Supplementary Material for "Optimized Auxiliary Particle Filters"

1 THEORETICAL PROPERTIES OF THE OAPF ESTIMATORS

The theoretical properties of the estimators OAPF are analized from the importance sampling perspective. In the case of the mixture proposals ψ_t, we assume that each time t, the support of ψ_t is a superset of the support of $p(\mathbf{x}_t|\mathbf{y}_{1:t})$, i.e., that $\psi_t(\mathbf{x}_t) > 0$ for all $\mathbf{x}_t$ where $p(\mathbf{x}_t|\mathbf{y}_{1:t}) > 0$. Let us define the *partial* normalizing constants as $Z_t \triangleq p(\mathbf{y}_t|\mathbf{y}_{1:t-1})$, the *joint* normalizing constant as $Z_{1:t} \triangleq p(\mathbf{y}_{1:t})$, and also $Z_{t-h:t} \triangleq p(\mathbf{y}_{t-h:t}|\mathbf{y}_{1:t-h-1})$. In the OAPF framework, we can build estimator of those quantities, e.g., the partial estimator $\widehat{Z}_\tau \triangleq \frac{1}{M}\sum_{m=1}^{M} \widetilde{w}_\tau^{(m)}$, the joint estimator $\widehat{Z}_{1:t} = \prod_{\tau=1}^{t} \widehat{Z}_\tau$, and also the estimator $\widehat{Z}_{t-h:t} = \prod_{\tau=t-h}^{t} \widehat{Z}_\tau$, with and the estimator $\widehat{Z}_t \triangleq \frac{1}{M}\sum_{m=1}^{M} \widetilde{w}_t^{(m)}$. We also assume that the estimators of all the partial normalizing constants have finite variance (see for instance [1, 2]). We define the set of weighted samples at time t as $\mathcal{A}_t \triangleq \{\mathbf{x}_t^{(m}, \widetilde{w}_t^{(m}\}_{m=1}^{M}$. In order to avoid ambiguities when evaluating pdfs, we define the functions $g(\mathbf{y}_t|\mathbf{x}_t) \triangleq p(\mathbf{y}_t|\mathbf{x}_t)$, $g(\mathbf{y}_t|\mathbf{x}_{t-1}) \triangleq p(\mathbf{y}_t|\mathbf{x}_{t-1})$, $g(\mathbf{y}_t, \mathbf{x}_t|\mathbf{x}_{t-1}) \triangleq p(\mathbf{y}_t, \mathbf{x}_t|\mathbf{x}_{t-1})$ and $g(\mathbf{y}_{t-h:t}, \mathbf{x}_t|\mathbf{x}_{t-1}) \triangleq p(\mathbf{y}_{t-h:t}, \mathbf{x}_t|\mathbf{x}_{t-1})$.

In the following, we show that OAPF provides an unbiased estimator of the normalizing constant $p(\mathbf{y}_{1:t})$, which follows a proof by induction, in a similar spirit as in [3], but with more generic results. In particular, here the (approximate) filtering distribution is the marginalized version of the one in [3] and is constituted by a mixture in the numerator of the importance weights (see [4] for an explanation). In OAPF the proposal density can be any mixture $\psi_t(\mathbf{x}_t)$ fulfilling the standard regularity conditions described above, hence in the denominator of the importance weights, a second mixture appears. Theorem 1 is here the main result, and is supported by Lemmas 1 and 2 which we present first.

Lemma 1 *We have that*

$$\mathrm{E}\left[\widehat{Z}_t|\mathcal{A}_{t-1}\right] = \sum_{m=1}^{M} w_{t-1}^{(m)} g(\mathbf{y}_t|\mathbf{x}_{t-1}^{(m)}). \tag{1}$$

