach a value

$$\langle \frac{1}{2}mv^2 \rangle = \frac{1}{2}kT, \tag{1.17}$$

where T is the temperature and k is the Boltzmann's constant. Averaging Equation (1.16) over a large number of different particles and using the previous fact one obtain an equation for $\langle x^2 \rangle$

$$\frac{m}{2}\frac{\mathrm{d}^2\langle x^2 \rangle}{\mathrm{d}t^2} + 3\pi\eta a \frac{\mathrm{d}\langle x^2 \rangle}{\mathrm{d}t} = kT \tag{1.18}$$

where the term $\langle x\xi\rangle$ has been set equal to zero because it is fair to assume the independence of the force from the position of the particle and because of Equation (1.15). One then finds the general solution

$$\frac{\mathrm{d}x^2}{\mathrm{d}t} = \frac{kT}{3\pi\eta a} + C\exp\left(\frac{-6\pi\eta a}{m}t\right), \tag{1.19}$$

where C is an arbitrary constant. Inspecting the decay of the exponential, it approaches zero with a time constant of the order of 10^{-8} s, which is essentially immediately for a practical observation time. Thus, neglecting this term and integrating once more it follows

$$\langle x^2\rangle - \langle x_0^2\rangle = \frac{kT}{3\pi\eta a}t. \tag{1.20}$$

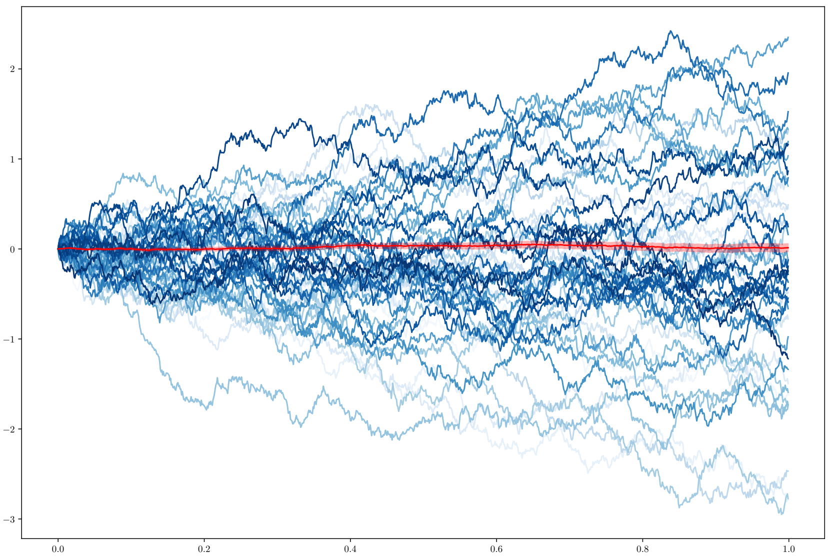

Figure 1.4: Simulation of N=50 one-dimensional Brownian motion.

Provided the identification

$$D = \frac{kT}{6\pi\eta a}, \tag{1.21}$$

it corresponds to Equation (1.11) derived by independent means by Einstein. The method of Langevin gives a very natural way of generalizing a dynamical

equation to a probabilistic equation. An adequate mathematical grounding for this approach, however, was not available until Itô formulated his concept of stochastic calculus. Still, Langevin's equation was the first example of stochastic differential equation, a differential equation with a noisy term ξ and hence whose solution is a random function.
Figure 1.4 shows multiple realization of the simplified equation $dX = \xi\, dt$. By averaging over the obtained values at the same instant, we can deduce the mean of the solution, which is represented by the red line. The error bar represents one unit of standard deviation which is calculated from the sample.

1.3 The Brownian motion

To properly justify the heuristic arguments given in the previous section, it's necessary to point out a precise definition to what is a Brownian motion $W(t)$. In the following, a slightly generalization will be done considering an n-dimensional analog. To construct $\{W(t)\}_{t\geq 0}$ it suffices to specify the family $\{\nu_{t_1,\dots,t_k}\}$ of probability measures on R^{kn} such that, given $x \in \mathsf{R}^n$ and

$$p(t,x,y) = (2\pi t)^{-n/2} \exp\left(\frac{-|x-y|^2}{2t}\right), \quad \text{for } y \in \mathsf{R}^n,\ t > 0, \tag{1.22}$$

it holds

$$\begin{aligned} &\nu_{t1,\dots,t_k}(F_1 \times \cdots \times F_k) = \\ &= \int_{F_1\times\cdots\times F_k} p(t_1,x,x_1)\, p(t_2-t_1,x_1,x_2)\dots p(t_k-t_{k-1},x_{k-1},x_k)\mathrm{d}x_1\dots\mathrm{d}x_k \end{aligned} \tag{1.23}$$

where it is adopted the convention that $p(0,x,y)\,\mathrm{d}y = \delta_x(y)$. The Kolmogorov's theorem assure the existence of a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $\{W(t)\}_{t\geq 0}$ on Ω such that the finite-dimensional distributions of W_t are given by

$$\begin{aligned} &P(B(t_1) \in F_1 \times \cdots \times W(t_k) \in F_k) = \\ &= \int_{F_1\times\cdots\times F_k} p(t_1,x,x_1)\, p(t_2-t_1,x_1,x_2)\dots p(t_k-t_{k-1},x_{k-1},x_k)dx_1\dots\mathrm{d}x_k. \end{aligned} \tag{1.24}$$

The Brownian motion thus defined is not unique, i.e. there exist many quadruples $(W(t), \Omega, \mathcal{F}, P)$ such that this definition holds. However, one can choose any version to work with, therefore it is convenient to identify $\omega \in \Omega$ with a continuous function $t \mapsto W(t, \omega)$ from $[0, \infty]$ into R^n. Thus, for the following pages, the Brownian motion is just the space $C([0, \infty], \mathsf{R}^n)$ equipped with the probability measures P given by (1.24). This version is called the *canonical* Brownian motion. From this definition it is possible to derive some of its basic properties [see Øks13, Section 2.2]:

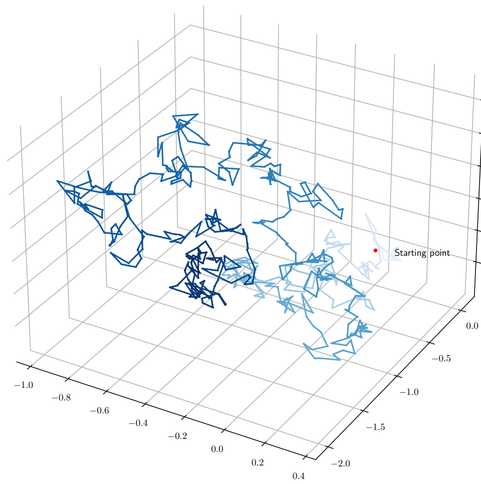

Figure 1.5: Simulation of a three dimensional Brownian motion.

1. $W(t)$ is a Gaussian process, i.e. for $0 \le t_0 < t_1 < \cdots < t_k$ the random variable $Z = (W(t_1), \ldots, W(t_k)) \in \mathsf{R}^{nk}$ has multi-normal distribution. This means that there exists a vector $M \in \mathsf{R}^{nk}$ and a positive definite matrix $C \in \mathsf{R}^{nk \times nk}$ such that its characteristic function $\phi_Z :\in \mathsf{R}^{nk} \mapsto \mathsf{C}$ is

$$\begin{aligned}\phi_Z(u_1, \ldots, u_n) &= \mathrm{E}\left[\exp\left(i \sum_j^{nk} u_j Z_j\right)..\right] \\ &= \exp\left(-\frac{1}{2}\sum_{jm} u_j c_{jm} u_m + i \sum_j u_j M_j\right)\end{aligned}$$

for all $u = (u_1, \dots, u_{nk}) \in \mathsf{R}^{nk}$, where E denotes expectation with respect to P. Moreover, M and C are respectively the mean and the covariance matrix of Z.

2. $W(t)$ has independent increments, i.e. $W(t_2) - W(t_1), \dots, W(t_k) - W(t_{k-1})$ are independent for all $0 \leq 0 < t_1 < \cdots < t_k$.

3. Brownian motion satisfies Kolmogorov's continuity theorem which ensures the existence of a continuous modification for $W(t)$. From now on, the symbol $W(t)$ will refer to such continuous version.

1.4 Stochastic differential equation

This section has the aim to find a reasonable mathematical interpretation of the peculiar term ξ in Equation (1.16). For notation purposes it is better to refer to the generalized equation of the form

$$\frac{\mathrm{d}X}{\mathrm{d}t} = b(t, X_t) + \sigma(t, X_t)\,\xi_t \tag{1.25}$$

To construct a solution, let $0 < t_0 < t_1, \cdots < t_m = t$ and consider the discrete version

$$X_{k+1} - X_k = b(t_k, X_k)\Delta t_k + \sigma(t_k, X_k)\xi_k \Delta t_k, \tag{1.26}$$

where

$$X_j = X(t_j), \quad \xi_j = \xi(t_j), \quad \Delta t_j = t_{j+1} - t_j. \tag{1.27}$$

Replace $\xi_k \Delta t_k$ by $\Delta V_k = V(t_{k+1} - V(t_k))$, where $\{V_t\}_{t>0}$ is some suitable stochastic process. Throughout the consideration made in the previous section such a stochastic process should respect the same properties that appears in the definition of the Brownian motion $W(t)$. Thus, provided the identification $V(t) = W(t)$ it follows

$$X_k = X_0 + \sum_{j=0}^{k-1} b(t_j, X_j)\Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_j)\Delta W_j. \tag{1.28}$$

Thus, the remainder of this chapter is dedicated to prove the existence of the limit of the right hand side for $\Delta t_k \to 0$. This means to give a sense to the formal expression

$$\int_0^t f(s, \omega)\,\mathrm{d}W_s(\omega), \tag{1.29}$$

where $B_t(\omega)$ is a one dimensional Brownian motion, for the widest class of function $f : [0, \infty] \times \Omega \to \mathsf{R}$. The natural interpretation of the Equation (1.29) in a Riemann-Stieltjes sense, however, is not possible since one can show that the trajectories of W_t are of locally unbounded variation [see Kar88, Theorem 9.18]. This has some peculiar consequences about the uniqueness of the summation limit. Consider the approximation of $f(t, \omega)$ by

$$\sum_j f(\bar{t}_j, \omega)\, \chi_{t_j, t_{j+1}}(t), \tag{1.30}$$

where the points $\bar{t}_j$ belong to the intervals $[t_j, t_{j+1}]$ and χ denotes the characteristic function. Unlike the Riemann-Stieltjes integral, it does make a difference here what points $\bar{t}_j$ is chosen: $\bar{t}_j = t_j$ leads to the Itô integral, while $\bar{t}_j = (t_j + t_{j+1})/2$ leads to the Stratonovich integral. For the purposes of this thesis, the Itô integral will be most convenient for the "not looking into the future" nature that will arise going trough its properties, and any further discussion from now on will be based on this choice.

1.4.1 Itô integral

The construction of the integral follows these steps: first of all one define the integral for a simple class of functions ϕ that can be used to approximate f in an appropriate sense; secondly, it is needed to define a class of well-behaved function on which the approximation procedure indicated above will work out successfully; lastly, one define the integral of f as the limit of the integral of the ϕ's when $\phi \to f$.
In some sense, to allow such a construction, the special class of function $f(t, \omega)$ must have the property that each of the function $\omega \mapsto f(t_j, \omega)$ only depends on the behavior of $W_s(\omega)$ up to time t_j (see Appendix A).

Definition 1.4.1. Let $\Upsilon(S, T)$ be the class of functions

$$f(t, \omega) : [0, \infty) \times \Omega \to \mathsf{R} \tag{1.31}$$

such that

1. $(t, \omega) \mapsto f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$.

2. There exists an increasing family of σ-algebras $\mathcal{H}_t$ for $t \geq 0$ such that B_t is a martingale with respect to $\mathcal{H}_t$ and f_t is $\mathcal{H}_t$-adarpted.

3. $\mathsf{P}\left[\int_S^T f(t,\omega)^2 \,\mathrm{d}t\right] < \infty$

Note that the measurability condition allows f_t to depend on more than $\mathcal{F}_t \subset \mathcal{H}_t$ as long as B_t remains a martingale with respect to the "history" of f_s with $s \leq t$.
Moreover, define $\phi \in \Upsilon$ to be elementary if it has the form

$$\phi(t,\omega) = \sum_j e_j(\omega)\, \chi_{[t_j,t_{j+1})}(t). \tag{1.32}$$

For such a function $\phi(t,\omega)$ define the integral as

$$\int_S^T \phi(t,\omega)\,\mathrm{d}W_t(\omega) = \sum_j e_j(\omega)\left[W_{t_{j+1}} - W_{t_j}\right](\omega). \tag{1.33}$$

It is now possible to enunciate the theorem summarizing the procedure briefly mentioned above [see Øks13, pp. 26–29].

Theorem 1.4.1. *Let $f \in \Upsilon(S,T)$. Then the Itô integral of f is defined by*

$$\int_S^T f(t,\omega)\,dW_t(\omega) = \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)\,\mathrm{d}W_t(\omega), \tag{1.34}$$

where the limit is intended in $L^2(P)$ sense and $\{\phi_n\}$ is a sequence of functions such that

$$\mathrm{E}\left[\int_S^T \left(f(t,\omega) - \phi_n(t,\omega)\right)^2 \,\mathrm{d}t\right] \to \quad \text{as } n \to \infty, \tag{1.35}$$

Note that the limit in (1.34) does not depend on the actual choice of $\{\phi_n\}$. It is worth to point out some relevant properties of the Itô integral. The following ones holds clearly for all elementary functions, so by taking limits one can obtain this for all $f, g \in \Upsilon$.

Corollary. *Let $f, g \in \Upsilon(0,T)$ and let $0 \leq S < U < T$. Then*

1. $\int_S^T f\,\mathrm{d}W_t = \int_S^U f\,\mathrm{d}W_t + \int_U^T f\,\mathrm{d}W_t$ *for almost all ω;*

2. $\int_S^T (cf + g)\,\mathrm{d}W_t = c\int_S^T f\,\mathrm{d}W_t + \int_S^T g\,\mathrm{d}W_t$ *for almost all ω and $c \in \mathsf{R}$;*

3. $\mathrm{E}\left[\int_S^T f\,\mathrm{d}W_t\right] = 0;$

4. $\displaystyle\int_S^T f \,\mathrm{d}W_t$ *is* $\mathcal{F}_t$*-measurable;*

As stated previously, an peculiar property of Itô integral is that it is a martingale

Corollary. *Let* $f(t,\omega) \in \Upsilon(0,T)$ *for all* T*. Then*

$$M_t(\omega) = \int_0^t f(s,\omega)\,\mathrm{d}W_s \tag{1.36}$$

is a martingale with respect to $\mathcal{F}_t$*.*

Lastly, one can define the multidimensional Itô integral as follows:

Definition 1.4.2. Let $W = (W_1, W_2, \dots, W_n)$ be a n-dimensional Brownian motion. Then $\Upsilon_{\mathcal{H}}^{m\times n}(S,T)$ denotes the set of $m \times n$ matrices $v = [v_{ij}(t,\omega)]$ where each entry $v_{ij}(t,\omega)$ satisfies the conditions of Definition (1.4.1) with respect to some filtration $\mathcal{H} = \{\mathcal{H}_t\}_{t\geq 0}$.
Let $v \in \Upsilon_{\mathcal{H}}^{m\times n}(S,T)$ and define

$$\int_S^T v\,\mathrm{d}W = \int_S^T \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nn} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_1 \\ \vdots \\ \mathrm{d}W_n \end{pmatrix} \tag{1.37}$$

to be the m-dimensional vector whose i'th component is the following sum of one dimensional Itô integrals

$$\sum_{j=1}^n \int_S^T v_{ij}(s,\omega)\,\mathrm{d}W_j(s,\omega). \tag{1.38}$$

1.4.2 Itô formula

Having defined the stochastic integral, the next ingredient must be some rules by which it can be manipulated. In this context however, there is no differentiation theory, so no analogue to the situation for ordinary Riemann's integrals is possible, where the fundamental theorem of calculus plus the chain rule comes in handy to evaluate the explicit calculations. Instead, the main tool of stochastic calculus is the change of variable formula knows as Itô's formula. It turns out that the family of well-defined stochastic integrals, following the prescription enunciated in the previous section, is stable under smooth maps.

Let $W(t,\omega) = (W_1(t,\omega), \dots, W_m(t,\omega))$ denote an m-dimensional Brownian motion. If each of the processes $v_{ij}(t,\omega) \in \Upsilon_{\mathcal{H}}$ satisfies the following condition

$$\mathsf{P}\left[\int_0^t v_{ij}^2(s,\omega)\,\mathrm{d}s < \infty\right] = 1 \quad \text{for all } t \geq 0. \tag{1.39}$$

In addition suppose that $u_i(t,\omega)$ is $\mathcal{H}_t$-adapted and

$$\mathsf{P}\left[\int_0^t |u_i(s,\omega)|\,\mathrm{d}s < \infty\right] = 1 \quad \text{for all } t \geq 0. \tag{1.40}$$

Then denote the n-dimensional stochastic integral

$$\begin{cases} \mathrm{d}X_1 = u_1\,\mathrm{d}t + v_{11}\,\mathrm{d}W_t + \dots + v_{1m}\,\mathrm{d}W_m \\ \quad\vdots \\ \mathrm{d}X_n = u_n\,\mathrm{d}t + v_{n1}\,\mathrm{d}W_t + \dots + v_{nm}\,\mathrm{d}W_m \end{cases}$$

in matrix notation as

$$\mathrm{d}X_t = u\,\mathrm{d}t + v\,\mathrm{d}W_t, \tag{1.41}$$

where

$$X_t = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix},\ u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},\ v = \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nn} \end{pmatrix},\ \mathrm{d}W_t = \begin{pmatrix} \mathrm{d}W_1(t) \\ \vdots \\ \mathrm{d}W_m(t) \end{pmatrix}.$$

Such a process X_t is called an n-dimensional Itô process.
In the following, it is given the actual formula in the general n-dimensional case: this preparatory choice will be motivated when considering the need to have more than one source of randomness to recreate a more realistic model. In particular, the correlation between two rates with different maturities can be tuned to adopt the market behavior only if two or more instantaneously correlated Brownian motions are supposed to lead the dynamic of the model.

Theorem 1.4.2. *Let X_t be a n-dimensional Itô process given as above. Let $f(t,x) = (f_1(t,x), \dots, f_p(t,x))$ be a twice continuously differentiable map from $[0,\infty) \times \mathsf{R}^n$ into R^p. Then the process*

$$Y_t = f(t, X_t) \tag{1.42}$$

is again an Itô process, whose component Y_k is given by

$$\mathrm{d}Y_k = \frac{\mathrm{d}f_k}{\mathrm{d}t}\,dt + \sum_i \frac{\partial f_k}{\partial x_i}\,\mathrm{d}X_t + \frac{1}{2}\sum_{ij} \frac{\partial^2 f_k}{\partial x_i \partial x_j}\,\mathrm{d}X_i \mathrm{d}X_j, \tag{1.43}$$

where $\mathrm{d}W_i \mathrm{d}W_j = \delta_{ij}\,\mathrm{d}t$ and $\mathrm{d}W_i\,\mathrm{d}t = 0$.

See [Øks13, Theorem 4.2.1] for a complete proof.

1.5 Solving a stochastic differential equation

It is the Itô formula that is the key to the solution of many stochastic differential equation. We show in the following some examples of application of Itô formula, that will be highly used in the work. First, turn to the existence and uniqueness natural question: the result is constructed by a similar Picard-Lindelöf procedure to obtain a unique solution in the standard view to ordinary differential equation [KS07, pp. 313–317].

Suppose that X_t takes values in the d-dimensional space, while W_t is an r-dimensional Brownian motion relative to a filtration $\mathcal{H}_t$. Let u be a measurable function from $\mathsf{R}^+ \times \mathsf{R}^d$ to R^d and v be a measurable function from $\mathsf{R}^+ \times \mathsf{R}^d$ to the space of $d \times r$ matrices.

Definition 1.5.1. Given the above assumptions, a process X_t defined on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is called a strong solution to the stochastic differential equation

$$\mathrm{d}X_t^i = u_i(t, X_t)\,\mathrm{d}t + \sum_{j=1}^{r} v_{ij}(t, X_t)\,\mathrm{d}W_t^i, \quad 1 \le i \le d, \tag{1.44}$$

with the initial condition Z, which is a $\mathcal{F}_r$-measurable d-dimensional vector, if the following conditions hold:

1. X_t is adapted to the filtration $\mathcal{F}_t$.

2. $X_0 = Z$ almost surely.

3. For every $t \in [0, \infty)$, $1 \le i \le d$ and $1 \le j \le r$,

$$\int_0^t \left(|u_i(s, X_s)| + |v_{ij}(s, X_s)|^2\right)\,\mathrm{d}s < \infty \quad \text{almost surely,} \tag{1.45}$$

4. For every $0 \le t < \infty$ and $1 \le i \le d$

$$X_t^i = X_0^i + \int_0^t u_i(s, X_s)\,\mathrm{d}s + \sum_{j=1}^{r} \int_0^t v_{ij}(s, X_s)\,\mathrm{d}W_s^j \tag{1.46}$$

holds almost surely.

