Chapter 6

Eigenvalues and Eigenvectors

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6.1 Introduction to eigenvalues

Motivations

- The **static** system problem of $A\boldsymbol{x} = \boldsymbol{b}$ has now been solved, e.g., by Gauss-Jordan method or Cramer's rule.

- However, a **dynamic** system problem such as

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

cannot be solved by the **static** system method.

- To solve the **dynamic** system problem, we need to find the **static** feature of A that is "unchanged" with the mapping A. In other words, $A\boldsymbol{x}$ maps to itself with possibly some stretching ($\lambda > 1$), shrinking ($0 < \lambda < 1$), or being reversed ($\lambda < 0$).

- These invariant characteristics of A are the **eigenvalues** and **eigenvectors**.

$A\boldsymbol{x}$ maps a vector $\boldsymbol{x}$ to its column space $\boldsymbol{C}(A)$. We are looking for a $\boldsymbol{v} \in \boldsymbol{C}(A)$ such that $A\boldsymbol{v}$ aligns with $\boldsymbol{v}$. The collection of all such vectors is the set of eigenvectors.

6.1 Eigenvalues and eigenvectors

Conception (Eigenvalues and eigenvectors): The eigenvalue-eigenvector pair $(\lambda_i, \boldsymbol{v}_i)$ of a square matrix A satisfies

$$A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$$

where $\boldsymbol{v}_i \neq \mathbf{0}$ (but λ_i can be zero).

Invariance of eigenvectors and eigenvalues.

- Property 1: The eigenvectors stay the same for every power of A.
  The eigenvalues equal the same power of the respective eigenvalues.

  I.e., $A^n \boldsymbol{v}_i = \lambda_i^n \boldsymbol{v}_i$.

$$\boxed{A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i} \implies \boxed{A^2 \boldsymbol{v}_i = A(\lambda_i \boldsymbol{v}_i) = \lambda_i A \boldsymbol{v}_i = \lambda_i^2 \boldsymbol{v}_i}$$

- Property 2: If the null space of $A_{n\times n}$ consists of non-zero vectors, then 0 is the eigenvalue of A.

  Accordingly, there exists non-zero $\boldsymbol{v}_i$ to satisfy $A\boldsymbol{v}_i = 0 \cdot \boldsymbol{v}_i = \mathbf{0}$.

- Property 3: Assume with no loss of generality $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_k|$. For any vector $\boldsymbol{x} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_k\boldsymbol{v}_k$ that is the linear combination of all **eigenvectors**, the normalized mapping $P = \frac{1}{\lambda_1}A$ (namely, $P\boldsymbol{v}_1 = \boldsymbol{v}_1$) (when being applied repeatedly) converges to the eigenvector with the largest absolute eigenvalue.

I.e.,

$$
\begin{aligned}
\lim_{k\to\infty} P^k\boldsymbol{x} = \lim_{k\to\infty}\frac{1}{\lambda_1^k}A^k\boldsymbol{x} &= \lim_{k\to\infty}\frac{1}{\lambda_1^k}\left(a_1A^k\boldsymbol{v}_1 + a_2A^k\boldsymbol{v}_2 + \cdots + a_kA^k\boldsymbol{v}_k\right)\\
&= \lim_{k\to\infty}\frac{1}{\lambda_1^k}\left(\underbrace{a_1\lambda_1^k\boldsymbol{v}_1}_{\text{steady state}} + \underbrace{a_2\lambda_2^k\boldsymbol{v}_2 + \cdots + \lambda_k^kA^k\boldsymbol{v}_k}_{\text{transient states}}\right)\\
&= a_1\boldsymbol{v}_1.
\end{aligned}
$$

6.1 How to determine the eigenvalues?

We wish to find a non-zero vector $\boldsymbol{v}$ to satisfy $A\boldsymbol{v} = \lambda\boldsymbol{v}$; then

$$(A - \lambda I)\boldsymbol{v} = 0 \quad \Leftrightarrow \quad \texttt{det}(A - \lambda I) = 0.$$

So by solving $\texttt{det}(A - \lambda I) = 0$, we can obtain **all** the **eigenvalues** of A.

Example. Find the eigenvalues of $A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$.

Solution.

$$\begin{aligned} \texttt{det}(A - \lambda I) &= \texttt{det}\left(\begin{bmatrix} 0.5 - \lambda & 0.5 \\ 0.5 & 0.5 - \lambda \end{bmatrix}\right) \\ &= (0.5 - \lambda)^2 - 0.5^2 = \lambda^2 - \lambda = \lambda(\lambda - 1) = 0 \\ &\Rightarrow \lambda = 0 \quad \text{or} \quad 1. \end{aligned}$$

6.1 How to determine the eigenvalues?

Proposition: Projection matrix (defined in Section 4.2) has eigenvalues either 1 or 0.

Proof:

- A projection matrix always satisfies $P^2 = P$. So $P^2\boldsymbol{v} = P\boldsymbol{v} = \lambda\boldsymbol{v}$.
- By definition of eigenvalues and eigenvectors, we have $P^2\boldsymbol{v} = \lambda^2\boldsymbol{v}$.
- Hence, $\lambda\boldsymbol{v} = \lambda^2\boldsymbol{v}$ for non-zero vector $\boldsymbol{v}$, which immediately implies $\lambda = \lambda^2$.
- Accordingly, λ is either 1 or 0. $\square$

Proposition: Permutation matrix has eigenvalues satisfying $\lambda^k = 1$ for some integer k.

Proof:

- A purmutation matrix always satisfies $P^{k+1} = P$ for some integer k.

  Example. $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then, $P\boldsymbol{v} = \begin{bmatrix} v_3 \\ v_1 \\ v_2 \end{bmatrix}$ and $P^3\boldsymbol{v} = \boldsymbol{v}$. Hence, $k = 3$.

- Accordingly, $\lambda^{k+1}\boldsymbol{v} = \lambda\boldsymbol{v}$, which gives $\lambda^k = 1$ since the eigenvalue of P cannot be zero. $\square$

6.1 How to determine the eigenvalues?

Proposition: Matrix

$$\alpha_m A^m + \alpha_{m-1} A^{m-1} + \cdots + \alpha_1 A + \alpha_0 I$$

has the same eigenvectors as A, but its eigenvalues become

$$\alpha_m \lambda^m + \alpha_{m-1} \lambda^{m-1} + \cdots + \alpha_1 \lambda + \alpha_0,$$

where λ is the eigenvalue of A.

Proof:

- Let $\boldsymbol{v}_i$ and λ_i be the eigenvector and eigenvalue of A. Then,

$$\left(\alpha_m A^m + \alpha_{m-1} A^{m-1} + \cdots + \alpha_0 I\right) \boldsymbol{v}_i = \left(\alpha_m \lambda^m + \alpha_{m-1} \lambda^{m-1} + \cdots + \alpha_0\right) \boldsymbol{v}_i$$

- Hence, $\boldsymbol{v}_i$ and $\left(\alpha_m \lambda^m + \alpha_{m-1} \lambda^{m-1} + \cdots + \alpha_0\right)$ are the eigenvector and eigenvalue of the polynomial matrix. $\square$

Theorem (Cayley-Hamilton): (Suppose A has linearly independent eigenvectors.)

$$f(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_0 I = \text{all-zero matrix}$$

if

$$f(\lambda) = \texttt{det}(A - \lambda I) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0.$$

Proof: The eigenvalues of $f(A)$ are **all zeros** and the eigenvectors of A remain the same as A. By definition of eigen-system, we have

$$f(A)\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\end{bmatrix}\begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_n)\end{bmatrix}.$$

□

Corollary (Cayley-Hamilton): (Suppose A has linearly independent eigenvectors.)

$$(\lambda_1 I - A)(\lambda_2 I - A)\cdots(\lambda_n I - A) = \text{all-zero matrix}$$

Proof: $f(\lambda)$ can be re-written as $f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$. □

6.1 How to determine the eigenvalues?

(Problem 11, Section 6.1) Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector $\boldsymbol{x}_2$. Any idea why this should be?

Hint: $(\lambda_1 I - A)(\lambda_2 I - A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ implies

$$(\lambda_1 I - A)\boldsymbol{w}_1 = \mathbf{0} \quad \text{and} \quad (\lambda_1 I - A)\boldsymbol{w}_2 = \mathbf{0}$$

where $(\lambda_2 I - A) = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 \end{bmatrix}$. Hence,

$\boxed{\text{the columns of } (\lambda_2 I - A) \text{ give the eigenvectors of } \lambda_1 \text{ if they are non-zero vectors}}$

and

$\boxed{\text{the columns of } (\lambda_1 I - A) \text{ give the eigenvectors of } \lambda_2 \text{ if they are non-zero vectors}}$.

So, **the (non-zero) columns of $A - \lambda_1 I$ are (multiples of) the eigenvector $\boldsymbol{x}_2$.**

6.1 Why Gauss-Jordan cannot solve $A\boldsymbol{x} = \lambda\boldsymbol{x}$?

- The forward elimination may change the eigenvalues and eigenvectors?

Example. Check eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$.

Solution.

- The eigenvalues of A satisfy $\mathtt{det}(A - \lambda I) = (\lambda - 3)^2 = 0$.
- $A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 9 \end{bmatrix}$. The eigenvalues of U apparently satisfy

$$\mathtt{det}(U - \lambda I) = (1 - \lambda)(9 - \lambda) = 0.$$

- Suppose $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the eigenvectors of U, respectively corresponding to eigenvalues 1 and 9. Then, they cannot be the eigenvectors of A since if they were,

$$\begin{cases} 3\boldsymbol{u}_1 = A\boldsymbol{u}_1 = LU\boldsymbol{u}_1 = L\boldsymbol{u}_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \boldsymbol{u}_1 \implies \boldsymbol{u}_1 = \mathbf{0} \\ 3\boldsymbol{u}_2 = A\boldsymbol{u}_2 = LU\boldsymbol{u}_2 = 9L\boldsymbol{u}_2 = 9\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \boldsymbol{u}_2 \implies \boldsymbol{u}_2 = \mathbf{0}. \end{cases}$$

□

Hence, the **eigenvalues** are nothing to do with **pivots** (except for a triangular A).

6.1 How to determine the eigenvalues? (revisited)

$$\boxed{\text{Solve } \mathtt{det}(A - \lambda I) = 0.}$$

- $\mathtt{det}(A - \lambda I)$ is a polynomial of order n.

$$f(\lambda) = \mathtt{det}(A - \lambda I) = \mathtt{det}\left(\begin{bmatrix} a_{1,1} - \lambda & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} - \lambda & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} - \lambda & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} - \lambda \end{bmatrix}\right)$$

$$= (a_{1,1} - \lambda)(a_{2,2} - \lambda)\cdots(a_{n,n} - \lambda) + \cdots \text{ (By Leibniz formula)}$$

$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

Observations

- The coefficient of λ^n is $(-1)^n$.
- The coefficient of λ^{n-1} is $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n a_{i,i} = \mathtt{trace}(A)$.
- $\ldots$
- The coefficient of λ^0 is $\prod_{i=1}^n \lambda_i = f(0) = \mathtt{det}(A)$.

6.1 How to determine the eigenvalues? (revisited)

These observations make easy the finding of the eigenvalues of 2×2 matrix.

Example. Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

Solution.

- $$\begin{cases} \lambda_1 + \lambda_2 = 1 + 1 = 2 \\ \lambda_1 \lambda_2 = 1 - 4 = -3 \end{cases} \implies (\lambda_1 - \lambda_2)^2 = (\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2 = 16$$
  $$\implies \lambda_1 - \lambda_2 = 4$$
  $$\implies \lambda = 3, -1.$$

$\square$

Example. Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

Solution.

- $$\begin{cases} \lambda_1 + \lambda_2 = 3 \\ \lambda_1 \lambda_2 = 0 \end{cases} \implies \lambda = 3, 0.$$

$\square$

6.1 Imaginary eigenvalues

In some cases, we have to allow **imaginary** eigenvalues.

- In order to solve polynomial equations $f(\lambda) = 0$, the mathematician was forced to **image** that there exists a number x satisfying $\boxed{x^2 = -1}$.

- By this technique, a polynomial equations of order n have exactly n (possibly **complex**, not **real**) solutions.

  Example. Solve $\lambda^2 + 1 = 0$. $\Longrightarrow$ $\lambda = \pm i$. □

- Based on this, to solve the **eigenvalues**, we were forced to accept **imaginary** eigenvalues.

  Example. Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

  Solution. $\texttt{det}(A - \lambda I) = \lambda^2 + 1 = 0 \Longrightarrow \lambda = \pm i$. □

6.1 Imaginary eigenvalues

Proposition: The eigenvalues of a symmetric matrix A (with real entries) are real, and the eigenvalues of a skew-symmetric (or antisymmetric) matrix B are pure imaginary.

Proof:

- Suppose $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Then,

$$\begin{aligned}
& A\boldsymbol{v}^* = \lambda^*\boldsymbol{v}^* \quad (A \text{ real}) \\
\Longrightarrow \quad & (A\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \\
\Longrightarrow \quad & (\boldsymbol{v}^*)^{\mathsf{T}}A^{\mathsf{T}}\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \\
\Longrightarrow \quad & (\boldsymbol{v}^*)^{\mathsf{T}}A\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \quad (\text{symmetry means } A^{\mathsf{T}} = A) \\
\Longrightarrow \quad & (\boldsymbol{v}^*)^{\mathsf{T}}\lambda\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \quad (A\boldsymbol{v} = \lambda\boldsymbol{v}) \\
\Longrightarrow \quad & \lambda\|\boldsymbol{v}\|^2 = \lambda^*\|\boldsymbol{v}\|^2 \quad (\text{eigenvector must be non-zero, i.e., } \|\boldsymbol{v}\|^2 \neq 0) \\
\Longrightarrow \quad & \lambda = \lambda^* \\
\Longrightarrow \quad & \lambda \text{ real}
\end{aligned}$$

6.1 Imaginary eigenvalues

- Suppose $B\boldsymbol{v} = \lambda\boldsymbol{v}$. Then,

$$
\begin{aligned}
& B\boldsymbol{v}^* = \lambda^*\boldsymbol{v}^* \quad (B \text{ real}) \\
\Longrightarrow\ & (B\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} = (\lambda^*\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \\
\Longrightarrow\ & (\boldsymbol{v}^*)^{\mathsf{T}}B^{\mathsf{T}}\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \\
\Longrightarrow\ & (\boldsymbol{v}^*)^{\mathsf{T}}(-B)\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \quad (\text{skew-symmetry means } B^{\mathsf{T}} = -B) \\
\Longrightarrow\ & (\boldsymbol{v}^*)^{\mathsf{T}}(-\lambda)\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \quad (B\boldsymbol{v} = \lambda\boldsymbol{v}) \\
\Longrightarrow\ & (-\lambda)\|\boldsymbol{v}\|^2 = \lambda^*\|\boldsymbol{v}\|^2 \quad (\text{eigenvector must be non-zero, i.e., } \|\boldsymbol{v}\|^2 \neq 0) \\
\Longrightarrow\ & -\lambda = \lambda^* \\
\Longrightarrow\ & \lambda \text{ imaginary}
\end{aligned}
$$

$\square$

6.1 Eigenvalues/eigenvectors of inverse

For invertible A, the relation between eigenvalues and eigenvectors of A and A^{-1} can be well-determined.