Proof:

$$
\begin{aligned}
\mathrm{E}\left[\widehat{Z}_t|\mathcal{A}_{t-1}\right] &= \mathrm{E}\left[\tfrac{1}{M}\textstyle\sum_{m=1}^{M} \widetilde{w}_t^{(m)}|\mathcal{A}_{t-1}\right] \\
&= \mathrm{E}\left[\tfrac{1}{M}\textstyle\sum_{m=1}^{M} \tfrac{g(\mathbf{y}_t|\mathbf{x}_t^{(m)})\sum_{j=1}^{M} w_{t-1}^{(j)} f(\mathbf{x}_t^{(m)}|\mathbf{x}_{t-1}^{(j)})}{\psi(\mathbf{x}_t^{(m)})}|\mathcal{A}_{t-1}\right] \\
&= \tfrac{1}{M}\textstyle\sum_{m=1}^{M} \mathrm{E}\left[\tfrac{g(\mathbf{y}_t|\mathbf{x}_t^{(m)})\sum_{j=1}^{M} w_{t-1}^{(j)} f(\mathbf{x}_t^{(m)}|\mathbf{x}_{t-1}^{(j)})}{\psi(\mathbf{x}_t^{(m)})}|\mathcal{A}_{t-1}\right].
\end{aligned}
\tag{2}
$$

Now, since given $\mathcal{A}_{t-1}$ the particles at time t are conditionally independent with pdf $\psi_t(\mathbf{x}_t)$, then we have that the integrals within (2) are identical:

$$
\begin{aligned}
\mathrm{E}\left[\widehat{Z}_t|\mathcal{A}_{t-1}\right] &= \int \tfrac{g(\mathbf{y}_t|\mathbf{x}_t)\sum_{j=1}^{M} w_{t-1}^{(j)} f(\mathbf{x}_t|\mathbf{x}_{t-1}^{(j)})}{\psi(\mathbf{x}_t)}\psi(\mathbf{x}_t)d\mathbf{x}_t \\
&= \int g(\mathbf{y}_t|\mathbf{x}_t)\textstyle\sum_{j=1}^{M} w_{t-1}^{(j)} f(\mathbf{x}_t|\mathbf{x}_{t-1}^{(j)})d\mathbf{x}_t \\
&= \textstyle\sum_{j=1}^{M} w_{t-1}^{(j)} \int g(\mathbf{y}_t, \mathbf{x}_t|\mathbf{x}_{t-1}^{(j)})d\mathbf{x}_t \\
&= \textstyle\sum_{j=1}^{M} w_{t-1}^{(j)} g(\mathbf{y}_t|\mathbf{x}_{t-1}^{(j)}).
\end{aligned}
\tag{3}
$$

$\square$

Lemma 2 *For any $h \in \{1, ..., t-1\}$ we have that*

$$\mathrm{E}\left[\widehat{Z}_{t-h:t}|\mathcal{A}_{t-h-1}\right] = \sum_{m=1}^{M} w_{t-h-1}^{(m)} g(\mathbf{y}_{t-h:t}|\mathbf{x}_{t-h-1}^{(m)}) \tag{4}$$

Proof: We follow a proof by induction. First, note that (4) is true for $h = 0$ due to Lemma 1. Then, we assume that (4) holds for a given h and we will prove that it then holds for $h+1$. Let us start developing the left-hand side of (4) for $h+1$ by first noting that $\widehat{Z}_{t-h-1:t} = \widehat{Z}_{t-h:t}\widehat{Z}_{t-h-1}$. Then,

$$\mathrm{E}\left[\widehat{Z}_{t-h-1:t}|\mathcal{A}_{t-h-2}\right] = \mathrm{E}\left[\mathrm{E}\left[\widehat{Z}_{t-h:t}|\mathcal{A}_{t-h-1}\right]\widehat{Z}_{t-h-1}|\mathcal{A}_{t-h-2}\right] \tag{5}$$

$$= \mathrm{E}\left[\left[\sum_{m=1}^{M} g(\mathbf{y}_{t-h:t}|\mathbf{x}_{t-h-1}^{(m)})w_{t-h-1}^{(m)}\right]\widehat{Z}_{t-h-1}|\mathcal{A}_{t-h-2}\right] \tag{6}$$

