Chapter 3

Vector Spaces and Subspaces

Po-Ning Chen, Professor

Department of Electrical Engineering

National Chiao Tung University

Hsin Chu, Taiwan 30010, R.O.C.

3.1 Spaces of vectors

- What is a **Space**?

Answer: An area that is free to occupy.

- What is the **Space** of vectors?

Answer: An area that is free to occupy by vectors.

Example. $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ can occupy a 2-dimensional plane.

- What is a **vector space**?

Answer: A **vector space** is not just a space occupied by vectors. It is a **mathematical structure** formed by vectors and rules (axioms) to handle these vectors. In other words, this term is reserved specifically for this use in mathematics!

3.1 Spaces of vectors

Definition (Vector space): A vector space is a space of vectors in $\mathbb{V}$ and scalars in $\mathbb{F}$ (named *field*) with operations:

- vector addition: $\boxed{\boldsymbol{v} + \boldsymbol{w}}$
- scalar-to-vector multiplication: $\boxed{a\boldsymbol{v}}$
- scalar addition: $\boxed{a + b}$

that satisfy the following axioms:

1. Commutativity of addition: $\boxed{\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}}$
2. Associativity of addition: $\boxed{\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}}$
3. Identity element of addition: There exists a unique "zero vector" $\mathbf{0}$ such that $\boxed{\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}}$ for all vector $\boldsymbol{v}$ in the space
4. Inverse elements of addition: For every $\boldsymbol{v}$, there exists a unique $-\boldsymbol{v}$ such that $\boxed{\boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}}$
5. Identity element of scalar multiplication: There exists a unique scalar 1 such that $\boxed{1\boldsymbol{v} = \boldsymbol{v}}$

6. Compatibility of scalar multiplication with scalar multiplication: $\boxed{a(b\boldsymbol{v}) = (ab)\boldsymbol{v}}$
7. Distributivity of scalar multiplication with respect to vector addition: $\boxed{a(\boldsymbol{v} + \boldsymbol{w}) = a\boldsymbol{v} + a\boldsymbol{w}}$
8. Distributivity of scalar multiplication with respect to scalar addition: $\boxed{(a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}}$

Example of a vector space.

1. $\mathbb{V} = \Re^n$ and $\mathbb{F} = \Re$ (with usual real addition and multiplication).
2. $\mathbb{V} =$The set of all real-valued function $f(t)$ and $\mathbb{F} = \Re$.

   Note: We only need **scalar and vector additions** and **scalar-to-vector multiplication**. Hence, $\{f(t)\}$ satisfies such demand.
3. $\mathbb{V} =$The set of all 2×2 matrices and $\mathbb{F} = \Re$ (with usual matrix addition and scalar-to-matrix multiplication).

3.1 Spaces of vectors

Some important notes:

- A **vector** is not necessary of the form $\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.
- A **vector** (in its most general form) is an element of a **vector space**.

Example of a vector space.

1. $\mathbb{V} = \Re^n$ and $\mathbb{F} = \Re$. $\Rightarrow$ **vector**$=\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$.
2. $\mathbb{V}$ =The set of all real-valued function $f(t)$ and $\mathbb{F} = \Re$. $\Rightarrow$ **vector**$=f(t)$.
3. $\mathbb{V}$ =The set of all 2×2 matrices and $\mathbb{F} = \Re$. $\Rightarrow$ **vector**$=\begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix}$.

In fact, it "makes no difference" in placing a (column) vector in $\Re^4$ into $\begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix}$ as long as the operation defined is term-wisely based.

3.1 Sub-vector space of a vector space

- In terminology, we usually term sub(-vector)space of a (vector) space for simplicity.

Definition (Subspace): A sub(-vector)space $\mathbb{S}$ of a (vector) space $\mathbb{V}$ satisfies

1. that all the vectors it contains belong to the vector space, and
2. the eight axioms/rules.

Equivalently,

– $\boldsymbol{v} \in \mathbb{S}$ and $\boldsymbol{w} \in \mathbb{S} \Rightarrow \boldsymbol{v} + \boldsymbol{w} \in \mathbb{S}$, and

– $\boldsymbol{v} \in \mathbb{S}$ and $a \in \mathbb{F} \Rightarrow a\boldsymbol{v} \in \mathbb{S}$

- The two equivalent conditions imply the validity of Conditions 1. and 2.

Exercise. Prove that the zero vector should belong to subspace $\mathbb{S}$ of the three-dimensional real vector space $\Re^3$ with field $\Re$.

Proof: $a\boldsymbol{v}$ should belong to $\mathbb{S}$ for any $a \in \mathbb{F} = \Re$. The proof is completed by taking $a = 0$. $\square$

3.1 Sub-vector space of a vector space

Examples of subspace of $\Re^3$*:*

- A single origin point. (A space always contains zero point.)
- Any line passes through the origin.
- Any plane passes through the origin.
- Choose any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, the set that consists of all the linear combination of the two vectors, i.e., $a\boldsymbol{v} + b\boldsymbol{w}$.

This is exactly **equivalent** to the two equivalent definitions of the subspace. I.e.,

$$\left.\begin{array}{l} - \ \boldsymbol{v} \in \mathbb{S} \text{ and } \boldsymbol{w} \in \mathbb{S} \Rightarrow \boldsymbol{v} + \boldsymbol{w} \in \mathbb{S}, \text{ and} \\ - \ \boldsymbol{v} \in \mathbb{S} \text{ and } a \in \mathbb{F} \Rightarrow a\boldsymbol{v} \in \mathbb{S} \end{array}\right\} \text{This will be useful in “proving the legitimacy of a subspace”!}$$

3.1 Column space

Following the last example on the previous slide:

$$\begin{aligned}\mathbb{S} &= \left\{\boldsymbol{b} \in \Re^3 : a\boldsymbol{v} + b\boldsymbol{w} = \boldsymbol{b} \text{ for some } a, b \in \Re\right\} \\ &= \left\{\boldsymbol{b} \in \Re^3 : \begin{bmatrix}\boldsymbol{v} & \boldsymbol{w}\end{bmatrix}\begin{bmatrix}a \\ b\end{bmatrix} = \boldsymbol{b} \text{ for some } a, b \in \Re\right\} \\ &= \left\{\boldsymbol{b} \in \Re^3 : \boxed{A\boldsymbol{x} = \boldsymbol{b}} \text{ for some } \boldsymbol{x} \in \Re^2\right\}\end{aligned}$$

where $A = \begin{bmatrix}\boldsymbol{v} & \boldsymbol{w}\end{bmatrix}$.

Every subspace of a space $\Re^n$ can be of the form:

