Useful information

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1 Relations between elastic constants

	(K, E)	(K, λ)	(K, G)	(K, ν)	(E, G)	(E, ν)	(λ, G)	(ν, λ)	(G, ν)	(G, M)
$K =$	K	K	K	K	$\frac{EG}{3(3G-E)}$	$\frac{E}{3(1-2\nu)}$	$\lambda + \frac{2G}{3}$	$\frac{\lambda(1+\nu)}{3(1-2\nu)}$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$M - \frac{4G}{3}$
$E =$	E	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{9KG}{2K+G}$	$3K(1-2\nu)$	E	E	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$2G(1+\nu)$	$\frac{G(3M-4G)}{M-2G}$
$\lambda =$	$\frac{3K(3KE)}{9K-E}$	λ	$K - \frac{2G}{3}$	$\frac{3K\nu}{1+\nu}$	$\frac{G(E-2G)}{3G-E}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	λ	λ	$\frac{2G\nu}{1-2\nu}$	$M - 2G$
$G =$	$\frac{3KE}{9K-E}$	$\frac{3(K-\lambda)}{2}$	G	$\frac{3K(1-2\nu)}{2(1+\nu)}$	G	$\frac{E}{2(1+\nu)}$	G	$\frac{\lambda(1-2\nu)}{2\nu}$	G	G
$\nu =$	$\frac{3K-E}{6K}$	$\frac{\lambda}{3K-\lambda}$	$\frac{3K-2G}{2(3K+G)}$	ν	$\frac{E}{2G} - 1$	ν	$\frac{\lambda}{2(\lambda+G)}$	ν	ν	$\frac{M-2G}{2(M-G)}$
$M =$	$\frac{3K(3K+E)}{9K-E}$	$3K - 2\lambda$	$K + \frac{4G}{3}$	$\frac{3K(1-\nu)}{1+\nu}$	$\frac{G(4G-E)}{3G-E}$	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$\lambda + 2G$	$\frac{\lambda(1-\nu)}{\nu}$	$\frac{2G(1-\nu)}{1-2\nu}$	M

K: Bulk modulus, λ: Lamé's first parameter, E: Young's modulus, G: Shear modulus, ν: Poisson's ratio, M: P-wave modulus.

2 Relations for elastic wave speeds

The P-wave is a dilatational wave with speed α given by

$$\alpha^2 = \frac{\lambda + 2G}{\rho}, \qquad \alpha^2 = \frac{G(1-\nu)}{\rho},$$
$$\alpha^2 = \frac{M}{\rho}, \qquad \alpha^2 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho},$$
$$\alpha^2 = \frac{2\beta^2(1-\nu)}{1-2\nu}.$$

The S-wave is a distorsional wave with speed β given by

$$\beta^2 = \frac{G}{\rho}$$
$$\beta^2 = \frac{E}{2(1+\nu)\rho},$$
$$\beta^2 = \frac{\alpha^2(1-2\nu)}{2(1-\nu)}.$$

Some particular values for the ratio

$$\frac{\alpha^2}{\beta^2} = \frac{2(1-\nu)}{1-2\nu}$$

are

$$\frac{\alpha^2}{\beta^2} = \frac{4}{3} \quad \text{for } \nu = -1 \ ,$$
$$\frac{\alpha^2}{\beta^2} = 2 \quad \text{for } \nu = 0 \ ,$$
$$\frac{\alpha^2}{\beta^2} = 4 \quad \text{for } \nu = \frac{1}{3} \ ,$$
$$\frac{\alpha^2}{\beta^2} \to \infty \quad \text{when } \nu \to \frac{1}{2} \ .$$