Recall that the fact that X is $\mathcal{F}_t$-adapted means that for each t, the solution X_t up to that time is a functional of the Brownian path on the interval $[0, t]$, as a transformation induced from the space of the continue function to itself where the trajectory $W(\cdot, \omega)$ is mapped to the corresponding solution $X(\cdot, \omega)$. In most cases, this transformation is far from being easy to study, and it is extremely rare that one can solve a stochastic differential equation in some explicit manner and some numerical methods have to be introduced.
The next result establish whether or not a given stochastic differential equation admit a strong solution. As with ordinary differential equation, it shall be required the Lipschitz continuity of the coefficients and certain growth estimates at infinity, which guarantee respectively the local uniqueness of the solution and its global existence.

Theorem 1.5.1. *Suppose the same conditions of the previous definition hold. Require that there exist some constant c_1 such that, for all $t \in [0, \infty)$ and for all $x, y \in \mathsf{R}^d$,*

$$|u_i(t,x) - u_i(t,y)| \leq c_1 \|x - y\|, \quad 1 \leq i \leq d, \tag{1.47}$$

$$|v_{ij}(t,x) - v_{ij}(t,y)| \leq c_1 \|x - y\|, \quad 1 \leq i \leq d, 1 \leq j \leq r. \tag{1.48}$$

Assume also the coefficients do not grow faster than linearly, that is,

$$|u_i(t,x)| \leq c_1(1 + \|x\|), \quad |v_{ij}(t,x)| \leq c_2(1 + \|x\|), \quad 1 \leq i \leq d, 1 \leq j \leq r. \tag{1.49}$$

for some constant c_2. If Z is a $\mathcal{F}_0$-measurable R^d-valued vector that satisfies

$$\mathrm{E}\|Z\|^2 < \infty, \tag{1.50}$$

then there exists a strong solution to equation (1.44) *with the initial condition Z.*

In finance applications the existence of a strong solution is not strictly necessary since the use of a stochastic dynamics is to generate a model for some market value and there is no interpretation for the input Brownian motion. However, it is convenient to work in the context where both uniqueness and existence hold to make use of the martingale representation results, which is the most used framework in pricing evaluation.

1.5.1 Geometric Brownian motion

An achievement worth showing is the solution to a dynamics where the diffusion coefficient is a first order homogeneous polynomial in the underlying variable. This stochastic differential equation can be obtained as an exponential of a linear equation with deterministic coefficient and it's known by the name of lognormal model. Its evolution is defined according to

$$\mathrm{d}X_t = aX_t\,\mathrm{d}t + bX_t\,\mathrm{d}W_t, \tag{1.51}$$

where a and b are positive constants. This is a particular model widely used to describe distributions of financial assets such as share prices and from the seminal work of Black and Scholes in 1973, processes of this type are frequently used in option pricing theory to model general underlying dynamics. It is important to emphasize that this is just a particular choice for the stock price behavior, which may or may not be reasonable depending on the situation. To check that X is indeed a lognormal process, one can compute $d\ln(X_t)$ via Itô formula and obtain

$$X_t = X_0 \exp\left[(a - \frac{1}{2}b^2)t + b\,W_t\right], \tag{1.52}$$

so that X_t is indeed the exponential of a Brownian motion.

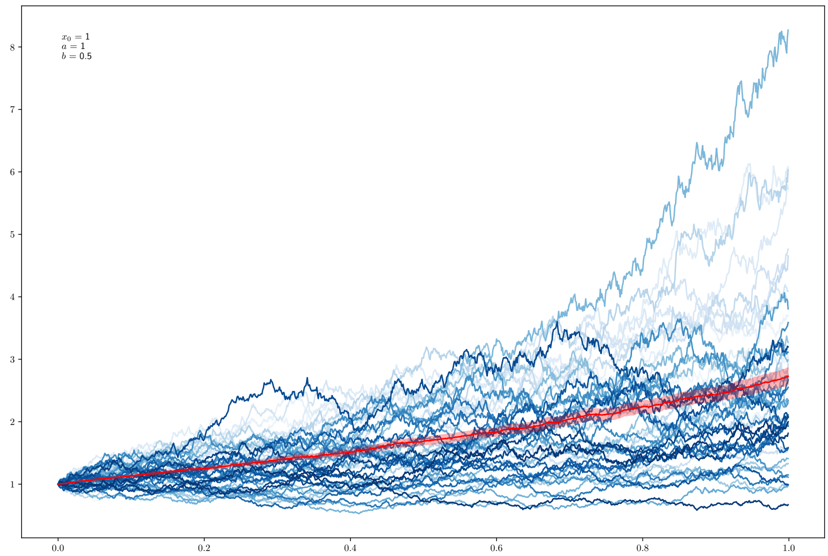

Figure 1.6: Simulation of N=50 one-dimensional geometric Brownian motion.

1.5.2 The Ornstein–Uhlenbeck process

We have finally all the theoretical tools needed to perform the correct analysis of the Langevin Equation (1.14) presented as the first noisy problem of this chapter. From a classic Newton approach, we managed to write the following stochastic differential equation

$$dX_t = -aX_t\,dt + b\,dW_t \tag{1.53}$$

where the notation has been changed a bit to better fit the new framework. Recall that the process $X(t)$ represent the velocity of the particle at the instant t and the coefficients depends on some of the system's properties: the temperature and the viscosity of the fluid and the mass of the particles. This dynamics in known by the name of Ornstein-Uhlenbeck process. To solve this equation, one can apply the Itô formula to the function $X_t e^{at}$ and obtain

$$X_t = X_0 e^{-at} + b\int_0^t e^{-a(t-s)}\,dW_s. \tag{1.54}$$

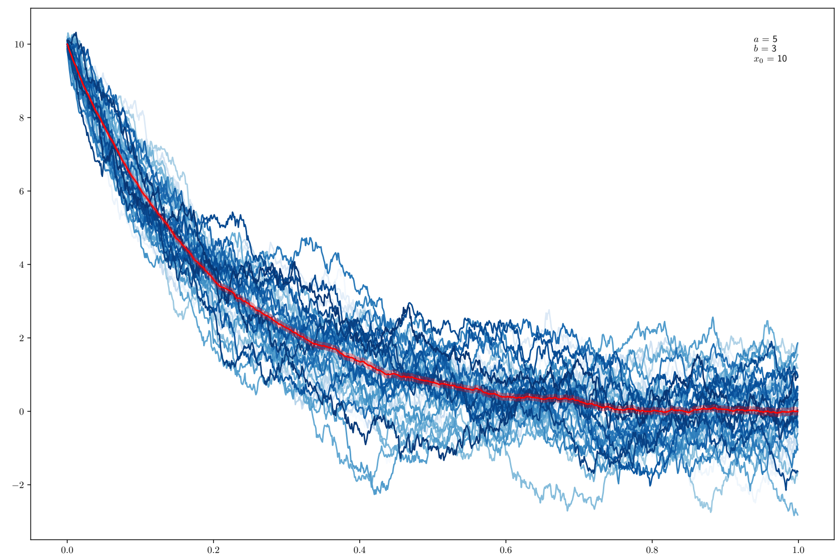

Figure 1.7: Simulation of N=50 one-dimensional Ornstein–Uhlenbeck processes.

Using Item 3 in Corollary 1.4.1, it is easy to show that the mean value of X_t is simply $X_0\,e^{-at}$. By mean of the same property we can compute the variance of the Ornstein-Uhlenbeck process resulting in

$$\mathrm{Var}[X_t] = \left\langle X_t^2 \right\rangle - \left\langle X_t \right\rangle^2 = \frac{b^2}{2a}\left(1 - e^{-2at}\right) \tag{1.55}$$

The above results tells us that, by waiting a sufficient long time, we expect to find the velocity of the particle to be zero with an asymptotically constant variance equal to $\mathrm{Var}[X_\infty] = b^2/2a$. The interpretation is fairly straightforward: the Brownian particle in the Langevin picture reaches the equilibrium with a characteristic time equal to the inverse of a and then its velocity fluctuates near the zero.
In the next chapter, we will see how this dynamics is used to model the spot rate behavior due to its property of being bounded, in contrast to the geometric Brownian motion in which the variance grows proportionally to the time passed. For this instance, we will use a more general dynamics with a value to which the process tends in the long run, being different from zero.

Chapter 2

Interest rates: basic toolkit

The key concepts in option pricing are the absence of arbitrage and portfolio replication. Basically, an arbitrage is a trading opportunity to generate profits from nothing with a positive probability. As a consequence, two portfolios having the same payoff at a given future date must have the same price today. Any economy we postulate must not allow arbitrage strategy to exist, so that we can evaluate the price of a derivative by finding a suitable portfolio of assets which is guaranteed to pay an amount identical to the payout of the derivative product.
These fundamental economic assumption was first established in the seminal work by Black and Scholes, which immediately led to their famous partial differential equation and, through its solution, to their option pricing formula. The same argument was subsequently used by Vasicek to develop a model for the evolution of the term structure of interest rates and for the pricing of interest rate derivatives. This procedure to option pricing is commonly used as the partial differential equations that appears have been extensively studied and are well understood. However, we will focus more on the martingale approach, allowing us to work within a more general framework that does not rely on regularity conditions and can be used to price more general derivatives, in particular American products. Also the underlying asset price dynamics remain quite general.

2.1 Portfolio strategies

We define an economy by specifying a model for the evolution of its assets. In the given economy, n non dividend paying securities are traded continuously from time 0 until time T. The price process is defined as the n dimensional process $P_t = (P^{(1)}, P^{(2)}, \dots, P^{(n)})_t$, where we assume it to be continuous and almost surely finite. The initial prices P_0 is some constant known at time zero and P satisfies

$$dP_t = a(P_t, t)\, dt + \sigma(P_t, t)\, dW_t, \tag{2.1}$$

where $B = (B^{(1)}, B^{(2)}, \dots, B^{(n)})$ is a d dimensional brownian motion on some filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathsf{P})$. The coefficients will be taken to be locally Lipschitz and subject to a linear growth condition, ensuring us the existence of a strong solution.
The main applications treated in the next chapters are pricing problems for interest rate derivatives, in which case the underlying assets P are zero coupon bonds, i.e. a contract paying a unit cash amount on its maturity date. Thus, one can define a continuum of assets, one for each and every maturity date, and the above setup may seems inadequate. In practice, most derivatives requires a known finite set of bonds in order to price and hedge the product so that the discrete approach will be sufficient.

Given a model for the asset price evolution, we can proceed to define an admissible trading strategy as the process $\phi_t = (\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(n)})$, where each component is interpreted as the number of units of assets held at time t. Any change in the strategy must occur as a consequence of the information available at the given current time t, thus ϕ is supposed to be predictable with respect to the natural filtration $\{\mathcal{F}_t^P\}$ generated by P.
The value associated with the strategy ϕ at time t is defined as the product $V_t = \phi_t \cdot P_t$. We will further assume that no money is either injected into or removed from the portfolio other than the initial amount, so that its value changes only due to modifications in the asset prices. Such a strategy is called self financing. Consequently, we can write

$$V_t = \phi_0 \cdot P_0 + \int_0^t \phi_s \cdot dP_s, \tag{2.2}$$

The above equation describes the law of the asset price P_t in some arbitrarily chosen currency such as euros or dollars but we may be interested in defining the underlying dynamics with respect to any chosen unit, including some that evolve randomly through time. We call this unit numeraire N, i.e. the value of a portfolio which is traded according to a self financing strategy that can be used as a reference asset to normalize all others with respect to it.
Whenever we change the units in which we describe the prices of the assets in the economy we are just presenting the information within the price in a different way, and so this should have no economic effect. In fact, it can be formally shown that the property of a portfolio to be self financing does not depend on the choice of the numeraraire [HK07, Theorem 7.13]. The

concept of numeraire is central to deciding whenever the economy satisfies the necessary condition to properly allow a derivative to be uniquely priced, and choosing the proper unit to express our model dynamics can lead to easier calculation. We begin with the following definition:

Definition 2.1.1. The measure N defined on the filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathsf{P})$ is a martingale measure for the economy if it is equivalent to P and there exists a numeraire N such that the process $P^N = N^{-1}P$ is a $\{\mathcal{F}_t^P\}$ martingale under N.

Recall that two measures P and N are said to be equivalent if both admit the same sets of possible paths, i.e. for any $A \in \mathcal{F}_t$ it holds $\mathsf{P}(A) = 0$ i.f.f. $\mathsf{N}(A) = 0$. This feature comes particularly in handy for our purposes, because whenever we operate a change of measure we would like to keep unaltered the set of realization that the economy can exhibit. The question about the existence of such a pair (N, N) does not concern us, as we usually specify a model already in some martingale measure. In other terms, we require P^N to follow a given equation for the dynamics and we assume P to be such that the appropriate martingale measure N exists.
However, we must ensure to maintain the economy free from any arbitrage strategies, being a fundamental prerequisite that allows us to price derivatives. This imposes condition both on the law for the evolution of the asset and the set of strategies which are allowed, as we will see in the next section.

2.1.1 No-arbitrage pricing

As briefly mentioned in the introduction, an arbitrage is a self financing portfolio strategy such that its value generates profit with positive probability but cannot generate a loss. If we impose the economy to be arbitrage free, then the price of a derivative must coincide with the value of a portfolio that replicates it. To prove this, suppose that the derivatives costs more than the replicating portfolio, then we can sell the first to buy the second and use the portfolio value to meet our obligation under the derivative contract. In the sale process, we are left with some free cash, thus we have constructed an arbitrage.
To ensure the absence of arbitrage in the formulation of a model, we must restrict the set of trading strategies

Definition 2.1.2. The self-financing strategy ϕ is admissible in the economy if, for every martingale measure N, the process

$$\frac{G_t^\phi}{N^t} = \int_0^t \phi_s \cdot dP_s^N$$

which represent the cumulative gains realized by the investor until time t, normalized to the numeraire N, is a $\{\mathcal{F}_t^P\}$ - martingale under N.

This additional constraint tells us that trading assets which have zero expected change in price over any time interval has zero expected revenue, which is intuitively what one would expect always to happen but it is not true in general. It is worth to notice that Definition2.1.2 is independent of the original measure P. Indeed, we are only interested in the sample of paths allowed by the economy and not on their actual probability and as long as we change from a numeraire pair to another, this implies that the set of admissible strategies remains the same. Moreover, when it comes to modeling in practice the problem of existence of such set does not concern us, as we prescribe the initial model directly in a martingale measure, so that at least the act of simply buying a unit asset and holding it upon some future time is an admissible strategy.
The existence of a martingale measure along with our definition of an admissible trading strategy ensures that the economy is arbitrage-free [HK07, Theorem 7.32]. In fact, we can always write the portfolio value in the existing martingale measure to check that any future expected income or loss would be null.

Theorem 2.1.1. *If the economy does not admit other then the set of admissible strategies, then it is arbitrage free.*

2.1.2 The fundamental pricing formulas

We have all the instruments needed to states and comprehend the main results in the formal aspects of the theory. The following theorem combines all the ideas encountered throughout the course of this chapter. The detailed proofs of the following statements can be found in [Bjö98, pp. 137–147]

Theorem 2.1.2. *Consider a price process V_t in an arbitrage free economy. Let V_T be the value of a derivative at time T. If V_T is attainable, meaning that we can find an admissible strategy ϕ such that*

$$V_T = \phi_0 \cdot P_0 + \int_0^T \phi_s \cdot dP_s \text{ almost surely,} \tag{2.3}$$

then the value of this derivative at time t is given by V_t.

As an immediate consequence, it follows from the martingale property of an admissible strategy what is known by the name of fundamental theorem of mathematical finance.

Proposition 2.1.3. *Assume the existence of the martingale measure* N *and the corresponding numeraire* N*. If* V_T *is the value of an attainable derivative, it holds*

$$V_t = N_t \mathrm{E}\left[V_T^N | \mathcal{F}_t^P\right]. \tag{2.4}$$

These results provides a nice pricing formula for any derivatives, but it may lack in consistence if we cannot provide a suitable portfolio strategy ϕ that achieve the desired value V_T. If every derivative admits a replicating portfolio, i.e. if it is attainable, we say that the market is complete. The following theorem gives a precise characterization of when a model satisfy this requirement.

Theorem 2.1.4. *The economy is complete if and only if, given two martingale measures* N_1 *and* N_2 *with common numeraire* N*, those are equivalent on* $\mathcal{F}_t^P$*.*

The existence of a unique martingale measure, therefore, not only makes the market arbitrage free, but also allows the derivation of a unique price associated with any derivative. Thus, within this setup, we perform the pricing by either finding the value of the associated replicating strategy, or by the expectation value of the claim payoff in a martingale measure. For most products encountered in practice such value will be determined by the joint distribution of a finite number of asset prices in a martingale measure. So the question arises as the how we should incorporate the information available on the financial market in a model which in turn yields a reasonable price for the considered derivative. The main ingredient we use is the price of traded instruments, and our model parameters will be estimated from this data.

2.2 Definitions and notation

This section is devoted to present the standard concepts and definitions in the interest rate world, starting with an overview of the major products in modern finance. The key concept is the following: when depositing a certain amount of money in a bank account, one expects that the amount grows at some rate as time goes by. This is due to the fact that lending money must be rewarded somehow, and one of the purposes of the following theoretical apparatus is to give some objective measurement to this concept of fairness. Note that as a first consequence it follows that money does have an intrinsic time-dependent value, so that receiving a given amount of money tomorrow is not equivalent to receiving exactly the same amount today.

2.2.1 The bank account and the short-rate

The first definition regards the money-market account, or simply bank account. This represents a risk-less investments, where profit is accrued continuously at some risk free rate prevailing in the market at every instant.

Definition 2.2.1. Let $B(t)$ be the value of a bank account at time $t \geq 0$, normalized such that $B(0) = 1$. It is assumed that it evolves according to the following dynamic:

$$dB(t) = r_t B(t)\, dt, \quad B(0) = 1, \tag{2.5}$$

where r_t is a function of time that be either positive or negative. As a consequence, the bank account is given by

$$B(t) = \exp\left(\int_0^t r_s\, ds\right). \tag{2.6}$$

This means that investing a unit amount at the initial time $t = 0$ yields at time t a value depending on the instantaneous rate r_t at which the bank account accrues.

If the spot rate follows a deterministic pattern so that $B(t)$ is known exactly, it is possible to evaluate any given amount of currency available in the future at the present time. As an example, forget its probabilistic nature and consider the case of agreement between two parties in which one pays the other a cash amount A at time 0 and in return receives this money back at some pre-agreed future date T, with an additional payment of interest $A \times B(T)$. If at time T this amount is equal to exactly one unit of currency, the initial amount to invest would be $A = 1/B(T)$. Hence, at time $t > 0$ the value of the amount A invested at the initial time is

$$A\, B(t) = \frac{B(t)}{B(T)}.$$

However, when dealing with interest rate products, the main variability that matters is obvious that of the interest rates themselves, so that is necessary to model its evolution in time through a stochastic process. Coming back to the initial assumption of a general r_t, this lead to the following definition.

Definition 2.2.2. The stochastic discount factor $D(t, T)$ is the amount at time t that is equivalent to one unit of currency in the future at time T, given by

$$D(t, T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_s\, ds\right). \tag{2.7}$$

Every upcoming movement of money can be valued currently considering this opportunity to invest it, given no risks, and receive some greater amount in the future, i.e. it must coincide to the accrued value of its present worth.

2.2.2 Zero-coupon bonds and spot interest rates

The next objects of investigation are the so called pure discount bonds of various maturity, also known as zero coupon bond. This type of debt does not pay interest and thus does not provide the owner a payment stream, but instead it's sold at a discount to its full face value, rendering the difference as a profit at maturity.

Definition 2.2.3. A T-maturity zero-coupon bond ZCB is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t, T)$.

We stress the subtle distinction between the above definition and the discount factor $D(t, T)$: the zero coupon bond price $P(t, T)$, being the value of a contract, has to be known at current time t and thus it is deterministic, while if the rates are stochastic $D(t, T)$ is a random quantity at time t depending on the future evolution of r_t up to time T. Still, one can think of the value $P(t, T)$ today to be such that the compounded short-rate on this value allows the seller of the bond to receive on average one unit of currency at maturity. We can show, in fact, how $P(t, T)$ is exactly the expected value of the random variable $D(t, T)$ under a particular probability measure.
Indeed, recalling Equation (2.4) and choosing the bank account $B(t)$ as numeraire, we can calculate the present value of V_T as

$$V_t = B(t)\,\mathrm{E}^{\mathsf{B}}\left[\frac{V_T}{B(T)}\,\middle|\,\mathcal{F}_t\right] = \mathrm{E}^{\mathsf{B}}\left[D(t, T)V_T \,|\, \mathcal{F}_t\right], \tag{2.8}$$

where the measure B is referred to as the risk neutral measure. In particular, the ZCB pays one unit at time T, so that $V_T = 1$ and we obtain

$$P(t, T) = \mathrm{E}^{\mathsf{B}}\left[D(t, T) \,|\, \mathcal{F}_t\right]. \tag{2.9}$$

The importance of this types of instrument relies in its feature to be used as basic auxiliary quantities from which all rates can be recovered, and in turn zero coupon bond prices can be defined in terms of any given family of interest rates. In moving from one to the other, it is needed to specify how the amount of interest is calculated from the time duration of the debt and the actual rate of interest. This feature is called compounding type, but for our purposes, we will suppose the accruing to be proportionally to the investment time, leading to the following definition:

Definition 2.2.4. The spot linearly compounded interest rate at time t for the maturity T, denoted by $L(t, T)$ is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t,T)$ units at time t

$$L(t,T) := \frac{1 - P(t,T)}{\tau(t,T)P(t,T)}, \tag{2.10}$$

where the $\tau(t,T)$ is the time measure between t and T.