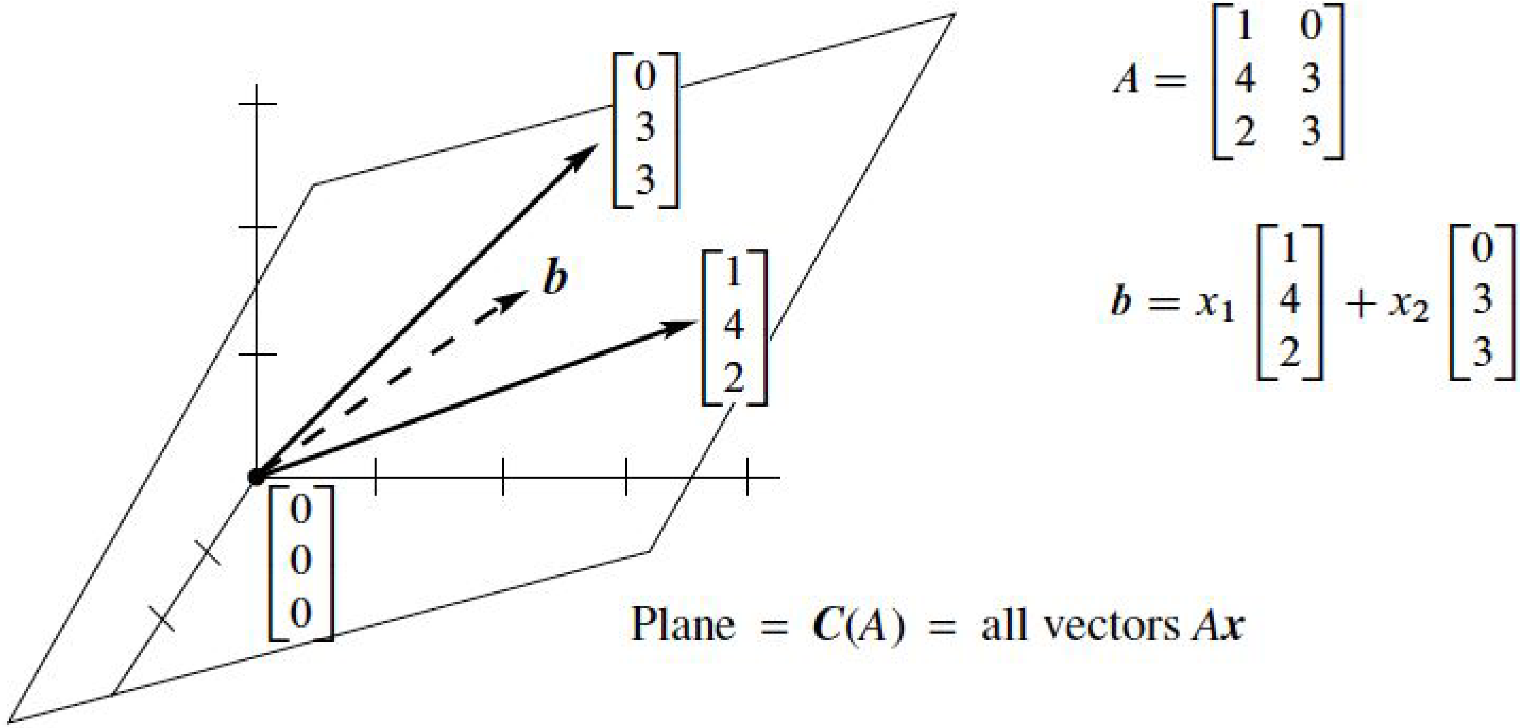
$$\left\{\boldsymbol{b} \in \Re^m : \boxed{A_{m\times n}\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n\right\}$$

Since it is a **linear combination** of the **column vectors** of A, it is also termed **column space** (denoted by $\boldsymbol{C}(A)$).

- So, given A, a subspace is then defined (through the old friend $A\boldsymbol{x} = \boldsymbol{b}$).
- If $A_{m\times m}$ invertible, $\mathbb{S} = \mathbb{V} = \Re^m$. *Example.* $A = I$.
- If $A_{m\times m}$ non-invertible, $\mathbb{S} \subset \mathbb{V} = \Re^m$. *Example.* $A =$ all-zero matrix.

3.1 Column space

- $A\boldsymbol{x} = \boldsymbol{b}$ is solvable if, and only if, $\boldsymbol{b} \in \boldsymbol{C}(A)$.



- $\boldsymbol{C}(A)$ is sometimes viewed as the subspace spanned by column vectors of A.

3.1 Column space

Examples of $\boldsymbol{C}(A)$:

- If $A_{n\times n}$ invertible, $\boldsymbol{C}(A) = \Re^n$.
- If $A_{n\times 1}$, $\boldsymbol{C}(A)$ is a line in $\Re^n$ space.
- If $A = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{a} & \cdots & \boldsymbol{a} \end{bmatrix}_{n\times n}$, $\boldsymbol{C}(A)$ is a line in $\Re^n$ space.

Exercise: Does there exist an A such that

$$\boldsymbol{C}(A) = \left\{ \boldsymbol{b} \in \Re^3 : \boxed{A\boldsymbol{x} = \boldsymbol{b}} \text{ for some } \boldsymbol{x} \in \Re^2 \right\}$$

is an empty set?

(Hint: What is the element that must be contained in all subspaces? The subspace that only consists of this element is denoted by $\mathbb{Z}$, and is called *zero subspace*.)

3.1 Column space

Exercise: If $\boldsymbol{C}(A_{n\times n}) = \Re^n$, is it possible that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution for some $\boldsymbol{b} \in \Re^n$?

(Hint: $\boldsymbol{C}(A) = \left\{\boldsymbol{b} \in \Re^n : \boxed{A_{n\times n}\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{n\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n\right\}$ is the set of $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ admits solutions.)

Note from the textbook:

- We can also find the subspace of a subspace $\mathbb{S}$.
- In textbook, it is denoted by $\mathbb{SS}$.
- We can certainly have $\mathbb{SSS}$, the subspace of the subspace $\mathbb{SS}$ of a subspace $\mathbb{S}$ of a space $\mathbb{V}$.

3.2 Nullspace of a matrix A

Before introducing the **nullspace**, we give a formal definition of the **column space**.

Definition (Column space): A column space of a matrix A, denoted by $\boldsymbol{C}(A)$, consists of all the vectors that are linear combinations of column vectors of A. It can always be represented as:

$$\boldsymbol{C}(A) = \left\{ \boldsymbol{b} \in \Re^m : \boxed{A_{m\times n}\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n \right\}$$

Definition (Nullspace): A nullspace of a matrix A, denoted by $\boldsymbol{N}(A)$, consists of all the vectors that "nullify" the linear combinations of column vectors of A. It can always be represented as:

$$\boldsymbol{N}(A) = \left\{ \boldsymbol{x} \in \Re^n : \boxed{A_{m\times n}\boldsymbol{x}_{n\times 1} = \mathbf{0}_{m\times 1}} \right\}$$

- $\boldsymbol{N}(A)$ is a vector space.
- $\boldsymbol{N}(A)$ is a sub-space of $\Re^n$.
- However, nullspace $\boldsymbol{N}(A)$ is not necessarily (often, not) a sub-space of column space $\boldsymbol{C}(A)$.

3.2 Determination of nullspace

- How to easily determine the nullspace of A?

Recall that in linear equations $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x}$ is called the **solutions**.

Specially when $\boldsymbol{b} = 0$, $\boldsymbol{x} \neq 0$ is called the **special solutions**.

Answer: **Method of special solution.**

– The subspace of $\Re^3$ can only be:

A single point $\mathbf{0}$	Determined by no **special solutions**
A line passing through $\mathbf{0}$	Determined by one **special solutions**
A plane passing through $\mathbf{0}$	Determined by two **special solutions**
$\Re^3$ itself	Determined by three **special solutions**

3.2 Determination of nullspace

Example. $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

$$\boldsymbol{N}(A) = \left\{ \boldsymbol{x} \in \Re^3 : \boxed{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{0}} \right\}$$

- The second row of A indicates $x_2 = 0$.
- The first row of A (i.e., $x_1 + 2x_3 = 0$) indicates $\boldsymbol{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ is a special solution.

We then know (by intuition) that $\boldsymbol{N}(A)$ is a line passing through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

□

Question is "Can we always rely on our intuition?"
Is there a systematic method to determine $\boldsymbol{N}(A)$?

3.2 Determination of nullspace

Example (continued). The solution of $A\boldsymbol{x} = \mathbf{0}$ does not change by multiplying a proper matrix such as

$$EA\boldsymbol{x} = E\mathbf{0} = \mathbf{0}.$$

So we do **forward** and **backward eliminations** (with no **row exchange**) and **pivot normalization** to obtain:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

Recall that this is called **reduced row echelon form**.

We then have

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 = 0 \end{cases}$$

We conclude that $\boldsymbol{N}(A)$ should be the intersection of two non-parallel planes. $\square$

What will happen if we need to perform row exchanges during forward elimination!

3.2 Determination of nullspace

Example. $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

$$\boldsymbol{N}(A) = \left\{ \boldsymbol{x} \in \Re^3 : \boxed{\begin{bmatrix} 0 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{0}} \right\}$$

- The first row of A indicates $x_2 = 0$.
- The second row of A (i.e., $x_1 + 2x_3 = 0$) indicates $\boldsymbol{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ is a special solution.

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Question is "Can we always rely on our intuition?"
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3.2 Determination of nullspace

Example (continued). The solution of $A\boldsymbol{x} = \mathbf{0}$ does not change by multiplying a proper matrix such as

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So we do **forward** and **backward eliminations** (with **row exchange**) and **pivot normalization** to obtain:

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Recall that this is called **reduced row echelon form**.

We then have

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 = 0 \end{cases}$$

We conclude that $\boldsymbol{N}(A)$ should be the intersection of two non-parallel planes. □

What will happen if $\boldsymbol{N}(A)$ is a plane!