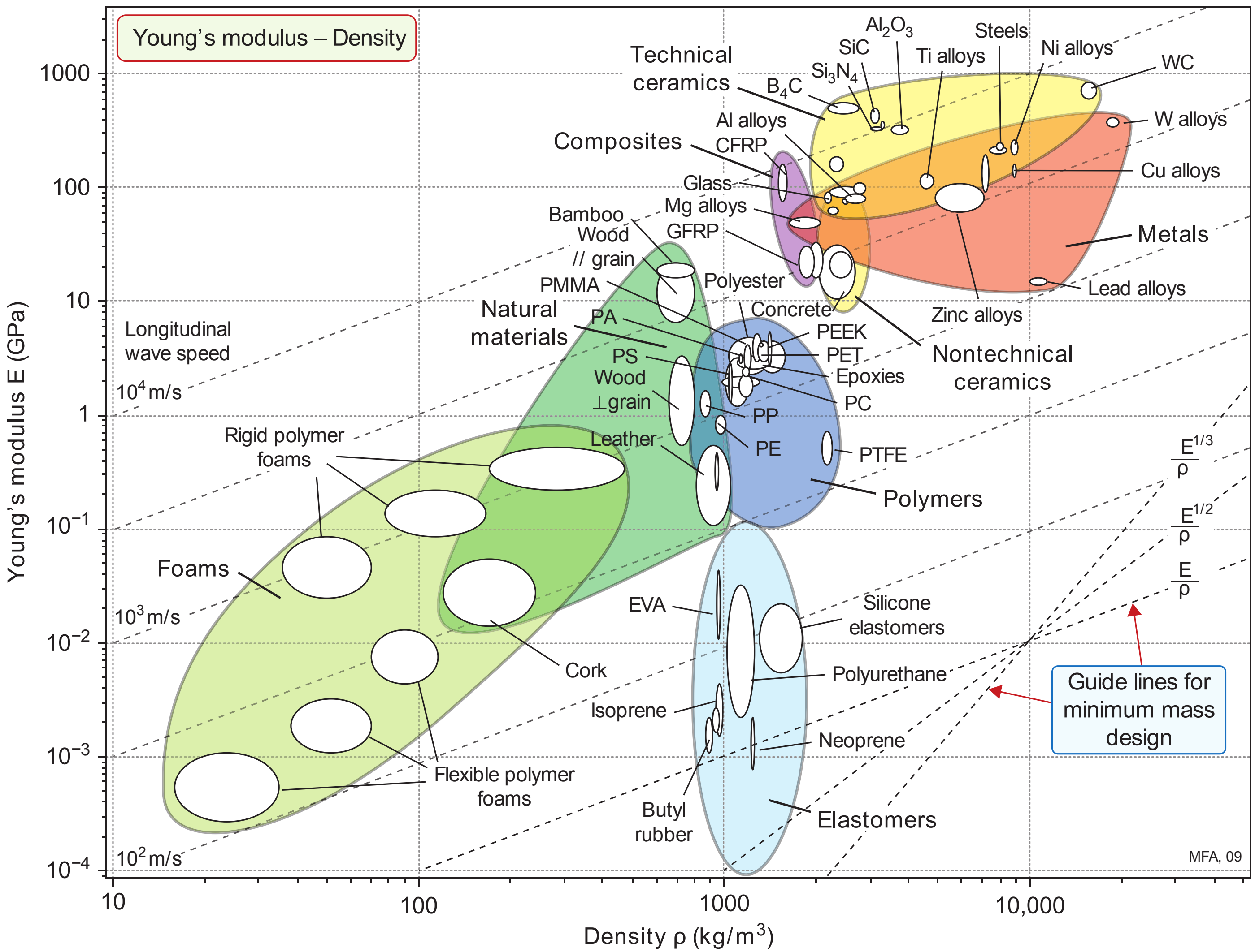

Figure 1: Ashby chart for Young Modulus vs density. The lines show the *sound speed*, that is the speed for a wave in a rod made of this material and is between the longitudinal and shear wave speeds –for Poisson ratios in (-0.5,0.5). Taken from *Ashby, Michael F. "Materials selection in mechanical design." MRS BULLETIN 30 (2005): 995.*

3 Courant–Friedrichs–Lewy condition

The Courant–Friedrichs–Lewy condition (CFL condition) is a necessary condition for convergence while solving certain partial differential equations (usually hyperbolic PDEs) numerically by the method of finite differences. It arises when explicit time-marching schemes are used for the numerical solution. The condition is named after Richard Courant, Kurt Friedrichs, and Hans Lewy who described it in their 1928 paper [1].

The criterion could be stated as

$$
\begin{aligned}
C = v_x \frac{\Delta t}{\Delta x} &\leq C_{max} \qquad \text{in 1D} \;\; ; \\
C = v_x \frac{\Delta t}{\Delta x} + v_y \frac{\Delta t}{\Delta y} &\leq C_{max} \qquad \text{in 2D} \;\; ; \\
C = v_x \frac{\Delta t}{\Delta x} + v_y \frac{\Delta t}{\Delta y} + v_z \frac{\Delta t}{\Delta z} &\leq C_{max} \qquad \text{in 3D} \;\; ;
\end{aligned}
$$

Where v_{x_i} is the phase speed (for wave phenomena) in the x_i direction, Δx_i is the minimum spatial discretization in x_i direction, Δt is the time step and C_{max} is the maximum allowable value for C, which depends on the time discretization scheme but should be less than 1.

In the case of elastodynamics and for a spatial discretization with the finite element method, the criterion could be re-stated as

$$
\begin{aligned}
C \leq \alpha \frac{\Delta t}{h} &\leq C_{max} \qquad \text{in 1D} \;\; ; \\
C \leq 2\alpha \frac{\Delta t}{h} &\leq C_{max} \qquad \text{in 2D} \;\; ; \\
C \leq 3\alpha \frac{\Delta t}{h} &\leq C_{max} \qquad \text{in 3D} \;\; ;
\end{aligned}
$$

where α is the phase speed for the P-wave and h is the minimum distance between consecutive nodes. This give us the maximum allowable timestep as

$$\Delta t \leq C_{max} \frac{h}{\alpha} \qquad \text{in 1D} \;\; ; \tag{1}$$

$$\Delta t \leq \frac{C_{max}}{2} \frac{h}{\alpha} \qquad \text{in 2D} \;\; ; \tag{2}$$

$$\Delta t \leq \frac{C_{max}}{3} \frac{h}{\alpha} \qquad \text{in 3D} \;\; . \tag{3}$$

4 Nyquist criterion

The Nyquist–Shannon (also Nyquist–Shannon–Whittaker theorem) sampling theorem, after Harry Nyquist and Claude Shannon, in the literature more commonly referred to as the Nyquist sampling theorem or simply as the sampling theorem, is a fundamental result in the field of information theory, in particular telecommunications and signal processing. Sampling is the process of converting a signal (for example, a function of continuous time or space) into a numeric sequence (a function of discrete time or space). Shannon's version of the theorem states [5, 3, 2, 4]:

> If a function $x(t)$ contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $1/(2B)$ seconds apart.

This theorem implies for us in the numerical simulation of wave propagation that

$$h \leq \frac{\lambda}{2} \ ,$$

where h is the maximum distance between consecutive nodes and λ is the shortest wavelength that want to be sampled. So, the selection of h is commonly

$$h = \frac{\lambda}{k} \ ,$$

where $k > 2$ is a factor that depends on the numerical method. For finite element methods k is commonly 10, and for spectral element methods k is reduced to 5 [6].

5 Information for different pulse signals

In wave propagation problems we must work with time signals that are finite in time, let's say *wave pulses* (or wave packets in the Quantum Mechanics jargon). Some times we are interested in wave pulses because they are useful as wavefronts representation. Furthermore, wave packets are important since we cannot store information in a wave with a single frequency[1]. There is a trade off between the localization in time (duration of the pulse) and frequency (concentration of the energy around a specific value). This fact is well discussed in Quantum Mechanics texts since it reflects the discussion about wave–particle duality and uncertainty principle ([9]). When we have finite time signals its corresponding Fourier transform will be in the whole frequency domain. In general terms, while more concentrated in the time domain, more spread in the frequency domain. From a numerical point of view we need to truncate the maximum frequency allowed for our simulations.