$$\tag{7}$$

where we have simply substituted Eq. (4) that we assume to hold for h. Next,

$$\mathrm{E}\left[\widehat{Z}_{t-h-1:t}|\mathcal{A}_{t-h-2}\right] = \mathrm{E}\left[\left[\sum_{m=1}^{M} g(\mathbf{y}_{t-h:t}|\mathbf{x}_{t-h-1}^{(m)})\frac{\widetilde{w}_{t-h-1}^{(m)}}{\sum_{j=1}^{M}\widetilde{w}_{t-h-1}^{(j)}}\right]\frac{1}{M}\sum_{j=1}^{M}\widetilde{w}_{t-h-1}^{(j)}|\mathcal{A}_{t-h-2}\right] \tag{8}$$

$$= \mathrm{E}\left[\frac{1}{M}\sum_{m=1}^{M} g(\mathbf{y}_{t-h:t}|\mathbf{x}_{t-h-1}^{(m)})\frac{g(\mathbf{y}_{t-h-1}|\mathbf{x}_{t-h-1}^{(m)})\sum_{j=1}^{M}w_{t-h-2}^{(j)}f(\mathbf{x}_{t-h-1}^{(m)}|\mathbf{x}_{t-h-2}^{(j)})}{\psi_{t-h-1}(\mathbf{x}_{t-h-1}^{(m)})}|\mathcal{A}_{t-h-2}\right] \tag{9}$$

$$= \frac{1}{M}\sum_{m=1}^{M}\mathrm{E}\left[g(\mathbf{y}_{t-h:t}|\mathbf{x}_{t-h-1}^{(m)})\frac{g(\mathbf{y}_{t-h-1}|\mathbf{x}_{t-h-1}^{(m)})\sum_{j=1}^{M}w_{t-h-2}^{(j)}f(\mathbf{x}_{t-h-1}^{(m)}|\mathbf{x}_{t-h-2}^{(j)})}{\psi(\mathbf{x}_{t-h-1}^{(m)})}|\mathcal{A}_{t-h-2}\right] \tag{10}$$

where we have substituted with the importance weights $\widetilde{w}_{t-h-1}^{(m)}$ of Eq. 7 of the manuscript. Since, given $\mathcal{A}_{t-h-2}$, the particles at time t are conditionally independent with pdf $\psi_{t-h-1}(\mathbf{x}_{t-h-1})$, all M expectations are identical:

$$\mathrm{E}\left[\widehat{Z}_{t-h-1:t}|\mathcal{A}_{t-h-2}\right] = \int g(\mathbf{y}_{t-h:t}|\mathbf{x}_{t-h-1})\frac{g(\mathbf{y}_{t-h-1}|\mathbf{x}_{t-h-1})\sum_{j=1}^{M}w_{t-h-2}^{(j)}f(\mathbf{x}_{t-h-1}|\mathbf{x}_{t-h-2}^{(j)})}{\psi(\mathbf{x}_{t-h-1})}\psi(\mathbf{x}_{t-h-1})d\mathbf{x}_{t-h-1} \tag{11}$$

$$= \int g(\mathbf{y}_{t-h:t}|\mathbf{x}_{t-h-1})g(\mathbf{y}_{t-h-1}|\mathbf{x}_{t-h-1})\sum_{j=1}^{M}w_{t-h-2}^{(j)}f(\mathbf{x}_{t-h-1}|\mathbf{x}_{t-h-2}^{(j)})d\mathbf{x}_{t-h-1} \tag{12}$$

$$= \int g(\mathbf{y}_{t-h-1:t}|\mathbf{x}_{t-h-1})\sum_{j=1}^{M}w_{t-h-2}^{(j)}f(\mathbf{x}_{t-h-1}|\mathbf{x}_{t-h-2}^{(j)})d\mathbf{x}_{t-h-1} \tag{13}$$