3.2 Determination of nullspace

Example. $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$

Let's begin to work on **forward elimination** on the first column. We then obtain:

$$\begin{bmatrix} \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \qquad \begin{matrix} \Rightarrow \text{1st pivot} \\ \\ \\ \end{matrix}$$

We continue to work on the second row
(by choosing the left-most non-zero entry as the next pivot)!

$$\begin{bmatrix} \mathbf{1} & 1 & 2 & 3 \\ 0 & 0 & \mathbf{4} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} \Rightarrow \text{1st pivot} \\ \Rightarrow \text{2nd pivot} \\ \\ \end{matrix}$$

Since we only have **two pivots**, the nullspace $\boldsymbol{N}(A)$ should be an intersection of **two** $\Re^4$ hyperplanes, which can be represented as *linear combinations of* ***two*** *special solutions.*

3.2 Determination of nullspace

Proceed with the back elimination:

$$\begin{bmatrix} \mathbf{1} & 1 & 0 & 1 \\ 0 & 0 & \mathbf{4} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \Rightarrow \text{1st pivot} \\ \Rightarrow \text{2nd pivot} \\ \end{array}$$

and **pivot normalization**:

$$\begin{bmatrix} \mathbf{1} & 1 & 0 & 1 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \Rightarrow \text{1st pivot} \\ \Rightarrow \text{2nd pivot} \\ \end{array}$$

So we know that

$$\begin{cases} x_1, x_3 & \text{pivot variables} \\ x_2, x_4 & \text{free variables} \quad \text{(free to choose its values for special solutions)} \end{cases}$$

We then assign $(x_2, x_4) = (1, 0)$ and $(x_2, x_4) = (0, 1)$ and obtain

$$\boldsymbol{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Then, $\boldsymbol{N}(A)$ consists of all linear combinations of the above two vectors. $\square$

3.2 Determination of nullspace

Remark.

- If we have three free variables, we may assign $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ for special solutions.

  Exercise 1. How about four free variables ?

  Exercise 2. How about zero free variables ?

3.2 Determination of nullspace

In MATLAB:

- The **special solutions** for $\boldsymbol{N}(A)$ can be determined by:

```
null(A);  % Take A from the previous example.
```

  Note that the special solutions are not unique!

  The result is:

```
answer =
        -0.4059  -0.6597
         0.7623   0.1373
         0.3564  -0.5225
        -0.3564   0.5225
```

- Recall that the reduced row echelon form can be obtained by

```
rref(A)
```

  The result is:

```
answer =
        1  1  0  1
        0  0  1  1
        0  0  0  0
```

3.2 Echelon matrix

- It is very useful to find the (row) **echelon matrix** of a matrix A.

- **Row echelon form**: A form of matrices satisfying the below two conditions.
  - Every all-zero row is below the other non-all-zero rows.
  - The leading coefficient (the first nonzero number from the left, also called the **pivot**) of a non-all-zero row is always strictly to the right of the leading coefficient of the row above it.

- **(row) Reduced echelon form**: A form of matrices in **row echelon form** with the leading coefficient being one and also being the only non-zero entry in the same column.

$$\textit{Example.}\ A \Rightarrow U = \underbrace{\begin{bmatrix} \boldsymbol{p} & x & x & x & x & x & x \\ 0 & \boldsymbol{p} & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{p} & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{row echelon form}} \Rightarrow R = \underbrace{\begin{bmatrix} \mathbf{1} & 0 & x & x & x & 0 & x \\ 0 & \mathbf{1} & x & x & x & 0 & x \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{row reduced echelon form}}$$

- In the above example, $\boldsymbol{N}(A) = \boldsymbol{N}(U) = \boldsymbol{N}(R)$ is a 4-dimensional hyperplane (in a 7-dimensional space) since there are **four** free variables.

3.2 Echelon matrix

- If $A_{n\times n}$ is invertible, then $R = I$.

- *Exercise.* Is the all-zero matrix in the row echelon form?

Example (Problem 20). Suppose column 1 + column 3 + column 5 = $\mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

Answer:

- Since column 1 + column 3 + column 5 = $\mathbf{0}$, column 5 has no pivot.

  It is not possible for columns 1 and 3 having no pivot. Check the row reduced echelon matrix R under the assumption that column 1 (or column 3) has no pivot. A contradiction will be obtained!

- Since column 5 has no pivot, the fifth variable is free.

- Since there are four pivots, column 5 has no pivot, and column 1 + column 3 + column 5 = $\mathbf{0}$, the row reduced echelon matrix is

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

3.2 Echelon matrix

- The special solution is of the form $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 1 \end{bmatrix}$ satisfying $R\boldsymbol{x} = \mathbf{0}$.

  Hence, $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

- The nullspace contains all multiples of $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ (a line in $\Re^5$). $\square$

Example (Problem 21). Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

Answer:

- Two special solutions $\Rightarrow$ two free variables and two pivots.
- R is an $m \times 4$ matrix, where $m \geq 2$. Take $m = 2$ for simplicity.
- R is of the form:

$$R = \begin{bmatrix} 1 & 0 & r_{1,3} & r_{1,4} \\ 0 & 1 & r_{2,3} & r_{2,4} \end{bmatrix}$$

- The two special solutions ($R\boldsymbol{x} = \mathbf{0}$) give:

$$R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

□

The answer is not unique ! For example,

$$R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example (Problem 22). Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.

Answer:

- One special solution $\Rightarrow$ one free variables and three pivots.
- R is a $m \times 4$ matrix, where $m \geq 3$. Take $m = 3$ for simplicity.
- R is of the form:

$$R = \begin{bmatrix} 1 & 0 & 0 & r_{1,4} \\ 0 & 1 & 0 & r_{2,4} \\ 0 & 0 & 1 & r_{3,4} \end{bmatrix}$$

- The special solution ($R\boldsymbol{x} = \mathbf{0}$) give:

$$R = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$\square$

Example (Problem 23). Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

Answer:

- Linear combination of column vectors $c_1\boldsymbol{a}_1 + c_1\boldsymbol{a}_2 + c_3\boldsymbol{a}_3$ can be made equal to $(1, 1, 5)$ and $(0, 3, 1)$. So, let $c_1 = 1$ and $c_2 = c_3 = 0$ for the first solution, and let $c_1 = c_3 = 0$ and $c_2 = 1$ for the second solution. We then obtain

$$A = \begin{bmatrix} 1 & 0 & a_{1,3} \\ 1 & 3 & a_{2,3} \\ 5 & 1 & a_{3,3} \end{bmatrix}.$$

- Solving $A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \mathbf{0}$ gives

$$A = \begin{bmatrix} 1 & 0 & -0.5 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}.$$

□

3.2 Echelon matrix

- What is the relationship between column space and nullspace of a **symmetric** matrix A?

  Answer: Their vectors are **perpendicular** to each other. I.e.,

  $$\boxed{\boldsymbol{a} \in \boldsymbol{N}(A) \text{ and } \boldsymbol{b} \in \boldsymbol{C}(A) \;\Rightarrow\; \boldsymbol{a} \cdot \boldsymbol{b} = 0.}$$

  Proof:

  – $\boldsymbol{a} \in \boldsymbol{N}(A)$ implies $A\boldsymbol{a} = \boldsymbol{0}$.

  – $\boldsymbol{b} \in \boldsymbol{C}(A)$ implies $A\boldsymbol{x} = \boldsymbol{b}$ for some $\boldsymbol{x}$.

  – Hence, by $A^{\mathsf{T}} = A$, we obtain

  $$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b}^{\mathsf{T}}\boldsymbol{a} = \boldsymbol{x}^{\mathsf{T}}A^{\mathsf{T}}\boldsymbol{a} = \boldsymbol{x}^{\mathsf{T}}A\boldsymbol{a} = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{0} = \boldsymbol{0}.$$

  □

- If $A_{m\times n}$ for $n \neq m$, then column space of A is a vector space of m-dimensional vectors, but nullspace of A is a vector space of n-dimensional vectors.
- So, we can talk about the relation of column space and nullspace only when A is a square matrix.
- The text then also introduces row space.

3.2 Echelon matrix

Definition (Row space): A row space of a matrix A, denoted by $\boldsymbol{R}(A)$, consists of all the vectors that are linear combinations of row vectors of A. It can always be represented as:

$$\boldsymbol{R}(A) = \left\{ \boldsymbol{b} \in \Re^n : \boxed{A^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{b}_{n\times 1}} \text{ for some } \boldsymbol{x} \in \Re^m \right\}$$

- What is the relationship between row space and nullspace of a matrix A?