[1] That's why phase speed of waves can be higher than speed of light. For non-dissipative media, the energy (and so the information) travel with the group velocity. [7]

In the next *subsections* we exposed some truncation tips for different wave pulses. We will refer to the time function as $f(t)$, and to its Fourier transform as $\hat{f}(\omega)$. Since the energy carried for the wave per frequency is proportional to the square of the absolute value of the Fourier transform $\hat{f}^2(\omega)$, we can compare the energy in the frequency interval $[0, \omega_{max}]$ to the energy in $[0, \infty)$[2]. So, we define

$$R(\omega_{max}) = \frac{\int\limits_0^{\omega_{max}} \hat{f}^2(\omega) d\omega}{\int\limits_0^{\infty} \hat{f}^2(\omega) d\omega} \times 100\% \ , \tag{4}$$

5.1 Ricker wavelet

It is common to use a Ricker wavelet as the input signal in simulations since it has the energy concentrated around a circular frequency ω_c. Let's write the signal as

$$f(t) = \left(\frac{12t^2}{b^2} - 1 \right) e^{-\frac{6t^2}{b^2}} \ ,$$

where b is the elapsed time between the peaks in the time domain [8]. The Fourier transform for this signal is

$$\hat{f}(\omega) = -\frac{\sqrt{\pi} b^3 \omega^2 e^{-\frac{b^2 \omega^2}{24}}}{2 \ 6^{3/2}} \ ,$$

with a characteristic (maximum) circular frequency

$$\omega_c = \frac{2\sqrt{6}}{b}.$$

To satisfy the sampling theorem we need to design our simulations thinking about the shortest wavelength, so we need to think about the maximum frequency. We can compute the ratio defined in 4. Some particular values are :

$$R(1.0\omega_c) = 45.06\% \ ; \tag{5}$$
$$R(1.5\omega_c) = 89.09\% \ ; \tag{6}$$
$$R(2.0\omega_c) = 99.32\% \ ; \tag{7}$$
$$R(3.0\omega_c) = 99.99\% \ ; \tag{8}$$
$$R(4.0\omega_c) = 100.00\% \ . \tag{9}$$

This can be depicted in 2.

[2]We just need to compute one halfspace because the pulse treated are symmetric.

5.2 Gaussian wavelet

One option to obtain a wavelet with the energy concentrated in a single frequency value is to use a signal which spectrum is a Gaussian function (Normal distribution) of the form [3]

$$\hat{f}(\omega) = \frac{1}{\omega_\sigma\sqrt{2\pi}}\left[e^{-\frac{(\omega-\omega_0)^2}{2\omega_\sigma^2}} + e^{-\frac{(\omega+\omega_0)^2}{2\omega_\sigma^2}}\right] \quad , \tag{10}$$

where ω is the angular frequency, ω_0 is the value of angular frequency for the highest energy density (mean value), ω_σ is the value around ω_0 to have roughly 80% of the energy (standard deviation). In the time domain, the signal looks like

$$f(t) = 2e^{-\frac{(\sigma t)^2}{2}}\cos\omega_0 t \ \ , \tag{11}$$

that is basically an harmonic function modulated by a Gaussian.

If we consider that $\omega_0/\omega_\sigma > 2$ we can have the expressions

$$\begin{aligned} R(\omega_0) &= 84.27\%;\\ R(\omega_0+\omega_\sigma) &= 84.27\%;\\ R(\omega_0+2\omega_\sigma) &= 97.73\%;\\ R(\omega_0+3\omega_\sigma) &= 99.87\%;\\ R(\omega_0+4\omega_\sigma) &= 100.00\%. \end{aligned}$$

This can be depicted in 3.

5.3 Square pulse

The square pulse is given by (with duration Δt)

$$f(t/\Delta t) = \begin{cases} 0 & \text{if } |t| > \Delta t/2 \\ 1/2 & \text{if } |t| = \Delta t/2 \\ 1 & \text{if } |t| < \Delta t/2 \end{cases}$$

And the Fourier transform is given by

$$\hat{f}(\omega\Delta t) = \frac{\sin\left(\frac{\omega\Delta t}{2}\right)}{\left(\frac{\omega\Delta t}{2}\right)} \ \ ,$$

[3]Both, the positive and negative frequencies are *valid*. That's why we should include the second term in brackets.

for this case the ratios R are:

$$\begin{aligned} R(\pi/2\Delta t) &= 77.37\%; \\ R(\pi/\Delta t) &= 90.28\%; \\ R(2\pi/\Delta t) &= 94.99\%; \\ R(3\pi/\Delta t) &= 96.64\%. \end{aligned}$$

This can be depicted in 4.

5.4 Triangle pulse

The square pulse is given by the convolution of two square pulses, and the Fourier transform is

$$\hat{f}(\omega) = \left[\frac{\sin\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)} \right]^2 .$$

For this case the ratios R are:

$$\begin{aligned} R(\pi/2\Delta t) &= 65.77\%; \\ R(\pi/\Delta t) &= 94.98\%; \\ R(2\pi/\Delta t) &= 99.71\%; \\ R(3\pi/\Delta t) &= 99.87\%. \end{aligned}$$

This can be depicted in 5.