The particular choice that is made to measure the time between two dates is known as day count convention and the amount $\tau(t,T)$ is called accrual factor. Different markets use different conventions to calculate this quantity, but having no impacts on our analysis. All that matters is that the interest payable is calculated via an accrual factor which is a known number.

The market spot LIBOR, the London Interbank Offer Rates, follows the above rules for compounding, which motivates the notation. A LIBOR is the rate of interest that one London bank will offer to pay on a deposit by another, given a certain maturity. There exist analogous interbank rates fixing in other markets (e.g. the EURIBOR, fixing in Brussels), but it is common practice to use the word LIBOR to intend any of these interbank rates. These are not risk-free rates, as there is always some chance that the bank borrowing the money will become either illiquid or insolvent. Due to the credit concerns, after the 2007 crisis banks became increasingly reluctant to lend to each other and the use of LIBOR to value derivatives was called into question. Moreover, the reliability among huge financial institutions became a primary factor, leading to a segmentation of the interest rate market corresponding to different lending duration. In fact, when the presence of credit and liquidity risks invalidated the possibility of linking LIBOR indexed deposits with risk-free bonds prices P(t,T), it became practice to model LIBOR of difference accrual period as different assets by means of a family of distinct stochastic processes [Hen14].
However, we will not focus on these details and in the following we will assume equation (2.10) to hold for every pair of t and T. This is a legitimate choice considering we will work later on in the context of the LIBOR transition to the new risk free interest rate reference.
The map $T \mapsto L(t,T)$ defines the so called term structure of interest rate at time t. An example of such a curve is shown in Figure 2.1.

2.2.3 Forward rates

We now move to the definition of forward rates. As the name suggests, these are the interest rates that can be locked in today but refers to an accrual

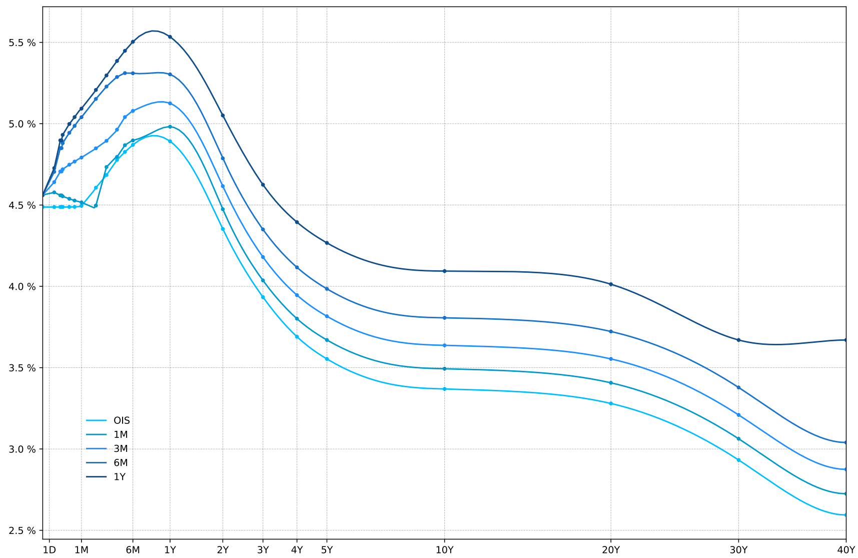

Figure 2.1: USD LIBOR ZC Curve on Dec 20 2022

period starting in a future instant. It is practice to characterize a forward rate by meaning of a prototypical contract called forward rate agreement, or FRA. This contract allows its holder to fix at current time t the interest rate between two future dates T and S at a desired value K. These two dates are called respectively expiry and maturity of the contract. At maturity time S, one counterparty agrees to pay the other $N\tau(S,T)K$ units of currency while receiving in return the amount $N\tau(S,T)L(T,S)$, where N is the contract nominal value and the LIBOR rate is set on date T.
The situation is summarized in Figure 2.2. The horizontal axis represents the time of cashflows, those below the axis are ones which we pay, while those above are ones that we receive.

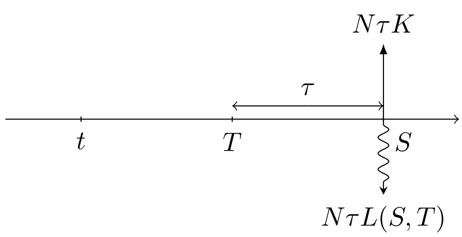

Figure 2.2: Cashflow diagram of a FRA

Is is straightforward to evaluate the contract at maturity time S as the amount equal to the difference of the cashflows exchanged at the payment day

$$\mathbf{FRA}_S(T,S,K) = N\tau\,(K - L(T,S)) \tag{2.11}$$

where we have assumed both rates have the same day-count convention. Notice, however, that this amount change its value back through time, as the fixing of $L(S,T)$ occur at T and it is unknown at any previous time. Furthermore, when the two counterparties enter the FRA, they do so at zero cost by choosing a proper value for the fixing rate K. To see this, suppose we hold at time t an amount of cash

$$V_t = P(t,T) - (1+\tau K)P(t,S), \tag{2.12}$$

and consider the following investment strategy. We immediately buy one unit of the ZCB with maturity at time T and sell $(1+\tau K)$ units of the ZCB maturing at time S. At time T we receive a unit payment from the first bond and we deposit this until time S. Thus, at time S, the deposit and the ZCB maturity give a net payment of

$$(1+\tau L(T,S)) - (1+\tau K), \tag{2.13}$$

which is precisely the payoff of the FRA. Hence, assuming that the economy does not admit arbitrage, V_t is the value of the FRA at time t. From Equation (2.12), we see that there is just one value of K that renders the contract fair at time t, i.e. such that the contract value is null to both counterparties. Equating to zero the FRA value, we find

Definition 2.2.5. The forward interest rate prevailing at time t for the expiry $T > t$ and maturity $S > T$ is defined by

$$F(t,T,S) := \frac{1}{\tau}\left(\frac{P(t,T)}{P(t,S)} - 1\right). \tag{2.14}$$

By mean of the above definition we can rewrite the value of a FRA as

$$\mathbf{FRA}_t(T,S,K) = P(t,S)\tau(K - F(t,T,S)), \tag{2.15}$$

where we set from now on every contract nominal value to one. Thus, the forward rate $F_t(T,S)$ may be viewed as a prior estimate of the future spot rate $L(T,S)$ based on market conditions at time t.
A requirement for any reasonable discount curve model is to be almost everywhere differentiable with respect to the maturity parameter T. This allows us define the term structure via the instantaneous forward rate, which intuitively represents a forward rate whose maturity is extremely close to its expiry.

Definition 2.2.6. The instantaneous forward interest rate $f(t,T)$ at time t for the maturity $T > t$ is defined as

$$f(t,T) := -\frac{\partial \ln(P(t,T))}{\partial T} \tag{2.16}$$

It corresponds to the rate that one can contract for at time t, on a riskless loan that begins at date T and is returned an instant later. Indeed, from Definition 2.2.5, consider the limit

$$\begin{aligned}\lim_{\tau\to 0} F(t,T,T+\tau) &= \lim_{\tau\to 0} \frac{P(t,T+\tau)-P(t,T)}{\tau P(t,T+\tau)}\\ &= -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T}\\ &= -\frac{\partial \ln(P(t,T))}{\partial T}.\end{aligned}$$

This confirm the view of $f(t,T)$ as the forward rate referred to a deposit whose maturity collapses towards its expiry.

2.2.4 Interest rate swaps

A generalization of the FRA is the interest rate swap IRS. Such a contract exchange payments between two different indexed source, called legs, starting from a common future date T_α up to T_β. Each leg is characterized by a set of payment dates. The fixed leg pays the fixed amount

$$\tau_i' K,$$

at each T_i' in $\mathcal{T}' = \left\{T'_{\alpha+1}, \dots, T_\beta\right\}$, corresponding to an accrual factor $\tau_i' = \tau(T'_{i-1}, T'_i)$ and a fixed rate K. Given the set of dates $\mathcal{T} = \{T_{\alpha+1}, \dots, T_\beta\}$ the floating leg pays the amount

$$\tau_i L(T_{i-1}, T_i),$$

being fixed at each T_i with an accrual factor $\tau_i = \tau(T_{i-1}, T_i)$. In general, a typical IRS in the market has a fixed leg with annual payments whereas the floating payments are six monthly or quarterly separated. When the fixed leg is paid the IRS is termed Payer IRS, whereas in the opposite case we call it a Receiver IRS. In order to value the swap, we can consider each of the two legs in turn. The fixed leg, being known at time t, has a value given by

$$V_t = K \sum_{i=\alpha+1}^{\beta} \tau_i' P(t, T_i).$$

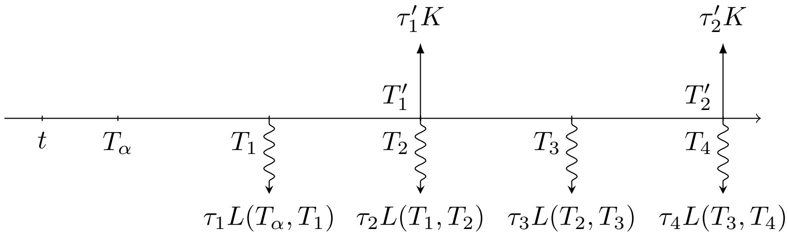

Figure 2.3: Cashflow diagram of a IRS

The floating leg can be priced by finding a strategy which replicates its payments. Suppose we hold at time t two ZCB for a total value of

$$P(t, T_\alpha) - P(t, T_\beta).$$

At T_α take the unit paid by the first and deposit it at LIBOR until time $T_{\alpha+1}$ to receive at that date the amount $1+\tau_{i+1}L(T_\alpha, T_{\alpha+1})$. The term $\tau_{i+1}L(T_\alpha, T_{\alpha+1})$ is needed for the swap replication, the unit of principal is used to make another deposit until time $T_{\alpha+2}$. Thus, we repeat this method at each floating payment date until the last date when we receive $1 + \tau_{i+1}L(T_{\beta-1}, T_\beta)$ and we can use the unit of currency to repay the amount owed on the ZCB sold at time t. Then, the value of a payers swap is given by the difference

$$\mathbf{IRS}_t(T_\alpha, \mathcal{T}', K) = P(t, T_\alpha) - P(t, T_\beta) - K \sum_{i=\alpha+1}^{\beta} \tau_i' P(t, T_i') \tag{2.17}$$

Notice that the price of a swap is independent of the floating payment structure. Swaps are usually entered at zero initial cost to both counterparties and the value of K that guarantee this property is called forward swap rate.

Definition 2.2.7. The forward swap rate at time t for the sets of dates $\mathcal{T}$ and $\mathcal{T}'$ is the fix rate in the fixed leg that makes the IRS a fair contract at present time. It is given by

$$S_t(T_\alpha, \mathcal{T}') = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i' P(t, T_i')}. \tag{2.18}$$

From Equation (2.14) we can derive the following identity

$$\frac{P(t, T_k)}{P(t, T_\alpha)} = \prod_{j=\alpha+1}^{k} \frac{P(t, T_j)}{P(t, T_{j-1})} = \prod_{j=\alpha+1}^{k} \frac{1}{1 + \tau_i F(t, T_{j-1}, T_j)}, \tag{2.19}$$

for all $i > \alpha$. Equation (2.18) can then be written in terms of forward rate by dividing both the numerator and the denominator by $P(t, T_\alpha)$

$$S_t(T_\alpha, \mathcal{T}') = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i' \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j' F_j'(t)}}. \tag{2.20}$$

where we set $F_j'(t) = F(t, T_{j-1}', T_j')$ and $F_j(t) = F(t, T_{j-1}, T_j)$. We will make use of the above expression in the last chapter.

2.2.5 Caps and Floors

Caps and floors offers a solution to the individuals which would like to buy some sort of protection against interest rate exposure, by locking the future payments corresponding to a maximum rate K. Indeed, a cap can be viewed as a floater leg whose payments are executed only if the corresponding rate is greater then K. A cap contract can be decomposed additively as the sum of individual payments that are termed caplets. The floorlets payments are defined in an analogous way. Given a start date T_α and the set of payment dates $\mathcal{T} = \{T_{\alpha+1}, \ldots, T_\beta\}$, the holder of a cap receives an amount

$$\tau_j \left[L(T_{j-1}, T_j) - K\right]^+ , \tag{2.21}$$

where $[x]^+ := \max(x, 0)$ indicates the positive part operator. Analogously, a floor provides the payment amount

$$\tau_j \left[K - L(T_{j-1}, T_j)\right]^+ ,$$

The value of K is specified in the contract and is called strike of the option. Since caps and floors consist in a linear combination of caplets and floorlets, we can proceed by evaluating each of these single payments individually. Consider the caplet payoff given by Equation (2.21). From Proposition 2.1.3, its current value corresponds to

$$V_t = N_t \,\mathrm{E}^{\mathsf{N}} \left[N_T^{-1} \tau_i \left(L(T_{j-1}, T_j) - K\right)^+ \,\middle|\, \mathcal{F}_t \right], \tag{2.22}$$

for some numeraire N and a martingale measure N. To calculate this price, it's convenient to choose $N_t = P(t, T_j)$, i.e. the discount bond maturing on the payment date T_j. The corresponding measure N is often referred to as the forward measure, denoted in this case by $\mathsf{T_j}$. This choice guarantees that $N_T = P(T, T) = 1$, so that this disappears from Equation (2.22).
Moreover, recalling Definition 2.2.5, we see that

$$F(t, T_{j-1}, T_j) P(t, T_j) = \frac{1}{\tau_j} \left(P(t, T_j) - P(t, T_{j-1})\right), \tag{2.23}$$

is the price at time t of a traded asset, since it is a fraction of the difference of two bonds. Therefore, from Definition 2.1.1,

$$F(t, T_{j-1}, T_j) = \frac{F(t, T_{j-1}, T_j)P(t, T_j)}{P(t, T_j)}$$

must be a martingale under the forward measure. Thus, since by Definition 2.2.4, $F(T_{j-1}, T_{j-1}, T_j) = L(T_{j-1}, T_j)$, it follows

$$F(t, T_{j-1}, T_j) = \mathrm{E}^{\mathsf{T_j}}\left[L(T_{j-1}, T_j) \,|\, \mathcal{F}_t\right]. \tag{2.24}$$

As long as we model F as a martingale under $\mathsf{T_j}$ we will have a model that

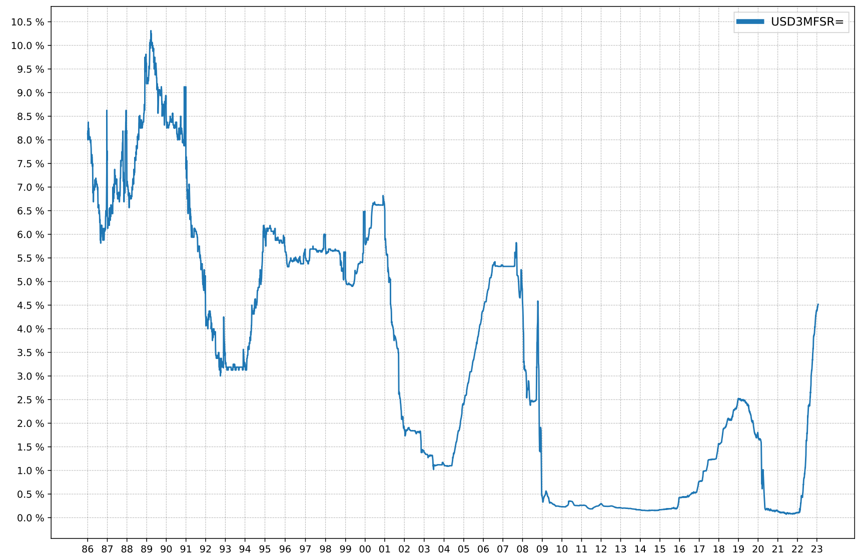

Figure 2.4: USD 3M LIBOR from 2 Gen 1986 to 25 Gen 2023

is arbitrage free and the caplet value will be given by Equation (2.22). It is market practice to use a log normal martingale, meaning

$$dF^j(t) = \sigma_t F^j(t)\, dW_t \tag{2.25}$$

for some volatility σ and a Brownian Motion W. From Theorem 1.4.2, this yields the solution

$$F^j(t) = F^j(0) \exp\left(\int_0^t \sigma_s \, dW_s - \frac{1}{2}\int_0^t \sigma_s^2 \, ds\right). \tag{2.26}$$

Substituting into Equation (2.22) we obtain

$$
\begin{aligned}
V_t &= \tau_i P(t,T_j)\,\mathrm{E}^{\mathsf{T_j}}\left[(L(T_{j-1},T_j)-K)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= \tau_i P(t,T_j)\,\mathrm{E}^{\mathsf{T_j}}\left[\left(F^j(t)\exp\left(\int_t^{T_{j-1}}\sigma_s\,dB_s-\frac{1}{2}\int_t^{T_{j-1}}\sigma_s^2\,ds\right)-K\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= \tau_i P(t,T_j)\left(F^j(t)\Phi(d_+)-K\Phi(d_-)\right)
\end{aligned}
$$

where Φ is the standard gaussian cumulative distribution function and

$$
d_\pm = \frac{\log(F^j(t)/K)}{\nu_{T_{j-1}}\sqrt{T_{j-1}-t}} \pm \frac{1}{2}\nu_{T_{j-1}}\sqrt{T_{j-1}-t}, \quad \nu_{T_{j-1}}^2 := \frac{1}{T_{j-1}-t}\int_t^{T_{j-1}}\sigma_s^2\,ds. \tag{2.27}
$$

This is precisely the famous Black's formula and it is market practice to price a cap by mean of this equation. The volatility parameter ν can be retrieved from market quotes by first pricing market instruments with the chosen model and then inverting the Black formula to retrieve the implied volatility associated with each maturity and strike. The relation one to one between the price and the volatility allows the market to quote, instead of the price, the volatility parameter itself that enters such formulas.
At any time t before the start, the amount $\tau_j\left[F^j(t)-K\right]^+$ is referred to as the intrinsic value of the caplet. If this quantity is positive then the option is said to be in the money, whereas if it is null the option is out of the money. When the strike equals $F^j(t)$ the caplet is said to be at the money (ATM). We will refer to the following notation

$$
\mathbf{CAPL}_t(T_{i-1},T_i,K) = \tau_i P(t,T_i)\,\mathrm{Black}(K,F^i(0),\nu_{T_{i-1}},1), 54 \tag{2.28}
$$

where

$$
\mathrm{Black}(K,F,\nu,\omega) = F\omega\Phi(\omega d_+) - K\omega\Phi(\omega d_-), \tag{2.29}
$$

and the parameters entering the formula are calculated following Equation (2.27). In conclusion, the price of a cap can be achieved through the sum

$$
\mathbf{CAP}_t(T_\alpha,\mathcal{T},K) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)\,\mathrm{Black}(K,F^i(0),\nu_{\alpha\beta},1), \tag{2.30}
$$

Analogously, we can derive a formula for the price of a floor and obtain

$$
\mathbf{FLR}_t(T_\alpha,\mathcal{T},K) = \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)\,\mathrm{Black}(K,F^i(0),\nu_{\alpha\beta},-1). \tag{2.31}
$$

Notice, however, that the Formula (2.30) assumes the same volatility for each caplet concurring to the given cap, which is then set to a value matching the market price. On the other hand, caplet volatilities can be stripped from the market cap volatilities along a procedure that we will not cover here [HW13b]. However, the caplet volatility is a fundamental quantity to retrieve as it contains the information on the parameter σ_t entering the forward rate dynamics.

2.2.6 Swaptions

Finally, we turn to the main ingredient to which the last chapter analysis is dedicated. A European swaption is an option on a swap, giving the right but no obligation to enter the contract at a given future time called maturity. Usually it coincides with the start of the underlying IRS, while $T_\beta - T_\alpha$ defines the tenor of the swaption.
There are two types of swaptions, depending if the underlying IRS is either a payer or receiver. Recalling Equation (2.17) and Definition 2.2.7, we obtain the alternative formula

$$\mathbf{IRS}_t(T_\alpha, \mathcal{T}', K) = \sum_{i=\alpha+1}^{\beta} \tau_i' P(t, T_i') \left(S_t(T_\alpha, \mathcal{T}') - K \right). \tag{2.32}$$

The option will be exercised only if this value is positive at the swap first reset date T_α, which is also assumed to be the swaption maturity. Thus, the payer swaption payoff is equal to

$$V_{T_\alpha} = \sum_{i=\alpha+1}^{\beta} \tau_i' P(T_\alpha, T_i') \left[S_{T_\alpha}(T_\alpha, \mathcal{T}') - K \right]^+ . \tag{2.33}$$

At any time $t < T_\alpha$ we use the terminology in the money, out of the money, at the money as we did for caplets and floorlets depending on the sign of $S_{T_\alpha}(T_\alpha, \mathcal{T}') - K$. Contrary to the cap case, we cannot proceed to price each term in the sum individually, because of the non linearity of the positive part operator. This implies that the above quantity depends on the joint action of the rates involved in the contract payoff, and their correlations are fundamental in handling their price.