Proposition: The eigenvalues and eigenvectors of A^{-1} are

$$\left(\frac{1}{\lambda_1}, \boldsymbol{v}_1\right), \left(\frac{1}{\lambda_2}, \boldsymbol{v}_2\right), \ldots, \left(\frac{1}{\lambda_n}, \boldsymbol{v}_n\right),$$

where $\{(\lambda_i, \boldsymbol{v}_i)\}_{i=1}^n$ are the eigenvalues and eigenvectors of A.

Proof:

- The eigenvalues of invertible A must be non-zero because $\texttt{det}(A) \neq 0$.
- Suppose $A\boldsymbol{v} = \lambda\boldsymbol{v}$, where $\boldsymbol{v} \neq \mathbf{0}$ and $\lambda \neq 0$. (I.e., λ and $\boldsymbol{v}$ are eigenvalue and eigenvector of A.)
- So, $A\boldsymbol{v} = \lambda\boldsymbol{v} \implies A^{-1}(A\boldsymbol{v}) = A^{-1}(\lambda\boldsymbol{v}) \implies \boldsymbol{v} = \lambda A^{-1}\boldsymbol{v} \implies \frac{1}{\lambda}\boldsymbol{v} = A^{-1}\boldsymbol{v}$. $\Box$

Note: The eigenvalues of A^k and A^{-1} are λ^k and λ^{-1} with the same eigenvectors as A, where λ is the eigenvalue of A.

6.1 Eigenvalues/eigenvectors of inverse

Proposition: The eigenvalues of A^{T} are the same as the eigenvalues of A. (But they may have different **eigenvectors**.)

Proof: $\texttt{det}(A-\lambda I) = \texttt{det}\left((A-\lambda I)^{\mathsf{T}}\right) = \texttt{det}\left(A^{\mathsf{T}} - \lambda I^{\mathsf{T}}\right) = \texttt{det}\left(A^{\mathsf{T}} - \lambda I\right)$. □

Corollary: The eigenvalues of an invertible matrix A satisfying $A^{-1} = A^{\mathsf{T}}$ are on the unit circle of the complex plain.

Proof: Suppose $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Then,

$$
\begin{aligned}
& A\boldsymbol{v}^* = \lambda^*\boldsymbol{v}^* \quad (A \text{ real}) \\
\Longrightarrow\ & (A\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \\
\Longrightarrow\ & (\boldsymbol{v}^*)^{\mathsf{T}}A^{\mathsf{T}}\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \\
\Longrightarrow\ & (\boldsymbol{v}^*)^{\mathsf{T}}A^{-1}\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \quad (A^{\mathsf{T}} = A^{-1}) \\
\Longrightarrow\ & (\boldsymbol{v}^*)^{\mathsf{T}}\tfrac{1}{\lambda}\boldsymbol{v} = \lambda^*(\boldsymbol{v}^*)^{\mathsf{T}}\boldsymbol{v} \quad (A^{-1}\boldsymbol{v} = \tfrac{1}{\lambda}\boldsymbol{v}) \\
\Longrightarrow\ & \tfrac{1}{\lambda}\|\boldsymbol{v}\|^2 = \lambda^*\|\boldsymbol{v}\|^2 \quad (\text{eigenvector must be non-zero, i.e., } \|\boldsymbol{v}\|^2 \neq 0) \\
\Longrightarrow\ & \lambda\lambda^* = |\lambda|^2 = 1
\end{aligned}
$$

□

Example. Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ that satisfies $A^{\mathsf{T}} = A^{-1}$.

Solution. $\texttt{det}(A - \lambda I) = \lambda^2 + 1 = 0 \implies \lambda = \pm i$. □

- After the identification of **eigenvalues** via $\texttt{det}(A-\lambda I)=0$, we can determine the respective eigenvectors using the **null space technique**.

- Recall (from slide 3-41) how we completely solve

$$B_{n\times n}\boldsymbol{v}_{n\times 1} = (A_{n\times n} - \lambda I_{n\times n})\boldsymbol{v}_{n\times 1} = \mathbf{0}_{n\times 1}.$$

Answer:

$$R = \texttt{rref}(B) = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathbf{0}_{(n-r)\times r} & \mathbf{0}_{(n-r)\times(n-r)} \end{bmatrix} \quad \text{(with no row exchange)}$$

$$\Rightarrow \quad N_{n\times(n-r)} = \begin{bmatrix} -F_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$$

Then, every solution $\boldsymbol{v}$ for $B\boldsymbol{v} = \mathbf{0}$ is of the form

$$\boldsymbol{v}^{(\text{null})}_{n\times 1} = N_{n\times(n-r)}\boldsymbol{w}_{(n-r)\times 1} \text{ for any } \boldsymbol{w} \in \Re^{n-r}.$$

Here, we usually find the $(n-r)$ **orthonormal bases** for the null space as the representative eigenvectors, which are exactly the $(n-r)$ columns of $N_{n\times(n-r)}$ (with proper normalization).

(Problem 12, Section 6.1) Find three eigenvectors for this matrix P (projection matrices have $\lambda = 1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

Solution.

- $\texttt{det}(P - \lambda I) = 0$ gives $\lambda(1 - \lambda^2) = 0$; so $\lambda = 0, 1$.
- $\lambda = 1$: $\texttt{rref}(P - I) = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

  so $F_{1\times 2} = \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix}$, and $N_{3\times 2} = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, which implies eigenvectors $= \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

6.1 Determination of eigenvectors

- $\lambda = 0$: $\texttt{rref}(P) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (with no row exchange);

  so $F = \begin{bmatrix} 2 \\ - \\ 0 \end{bmatrix}$, and $N_{3\times 1} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, which implies eigenvector $= \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

$\square$

6.1 MATLAB revisited

In MATLAB:

- We can find the eigenvalues and eigenvectors of a matrix A by:

```
[V D]= eig(A);  % Find the eigenvalues/vectors of A
```

- The columns of V are the eigenvectors.
- The diagonals of D are the eigenvalues.

Proposition:

- The eigenvectors corresponding non-zero eigenvalues are in $\boldsymbol{C}(A)$.
- The eigenvectors corresponding zero eigenvalues are in $\boldsymbol{N}(A)$.

Proof: The first one can be seen from $A\boldsymbol{v} = \lambda\boldsymbol{v}$ and the second one can be proved by $A\boldsymbol{v} = \boldsymbol{0}$. $\square$

6.1 Some discussions on problems

(Problem 25, Section 6.1) Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same **independent** eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Then $A = B$. *Reason:* Any vector $\boldsymbol{x}$ is a combintion $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. What is $A\boldsymbol{x}$? What is $B\boldsymbol{x}$?

Thinking over Problem 25: Suppose A and B have the same eigenvalues and eigenvectors (not necessarily **independent**). Can we claim that $A = B$.

Answer to the thinking: Not necessarily.

As a counterexample, both $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ have eigenvalues $0, 0$ and single eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ but they are not equal.

If however the eigenvectors span the n-dimensional space (such as there are n of them and they are linearly independent), then $A = B$.

Hint for Problem 25: $A \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 & \cdots & \boldsymbol{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$.

This important fact will be re-emphasized in Section 6.2.

6.1 Some discussions on problems

(Problem 26, Section 6.1) The block B has eigenvalues $1, 2$ and C has eigenvalues $3, 4$ and D has eigenvalues $5, 7$. Find the eigenvalues of the 4 by 4 matrix A:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \left[\begin{array}{rr|rr} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ \hline 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{array}\right].$$

Thinking over Problem 26: The eigenvalues of $\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ are the eigenvalues of B and D because we can show

$$\texttt{det}\left(\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}\right) = \texttt{det}(B) \cdot \texttt{det}(D).$$

(See *Problems 23* and *25* in Section 5.2.)

6.1 Some discussions on problems

(Problem 23, Section 5.2) With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A|\,|D| \quad \text{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D| - |C|\,|B|.$$

(a) Why is the first statement true? Somehow B doesn't enter.

Hint: Leibniz formula.

(b) Show by example that equality fails (as shown) when C enters.

(c) Show by example that the answer $\texttt{det}(AD - CB)$ is also wrong.

6.1 Some discussions on problems

(Problem 25, Section 5.2) Block elimination subtracts CA^{-1} times the first row $\begin{bmatrix} A & B \end{bmatrix}$ from the second row $\begin{bmatrix} C & D \end{bmatrix}$. This leaves the *Schur complement* $D - CA^{-1}B$ in the corner:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Take determinants of these block matrices to prove correct rules if A^{-1} exists:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A|\,|D - CA^{-1}B| = |AD - CB| \quad \text{provided} \quad AC = CA.$$

Hint: $\texttt{det}(A) \cdot \texttt{det}(D - CA^{-1}B) = \texttt{det}\left(A(D - CA^{-1}B)\right)$.

6.1 Some discussions on problems

(Problem 37, Section 6.1)

(a) Find the eigenvalues and eigenvectors of A. They depend on c:

$$A = \begin{bmatrix} .4 & 1-c \\ .6 & c \end{bmatrix}.$$

(b) Show that A has just one line of eigenvectors when $c = 1.6$.

(c) This is a Markov matrix when $c = .8$. Then A^n will approach what matrix A^∞?

Definition (Markov matrix): A Markov matrix is a matrix with positive entries, for which every column adds to one.

- Note that some researchers define the Markov matrix by replacing positive by non-negative; thus, they will use the term *positive Markov matrix*. The below observations hold for Markov matrices with "non-negative entries".

6.1 Some discussions on problems

- 1 must be one of the eigenvalues of a Markov matrix.

  Proof: A and A^{T} have the same eigenvalues, and $A^{\mathsf{T}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. □

- The eigenvalues of a Markov matrix satisfy $|\lambda| \leq 1$.

  Proof: $A^{\mathsf{T}}\boldsymbol{v} = \lambda\boldsymbol{v}$ implies $\sum_{i=1}^{n} a_{i,j} v_i = \lambda v_j$; hence, by letting v_k satisfying $|v_k| = \max_{1 \leq i \leq n} |v_i|$, we have

$$|\lambda v_k| = \left| \sum_{i=1}^{n} a_{i,k} v_i \right| \leq \sum_{i=1}^{n} a_{i,k} |v_i| \leq \sum_{i=1}^{n} a_{i,k} |v_k| = |v_k|$$

  which implies the desired result. □

6.1 Some discussions on problems

(Problem 31, Section 6.1) If we exchange rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and B for $\lambda_1 = 11$. *Rank one gives* $\lambda_2 = \lambda_3 = 0$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad B = PAP^{\mathsf{T}} = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

Thinking over Problem 31: This is sometimes useful in determining the eigenvalues and eigenvectors.

- If $A\boldsymbol{v} = \lambda\boldsymbol{v}$ and $B = PAP^{-1}$ (for any invertible P), then λ and $\boldsymbol{u} = P\boldsymbol{v}$ are the eigenvalue and eigenvector of B.

  Proof: $B\boldsymbol{u} = PAP^{-1}\boldsymbol{u} = PA\boldsymbol{v} = \lambda P\boldsymbol{v} = \lambda\boldsymbol{u}$. $\square$

For example, $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In such case, $P^{-1} = P = P^{\mathsf{T}}$.

6.1 Some discussions on problems

With the fact above, we can further claim that

Proposition. The eigenvalues of AB and BA are equal if one of A and B is invertible.

Proof: This can be proved by $(AB) = A(BA)A^{-1}$ or $(AB) = B^{-1}(BA)B$. Note that AB has eigenvector $A\boldsymbol{v}$ or $B^{-1}\boldsymbol{v}$ if $\boldsymbol{v}$ is the eigenvector of BA. $\square$

The eigenvectors of a diagonal matrix $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ are apparently

$$\boldsymbol{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{position } i \ , \quad i = 1, 2, \ldots, n$$

Hence, $A = S\Lambda S^{-1}$ have the same eigenvalues as Λ and have eigenvectors $\boldsymbol{v}_i = S\boldsymbol{e}_i$. This implies that

$$S = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}$$

where $\{\boldsymbol{v}_i\}$ are the eigenvectors of A.

What if S is not invertible? Then, we have the next theorem.

6.2 Diagnolizing a matrix

A convenience for the eigen-system analysis is that we can **diagonalize** a matrix.

Theorem. A matrix A can be written as

$$A \underbrace{\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}}_{=S} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}}_{=\Lambda}$$

where $\{(\lambda_i, \boldsymbol{v}_i)\}_{i=1}^n$ are the eigenvalue-eigenvector pairs of A.

Proof: The theorem holds by definition of eigenvalues and eigenvectors. □

Corollary. If S is invertible, then

$$S^{-1}AS = \Lambda \quad \text{equivalently} \quad A = S\Lambda S^{-1}.$$

6.2 Diagnolizing a matrix

This makes **easy** the computation of polynomial with argument A.

Proposition (recall slide 6-6): Matrix

$$\alpha_m A^m + \alpha_{m-1} A^{m-1} + \cdots + \alpha_0 I$$

has the same eigenvectors as A, but its eigenvalues become

$$\alpha_m \lambda^m + \alpha_{m-1} \lambda^{m-1} + \cdots + \alpha_0,$$

where λ is the eigenvalue of A.

Proposition:

$$\alpha_m A^m + \alpha_{m-1} A^{m-1} + \cdots + \alpha_0 I = S \left(\alpha_m \Lambda^m + \alpha_{m-1} \Lambda^{m-1} + \cdots + \alpha_0 \right) S^{-1}$$

Proposition:

$$A^m = S \Lambda^m S^{-1}$$

Exception

- It is possible that an $n \times n$ matrix does not have n eigenvectors.

Example. $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

Solution.

- $\lambda_1 = 0$ is apparently an eigenvalue for a non-invertible matrix.
- The second eigenvalue is $\lambda_2 = \texttt{trace}(A) - \lambda_1 = [1 + (-1)] - \lambda_1 = 0$.
- This matrix however only has one eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (or some may say "two repeated eigenvectors").
- In such case, we still have

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Lambda}$$

but S has no inverse. □

In such case, we cannot preform SAS^{-1}; so we say A cannot be **diagonalized**.

When will S be guaranteed to be invertible?

One possible answer: When all eigenvalues are distinct.

Proof: Suppose for a matrix A, the first k eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ are linearly independent, but the $(k+1)$th eigenvector is dependent on the previous k eigenvectors. Then, for some unique $a_1, \ldots, a_k$,

$$\boldsymbol{v}_{k+1} = a_1\boldsymbol{v}_1 + \cdots + a_k\boldsymbol{v}_k,$$

which implies

$$\begin{cases} \lambda_{k+1}\boldsymbol{v}_{k+1} = A\boldsymbol{v}_{k+1} = a_1 A\boldsymbol{v}_1 + \ldots + a_k A\boldsymbol{v}_k = a_1\lambda_1\boldsymbol{v}_1 + \ldots + a_k\lambda_k\boldsymbol{v}_k \\ \lambda_{k+1}\boldsymbol{v}_{k+1} = a_1\lambda_{k+1}\boldsymbol{v}_1 + \ldots + a_k\lambda_{k+1}\boldsymbol{v}_k \end{cases}$$

Accordingly, $\lambda_{k+1} = \lambda_i$ for $1 \leq i \leq k$; i.e., that $\boldsymbol{v}_{k+1}$ is linearly dependent on $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ only occurs when they have the same eigenvalues. (The proof is not yet completed here! See the discussion and Example below.)

