Useful information

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1 Relations between elastic constants

	(K,E)	(K,λ)	(K,G)	(K,ν)	(E,G)	(E,ν)	(ν,G)	(ν,λ)	(G,λ)	(G,M)
$K =$	K	K	K	K	$\frac{EG}{3(3G-E)}$	$\frac{E}{3(1-2\nu)}$	$\lambda + \frac{2G}{3}$	$\frac{\lambda(1+\nu)}{3(1-2\nu)}$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$M - \frac{4G}{3}$
$E =$	E	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{9KG}{2K+G}$	$3K(1-2\nu)$	E	E	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$2G(1+\nu)$	$\frac{G(3-M-4G)}{M-2G}$
$\lambda =$	$\frac{3K(3KE)}{9K-E}$	λ	$K - \frac{2G}{3}$	$\frac{3K\nu}{1+\nu}$	$\frac{G(E-2G)}{EG-E}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	λ	λ	$\frac{2G\nu}{1-2\nu}$	$M - 2G$
$G =$	$\frac{3KE}{9K-E}$	$\frac{3(K-\lambda)}{2}$	G	$\frac{3K(1-2\nu)}{2(1+\nu)}$	G	$\frac{E}{2(1+\nu)}$	G	$\frac{\lambda(1-2\nu)}{2\nu}$	G	G
$\nu =$	$\frac{3K-E}{6K}$	$\frac{\lambda}{3K-\lambda}$	$\frac{3K-2G}{2(3K+G)}$	ν	$\frac{E}{2G} - 1$	ν	$\frac{\lambda}{2(\lambda+G)}$	ν	ν	$\frac{M-2G}{2(M-G)}$
$M =$	$\frac{3K(3K+E)}{9K-E}$	$3K - 2\lambda$	$K + \frac{4G}{3}$	$\frac{3K(1-\nu)}{1+\nu}$	$\frac{G(4G-E)}{3G-E}$	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$\lambda + 2G$	$\frac{\lambda(1-\nu)}{\nu}$	$\frac{2G(1-\nu)}{1-2\nu}$	M

K: Bulk modulus, λ: Lamé's first parameter, E: Young's modulus, G: Shear modulus, ν: Poisson's ratio, M: P-wave modulus.

2 Relations for elastic wave speeds

The P-wave is a dilatational wave with speed α given by

$$\alpha^2 = \frac{\lambda + 2G}{\rho}, \qquad \alpha^2 = \frac{G(1-\nu)}{\rho},$$
$$\alpha^2 = \frac{M}{\rho}, \qquad \alpha^2 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho},$$
$$\alpha^2 = \frac{2\beta^2(1-\nu)}{1-2\nu}.$$

The S-wave is a distorsional wave with speed β given by

$$\beta^2 = \frac{G}{\rho}$$
$$\beta^2 = \frac{E}{2(1+\nu)\rho},$$
$$\beta^2 = \frac{\alpha^2(1-2\nu)}{2(1-\nu)}.$$

Some particular values for the ratio

$$\frac{\alpha^2}{\beta^2} = \frac{2(1-\nu)}{1-2\nu}$$

are

$$\frac{\alpha^2}{\beta^2} = \frac{4}{3} \quad \text{for } \nu = -1 \ ,$$
$$\frac{\alpha^2}{\beta^2} = 2 \quad \text{for } \nu = 0 \ ,$$
$$\frac{\alpha^2}{\beta^2} = 4 \quad \text{for } \nu = \frac{1}{3} \ ,$$
$$\frac{\alpha^2}{\beta^2} \to \infty \quad \text{when } \nu \to \frac{1}{2} \ .$$

3 Courant-Friedrichs-Lewy condition

The Courant-Friedrichs-Lewy condition (CFL condition) is a necessary condition for convergence while solving certain partial differential equations (usually hyperbolic PDEs) numerically by the method of finite differences. It arises when explicit time-marching schemes are used for the numerical solution. The condition is named after Richard Courant, Kurt Friedrichs, and Hans Lewy who described it in their 1928 paper [1].

The criterion could be stated as

$$\begin{aligned} C = v_x \frac{\Delta t}{\Delta x} &\leq C_{max} \qquad \text{in 1D} \ ; \\ C = v_x \frac{\Delta t}{\Delta x} + v_y \frac{\Delta t}{\Delta y} &\leq C_{max} \qquad \text{in 2D} \ ; \\ C = v_x \frac{\Delta t}{\Delta x} + v_y \frac{\Delta t}{\Delta y} + v_z \frac{\Delta t}{\Delta z} &\leq C_{max} \qquad \text{in 3D} \ ; \end{aligned}$$

Where v_{x_i} is the phase speed (for wave phenomena) in the x_i direction, Δx_i is the minimum spatial discretization in x_i direction, Δt is the time step and C_{max} is the maximum allowable value for C, which depends on the time discretization scheme but should be less than 1.