To evaluate Equation (2.33) we choose the quantity $\sum_{i=\alpha+1}^{\beta} \tau_i' P(T_\alpha, T_i')$ to be the numeraire, and we call swaption measure S the corresponding martingale measure. From Definition 2.2.7, dropping the $(T_\alpha, \mathcal{T})$ notation, it

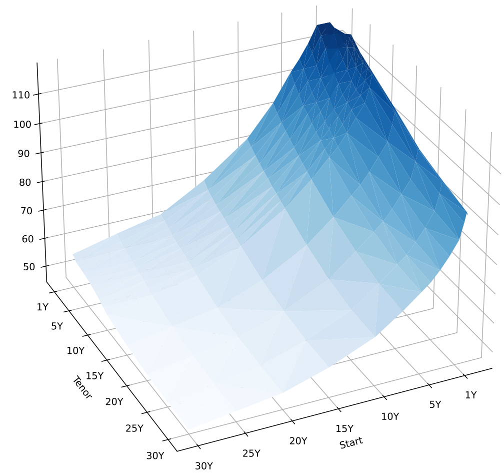

Figure 2.5: SB/3M LIBOR Swaption volatility surface (bp) on 26 Jan 2023.

is easy to see that S_t is the ratio of asset prices over the numeraire, so must be a martingale under S. By choosing a log normal model of the form

$$dS(t) = \sigma_t S(t)\, dW_t, \tag{2.34}$$

we are back in the same framework as when we priced the caplets, so that the price of the swaption follows the Black's Formula (2.29)

$$\mathbf{PS}_t(T_\alpha, \mathcal{T}', K) = \sum_{i=\alpha+1}^{\beta} \tau_i' P(t, T_i')\mathrm{Black}(K, S(0), \nu_{\alpha\beta}, 1). \tag{2.35}$$

A similar formula is used for a receiver swaption, which gives the holder the right to enter at time T_α a receiver IRS

$$\mathbf{RS}_t(T_\alpha, \mathcal{T}', K) = \sum_{i=\alpha+1}^{\beta} \tau_i' P(t, T_i')\mathrm{Black}(K, S(0), \nu_{\alpha\beta}, -1). \tag{2.36}$$

Again, the quantity ν_{T_α} is a parameter quoted in the market, and it will depends in general on the swaption tenor and maturity. An example of the implied volatility is represented in Figure 2.5.

Chapter 3

From short-rate models to the Libor Market Model

In this chapter, we introduce the first dynamics model for the term structure. Ideally, what we aim to achieve is a complete knowledge about the evolution of the set $\{P(t,T)\,|\,T > t\}$ for any future time instant t. The whole evolution of the discount curve can be characterized by a diffusion term, with a arbitrary dimensionality which we refer to as the number of factor. This degree of freedom can be chosen to adapt our model to the correlations that the market shows between different rates of the yield curve at a certain time.

3.1 One-factor short-rate models

The theory of interest rate models originally relied on the assumption of a specific one dimensional dynamic structure for the instantaneous rate r introduced in Definition 2.2.1. The pioneering approach proposed by Vasicek allows the derivation of an arbitrage free price for any interest rate derivative and it is based on the assumption of a particular dynamic for the quantity $r(t)$. His brilliant derivation followed the Black and Scholes argument for the construction of a suitable riskless portfolio, while taking into account some slightly modifications due to the non tradable nature of interest rates. The model for the asset price processes can be formulated by looking at the current information available on the market quotes. The initial measure P under which we specify the chosen dynamics is then referred to as the real world measure. We start by modeling the short-rate r as the solution of the following SDE in the P measure

$$dr(t) = \mu(t, r(t))\,dt + \sigma(t, r(t))\,dW_t. \tag{3.1}$$

The only exogenously given asset in our current economy is the bank account

$$B(t) = \exp\left(\int_0^t r(t)\,dt\right)$$

from which we desire to retrieve the value of all possible bonds as derivatives of the underlying short-rate r. The Vasicek's result shows the relationship which must hold between the price processes of bonds with different maturities coherently with the arbitrage free assumption. Firstly, it is possible to show that our currently model prescriptions does not uniquely determine the bond prices. Let $P_T(t) := P(t, T, r(t))$ be a generic function that represents our desired relation between r and the bond price $P(t, T)$. By Itô Formula 1.4.2 and Equation (3.1), we have

$$dP_T(t) = P_T(t)\alpha_T(t)\,dt + P_T(t)\sigma_T(t)\,dW_t, \tag{3.2}$$

provided the identifications

$$\alpha_T(t) = \frac{1}{P_T(t)}\left(P_T(t) + \mu\frac{\partial P_T(t)}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P_T(t)}{\partial r^2}\right) \qquad \sigma_T(t) = \frac{\sigma}{P_T(t)}\frac{\partial P_T(t)}{\partial r}.$$

Denote by ϕ_T the fraction of a traded bond with maturity T and suppose we pursue the trading strategy (ϕ_T, ϕ_S) with relative value increment given by

$$\frac{dV}{V} = \phi_T\frac{dP_T}{P_T} + \phi_S\frac{dP_S}{P_S}$$

We can obtain a deterministic rate of return by choosing an appropriate fraction of bonds to hold such that the diffusion term vanishes and we are left with a completely riskless portfolio. By substituting Equation (3.2) in the above formula, provided the choice

$$\phi_T = -\frac{\sigma_S}{\sigma_T - \sigma_S}, \quad \phi_S = \frac{\sigma_T}{\sigma_T - \sigma_S}$$

we obtain

$$dV = V\left(\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S}\right)dt$$

to be the value of our risk less portfolio. Now, the assumption of arbitrage absence implies that this portfolio must return exactly as the bank account. Thus, we have the condition

$$\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S} = r(t)$$

or, equivalently

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}$$

Notice how each side of the above equation depends respectively on S and T singularly. This implies that the process

$$\lambda(t) := \frac{\alpha_T(t) - r(t)}{\sigma_T(t))},$$

must be independent from the choice of T. This quantity is called market price of risk, as it measures in volatility unit the exceeding rate of return for a T-maturity bond over the risk less rate of return. Regardless of the maturity, all bonds must have in common the same market price of risk. By mean of λ we can rewrite the bond price dynamics and by further manipulating it we obtain the equation for the term structure evolution. In order to be able to solve it, however, it is needed to specify the market price of risk as a ulterior degree of freedom that enters our model. In particular this means that the choice of λ implies an assumption concerning the aggregate risk aversion on the market.

For what concern our scope, instead of specifying λ under the objective probability measure P we will directly assume the dynamics of the short-rate r under the risk free measure B. In fact, it can be shown that, when there are no arbitrage opportunities, choosing a particular model in a given martingale measure is equivalent to the choice of an appropriate market price of risk. Indeed, when a price process is expressed with respect to a numeraire, and thus it is a martingale, we set λ equal to the volatility of the chosen numeraire and the model is said to be forward risk neutral with respect to it [Hul06, pp. 483–491].

3.1.1 The Vasicek model

The Vasiceck model is defined, under the risk neutral measure B, by the dynamics

$$dr(t) = k\big[\theta - r(t)\big]\,dt + \sigma\,dW_t, \quad r(0) = r_0, \tag{3.3}$$

where r_0, k, θ and σ are positive constants. We have already encountered and solved a similar model in Section1.5.2. Integrating Equation (3.3) yields

$$r(t) = r_0 e^{-kt} + \theta\left(1 - e^{-kt}\right) + \sigma \int_0^t e^{-k(t-s)}\,dW_s,$$

which is a normally distributed process with mean and variance given by

$$\mathrm{E}\left[r(t)\right] = r_0 e^{-kt} + \theta\left(1 - e^{-kt}\right), \quad \mathrm{Var}\left[r(t)\right] = \frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right).$$

It maintain the property of being mean reverting in the sense that it will

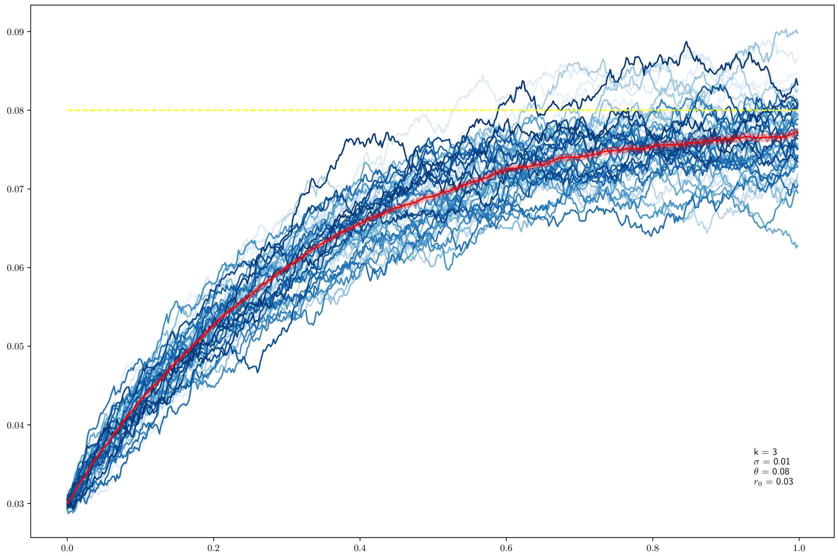

Figure 3.1: Simulation of N=100 paths of the Vasicek model.

tend to revert to the mean level θ for t going to infinity. The drift term in indeed positive whenever r is less then θ and negative otherwise, so that r goes, at every instant, closer on average to θ. Such a feature is desirable as it allows to capture monetary authority's behavior of setting target rates. The term structure can be recovered either by solving Equation (2.9) or via Equation (3.2). We obtain

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{3.4}$$

where

$$\begin{aligned} A(t,T) &= \exp\left[\left(\theta - \frac{\sigma^2}{2k^2}\right)\left(B(t,T) - T + t\right) - \frac{\sigma^2}{4k}B(t,T)^2\right], \\ B(t,T) &= \frac{1}{k}\left(1 - e^{-k(T-t)}\right). \end{aligned}$$

From the bond prices we can compute the prices through the expectation under the the risk neutral measure. The parameters k, θ, σ involved can found by observing the market price quotes and performing a calibration procedure. In particular, we will make large use of the initial discount curve

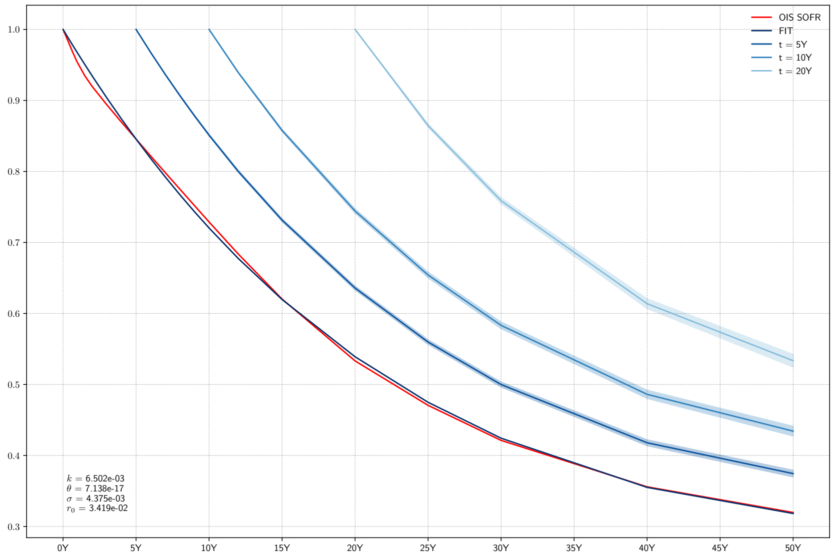

Figure 3.2: ZC Curve evolution through the Vasicek Model.

and option prices, either caps and floors or swaptions. Figure 3.2 shows an example of a calibration procedure on Jan 26th 2023 data and the subsequent bond price evolution. Since the Vasicek model does not allows a direct use of the initial curve values, all the model's parameters are fitted through a least squares method.

3.1.2 The Hull-White model

The major disadvantage of the presented model is that it does not provide a perfect fit to the initial term structure of interest rate. An improvement of the model should, however, maintain the analytical tractability in order to avoid any further computational stress. The Hull and White extension of the Vasicek model introduces a time dependency in the drift parameter in order to reflect the expected trends of the macroeconomic variables, allowing a more precise calibration to the market data. Thus, we assume that the instantaneous short-rate process evolves according to

$$dr(t) = \big[\theta(t) - ar(t)\big]\, dt + \sigma\, dW_t$$

where now θ is a function of time chosen as to exactly fit the term structure currently observed in the market, whereas a and σ are chosen in order to obtain a realistic volatility structure. Denote by $f(0,T)$ and $P(0,T)$ respectively the market instantaneous forward rate and the market discount factor for the maturity T. Then, Equation (3.4) still holds with the following prescription

$$A(t,T) = \exp\left[\int_t^T \left(\frac{1}{2}B(s,T)^2 - \theta(s)B(s,T)^2\right) ds\right], \tag{3.5}$$

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right). \tag{3.6}$$

Given an arbitrary bond curve $\{P(0,T) \,|\, T > 0\}$ the unknown function θ that allows the bond prices of our model to fit the market data must be of the form [Bjö98, pp. 336–337]

$$\theta(t) = \left.\frac{\partial f(t,T)}{\partial T}\right|_{t=0} + af(0,T) + \frac{\sigma^2}{2a}\left(1 - e^{-2aT}\right),$$

Once this choice of θ has been made, we use Equation (3.5) and Equation (3.4) to obtain the bond prices as

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[B(t,T)f(0,t) - \frac{\sigma^2}{4a}B(t,T)^2\left(1 - e^{-2at}\right) - B(t,T)r(t)\right]$$

3.2 The Heath-Jarrow-Morton framework

The first historical alternative to short-rate models change the fundamental quantity to model into the instantaneous forward rate, introduced in Definition 2.2.6. This framework has the advantage to fully characterize the stochastic evolution of the yield curve by mean of the instantaneous volatility structure of the forward rate dynamics. Precisely, we will see that the drift term in the f's dynamics is determined as a suitable transformation of its diffusion coefficient, as a consequence to the no arbitrage condition.
Assume the discount curve to be absolutely continuous with respect to Lebesgue measure. Then we have the representation

$$P(t,T) = \exp\left(-\int_t^T f(t,S)\, dS\right), \tag{3.7}$$

where $f(t,S)$ are the instantaneous forward rates for the maturity S. The idea of Heath *et al.* was to exploit this fact and formulate a model by specifying the process $\{f(\cdot,T) \,|\, T > 0\}$. In order to be able to move from a particular choice of the SDE governing the instantaneous forward rates to the discount bound price, we need to satisfy certain regularity condition which allows the application of the stochastic Fubini theorem, a result that allows to interchange the Lebesgue integral and the stochastic integral [see Bax77, Theorem 3].

3.2.1 The forward-rate dynamics

Suppose that the instantaneous forward rate $f(t,T)$, given a fixed maturity T, evolves according the following stochastic dynamics

$$df(t,T) = \mu(t,T)\,dt + \sigma(t,T)\,dW_t \tag{3.8}$$

with $\{f(0,T)\,|\,T>0\}$ being a fixed, non random initial forward rate curve. Here, the stochastic fluctuation of the entire forward structure is determined by N independent Brownian motions, each one concurring in the process with differing volatility coefficients. Both the drift and the diffusion are left unspecified, except for needful integrability conditions. The only substantive economic restrictions imposed at this point are that the forward rate process have continuous paths and their evolution is led by a finite number of stochastic shocks. The reason behind the latter modeling choice relies in some weaknesses that appear in the one dimensional models of the previous section. Indeed, interest rates are known to exhibit some different responses to economic shocks depending on their maturity, so that, for example, the thirty year interest rate is not perfectly correlated with the three month rate at the same instant. However, one factor models imply at every time that any two interest rates have correlation equals to one. In this situation, any shock to the interest rate curve is transmitted equally through all maturities and moves the curve rigidly in the same direction. The approximation might still be acceptable in some cases, when the rates that jointly influence the payoff at every instant are close, but in general if we require a higher accuracy we have to move to the set of multi factor models.

In order to guarantee the existence of a unique martingale measure, making all bonds prices respect to the bank account martingales, the following restriction must hold [HJM92, Proposition 3].

$$\mu(t,T) = \sigma(t,T)\int_t^T \sigma(t,S)\,dS. \tag{3.9}$$

To value a derivative V_T, we need to know the dynamics for $r(t)$ and $P(t,T_i)$, where $\{T_i\}_{i=\alpha,\dots,\beta}$ are the dates involved in the payoff. Notice that, by definition, $r(t) = f(t,t)$, we have the following equation

$$r(t) = f(0,t) + \int_0^t \sigma(u,t)\int_u^t \sigma(u,s)\,ds\,du + \int_0^t \sigma(s,t)\,dW_s \tag{3.10}$$

$$= f(0,t) + \sum_{i=1}^N \int_0^t \sigma_i(u,t)\int_u^t \sigma_i(u,s)\,ds\,du + \sum_{i=1}^N \int_0^t \sigma_i(s,t)\,dW_s \tag{3.11}$$

where we have made explicit the dependencies from the multidimensional brownian motion using Definition 1.4.2. Given the above result, application of Itô's lemma given the following dynamics of the zero coupon bond price:

$$dP(t,T) = P(t,T)\left[r(t)\,dt + \left(\int_t^T \sigma(t,s)\,ds\right)\,dW_t\right] \tag{3.12}$$

Requiring the absence of the market prices of risk, we are now dealing with an expression involving only the instantaneous volatility across different maturities of the forward rates. This set is often called the term structure of volatilities.

3.3 Market Models

We now turn to the most recent families in the interest rate modeling, meaning the market models. The cause behind their increasing popularity relies in the agreement between such models and the established Black's formula for the pricing in the cap and swaption, being the two main market in the interest rate options world. Until now, we dealt with models based on the infinitesimal quantities like the instantaneous short-rate and the instantaneous forward rate, which offer an easy mathematical tractability in exchange to a complicated calibration procedure from a numerical point of view. In fact, neither $r(t)$ or $f_T(t)$ are objects observed in real like. To overcome this aspect of the theory, the market models has been developed. The question was pioneered by Brace et al. [BGM97], Miltersen et al. [MSS97] and Jamshidian [Jam97]. The essential new and defining feature is that they model discrete market rates like LIBOR rates or forward swap rates, and by construction are very easy to calibrate to caps/floors and swaptions market data respectively. Moreover, they can be used to price more exotic products through some general numerical methods, even though path dependencies need to be carefully treated because the consequent short-rate dynamics does not lead to a recombining lattice in a tree approach. Despite their benefits, the market model suffer from highly dimension problems as the number of the rates involved increases rapidly with the contract characteristic time. From a practice point of view, dealing for example with a ten years cap paying quarterly coupons requires a thirty dimensional process to be carried on through simulation, leading to a multitude of parameters that makes the implementation infeasible.

3.3.1 The Libor Market Model

The first case of study is the Libor market models (LMM), i.e. a modeling setup for the LIBOR rates through a lognormal distribution under a particular measure. Indeed, consider the set of dates $\mathcal{T} = \{T_0 = 0, \ldots, T_M\}$ and denote by $\tau_i \in \{\tau_1, \ldots, \tau_M\}$ the year fraction associated with the pair T_{i-1} and T_i for $i > 0$. Those accruing factors corresponds to a set of forward rate $F(t) = (F_1(t), \ldots, F_M(t))$ where the generic $F_i(t) = F(t, T_{i-1}, T_i)$, $i = 1, \ldots, M$ satisfies $F_i(T_{i-1}) = L(T_{i-1}, T_i)$. Each of the forward rate evolves randomly upon its expiry, when it coincides with the spot rate prevailing at that moment for the maturity given by the following date in $\mathcal{T}$. Recall that, formally speaking, each forward rate defines a call option through Equation (2.28), where it is implicit in the Black formula that the forward rates dynamics are taken lognormal.

Start by specifying a SDE for the forward LIBORs of the form

$$dF_i(t) = \mu_i\big(F(t), t\big) F_i(t)\, dt + \sigma_i\big(F(t), t\big) F_i(t)\, dW_i(t) \tag{3.13}$$

where $W(t) = (W_1(t), \ldots, W_M(t))$ is a M-dimensional brownian motion whose correlation is determined by the covariance matrix $\rho = (\rho_{ij})$, $i, j = 1, \ldots, M$ through the relation $dW_i dW_j = \rho_{ij} dt$.