  Answer: Their vectors are **perpendicular** to each other. I.e.,

$$\boxed{\boldsymbol{a} \in \boldsymbol{N}(A) \text{ and } \boldsymbol{b} \in \boldsymbol{R}(A) \;\Rightarrow\; \boldsymbol{a} \cdot \boldsymbol{b} = 0.}$$

  Proof:

  - $\boldsymbol{a} \in \boldsymbol{N}(A)$ implies $A\boldsymbol{a} = \boldsymbol{0}$.
  - $\boldsymbol{b} \in \boldsymbol{R}(A)$ implies $A^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{b}$ for some $\boldsymbol{x}$.
  - Hence, we obtain

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{a} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{0} = \boldsymbol{0}.$$

$\square$

For a symmetric matrix, row space = column space.
The row space can be defined in terms of reduced row echelon matrix R.

Exercise. What is the relationship between column space of A and nullspace of $A^{\mathtt{T}}$?

3.3 The rank and row reduced form

Now we turn to another important quantitative index of matrix — **rank**.

Definition (Rank): The rank of a matrix A is the number of non-zero pivots.

- How to determine the rank?

  Answer: By elimination,

  $$A \;\Rightarrow\; \textit{upper triangular } U \Rightarrow \textit{Row reduced echelon } R.$$

- The rank r is
  - the dimension of the row space

    The non-pivot rows should be linear combinations of the pivot rows!
  - the dimension of the column space

◇ First note that

$$\boldsymbol{C}(A) = \left\{\boldsymbol{b} \in \Re^m : \boxed{A_{m\times n}\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n\right\}$$

and

$$\left\{\boldsymbol{a} \in \Re^m : \boxed{E_{m\times m}\boldsymbol{b}_{m\times 1} = \boldsymbol{a}_{m\times 1}} \text{ for some } \boldsymbol{b} \in \boldsymbol{C}(A)\right\}$$

has the same dimension if E is invertible.

◇ So, $\boldsymbol{C}(A)$ and $\boldsymbol{C}(R)$ has the same dimension as $R = EA$ for some invertible E.

◇ Apparently, the non-pivot columns of R are linear combinations of pivot columns (cf., slide 3-21).

– the number of independent rows in A

– the number of independent columns in A

- The dimension of the nullspace of A is $(n - r)$. I.e., $(n - r)$ = the number of free variables = the number of special solutions (for $A\boldsymbol{x} = \mathbf{0}$).

Definition (Independence): A set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is said to be *linearly independent* if the linear combination of them $(a_1\boldsymbol{v}_1 + \cdots + a_m\boldsymbol{v}_m)$ equals $\mathbf{0}$ only when the coefficients are zero $(a_1 = a_2 = \cdots = a_m = 0)$.

3.3 Pivot columns

- How to locate the r pivot columns in $\boldsymbol{A}$?

  Answer: $R =$`rref`(A).

  The r pivots columns of $R = EA$ and A are exactly at the same positions.

- By $R = EA$, an interesting observation is that the first r columns of E^{-1} is exactly the r pivot columns of A!

  This is because

  $$\boxed{\text{the } r \text{ pivot columns of } R \text{ form an } r \times r \text{ identity matrix.}}$$

$$\textit{Example.}\ A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 4 & -3 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad A = E^{-1}R = \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{bmatrix} \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 4 & -3 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{d}_{1,1} & - & \boldsymbol{d}_{1,2} & - & - \\ \boldsymbol{d}_{2,1} & - & \boldsymbol{d}_{2,2} & - & - \\ \boldsymbol{d}_{3,1} & - & \boldsymbol{d}_{3,2} & - & - \end{bmatrix} \square$$

3.3 Pivot columns

- Can we find the special solutions directly using the fact that

$$\boxed{\text{the } r \text{ pivot columns of } R \text{ form an } r \times r \text{ identity matrix.}}$$

Answer: Of course.

- For a matrix $A_{m\times n}$, there are $(n-r)$ special solutions.
- These special solutions satisfies

$$R\boldsymbol{x}_1 = \mathbf{0}, \quad R\boldsymbol{x}_2 = \mathbf{0}, \quad \cdots \quad R\boldsymbol{x}_{n-r} = \mathbf{0}.$$

  Equivalently,

$$R_{m\times n}\begin{bmatrix}\boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_{n-r}\end{bmatrix}_{n\times(n-r)} = R_{m\times n}N_{n\times(n-r)} = \mathbf{0}_{m\times(n-r)}.$$

- By reordering the columns of R, it can be of the form:

$$R = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathbf{0}_{(n-r)\times r} & \mathbf{0}_{(n-r)\times(n-r)} \end{bmatrix} \quad \Rightarrow \quad N_{n\times(n-r)} = \begin{bmatrix} -F_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$$

- So, the special solutions just to collect the **free-variable columns**, negate them, and add an identity matrix.

3.3 Pivot columns

Example. $A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 4 & -3 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{bmatrix}$

$$\Rightarrow \; N = \underbrace{\begin{bmatrix} - & - & - \\ 1 & 0 & 0 \\ - & - & - \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Put identity matrix}} = \underbrace{\begin{bmatrix} -3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Put } -F}$$

linear independent pivot column of A

2nd column of A is a linear combination of 1st column of A

linear independent pivot column of A

4th column of A is a linear combination of 1st & 3rd columns of A

5th column of A is a linear combination of 1st & 3rd columns of A

$\square$

Example. $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad \Rightarrow \quad R = \begin{bmatrix} \mathbf{1} & 2 & 3 \end{bmatrix}$

$$\Rightarrow \; N = \underbrace{\begin{bmatrix} - & - \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Put identity matrix}} = \underbrace{\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Put } -F}$$

$\square$

3.3 Pivot columns

In summary:

- The **pivot** columns are **not** linear combination of **earlier** columns.
- The **free** columns are linear combination of **earlier pivot** columns.

Final question:

- How to find the matrix E such that $R = \texttt{rref}(A) = EA$?

Answer: Forward and backward eliminations (with possibly row exchanges) on matrix

$$\begin{bmatrix} A_{m\times n} & I_{m\times m} \end{bmatrix}_{m\times(n+m)}$$

will yield

$$E_{m\times m}\begin{bmatrix} A_{m\times n} & I_{m\times m} \end{bmatrix}_{m\times(n+m)} = \begin{bmatrix} R_{m\times n} & E_{m\times m} \end{bmatrix}_{m\times(n+m)}.$$

Example. $A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix}$

$$\Rightarrow \quad R = \texttt{rref}\left(\begin{bmatrix} 1 & 3 & 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -3 & 0 & 1 & 0 \\ 1 & 3 & 1 & 6 & -4 & 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 & -1 & 0 & -1 & 1 \\ \mathbf{0} & 0 & \mathbf{1} & 4 & -3 & 0 & 1 & 0 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \quad \square$$

Example. $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

$$\Rightarrow \quad R = \texttt{rref}\left(\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} \mathbf{1} & \mathbf{0} & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{1} & 0 & 1 & 0 \end{bmatrix} \qquad \square$$

3.3 Rank revisited

- The matrix A can be decomposed into sum of outer products of r pairs of vectors.
  - I.e., $A = \sum_{i=1}^{r} \boldsymbol{u}_i \boldsymbol{v}_i^{\mathrm{T}}$
  - How to prove this? Or how to determine these $\{\boldsymbol{u}_i, \boldsymbol{v}_i\}_{i=1}^{r}$ pairs?