The above proof said as long as we have the $(k+1)$th eigenvector, then it is linearly dependent on the previous k eigenvectors only when all eigenvalues are equal! But sometimes, we do not guarantee to have the $(k+1)$th eigenvector.

6.2 Diagnolizing a matrix

Example. $A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}$.

Solution.

- We have **four** eigenvalues $1, 2, 4, 4$.
- But we can only have **three** eigenvectors for $1, 2$ and 4.
- From the proof in the previous page, the eigenvectors for 1, 2 and 4 are linearly **indepedent** because 1, 2 and 4 are **not equal**. $\square$

Proof (Continue): One condition that guarantees to have n eigenvectors is that we have n distinct eigenvalues, which now complete the proof. $\square$

Definition (GM and AM): The number of linearly independent eigenvectors for an eigenvalue λ is called its **geometric multiplicity**; the number of its appearance in solving $\texttt{det}(A - \lambda I)$ is called its **algebraic multiplicity**.

Example. In the above example, GM=1 and AM=2 for $\lambda = 4$.

6.2 Fibonacci series

Eigen-decomposition is useful in solving Fibonacci series

Problem: Suppose $F_{k+2} = F_{k+1} + F_k$ with initially $F_1 = 1$ and $F_0 = 0$. Find F_{100}.

Answer:

- $\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \implies \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$
- So,

$$\begin{aligned}
\begin{bmatrix} F_{100} \\ F_{99} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{99} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = S\Lambda^{99}S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{99} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{99} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{99} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{99} \end{bmatrix} \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{100} & \left(\frac{1-\sqrt{5}}{2}\right)^{100} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{99} & \left(\frac{1-\sqrt{5}}{2}\right)^{99} \end{bmatrix} \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{100} - \left(\frac{1-\sqrt{5}}{2}\right)^{100} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{99} - \left(\frac{1-\sqrt{5}}{2}\right)^{99} \end{bmatrix}.$$

$\square$

6.2 Generalization to $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$

- A generalization of the solution to Fibonacci series is as follows.

Suppose $\boldsymbol{u}_0 = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n$.
Then

$$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = a_1A^k\boldsymbol{v}_1 + a_2A^k\boldsymbol{v}_2 + \cdots + a_nA^k\boldsymbol{v}_n$$
$$= a_1\lambda_1^k\boldsymbol{v}_1 + a_2\lambda_2^k\boldsymbol{v}_2 + \ldots + a_n\lambda_n^k\boldsymbol{v}_n.$$

Examination:

$$\boldsymbol{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = A^k\boldsymbol{u}_0$$

and

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} - \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

So

$$\boldsymbol{u}_{99} = \begin{bmatrix} F_{100} \\ F_{99} \end{bmatrix} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{99}}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} - \frac{\left(\frac{1-\sqrt{5}}{2}\right)^{99}}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}.$$

(Problem 30, Section 6.2) Suppose the same S diagonalizes both A and B. They have the same eigenvectors in $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Prove that $AB = BA$.

Proposition: Suppose both A and B can be diagnosed, and one of A and B have distinct eigenvalues. Then, A and B have the same eigenvectors if, and only if, $AB = BA$.

Proof:

- *(See Problem 30 above)* If A and B have the same eigenvectors, then

  $$AB = (S\Lambda_A S^{-1})(S\Lambda_B S^{-1}) = S\Lambda_A\Lambda_B S^{-1} = S\Lambda_B\Lambda_A S^{-1} = (S\Lambda_B S^{-1})(S\Lambda_A S^{-1}) = BA.$$

- Suppose without loss of generality that the eigenvalues of A are all distinct.

  Then, if $AB = BA$, we have for given $A\boldsymbol{v}_\ell = \lambda_\ell \boldsymbol{v}_\ell$ and $\boldsymbol{u}_\ell = B\boldsymbol{v}_\ell$,

  $$A\boldsymbol{u}_\ell = A(B\boldsymbol{v}_\ell) = (AB)\boldsymbol{v}_\ell = (BA)\boldsymbol{v}_\ell = B(A\boldsymbol{v}_\ell) = B(\lambda_\ell \boldsymbol{v}_\ell) = \lambda_\ell(B\boldsymbol{v}_\ell) = \lambda_\ell \boldsymbol{u}_\ell.$$

  Hence, $\boldsymbol{u}_\ell = B\boldsymbol{v}_\ell$ and $\boldsymbol{v}_\ell$ are both the eigenvectors of A corresponding to the same eigenvalue λ_ℓ.

6.2 More applications of eigen-decomposition

Let $\boldsymbol{u}_\ell = \sum_{i=1}^n a_i \boldsymbol{v}_i$, where $\{\boldsymbol{v}_i\}_{i=1}^n$ are linearly independent eigenvectors of A.

$$\begin{cases} A\boldsymbol{u}_\ell = \sum_{i=1}^n a_i A\boldsymbol{v}_i = \sum_{i=1}^n a_i \lambda_i \boldsymbol{v}_i \\ A\boldsymbol{u}_\ell = \lambda_\ell \boldsymbol{u}_\ell \qquad\qquad = \sum_{i=1}^n a_i \lambda_\ell \boldsymbol{v}_i \end{cases} \implies a_i(\lambda_i - \lambda_\ell) = 0 \text{ for } 1 \le i \le n.$$

This implies

$$\boldsymbol{u}_\ell = \sum_{i:\lambda_i=\lambda_\ell} a_i \boldsymbol{v}_i = \boxed{a_\ell \boldsymbol{v}_\ell = B\boldsymbol{v}_\ell}.$$

Thus, $\boldsymbol{v}_\ell$ is an eigenvector of B. $\square$

6.2 Problem discussions

(Problem 32, Section 6.2) Substitute $A = S\Lambda S^{-1}$ into the product $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \texttt{det}(A-\lambda I)$. The **Cayley-Hamilton Theorem** says that this product is always $p(A) =$*zero matrix*, even if A is not diagonalizable.

Thinking over Problem 32: The Cayley-Hamilton Theorem can be easily proved if A is diagonalizable.

Corollary (Cayley-Hamilton): Suppose A has linearly independent eigenvectors.

$$(\lambda_1 I - A)(\lambda_2 I - A)\cdots(\lambda_n I - A) = \text{all-zero matrix}$$

Proof:

$$\begin{aligned}
&(\lambda_1 I - A)(\lambda_2 I - A)\cdots(\lambda_n I - A)\\
&= S(\lambda_1 I - \Lambda)S^{-1}S(\lambda_2 I - \Lambda)S^{-1}\cdots S(\lambda_n I - \Lambda)S^{-1}\\
&= S\ \underbrace{(\lambda_1 I - \Lambda)}_{(1,1)\text{th entry}=0}\ \underbrace{(\lambda_2 I - \Lambda)}_{(2,2)\text{th entry}=0}\ \cdots\ \underbrace{(\lambda_n I - \Lambda)}_{(n,n)\text{th entry}=0}\ S^{-1}\\
&= S\left[\text{all-zero entries}\right]S^{-1} = \left[\text{all-zero entries}\right]
\end{aligned}$$

$\square$

6.3 Applications to differential equations

We can use eigen-decomposition to solve

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \boxed{\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}} = \begin{bmatrix} a_{1,1}u_1(t) + a_{1,2}u_2(t) + \cdots + a_{1,n}u_n(t) \\ a_{2,1}u_1(t) + a_{2,2}u_2(t) + \cdots + a_{2,n}u_n(t) \\ \vdots \\ a_{n,1}u_1(t) + a_{n,2}u_2(t) + \cdots + a_{n,n}u_n(t) \end{bmatrix}$$

where A is called the **companion** matrix.

By differential equation technique, we know that the solution is of the form

$$\boxed{\boldsymbol{u} = e^{\lambda t}\boldsymbol{v}}$$

for some constant λ and constant vector $\boldsymbol{v}$.

Question: What are all λ and $\boldsymbol{v}$ that satisfy $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$?

Solution: Take $\boldsymbol{u} = e^{\lambda t}\boldsymbol{v}$ into the equation: $\dfrac{d\boldsymbol{u}}{dt}\left(= \lambda e^{\lambda t}\boldsymbol{v}\right) = A\boldsymbol{u}\left(= Ae^{\lambda t}\boldsymbol{v}\right)$
$\Longrightarrow$ $\lambda\boldsymbol{v} = A\boldsymbol{v}$.

The answer are all eigenvalues and eigenvectors.

6.3 Applications to differential equations

Since $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$ is a linear system, a linear combination of solutions is still a solution.

Hence, if there are n eigenvectors, the complete solution can be represented as

$$\boxed{c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2 + \cdots + c_n e^{\lambda_n t}\boldsymbol{v}_n}$$

where $c_1, c_2, \ldots, c_n$ can be determined by the initial conditions.

Here, for convenience of discussion at this moment, we assume that A gives us exactly n eigenvectors.

6.3 Applications to differential equations

It is sometimes **convenient** to re-express the solution of $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$ is $e^{At}\boldsymbol{u}(0)$ for a given initial condition $\boldsymbol{u}(0)$ as $e^{At}\boldsymbol{u}(0)$, i.e.,

$$
\begin{aligned}
&c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2 + \cdots + c_n e^{\lambda_n t}\boldsymbol{v}_n \\
&= \underbrace{\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}}_{S} \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = e^{At}\boldsymbol{u}(0)
\end{aligned}
$$

where we define

$$
\begin{aligned}
e^{At} &\triangleq \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} (S\Lambda^k S^{-1}) = S \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \Lambda^k \right) S^{-1} \\
&= S \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda_1^k & 0 & \cdots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda_n^k \end{bmatrix} S^{-1} = S \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} S^{-1}.
\end{aligned}
$$

and hence $\boldsymbol{u}(0) = S\boldsymbol{c}$.

Key to remember: Again, we define by convention that

$$f(A) = S \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_n) \end{bmatrix} S^{-1}.$$

So,
$$e^{At} = S \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} S^{-1}.$$

6.3 Applications to differential equations

We can solve the second-order equation in the same manner.

Example. Solve $my'' + by' + ky = 0$.

Answer:

- Let $z = y'$. Then, the problem is reduced to

$$\begin{cases} y' = z \\ z' = -\frac{k}{m}y - \frac{b}{m}z \end{cases} \implies \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$$

  with $\boldsymbol{u} = \begin{bmatrix} y \\ z \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$.

- The complete solution for $\boldsymbol{u}$ is therefore

$$c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2.$$

$\square$

Example. Solve $y'' + y = 0$ with initial $y(0) = 1$ and $y'(0) = 0$.

Solution.

- Let $z = y'$. Then, the problem is reduced to

$$\begin{cases} y' = z \\ z' = -y \end{cases} \implies \frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u}$$

with $\boldsymbol{u} = \begin{bmatrix} y \\ z \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

- The eigenvalues and eigenvectors of A are

$$\left(\imath, \begin{bmatrix} -\imath \\ 1 \end{bmatrix}\right) \quad \text{and} \quad \left(-\imath, \begin{bmatrix} \imath \\ 1 \end{bmatrix}\right).$$

- The complete solution for $\boldsymbol{u}$ is therefore

$$\begin{bmatrix} y \\ y' \end{bmatrix} = c_1 e^{\imath t} \begin{bmatrix} -\imath \\ 1 \end{bmatrix} + c_2 e^{-\imath t} \begin{bmatrix} \imath \\ 1 \end{bmatrix} \text{ with initially } \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} -\imath \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \imath \\ 1 \end{bmatrix}.$$

So, $c_1 = \frac{\imath}{2}$ and $c_2 = -\frac{\imath}{2}$. Thus,

$$y(t) = \frac{\imath}{2} e^{\imath t}(-\imath) - \frac{\imath}{2} e^{-\imath t}\imath = \frac{1}{2} e^{\imath t} + \frac{1}{2} e^{-\imath t} = \cos(t).$$

$\square$

6.3 Applications to differential equations

In practice, we will use a **discrete approximation** to approximate a continuous function. There are however three different discrete approximations.

For example, how to approximate $y''(t) = -y(t)$ by a discrete system?

$$\frac{Y_{n+1} - 2Y_n + Y_{n-1}}{(\Delta t)^2} = \frac{\frac{Y_{n+1}-Y_n}{\Delta t} - \frac{Y_n - Y_{n-1}}{\Delta t}}{\Delta t} = \begin{array}{ll} -Y_{n-1} & \textbf{Forward approximation} \\ -Y_n & \textbf{Centered approximation} \\ -Y_{n+1} & \textbf{Backward approximation} \end{array}$$

Let's take **forward** approximation as an example, i.e.,

$$\frac{\underbrace{\frac{Y_{n+1}-Y_n}{\Delta t}}_{Z_n} - \underbrace{\frac{Y_n - Y_{n-1}}{\Delta t}}_{Z_{n-1}}}{\Delta t} = -Y_{n-1}.$$

Thus,

$$\begin{cases} y'(t) = z(t) \\ z'(t) = -y(t) \end{cases} \approx \begin{cases} \dfrac{Y_{n+1} - Y_n}{\Delta t} = Z_n \\ \dfrac{Z_{n+1} - Z_n}{\Delta t} = -Y_n \end{cases} \approx \underbrace{\begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix}}_{\boldsymbol{u}_{n+1}} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}}_{\boldsymbol{u}_n}$$

Then, we obtain

$$\boldsymbol{u}_n = A^n \boldsymbol{u}_0 = \begin{bmatrix} 1 & 1 \\ \imath & -\imath \end{bmatrix} \begin{bmatrix} (1+\imath\Delta t)^n & 0 \\ 0 & (1-\imath\Delta t)^n \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\imath}{2} \\ \frac{1}{2} & \frac{\imath}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$Y_n = \frac{(1+\imath\Delta t)^n + (1-\imath\Delta t)^n}{2} = \left[1 + (\Delta t)^2\right]^{n/2} \cos(n \cdot \Delta\theta)$$

where $\Delta\theta = \tan^{-1}(\Delta t)$.

Problem: $Y_n \to \infty$ as n large.