It will be useful to further develop the model in the forward measure $\mathsf{T_M}$ corresponding to taking the bond $P_M(\cdot) := P(\cdot, T_M)$ as numeraire. Then, in order to avoid arbitrage opportunities, the process

$$P_i^{(M)}(t) := \frac{P_i(t)}{P_M(t)} \tag{3.14}$$

must be a martingale under this measure. We can exploit this fact to establish a relation between the diffusion coefficients σ_i and the drifts μ_i. We proceed by recursion, starting from the following relation on subsequent rates, which can be immediately be derived from Definition 2.2.5

$$P_i^{(M)}(t) = \big(1 + \tau_{i+1} F_{i+1}(t)\big) P_{i+1}^{(M)}(t). \tag{3.15}$$

From now on, we adopt the convention that each quantity whose index exceeds M is set to zero. Applying Theorem 1.4.2 to the above equation yields

$$\begin{aligned} dP_i^{(M)} &= (1 + \tau_{i+1} F_{i+1})\ dP_{i+1}^{(M)} + \sigma_{i+1}\tau_{i+1} P_{i+1}^{(M)} F_{i+1}\, dW_{i+1} \\ &\quad + \mu_{i+1}\tau_{i+1} P_{i+1}^{(M)} F_{i+1}\, dt + \sigma_{i+1}\tau_{i+1} F_{i+1}\, dW_{i+1} dP_{i+1}^{(M)}. \end{aligned} \tag{3.16}$$

where we have dropped the dependencies on t and F_t to maintain a clever notation. The two last terms of the right hand side of Equation (3.16) contains the drift terms and thus they must cancel each other out, i.e.

$$\mu_{i+1}\tau_{i+1} P_{i+1}^{(M)} F_{i+1}\, dt + \sigma_{i+1}\tau_{i+1} F_{i+1}\, dW_{i+1} dP_{i+1}^{(M)} = 0, \tag{3.17}$$

for $i = 0, \ldots, M-1$. We are left with the following dynamics

$$dP_i^{(M)} = (1 + \tau_{i+1} F_{i+1})\, dP_{i+1}^{(M)} + P_{i+1}^{(M)} \tau_{i+1} F_{i+1} \sigma_{i+1}\, dW_{i+1}. \tag{3.18}$$

Define

$$\Pi_i(t) := \begin{cases} \prod_{j=1}^{i} (1 + \tau_j F_j(t)) & \text{for } i = 1, \ldots, M \\ 1 & \text{for i=0} \end{cases}$$

and multiply both sides of Equation (3.18) by $\Pi_i(t)$ to obtain

$$\Pi_i\, dP_i^{(M)} = \Pi_{i+1}\, dP_{i+1}^{(M)} + \Pi_{i+1} P_{i+1}^{(M)} \left(\frac{\tau_{i+1} F_{i+1}}{1 + \tau_{i+1} F_{i+1}} \right) \sigma_{i+1}\, dW_{i+1}. \tag{3.19}$$

Applying this inductively up to M yields, for $i = 1, \ldots, M-1$

$$\Pi_i\, dP_i^{(M)} = \sum_{j=i+1}^{M} \Pi_j P_j^{(M)} \left(\frac{\tau_j F_j}{1 + \tau_j F_j} \right) \sigma_j\, dW_j,$$

and thus

$$\begin{aligned} dP_i^{(M)} &= P_i^{(M)} \sum_{j=i+1}^{M} \frac{\Pi_j P_j^{(M)}}{\Pi_i P_i^{(M)}} \left(\frac{\tau_j F_j}{1 + \tau_j F_j} \right) \sigma_j\, dW_j \\ &= P_i^{(M)} \sum_{j=i+1}^{M} \left(\frac{\tau_j F_j}{1 + \tau_j F_j} \right) \sigma_j\, dW_j \end{aligned} \tag{3.20}$$

From Equation (3.17), substituting Equation (3.20) recursively, it follows that

$$\mu_i = -\sum_{j=i+1}^{M} \left(\frac{\tau_j F_j}{1 + \tau_j F_j} \right) \sigma_i \sigma_j \rho_{ij}.$$

Finally, Equation (3.13) becomes

$$dF_i(t) = -\sigma_i F_i(t) \sum_{j=i+1}^{M} \left(\frac{\rho_{ij} \tau_j \sigma_j F_j(t)}{1 + \tau_j F_j(t)} \right) dt + \sigma_i F_i(t)\, dW_i(t), \tag{3.21}$$

where again we ease the notation by avoiding to write the explicit dependencies on t and F_t of the diffusion coefficients. By mean of a similar argument, one can infer the relationship between the drifts and the diffusion coefficients when the chosen numeraire is any of the other ZCB with maturity in $\mathcal{T}$ [BM01, pp. 213–215].

Proposition 3.3.1. *The dynamics of F_i under the forward measure $\mathsf{T_k}$ is given by*

$$
\begin{aligned}
dF_i(t) &= \sigma_i F_i(t) \sum_{j=i+1}^{k} \left(\frac{\rho_{ij}\tau_j\sigma_j F_j(t)}{1+\tau_j F_j(t)} \right) dt + \sigma_i F_i(t)\, dW_i(t) && \text{for } i > k,\ t \le T_k, \\
dF_i(t) &= \sigma_i F_i(t)\, dW_i(t) && \text{for } i = k,\ t \le T_{i-1}, \\
dF_i(t) &= -\sigma_i F_i(t) \sum_{j=i+1}^{k} \left(\frac{\rho_{ij}\tau_j\sigma_j F_j(t)}{1+\tau_j F_j(t)} \right) dt + \sigma_i\, dW_i(t) && \text{for } i < k,\ t \le T_{i-1}.
\end{aligned}
$$

Moreover, all of the above equation admit a unique strong solution if the coefficients $\sigma(\cdot)$ are bounded.

Notice that for $i = k$ the distribution of F_i is lognormal, hence the name of the model. The above dynamics does not admit an analytical expression for the distribution of the forward rates' increments and thus no simple numeric integration can be used. Moreover, to obtain a path's value at a specific instant it is necessary to proceed by dividing the time into discrete sufficiently small steps Δt, generate a realization from the well known distribution of $W(t + \Delta t) - W(t)$ and compute the desired quantity through a simulation scheme.

3.3.2 Pricing through Monte Carlo

The simplest way to discretize Equation (3.21) is to introduce a time grid $\{t_i\}_{i=0,\dots,N}$ and proceed by carrying on the value of each rate step by step consistently with Proposition 3.3.1.
Assume we need to value a payoff $\Pi(T)$ which depends on the realization of different forward LIBOR rates at one or more specified instants. For example, it could be the payoff of a payer swaption with maturity T_α which would grant us payment up to T_β. For simplicity suppose that we are standing at time $t = 0$. The initial set $\mathcal{T}$ is given by the set of payments date of the two legs involved in the contract. Recall Equation (2.33) and Proposition 2.1.3 so that we can write the swaption price as

$$
P_\alpha(0)\,\mathrm{E}^{T_\alpha}\left[\Pi(T_\alpha)\right] = P_\alpha(0)\,\mathrm{E}^{T_\alpha}\left[\sum_{i=\alpha+1}^{\beta} \tau_i P_i(T_\alpha)\left(S_{\alpha\beta}(T_\alpha) - K\right)^+\right], \tag{3.22}
$$

where $S_{\alpha\beta}(\cdot)$ is the forward swap rate introduced in Definition 2.2.7. In this case, the LMM provides all of the quantities contained in the above expression

which can be expressed in terms of the forward rates at time T_α through Equation (2.20) and Equation (2.19).
In order to evaluate Equation (3.22) through a Monte Carlo approach, we need to generate n realization of the $\beta - \alpha$ forward rates. Recall their dynamics under T_α

$$dF_i(t) = \sigma_i F_i(t) \sum_{j=i+1}^{k} \left(\frac{\rho_{ij}\tau_j\sigma_j F_j(t)}{1+\tau_j F_j(t)} \right) dt + \sigma_i F_i(t)\, dW_i(t), \tag{3.23}$$

which thanks to Theorem 1.4.2 can be written as

$$d\ln F_i(t) = \sigma_i \left[\sum_{j=i+1}^{k} \left(\frac{\rho_{ij}\tau_j\sigma_j F_j(t)}{1+\tau_j F_j(t)} \right) - \frac{\sigma_i}{2} \right] dt + \sigma_i\, dW_i(t). \tag{3.24}$$

The point of this change of variable is that we now have a deterministic diffusion term, which allows for a better convergence of the corresponding discrete scheme [KP95, Section 10.3]. It is worth to point out that our choice of martingale measure actually breaks the property of non arbitrage through the discretization process but the estimate of the precision loss is found to be negligible [Gla03, pp. 174–180]. Given the time step Δt the simulation proceeds by iterating the Euler discretized version of Equation (3.24) which reads

$$d\ln F_i(t+\Delta t) = \sigma_i \left[\sum_{j=i+1}^{k} \left(\frac{\rho_{ij}\tau_j\sigma_j F_j(t)}{1+\tau_j F_j(t)} \right) - \frac{\sigma_i}{2} \right] \Delta t + \sigma_i\, \Delta W_i(t).$$

where $\Delta W_i(t) = W_i(t+\Delta t) - W_i(t)$ can be taken as the component of independent draws from a multivariate normal distribution $\mathcal{N}(0,\rho)$ multiplied by Δt. The above scheme for the rates entering the payoff Π provides us with the needed quantities to evaluate a path which we denote by a subscript. Then the Monte Carlo price given the payoff Π entering the expected value in Equation (3.22) is computed through the sample mean

$$P_\alpha(0)\mathrm{E}^{\mathsf{T}_\alpha}\left[\Pi(T)\right] \approx P_\alpha(0) \sum_{j=1}^{n} \frac{\Pi_j(T)}{n} \tag{3.25}$$

where all the floating quantities entering the payoff have been simulated under the T_α measure. The measure of the error on the above estimate is given by the sample variance, where the $\Pi_j(T)$ belongs from a sample of independent identically distributed random variables. Therefore, we can write

$$\mathrm{Var}\left[\frac{1}{n} \sum_{j=1}^{n} \Pi_j \right] = \frac{1}{n^2} \sum_{j=1}^{n} \mathrm{Var}\left[\Pi_j\right] = \frac{s^2}{n}, \tag{3.26}$$

where s is the standard deviation of the sample distribution, which can be computed from the simulated paths. The central limit theorem helps us define the confidence intervals which can be computed through the Gaussian cumulative function for large n, surely reached by our simulations [RPP22, pp. 192–194]. It follows that

$$\mathsf{P}\left[\left|\frac{\sum_{j=1}^{n}\Pi_j}{n}-\mathrm{E}\left(\Pi\right)\right|<\epsilon\right]=2\Phi\left(\epsilon\frac{\sqrt{n}}{s}\right)-1,$$

where the above probabilities are taken under the T_α measure. If we set this probability to 0.99, the value of the quantity inside the Gaussian cumulative function must be equal to 2.58. Thus, we find the value of ϵ to be

$$\epsilon = 2.58\frac{s}{\sqrt{n}}$$

In conclusion, the Monte Carlo simulation provides us the window

$$\frac{1}{n}\sum_{j=1}^{n}\Pi_j \pm 2.58\,\frac{s}{\sqrt{n}},$$

which does not cover the desired expectation value less than a fraction 0.01 of repeated evaluations. However, given the rate at which the interval shrinks, if we want to improve our accuracy to, for example, one tenth, we have to increase the number of paths by a factor of a hundred. For practical purposes, there exists a limit to the number of simulation we can run before it become too much time consuming, but we can still increase our overall performances by mean of the so called variance reduction techniques [BM01, pp. 269–271].

3.3.3 The Term Structure of Volatility

In the previous discussion we have taken the instantaneous volatility $\sigma_i(t)$ without making any structural assumption about their functional shape. These coefficients are chosen to respect the constraints given by the market quotes of the implied volatility for the caps and floors. In fact, the calibration to these contracts' prices is almost automatic, since Equation (2.28) still holds in the LMM framework, because its validity relies only on the lognormal assumption for the forward rates in their corresponding forward measure, which is a requirement surely satisfied by construction. Therefore, given the set of implied volatilities ν_{T_j} if we choose $\sigma_i(t)$ to be a deterministic function satisfying Equation (2.27) then the model is calibrated to the market price of the this caplet. By imposing this constraint for each $j = 1, \ldots, M$ we ensure

that the model volatility structure is calibrated to all caplet prices. There are many available specifications to the actual structure of the instantaneous volatility, most of them involving time depending functions which are piece wise constant over the time period between maturity dates. Let us stress that even though Equation (3.21) allows us to use a different volatility function σ_i for each of the forward rates $F_i(t)$ giving us great flexibility in specifying the structure of the curve evolution, in practice will require us to impose simply constraint to ensure computationally efficiency and to avoid an excess of parameters. For this reason we will assume each volatility to take constant values for the entire lifespan of the corresponding forward rate.
The important feature of this model is that a good deal of calibration can be achieved through closed form expressions, which make the overall procedure fast. In the absence of such a direct interpretation in the parameters involved, the calibration would be an iterative procedure requiring repeated simulations at various parameter values until the match of the model prices with the market ones. This approach is generally onerous, because computing each simulation can be quite time consuming.
In Figure 3.3 are shown the results of a stripping algorithm used for recovering

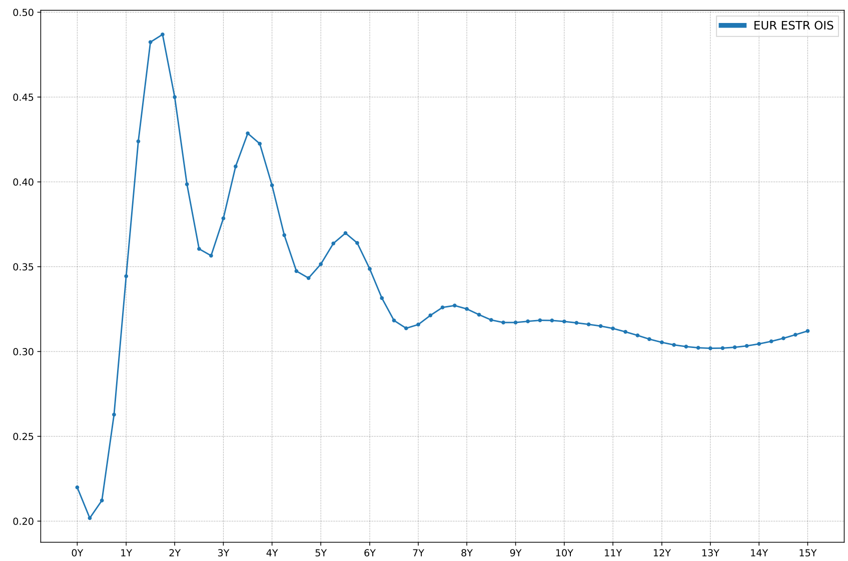

Figure 3.3: Caplet implied volatility on March 2, 2023 quoted by TRDL on Reuters platform.

the caplet volatility from the at-the-money cap prices. The resulting curve

$T \mapsto \nu_T$ is typically referred to as term structure of volatility. Again, the parameters σ_i can be used to exactly fit the squares of the market caplet volatilities multiplied by time $\nu_{T_{i-1}}$ so that the only degree of freedom left to tackle calibration are the instantaneous correlations coefficients.

3.3.4 Instantaneous Correlation

Taking $M = \beta - \alpha$ as the number of evolving rates, a $M \times M$ full rank correlation matrix is generally characterized by $M(M-1)/2$ entries, given symmetry and the ones in the diagonal. However, this number of parameters can become easily too large to handle for practical purposes, as the time period covered by our model increase. Therefore, one can look up for a parametric form for the correlation coefficients, based on a reduced number of independent entries.
Notice that being ρ a real valued symmetric matrix, it is surely diagonalizable, i.e. there exist the orthogonal matrix P such that

$$\rho = PHP^{\intercal}$$

where H is a diagonal matrix. Let Λ be such that $H = \Lambda^2$, so that if we set $A = P\Lambda$ we can write

$$\rho = AA^{\intercal}.$$

Following Rebonato [Reb99, Section 10.2], our goal is to mimic the above decomposition by means of a suitable $M \times n$ matrix B with $n << M$, in order to reduce the number of noise factors. The i-th row of B is given by

$$\begin{aligned} b_{ik} &= \cos(\theta_{ik}) \prod_{j=1}^{k-1} \sin(\theta_{ij}), \quad && \text{for } k = 1, \ldots, n-1 \\ b_{ik} &= \prod_{j=1}^{k-1} \sin(\theta_{ij}), && \text{for } k = n \end{aligned}$$

where the productory over an empty set is set to one. This assumption ensure that the resulting ρ is positive semi definite and its diagonal terms are ones. The total number of parameters is thus reduced to $M(n-1)$. Moreover, we will consider the simple two factor scenario, with $n = 2$, consisting of M parameters so that we can simply write

$$\rho_{ij} = b_{i1}b_{j1} + b_{i2}b_{j2} = \cos(\theta_{i1} - \theta_{j1}).$$

Another advantage of doing so is that we can abandon the use of the correlated n dimensional brownian motion W and replace it with the uncorrelated noise term Z such that $dW = B\,dZ$, since

$$\begin{aligned} dW_i dW_j &= \sum_{k,l=1}^{n} B_{ik} dZ_k B_{jl} dZ_l = \sum_{k,l=1}^{n} B_{ik} B_{jl} \delta_{kl}\, dt \\ &= \sum_{k=1}^{n} B_{ik} B_{jk}\, dt = (BB^{\intercal})_{ij}\ dt = \rho_{ij}\, dt \end{aligned}$$

3.3.5 Calibration to Swaption prices

The left unanswered question concern the actual methods to fit the angles in an efficient way. This is done by calibrating the model to swaption prices quoted in the market. These contracts are priced with through a model approximating formula which depends on the correlation parameters $\{\theta_i\}_{i=1,\dots,M}$. Such functions of ρ are then forced to match the corresponding market prices so that the resulting parameters values implied by the market are found.
Recall from Section 2.2.6 the lognormal dynamics for the forward swap rate leading to Black's formula

$$dS_{\alpha\beta}(t) = \sigma_{\alpha\beta}(t) S_{\alpha,\beta}(t)\, dW_{\alpha,\beta}(t) \tag{3.27}$$

under the swaption measure S. Here, we use the subscript $\alpha\beta$ to keep track of the underlying swap structure to which the forward swap rate is referred. As previously seen, the market quotes the swaption prices through its implied volatilities, computed by requiring the resulting Black's price to equate the market one. From Equation (2.20) we can write a compact expression for $S_{\alpha\beta}$ in terms of the forward rates

$$S_{\alpha\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i^{\alpha\beta}(t) F_i(t), \tag{3.28}$$

where

$$w_i^{\alpha\beta}(t) = \frac{\tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}{\sum_{k=\alpha+1}^{\beta} \tau_k' \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j' F_j'(t)}}$$

In order to manage the above formula we can proceed by neglecting part of the time dependencies introducing the approximation $w_i^{\alpha\beta}(t) \approx w_i^{\alpha\beta}(t)$ while maintaining the external $F_i(t)$ free to evolve. This can be justified numerically, as the main source of variability in the above expression is given by the second factor of each term in the summation. By differentiating both sides we obtain

$$dS_{\alpha\beta}(t) \approx \sum_{i=\alpha+1}^{\beta} w_i^{\alpha\beta}(0)\, dF_i(t) = (\dots)\ dt + \sum_{i=\alpha+1}^{\beta} w_i^{\alpha\beta}(0) \sigma_i(t) F_i(t) dW_i(t)$$

where we have used Proposition 3.21. The fact that we ignored the drift coefficient is justified by our interest in evaluating the quadratic variation of $S_{\alpha\beta}$ which does not involve the first part of the above expression. Thus,

$$dS_{\alpha\beta}(t)dS_{\alpha\beta}(t) = \sum_{i,j=\alpha+1}^{\beta} w_i^{\alpha\beta}(0)w_j^{\alpha\beta}(0)\sigma_i(t)\sigma_j(t)F_i(t)F_j(t)\rho_{ij}\,dt.$$

From Equation (3.27) and Equation(2.27) notice that we can find a closed expression for the Black's volatility of the forward swap rate in terms of the LMM parameters

$$\begin{aligned}\nu_{\alpha\beta}^2 &= \frac{1}{T_\alpha}\int_0^{T_\alpha} \sigma_{\alpha\beta}^2(t)\,dt = \frac{1}{T_\alpha}\int_0^{T_\alpha} \big(\ln S_{\alpha\beta}(t)\big)\big(\ln S_{\alpha\beta}(t)\big) \\ &= \frac{1}{T_\alpha}\sum_{i,j=\alpha+1}^{\beta} \rho_{ij} w_i^{\alpha\beta}(0)w_j^{\alpha\beta}(0)\int_0^{T_\alpha} \frac{\sigma_i(t)\sigma_j(t)F_i(t)F_j(t)}{S_{\alpha\beta}^2(t)}\,dt\end{aligned}$$

As done earlier, we can introduce a further approximation by fixing all the forward rates contained in the above quantities to their initial time values. Again, this is indeed a good assumption and it can be verified through numerical analysis. What we are left with is an expression linking the swaptions volatilities to our model parameters from which we can retrieve their implied values.

$$\nu_{\alpha\beta}^2 = \sum_{i,j=\alpha+1}^{\beta} \frac{\rho_{ij} w_i^{\alpha\beta}(0)w_j^{\alpha\beta}(0)F_i(0)F_j(0)}{T_\alpha S_{\alpha\beta}^2(0)} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)\,dt \qquad (3.29)$$

Another assumption which is not directly noticeable lays on the incompatibility between the lognormal dynamics chosen for both the processes $S_{\alpha\beta}$ and $\{F_i\}_{i=\alpha+1,\ldots,\beta}$. In fact pricing the swaption through the expectation value that led to Equation (2.35) means that we are basing our computations directly on the dynamics assumption of Equation (3.27). However, we previously assumed the forward rate dynamics such as in Preposition 3.3.1. These forward LIBOR rates define a swap rate through Equation (2.20) and there is no hope for its resulting distribution to be lognormal. This results in the two models being theoretically incompatible, and so are the respective Black's formulas. However, Brigo and Mercurio [BM01, Chapter 8] investigated numerically these approximations and stated their general validity.