In the case of elastodynamics and for a spatial discretization with the finite element method, the criterion could be re-stated as

$$\begin{aligned} C \leq \alpha \frac{\Delta t}{h} &\leq C_{max} \qquad \text{in 1D} \ ; \\ C \leq 2\alpha \frac{\Delta t}{h} &\leq C_{max} \qquad \text{in 2D} \ ; \\ C \leq 3\alpha \frac{\Delta t}{h} &\leq C_{max} \qquad \text{in 3D} \ ; \end{aligned}$$

where α is the phase speed for the P-wave and h is the minimum distance between consecutive nodes. This give us the maximum allowable timestep as

$$\Delta t \leq C_{max} \frac{h}{\alpha} \qquad \text{in 1D} \ ; \tag{1}$$

$$\Delta t \leq \frac{C_{max}}{2} \frac{h}{\alpha} \qquad \text{in 2D} \ ; \tag{2}$$

$$\Delta t \leq \frac{C_{max}}{3} \frac{h}{\alpha} \qquad \text{in 3D} \ . \tag{3}$$

4 Nyquist criterion

The NyquistShannon sampling theorem, after Harry Nyquist and Claude Shannon, in the literature more commonly referred to as the Nyquist sampling theorem or simply as the sampling theorem, is a fundamental result in the field of information theory, in particular telecommunications and signal processing. Sampling is the process of converting a signal (for example, a function of continuous time or space) into a numeric sequence (a function of discrete time or space). Shannon's version of the theorem states [2]:

> If a function $x(t)$ contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $1/(2B)$ seconds apart.

This theorem implies for us in the numerical simulation of wave propagation that

$$h \leq \frac{\lambda}{2} \ ,$$

where h is the maximum distance between consecutive nodes and λ is the shortest wavelength that want to be sampled. So, the selection of h is commonly

$$h = \frac{\lambda}{k} \ ,$$

where $k > 2$ is a factor that depends on the numerical method. For finite element methods k is commonly 10, and for spectral element methods k is reduced to 5 [3].

5 Ricker wavelet

It is common to use a Ricker wavelet as the input signal in simulations since it has the energy concentrated around a circular frequency ω_c. Let's write the signal as

$$f(t) = \left(\frac{12t^2}{b^2} - 1\right) e^{-\frac{6t^2}{b^2}} \ ,$$

where b is the elapsed time between the peaks in the time domain [4]. The Fourier transform for this signal is

$$\hat{f}(\omega) = -\frac{\sqrt{\pi} b^3 \omega^2 e^{-\frac{b^2\omega^2}{24}}}{2\ 6^{3/2}} \ ,$$

with a characteristic (maximum) circular frequency

$$\omega_c = \frac{2\sqrt{6}}{b}.$$

To satisfy the sampling theorem we need to design our simulations thinking about the shortest wavelength, so we need to think about the maximum frequency. To know where to trunk the signal (in the frequency domain) we could compute how much energy is in $[0, \omega_{max}]$ and compare this value with the energy stored in the interval $[0, \infty)$. So, the ratio

$$R(\omega_{max}) = \frac{\int\limits_0^{\omega_{max}} \hat{f}(\omega) d\omega}{\int\limits_0^{\infty} \hat{f}(\omega) d\omega} \times 100\% \ ,$$

tell us the proportion of energy taken into account if we neglect circular frequencies higher than ω_{max}. Some particular values are :

$$R(1.0\omega_c) = 42.76\% \ ; \tag{4}$$
$$R(1.5\omega_c) = 78.77\% \ ; \tag{5}$$
$$R(2.0\omega_c) = 95.40\% \ ; \tag{6}$$
$$R(3.0\omega_c) = 99.96\% \ ; \tag{7}$$
$$R(4.0\omega_c) = 99.99\% \ . \tag{8}$$

References

[1] Courant, R.; Friedrichs, K.; Lewy, H. (1928), *Über die partiellen Differenzengleichungen der mathematischen Physik* (in German), Mathematische Annalen 100 (1): 32–74.

[2] C. E. Shannon, *Communication in the presence of noise*, Proc. Institute of Radio Engineers, vol. 37, no. 1, pp. 10–21, Jan. 1949. Reprint as classic paper in: Proc. IEEE, vol. 86, no. 2, (Feb. 1998).

[3] D. Komatitsch and J. Tromp,*Introduction to the spectral element method for three-dimensional seismic wave propagation*, Geophysical Journal International, 1999, (139): 806–822.

[4] A. Papageorgiou and J. Kim,*Stufy of the propagation and amplification of seismic waves in Caracas Valley with reference to the 29 July 1967 earthquake: SH waves*, Bulletin of the Seismological Society of America, 1991, 81 (6): 2214–2233.