Answer: (Denote $\tilde{E} = E^{-1}$ for convenience.)

$$A_{m\times n} = \tilde{E}_{m\times m} R_{m\times n} = \begin{bmatrix} \tilde{\boldsymbol{e}}_1 & \tilde{\boldsymbol{e}}_2 & \cdots & \tilde{\boldsymbol{e}}_m \end{bmatrix}_{m\times m} \begin{array}{l} \begin{bmatrix} \boldsymbol{r}_1^{\mathrm{T}} \\ \vdots \\ \boldsymbol{r}_r^{\mathrm{T}} \\ \boldsymbol{0}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{0}^{\mathrm{T}} \end{bmatrix}_{m\times n} \end{array} \begin{array}{r} \ldots \text{ 1st pivot row} \\ \vdots \\ \ldots\ r\text{th pivot row} \\ \ldots \text{ all-zero row} \\ \vdots \\ \ldots \text{ all-zero row} \end{array}$$

$$= \begin{bmatrix} \tilde{\boldsymbol{e}}_1 & \tilde{\boldsymbol{e}}_2 & \cdots & \tilde{\boldsymbol{e}}_m \end{bmatrix}_{m\times m} \left(\begin{bmatrix} \boldsymbol{r}_1^{\mathrm{T}} \\ \boldsymbol{0}^{\mathrm{T}} \\ \boldsymbol{0}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{0}^{\mathrm{T}} \\ \boldsymbol{0}^{\mathrm{T}} \\ \boldsymbol{0}^{\mathrm{T}} \end{bmatrix}_{m\times n} + \cdots + \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{0}^{\mathrm{T}} \\ \boldsymbol{r}_r^{\mathrm{T}} \\ \boldsymbol{0}^{\mathrm{T}} \\ \vdots \\ \boldsymbol{0}^{\mathrm{T}} \end{bmatrix}_{m\times n} \right)$$

3.3 Rank revisited

$$= \begin{bmatrix} \tilde{\boldsymbol{e}}_1 & \cdots & \tilde{\boldsymbol{e}}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_1^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} \\ \vdots \\ \mathbf{0}^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} \end{bmatrix} + \cdots + \begin{bmatrix} \tilde{\boldsymbol{e}}_1 & \cdots & \tilde{\boldsymbol{e}}_m \end{bmatrix} \begin{bmatrix} \mathbf{0}^{\mathrm{T}} \\ \vdots \\ \mathbf{0}^{\mathrm{T}} \\ \boldsymbol{r}_r^{\mathrm{T}} \\ \mathbf{0}^{\mathrm{T}} \\ \vdots \\ \mathbf{0}^{\mathrm{T}} \end{bmatrix}$$

$$= \tilde{\boldsymbol{e}}_1 \boldsymbol{r}_1^{\mathrm{T}} + \cdots + \tilde{\boldsymbol{e}}_r \boldsymbol{r}_r^{\mathrm{T}}$$

□

- The determination of $\tilde{E}$ and R (with $A = \tilde{E}R$) gives the "outer-product decomposition" of A.

Exercise 1 (Problem 25): Show that every m-by-n matrix of rank r reduces to (m by r) times (r by n). Also, show that the column of the m by r matrix equal the pivot columns of A.

$$\textit{Hint: } A_{m\times n} = \begin{bmatrix}\tilde{\boldsymbol{e}}_1 & \cdots & \tilde{\boldsymbol{e}}_r\end{bmatrix}_{m\times r}\begin{bmatrix}\boldsymbol{r}_1^{\mathsf{T}}\\ \vdots \\ \boldsymbol{r}_r^{\mathsf{T}}\end{bmatrix}_{r\times n}$$

Exercise 2 (Problem 16): If $A_{m\times n} = \tilde{\boldsymbol{e}}_A\boldsymbol{r}_A^{\mathsf{T}}$ and $B_{n\times k} = \tilde{\boldsymbol{e}}_B\boldsymbol{r}_B^{\mathsf{T}}$ are two rank-1 matrices, prove that the rank of AB is 1 if, and only if, $\boldsymbol{r}_A^{\mathsf{T}}\tilde{\boldsymbol{e}}_B \neq 0$.

Exercise 3: If $A = \sum_{i=1}^{r}\tilde{\boldsymbol{e}}_i\boldsymbol{r}_i^{\mathsf{T}}$ is a rank-r matrix, and $B = \sum_{j=1}^{\bar{r}}\bar{\boldsymbol{e}}_j\bar{\boldsymbol{r}}_j^{\mathsf{T}}$ is a rank-$\bar{r}$ matrix, show that

$$AB = \underbrace{\begin{bmatrix}A\bar{\boldsymbol{e}}_1 & \cdots & A\bar{\boldsymbol{e}}_{\bar{r}}\end{bmatrix}}_{=\tilde{E}_{AB}}\bar{R}$$

What is the condition under which AB remains rank $\bar{r}$? Answer: $\tilde{E}_{AB}$ has rank $\bar{r}$, or $A\bar{\boldsymbol{e}}_1, \cdots, A\bar{\boldsymbol{e}}_{\bar{r}}$ are linearly independent.

Exercise 4: Prove that $\texttt{rank}(AB) \leq \texttt{rank}(B)$.

Exercise 5: Prove that $\texttt{rank}(AB) \leq \texttt{rank}(A)$. *Hint:* $\texttt{rank}(A) = \texttt{rank}(A^{\mathsf{T}})$.

3.4 The complete solution of $A\boldsymbol{x} = \boldsymbol{b}$

- Recall how we completely solve $A_{m\times n}\boldsymbol{x}_{n\times 1} = \mathbf{0}_{m\times 1}$.

  (Here, "complete" means that we wish to find all solutions.)

Answer:

$$R = \texttt{rref}(A) = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} \quad \Rightarrow \quad N_{n\times(n-r)} = \begin{bmatrix} -F_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$$

Then, every solution $\boldsymbol{x}$ for $A\boldsymbol{x} = \mathbf{0}$ is of the form

$$\boldsymbol{x}^{(\text{null})}_{n\times 1} = N_{n\times(n-r)}\boldsymbol{v}_{(n-r)\times 1} \text{ for any } \boldsymbol{v} \in \Re^{n-r}.$$

- How to completely solve $A_{m\times n}\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{m\times 1}$?

Answer: Identify one particular solution $\boldsymbol{x}^{(\text{p})}$ to $A_{m\times n}\boldsymbol{x}^{(\text{p})}_{n\times 1} = \boldsymbol{b}_{m\times 1}$.
Then, every solution $\boldsymbol{x}$ for $A\boldsymbol{x} = \boldsymbol{b}$ is of the form

$$\boldsymbol{x}^{(\text{p})} + \boldsymbol{x}^{(\text{null})}.$$

3.4 The complete solution of $A\boldsymbol{x} = \boldsymbol{b}$

Proof:

- Suppose that $\boldsymbol{w}$ satisfies $A\boldsymbol{w} = \boldsymbol{b}$ but cannot be expressed as $\boldsymbol{x}^{(\mathrm{p})} + \boldsymbol{x}^{(\mathrm{null})}$.
- Then, $A(\boldsymbol{w} - \boldsymbol{x}^{(\mathrm{p})}) = A\boldsymbol{w} - A\boldsymbol{x}^{(\mathrm{p})} = \boldsymbol{b} - \boldsymbol{b} = \mathbf{0}$. Hence, $(\boldsymbol{w} - \boldsymbol{x}^{(\mathrm{p})}) \in \boldsymbol{N}(A)$.
- This implies $\boldsymbol{w} = \boldsymbol{x}^{(\mathrm{p})} + \boldsymbol{x}^{(\mathrm{null})}$ for some $\boldsymbol{x}^{(\mathrm{null})} \in \boldsymbol{N}(A)$; a contradiction is accordingly obtained. $\square$

Exercise: When will there be only one solution to $A\boldsymbol{x} = \boldsymbol{b}$?

Answer: When $\boldsymbol{N}(A)$ consists of only $\mathbf{0}$, and $\boldsymbol{x}^{(\mathrm{p})}$ exists!
Note that $\boldsymbol{x}^{(\mathrm{p})}$ exists if, and only if, $\boldsymbol{b} \in \boldsymbol{C}(A)$.

3.4 The complete solution of $A\boldsymbol{x} = \boldsymbol{b}$

- How to find (one of) $\boldsymbol{x}^{(\mathrm{p})}$?

Answer:

$$[R \;\; \boldsymbol{d}] = \texttt{rref}([A \;\; \boldsymbol{b}]) = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} & \begin{matrix} d_1 \\ \vdots \\ d_r \end{matrix} \\ \boldsymbol{0}_{(m-r)\times r} & \boldsymbol{0}_{(m-r)\times(n-r)} & \boldsymbol{0}_{(m-r)\times 1} \end{bmatrix}.$$

It is the solution of $A\boldsymbol{x} = \boldsymbol{b}$ (or $R\boldsymbol{x} = EA\boldsymbol{x} = E\boldsymbol{b} = \boldsymbol{d}$) by setting all free variables equal to zeros, and the remaining variables equal to $d_1, d_2, \ldots, d_r$.