References

[1] Courant, Richard and Kurt Friedrichs, and Hans Lewy. (1967) [1928] *On the partial difference equations of mathematical physics.* IBM Journal of Research and Development 11.2: 215–234. Translated from German by Phyllis Fox, url(accessed May 22, 2013)

[2] C. E. Shannon, *Communication in the presence of noise*, Proc. Institute of Radio Engineers, vol. 37, no. 1, pp. 10–21, Jan. 1949. Reprint as classic paper in: Proc. IEEE, vol. 86, no. 2, (Feb. 1998), url (accessed May 22, 2013)

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[4] E. T. Whittaker, *On the Functions Which are Represented by the Expansions of the Interpolation Theory*, Proc. Royal Soc. Edinburgh, Sec. A, vol.35, pp. 181–194, 1915

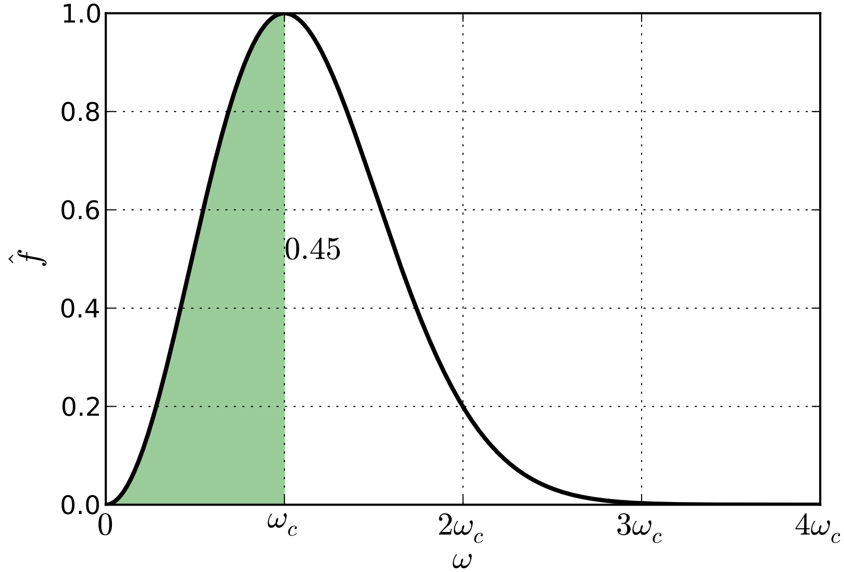
[5] Wikipedia contributors, *Nyquist–Shannon sampling theorem*, Wikipedia, The Free Encyclopedia, url (accessed May 22, 2013).

[6] D. Komatitsch and J. Tromp,*Introduction to the spectral element method for three-dimensional seismic wave propagation*, Geophysical Journal International, 1999, (139): 806-822.

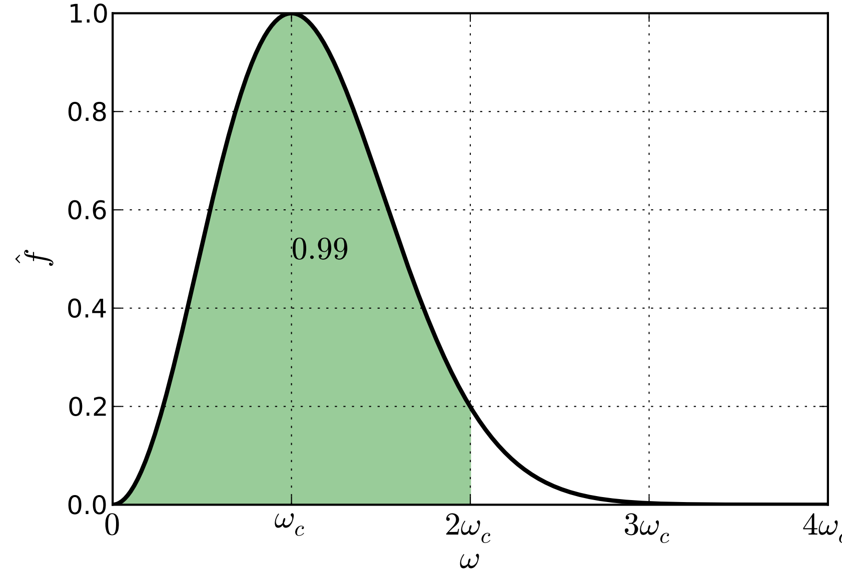
[7] Brillouin, León, and Arnold Sommerfeld. Wave propagation and group velocity. Vol. 960. New York: Academic Press, 1960.

[8] A. Papageorgiou and J. Kim,*Study of the propagation and amplification of seismic waves in Caracas Valley with reference to the 29 July 1967 earthquake: SH waves*, Bulletin of the Seismological Society of America, 1991, 81 (6): 2214-2233.

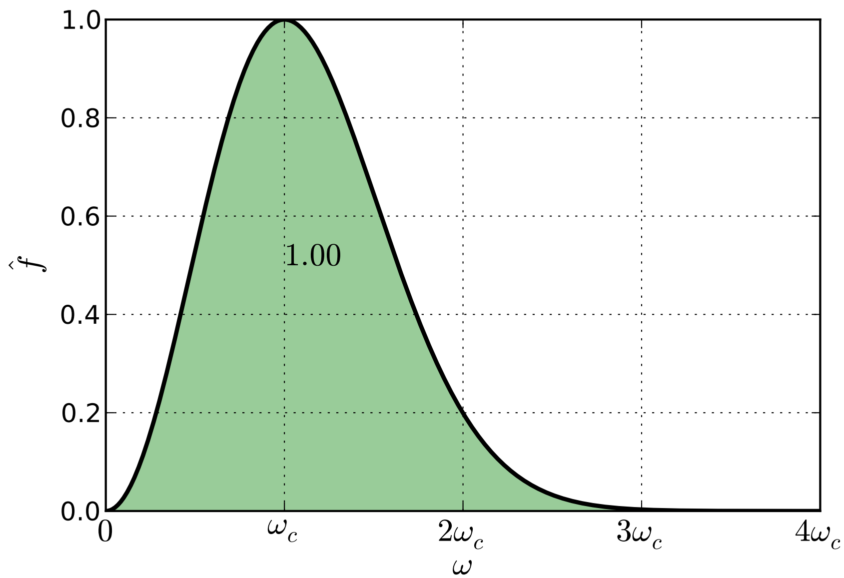
[9] Zettili, Nouredine. Quantum mechanics: concepts and applications. Wiley, 2009.