Backward approximation will leave to $Y_n \to 0$ as n large.

$$\begin{cases} y'(t) = z(t) \\ z'(t) = -y(t) \end{cases} \approx \begin{cases} \dfrac{Y_{n+1} - Y_n}{\Delta t} = Z_{n+1} \\ \dfrac{Z_{n+1} - Z_n}{\Delta t} = -Y_{n+1} \end{cases} \approx \underbrace{\begin{bmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{bmatrix}}_{\tilde{A}^{-1}} \underbrace{\begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix}}_{\boldsymbol{u}_{n+1}} = \underbrace{\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}}_{\boldsymbol{u}_n}$$

A solution to the problem: Interleave the **forward approximation** with the **backward approximation**.

$$\begin{cases} y'(t) = z(t) \\ z'(t) = -y(t) \end{cases} \approx \begin{cases} \dfrac{Y_{n+1} - Y_n}{\Delta t} = Z_n \\ \dfrac{Z_{n+1} - Z_n}{\Delta t} = -Y_{n+1} \end{cases} \approx \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \underbrace{\begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix}}_{\boldsymbol{u}_{n+1}} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}}_{\boldsymbol{u}_n}$$

We then perform this so-called **leapfrog method**. (See *Problems 28 and 29*.)

$$\boxed{\underbrace{\begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix}}_{\boldsymbol{u}_{n+1}} = \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}}_{\boldsymbol{u}_n} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 - (\Delta t)^2 \end{bmatrix}}_{|\text{eigenvalues}|=1 \text{ if } \Delta t \leq 2} \underbrace{\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}}_{\boldsymbol{u}_n}}$$

(Problem 28, Section 6.3) Centering $y'' = -y$ in Example 3 will produce $Y_{n+1} - 2Y_n + Y_{n-1} = -(\Delta t)^2 Y_n$. This can be written as a one-step difference equation for $U = (Y, Z)$:

$$\begin{aligned} Y_{n+1} &= Y_n + \Delta t\, Z_n \\ Z_{n+1} &= Z_n - \Delta t\, Y_{n+1} \end{aligned} \qquad \begin{bmatrix} 1 & 0 \\ \Delta t & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}.$$

Invert the matrix on the left side to write this as $U_{n+1} = AU_n$. Show that $\texttt{det} A = 1$. Choose the large time step $\Delta t = 1$ and find the eigenvalues λ_1 and $\lambda_2 = \bar{\lambda}_1$ of A:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \text{ has } |\lambda_1| = |\lambda_2| = 1. \text{ Show that } A^6 \text{ is exactly } I.$$

After 6 steps to $t = 6$, U_6 equals U_0. The exact $y = \cos(t)$ returns to 1 at $t = 2\pi$.

(Problem 29, Section 6.3) That centered choice (*leapfrog method*) in Problem 28 is very successful for small time steps Δt. But find the eigenvalues of A for $\Delta t = \sqrt{2}$ and 2:

$$A = \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix} \quad \text{and } A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}.$$

Both matrices have $|\lambda| = 1$. Compute A^4 in both cases and find the eigenvectors of A. That value $\Delta t = 2$ is at the border of instability. Time steps $\Delta t > 2$ will lead to $|\lambda| > 1$, and the powers in $U_n = A^n U_0$ will explode.

Note You might say that nobody would compute with $\Delta t > 2$. But if an atom vibrates with $y'' = -1000000y$, then $\Delta t > .0002$ will give instability. Leapfrog has a very strict stability limit. $Y_{n+1} = Y_n + 3Z_n$ and $Z_{n+1} = Z_n - 3Y_{n+1}$ will explode because $\Delta t = 3$ is too large.

A better solution to the problem: Mix the **forward approximation** with the **backward approximation**.

$$\begin{cases} y'(t) = z(t) \\ z'(t) = -y(t) \end{cases} \approx \begin{cases} \dfrac{Y_{n+1} - Y_n}{\Delta t} = \dfrac{Z_{n+1} + Z_n}{2} \\ \dfrac{Z_{n+1} - Z_n}{\Delta t} = -\dfrac{Y_{n+1} + Y_n}{2} \end{cases} \approx \begin{bmatrix} 1 & -\frac{\Delta t}{2} \\ \frac{\Delta t}{2} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix}}_{\boldsymbol{u}_{n+1}} = \begin{bmatrix} 1 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}}_{\boldsymbol{u}_n}$$

We then perform this so-called **trapezoidal method**. (See *Problem 30*.)

$$\boxed{\underbrace{\begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix}}_{\boldsymbol{u}_{n+1}} = \begin{bmatrix} 1 & -\frac{\Delta t}{2} \\ \frac{\Delta t}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}}_{\boldsymbol{u}_n} = \underbrace{\frac{1}{1 + \left(\frac{\Delta t}{2}\right)^2} \begin{bmatrix} 1 - \left(\frac{\Delta t}{2}\right)^2 & \Delta t \\ -\Delta t & 1 - \left(\frac{\Delta t}{2}\right)^2 \end{bmatrix}}_{|\text{eigenvalues}|=1 \text{ for all } \Delta t>0} \underbrace{\begin{bmatrix} Y_n \\ Z_n \end{bmatrix}}_{\boldsymbol{u}_n}}$$

(Problem 30, Section 6.3) Another good idea for $y'' = -y$ is the trapezoidal method (half forward/half back): This may be the best way to keep (Y_n, Z_n) exactly on a circle.

$$\textbf{Trapezoidal} \quad \begin{bmatrix} 1 & -\Delta t/2 \\ \Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_{n+1} \\ Z_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t/2 \\ -\Delta t/2 & 1 \end{bmatrix} \begin{bmatrix} Y_n \\ Z_n \end{bmatrix}.$$

(a) Invert the left matrix to write this equation as $U_{n+1} = AU_n$. *Show that A is an orthogonal matrix: $A^TA = I$.* **These points U_n never leave the circle.** $A = (I - B)^{-1}(I + B)$ is always an orthogonal matrix if $B^{\mathrm{T}} = -B$ (See the proof on next page).

(b) (Optional `MATLAB`) Take 32 steps from $U_0 = (1, 0)$ to U_{32} with $\Delta t = 2\pi/32$. Is $U_{32} = U_0$? I think there is a small error.

Proof:

$$
\begin{aligned}
& A = (I - B)^{-1}(I + B) \\
& \Rightarrow (I - B)A = (I + B) \\
& \Rightarrow A^{\mathsf{T}}(I - B)^{\mathsf{T}} = (I + B)^{\mathsf{T}} \\
& \Rightarrow A^{\mathsf{T}}(I - B^{\mathsf{T}}) = (I + B^{\mathsf{T}}) \\
& \Rightarrow A^{\mathsf{T}}(I + B) = (I - B) \\
& \Rightarrow A^{\mathsf{T}} = (I - B)(I + B)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow A^{\mathsf{T}}A &= (I - B)(I + B)^{-1}(I - B)^{-1}(I + B) \\
&= (I - B)[(I - B)(I + B)]^{-1}(I + B) \\
&= (I - B)[(I + B)(I - B)]^{-1}(I + B) \\
&= (I - B)(I - B)^{-1}(I + B)^{-1}(I + B) \\
&= I
\end{aligned}
$$

Question: What if the number of eigenvectors is smaller than n?

Recall that it is **convenient** to say that the solution of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ is $e^{At}\boldsymbol{u}(0)$ for some constant vecor $\boldsymbol{u}(0)$.

Conveniently, we can presume that $e^{At}\boldsymbol{u}(0)$ is the solution of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ even if A does not have n eigenvectors.

Example. Solve $y'' - 2y' + y = 0$ with initially $y(0) = 1$ and $y'(0) = 0$.

Answer.

- Let $z = y'$. Then, the problem is reduced to

$$\begin{cases} y' = z \\ z' = -y + 2z \end{cases} \implies \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} y \\ z \end{bmatrix} = A\boldsymbol{u}$$

- The solution is still $e^{At}\boldsymbol{u}(0)$ but

$$e^{At} \neq Se^{\Lambda t}S^{-1}$$

because there is only one eigenvector for A (it doesn't matter whether we regard this case as "S does not exist" or we regard this case as "S exists but has no inverse").

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{S}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\Lambda} \quad \text{and} \quad e^{At} = e^{\lambda It}e^{(A-\lambda I)t} = e^{\lambda It}e^{(A-I)t}$$

$$\begin{aligned} e^{At} &= e^{It}e^{(A-I)t} \quad \text{(since } (It)((A-I)t) = ((A-I)t)(It)\text{. See below.)} \\ &= \left(\sum_{k=0}^{\infty}\frac{1}{k!}I^k t^k\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!}(A-I)^k t^k\right) \\ &= \left(\left(\sum_{k=0}^{\infty}\frac{1}{k!}t^k\right)I\right)\left(\sum_{k=0}^{1}\frac{1}{k!}(A-I)^k t^k\right) \end{aligned}$$

where we know $I^k = I$ for every $k \geq 0$ and $(A-I)^k$ =all-zero matrix for $k \geq 2$. This gives

$$e^{At} = \left(e^t I\right)\left(I + (A-I)t\right) = e^t\left(I + (A-I)t\right)$$

and

$$\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0) = e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^t\left(I + (A-I)t\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^t\begin{bmatrix} 1-t \\ -t \end{bmatrix}$$

□

Note that $e^A e^B$, $e^B e^A$ and e^{A+B} may not be equal except $AB = BA$!

Properities of e^{At}

- The eigenvalues of e^{At} are $e^{\lambda t}$ for all eigenvalues λ of A.
  For example, $e^{At} = e^t\,(I + (A - I)t)$ has repeated eigenvalues e^t, e^t.
- The eigenvectors of e^{At} remains the same as A.
  For example, $e^{At} = e^t\,(I + (A - I)t)$ has only one eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- The inverse of e^{At} always exist (hint: $e^{\lambda t} \neq 0$), and is equal to e^{-At}.
  For example, $e^{-At} = e^{-It}e^{(I-A)t} = e^{-t}(I - (A - I)t)$.
- The **transpose** of e^{At} is

$$\left(e^{At}\right)^{\mathsf{T}} = \left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k\right)^{\mathsf{T}} = \left(\sum_{k=0}^{\infty} \frac{1}{k!} (A^{\mathsf{T}})^k t^k\right) = e^{A^{\mathsf{T}} t}.$$

  Hence, if $A^{\mathsf{T}} = -A$ (skew-symmetric), then $\left(e^{At}\right)^{\mathsf{T}} e^{At} = e^{A^{\mathsf{T}} t} e^{At} = I$; so, e^{At} is an orthogonal matrix (Recall that a matrix Q is **orthogonal** if $Q^{\mathsf{T}} Q = I$).

6.3 Quick tip

- For a 2×2 matrix $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$, the eigenvector corresponding to the eigenvalue λ_1 is $\begin{bmatrix} a_{1,2} \\ \lambda_1 - a_{1,1} \end{bmatrix}$.

$$(A - \lambda_1 I) \begin{bmatrix} a_{1,2} \\ \lambda_1 - a_{1,1} \end{bmatrix} = \begin{bmatrix} a_{1,1} - \lambda_1 & a_{1,2} \\ a_{2,1} & a_{2,2} - \lambda_1 \end{bmatrix} \begin{bmatrix} a_{1,2} \\ \lambda_1 - a_{1,1} \end{bmatrix} = \begin{bmatrix} 0 \\ ? \end{bmatrix}$$

where $\boxed{? = 0}$ because the two row vectors of $(A - \lambda_1 I)$ are parallel.

This is especially useful when solving the differential equation because $a_{1,1} = 0$ and $a_{1,2} = 1$; hence, the eigenvector $\boldsymbol{v}_1$ corresponding to the eigenvalue λ_1 is

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}.$$

Solve $y'' - 2y' + y = 0$. Let $z = y'$. Then, the problem is reduced to

$$\begin{cases} y' = z \\ z' = -y + 2z \end{cases} \implies \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} y \\ z \end{bmatrix} = A\boldsymbol{u} \implies \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 6.3B and *Problem 31:* A convenient way to solve $d^2\boldsymbol{u}/dt = A\boldsymbol{u}$.

(Problem 31, Section 6.3) The **cosine of a matrix** is defined like e^A, by copying the series for $\cos t$:

$$\cos t = 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \cdots \qquad \cos A = I - \frac{1}{2!}A^2 + \frac{1}{4!}A^4 - \cdots$$

(a) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, multiply each term times $\boldsymbol{x}$ to find the eigenvalue of $\cos A$.

(b) Find the eigenvalues of $A = \begin{bmatrix} \pi & \pi \\ \pi & \pi \end{bmatrix}$ with eigenvectors $(1, 1)$ and $(1, -1)$. From the eigenvalues and eigenvectors of $\cos A$, find that matrix $C = \cos A$.

(c) The second derivative of $\cos(At)$ is $-A^2\cos(At)$.

$$\boldsymbol{u}(t) = \cos(At)\boldsymbol{u}(0) \text{ solve } \frac{d^2\boldsymbol{u}}{dt^2} = -A^2\boldsymbol{u} \text{ starting from } \boldsymbol{u}'(0) = 0.$$

Construct $\boldsymbol{u}(t) = \cos(At)\boldsymbol{u}(0)$ by the usual three steps for that specific A:

1. Expand $\boldsymbol{u}(0) = (4, 2) = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ in the eigenvectors.
2. Multiply those eigenvectors by _______ and _______ (instead of $e^{\lambda t}$).
3. Add up the solution $\boldsymbol{u}(t) = c_1\_\_\_\_\boldsymbol{x}_1 + c_2\_\_\_\_\boldsymbol{x}_2$. (Hint: See slide 6-45.)

(Problem 6.3B, Section 6.3) Find the eigenvalues and eigenvectors of A and write $\boldsymbol{u}(0) = (0, 2\sqrt{2}, 0)$ as a combination of the eigenvectors. Solve both equations $\boldsymbol{u}' = A\boldsymbol{u}$ and $\boldsymbol{u}'' = A\boldsymbol{u}$:

$$\frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \boldsymbol{u} \quad \text{and} \quad \frac{d^2\boldsymbol{u}}{dt^2} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \boldsymbol{u} \text{ with } \frac{d\boldsymbol{u}}{dt}(0) = \mathbf{0}.$$

The $1, -2, 1$ diagonals make A into a *second difference matrix* (like a second derivative).

$\boldsymbol{u}' = A\boldsymbol{u}$ is like the heat equation $\partial \mathbf{u}/\partial t = \partial^2 \mathbf{u}/\partial x^2$.

Its solution $\boldsymbol{u}(t)$ will decay (negative eigenvalues).

$\boldsymbol{u}'' = A\boldsymbol{u}$ is like the wave equation $\partial^2 \mathbf{u}/\partial t^2 = \partial^2 \mathbf{u}/\partial x^2$.

Its solution will oscillate (imaginary eigenvalues).

Thinking over Problem 6.3B and Problem 31: Solve $\frac{d^2\boldsymbol{u}}{dt} = A\boldsymbol{u}$.

Solution. The answer should be $e^{A^{1/2}t}\boldsymbol{u}(0)$. Note that $A^{1/2}$ has the same eigenvectors as A but its eigenvalues are the square root of A.

$$e^{A^{1/2}t} = S \begin{bmatrix} e^{\lambda_1^{1/2}t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2^{1/2}t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n^{1/2}t} \end{bmatrix} S^{-1}.$$

As an example from Problem 6.3B, $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$.

The eigenvalues of A are -2 and $-2 \pm \sqrt{2}$.
So,

$$e^{A^{1/2}t} = S \begin{bmatrix} e^{\imath\sqrt{2}\ t} & 0 & \\ 0 & e^{\imath\sqrt{2-\sqrt{2}}\ t} & 0 \\ 0 & 0 & e^{\imath\sqrt{2+\sqrt{2}}\ t} \end{bmatrix} S^{-1}.$$

A more direct solution is $\boldsymbol{u}(t) = \cos(At)\boldsymbol{u}(0)$. See *Problem 31*.