Step (12) to (13) is justified since $\mathbf{y}_{t-h:t} \perp\!\!\!\perp \mathbf{y}_{t-h-1}|\mathbf{x}_{t-h-1}$, so we can replace $g(\mathbf{y}_{t-h:t}|\mathbf{x}_{t-h-1})$ in 12 with $g(\mathbf{y}_{t-h:t}|\mathbf{y}_{t-h-1}, \mathbf{x}_{t-h-1})$ and then $g(\mathbf{y}_{t-h-1:t}|\mathbf{x}_{t-h-1}) = g(\mathbf{y}_{t-h:t}|\mathbf{x}_{t-h-1})g(\mathbf{y}_{t-h-1}|\mathbf{x}_{t-h-1})$ follows by the chain rule. Next,

$$= \sum_{j=1}^{M}w_{t-h-2}^{(j)}\int g(\mathbf{y}_{t-h-1:t}, \mathbf{x}_{t-h-1}|\mathbf{x}_{t-h-2}^{(j)})d\mathbf{x}_{t-h-1} \tag{14}$$

$$= \sum_{j=1}^{M}w_{t-h-2}^{(j)}g(\mathbf{y}_{t-h-1:t}|\mathbf{x}_{t-h-2}^{(j)}) \tag{15}$$

$$\tag{16}$$

which is the right-hand side of (4). $\qquad\square$

Theorem 1 *The OAPF estimator of the normalizing constant is unbiased, i.e.,* $\mathrm{E}[\widehat{Z}_{1:t}] = p(\mathbf{y}_{1:t})$.

Proof: The unbiasedness is a consequence of Lemma 2 with $h = t - 1$. □

Now we look at the variance of the normalizing constant estimators. First, we establish a superiority in performance (i.e., equal or less variance) of the OAPF importance weights. This result is also used below to prove the convergence of the estimators by standard results in particle filtering.

Let us particularize importance weights in OAPF for the case with $K = M$ as

$$\widetilde{w}_t^{(m)} = \frac{g(\mathbf{y}_t | \mathbf{x}_t^{(m)}) \sum_{i=1}^M w_{t-1}^{(i)} f(\mathbf{x}_t^{(m)} | \mathbf{x}_{t-1}^{(i)})}{\sum_{i=1}^M \lambda_t^{(i)} q_t^{(i)}(\mathbf{x}_t^{(m)} | \bar{\mathbf{x}}_{t-1}^{(i)})}. \tag{17}$$

We also consider the generalized APF weights given by

$$\widetilde{v}_t^{(m)} = \frac{g(\mathbf{y}_t | \mathbf{x}_t^{(m)}) w_{t-1}^{(m)} f(\mathbf{x}_t^{(m)} | \mathbf{x}_{t-1}^{(m)})}{\lambda_t^{(m)} q_t^{(m)}(\mathbf{x}_t^{(m)} | \bar{\mathbf{x}}_{t-1}^{(m)})}. \tag{18}$$

These are generalized in the sense that the concrete APF described in the main paper is obtained by setting $\lambda_t^{(m)} \propto w_{t-1}^{(m)} g(\mathbf{y}_t | \boldsymbol{\mu}_t^{(m)})$ and propagating particles with transition kernels $f(\cdot)$, thus our following discussion holds for any choice of $\lambda_t^{(m)}$.

Lemma 3 *The conditional variance of* $\widehat{Z}_t^{OAPF}$ *using the OAPF weights in* (17) *is always less or equal than the same estimator* $\widehat{Z}_t^{APF}$ *using the APF weights in* (18).

Proof: First, note that $\widetilde{v}_t^{(m)}$ can be interpreted as an importance weight in an extended space on $\mathbf{x}_t$ and the auxiliary variable m (see for instance [4, Section 3.1] and [5, 6]). Next, $\widetilde{w}_t^{(m)}$ can be interpreted as a version of $\widetilde{v}_t^{(m)}$ where both in the numerator (approximate filtering pdf) and denominator (proposal pdf), the auxiliary variable has been marginalized. Then, the variance inequality for each importance weight holds from the application of the variance decomposition lemma (also known as law of total variance). This proof generalizes the result in [4] for any set of mixture weights $\{\lambda_t^{(m)}\}_{m=1}^M$, with $\sum_{j=1}^M \lambda_t^{(j)}$ and $\lambda_t^{(m)} \geq 0$, for all m. Finally, since both $\hat{Z}_t^{\mathrm{OAPF}}$ and $\hat{Z}_t^{\mathrm{APF}}$ are constructed as the average of the OAPF and APF weights, respectively, the conditional variance of $\hat{Z}_t^{\mathrm{OAPF}}$ is necessarily upper-bounded by that of $\hat{Z}_t^{\mathrm{APF}}$. □

We now address the consistency of the normalizing constant, $\widehat{Z}_{1:t}$, and the self-normalized IS (SNIS) estimator $\widehat{I}(h_t) = \sum_{m=1}^M w_t^{(m)} h_t(\mathbf{x}_t^{(m)})$.