In the remaining chapter we present a numerical example concerning the described calibration procedure of the forward Libor Market Model to both

Table 3.1: ESTR term-structure expressed in term of ACT/ACT zero rates quoted on Reuters platform on 2nd March 2023.

Date	Rate	Date	Rate
Mar 03, 2023	2.4610	Dec 06, 2024	3.6799
Mar 13, 2023	2.4604	Mar 06, 2025	3.6442
Mar 20, 2023	2.4595	Mar 06, 2026	3.4533
Mar 27, 2023	2.5632	Mar 08, 2027	3.3122
Apr 06, 2023	2.6779	Mar 06, 2028	3.2190
May 08, 2023	2.8275	Mar 06, 2029	3.1595
Jun 06, 2023	2.9833	Mar 06, 2030	3.1174
Jul 06, 2023	3.1196	Mar 06, 2031	3.0942
Aug 07, 2023	3.2493	Mar 08, 2032	3.0857
Sep 06, 2023	3.3442	Mar 07, 2033	3.0896
Oct 06, 2023	3.4221	Mar 06, 2034	3.0968
Nov 06, 2023	3.4859	Mar 06, 2035	3.1025
Dec 06, 2023	3.5376	Mar 08, 2038	3.0961
Jan 08, 2024	3.5852	Mar 06, 2043	2.9490
Feb 06, 2024	3.6150	Mar 06, 2048	2.7633
Mar 06, 2024	3.6546	Mar 06, 2053	2.5948
Jun 06, 2024	3.7010	Mar 06, 2063	2.3995
Sep 06, 2024	3.7039	Mar 06, 2073	2.2449

caps and swaptions markets, based on quoted data. The market organizes swaption prices in a table similar to the one shown in Table 3.2, where each row is indexed by the swaption maturity T_α and each columns refers to the underlying swap length between T_α and T_β.
The Table 3.1 provides the discount curve $\{P_T(0)\}_{T>0}$ as well as the quarterly forward rates $\{F(0,0,0.25), F(0,0.25,0.5), \ldots, F(0,29.75,30)\}$, which can be both retrieved using the tools explained in Section 2.2. Notice that we do only have access to a limited quantities corresponding to a discrete time structure, so that we are forced to fill the missing curve by interpolating the available points. The fit procedure proceeds as follows. The volatility structure is assumed to be constant so that $\{\sigma_1, \ldots, \sigma_M\}$ is taken directly from caplet volatilities, shown in Figure 3.3, whereas the correlation parameters $\{\theta_1, \ldots, \theta_M\}$ are forced to match the swaption volatility through Equation (3.29). The matrix of percentage errors in the swaption calibration is reported in Figure 3.3. The results are quite promising, considering that we are trying to fit forward rates over a thirty years time span. However the model lacks in simplicity, and requires a considerable amount of calculation efforts in order to calibrate itself. Moreover, we still need to find a method to extend the model to being able to price derivatives which contains rates out of the initial set.

Table 3.2: At the money swaption prices quoted by ICAP on Reuters platform on 2nd March 2023. Underlying swap's tenor on columns, its starting time on rows. Entries are expressed as percentage.

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y
1Y	28.2	31.2	32.9	33.7	34.1	34.3	34.4	34.1	33.8	33.4	32.8
2Y	33.6	34.5	34.9	34.8	34.5	34.4	34.2	33.9	33.6	33.2	32.3
3Y	35.5	35.2	34.8	34.3	33.9	33.6	33.2	32.8	32.5	32.2	31.6
4Y	35.2	34.6	33.9	33.4	32.8	32.4	32.0	31.6	31.3	31.0	30.8
5Y	34.5	33.8	33.0	32.1	31.5	30.9	30.5	30.2	30.0	29.8	30.0
6Y	33.3	32.7	31.4	30.5	29.9	29.5	29.2	29.1	29.0	29.0	29.5
7Y	31.3	31.1	30.0	29.2	28.8	28.3	28.1	28.1	28.2	28.4	29.1
8Y	29.3	29.6	28.8	28.2	27.8	27.6	27.6	27.6	27.9	28.2	29.0
9Y	27.9	28.7	28.0	27.5	27.3	27.2	27.4	27.6	28.0	28.3	29.1
10Y	27.1	28.2	27.6	27.3	27.2	27.4	27.6	27.9	28.3	28.6	29.3
15Y	28.7	31.4	31.9	32.5	32.6	33.0	33.1	33.3	33.5	33.8	33.3

Table 3.3: Calibration results under the volatility approximation expressed in Equation (3.29). Entries are expressed as percentage difference from quoted values.

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y
1Y	8.127	3.620	0.805	1.223	2.820	4.427	6.416	7.692	9.124	10.911	16.134
2Y	6.588	4.703	2.616	1.233	0.274	1.853	4.037	5.991	8.543	9.362	13.345
3Y	5.826	4.839	3.849	2.820	1.139	1.123	3.124	5.804	6.922	8.166	11.500
4Y	5.840	5.227	4.558	2.903	1.269	0.966	4.003	4.777	6.040	6.898	9.606
5Y	5.954	5.008	3.996	3.042	1.229	1.568	2.188	3.426	4.476	3.294	7.665
6Y	6.292	4.346	4.447	3.487	0.277	0.037	1.186	2.366	0.933	3.751	6.081
7Y	7.623	3.855	3.622	0.585	0.912	0.735	0.179	2.048	1.472	2.230	5.410
8Y	8.580	3.581	0.191	1.872	1.590	1.451	4.315	0.665	0.300	1.033	5.324
9Y	8.982	1.119	2.361	2.260	2.757	7.342	2.168	1.793	0.829	0.410	5.256
10Y	3.366	2.221	2.066	4.051	10.529	4.004	3.743	3.343	2.777	2.114	4.469
15Y	0.388	0.540	0.613	0.606	0.757	1.492	1.389	1.776	2.147	1.754	0.329

Chapter 4

Work at LIST: theory and results

In this last chapter we sum up the results of the internship at the List's Financial Engineering Department. The goal of this experience is to provide a theoretical and practice framework to the LIBOR transition capable of managing the newly introduced Risk Free Rates (RFRs) as their alternative benchmarks. Following the crisis that affected financial markets in 2008, the Financial Stability Board (FSB) initiated a complex regulatory process reviewing the major interest rate benchmarks and recommended developing alternative risk free references that better suit certain financial transactions. The intent above these efforts followed the widespread attempts to manipulate LIBOR by banks falsely submitting information about the cost of funds in the interbank market, as the credit crisis significantly decreased the number of trades. This process culminated in the decision by the sector authorities to no longer require, starting from the end of 2021, the partecipation of contributing banks in the IBORs publication procedure. By now, new references have been selected in all major economies. The Euro zone chose a new unsecured overnight rate called ESTER (Euro Short-Term Rate). The RFRs all reference the daily compounded rate based on the corresponding overnight benchmark and to full provide a proper replacement for LIBORs in both new and existing contracts they first need to be converted into term rates. We follow Lyashenko and Mercurio [LM19a] and construct a modeling framework to extend the single curve Libor Market Model. Then we proceed by defining a finite dimensional HJM model which produces an equivalent dynamics while requiring less computational effort.

4.1 Extended T-forward measure

We start by defining the usual risk-free economy, given by the bank account $B(t)$, with $B(0) = 1$, whose accruing is compounded following an instantaneous risk free rate $r(t)$. We assume the existence of a risk neutral measure B associated with the numeraire $B(t)$ so that we can write the price at time t of a ZCB with maturity T as

$$P(t,T) = \mathrm{E}^{\mathsf{B}}\left[\exp\left(-\int_t^T r(s)\,ds\right)\middle|\mathcal{F}_t\right],$$

for $t < T$. The corresponding trading strategy, consisting of simply holding the above contract at time t, can be extended in time by reinvesting the proceeds of the bond's unit notional received at T at the risk free rate $r(t)$ from the bond maturity onwards. Denoting by $\phi(t)$ the value of this strategy at time t, we have

$$\phi_T(t) = \begin{cases} P(t,T) & \text{for } t \leq T \\ \exp\left(\int_T^t r(s)\,ds\right) = \frac{B(t)}{B(T)} & \text{for } t > T. \end{cases}$$

This expression define the so called extended bond price and we will simply refer to it using the notation $P(t,T)$ which now holds for all times t, including those past T. Notice that it represents the value of a strictly positive strategy which is also self financing and it defines a viable numeraire. The martingale measure associated with this asset is called extended T-forward measure and it will be denoted by the symbol T as its original counterpart. Provided the identification $f(t,s) = r(t)$ when $t \geq s$, the extended bond price can be written as

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right), \tag{4.1}$$

for all t.

4.1.1 The backward-looking forward rates

The RFRs are overnight rates which are published the day after the transaction must be repaid. Given this feature, in order to determine payments for contracts and obligations indexed to higher level tenors, the actual daily compounded rate for the interval $[T,S]$, denoted by $R(S,T)$, is given by the average

$$R(T,S) = \frac{1}{\tau\,(T,S)}\left[\prod_{i=1}^{n}(1 + r_i\delta_i) - 1\right], \tag{4.2}$$

where the product is over the business day and r_i is the RFR fixing on date i with associated time fraction δ_i. Since typically δ_i is sufficiently small compared to the model scale, we can take the continuous time approximation replacing the above discrete calculation. Consider the same time structure $T_0, T_1, \ldots, T_M$ as in the previous chapter. For each $j = 1, \ldots, M$ we approximate the rate for the interval $[T_{j-1}, T_j]$ as

$$R\left(T_{j-1}, T_j\right) = \frac{1}{\tau_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)\mathrm{d}u} - 1\right] = \frac{1}{\tau_j}\left[\frac{B\left(T_j\right)}{B\left(T_{j-1}\right)} - 1\right] = \frac{1}{\tau_j}\left[P_{j-1}(T_j) - 1\right],$$

where we used the short hand notation $P_j(t)$ to denote the bond price $P(t, T_j)$ and where t_j is the year fraction for the interval $[T_{j-1}, T_j]$. The RFRs are referred to as backward looking because one has to wait until the end of their accrual period to know their fixing value, and so they differs from the LIBORs which are set at the beginning of their application period and thus called forward looking. For the latter, we keep the initial notation $F(T_{j-1}, T_j)$ which define the fixed rate to be exchanged that nullifies the swap value at time T_{j-1}. Clearly, it must hold

$$F(T_{j-1}, T_j) = \mathrm{E}^{T_j}\left[R(T_{j-1}, T_j) \,\middle|\, \mathcal{F}_{T_{j-1}}\right], \tag{4.3}$$

for each $j = 1, \ldots, M$. By mean of the same argument of Section 2.2.5 we can define its time t value $R_j(t)$ as

$$R_j(t) = \mathrm{E}^{T_j}\left[R\left(T_{j-1}, T_j\right) \mid \mathcal{F}_t\right] = \frac{1}{\tau_j}\left[\frac{P\left(t, T_{j-1}\right)}{P\left(t, T_j\right)} - 1\right], \tag{4.4}$$

where the second equality has been obtained from the following identity

$$\begin{aligned}
1 + \tau_j R_j(t) &= \mathrm{E}^{T_j}\left[\exp\left(\int_{T_{j-1}}^{T_j} r(u)\,\mathrm{d}u\right) \,\middle|\, \mathcal{F}_t\right] \\
&= \frac{1}{P_j(t)}\mathrm{E}\left[\exp\left(-\int_t^{T_j} r(u)\,\mathrm{d}u\right)\exp\left(\int_{T_{j-1}}^{T_j} r(u)\,\mathrm{d}u\right) \,\middle|\, \mathcal{F}_t\right] \\
&= \frac{1}{P_j(t)}\mathrm{E}\left[\exp\left(-\int_t^{T_{j-1}} r(u)\,\mathrm{d}u\right) \,\middle|\, \mathcal{F}_t\right] \\
&= \frac{P_{j-1}(t)}{P_j(t)}.
\end{aligned}$$

Equation (4.4) coincides with Equation (2.14) for the forward rate, but it holds for all times t. By no arbitrage, we have, for $t < T_{j-1}$

$$\begin{aligned}
F_j(t) &= \mathbb{E}^{T_j}\left[F(T_{j-1}, T_j) \mid \mathcal{F}_t\right] \\
&= \mathbb{E}^{T_j}\left\{\mathbb{E}^{T_j}\left[R(T_{j-1}, T_j) \mid \mathcal{F}_{T_{j-1}}\right] \mid \mathcal{F}_t\right\} \\
&= \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j) \mid \mathcal{F}_t\right] \\
&= R_j(t).
\end{aligned}$$

The evolution of the two differs from T_{j-1}, when $F_j(t)$ fixes whereas $R_j(t)$ continues to evolve until T_j. Thus, slightly abusing the notation we will designate up to the first fix the backward looking and the forward looking rates by the common symbol $R_j(t)$. In the following, we can use this fact and introduce a clever extension of the Libor Market Model to the case where rates are backward looking and evolves up to the payment date.

4.1.2 The generalized Forward-Market-Model

From Equation (4.4) we see that for every $j = 1, \dots, M$ as its forward looking counterpart $R_j(t)$ is a martingale under the corresponding T_j-forward measure. Thus, we choose its dynamics to be

$$\mathrm{d}R_j(t) = \sigma_j(t)\,\mathrm{d}W_j(t),$$

under the extended T_j forward measure, for all t even past T_j. We keep the notation simple, but the volatility coefficient is intended to vanish as t exceed T_j, keeping $R_j(t)$ constant after its fixing date. Moreover, a key point in the choice of a proper dynamics for the forward rate lays in its behavior in the accrual time $[T_j, T_j]$. In particular, a decay rate of the σ function found to be reasonable is obtained by introducing a deterministic function $g(t)$ which gradually suppress the volatility as the time goes by. Assuming a linear decay, the function $g_j(t)$ takes the form

$$g_j(t) = \min\left[\frac{(T_j - t)^+}{T_j - T_{j-1}}, 1\right]. \tag{4.5}$$

Then the dynamics of $R_j(t)$ is given by

$$\mathrm{d}R_j(t) = \sigma_j(t) g_j(t)\,\mathrm{d}W_j(t). \tag{4.6}$$

To allow analytical tractability we derive the dynamics of each forward under the common probability measure B. Analogous representations are not available in the Libor Market Model, without having to introduce further assumption. We obtain

$$\mathrm{d}R_j(t) = \sigma_j(t)g_j(t)\sum_{i=1}^{j}\rho_{i,j}\frac{\tau_i\sigma_i(t)g_i(t)}{1+\tau_iR_i(t)}\,\mathrm{d}t + \sigma_j(t)g_j(t)\,\mathrm{d}W_j. \tag{4.7}$$

for all t and for every $j = 1, \dots, M$. Equation (4.7) defines the Forward Market Model (FMM) which offers a viable replacement to the Libor Market Model from an analytical prospective without any significant complexities. In fact the model preserves its founding properties while gaining the backward looking rates tractability.
Following the procedure explained in the previous chapter, we once again show the calibration results to both cap and swaption market quotes. The swaption volatility surface showed in Figure 4.1 is obtained by inverting the pricing formula under the assumption of a normal dynamics. The resulting equation is typically referred to as Bachelier formula. Such a choice can be justified by looking at the values of the interest rates which are often below zero, and thus not compatible with a lognormal dynamics.

Table 4.1: At the money swaption prices quoted by ICAP on Reuters platform on 25th May 2022. Underlying swap's tenor on columns, its starting time on rows. Entries are expressed as basis point.

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1Y	101.21	101.71	102.25	93.74	94.09	88.37	87.42	83.69	83.11	81.99	76.73	71.88	68.36	65.20
2Y	105.54	105.26	92.25	93.12	86.57	86.02	81.96	81.59	80.59	79.71	74.03	69.89	66.93	64.23
3Y	106.28	88.15	91.71	84.54	83.98	79.94	79.71	79.08	78.35	77.85	71.79	68.60	65.42	63.47
4Y	82.18	90.32	82.88	81.31	77.44	77.34	77.31	76.71	76.42	75.53	70.32	66.99	64.09	62.83
5Y	102.19	84.81	83.44	78.00	78.37	78.15	77.58	77.31	76.46	76.02	70.06	66.51	64.05	62.56
7Y	92.55	74.81	78.53	77.88	77.68	77.71	77.16	76.55	73.89	72.00	68.09	65.47	63.04	62.26
10Y	88.30	83.03	82.59	80.80	80.29	76.79	74.66	73.25	72.30	71.60	67.76	65.38	64.17	63.52
15Y	83.43	81.02	79.81	72.55	70.32	70.06	69.31	68.97	68.38	68.01	66.60	66.24	66.17	64.16
20Y	76.27	72.52	73.97	72.05	71.58	72.28	72.04	70.92	69.83	70.43	70.51	70.85	68.26	68.20
25Y	85.78	79.20	74.75	72.26	73.92	72.84	73.84	74.46	75.41	74.70	75.36	72.11	72.09	71.79
30Y	92.04	79.69	80.00	82.14	80.12	81.43	81.64	81.78	82.33	80.82	76.24	76.06	76.02	76.57

The caplet volatility can be retrieved by the cap quotes showed in Figure 4.2 by mean of the same procedure we cited in the previous chapter. However, notice that the market quotes the implied volatility as a function of the option strike, which is a surprising feature considering that in its formulation such a dependence does not show up mathematically. In fact, the instantaneous volatility is a characteristic of the forward rate underlying the option, and it is reasonable to expect that it has nothing to do with the nature of the contract itself. Nevertheless, market caplet do not behave like this and usually one requires two different implied volatilities to match the observed market prices. Since the function $K \mapsto \nu_T$ usually exhibit a convex structure, it takes the name of volatility smile, in which the minimum point correspond

Table 4.2: Cap prices quoted by ICAP on Reuters platform on 25th May 2022. Cap's tenor on columns, strike values on rows. Entries are expressed as basis point.

	-1.5	-1.0	-0.5	0.0	0.3	0.5	0.8	1.0	1.5	2.0	3.0	5.0	10.0
1Y	96.5	84.6	70.1	51.7	58.2	67.0	74.2	81.1	95.2	109.6	137.7	189.4	303.2
2Y	104.2	96.5	87.9	77.0	77.3	81.3	85.8	90.9	102.4	114.9	140.0	187.5	293.3
3Y	105.3	99.8	94.9	90.8	89.4	89.4	91.3	95.0	104.8	115.6	137.5	179.2	272.3
4Y	102.4	98.0	94.4	91.8	91.0	91.3	93.0	96.0	104.3	113.6	132.7	169.7	252.5
5Y	96.2	93.3	91.3	90.2	90.1	90.8	92.5	95.3	102.6	110.7	127.4	159.4	238.6
6Y	90.9	89.2	88.3	88.4	88.7	89.7	91.5	94.1	100.6	107.8	122.3	150.4	236.7
7Y	87.1	86.0	85.8	86.4	87.1	88.2	90.0	92.4	98.4	104.8	117.8	142.8	229.4
8Y	83.7	83.0	83.3	84.3	85.1	86.3	88.0	90.3	95.7	101.6	113.2	135.6	219.5
9Y	80.9	80.5	81.0	82.2	83.1	84.3	86.0	88.2	93.2	98.5	109.1	129.4	210.1
10Y	78.5	78.2	78.8	80.2	81.2	82.4	84.0	86.1	90.7	95.6	105.2	123.8	201.3
12Y	74.4	74.4	75.1	76.7	77.7	78.9	80.5	82.3	86.4	90.6	98.8	114.9	185.6
15Y	69.1	69.5	70.5	72.3	73.4	74.6	76.1	77.8	81.4	85.0	92.1	106.1	168.4
20Y	62.8	63.6	65.1	67.2	68.4	69.7	71.1	72.6	75.8	79.0	85.0	97.0	149.6
25Y	58.5	59.6	61.4	63.7	64.9	66.2	67.6	69.1	72.2	75.1	80.7	91.5	137.9
30Y	55.8	57.0	58.9	61.2	62.4	63.7	65.1	66.6	69.5	72.2	77.5	87.6	128.7

to that value of strike that makes the contract at the money, i.e. when it coincides with the forward rate. The dynamics entering Equation (4.7) does not features this metric, so that we cannot feed our model with the entire available information entering the cap matrix and we are limited to consider only ATM volatilities.
The last piece of data we need to calibrate the model is the discount curve, which is shown in Figure 4.3. Again, we can proceed by forcing the match between the swaption volatilities calculated through Equation (3.29) where we have repeated the calculation assuming a normal dynamics for both $S_{\alpha\beta}$ and where instead of Equation (3.21) we adopt Equation (4.7). Since we are now considering the full swaption matrix, we consider the set $\{R(0, 0, 0.5), \ldots, R(0, 59.5, 60)\}$ of forward rates with an half yearly tenor to decrease the number of free parameters.