Final reminder on the approach delivered in the textbook

- In solving $\dfrac{d\boldsymbol{u}(t)}{dt} = At$ with initial $\boldsymbol{u}(0)$, it is convenient to find the solution as

$$\begin{cases} \boldsymbol{u}(0) = S\boldsymbol{c} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n \\ \boldsymbol{u}(t) = Se^{\Lambda t}\boldsymbol{c} = c_1e^{\lambda_1}\boldsymbol{v}_1 + \cdots + c_ne^{\lambda_n}\boldsymbol{v}_n \end{cases}$$

  This is what the textbook always does in Problems.

6.3 Problem discussions

You should practice these problems by yourself!

Problem 15: How to solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} - \boldsymbol{b}$ (for invertible A)?

(Problem 15, Section 6.3) A particular solution to $d\boldsymbol{u}/dt = A\boldsymbol{u} - \boldsymbol{b}$ is $\boldsymbol{u}_p = A^{-1}\boldsymbol{b}$, if A is invertible. The usual solutions to $d\boldsymbol{u}/dt = A\boldsymbol{u}$ give $\boldsymbol{u}_n$. Find the complete solution $\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_n$:

$$(a)\ \frac{du}{dt} = u - 4 \qquad\qquad (b)\ \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{u} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}.$$

Problem 16: How to solve $\dfrac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} - e^{ct}\boldsymbol{b}$ (for non-eigenvalue c of A)?

(Problem 16, Section 6.3) If c is not an eigenvalue of A, substitute $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ and find a particular solution to $d\boldsymbol{u}/dt = A\boldsymbol{u} - e^{ct}\boldsymbol{b}$. How does it break down when c is an eigenvalue of A? The "nullspace" of $d\boldsymbol{u}/dt = A\boldsymbol{u}$ contains the usual solutions $e^{\lambda_i t}\boldsymbol{x}_i$.
Hint: The particular solution $\boldsymbol{v}_p$ satisfies $(A - cI)\boldsymbol{v}_p = \boldsymbol{b}$.

Problem 23: $e^A e^B$, $e^B e^A$ and e^{A+B} are not necessarily equal. (They are equal when $AB = BA$.)

(Problem 23, Section 6.3) Generally $e^A e^B$ is different from $e^B e^A$. They are both different from e^{A+B}. Check this using Problems 21-2 and 19. (If $AB = BA$, all these are the same.)

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \quad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 27: **An interesting brain-storming problem!**

(Problem 27, Section 6.3) Find a solution $x(t)$, $y(t)$ that gets large as $t \to \infty$. To avoid this instability a scientist exchanges the two equations:

$$\begin{array}{rcrcr} dx/dt & = & 0x & - & 4y \\ dy/dt & = & -2x & + & 2y \end{array} \quad \text{becomes} \quad \begin{array}{rcrcr} dy/dt & = & -2x & + & 2y \\ dx/dt & = & 0x & - & 4y \end{array}$$

Now the matrix $\begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix}$ is stable. It has negative eigenvalues. How can this be?

Hint: The matrix is not the right one to be used to describe the transformed linear equations.

6.4 Symmetric matrices

Definition (Symmetric matrix): A matrix A is symmetric if $A = A^{\mathsf{T}}$.

- When A is **symmetric**,
  - its eigenvalues are all **reals**;
  - it has n **orthogonal** eigenvectors (so it can be diagonalize). We can then normalize these orthogonal eigenvectors to obtain an orthonormal basis.

Definition (Symmetric diagonalization): A symmetric matrix A can be written as

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathsf{T}} \text{ with } Q^{-1} = Q^{\mathsf{T}}$$

where Λ is a diagonal matrix with real eigenvalues at the diagonal.

6.4 Symmetric matrices

Theorem (Spectral theorem or principal axis theorem) A symmetric matrix A with distinct eigenvalues can be written as

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathsf{T}} \text{ with } Q^{-1} = Q^{\mathsf{T}}$$

where Λ is a diagonal matrix with real eigenvalues at the diagonal.

Proof:

- We have proved on slide 6-13 that a **symmetric** matrix has **real** eigenvalues and a **skew-symmetric** matrix has **pure imaginary** eigenvalues.
- Next, we prove that eigenvectors are orthogonal, i.e., $\boldsymbol{v}_1^{\mathsf{T}}\boldsymbol{v}_2 = 0$, for unequal eigenvalues $\lambda_1 \neq \lambda_2$.

$$\begin{aligned} & A\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1 \text{ and } A\boldsymbol{v}_2 = \lambda_2\boldsymbol{v}_2 \\ \Longrightarrow\ & (\lambda_1\boldsymbol{v}_1)^{\mathsf{T}}\boldsymbol{v}_2 = (A\boldsymbol{v}_1)^{\mathsf{T}}\boldsymbol{v}_2 = \boldsymbol{v}_1^{\mathsf{T}}A^{\mathsf{T}}\boldsymbol{v}_2 = \boldsymbol{v}_1^{\mathsf{T}}A\boldsymbol{v}_2 = \boldsymbol{v}_1^{\mathsf{T}}\lambda_2\boldsymbol{v}_2 \end{aligned}$$

which implies

$$(\lambda_1 - \lambda_2)\boldsymbol{v}_1^{\mathsf{T}}\boldsymbol{v}_2 = 0.$$

Therefore, $\boldsymbol{v}_1^{\mathsf{T}}\boldsymbol{v}_2 = 0$ because $\lambda_1 \neq \lambda_2$. $\square$

- $A = Q\Lambda Q^{\mathsf{T}}$ implies that

$$\begin{aligned} A &= \lambda_1 \boldsymbol{q}_1 \boldsymbol{q}_1^{\mathsf{T}} + \cdots + \lambda_n \boldsymbol{q}_n \boldsymbol{q}_n^{\mathsf{T}} \\ &= \lambda_1 P_1 + \cdots + \lambda_n P_n \end{aligned}$$

Recall from slide 4-37, the projection matrix P_i onto a unit vector $\boldsymbol{q}_i$ is

$$P_i = \boldsymbol{q}_i(\boldsymbol{q}_i^{\mathsf{T}}\boldsymbol{q}_i)^{-1}\boldsymbol{q}_i^{\mathsf{T}} = \boldsymbol{q}_i\boldsymbol{q}_i^{\mathsf{T}}$$

So, $A\boldsymbol{x}$ is the sum of projections of vector $\boldsymbol{x}$ onto each eigenspace.

$$A\boldsymbol{x} = \lambda_1 P_1 \boldsymbol{x} + \cdots + \lambda_n P_n \boldsymbol{x}.$$

The **spectral theorem** can be extended to a symmetric matrix with **repeated** eigenvalues.

To prove this, we need to introduce the famous **Schur's theorem**.

Theorem (Schur's theorem): Every square matrix A can be factorized into

$$A = QTQ^{-1} \text{ with } T \text{ upper triangular and } Q^\dagger = Q^{-1},$$

where "$\dagger$" denotes Hermisian transpose operation.

Further, if A has real eigenvalues (and hence has real eigenvectors), then Q and T can be chosen real.

Proof: The existence of Q and T such that $A = QTQ^\dagger$ and $Q^\dagger = Q^{-1}$ can be proved by induction.

- The theorem trivially holds when A is a 1×1 matrix.
- Suppose that Schur's Theorem is valid for all $(n-1) \times (n-1)$ matrices. Then, we claim that Schur's Theorem will hold true for all $n \times n$ matrices.
  - This is because we can take $t_{1,1}$ and $\boldsymbol{q}_1$ to be the eigenvalue and eigenvector of $A_{n\times n}$ (as there must exist at least one pair of eigenvalue and eigenvector for $A_{n\times n}$). Then, choose any $\boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$ such that they together with $\boldsymbol{q}_1$ span the n-dimensional space, and

    $$P = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_n \end{bmatrix}$$

    is a orthonormal matrix.

– We derive

$$\begin{aligned}
P^\dagger A P &= \begin{bmatrix} \boldsymbol{q}_1^\dagger \\ \boldsymbol{p}_2^\dagger \\ \vdots \\ \boldsymbol{p}_n^\dagger \end{bmatrix} \begin{bmatrix} A\boldsymbol{q}_1 & A\boldsymbol{p}_2 & \cdots & A\boldsymbol{p}_n \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{q}_1^\dagger \\ \boldsymbol{p}_2^\dagger \\ \vdots \\ \boldsymbol{p}_n^\dagger \end{bmatrix} \begin{bmatrix} t_{1,1}\boldsymbol{q}_1 & A\boldsymbol{p}_2 & \cdots & A\boldsymbol{p}_n \end{bmatrix} \quad (\text{since } A \underbrace{\boldsymbol{q}_1}_{\text{eigenvector}} = \underbrace{t_{1,1}}_{\text{eigenvalue}} \boldsymbol{q}_1) \\
&= \begin{bmatrix} t_{1,1} & t_{1,2} \cdots t_{1,n} \\ 0 & \\ \vdots & \tilde{A}_{(n-1)\times(n-1)} \\ 0 & \end{bmatrix}
\end{aligned}$$

where $t_{1,j} = \boldsymbol{q}_1^\dagger A \boldsymbol{p}_j$ and

$$\tilde{A}_{(n-1)\times(n-1)} = \begin{bmatrix} \boldsymbol{p}_2^\dagger A\boldsymbol{p}_2 & \cdots & \boldsymbol{p}_2^\dagger A\boldsymbol{p}_n \\ \vdots & \ddots & \vdots \\ \boldsymbol{p}_n^\dagger A\boldsymbol{p}_2 & \cdots & \boldsymbol{p}_n^\dagger A\boldsymbol{p}_n \end{bmatrix}.$$

– Since Schur's Theorem is true for any $(n-1)\times(n-1)$ matrix, we can find $\tilde{Q}_{(n-1)\times(n-1)}$ and $\tilde{T}_{(n-1)\times(n-1)}$ such that

$$\tilde{A}_{(n-1)\times(n-1)} = \tilde{Q}_{(n-1)\times(n-1)}\tilde{T}_{(n-1)\times(n-1)}\tilde{Q}^{\dagger}_{(n-1)\times(n-1)}$$

and

$$\tilde{Q}^{\dagger}_{(n-1)\times(n-1)} = \tilde{Q}^{-1}_{(n-1)\times(n-1)}.$$

– Finally, define

$$Q_{n\times n} = P\begin{bmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \tilde{Q}_{(n-1)\times(n-1)} \end{bmatrix}$$

and

$$T_{n\times n} = \begin{bmatrix} t_{1,1} & t_{1,2}\cdots t_{1,n} \\ 0 & \\ \vdots & \tilde{T}_{(n-1)\times(n-1)} \\ 0 & \end{bmatrix}$$

satisfy that

$$Q^{\dagger}_{n\times n}Q_{n\times n} = \begin{bmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \tilde{Q}^{\dagger}_{(n-1)\times(n-1)} \end{bmatrix} P^{\dagger}P \begin{bmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \tilde{Q}_{(n-1)\times(n-1)} \end{bmatrix} = I_{n\times n}$$

and

$$\begin{aligned}
\boxed{Q_{n\times n}T_{n\times n}} &= P_{n\times n}\begin{bmatrix}1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \tilde{Q}_{(n-1)\times(n-1)}\end{bmatrix}\begin{bmatrix}t_{1,1} & t_{1,2}\cdots t_{1,n} \\ 0 & \\ \vdots & \tilde{T}_{(n-1)\times(n-1)} \\ 0 & \end{bmatrix} \\
&= P_{n\times n}\begin{bmatrix}t_{1,1} & t_{1,2}\cdots t_{1,n} \\ 0 & \\ \vdots & \tilde{Q}_{(n-1)\times(n-1)}\tilde{T}_{(n-1)\times(n-1)} \\ 0 & \end{bmatrix} \\
&= P_{n\times n}\begin{bmatrix}t_{1,1} & t_{1,2}\cdots t_{1,n} \\ 0 & \\ \vdots & \tilde{A}_{(n-1)\times(n-1)}\tilde{Q}_{(n-1)\times(n-1)} \\ 0 & \end{bmatrix} \\
&= P_{n\times n}\begin{bmatrix}t_{1,1} & t_{1,2}\cdots t_{1,n} \\ 0 & \\ \vdots & \tilde{A}_{(n-1)\times(n-1)} \\ 0 & \end{bmatrix}\begin{bmatrix}1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \tilde{Q}_{(n-1)\times(n-1)}\end{bmatrix} \\
&= P_{n\times n}(P^{\dagger}_{n\times n}A_{n\times n}P_{n\times n})\begin{bmatrix}1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \tilde{Q}_{(n-1)\times(n-1)}\end{bmatrix} = \boxed{A_{n\times n}Q_{n\times n}}
\end{aligned}$$

- It remains to prove that if A has real eigenvalues, then Q and T can be chosen real, which can be similarly proved by induction. Suppose "If A has real eigenvalues, then Q and T can be chosen real." is true for all $(n-1)\times(n-1)$ matrices. Then, the claim should be true for all $n \times n$ matrices.
  - For a given $A_{n\times n}$ with real eigenvalues (and real eigenvectors), we can certainly have the real $t_{1,1}$ and $\boldsymbol{q}_1$, and so are $\boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$. This makes real the resultant $\tilde{A}_{(n-1)\times(n-1)}$ and $t_{1,2}, \ldots, t_{1,n}$.

The eigenvector associated with a real eigenvalue can be chosen real. For complex $\boldsymbol{v}$ and real λ and A, by its definition,

$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

is equivalent to

$$A \cdot \mathbf{Re}\{\boldsymbol{v}\} = A \cdot \lambda \cdot \mathbf{Re}\{\boldsymbol{v}\} \text{ and } A \cdot \mathbf{Im}\{\boldsymbol{v}\} = A \cdot \lambda \cdot \mathbf{Im}\{\boldsymbol{v}\}.$$

– The proof is completed by noting that the eigenvalues of $\tilde{A}_{(n-1)\times(n-1)}$ satisfying

$$P^\dagger A P = \begin{bmatrix} t_{1,1} & t_{1,2}\cdots t_{1,n} \\ 0 & \\ \vdots & \tilde{A}_{(n-1)\times(n-1)} \\ 0 & \end{bmatrix}$$

are also the eigenvalues of $A_{n\times n}$; hence, they are all reals. So, by the validity of the claimed statement for $(n-1)\times(n-1)$ matrices, the existence of real $\tilde{Q}_{(n-1)\times(n-1)}$ and real $T_{(n-1)\times(n-1)}$ satisfying

$$\tilde{A}_{(n-1)\times(n-1)} = \tilde{Q}_{(n-1)\times(n-1)}\tilde{T}_{(n-1)\times(n-1)}\tilde{Q}^\dagger_{(n-1)\times(n-1)}$$

and

$$\tilde{Q}^\dagger_{(n-1)\times(n-1)} = \tilde{Q}^{-1}_{(n-1)\times(n-1)}$$

is confirmed. $\square$

Two important facts:

- $P^\dagger AP$ and A have the same eigenvalues but possibly different eigenvectors. A simple proof is that for $\boldsymbol{v} = P\boldsymbol{v}'$,

$$(P^\dagger AP)\boldsymbol{v}' = P^\dagger A\boldsymbol{v} = P^\dagger(\lambda\boldsymbol{v}) = \lambda P^\dagger\boldsymbol{v} = \lambda\boldsymbol{v}'.$$

- (*Section 6.1: Problem 26* or *see slide 6-22*)

$$\texttt{det}(A) = \texttt{det}\left(\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}\right) = \texttt{det}(B) \cdot \texttt{det}(C)$$

Thus,

$$P^\dagger AP = \begin{bmatrix} t_{1,1} & t_{1,2} \cdots t_{1,n} \\ 0 & \\ \vdots & \tilde{A}_{(n-1)\times(n-1)} \\ 0 & \end{bmatrix}$$

So the eigenvalues of $\tilde{A}$ should be the eigenvalues of $P^\dagger AP$.