Corollary 1 *The OAPF estimator of the normalizing constant* $\widehat{Z}_{1:t}$ *and the SNIS estimator* $\widehat{I}(h_t)$ *are consistent, i.e.,* $\lim_{M \to \infty} \widehat{Z}_{1:t} = p(\mathbf{y}_{1:t})$ *and* $\lim_{M \to \infty} \widehat{I}(h_t) = I(h_t)$ *a.s. (almost surely) for a finite* t.

Proof: The consistency of $\widehat{Z}_{1:t}$ is a consequence of its unbiasedness, proved in Theorem 1, and the variance inequality in Lemma 3, which ensures the variance convergence to zero a.s. when $N \to \infty$ since the APF, which upper-bounds its variance, is also consistent [7, Section 3.6]. A similar argumentation can be done for the SNIS estimator $\widehat{I}(h_t)$. Note that the SNIS estimator can be re-expressed as $\widehat{I}(h_t) = \sum_{m=1}^M \frac{\widetilde{w}_t^{(m)}}{M\widehat{Z}_t} h_t(\mathbf{x}_t^{(m)}) = \frac{1}{M} \sum_{m=1}^M \frac{\widetilde{w}_t^{(m)}}{\widehat{Z}_t} h_t(\mathbf{x}_t^{(m)})$. Since $\widehat{Z}_t$ is a consistent estimator of $p(\mathbf{y}_t | \mathbf{y}_{1:t-1})$, the denominator converges to $p(\mathbf{y}_t | \mathbf{y}_{1:t-1})$ while the numerator converges to $p(\mathbf{y}_t | \mathbf{y}_{1:t-1}) I(h_t)$, when $N \to \infty$. Therefore, the ratio converges to $I(h_t)$ a.s. □

2 ADDITIONAL EXPERIMENTS AND RESULTS

2.1 Experiment 1

We provide all necessary parameters to reproduce Figure 1 in the main paper. We recall that in this toy example we do the Bayesian recursion from $t-1$ to t with $M = 4$ particles. In Figure 1(a), we have set the particles $\{\bar{\mathbf{x}}_{t-1}^{(m)}\}_{m=1}^{M=4} = \{2, 2.5, 3, 3.5\}$, the normalized weights $\{3/10, 3/10, 1/5, 1/5\}$, likelihood centered at 3,

and $\sigma_{\text{lik}} = 0.8$, and $\sigma_{\text{kern}} = 0.5$.

In Figure 1(b), $\{\bar{\mathbf{x}}_{t-1}^{(m)}\}_{m=1}^{M=4} = \{2, 2.5, 5, 5.5\}$, the normalized weights are $\{7/22, 1/11, 1/2, 1/11\}$, the likelihood is centered at 3.5, and $\sigma_{\text{lik}} = 1.2$, and $\sigma_{\text{kern}} = 0.5$. The proposals of all algorithms are then calculated as:

$$\sum_{m=1}^{4} \lambda_t^{(m)} f(\mathbf{x}_t | \bar{\mathbf{x}}_{t-1}^{(m)}), \tag{19}$$

where the mixture weights $\lambda_t^{(m)}$ for BPF are $w_{t-1}^{(m)}$, for APF are $\propto w_{t-1}^{(m)} g(\mathbf{y}_t | \boldsymbol{\mu}_t^{(m)})$, for IAPF $\propto g(\mathbf{y}_t | \boldsymbol{\mu}_t^{(m)}) \sum_{m=1}^{M} w_{t-1}^{(m)} f(\boldsymbol{\mu}_t^{(m)} | \bar{\mathbf{x}}_{t-1}^{(m)}) / \sum_{m=1}^{M} f(\boldsymbol{\mu}_t^{(m)} | \bar{\mathbf{x}}_{t-1}^{(m)})$ and finally for OAPF they are the solution to the NNLS optimization problem.

