Trajectory planning in joint space

- $q = q(t)$ in time or $q = q(\lambda)$ in space (then with $\lambda = \lambda(t)$)
- it is sufficient to work component-wise (q_i in vector q)
- an implicit definition of the trajectory, by solving a problem with specified boundary conditions in a given class of functions
- typical classes: polynomials (cubic, quintic,…), (co)sinusoids, clothoids, …
- imposed conditions
  - passage through points = interpolation
  - initial, final, intermediate velocity (or geometric tangent for paths)
  - initial, final acceleration (or geometric curvature)
  - continuity up to the k-th order time (or space) derivative: class $\mathbf{C}^k$

many of the following methods and remarks can be directly applied also to Cartesian trajectory planning (and vice versa)!

Cubic polynomial in space

$\boxed{q(0) = q_0}$ $\boxed{q(1) = q_1}$ $\boxed{q'(0) = v_0}$ $\boxed{q'(1) = v_1}$ ← 4 conditions

$$q(\lambda) = q_0 + \Delta q \, \underbrace{[a\lambda^3 + b\lambda^2 + c\lambda + d]}$$

$\Delta q = q_1 - q_0$

$\lambda \in [0,1]$

4 coefficients → "doubly normalized" polynomial $q_N(\lambda)$

$q_N(0) = 0 \Leftrightarrow d = 0$

$q_N(1) = 1 \Leftrightarrow a + b + c = 1$

$q_N'(0) = dq_N/d\lambda|_{\lambda=0} = c = v_0/\Delta q$

$q_N'(1) = dq_N/d\lambda|_{\lambda=1} = 3a + 2b + c = v_1/\Delta q$

special case: $v_0 = v_1 = 0$ (zero tangent)

$q_N'(0) = 0 \Leftrightarrow c = 0$

$$\left.\begin{aligned} q_N(1) = 1 &\Leftrightarrow a + b = 1 \\ q_N'(1) = 0 &\Leftrightarrow 3a + 2b = 0 \end{aligned}\right\} \Leftrightarrow \quad \begin{aligned} a &= -2 \\ b &= 3 \end{aligned}$$

Cubic polynomial in time

$\boxed{q(0) = q_{in}}$ $\boxed{q(T) = q_{fin}}$ $\boxed{\dot{q}(0) = v_{in}}$ $\boxed{\dot{q}(T) = v_{fin}}$ ← 4 conditions

$$q(\tau) = q_{in} + \Delta q\,\underbrace{[a\,\tau^3 + b\,\tau^2 + c\,\tau + d]}$$

$\Delta q = q_{fin} - q_{in}$

$\tau = t/T,\ \tau \in [0,1]$

4 coefficients → "doubly normalized" polynomial $q_N(\tau)$

$q_N(0) = 0 \;\Leftrightarrow\; d = 0$ $q_N(1) = 1 \Leftrightarrow a + b + c = 1$

$q_N'(0) = dq_N/d\tau|_{\tau=0} = c = v_{in}T/\Delta q$ $q_N'(1) = dq_N/d\tau|_{\tau=1} = 3a + 2b + c = v_{fin}T/\Delta q$

special case: $v_{in} = v_{fin} = 0$ (rest-to-rest)

$$q_N'(0) = 0 \;\Leftrightarrow\; c = 0$$

$$\left.\begin{array}{l} q_N(1) = 1 \;\Leftrightarrow\; a + b = 1 \\ q_N'(1) = 0 \;\Leftrightarrow\; 3a + 2b = 0 \end{array}\right\} \Leftrightarrow \begin{array}{l} a = -2 \\ b = 3 \end{array}$$

Quintic polynomial

$$q(\tau) = a\,\tau^5 + b\,\tau^4 + c\,\tau^3 + d\,\tau^2 + e\,\tau + f$$

6 coefficients

$\tau \in [0, 1]$

allows to satisfy 6 conditions, for example (in normalized time $\tau = t/T$)

$\boxed{q(0) = q_0}$ $\boxed{q(1) = q_1}$ $\boxed{q'(0) = v_0 T}$ $\boxed{q'(1) = v_1 T}$ $\boxed{q''(0) = a_0 T^2}$ $\boxed{q''(1) = a_1 T^2}$

$$\begin{aligned} q(\tau) = {} & (1 - \tau)^3 [q_0 + (3q_0 + v_0 T)\,\tau + (a_0 T^2 + 6 v_0 T + 12 q_0)\,\tau^2/2] \\ & + \tau^3\,[q_1 + (3q_1 - v_1 T)(1 - \tau) + (a_1 T^2 - 6 v_1 T + 12 q_1)\,(1 - \tau)^2/2] \end{aligned}$$

special case: $v_0 = v_1 = a_0 = a_1 = 0$

$$q(\tau) = q_0 + \Delta q\,[6\,\tau^5 - 15\,\tau^4 + 10\,\tau^3]$$

$\Delta q = q_1 - q_0$