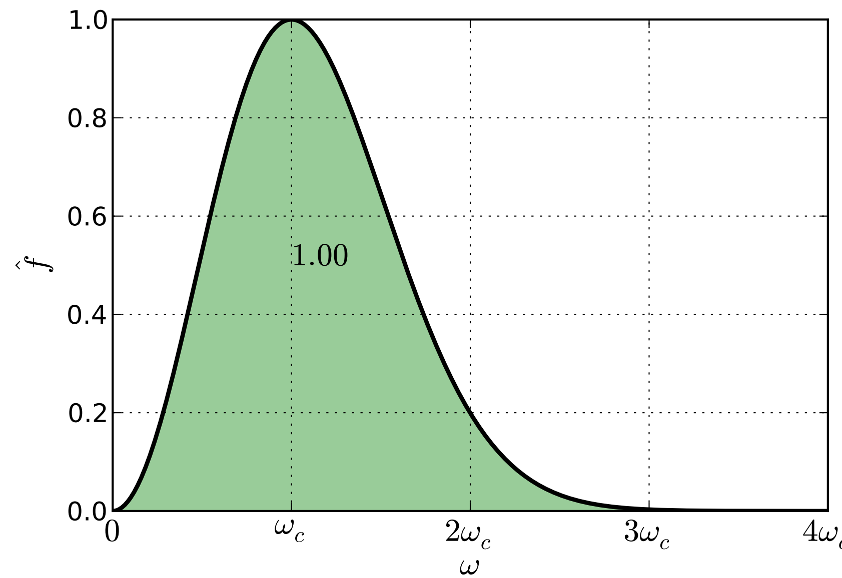
(a) Area under the curve in $[0, \omega_c]$