Again, the procedure is carried by a least square algorithm and the results are shown in Figure 4.4. As we could expect, the inaccuracy increases as the number of cash flows in the corresponding swaption grows, since it involves more forward rates and thus more parameters which contain the approximation errors. Moreover, as the model goes forward in time, it loses its efficacy up to fifty percent in the forward rate volatility. Nevertheless, through this method we are capable of modeling 120 forward rates covering each one a caplet with half yearly tenor for a total of 60 years of the market expectations.

4.1.3 Completing the curve using the HJM

Now that we have access to the forward rates dynamics based on the parameters obtained through the calibration procedure explained in the previous section, we can bring our attention to the actual simulation of the term

Table 4.3: ESTR term-structure expressed in term of ACT/ACT zero rates quoted on Reuters platform on 25th May 2022.

Date	Rate	Date	Rate
May 26, 2022	-0.5934	May 27, 2027	1.1125
May 27, 2022	-0.5929	Aug 27, 2027	1.1312
Jun 03, 2022	-0.5906	Nov 29, 2027	1.1490
Jun 27, 2022	-0.5836	Feb 28, 2028	1.1656
Jul 27, 2022	-0.5785	May 29, 2028	1.1828
May 26, 2022	-0.5934	Aug 28, 2028	1.1999
Aug 29, 2022	-0.4612	Nov 27, 2028	1.2170
Nov 28, 2022	-0.2009	Feb 27, 2029	1.2342
Feb 27, 2023	0.0257	May 28, 2029	1.2505
May 29, 2023	0.2333	Aug 27, 2029	1.2663
Aug 28, 2023	0.3965	Nov 27, 2029	1.2821
Nov 27, 2023	0.5297	Feb 27, 2030	1.2984
Feb 27, 2024	0.6317	May 27, 2030	1.3149
May 27, 2024	0.7108	Aug 27, 2030	1.3331
Aug 27, 2024	0.7712	Nov 27, 2030	1.3520
Nov 27, 2024	0.8181	Feb 27, 2031	1.3710
Feb 27, 2025	0.8574	May 27, 2031	1.3888
May 27, 2025	0.8918	Aug 27, 2031	1.4064
Aug 27, 2025	0.9258	Nov 27, 2031	1.4233
Nov 27, 2025	0.9584	Feb 27, 2032	1.4397
Feb 27, 2026	0.9893	May 27, 2032	1.4561
May 27, 2026	1.0174	May 29, 2034	1.5766
Aug 27, 2026	1.0445	May 27, 2037	1.6806
Nov 27, 2026	1.0695	May 27, 2042	1.6911
Feb 26, 2027	1.0921	May 27, 2047	1.6187

structure of the interest rates. Firstly, we proceed by evolving the set of initial values $\{R(0, T_0, T_1), \ldots, R(0, T_{M-1}, T_M)\}$ under the dynamics expressed in Equation (4.7) up to each fixing date to obtain, for each $i = 1, \ldots, M$ the realization $R_i(t) = R(t, T_{i-1}, T_i)$ for each $t \geq 0$ which is constant after the corresponding T_i. In Figure 4.1 it is shown an example of a realization of the forward rates corresponding to the fist four semi annual caplet. The effect of the decaying volatility in each accruing period is clearly visible. The correlation sub-matrix obtained from the calibration procedure is reported below and we can notice how it is reflected by the behavior of the forward rates.

$$\rho = \begin{pmatrix} 1 & 0.999 & 0.221 & 0.221 \\ 0.999 & 1 & 0.234 & 0.234 \\ 0.221 & 0.234 & 1 & 1 \\ 0.221 & 0.234 & 1 & 1 \end{pmatrix}.$$

Recall that the FMM constitutes an extension of the LMM introduced in the previous chapter in the sense that it models the backward looking forward rates $R_j(t)$ and simultaneously the dynamics of the forward looking forward rates $F_j(t)$ since it holds $F_j(t) = R_j(t)$ for every t before the expiry date T_{j-1}

Table 4.4: Calibration results under the volatility approximation expressed in the modified Equation (3.29). Entries are expressed as percentage difference from quoted values.

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1Y	2.162	2.105	1.678	0.936	1.569	1.369	0.013	0.391	0.597	1.191	8.997	14.961	21.419	28.227
2Y	3.829	1.002	1.855	1.725	2.106	0.307	0.752	0.777	1.176	1.669	7.547	13.771	18.764	23.508
3Y	3.182	1.318	1.562	2.274	0.848	0.757	0.779	1.554	2.041	1.915	6.515	11.664	15.897	19.968
4Y	3.678	1.205	1.830	0.489	1.417	1.452	1.279	1.843	1.681	1.240	5.062	10.439	14.435	18.258
5Y	2.651	4.278	1.240	2.883	3.385	0.034	0.291	0.600	0.157	0.895	2.034	7.365	12.182	16.270
7Y	0.253	5.662	7.459	3.321	3.242	3.630	4.325	5.242	1.723	0.784	0.048	3.765	7.957	11.849
10Y	0.169	0.404	4.656	7.419	9.655	4.413	3.665	4.364	5.361	7.060	4.356	1.032	3.206	8.080
15Y	0.054	0.226	0.411	4.029	9.284	5.794	4.936	4.979	5.072	7.473	7.446	4.653	0.150	3.496
20Y	0.193	0.313	0.465	0.595	5.895	4.118	4.285	4.338	4.303	5.327	5.234	3.378	0.100	3.994
25Y	0.098	0.097	0.264	0.279	1.699	1.705	2.436	2.976	3.372	1.360	2.250	1.901	0.449	3.683
30Y	0.295	0.708	0.181	0.108	0.820	0.146	1.803	0.006	0.008	0.502	2.104	2.560	3.051	0.046

of the latter. In this sense the same modeling framework sill holds for both cases, so that the implementation cost has significant benefits.
Moreover, the additional information about the rates dynamics within each accrual period provides a clever way to complete the yield curve, i.e. infer the evolution of generic term rates outside the initial set as well as the short-rate. In the LMM construct, this problem is extensively discussed in the available literature but the approach is often extemporaneous and based on unrealistic interpolations which are computationally pleasing and efficient but lacks of a founding theoretical basis [AP10, Chapter 15]. As discussed by Lyashenko and Mercurio [LM19b] it is possible to embed the FMM dynamics into the instantaneous forward rate evolution following the HJM approach described in Section 3.2. The resulting hybrid model contains the ZCB pricing formula and the bank account as a function of the forward rates which results from the simulation. Recall Equation (3.12) which can be now extended consistently at valuation times t following the its maturity T. Thus, we can write

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r(t)\mathrm{d}t + \left(\int_t^T \sigma_f(t,u)\mathrm{d}u\right)\mathrm{d}W(t), \tag{4.8}$$

where $\sigma(t,u)$ is null when $t > u$ and the subscript f is used to distinguish the volatility coefficient of the instantaneous forward rate from the parameters entering the FMM. From Equation (4.4) using Itô's Theorem 1.4.2 we obtain

$$\mathrm{d}R_j(t) = \ldots\mathrm{d}t + \left(R_j(t) + \frac{1}{\tau_j}\right)\left(\int_{T_{j-1}\vee t}^{T_j\vee t} \sigma(t,u)\mathrm{d}u\right)\mathrm{d}W(t).$$

where we have introduced the useful notation $t \vee s = \max(t,s)$ to keep track of the volatility behavior. In this sense, we are saying that the above integral must be performed over those instants which are greater t. Matching the

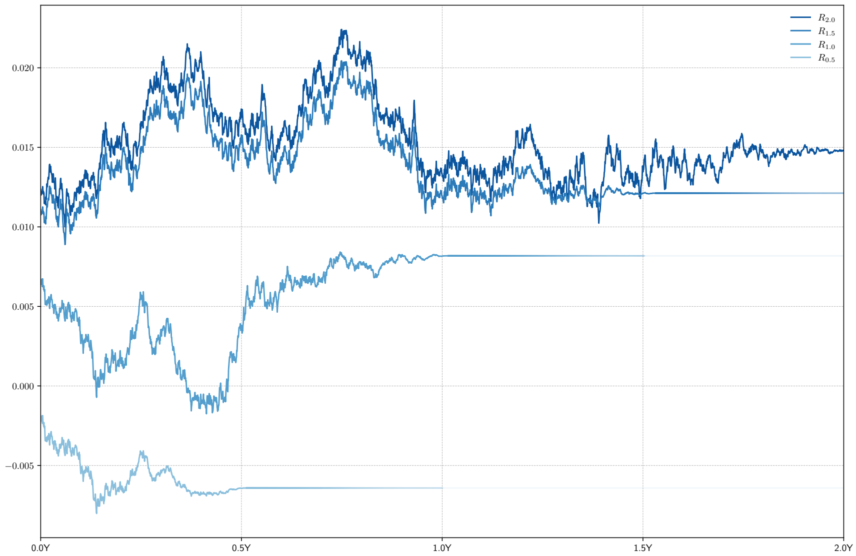

Figure 4.1: Simulation of the set of forward rates corresponding to a semiannual application period.

above dynamics with the one expressed in Equation (4.7) yields the following identity.

$$\sigma_j(t)g_j(t) = \left(R_j(t) + \frac{1}{\tau_j}\right)\int_{T_{j-1}\vee t}^{T_j\vee t}\sigma_f(t,u)\mathrm{d}u. \tag{4.9}$$

This condition guarantees that the generates dynamics of $\{R_j(t)\}_{i=1,\dots,M}$ are equivalent to the HJM ones. Assuming a separable structure for the volatility of the form

$$\sigma_f(t,T) = \sum_{k=1}^{M} a_k(t)b_k(T),$$

we obtain

$$\sigma_j(t)g_j(t) = a_j(t)B_j\left(T_{j-1}\vee t, T_j\vee t\right)\left(R_j(t) + \frac{1}{\tau_j}\right),$$

where

$$B_j(t,T) = \int_t^T b_j(s)\,\mathrm{d}s.$$

Imposing the arbitrary boundary condition $B_j(T_{j-1}, T_j) = 1$, a solution of Equation (4.9) is given by the pair

$$a_j(t) = \frac{\sigma_j(t)}{R_j(t) + \frac{1}{\tau_j}}, \quad b_j(t) = \begin{cases} -\frac{1}{\tau_j} & \text{for } T_{j-1} < t < T_j \\ 0 & \text{otherwise} \end{cases}.$$

where we have used Equation (4.5). By constructing an HJM model from the above volatility constraint we retrieve the postulated FMM dynamics. Therefore, we can rewrite the $f(t,T)$ dynamics by substituting the above results as

$$\begin{aligned} f(t,T) =& f(0,T) + \int_0^t \sigma(u,T) \int_u^T \sigma(u,s)\,\mathrm{d}s\,\mathrm{d}u + \int_0^t \sigma(s,T)\,\mathrm{d}W(s) \\ =& f(0,T) + \sum_{k=1}^{M} b_k(T) X_k(t) + \sum_{k,h=1}^{M} b_k(T) Y_{kh}(t) B_h(t,T), \end{aligned} \tag{4.10}$$

where we have defined, for each $h, k = 1, \ldots, M$

$$\begin{aligned} \mathrm{d}X_k(t) &= \sum_{h=1}^{M} b_h(t) Y_{kh}(t)\,\mathrm{d}t + a_k(t)\,\mathrm{d}W(t), \\ \mathrm{d}Y_{kh}(t) &= a_k(t) a_h(t)\,\mathrm{d}t, \end{aligned}$$

with $X_k(0) = Y_{hk}(0) = 0$ and $t < T$. Moreover, since $f(t,T) = r(T)$ for $t > T$ we have

$$r(T) = f(0,T) + \sum_{k=1}^{M} b_k(T) X_k(T).$$

Then we can write the extended bond price at time t with maturity T through Equation (4.1). We obtain, for $t < T$

$$\begin{aligned} P(t,T) =& \frac{P(0,T)}{P(0,t)} \\ & \times \exp\left(-\sum_{k=1}^{M} B_k(t,T) X_k(t) - \frac{1}{2} \sum_{k,h=1}^{M} B_k(t,T) Y(t)_{kh} B_h(t,T) \right), \end{aligned} \tag{4.11}$$

whereas, for $t > T$, it holds

$$P(t,T) = \exp\biggl(\int_t^T r(s)\,\mathrm{d}s \biggr), \tag{4.12}$$

where $r(t)$ can be evaluated through Equation (4.10). Notice, however, even though we are able to simulate all possible ZCB prices up to T_M, it may

be computationally unappealing since the additional processes $X_k(t)$ and Y_{hk} for each $k, j = 1, \dots, M$ require the simulated forward rates $\{R_j\}$ for $j = 1, \dots, M$. The state of variables to store grows as a quadratic function of M which may cause Equation (4.11) and Equation (4.12) difficult to implement in practice. This issues can be solved by noticing that the off-diagonal elements Y_{hk}, $h \neq k$ are redundant, and thus their costly dependence can be eliminated [LM19b, pp. 10–13].
For each time valuation t, we distinguish two cases depending on the general bond maturity T which may be before or after the closest future date in the initial set $\{T_0, \dots, T_M\}$ from t. For notation purpose, it is convenient introduce the quantity $\eta(t) = \min\left(j \,|\, T_j \geq t\right)$ and set $k = \eta(t)$. For $t < T \leq T_k$, the previous argument yield to

$$\begin{aligned} P(t,T) =& \frac{P(0,T)}{P(0,t)} \left(\frac{P(0,T_k)}{P(0,T_{k-1})} \right)^{-B_k(t,T)} \big(1 + \tau_k R_k(T_{k-1})\big)^{-B_k(t,T)} \\ & \times \exp \Bigg[-B_k(t,T) x_k(t) - \frac{1}{2} B_k^2(t,T) y_k(t) \\ & + \frac{1}{2} Y_{k,k}(T_{k-1}) \big(B_k(T,T_k) B_k(T_{k-1},T) - B_k(t,T_k) B_k(T_{k-1},t) \big) \Bigg], \end{aligned} \tag{4.13}$$

where we have defined the auxiliary processes $x_k(t)$ and $y_k(t)$ which evolve respectively as $X_k(t)$ and $Y_{kk}(t)$ but with the initial conditions $x(T_{k-1}) = 0$ and $y_k(T_{k-1}) = 0$.
For $T > T_k$ we can write

$$P\left(t,T\right) = P(t,T_k) \prod_{j=k+1}^{\eta(T)-1} \frac{1}{1 + \tau_j R_j(t)} \frac{P(t,T)}{P(t,T_{k-1})}, \tag{4.14}$$

where $P(t,T_k)$ can be calculated through Equation (4.13) and

$$\begin{aligned} \frac{P(t,T)}{P(t,T_{k-1})} =& \frac{P(0,T)}{P(0,T_{k-1})} \left(1 + \tau_k R_k(t)\right)^{-B_k(T_{k-1},T)} \left(\frac{P(0,T_k)}{P(0,T_{k-1})} \right)^{-B_k(T_{k-1},T)} \\ & \times \exp \left(\frac{1}{2} B_k\left(T_{k-1},T\right) B_k\left(T,T_k\right) Y_{k,k}(t) \right). \end{aligned}$$

The above framework gives us access to the whole yield curve solving the problem of the simulation of the forward rates that do not lie on the initial grid of times. The results of this setup applied to the data on which we based the calibration procedure of the previous section are shown in Figure 4.2. We are able to evaluate $P(t,T)$ for quarterly separated maturities, thus including points out of the range of the initial forward rates. Moreover, the curve covers the entire time span of the model, thanks to our definition of extended bond price so that the quantity $P(t,T)$ exists for every pairs (t,T).

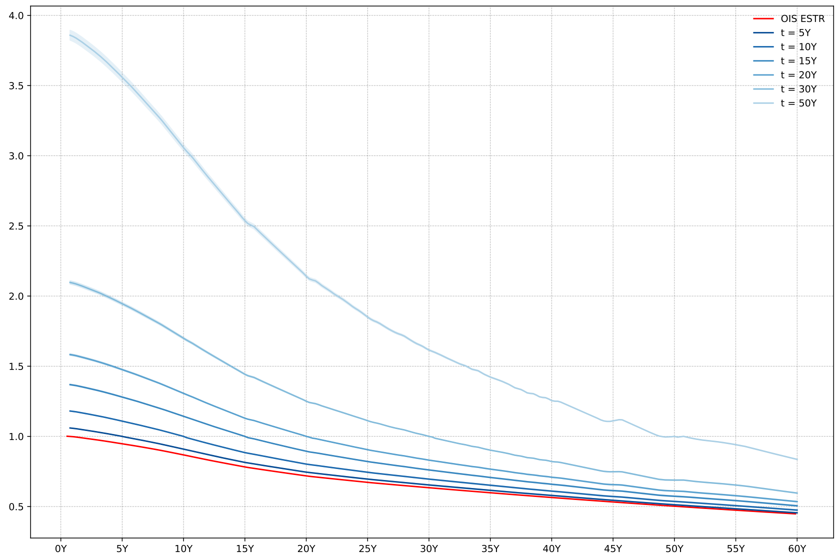

Figure 4.2: Simulation of the extended ZCB price.

4.2 The parsimonious HJM

The fine resolution provided by the HJM extension comes at the computational costs of simulating the auxiliary processes $X_(t)$ and $Y_(t)$ in addition to the initial set of forward rates. We present now an alternative model which maintains the property to handle the complexities of the new rates while being characterized by less state of variables, thus demanding a lower computational effort. We adopt the modeling choice from Moreni and Pallavicini [MP10]. We start by specifying the volatility process in Equation (4.8) via the following separable form.

$$(\sigma_f)_h\,(t,T) = \sum_{k=1}^{n} a_{hk} b_k(t,T), \quad h = 1,\ldots,n,$$

where the index h is summed over the n-dimensional source of randomness which component is denoted by $dW^{(h)}(t)$. Again, we stress that $\sigma_f(t,T)$

vanishes for $t > T$. Moreover, we suppose

$$b_k(t,T) = \exp\biggl(-\int_t^T \lambda_k(s)\,\mathrm{d}s\biggr), \quad k = 1,\dots,n, \tag{4.15}$$

where λ_k is a deterministic function. Plugging the above volatility into the dynamics of the instantaneous forward rate yields

$$f(t,T) = f(0,T) + \sum_{k=1}^{n} b_k(t,T)\left[X_k(t) + \sum_{l=1}^{n} Y_{kl}(t) B_k(t,t,T)\right], \quad t \le T \tag{4.16}$$

where we have defined, for $t \le T \le S$

$$B_k(t,T,S) = \int_T^S b_k(t,u)\,\mathrm{d}u, \quad k = 1,\dots,n, \tag{4.17}$$

and where auxiliary processes $\{X_k(t)\}_{k=1,\dots,n}$, $\{Y_{hk}(t)\}_{h,k=1,\dots,n}$ are specified by the following coupled SDE

$$\mathrm{d}X_k(t) = \left(\sum_{i=1}^{n} Y_{hk}(t) - \lambda_k(t) X_k(t)\right)\mathrm{d}t + \sum_{i=1}^{n} a_{ki}(t)\,\mathrm{d}W_i(t), \tag{4.18}$$

$$\mathrm{d}Y_{hk}(t) = \left[\sum_{i=1}^{n} a_{ih}(t) a_{ik}(t) - \left(\lambda_h(t) + \lambda_k(t)\right) Y_{hk}(t)\right]\mathrm{d}t. \tag{4.19}$$

When the valuation time exceeds the maturity, the expression for $f(t,T)$ reduces to the constant value

$$f(t,T) = f(0,T) + \sum_{k=1}^{n} X_k(T), \quad t > T. \tag{4.20}$$

We are left with the choice of specifying a suitable shape for the volatility function. We consider a two factor model having a square root volatility process of the form

$$a_{hk}(t) = \sqrt{V(t)} \sum_{i=1}^{2} \hat{\rho}_{ih} \hat{\sigma}_{ik},$$

where we have defined the constant values matrixes

$$\hat{\sigma} = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix}, \quad \hat{\rho} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}. \tag{4.21}$$

and $V(t)$ is a deterministic function satisfying the following differential equation, which might be in turn generalized further more introducing a stochastic term to approach the smile problem.

$$dV(t) = \omega(\theta - V(t))\,\mathrm{d}t, \quad V(0) = V_0.$$

Here ω, θ, V_0 are three positive real number. Furthermore, we assume $\{\lambda_1, \lambda_2\}$ in Equation (4.15) to be constants, so that we can write

$$b_k(t,T) = e^{-\lambda_k(t-T)}, \quad k = 1, 2.$$

Then Equation (4.17) reduces to

$$B_k(t,T,S) = \frac{1}{\lambda_k}\left(e^{-\lambda_k(T-t)} - e^{-\lambda_k(S-t)}\right), \quad k = 1, 2.$$