Example (Problem 3). $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$. Find $\boldsymbol{x}^{(\mathrm{p})}$.

$$\Rightarrow \texttt{rref}([A \;\; \boldsymbol{b}]) = \begin{bmatrix} 1 & 3 & 0 & d_1 = -2 \\ 0 & 0 & 1 & d_2 = 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Then, } \boldsymbol{x}^{(\mathrm{p})} = \begin{bmatrix} d_1 \\ 0 \\ d_2 \end{bmatrix} \text{ .... free variable} = 0 .$$

3.4 The complete solution of $A\boldsymbol{x} = \boldsymbol{b}$

Based on what we learn about rank, we can summarize the solutions of $A_{m\times n}\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{m\times 1}$ as follows.

	A	$A\boldsymbol{x} = \boldsymbol{b}$	$\boldsymbol{N}(A)$
$r = m \quad r = n$	square and invertible	1 solution	$\{\mathbf{0}_{n\times 1}\}$
$r = m \quad r < n$	rectangular (smaller height)	∞ solutions	combination of $(n-r)$ linearly independent n-by-1 vectors
$r < m \quad r = n$	rectangular (smaller width)	0 or 1 solution	$\{\mathbf{0}_{n\times 1}\}$
$r < m \quad r < n$	not full rank	0 or ∞ solutions	combination of $(n-r)$ linearly independent n-by-1 vectors

It may be easier to memorize this by the following equivalent table.

	A	$A\boldsymbol{x} = \boldsymbol{b}$	$\boldsymbol{N}(A)$
$r = m \quad r = n$	square and invertible	1 solution $\boldsymbol{C}(A) = \Re^m$	combination of $(n-r) = 0$ linearly independent n-by-1 vectors
$r = m \quad r < n$	rectangular (smaller height)	∞ solutions $\boldsymbol{C}(A) = \Re^m$	combination of $(n-r) > 0$ linearly independent n-by-1 vectors
$r < m \quad r = n$	rectangular (smaller width)	0 or 1 solution $\boldsymbol{C}(A) \subset \Re^m$	combination of $(n-r) = 0$ linearly independent n-by-1 vectors
$r < m \quad r < n$	not full rank	0 or ∞ solutions $\boldsymbol{C}(A) \subset \Re^m$	combination of $(n-r) > 0$ linearly independent n-by-1 vectors

3.4 Denote for convenience $\tilde{E} = E^{-1}$

		A	$A\boldsymbol{x} = \boldsymbol{b}$	$\boldsymbol{C}(A)$
(a)	$r = m \quad r = n$	square and invertible	1 solution	$\Re^m$
(b)	$r = m \quad r < n$	rectangular (smaller height)	∞ solutions	$\Re^m$
(c)	$r < m \quad r = n$	rectangular (smaller width)	0 or 1 solution	combination of r linearly independent m-by-1 vectors
(d)	$r < m \quad r < n$	not full rank	0 or ∞ solutions	combination of r linearly independent m-by-1 vectors

$$
\begin{aligned}
\boldsymbol{C}(A) &= \left\{ \boldsymbol{b} \in \Re^m : \boxed{A_{m\times n}\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n \right\} \\
&= \left\{ \boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times m} \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix} \boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n \right\} \\
&= \left\{ \boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times m} \begin{bmatrix} \boldsymbol{x}_{r\times 1} + F_{r\times(n-r)}\boldsymbol{x}_{(n-r)\times 1} \\ \mathbf{0}_{(m-r)\times 1} \end{bmatrix} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n \right\} \\
&= \begin{cases}
\left\{ \boldsymbol{b} \in \Re^r : \boxed{\tilde{E}_{r\times r}\boldsymbol{x}_{r\times 1} = \boldsymbol{b}_{r\times 1}} \text{ for some } \boldsymbol{x} \in \Re^r \right\} & (a) \\
\left\{ \boldsymbol{b} \in \Re^r : \boxed{\tilde{E}_{r\times r}\boldsymbol{x}_{r\times 1} + \tilde{E}_{r\times r}F_{r\times(n-r)}\boldsymbol{x}_{(n-r)\times 1} = \boldsymbol{b}_{r\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n \right\} & (b) \\
\left\{ \boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times r}\boldsymbol{x}_{r\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^r \right\} & (c) \\
\left\{ \boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times r}\boldsymbol{x}_{r\times 1} + \tilde{E}_{m\times r}F_{r\times(n-r)}\boldsymbol{x}_{(n-r)\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n \right\} & (d)
\end{cases}
\end{aligned}
$$

3.4 Denote for convenience $\tilde{E} = E^{-1}$

$$
\boldsymbol{C}(A) = \begin{cases}
\left\{\boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times r}\boldsymbol{x}_{r\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^r\right\} & (a) \\
\left\{\boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times r}\boldsymbol{x}_{r\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^r\right\} & (b) \\
\left\{\boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times r}\boldsymbol{x}_{r\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^r\right\} & (c) \\
\left\{\boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times r}\boldsymbol{x}_{r\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^r\right\} & (d)
\end{cases}
$$

3.4 Denote for convenience $\tilde{E} = E^{-1}$

Example. Find the condition on $\boldsymbol{b}$ under which $A\boldsymbol{x} = \boldsymbol{b}$ has solutions.

Answer:

$$\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \texttt{rref}(\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}) = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} & \begin{matrix} d_1 \\ \vdots \\ d_r \end{matrix} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} & \begin{matrix} d_{r+1} \\ \vdots \\ d_m \end{matrix} \end{bmatrix}.$$

We will have $(m-r)$ conditions, i.e., $\begin{cases} d_{r+1}(\boldsymbol{b}) = 0 \\ \vdots \\ d_m(\boldsymbol{b}) = 0 \end{cases}$

Example. Find the condition on $\boldsymbol{b}$ under which $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \boldsymbol{x} = \boldsymbol{b}$ has solutions.

Answer: $\texttt{rref}(\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}) = \begin{bmatrix} 1 & 2 & 0 & 2 & 4b_1 - \frac{3}{2}b_2 \\ 0 & 0 & 1 & 1 & -b_1 + \frac{1}{2}b_2 \\ 0 & 0 & 0 & 0 & -5b_1 + b_2 + b_3 \end{bmatrix} \quad \Rightarrow \quad d_3(\boldsymbol{b}) = -5b_1 + b_2 + b_3 = 0.$

3.4 Column space revisited

The above example gives an alternative way to define the column space of A.

$$\begin{aligned}
\boldsymbol{C}(A) &= \left\{\boldsymbol{b} \in \Re^m : \boxed{A_{m\times n}\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n\right\} \\
&= \left\{\boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times r}\boldsymbol{x}_{r\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^r\right\} \\
&= \left\{\boldsymbol{b} \in \Re^m : \boxed{\begin{cases} d_{r+1}(\boldsymbol{b}) = 0 \\ \vdots \\ d_m(\boldsymbol{b}) = 0 \end{cases}}\right\}
\end{aligned}$$

Recall that by $R = EA$, an interesting observation is that r pivot columns of $\tilde{E} = E^{-1}$ is exactly the r pivot columns of A! Hence, the above $\tilde{E}_{m\times r}$ are exactly the r pivot columns of A.

Example. Find the column space of $A_{3\times 4} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix}$.

Answer:

$$\begin{aligned}
\boldsymbol{C}(A) &= \left\{\boldsymbol{b} \in \Re^3 : \boxed{\begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 3 & 7 \end{bmatrix} \boldsymbol{x}_{2\times 1} = \boldsymbol{b}_{3\times 1}} \text{ for some } \boldsymbol{x} \in \Re^2\right\} \\
&= \left\{\boldsymbol{b} \in \Re^3 : \boxed{-5b_1 + b_2 + b_3 = 0}\right\}
\end{aligned}$$

□

3.5 Independence, basis and dimension

- From the previous section, we learn that the **dimension** of column space $\boldsymbol{C}(A_{m\times n})$ is rank r, i.e., it is a linear combination of r **linearly independent** $m \times 1$ vectors.
- These **linearly independent** vectors span the space $\boldsymbol{C}(A)$. So they can be the **basis** of the vector space $\boldsymbol{C}(A)$.

> **Definition (Basis):** Independent vectors that span the space are called the basis of the space.
>
> Note: "Span" = All vectors in the space can be represented as a **unique** linear combinations of the basis. (Please note again "unique" is an important key word here!) Uniqueness can be proved by "independence" as can be seen by the exercise on the bottom of this slide.