As specified in the main paper and can be seen from 19, we used transition kernels as proposal kernels for OAPF. Moreover, we used the centers of the transition kernels $\boldsymbol{\mu}_t^{(m)}$ as evaluation points, which in this case they correspond to the resampled particles $\{\bar{\mathbf{x}}_{t-1}^{(m)}\}_{m=1}^{M=4}$.

2.2 Experiment 2

We provide the equations of the linear dynamical model (LDM) and full results for Experiment 2 in the main paper. The model is given by

$$p(\mathbf{x}_0) = \mathcal{N}_{\mathbf{x}_0}(\mathbf{m}_0, \boldsymbol{\Sigma}_0) \tag{20}$$

$$f(\mathbf{x}_t \mid \mathbf{x}_{t-1}) = \mathcal{N}_{\mathbf{x}_t}(\mathbf{A}\mathbf{x}_{t-1} + \mathbf{c}, \mathbf{R}) \tag{21}$$

$$g(\mathbf{y}_t \mid \mathbf{x}_t) = \mathcal{N}_{\mathbf{y}_t}(\mathbf{C}\mathbf{x}_t + \mathbf{g}, \mathbf{Q}). \tag{22}$$

The posterior filtering distribution can be computed in closed form via the Kalman filter:

$$p(\mathbf{x}_t \mid \mathbf{y}_{1:t}) = \mathcal{N}_{\mathbf{x}_t}(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t) \tag{23}$$

$$\boldsymbol{\mu}_t = \overline{\boldsymbol{\mu}}_t + \mathbf{K}(\mathbf{y}_t - \mathbf{C}\overline{\boldsymbol{\mu}}_t - \mathbf{g}) \tag{24}$$

$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{KC})\overline{\boldsymbol{\Sigma}}_t \tag{25}$$

$$\overline{\boldsymbol{\mu}}_t = \mathbf{A}\boldsymbol{\mu}_{t-1} + \mathbf{c} \tag{26}$$

$$\overline{\boldsymbol{\Sigma}}_t = \mathbf{A}\boldsymbol{\Sigma}_{t-1}\mathbf{A}^\top + \mathbf{R} \tag{27}$$

$$\mathbf{K} = \overline{\boldsymbol{\Sigma}}_t\mathbf{C}^\top (\mathbf{C}\overline{\boldsymbol{\Sigma}}_t\mathbf{C}^\top + \mathbf{Q})^{-1}. \tag{28}$$

Moreover, $p(\mathbf{y}_{1:t})$ can also be computed in closed form from $p(\mathbf{y}_t | \mathbf{y}_{1:t-1})$. For numerical stability, one computes $\log p(\mathbf{y}_{1:t})$ and $\log p(\mathbf{y}_t | \mathbf{y}_{1:t-1})$, which are given by:

$$\log p(\mathbf{y}_{1:t}) = \log p(\mathbf{y}_1) + \sum_{\tau=2}^{t} \log p(\mathbf{y}_\tau | \mathbf{y}_{1:\tau-1}) \tag{29}$$

$$\log p(\mathbf{y}_\tau | \mathbf{y}_{1:\tau-1}) = -\frac{1}{2} \left[\log(|\mathbf{C}\overline{\boldsymbol{\Sigma}}_\tau\mathbf{C}^\top + \mathbf{Q}|) + (\mathbf{y}_\tau - \mathbf{C}\overline{\boldsymbol{\mu}}_\tau - \mathbf{g})^{\text{T}}(\mathbf{C}\overline{\boldsymbol{\Sigma}}_\tau\mathbf{C}^\top + \mathbf{Q})^{-1}(\mathbf{y}_\tau - \mathbf{C}\overline{\boldsymbol{\mu}}_\tau - \mathbf{g}) + d_{\mathbf{y}_\tau} \ln(2\pi) \right] \tag{30}$$