The particular example of the HJM model we have just defined allows us to retrieve the expressions for the dynamics of the forward rates and the swap rate, giving us the ability to price caps and swaptions through simple formulas. These can be in turn implemented into the calibration process as a proxy from the market data to the parameters entering the model.
Recall that the risk free forward rate $R(t, T-\tau, T)$ with maturity T and tenor τ is a martingale under the corresponding extended T-forward measure, so that we can assume, for every t, the following shifted lognormal dynamics

$$\frac{\mathrm{d}R(t,T-\tau,T)}{R(t,T-\tau,T) + 1/\tau} = \left(\int_{(T-\tau)\vee t}^{T\vee t} \sigma_f(t,u)\mathrm{d}u\right)\mathrm{d}W_t,$$

which is compatible with the negative values of most current yield curves. Substituting the expression for the volatility yields

$$\begin{aligned}\frac{\mathrm{d}R(t,T-\tau,T)}{R(t,T-\tau,T) + 1/\tau} &= \sqrt{V_t}\sum_{h,k=1}^{2}\frac{\hat{\sigma}_{kk}}{\lambda_k}\hat{\rho}_{hk}\left(e^{-\lambda_k((T-\tau)\vee t-t)} - e^{-\lambda_k(T\vee t-t)}\right)\mathrm{d}W_h(t)\\ &= \sqrt{V_t}\sum_{h,k=1}^{2}\hat{\rho}_{hk}\Lambda_k(t,T-\tau,T)\mathrm{d}W_h(t), \qquad (4.22)\end{aligned}$$

where we have defined

$$\Lambda_k(t,T,S) = \frac{\hat{\sigma}_{kk}}{\lambda_k}\left(e^{-\lambda_k(T\vee t-t)} - e^{-\lambda_k(S\vee t-t)}\right) \quad k = 1, 2. \qquad (4.23)$$

4.2.1 The valuation of a RFR swaption

Let us consider a swap where the floating leg pays at time $\{T_{\alpha+1}, \ldots, T_\beta\}$ the compounded daily fixing RFR in exchange for the fixed leg which payment dates occurs at $\left\{T'_{\alpha+1}, \ldots, T'_\beta\right\}$. The forward rate swap is then obtained as

$$S^{\alpha\beta}(t) = \frac{\sum_{i=\alpha+1}^{\beta} \tau_i P_i(t) R_i(t)}{\sum_{i=\alpha+1}^{\beta} \tau'_i P'_i(t)} = \sum_{i=\alpha+1}^{\beta} w_i^{\alpha\beta}(t) R_i(t), \tag{4.24}$$

where we have used the notation of Section 3.3.5. Being $R_i(t)$ defined for every t, $S^{\alpha\beta}(t)$ inherit the property too. For $t > T_\alpha$, the above quantity represents the value of a self-financing strategy in which all cash flows are reinvested into the bank account, whereas for $t \leq T_\alpha$ we interpret it the usual way for both the forward looking and the backward looking case. Analogously to Section 3.3.5, we adopt the approximation $w^{\alpha\beta}(t) \simeq w^{\alpha\beta}(0)$, so that Equation (4.24) lead to

$$\begin{aligned}
\mathrm{d}S^{\alpha\beta}(t) &\simeq \sum_{i=\alpha+1}^{\beta} w_i^{\alpha\beta}(0)\, \mathrm{d}R_i(0) \\
&= \sum_{i=\alpha+1}^{\beta} w_i^{\alpha\beta}(0) \left(R_i(t) + \frac{1}{\tau_i}\right) \sqrt{V_t} \sum_{h,k=1}^{2} \hat{\rho}_{hk} \Lambda_k\left(t, T_{i-1}, T_i\right) \mathrm{d}W_h(t) \\
&= \left(S^{\alpha\beta}(t) + \psi^{\alpha\beta}\right) \sqrt{V_t} \sum_{i=\alpha+1}^{\beta} \delta_i^{\alpha\beta} \sum_{h,k=1}^{2} \hat{\rho}_{hk} \Lambda_k\left(t, T_{i-1}, T_i\right) \mathrm{d}W_h(t),
\end{aligned}$$

where we have made use of Equation (4.22) and where we have defined

$$\psi^{\alpha\beta} = \frac{\sum_{j=\alpha+1}^{b} P_j(0)}{\sum_{j=\alpha+1}^{\beta} \tau'_j P'_j(0)}, \qquad \delta_i^{\alpha\beta} = \frac{P_{i-1}(0)}{\sum_{k=\alpha+1}^{\beta} P_{k-1}(0)}, \tag{4.25}$$

thanks to the last equality in Equation (4.4). From the last identity, we can write

$$\begin{aligned}
&\mathrm{d}\ln\left(S^{\alpha\beta}(t) + \psi^{\alpha\beta}\right) \mathrm{d}\ln\left(S^{\alpha\beta}(t) + \psi^{\alpha\beta}\right) \\
&= V_t \sum_{i,j=\alpha+1}^{\beta} \delta_i^{\alpha\beta} \delta_j^{\alpha\beta} \sum_{h,k,l,m=1}^{2} \hat{\rho}_{hk} \hat{\rho}_{lm} \Lambda_k\left(t, T_{i-1}, T_i\right) \Lambda_m\left(t, T_{j-1}, T_j\right) \, \mathrm{d}W_h(t) \mathrm{d}W_l(t) \\
&= V_t \sum_{k,m=1}^{2} \Lambda_k^{\alpha\beta}(t) \left(\hat{\rho}^\top \hat{\rho}\right)_{km} \Lambda_m^{\alpha\beta}(t)\, \mathrm{d}t,
\end{aligned}$$

with the useful notation

$$\Lambda_k^{\alpha\beta}(t) = \sum_{i=\alpha+1}^{\beta} \delta_i^{\alpha\beta} \Lambda_k\left(t, T_{i-1}, T_i\right).$$

The swaption volatility can now be evaluated by performing an integral of the above quantity. We get, after some minor simplifications

$$\begin{aligned}
\nu_{\alpha\beta}^2 &= \frac{1}{T_\alpha} \int_0^{T_\alpha} V_t \sum_{k,m=1}^{2} \Lambda_k^{\alpha\beta}(t) \left(\hat{\rho}^\top \hat{\rho}\right)_{km} \Lambda_m^{\alpha\beta}(t)\, \mathrm{d}t \\
&= \frac{1}{T_\alpha} \sum_{k,m=1}^{2} \left(\hat{\rho}^\top \hat{\rho}\right)_{km} \Lambda_k^{\alpha\beta}(0) \Lambda_m^{\alpha\beta}(0) \\
&\times \left[\frac{\theta}{\lambda_k + \lambda_m}\left(e^{(\lambda_k+\lambda_m)T_\alpha} - 1\right) - \frac{\theta - V_0}{\lambda_k + \lambda_m - \omega}\left(e^{(\lambda_k+\lambda_m-\omega)T_\alpha} - 1\right)\right].
\end{aligned}$$

Then, the price at time $t = 0$ of a payer swaption can be computed through the Black's Equation (2.35), which reads

$$\mathbf{PS}_0(T_\alpha, \mathcal{T}', K) \simeq \sum_{i=\alpha+1}^{\beta} \tau_i' P(0, T_i')\, \mathrm{Black}\big(K + \psi^{\alpha\beta}, S^{\alpha\beta}(0) + \psi^{\alpha\beta}, \nu_{\alpha\beta}, 1\big). \tag{4.26}$$

We would like to point out that, except for the the arbitrary modeling choice of the forward rate dynamics, the valuation formulas for swap and swaptions do not incur into formal differences when we switch from the forward looking to the backward looking case.

Table 4.5: Model parameters resulting from the calibration to quoted swaption prices.

Key	Value	Key	Value
λ_1	0.021391	ρ	−0.570510
λ_2	0.021389	ω	0.246728
σ_1	0.038836	θ	0.030466
σ_2	0.040088	V_0	0.093380

We have implemented the above setup and tested it on the market data used in the previous section. The initial curve and the swaption volatility matrix are respectively shown in Table 4.3 and Table 4.1. The model is calibrated through Equation (4.26) which grants us a link between the model parameters and quoted values. The optimization is performed again by mean of a least square procedure, using the Trust Region Reflective algorithm, which allows us to impose the constraint $-1 \le \rho \le 1$. We can check the accuracy of the procedure by evaluating the swaption prices from the calibrated parameters and confront them with the actual prices. The results are presented in Table 4.6 and the model parameters are listed in Table 4.5.

Table 4.6: Calibration results expressed as percentage difference between the swaption quoted prices and the model price.

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1Y	7.269	1.517	1.881	5.042	6.813	6.776	7.123	7.691	8.280	8.593	4.358	1.481	1.371	4.450
2Y	2.678	0.280	1.814	3.272	3.769	3.823	4.426	4.818	5.226	5.562	3.622	0.414	1.560	3.200
3Y	0.559	1.723	0.244	1.105	1.820	1.668	1.986	2.283	2.562	3.071	2.540	0.563	0.997	2.290
4Y	2.048	2.278	2.082	0.666	0.063	0.106	0.322	0.507	0.915	1.364	1.712	0.076	1.434	2.699
5Y	1.745	1.930	1.550	0.716	0.260	0.355	0.090	0.085	0.242	1.026	2.006	0.344	1.196	2.262
7Y	1.015	1.116	0.175	0.221	0.647	0.743	0.949	1.288	1.370	1.578	2.957	1.405	0.241	1.992
10Y	0.595	1.473	0.971	0.506	0.069	0.044	0.478	1.084	1.471	1.786	2.811	2.451	0.906	1.012
15Y	1.689	2.184	2.392	2.641	2.466	2.219	1.695	0.967	0.744	0.411	2.660	2.433	0.343	1.035
20Y	1.934	2.483	3.328	3.935	4.616	4.319	4.036	3.521	3.110	2.511	0.323	1.331	0.220	0.738
25Y	5.078	5.590	6.399	6.677	7.065	6.623	5.651	4.740	4.129	3.852	0.092	2.044	1.720	1.673
30Y	8.101	8.889	9.518	9.656	9.698	9.086	7.929	6.834	5.446	5.254	0.485	2.810	3.081	3.626

4.2.2 The valuation of a RFR caplet

Given the accrual time from $T-\tau$ to T, we can define two distinct caplets payoffs depending respectively on the forward looking or backward looking nature of the interest rate:

1. $\big[R(T-\tau, T-\tau, T) - K\big]^+$

2. $\big[R(T, T-\tau, T) - K\big]^+$

The first one is known at the beginning of the application period, whereas the second fixes at the end date of the caplet. At the time of writing, the Reuters' platform has made both pricing formulas available to customers. From Equation (4.22), we find the expression for the caplet volatility by mean of the usual technique.

$$\begin{aligned} &\mathrm{d}\ln\big(R(t,T-\tau,T)+1/\tau\big)\mathrm{d}\ln\big(R(t,T-\tau,T)+1/\tau\big) \\ &= V_t \sum_{h,k,l,m=1}^{2} \hat{\rho}_{hk}\hat{\rho}_{lm}\Lambda_k(t,T-\tau,T)\Lambda_m(t,T-\tau,T)\,\mathrm{d}W_h(t)\mathrm{d}W_l(t) \\ &= V_t \sum_{k,m=1}^{2} \Lambda_k(t,T-\tau,T)\left(\hat{\rho}^\top\hat{\rho}\right)_{km}\Lambda_m(t,T-\tau,T)\,dt. \end{aligned} \tag{4.27}$$

We can now proceed to calculate the volatility. Before stating the results, it is convenient to define the following quantity which will appears often into the calculations.

$$\Gamma(t,\theta) = \frac{1}{\theta}\left(e^{-\theta t} - 1\right).$$

For the forward looking caplet, we have

$$\begin{aligned}
\nu_{\mathrm{FL}}^2 &= \frac{1}{T-\tau}\int_0^{T-\tau} V_t \sum_{k,m=1}^{2} \Lambda_k(t,T-\tau,T)\left(\hat\rho^\top\hat\rho\right)_{km}\Lambda_m(t,T-\tau,T)\,dt \\
&= \frac{1}{T-\tau}\sum_{k,m=1}^{2}\left(\hat\rho^\top\hat\rho\right)_{km}\Lambda_k(0,T-\tau,T)\Lambda_m(0,T-\tau,T) \\
&\times\left[\theta\,\Gamma(\tau-T,\lambda_k+\lambda_m) - (\theta - V_0)\,\Gamma(\tau-T,\lambda_k+\lambda_m)\right].
\end{aligned}$$

Analogously, the volatility of the backward looking caplet is written as

$$\begin{aligned}
\nu_{\mathrm{BL}}^2 &= \frac{1}{T}\int_0^{T} V_t \sum_{k,m=1}^{2} \Lambda_k(t,T-\tau,T)\left(\hat\rho^\top\hat\rho\right)_{km}\Lambda_m(t,T-\tau,T)\,dt = \\
&= v_{\mathrm{FL}}^2 + \sum_{k,m=1}^{2}\frac{\hat\sigma_{kk}\hat\sigma_{mm}}{\lambda_k\lambda_m}\left(\hat\rho^\top\hat\rho\right)_{km} \\
&\times\Big[\theta\big(\tau + \Gamma(\tau,\lambda_k) + \Gamma(\tau,\lambda_m) - \Gamma(\tau,\lambda_k+\lambda_m)\big) - (\theta - V_0)\,e^{-\omega T} \\
&\times\big(\Gamma(\tau,\omega) + \Gamma(\tau,\lambda_k-\omega) + \Gamma(\tau,\lambda_m-\omega) + \Gamma(\tau,\lambda_k+\lambda_m-\omega)\big)\Big].
\end{aligned}$$

Thus, we can now deduce the prices of both contracts by applying Black's Equation (2.29). We obtain

$$\begin{aligned}
\mathbf{CPL}_0^{\mathrm{FL}}(T-\tau,T,K) &= P(0,T)\,\mathrm{Black}\left(R(0,T-\tau,T) + \frac{1}{\tau}, K + \frac{1}{\tau}, \nu_{\mathrm{FL}}^2(t)\right), \\
\mathbf{CPL}_0^{\mathrm{BL}}(T-\tau,T,K) &= P(0,T)\,\mathrm{Black}\left(R(0,T-\tau,T) + \frac{1}{\tau}, K + \frac{1}{\tau}, \nu_{\mathrm{BL}}^2(t)\right).
\end{aligned} \tag{4.28}$$

Finally, we point out that, since $\nu_{\mathrm{BL}} > \nu_{\mathrm{FL}}$, the backward looking caplet is always more expensive than the forward looking one.

Again, we have performed the calibration to the market data, requiring Equation (4.28) to match the quoted caplet prices. We used the cap volatility and the yield curve shown respectively in Table 4.2 and Table 4.3. The procedure follows exactly the same scheme explained in the previous section. The model involves 121 semi annual caplets, starting on November 25th 2022. Due to the large number of data to be matched, we report the value of the local minimum of the cost function

$$\sum_{i=1}^{N}\left(\mathbf{CPL}_{\mathrm{MKT}}^{(i)}-\mathbf{CPL}_{\mathrm{MODEL}}^{(i)}\right)^{2} \tag{4.29}$$

at the end of the calibration. It is equal to $1.86 \cdot 10^{-5}$. The resulting parameters are listed in Table 4.7.

Table 4.7: Model parameters resulting from the calibration to quoted cap prices.

Key	Value	Key	Value
λ_1	0.255246	ρ	0.376994
λ_2	0.255094	ω	0.322650
σ_1	1.178531	θ	0.127638
σ_2	1.179202	V_0	0.000048

4.2.3 Valuation of an American Swaption

Table 4.8: American swaptions prices evaluated with the trinomial tree in Hull-White model and Longstaff-Schwartz in HJM model.

Start	Tenor	Hull-White	HJM	Start	Tenor	Hull-White	HJM
1Y	1Y	0.348	0.419	15Y	1Y	0.809	0.788
1Y	5Y	2.826	2.814	15Y	5Y	3.909	3.823
1Y	10Y	6.491	6.575	15Y	10Y	7.455	7.290
1Y	15Y	10.096	10.180	15Y	15Y	10.811	10.514
1Y	20Y	13.185	13.265	15Y	20Y	14.042	13.535
1Y	30Y	18.821	18.660	15Y	30Y	19.962	19.167
5Y	1Y	0.756	0.748	20Y	1Y	0.785	0.759
5Y	5Y	3.873	3.849	20Y	5Y	3.846	3.664
5Y	10Y	7.597	7.605	20Y	10Y	7.412	7.063
5Y	15Y	10.886	10.934	20Y	15Y	10.758	10.251
5Y	20Y	13.951	13.926	20Y	20Y	13.932	13.234
5Y	30Y	19.701	19.484	20Y	30Y	19.737	18.694
10Y	1Y	0.819	0.814	30Y	1Y	0.763	0.713
10Y	5Y	3.927	3.932	30Y	5Y	3.668	3.442
10Y	10Y	7.524	7.468	30Y	10Y	7.085	6.640
10Y	15Y	10.843	10.688	30Y	15Y	10.277	9.621
10Y	20Y	13.996	13.674	30Y	20Y	13.251	12.364
10Y	30Y	19.847	19.323	30Y	30Y	18.584	17.230

We now turn to the problem of evaluating an American swaption. Such a contract allows the holder to enter a swap on a multiple set of dates up to its expiration. As a consequence, the date of the payment is not known a priori and the value of an American option is the value of the optimal exercise.

The best strategy for the holder is to compare the payoff from immediate exercise with the expected payoff from continuation, and then proceed if the latter is higher revisiting the decision at the next exercise time. Longstaff and Schwartz [LS98] proposed an algorithm based on backward induction which became one of the most popular among practitioners, as it offers a simple solution which is compatible when more than one factors affects the model.

We implemented the Longstaff-Schwartz algorithm and the outcomes are shown in Table 4.8. The HJM values are obtained by simulating the yield curve based on the value of the parameters shown in Table 4.5, whereas the label Hull-White refers to the model explained in Section 3.1.2 and calibrated on the market data presented in Table 4.1 and Table 4.3. Its values are obtained by mean of the trinomial tree technique [BM01, pp. 71–78]. For this case, we have assumed the set of exercise dates to be equivalent to the set of payment dates of the fixed leg. Consequently, American swaption having one year tenor are equivalent to the European case.

Conclusion

In this thesis, we discussed the insight of the interest-rates market to propose and implement a model which is capable to capture the dynamics of a backward looking structure of term rates and the existing Libor framework.

We started by establishing the analogy between an historical example from statistical physics and the trajectories of a generic financial assets in the market quotes. This idea is not capable to predict the trend of an asset, but serves as a reasonable modeling tool on which the market participants can agree on to assign a fair value on a contract. This approach let us take advantage of the nice visual interpretation we get of the entities featured in a stochastic differential equation.
We moved to the founding assumption of the pricing theory, which is the absence of arbitrage opportunities. An arbitrage is an investment strategy that costs nothing, does not expect a loss, and has a positive probability of generating a return. When two investments have the same risk but different returns, an arbitrage opportunity appear as buying where it costs less and selling where it costs more immediately generate a profit. If the market is arbitrage-free, this should not be possible. Under the risk-neutral probability measure, we diminished the returns to neutralize risks, and thus if the market is arbitrage-free, then the expected rate of return must be the same for all assets, including the riskless bank account whose corresponding interest rate is known. We analyzed arbitrage first, and then we investigated the hypothesis of market completeness, which grants all the securities to be actually tradable. Once the basics knowledge of pricing theory had been discussed, we moved into the analysis of the fundamental derivatives, whose evaluation depends on the initially given zero-coupon curve and the volatility structure. We briefly hinted the econometric approach and subsequently focused on the market calibration. For what concerns the historical one-factor Vasicek model, it can be calibrated only to the initial yield curve, without taking into account market quotes for the volatility. The Hull-White extension filled this gap by introducing a time dependent function which captures perfectly

the initial interest-rate values so that the remaining model parameters can be used to calibrate the volatility structure. A first alternative to short-rate models is achieved by the Heat-Jarrow-Morton (HJM) model, by choosing the instantaneous forward rate as the fundamental object to be modeled. We discussed the conditions which grant the no-arbitrage requirement on the drift term and derived an analytical formula for bond options. A further development is obtained by the Libor Market Model (LMM), which allows for an easy calibration to prices of actively traded instruments and it is expressed in terms of quantities directly observed in the market, the forward rates. Moreover, it has the desirable property to be compatible with Black's cap formula, which is the standard formula employed in the cap market. From a set of market data including zero-coupon curve, caps volatilities and swaptions volatilities, we performed the calibration by resorting to a parameterizations of the instantaneous correlations and through analytical approximations. Following the work by Lyashenko and Mercurio [LM19a], we extended the above framework to include the dynamics the Risk Free Rates (RFRs) which are backward looking in nature and they are set at the end of the corresponding application period. We showed how it can be extended to make it a complete term-structure model describing the evolution of all points on a yield curve, and not only the initial discrete set of forward rates. In order to limit the number of variables to simulate, that is, the given forward rates supplemented by the auxiliary processes in the HJM representation, we adopted the approach by Moreni and Pallavicini [MP10] and proposed a single-curve model specifically designed to be a flexible choice for pricing derivatives on the new RFRs benchmarks.

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