Example. The all-zero vector $\mathbf{0}$ is always dependent on other vectors. Why?

Exercise (Uniqueness). Suppose $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a basis, and a vector $\boldsymbol{v} = a_1\boldsymbol{v}_1 + \cdots + a_n\boldsymbol{v}_n = b_1\boldsymbol{v}_1 + \cdots + b_n\boldsymbol{v}_n$. Show that $a_i = b_i$ for $1 \le i \le n$. (Hint: See p. 172 of the text.)

3.5 Independence, basis and dimension

Example. Prove that if the columns of A are linearly independent, then $A\boldsymbol{x} = \boldsymbol{0}$ has a unique solution $\boldsymbol{x} = \boldsymbol{0}$.

Proof: Suppose $A\boldsymbol{x} = \sum_{i=1}^{m} x_i \boldsymbol{a}_i = \boldsymbol{0}$ for some non-zero $\boldsymbol{x}$. Then the columns of A are not linearly independent according to the definition in slide 3-32, which contradicts to the assumption that the columns of A are linearly independent! $\square$

- How to examine whether a group of vectors are independent or not?

Answer: Place them as column vectors of a matrix A.

- They are independent if, and only if, $A\boldsymbol{x} = \boldsymbol{0}$ does not have non-zero solution.
- They are independent if, and only if, $\boldsymbol{N}(A) = \boldsymbol{N}(R) = \{\boldsymbol{0}\}$.
- They are independent if, and only if, no free variables.
- They are independent if, and only if, $A_{m\times n}$ has full column rank, i.e., $r = n$.

3.5 Column space and row space of $A_{m\times n}$ with rank r

After the introduction of "independence," we can re-define column space and row space as follows.

- Column space = linear combinations of r independent columns.
  - These r independent columns are the basis of the column space.
  - The dimension of $\boldsymbol{C}(A)$ is r.
- Row space = linear combinations of r independent rows.
  - These r independent rows are the basis of the row space.
  - The dimension of $\boldsymbol{R}(A) = \boldsymbol{C}(A^{\mathsf{T}})$ is r.
- Null space = linear combinations of $n - r$ independent vectors, each of which is perpendicular to r independent rows of A.
  - These $n - r$ independent vectors are the basis of the null space.
  - The dimension of $\boldsymbol{N}(A)$ is $n - r$.

3.5 Column space and row space of $A_{m\times n}$ and $R_{m\times n}$

- Column space = linear combinations of r independent columns.
  - It is possible that $\boldsymbol{C}(A) \neq \boldsymbol{C}(R)$.
  - However, they have the same rank.
- Row space = linear combinations of r independent rows.
  - $\boldsymbol{R}(A) = \boldsymbol{R}(R)$. In fact, $\boldsymbol{R}(EA) = \boldsymbol{R}(A)$ for invertible E.
- Null space = linear combinations of $n - r$ independent vectors, each of which is independent of r independent rows.
  - $\boldsymbol{N}(A) = \boldsymbol{N}(R)$. In fact, $\boldsymbol{N}(EA) = \boldsymbol{N}(A)$ for invertible E.

Tip: $\boldsymbol{C}(AE) = \boldsymbol{C}(A)$ for invertible E.

- The number of basis for a vector space is unique.

Proof: Suppose $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are two bases of the same vector space, and suppose $n > m$. By definition of basis, $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ can be represented as linear combinations of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$. Hence,

$$W = \begin{bmatrix} \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_n \end{bmatrix} = \begin{bmatrix} V\boldsymbol{a}_1 & \cdots & V\boldsymbol{a}_n \end{bmatrix} = VA_{m\times n}$$

where

$$V = \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_m \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \end{bmatrix}.$$

Then, $m < n$ implies that $A\boldsymbol{x} = \boldsymbol{0}$ has non-zero solution (at least $n - m$ free variables). Consequently, $W\boldsymbol{x} = VA\boldsymbol{x} = V\boldsymbol{0} = \boldsymbol{0}$, i.e., a non-zero linear combination $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ equal $\boldsymbol{0}$, which implies $\boldsymbol{w}_n$ is linearly dependent on the other vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{n-1}$. A contradiction to the linear independence property of a basis is obtained.

- The number of basis is also called the *dimension* (維度) of the space.
- The number of basis is sometimes referred as the "degree of freedom" (自由度) of the space.

3.5 Basis

- This is an extension to the "dimension" or "degree of freedom" notion of the Euclidean space.

Example. Find a basis of the vector space of polynomial equations of order m, $a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$, where each $a_i \in \Re$.

Answer: In this vector space, $\mathbb{V}$ is a set of polynomial equations of order m and scalar field $\mathbb{F} = \Re$. One of the basis can be

$$\{1, x, x^2, \ldots, x^m\}.$$

All the vectors (i.e., polynomials) can be represented as a linear combination of the basis.

- Back to the Euclidean space, how to find the basis for a set of vectors?

Answer:

Approach 1) Put them as the rows of a matrix A. The r pivot rows of $R = \texttt{rref}(A)$ are the answers.

Approach 2) Put them as the columns of a matrix A. Determine the r pivot columns through R. Then, the r pivot columns of A (not R) are the answers.

3.5 Extension to matrix space and function space

- "Extension to Matrix space and Function space" is only for your reference, and is excluded from our focus (specifically is excluded from the exam).
- It is however an important concept in the area of communications. E.g., to find a basis of a group of communication signals.

Exercise. Can $\mathbf{0}$ be a part of the basis? Hint: $\mathbf{0}$ is linearly dependent on any vector, including itself.

Definition (Independence): A set of vectors $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$ is said to be *linearly independent* if the linear combination of them $(a_1\boldsymbol{v}_1 + \cdots + a_m\boldsymbol{v}_m)$ equals $\mathbf{0}$ only when the coefficients are zero $(a_1 = a_2 = \cdots = a_m = 0)$.

We may then modify the definition of basis as:

Definition (Basis): Independent **(non-zero)** vectors that span the space are called the basis of the space.
Note: "Span" = All vectors in the space can be represented as a **unique** linear combinations of the basis. (Please note again "unique" is an important key word here!)

Exercise. What is the basis for null space $\{\mathbf{0}\}$? What is the dimension of this null space?

Hint 1: Put $A = \begin{bmatrix}\mathbf{0}\end{bmatrix}$, and find the rank r of A.

Hint 2: If we choose $\mathbf{0}$ to be the basis of such a null space, can all vectors in the space $\{\mathbf{0}\}$ be represented as a **unique** linear combinations of the basis?

Answer: The dimension is 0 and no basis exists.

Example (Challenge). What are all the matrices that have the column space spanned by two vectors $\boldsymbol{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix}2\\3\\0\end{bmatrix}$?

Answer: All $3 \times n$ matrices with rank $r = 2$ and with all columns are linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. I.e., $\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2\end{bmatrix}\begin{bmatrix}c_{1,1} & c_{1,2} & \cdots & c_{1,n}\\ c_{2,1} & c_{2,2} & \cdots & c_{2,n}\end{bmatrix}$ with $n \geq 2$ and at least two columns of C are not equal.

Important notion: All the columns of A must lie in $\boldsymbol{C}(A)$, and the r pivot columns form a basis.

Example (Challenge). What are all the matrices that have the null space spanned by two vectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$?

Answer: All $m \times 3$ matrices with rank $r = 1$ and with all rows are linear combination of the vector $\boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (because $\boldsymbol{w}^{\mathsf{T}}\boldsymbol{v}_1 = \boldsymbol{w}^{\mathsf{T}}\boldsymbol{v}_2 = 0$). I.e., $\begin{bmatrix} c_{1,1} \\ \vdots \\ c_{m,1} \end{bmatrix} \boldsymbol{w}^{\mathsf{T}}$ with $m \geq 1$ and at least two rows of C are unequal.

Important notion: All the rows of A must lie in the dual space of $\boldsymbol{N}(A)$, i.e., row space $\boldsymbol{R}(A)$. Also, $(n - r) = 2$ is the dimension of $\boldsymbol{N}(A_{m \times n})$.

Example (Challenge). Can the matrices in each of the above two examples form a vector space?

3.6 Dimensions of the four subspaces

What are the four subspaces?