$$\log p(\mathbf{y}_1) = -\frac{1}{2} \left[\log(|\mathbf{C}\boldsymbol{\Sigma}_0\mathbf{C}^\top + \mathbf{Q}|) + (\mathbf{y}_\tau - \mathbf{C}\overline{\boldsymbol{\mu}}_1 - \mathbf{g})^{\text{T}}(\mathbf{C}\boldsymbol{\Sigma}_0\mathbf{C}^\top + \mathbf{Q})^{-1}(\mathbf{y}_\tau - \mathbf{C}\overline{\boldsymbol{\mu}}_1 - \mathbf{g}) + d_{\mathbf{y}_1} \ln(2\pi) \right]. \tag{31}$$

We set $\mathbf{R} = 5\mathbf{I}$ and $\mathbf{Q} = 2.5\mathbf{I}$. In this setting, the kernels overlap and the observations are very informative. The setting is particularly advantageous for IAPF, and hence it is more difficult to beat its performance. Moreover, $\mathbf{A} = \frac{1}{2}\mathbf{I}$ and $\mathbf{C} = \frac{1}{2}\mathbf{I}$. For $d_{\mathbf{x}_t} = 2$, then $\mathbf{c} = \mathbf{g} = (-2, 2)^\top$; for $d_{\mathbf{x}_t} = 5$, then $\mathbf{c} = \mathbf{g} = (-2, 2, -2, 2, -2)^\top$; similarly defined for $d_{\mathbf{x}_t} = 10$. The results for all $d_{\mathbf{x}_t}$ are shown in Table 1. We recall that in this experiment, OAPF ran with $K = 5, E = 5$. This implies large computational savings with respect to the IAPF (or similarly to any other algorithm which uses the full mixture in the denominator of the importance weights).

2.3 Experiment 3

In this experiment, we have used the standard parameters for the Lorenz model given by $(s, r, b) = (10, 28, 2.667)$. We set transition and observation noise as independent standard normally distributed random variables. In

Table 1: Full results for Experiment 2 in main paper. The Table shows normalized MSE as defined in the paper, with standard errors over 100 Monte Carlo runs. Recall that whenever $d_{\mathbf{x}_t} \in \{2,5\}$ then $M = 100$ and when $d_{\mathbf{x}_t} = 10$ then $M = 1000$.

Method	$d_{\mathbf{x}_t} = 2$	$d_{\mathbf{x}_t} = 5$	$d_{\mathbf{x}_t} = 10$
BPF	$3.19 \cdot 10^{-7} \pm 3.14 \cdot 10^{-8}$	$5.09 \cdot 10^{-7} \pm 4.40 \cdot 10^{-8}$	$1.539 \cdot 10^{-7} \pm 1.267 \cdot 10^{-8}$
APF	$3.51 \cdot 10^{-7} \pm 3.88 \cdot 10^{-8}$	$4.68 \cdot 10^{-7} \pm 4.49 \cdot 10^{-8}$	$1.333 \cdot 10^{-7} \pm 1.224 \cdot 10^{-8}$
IAPF	$2.15 \cdot 10^{-7} \pm 2.25 \cdot 10^{-8}$	$1.63 \cdot 10^{-7} \pm 1.58 \cdot 10^{-8}$	$6.330 \cdot 10^{-8} \pm 6.204 \cdot 10^{-9}$
OAPF	$\mathbf{1.35 \cdot 10^{-7} \pm 1.23 \cdot 10^{-8}}$	$\mathbf{9.67 \cdot 10^{-8} \pm 9.03 \cdot 10^{-9}}$	$\mathbf{4.771 \cdot 10^{-8} \pm 5.329 \cdot 10^{-9}}$

Table 2: Results with additional parameters for Experiment 4. Note that when $d_{\mathbf{x}_t} = 10$ then $M = 1000$. Averaged ESS and standard errors over 100 Monte Carlo runs.
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Method	$d_{\mathbf{x}_t} = 2$	$d_{\mathbf{x}_t} = 5$	$d_{\mathbf{x}_t} = 10$
BPF	63.5 ± 0.2	33.5 ± 0.2	108.7 ± 0.8
APF	63.5 ± 0.2	34.5 ± 0.2	107.2 ± 1.0
IAPF	73.0 ± 0.1	44.9 ± 0.2	203.5 ± 0.9
OAPF	$\mathbf{88.3 \pm 0.2}$	$\mathbf{63.5 \pm 0.2}$	$\mathbf{366.2 \pm 1.8}$