(b) Area under the curve in $[0, 2\omega_c]$

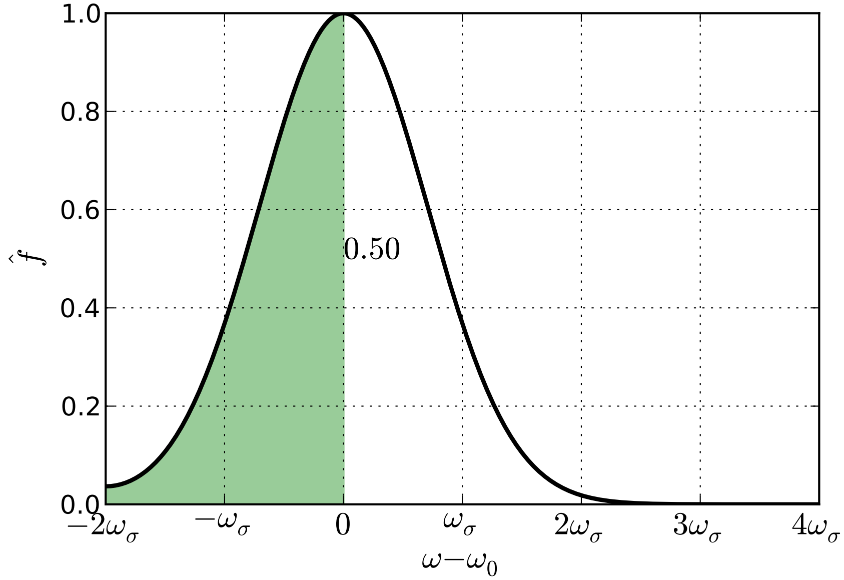


(c) Area under the curve in $[0, 3\omega_c]$

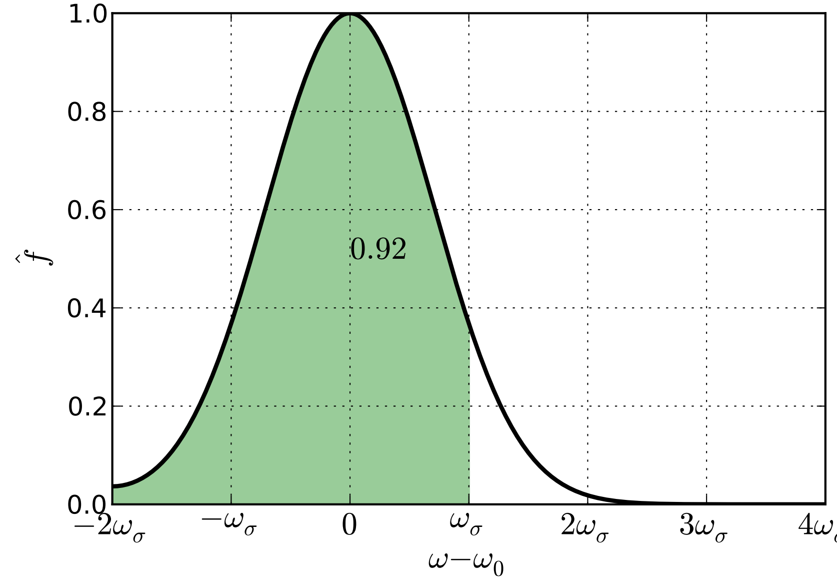


(d) Area under the curve in $[0, 4\omega_c]$

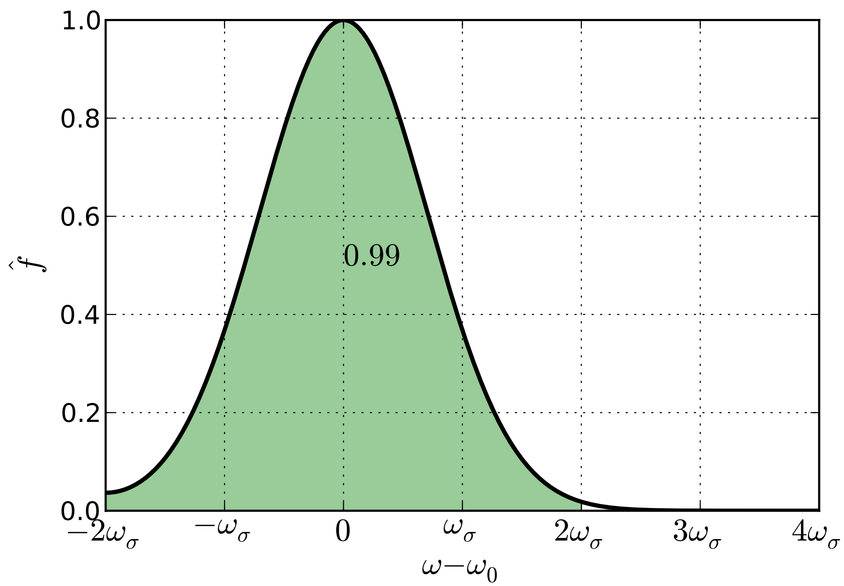
Figure 2: Energy distribution in the spectrum of the Ricker wavelet.



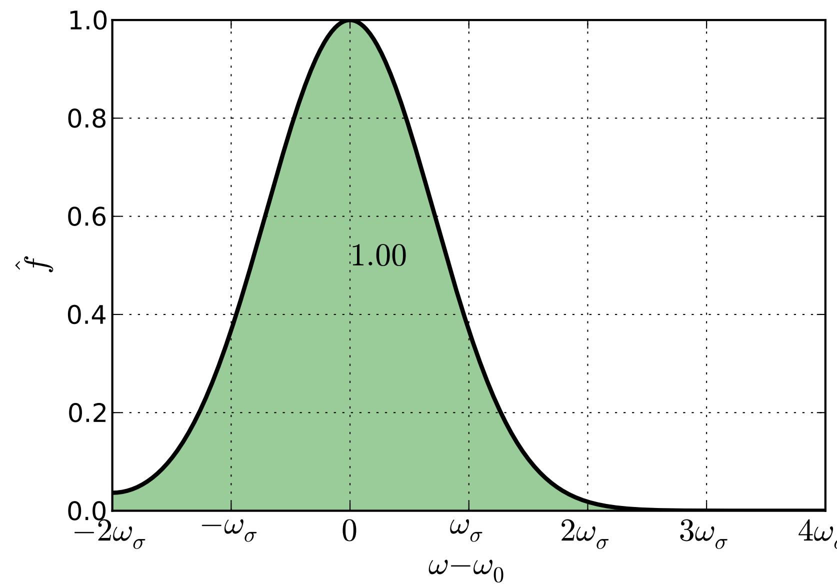
(a) Area under the curve in $[0, \omega_0]$