6.4 Symmetric matrices

Theorem (Spectral theorem or principal axis theorem) A symmetric matrix A (not necessarily with distinct eigenvalues) can be written as

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathsf{T}} \text{ with } Q^{-1} = Q^{\mathsf{T}}$$

where Λ is a diagonal matrix with real eigenvalues at the diagonal.

Proof:

- A symmetric matrix certainly satisfies Schur's Theorem.

$$\boxed{A = QTQ^{\dagger} \text{ with } T \text{ upper triangular and } Q^{\dagger} = Q^{-1}.}$$

- A has real eigenvalues. So, both T and Q are reals.
- By $A^{\mathsf{T}} = A$, we have

$$A^{\mathsf{T}} = QT^{\mathsf{T}}Q^{\mathsf{T}} = QTQ^{\mathsf{T}} = A.$$

  This immediately gives

$$T^{\mathsf{T}} = T$$

  which implies the off-diagonals are zeros.

- By $AQ = QT$, equivalently,

$$\begin{cases} A\boldsymbol{q}_1 = t_{1,1}\boldsymbol{q}_1 \\ A\boldsymbol{q}_2 = t_{2,2}\boldsymbol{q}_2 \\ \vdots \\ A\boldsymbol{q}_n = t_{n,n}\boldsymbol{q}_n \end{cases}$$

we know that Q is the matrix of eigenvectors (and there are n of them) and T is the matrix of eigenvalues. □

This result immediately indicates that a symmetric A can always be diagonalized.

Summary

- A symmetric matrix has **real** eigenvalues and n **real orthogonal** eigenvectors.

(Problem 21, Section 6.4)(Recommended) This matrix M is skew-symmetric and also ______. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($\|M\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for every $\boldsymbol{x}$ so $\|\lambda\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for eigenvectors.) Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \text{ can only have eigenvalues } \imath \text{ or } -\imath.$$

Thinking over Problem 14: The eigenvalues of an orthogonal matrix satisfies $|\lambda| = 1$.

Proof:

$$\|Q\boldsymbol{v}\|^2 = (Q\boldsymbol{v})^\dagger Q\boldsymbol{v} = \boldsymbol{v}^\dagger Q^\dagger Q\boldsymbol{v} = \boldsymbol{v}^\dagger\boldsymbol{v} = \|\boldsymbol{v}\|^2$$

implies

$$\|\lambda\boldsymbol{v}\| = \|\boldsymbol{v}\|.$$

□

$|\lambda| = 1$ and λ pure imaginary implies $\lambda = \pm\imath$.

6.4 Problem discussions

(Problem 15, Section 6.4) Show that A (**symmetric but complex**) has only one line of eigenvectors:

$$A = \begin{bmatrix} \imath & 1 \\ 1 & -\imath \end{bmatrix} \text{ is not even diagonalizable: eigenvalues } \lambda = 0, 0.$$

$A^{\mathrm{T}} = A$ is not such a special property for complex matrices. The good property is $\bar{A}^{\mathrm{T}} = A$ (Section 10.2). Then all λ's are real and eigenvectors are orthogonal.

Thinking over Problem 15: That a symmetric matrix A satisfying $A^{\mathrm{T}} = A$ has real eigenvalues and n orthogonal eigenvectors is only true for **real** symmetric matrices.

For a **complex** matrix A, we need to rephrase it as "A Hermitian symmetric matrix A satisfying $A^{\dagger} = A$ has real eigenvalues and n orthogonal eigenvectors."

Inner product and norm for complex vectors:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^{\dagger} \boldsymbol{w} \quad \text{and} \quad \|\boldsymbol{v}\|^2 = \boldsymbol{v}^{\dagger} \boldsymbol{v}$$

Proof of the red-color claim: Suppose $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Then,

$$\begin{aligned}
& A\boldsymbol{v} = \lambda\boldsymbol{v} \\
\Longrightarrow\quad & (A\boldsymbol{v})^\dagger\boldsymbol{v} = (\lambda\boldsymbol{v})^\dagger\boldsymbol{v} \quad \boxed{\text{(i.e., } (\bar{A}\bar{\boldsymbol{v}})^{\mathsf{T}}\boldsymbol{v} = (\bar{\lambda}\bar{\boldsymbol{v}})^{\mathsf{T}}\boldsymbol{v}\text{)}} \\
\Longrightarrow\quad & \boldsymbol{v}^\dagger A^\dagger\boldsymbol{v} = \lambda^*\boldsymbol{v}^\dagger\boldsymbol{v} \\
\Longrightarrow\quad & \boldsymbol{v}^\dagger A\boldsymbol{v} = \lambda^*\boldsymbol{v}^\dagger\boldsymbol{v} \\
\Longrightarrow\quad & \boldsymbol{v}^\dagger \lambda\boldsymbol{v} = \lambda^*\boldsymbol{v}^\dagger\boldsymbol{v} \\
\Longrightarrow\quad & \lambda\|\boldsymbol{v}\|^2 = \lambda^*\|\boldsymbol{v}\|^2 \\
\Longrightarrow\quad & \lambda = \lambda^*
\end{aligned}$$

□

(Problem 28, Section 6.4) For complex matrices, the symmetry $A^{\mathsf{T}} = A$ that produces real eigenvalues changes to $\bar{A}^{\mathsf{T}} = A$. From $\texttt{det}(A - \lambda I) = 0$, find the eigenvalues of the 2 by 2 "Hermitian" matrix $A = \begin{bmatrix}4 & 2+\imath; & 2-\imath & 0\end{bmatrix} = \bar{A}^{\mathsf{T}}$. To see why eigenvalues are real when $\bar{A}^{\mathsf{T}} = A$, adjust equation (1) of the text to $\bar{A}\bar{\boldsymbol{x}} = \bar{\lambda}\bar{\boldsymbol{x}}$. (See the green box above.)

6.4 Problem discussions

(Problem 27, Section 6.4) (`MATLAB`) Take two symmetric matrices with different eigenvectors, say $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 1 \\ 1 & 0 \end{bmatrix}$. Graph the eigenvalues $\lambda_1(A + tB)$ and $\lambda_2(A + tB)$ for $-8 < t < 8$. Peter Lax says on page 113 of *Linear Algebra* that λ_1 and λ_2 appear to be on a collision course at certain values of t. "Yet at the last minute they turn aside." How close do they come?

Correction for Problem 27: The problem should be "... Graph the eigenvalues λ_1 and λ_2 of $A + tB$ for $-8 < t < 8$. ...."

Hint: Draw the pictures of $\lambda_1(t)$ and $\lambda_2(t)$ with respect to t and check $\lambda_1(t) - \lambda_2(t)$.

(Problem 29, Section 6.4) **Normal matrices** have $\bar{A}^{\mathrm{T}}A = A\bar{A}^{\mathrm{T}}$. For real matrices, $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ includes symmetric, skew symmetric, and orthogonal matrices. Those have real λ, imaginary λ and $|\lambda| = 1$. Other normal matrices can have any complex eigenvalues λ.

Key point: ***Normal matrices have n orthonormal eigenvectors.*** Those vectors $\boldsymbol{x}_i$ probably will have complex components. In that complex case orthogonality means $\bar{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$ as Chapter 10 explains. Inner products (dot products) become $\bar{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{y}$.

The test for n orthonormal columns in Q becomes $\bar{Q}^{T}Q = I$ instead of $Q^{T}Q = I$.

A has n orthonormal eigenvectors ($A = Q\Lambda\bar{Q}^{\mathrm{T}}$) if and only if A is normal.

(a) Start from $A = Q\Lambda\bar{Q}^{\mathrm{T}}$ with $\bar{Q}^{\mathrm{T}}Q = I$. Show that $\bar{A}^{\mathrm{T}}A = A\bar{A}^{\mathrm{T}}$.

(b) Now start from $\bar{A}^{\mathrm{T}}A = A\bar{A}^{\mathrm{T}}$. Schur found $A = QT\bar{Q}^{\mathrm{T}}$ for every matrix A, with a triangular T. For normal matrices we must show (in 3 steps) that this T will actually be diagonal. Then $T = \Lambda$.

Step 1. Put $A = QT\bar{Q}^{\mathrm{T}}$ into $\bar{A}^{\mathrm{T}}A = A\bar{A}^{\mathrm{T}}$ to find $\bar{T}^{\mathrm{T}}T = T\bar{T}^{\mathrm{T}}$.

Step 2: Suppose $T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\bar{T}^{\mathrm{T}}T = T\bar{T}^{\mathrm{T}}$. Prove that $b = 0$.

Step 3: Extend Step 2 to size n. A normal triangular T must be diagonal.

6.4 Problem discussions

Important conclusion from Problem 29: A matrix A has n orthogonal eigenvectors if, and only if, A is normal.

Definition (Normal matrix): A matrix A is **normal** if $A^\dagger A = A^\dagger A$.

Proof:

- Schur's theorem: $A = QTQ^\dagger$ with T upper triangular and $Q^\dagger = Q^{-1}$.
- $A^\dagger A = AA^\dagger \implies T^\dagger T = TT^\dagger \implies T$ diagonal $\implies Q$ matrix of eigenvectors by $AQ = QT$. $\square$

6.5 Positive definite matrices

Definition (Positive definite): A **symmetric** matrix A is **positive definite** if its **eigenvalues** are all **positive**.

- The above definition only applies to a symmetric matrix because a non-symmetric matrix may have complex eigenvalues (which cannot be compared with zero)!

Properties of positive definite (symmetric) matrices

- **Equivalent Definition:** A is positive definite if, and only if, $\boldsymbol{x}^{\mathsf{T}}A\boldsymbol{x} > 0$ for all non-zero $\boldsymbol{x}$.

  $\boldsymbol{x}^{\mathsf{T}}A\boldsymbol{x}$ is usually referred to as the quadratic form!

  The proofs can be found in Problems 18 and 19.

  (Problem 18, Section 6.5) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\boldsymbol{x}^{\mathsf{T}}A\boldsymbol{x} =$ ____. If $\boldsymbol{x}^{\mathsf{T}}A\boldsymbol{x} > 0$, prove that $\lambda > 0$.

(Problem 19, Section 6.5) Reverse Problem 18 to show that *if all* $\lambda > 0$ *then* $\boldsymbol{x}^{\mathsf{T}} A\boldsymbol{x} > 0$. We must do this for *every* nonzero $\boldsymbol{x}$, not just the eigenvectors. So write $\boldsymbol{x}$ as a combination of the eigenvectors and *explain why all "cross terms" are* $\boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{x}_j = 0$. Then $\boldsymbol{x}^{\mathsf{T}} A\boldsymbol{x}$ is

$$(c_1\boldsymbol{x}_1+\cdots+c_n\boldsymbol{x}_n)^{\mathsf{T}}(c_1\lambda_1\boldsymbol{x}_1+\cdots+c_n\lambda_n\boldsymbol{x}_n) = c_1^2\lambda_1\boldsymbol{x}_1^{\mathsf{T}}\boldsymbol{x}_1+\cdots+c_n^2\lambda_n\boldsymbol{x}_n^{\mathsf{T}}\boldsymbol{x}_n > 0.$$

Proof (Problems 18):

– (only if *part*: Problem 19) $A\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i$ implies $\boldsymbol{v}_i^{\mathsf{T}} A\boldsymbol{v}_i = \lambda\boldsymbol{v}_i^{\mathsf{T}}\boldsymbol{v}_i > 0$ for all eigenvalues $\{\lambda_i\}_{i=1}^n$ and eigenvectors $\{\boldsymbol{v}_i\}_{i=1}^n$. The proof is completed by noting that with $\boldsymbol{x} = \sum_{i=1}^n c_i\boldsymbol{v}_i$ and $\{\boldsymbol{v}_i\}_{i=1}^n$ orthogonal,

$$\boldsymbol{x}^{\mathsf{T}}(A\boldsymbol{x}) = \left(\sum_{i=1}^n c_i\boldsymbol{v}_i^{\mathsf{T}}\right)\left(\sum_{j=1}^n c_j\lambda_j\boldsymbol{v}_j\right) = \sum_{i=1}^n c_i^2\lambda_i\|\boldsymbol{v}_i\|^2 > 0.$$

– (if *part*: Problem 18) Taking $\boldsymbol{x} = \boldsymbol{v}_i$, we obtain that $\boldsymbol{v}_i^{\mathsf{T}}(A\boldsymbol{v}_i) = \boldsymbol{v}_i^{\mathsf{T}}(\lambda\boldsymbol{v}_i) = \lambda_i\|\boldsymbol{v}_i\|^2 > 0$, which implies $\lambda_i > 0$. $\square$

- Based on the above proposition, we can conclude similar to two positive scalars (as "if a and b are both positive, so is $a + b$") that

Proposition: If A and B are both positive definite, so is $A + B$.

- The next property provides an easy way to construct positive definite matrices.

Proposition: If $A_{n\times n} = R^{\mathsf{T}}_{n\times m} R_{m\times n}$ and $R_{m\times n}$ has linearly independent columns, then A is positive definite.

Proof:

- $\boldsymbol{x}$ non-zero $\implies$ $R\boldsymbol{x}$ non-zero.
- Then, $\boldsymbol{x}^{\mathsf{T}}(R^{\mathsf{T}}R)\boldsymbol{x} = (R\boldsymbol{x})^{\mathsf{T}}(R\boldsymbol{x}) = \|R\boldsymbol{x}\|^2 > 0.$ $\square$

Since (symmetric) $A = Q\Lambda Q^{\mathsf{T}}$, we can choose $R = Q\Lambda^{1/2}Q^{\mathsf{T}}$, which requires R to be a square matrix. This observation gives another equivalent definition of positive definite matrices.

Equivalent Definition: $A_{n\times n}$ is positive definite if, and only if, there exists $R_{m\times n}$ with independent columns such that $A = R^{\mathsf{T}}R$.

- **Equivalent Definition:** $A_{n\times n}$ is positive definite if, and only if, all pivots are positive.