- Column space $\boldsymbol{C}(A)$
- Nullspace $\boldsymbol{N}(A)$
- Row space $\boldsymbol{R}(A) = \boldsymbol{C}(A^{\mathsf{T}})$
- Left nullspace $\boldsymbol{N}(A^{\mathsf{T}})$:

Solution $\boldsymbol{y}$ to $A^{\mathsf{T}}\boldsymbol{y} = \mathbf{0}$

Equivalently, solution $\boldsymbol{y}$ to $\underbrace{\boldsymbol{y}^{\mathsf{T}}}_{\text{on the left of } \boldsymbol{A}} A = \mathbf{0}^{\mathsf{T}}$.

It is therefore named *left nullspace*.

3.6 Determination of the left nullspace

Exercise (Review of key idea 5). The last $(m-r)$ rows of E are a basis of the **left nullspace** of A.

- Recall that the basis (special solutions, i.e., $\begin{bmatrix} -F_{r\times(n-r)} \\ I_{(n-r)\times(n-r)} \end{bmatrix}$) for the **nullspace** of A can be obtained through

$$R = \texttt{rref}(A) = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} \end{bmatrix}.$$

- This exercise tells you that the special solutions of the **left nullspace** is given by $E_{m\times m}$.

3.6 Determination of the left nullspace

Proof:

- That $E_{m\times m}A_{m\times n} = \begin{bmatrix} E_{r\times m} \\ E_{(m-r)\times m} \end{bmatrix} A_{m\times n} = R_{m\times n}$ and the last $(m-r)$ rows of R are all zeros imply

$$E_{(m-r)\times m}A_{m\times n} = \mathbf{0}_{(m-r)\times n}.$$

- Hence, $A^{\mathsf{T}}_{n\times m}E^{\mathsf{T}}_{m\times(m-r)} = \mathbf{0}_{n\times(m-r)}$.
- Observe that the left nullspace requires $A^{\mathsf{T}}_{n\times m}\boldsymbol{y}_{m\times 1} = \mathbf{0}_{n\times 1}$, and the columns of $E_{m\times(m-r)}$ are linearly independent (because E is invertible), and the dimension of the left null space is $(m-r)$. The proof is then completed.

$\square$

Recall from slide 3-48 that by $R = EA$, an interesting observation is that r pivot columns of $\tilde{E} = E^{-1}$ is exactly the r pivot columns of A! So these r columns are a basis of the **column space** of A.

Now we further show that the last $(m-r)$ rows of E are a basis of the **left nullspace** of A.

Example. Find the condition on $\boldsymbol{b}$ under which $A\boldsymbol{x} = \boldsymbol{b}$ has solutions.

Answer:

$$\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \texttt{rref}(\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}) = \begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} & \begin{matrix} d_1 \\ \vdots \\ d_r \end{matrix} \\ \mathbf{0}_{(m-r)\times r} & \mathbf{0}_{(m-r)\times(n-r)} & \begin{matrix} d_{r+1} \\ \vdots \\ d_m \end{matrix} \end{bmatrix}.$$

We will have $(m-r)$ conditions, i.e., $\begin{cases} d_{r+1}(\boldsymbol{b}) = 0 \\ \vdots \\ d_m(\boldsymbol{b}) = 0 \end{cases}$

Example. Find the condition on $\boldsymbol{b}$ under which $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \boldsymbol{x} = \boldsymbol{b}$ has solutions.

Answer: $\texttt{rref}(\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}) = \begin{bmatrix} 1 & 2 & 0 & 2 & 4b_1 - \frac{3}{2}b_2 \\ 0 & 0 & 1 & 1 & -b_1 + \frac{1}{2}b_2 \\ 0 & 0 & 0 & 0 & -5b_1 + b_2 + b_3 \end{bmatrix} \quad \Rightarrow \quad d_3(\boldsymbol{b}) = -5b_1 + b_2 + b_3 = 0.$

The above example gives an alternative way to define the column space of A.

$$\begin{aligned}
\boldsymbol{C}(A) &= \left\{\boldsymbol{b} \in \Re^m : \boxed{A_{m\times n}\boldsymbol{x}_{n\times 1} = \tilde{E}_{m\times m}R_{m\times n}\boldsymbol{x}_{n\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \boldsymbol{x} \in \Re^n\right\} \\
&= \left\{\boldsymbol{b} \in \Re^m : \boxed{\tilde{E}_{m\times r}\tilde{\boldsymbol{x}}_{r\times 1} = \boldsymbol{b}_{m\times 1}} \text{ for some } \tilde{\boldsymbol{x}} \in \Re^r\right\} \\
&= \left\{\boldsymbol{b} \in \Re^m : \boxed{\begin{cases} d_{r+1}(\boldsymbol{b}) = 0 \\ \vdots \\ d_m(\boldsymbol{b}) = 0 \end{cases}}\right\}
\end{aligned}$$

Recall that by $R = EA$, an interesting observation is that r pivot columns of $\tilde{E} = E^{-1}$ is exactly the r pivot columns of A! Hence, the above $\tilde{E}_{m\times r}$ are exactly the r pivot columns of A.

Example. Find the column space of $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix}$.

Answer:

$$\begin{aligned}
\boldsymbol{C}(A) &= \left\{\boldsymbol{b} \in \Re^3 : \boxed{\begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 3 & 7 \end{bmatrix} \boldsymbol{x}_{2\times 1} = \boldsymbol{b}_{3\times 1}} \text{ for some } \boldsymbol{x} \in \Re^2\right\} \\
&= \left\{\boldsymbol{b} \in \Re^3 : \boxed{-5b_1 + b_2 + b_3 = 0}\right\}
\end{aligned}$$

3.6 Column space and left nullspace (cf. slides 3-47 and 3-48)

$$\boldsymbol{C}(A) = \left\{ \boldsymbol{b} \in \Re^m : \boxed{\begin{cases} d_{r+1}(\boldsymbol{b}) = 0 \\ \vdots \\ d_m(\boldsymbol{b}) = 0 \end{cases}} \right\}$$ gives the $(m-r)$ basis of the left nullspace because the vectors in $\boldsymbol{C}(A)$ are orthogonal to those in $\boldsymbol{N}(A^{\mathrm{T}})$.

Example. Find the left nullspace of $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix}$ whose column space is written as

$$\boldsymbol{C}(A) = \left\{ \boldsymbol{b} \in \Re^3 : \boxed{-5b_1 + b_2 + b_3 = 0} \right\}$$

Answer:

$$\boldsymbol{N}(A^T) = \left\{ \boldsymbol{y} \in \Re^3 : \boldsymbol{y} = c \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} \text{ for some } c \in \Re \right\}$$

$\square$

$\begin{bmatrix} -5 & 1 & 1 \end{bmatrix}$ is exactly the last row of E (cf. slide 3-59).

3.6 Column spaces and row spaces

Tip. $A_{m\times n} = B_{m\times r}C_{r\times n}$ (and the rank of all matrices is r). Then,

$$\boldsymbol{C}(A_{m\times n}) = \boldsymbol{C}(B_{m\times r}) \quad \text{and} \quad \boldsymbol{R}(A_{m\times n}) = \boldsymbol{R}(C_{r\times n}).$$

If the rank of A is not r, then we can only have

$$\boldsymbol{C}(A_{m\times n}) \subset \boldsymbol{C}(B_{m\times r}) \quad \text{and} \quad \boldsymbol{R}(A_{m\times n}) \subset \boldsymbol{R}(C_{r\times n}).$$

For example, for $\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{C}$, then $\boldsymbol{C}(A) \subset \boldsymbol{C}(B)$. They are equal when A and B have the same rank.

Exercise. Give $A_{m\times n} = B_{m\times r}C_{r\times n}$. Then, the rank of A is no greater than r.

Example. $A_{m\times n} = \tilde{E}_{m\times m}R_{m\times n}$, where $R = \texttt{rref}(A)$. Then,

$$\boldsymbol{C}(A_{m\times n}) = \boldsymbol{C}(\tilde{E}_{m\times m}) = \boldsymbol{C}(A_{m\times r}) = \boldsymbol{C}(\tilde{E}_{m\times r}) \quad \text{and} \quad \boldsymbol{R}(A_{m\times n}) = \boldsymbol{R}(R_{m\times n}),$$

where $A_{m\times r} = \tilde{E}_{m\times r}$ consists of the r-pivot columns of A.