(b) Area under the curve in $[0, \omega_0 + \omega_\sigma]$

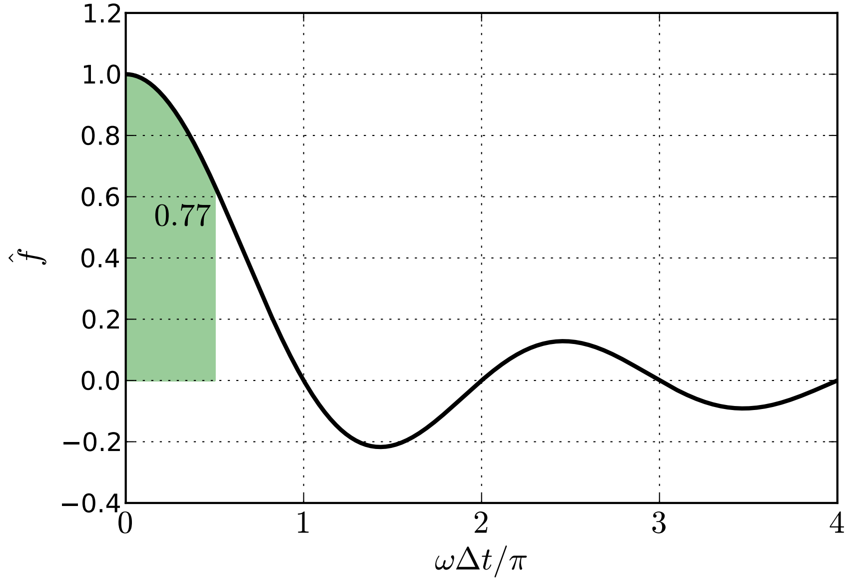


(c) Area under the curve in $[0, \omega_0 + 2\omega_\sigma]$

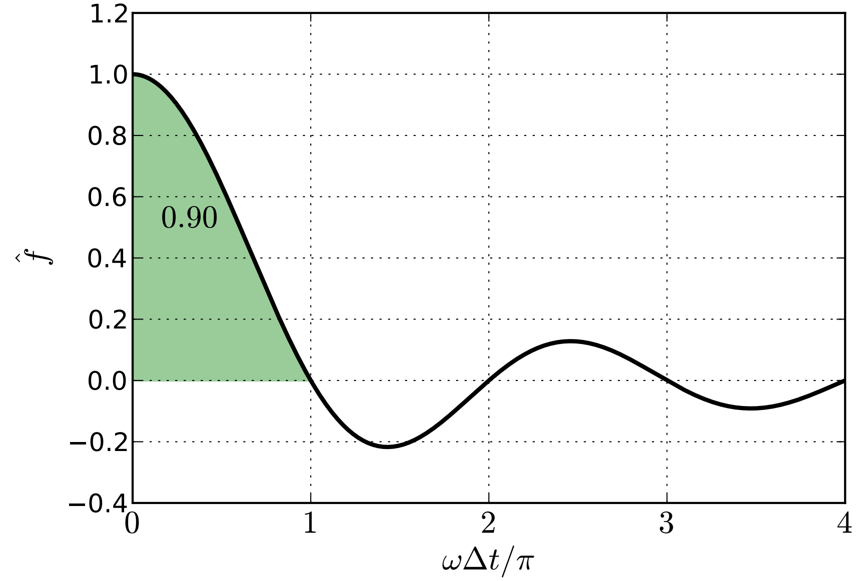


(d) Area under the curve in $[0, \omega_0 + 3\omega_\sigma]$

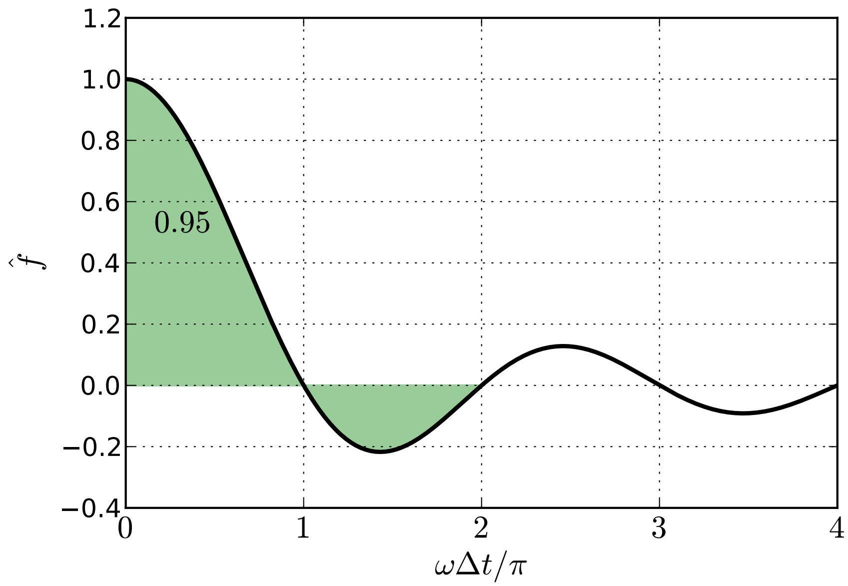
Figure 3: Energy distribution in the spectrum of the *Gaussian* wavelet.



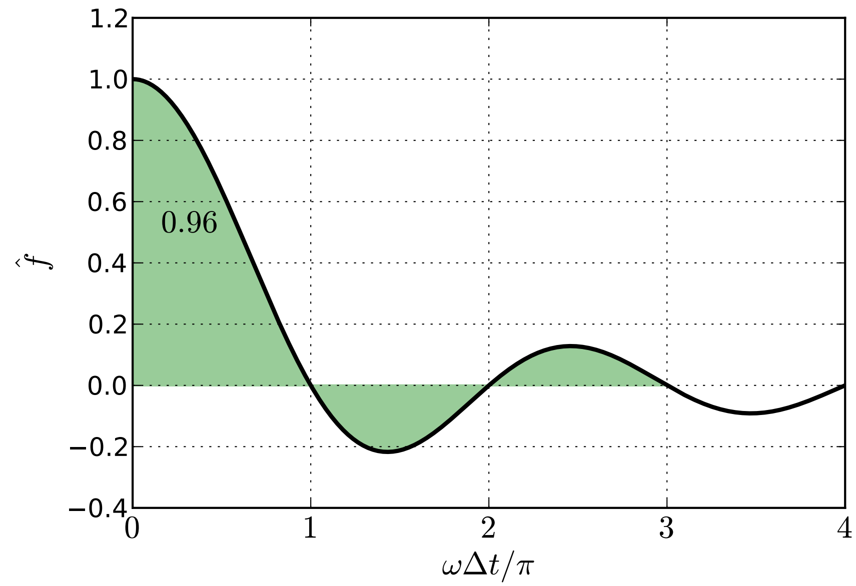
(a) Area under the curve in $[0, 0.5\pi/\Delta t]$