Proof:

– (By LDU decomposition,) $A = LDL^{\mathsf{T}}$, where D is a diagonal matrix with pivots as diagonals, and L is a lower triangular matrix with 1 as diagonals. Then,

$$\begin{aligned} \boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} &= \boldsymbol{x}^{\mathsf{T}}(LDL^{\mathsf{T}})\boldsymbol{x} = (L^{\mathsf{T}}\boldsymbol{x})^{\mathsf{T}} D (L^{\mathsf{T}}\boldsymbol{x}) \\ &= \boldsymbol{y}^{\mathsf{T}} D \boldsymbol{y} \quad \text{where } \boldsymbol{y} = L^{\mathsf{T}}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{l}_1^{\mathsf{T}}\boldsymbol{x} \\ \vdots \\ \boldsymbol{l}_n^{\mathsf{T}}\boldsymbol{x} \end{bmatrix} \\ &= d_1 y_1^2 + \cdots + d_n y_n^2 \\ &= d_1 \left(\boldsymbol{l}_1^{\mathsf{T}}\boldsymbol{x}\right)^2 + \cdots + d_n \left(\boldsymbol{l}_n^{\mathsf{T}}\boldsymbol{x}\right)^2. \end{aligned}$$

– So, if all pivots are positive, $\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} > 0$ for all non-zero $\boldsymbol{x}$. Conversely, if $\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} > 0$ for all non-zero $\boldsymbol{x}$, which in turns implies $\boldsymbol{y}^{\mathsf{T}} D \boldsymbol{y} > 0$ for all non-zero $\boldsymbol{y}$ (as L is invertible), then pivot d_i must be positive for the choice of $y_j = 0$ except for $j = i$. $\square$

An extension of the previous proposition is:

Suppose $A = BCB^T$ with B invertible and C diagonal. Then, A is positive definite if, and only if, diagonals of C are all positive! See *Problem 35*.

(Problem 35, Section 6.5) Suppose C is positive definite (so $\boldsymbol{y}^{\mathsf{T}} C \boldsymbol{y} > 0$ whenever $\boldsymbol{y} \neq \mathbf{0}$) and A has independent columns (so $A\boldsymbol{x} \neq \mathbf{0}$ whenever $\boldsymbol{x} \neq \mathbf{0}$). Apply the energy test to $\boldsymbol{x}^{\mathsf{T}} A^{\mathsf{T}} C A \boldsymbol{x}$ to show that $A^{\mathsf{T}} C A$ is positive definite: *the crucial matrix in engineering.*

6.5 Positive definite matrices

Equivalent Definition: $A_{n\times n}$ is positive definite if, and only if, the n upper left determinants (i.e., the n leading principle minors) are all positive.

Definition (Minors, Principle minors and leading principle minors):

– A minor of a matrix A is the determinant of some smaller square matrix, obtained by removing one or more of its rows or columns.

– The first-order (respectively, second-order, etc) minor is a minor, obtained by removing $(n-1)$ (respectively, $(n-2)$, etc) rows or columns.

– A principle minor of a matrix is a minor, obtained by removing the same rows and columns.

– A leading principle minor of a matrix is a minor, obtaining by removing the last few rows and columns.

The below example gives **three** upper left determinants (or three leading principle minors), $\mathtt{det}(A_{1\times 1})$, $\mathtt{det}(A_{2\times 2})$ and $\mathtt{det}(A_{3\times 3})$.

$$\boxed{\begin{matrix} \boxed{\begin{matrix} \boxed{a_{1,1}} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{matrix}} & \begin{matrix} a_{1,3} \\ a_{2,3} \end{matrix} \\ \begin{matrix} a_{3,1} & a_{3,2} \end{matrix} & a_{3,3} \end{matrix}}$$

Proof of the equivalent definition: This can be proved based on slide 5-18: By LU decomposition,

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{2,1} & 1 & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n,1} & l_{n,2} & l_{n,3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} d_1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & d_2 & u_{2,3} & \cdots & d_{2,n} \\ 0 & 0 & d_3 & \cdots & d_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$\Longrightarrow\ d_k = \frac{\mathtt{det}(A_{k\times k})}{\mathtt{det}(A_{(k-1)\times(k-1)})}$$

$\square$

Equivalent Definition: $A_{n\times n}$ is positive definite if, and only if, all $(2^n - 2)$ principle minors are positive.

- Based on the above equivalent definitions, a positive definite matrix cannot have either **zero** or **negative value** in its main diagonals. See *Problem 16*.

(Problem 16, Section 6.5) A positive definite matrix cannot have a zero (or even worse, a negative number) on its diagonal. Show that this matrix fails to have $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} > 0$:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (\quad,\quad,\quad).$$

With the above discussion of positive definite matrices, we can proceed to define similar notions like **negative definite**, **positive semidefinite**, **negative semidefnite** matrices.

6.5 Positive semidefinite (or nonnegative definite)

Definition (Positive semidefinite): A **symmetric** matrix A is **positive semidefinite** if its **eigenvalues** are all nonnegative.

Equivalent Definition: A is positive semidefinite if, and only if, $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} \geq 0$ for all non-zero $\boldsymbol{x}$.

Equivalent Definition: $A_{n\times n}$ is positive semidefinite if, and only if, there exists $R_{m\times n}$ (perhaps with dependent columns) such that $A = R^{\mathrm{T}}R$.

Equivalent Definition: $A_{n\times n}$ is positive semidefinite if, and only if, all pivots are nonnegative.

Equivalent Definition: $A_{n\times n}$ is positive semidefinite if, and only if, all principle minors are non-negative.

Example. Non-positive semidefinite $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ has $(2^3 - 2)$ principle minors.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} -1 \end{bmatrix}.$$

Note: It is not sufficient to define the positive semidefinite based on the non-negativity of leading principle minors. Check the above example.

6.5 Negative definite, negative semidefinite, indefinite

We can similarly define **negative definite**.

Definition (Negative definite): A **symmetric** matrix A is **negative definite** if its **eigenvalues** are all negative.

Equivalent Definition: A is negative definite if, and only if, $\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} < 0$ for all non-zero $\boldsymbol{x}$.

Equivalent Definition: $A_{n \times n}$ is negative definite if, and only if, there exists $R_{m \times n}$ with independent columns such that $A = -R^{\mathsf{T}} R$.

Equivalent Definition: $A_{n \times n}$ is negative definite if, and only if, all pivots are negative.

Equivalent Definition: $A_{n \times n}$ is negative definite if, and only if, all odd-order leading principle minors are negative and all even-order leading principle minors are positive.

Equivalent Definition: $A_{n \times n}$ is negative definite if, and only if, all odd-order principle minors are negative and all even-order principle minors are positive.

6.5 Negative definite, negative semidefinite, indefinite

We can also similarly define **negative semidefinite**.

Definition (Negative definite): A **symmetric** matrix A is **negative semidefinite** if its **eigenvalues** are all non-positive.

Equivalent Definition: A is negative semidefinite if, and only if, $\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} \leq 0$ for all non-zero $\boldsymbol{x}$.

Equivalent Definition: $A_{n\times n}$ is negative semidefinite if, and only if, there exists $R_{m\times n}$ (possibly with dependent columns) such that $A = -R^{\mathsf{T}} R$.

Equivalent Definition: $A_{n\times n}$ is negative semidefinite if, and only if, all pivots are non-positive.

Equivalent Definition: $A_{n\times n}$ is negative definite if, and only if, all odd-order principle minors are non-positive and all even-order principle minors are non-negative.

Note: It is not sufficient to define the negative semidefinite based on the non-positivity of leading principle minors.

Finally, if some of the eigenvalues of a symmetric matrix are positive, and some are negative, the matrix will be referred to as **indefinite**.

6.5 Quadratic form

- For a positive definite matrix $A_{2\times 2}$,

$$\begin{aligned} \boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} &= \boldsymbol{x}^{\mathsf{T}}(Q\Lambda Q^{\mathsf{T}})\boldsymbol{x} = (Q^{\mathsf{T}}\boldsymbol{x})^{\mathsf{T}}\Lambda(Q^{\mathsf{T}}\boldsymbol{x}) \\ &= \boldsymbol{y}^{\mathsf{T}}\Lambda\boldsymbol{y} \quad \text{where } \boldsymbol{y} = Q^{\mathsf{T}}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q}_1^{\mathsf{T}}\boldsymbol{x} \\ \boldsymbol{q}_2^{\mathsf{T}}\boldsymbol{x} \end{bmatrix} \\ &= \lambda_1 \left(\boldsymbol{q}_1^{\mathsf{T}}\boldsymbol{x}\right)^2 + \lambda_2 \left(\boldsymbol{q}_2^{\mathsf{T}}\boldsymbol{x}\right)^2 \end{aligned}$$

So, $\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} = c$ gives an **ellipse** if $c > 0$.

Example. $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

– $\lambda_1 = 9$ and $\lambda_2 = 1$, and $\boldsymbol{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

– $\boxed{\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} = \lambda_1 \left(\boldsymbol{q}_1^{\mathsf{T}}\boldsymbol{x}\right)^2 + \lambda_2 \left(\boldsymbol{q}_2^{\mathsf{T}}\boldsymbol{x}\right)^2 = 9\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 + \left(\frac{x_1 - x_2}{\sqrt{2}}\right)^2}$

– $\begin{cases} \boldsymbol{q}_1^{\mathsf{T}}\boldsymbol{x} \text{ is an axis perpendicular to } \boldsymbol{q}_1. \text{ (Not along } \boldsymbol{q}_1 \text{ as the textbook said.)} \\ \boldsymbol{q}_2^{\mathsf{T}}\boldsymbol{x} \text{ is an axis perpendicular to } \boldsymbol{q}_2. \end{cases}$

Tip: $A = Q\Lambda Q^{\mathsf{T}}$ is called the **principal axis theorem** (cf. slide 6-66) because $\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} = \boldsymbol{y}^{\mathsf{T}}\Lambda\boldsymbol{y} = c$ is an ellipse with axes along the eigenvectors.

6.5 Problem discussions

(Problem 13, Section 6.5) Find a matrix with $a > 0$ and $c > 0$ and $a + c > 2b$ that has a negative eigenvalue.

Missing point in Problem 13: The matrix to be determined is of the shape $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

6.6 Similar matrices

Definition (Similarity): Two matrices A and B are similar if $B = M^{-1}AM$ for some invertible M.

Theorem: A and $M^{-1}AM$ have the same eigenvalues.

Proof: For eigenvalue λ and eigenvector $\boldsymbol{v}$ of A, define $\boldsymbol{v}' = M^{-1}\boldsymbol{v}$ (hence, $\boldsymbol{v} = M\boldsymbol{v}'$). We then derive

$$(M^{-1}AM)\boldsymbol{v}' = M^{-1}A(M\boldsymbol{v}') = M^{-1}A\boldsymbol{v} = \lambda M^{-1}\boldsymbol{v} = \lambda\boldsymbol{v}'$$

So, λ is also the eigenvalue of $M^{-1}AM$ (associated with eigenvector $\boldsymbol{v}' = M^{-1}\boldsymbol{v}$). $\square$

Notes:

- The LU decomposition over a symmetric matrix A gives $A = LDL^{\mathsf{T}}$ but A and D (pivot matrix) apparently may have different eigenvalues. Why? Because

  $(L^{\mathsf{T}})^{-1} \neq L$. **Similarity is defined based on** M^{-1}, **not** M^{T}.

- The **converse** to the above theorem is **wrong**!. In other words, we cannot say that "two matrices with the same eigenvalues are similar."

  Example. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ have the same eigenvalues 0, 0, but they are not similar.

6.6 Jordan form

- Why introducing **matrix similarity**?

  Answer: We can then extend the **diagonalization** of a matrix with less than n eigenvectors to the **Jordan form**.

- For an un-diagonalizable matrix A, we can find invertible M such that

$$A = MJM^{-1},$$

where

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{bmatrix}$$

and s is the number of distinct eigenvalues, and the size of J_i is equal to the multiplicity of eigenvalue λ_i, and J_i is of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

6.6 Jordan form

- The idea behind the Jordan form is that A is similar to J.
- Based on this, we can now compute

$$A^{100} = MJ^{100}M^{-1} = M \begin{bmatrix} J_1^{100} & 0 & \cdots & 0 \\ 0 & J_2^{100} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s^{100} \end{bmatrix} M^{-1}.$$

- How to find M_i corresponding to J_i?

Answer: By $AM = MJ$ with $M = \begin{bmatrix} M_1 & M_2 & \cdots & M_s \end{bmatrix}$, we know

$$AM_i = M_i J_i \quad \text{for} \quad 1 \le i \le s.$$

Specifically, assume that the multiplicity of λ_i is two. Then,

$$A \begin{bmatrix} \boldsymbol{v}_i^{(1)} & \boldsymbol{v}_i^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_i^{(1)} & \boldsymbol{v}_i^{(2)} \end{bmatrix} \begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}$$

which is equivalent to

$$\begin{cases} A\boldsymbol{v}_i^{(1)} = \lambda_i \boldsymbol{v}_i^{(1)} \\ A\boldsymbol{v}_i^{(2)} = \boldsymbol{v}_i^{(1)} + \lambda_i \boldsymbol{v}_i^{(2)} \end{cases} \Longrightarrow \begin{cases} (A - \lambda_i I)\boldsymbol{v}_i^{(1)} = 0 \\ (A - \lambda_i I)\boldsymbol{v}_i^{(2)} = \boldsymbol{v}_i^{(1)} \end{cases}$$

6.6 Problem discussions

(Problem 14, Section 6.6) Prove that A^{T} is always similar to A (we know the λ's are the same):

1. For one Jordan block J_i: Find M_i so that $M_i^{-1} J_i M_i = J_i^{\mathsf{T}}$.
2. For any J with blocks J_i: Build M_0 from blocks so that $M_0^{-1} J M_0 = J^{\mathsf{T}}$.
3. For any $A = MJM^{-1}$: Show that A^{T} is similar to J^{T} and so to J and to A.

6.6 Problem discussions

Thinking over Problem 14: A^{T} and A are always similar.

Answer:

- It can be easily checked that

$$\begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n,1} & u_{n,2} & \cdots & u_{n,n} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \cdots & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{n,n} & \cdots & u_{n,2} & u_{n,1} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & u_{2,2} & u_{2,1} \\ u_{1,n} & \cdots & u_{1,2} & u_{1,1} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \cdots & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}$$

$$= V^{-1} \begin{bmatrix} u_{n,n} & \cdots & u_{n,2} & u_{n,1} \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & u_{2,2} & u_{2,1} \\ u_{1,n} & \cdots & u_{1,2} & u_{1,1} \end{bmatrix} V$$

where here V represents a matrix with zero entries except $v_{i,n+1-i} = 1$ for $1 \le i \le n$. We note that $V^{-1} = V^{\mathsf{T}} = V$.

So, we can use the proper size of $M_i = V$ to obtain

$$J_i^{\mathsf{T}} = M_i^{-1} J_i M_i.$$

Define

$$M = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_s \end{bmatrix}$$

We have

$$J^{\mathrm{T}} = MJM^{-1},$$

where

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{bmatrix}$$

and s is the number of distinct eigenvalues, and the size of J_i is equal to the multiplicity of eigenvalue λ_i, and J_i is of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

Finally, we know

- A is similar to J;
- A^{T} is similar to J^{T};
- J^{T} is similar to J.

So, A^{T} is similar to A. □

(Problem 19, Section 6.6) If A is 6 by 4 and B is 4 by 6, AB and BA have different sizes. But with blocks,

$$M^{-1}FM = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = G.$$

(a) What sizes are the four blocks (the same four sizes in each matrix)?

(b) This equation is $M^{-1}FM = G$, so F and G have the same eigenvalues. F has the 6 eigenvalues of AB plus 4 zeros; G has the 4 eigenvalues of BA plus 6 zeros. **AB has the same eigenvalues as BA plus ______ zeros.**

6.6 Problem discussions

Thinking over Problem 19: $A_{m\times n}B_{n\times m}$ and $B_{n\times m}A_{m\times n}$ have the same eigenvalues except for additional $(m-n)$ zeros.

Solution: The example shows the usefulness of the similarity.

- $\begin{bmatrix} I_{m\times m} & -A_{m\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix} \begin{bmatrix} A_{m\times n}B_{n\times m} & 0_{m\times n} \\ B_{n\times m} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} I_{m\times m} & A_{m\times n} \\ 0_{n\times m} & I_{n\times n} \end{bmatrix} = \begin{bmatrix} 0_{m\times m} & 0_{m\times n} \\ B_{n\times m} & B_{n\times m}A_{m\times n} \end{bmatrix}$
- Hence, $\begin{bmatrix} A_{m\times n}B_{n\times m} & 0_{m\times n} \\ B_{n\times m} & 0_{n\times n} \end{bmatrix}$ and $\begin{bmatrix} 0_{m\times m} & 0_{m\times n} \\ B_{n\times m} & B_{n\times m}A_{m\times n} \end{bmatrix}$ are similar and have the same eigenvalues.
- From Problem 26 in Section 6.1 (see slide 6-22), the desired claim is proved.$\square$

6.6 Problem discussions

(Problem 21, Section 6.6) If J is the 5 by 5 Jordan block with $\lambda = 0$, find J^2 and count its eigenvectors (are these the eigenvectors?) and find its Jordan form (there will be two blocks).

Problem 21: Find the Jordan form of $A = J^2$, where

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution.

- $A = J^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and the eigenvalues of A are 0, 0, 0, 0, 0.

- $$Av_1 = A \begin{bmatrix} v_{1,1} \\ v_{2,1} \\ v_{3,1} \\ v_{4,1} \\ v_{5,1} \end{bmatrix} = \begin{bmatrix} v_{3,1} \\ v_{4,1} \\ v_{5,1} \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \implies v_{3,1} = v_{4,1} = v_{5,1} = 0 \implies \boldsymbol{v}_1 = \begin{bmatrix} v_{1,1} \\ v_{2,1} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- $$A\boldsymbol{v}_2 = A \begin{bmatrix} v_{1,2} \\ v_{2,2} \\ v_{3,2} \\ v_{4,2} \\ v_{5,2} \end{bmatrix} = \begin{bmatrix} v_{3,2} \\ v_{4,2} \\ v_{5,2} \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{v}_1 \implies \begin{cases} v_{3,2} = v_{1,1} \\ v_{4,2} = v_{2,1} \\ v_{5,2} = 0 \end{cases} \implies \boldsymbol{v}_2 = \begin{bmatrix} v_{1,2} \\ v_{2,2} \\ v_{1,1} \\ v_{2,1} \\ 0 \end{bmatrix}$$

- $$A\boldsymbol{v}_3 = A \begin{bmatrix} v_{1,3} \\ v_{2,3} \\ v_{3,3} \\ v_{4,3} \\ v_{5,3} \end{bmatrix} = \begin{bmatrix} v_{3,3} \\ v_{4,3} \\ v_{5,3} \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{v}_2 \implies \begin{cases} v_{3,3} = v_{1,2} \\ v_{4,3} = v_{2,2} \\ v_{5,3} = v_{1,1} \\ v_{2,1} = 0 \end{cases} \implies \boldsymbol{v}_3 = \begin{bmatrix} v_{1,3} \\ v_{2,3} \\ v_{1,2} \\ v_{2,2} \\ v_{1,1} \end{bmatrix}$$

6.6 Problem discussions

- To summarize,

$$\boldsymbol{v}_1 = \begin{bmatrix} v_{1,1} \\ v_{2,1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \boldsymbol{v}_2 = \begin{bmatrix} v_{1,2} \\ v_{2,2} \\ v_{1,1} \\ v_{2,1} \\ 0 \end{bmatrix} \Longrightarrow \text{If } v_{2,1} = 0, \text{then } \boldsymbol{v}_3 = \begin{bmatrix} v_{1,3} \\ v_{2,3} \\ v_{1,2} \\ v_{2,2} \\ v_{1,1} \end{bmatrix}$$

$$\left\{ \begin{array}{l} \boldsymbol{v}_1^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \boldsymbol{v}_2^{(1)} = \begin{bmatrix} v_{1,2}^{(1)} \\ v_{2,2}^{(1)} \\ 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \boldsymbol{v}_3^{(1)} = \begin{bmatrix} v_{1,3}^{(1)} \\ v_{2,3}^{(1)} \\ v_{1,2}^{(1)} \\ v_{2,2}^{(1)} \\ 1 \end{bmatrix} \\ \boldsymbol{v}_1^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \boldsymbol{v}_2^{(2)} = \begin{bmatrix} v_{1,2}^{(2)} \\ v_{2,2}^{(2)} \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{array} \right.$$

Since we wish to choose each of $\boldsymbol{v}_2^{(1)}$ and $\boldsymbol{v}_2^{(2)}$ to be orthogonal to both $\boldsymbol{v}_1^{(1)}$ and $\boldsymbol{v}_1^{(2)}$,

$$v_{1,2}^{(1)} = v_{2,2}^{(1)} = v_{1,2}^{(2)} = v_{2,2}^{(2)} = 0,$$

i.e.,

$$\begin{cases} \boldsymbol{v}_1^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \boldsymbol{v}_2^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \implies \boldsymbol{v}_3^{(1)} = \begin{bmatrix} v_{1,3}^{(1)} \\ v_{2,3}^{(1)} \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ \boldsymbol{v}_1^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \boldsymbol{v}_2^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{cases}$$

Since we wish to choose $\boldsymbol{v}_3^{(1)}$ to be orthogonal to all of $\boldsymbol{v}_1^{(1)}$, $\boldsymbol{v}_1^{(2)}$, $\boldsymbol{v}_2^{(1)}$ and $\boldsymbol{v}_2^{(2)}$,

$$v_{1,3}^{(1)} = v_{2,3}^{(1)} = 0.$$

6.6 Problem discussions

As a result,

$$A\begin{bmatrix} \boldsymbol{v}_1^{(1)} & \boldsymbol{v}_2^{(1)} & \boldsymbol{v}_3^{(1)} & \boldsymbol{v}_1^{(2)} & \boldsymbol{v}_2^{(2)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1^{(1)} & \boldsymbol{v}_2^{(1)} & \boldsymbol{v}_3^{(1)} & \boldsymbol{v}_1^{(2)} & \boldsymbol{v}_2^{(2)} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

□

- Are all $\boldsymbol{v}_1^{(1)}, \boldsymbol{v}_2^{(1)}, \boldsymbol{v}_3^{(1)}, \boldsymbol{v}_1^{(2)}$ and $\boldsymbol{v}_2^{(2)}$ eigenvectors of A (satisfying $A\boldsymbol{v} = \lambda\boldsymbol{v}$)?

  Hint: A does not have 5 eigenvectors. More specifically, A only has 2 eigenvectors? Which two? Think of it.

- Hence, the problem statement may not be accurate as I mark "(are these the eigenvectors?)".

6.7 Singular value decomposition (SVD)

- Now we can represent all **square** matrix in the Jordan form:

$$A = M^{-1} J M.$$

- What if the matrix $A_{m\times n}$ is not a square one?

  – Problem: $A\boldsymbol{v} = \lambda\boldsymbol{v}$ is not possible! Specifically, $\boldsymbol{v}$ cannot have two different dimensionalities.

$$A_{m\times n}\boldsymbol{v}_{n\times 1} = \lambda\boldsymbol{v}_{n\times 1} \text{ infeasible if } n \neq m.$$

  – So, we can only have

$$A_{m\times n}\boldsymbol{v}_{n\times 1} = \sigma\boldsymbol{u}_{m\times 1}.$$

6.7 Singular value decomposition (SVD)

- If we can find enough numbers of orthogonal $\boldsymbol{u}$ and $\boldsymbol{v}$ such that

$$A_{m\times n}\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n\end{bmatrix}_{n\times n} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_m\end{bmatrix}_{m\times m}\begin{bmatrix}\sigma_1 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \sigma_r & 0 & \cdots & 0\\ 0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & 0 & \cdots & 0\end{bmatrix}_{m\times n}.$$

where r is the rank of A, then we can perform the so-called **singular value decomposition (SVD)**

$$\boxed{A = U\Sigma V^{-1}.}$$

Note again that the required enough number of orthogonal $\boldsymbol{u}$ and $\boldsymbol{v}$ may be impossible when A has repeated "eigenvalues."

6.7 Singular value decomposition (SVD)

- If V is chosen to be an orthogonal matrix satisfying $V^{-1} = V^{\mathsf{T}}$, then we have the so-called **reduced SVD**

$$\begin{aligned} A &= U\Sigma V^{\mathsf{T}} \\ &= \sum_{i=1}^{r} \lambda_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathsf{T}} \\ &= \begin{bmatrix} \boldsymbol{u}_1 \cdots & \boldsymbol{u}_r \end{bmatrix}_{m\times r} \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}_{r\times r} \begin{bmatrix} \boldsymbol{v}_1^{\mathsf{T}} \\ \vdots \\ \boldsymbol{v}_r^{\mathsf{T}} \end{bmatrix}_{r\times n} \end{aligned}$$

- Usually, we prefer to choose an orthogonal V (as well as orthogonal U).

- In the sequel, we will assume the found U and V are orthogonal matrices in the first place; later, we will confirm that orthogonal U and V can always be found.

6.7 How to determine U, V and $\{\sigma_i\}$?

- $A = U\Sigma V^{\mathsf{T}}$

$$\Longrightarrow \quad A^{\mathsf{T}}A = \left(U\Sigma V^{\mathsf{T}}\right)^{\mathsf{T}}\left(U\Sigma V^{\mathsf{T}}\right) = V\Sigma^{\mathsf{T}}U^{\mathsf{T}}U\Sigma V^{\mathsf{T}} = V\Sigma^2 V^{\mathsf{T}}.$$

  So, V is the (orthogonal) matrix of n eigenvectors of symmetric $A^{\mathsf{T}}A$.

- $A = U\Sigma V^{\mathsf{T}}$

$$\Longrightarrow \quad AA^{\mathsf{T}} = \left(U\Sigma V^{\mathsf{T}}\right)\left(U\Sigma V^{\mathsf{T}}\right)^{\mathsf{T}} = U\Sigma V^{\mathsf{T}}V\Sigma^{\mathsf{T}}U^{\mathsf{T}} = U\Sigma^2 U^{\mathsf{T}}.$$

  So, U is the (orthogonal) matrix of m eigenvectors of symmetric AA^{T}.

- Remember that $A^{\mathsf{T}}A$ and AA^{T} have the same eigenvalues except for additional $(m-n)$ zeros.

  Section 6.6, *Problem 19* (see slide 6-104): $A_{m\times n}B_{n\times m}$ and $B_{n\times m}A_{m\times n}$ have the same eigenvalues except for additional $(m-n)$ zeros.

  In fact, there are only r non-zero eigenvalues for $A^{\mathsf{T}}A$ and AA^{T}, which satisfy

$$\lambda_i = \sigma_i^2, \qquad \text{where } 1 \le i \le r.$$

6.7 How to determine U, V and $\{\sigma_i\}$?

Example (Problem 21 in Section 6.6): Find the SVD of $A = J^2$, where

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution.

- $A^{\mathsf{T}}A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and $AA^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
- So $V = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ for $\Lambda = \Sigma^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

6.7 How to determine U, V and $\{\sigma_i\}$?

- Hence,

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{V^{\mathrm{T}}}$$

$\square$

We may compare this with the Jordan form.

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{M^{-1}}$$

6.7 How to determine U, V and $\{\sigma_i\}$?

Remarks on the previous example

- In the above example, we only know $\Lambda = \Sigma^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
- Hence, $\Sigma = \begin{bmatrix} \pm 1 & 0 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
- However, by $A\boldsymbol{v}_i = \sigma_i \boldsymbol{u}_i = (-\sigma_i)(-\boldsymbol{u}_i)$, we can always choose **positive** σ_i by adjusting the sign of $\boldsymbol{u}_i$.

6.6 SVD

Important notes:

- In terminology, σ_i, where $1 \leq i \leq r$, is called the **singular value**. Hence, the name of **singular value decomposition** is used.

- The singular value is always **non-zero** (even though the eigenvalues of A can be **zeros**).

  $$\textit{Example.}\ A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ but } \Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- It is good to know that:

  The first r columns of $V \in$ row space $\boldsymbol{R}(A)$ and are **bases** of $\boldsymbol{R}(A)$

  The last $(n-r)$ columns of $V \in$ null space $\boldsymbol{N}(A)$ and are **bases** of $\boldsymbol{N}(A)$

  The first r columns of $U \in$ column space $\boldsymbol{C}(A)$ and are **bases** of $\boldsymbol{C}(A)$

  The last $(m-r)$ columns of $U \in$ left null space $\boldsymbol{N}(A^{\mathrm{T}})$ and are **bases** of $\boldsymbol{N}(A^{\mathrm{T}})$

  How useful the above facts are can be seen from the next example.

6.6 SVD

Example. Find the SVD of $A_{4\times 3} = \boldsymbol{x}_{4\times 1}\boldsymbol{y}^{\mathtt{T}}_{1\times 3}$.

Solution.

- A is a rank-1 matrix. So, $r = 1$ and $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, where $\sigma_1 \neq 0$. (Actually, $\sigma_1 = 1$.)
- The base of the row space is $\boldsymbol{v}_1 = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}$; pick up perpendicular $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ that span the null space.
- The base of the column space is $\boldsymbol{u}_1 = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$; pick up perpendicular $\boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ that span the left null space.
- The SVD is then

$$A_{4\times 3} = \begin{bmatrix} \underbrace{\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}}_{\text{column space}} & \underbrace{\boldsymbol{u}_2 \quad \boldsymbol{u}_3 \quad \boldsymbol{u}_4}_{\text{left null space}} \end{bmatrix}_{4\times 4} \Sigma_{4\times 3} \begin{bmatrix} \frac{\boldsymbol{y}^{\mathrm{T}}}{\|\boldsymbol{y}\|} \\ \boldsymbol{v}_2^{\mathrm{T}} \\ \boldsymbol{v}_3^{\mathrm{T}} \end{bmatrix}_{3\times 3} \begin{matrix} \text{row space} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{null space} \end{matrix}$$

$\square$