(b) Area under the curve in $[0, \pi/\Delta t]$

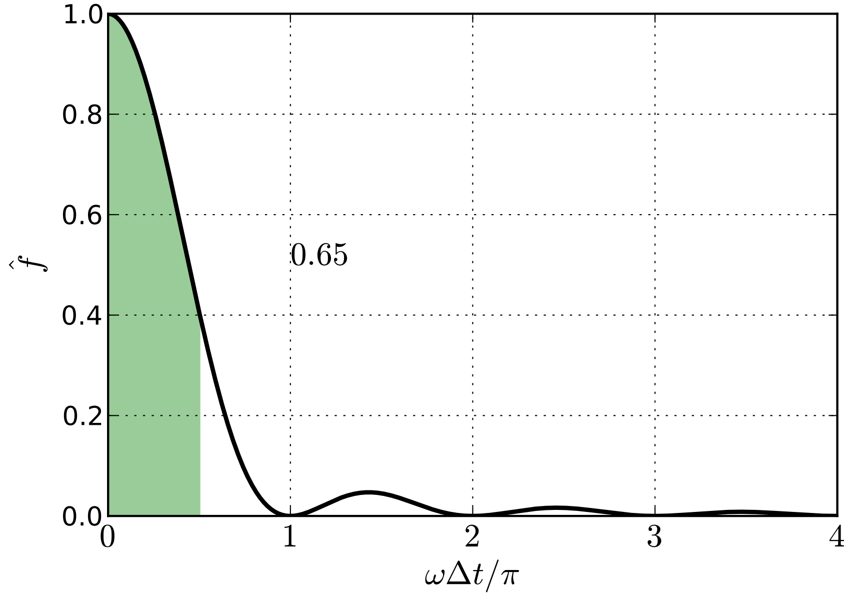


(c) Area under the curve in $[0, 2\pi/\Delta t]$

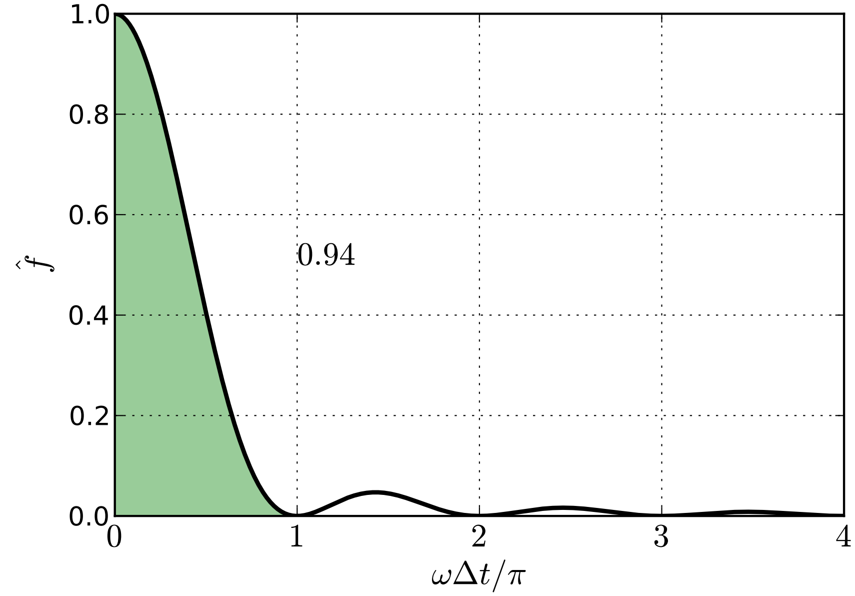


(d) Area under the curve in $[0, 3\pi/\Delta t]$

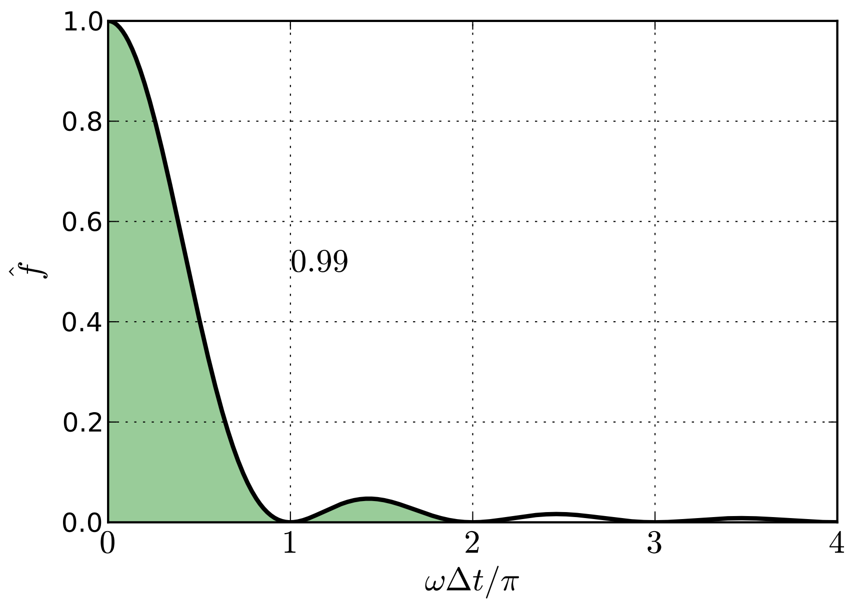
Figure 4: Energy distribution in the spectrum of the square wavelet.



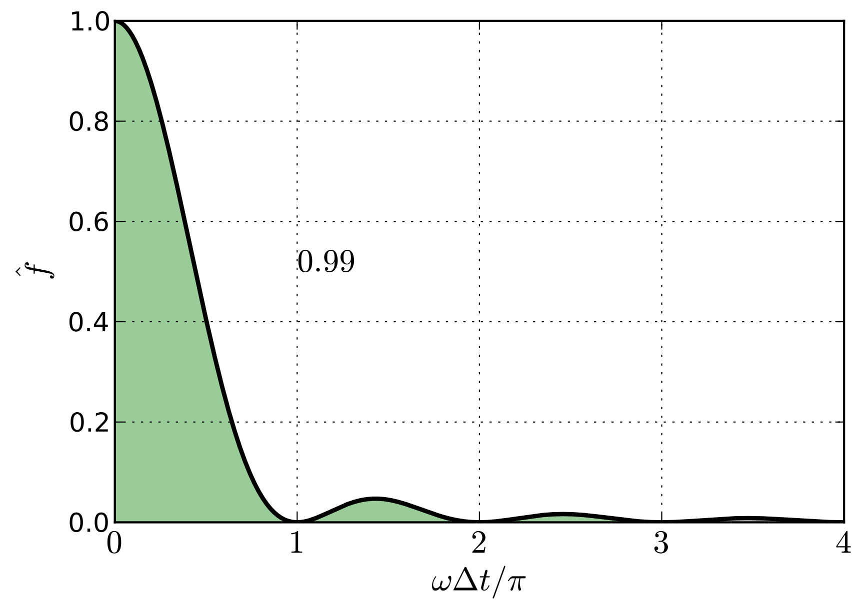
(a) Area under the curve in $[0, 0.5\pi/\Delta t]$



(b) Area under the curve in $[0, \pi/\Delta t]$



(c) Area under the curve in $[0, 2\pi/\Delta t]$



(d) Area under the curve in $[0, 3\pi/\Delta t]$

Figure 5: Energy distribution in the spectrum of the triangle wavelet.