Figure 1, we show visually, as stated in the main paper, how a small change in Δt can lead to very different trajectories of $\mathbf{x}_t$. The sensitivity to Δt, to the initialization, and even to the parameters (s, r, b), jointly with the strong non-linearity of the generated trajectories, make the Lorenz model particularly challenging.
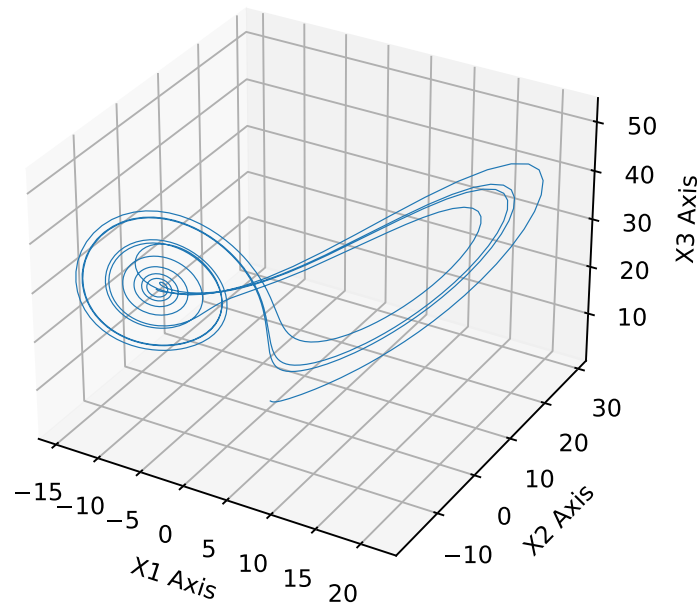
2.4 Experiment 4

In this experiment, we have used a challenging multivariate stochastic volatility model, which is common in related works (see for instance [8]). Additional results with parameters $\mathbf{m} = \mathbf{0}, \mathbf{U}_0 = \mathbf{I}, \mathbf{U} = \mathbf{I}, \phi = \frac{1}{2}\mathbf{1}$ are shown in Table 2.

Lorenz 63 noiseless trajectory
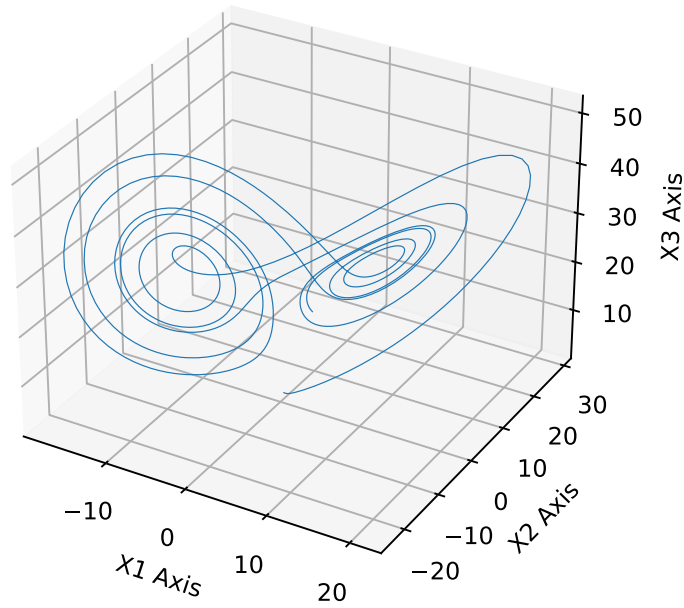
(a)

Lorenz 63 noiseless trajectory

(b)

Figure 1: In this figure, we show the noiseless versions of a trajectory for $\Delta t = 0.01$ (a) and $\Delta t = 0.008$ (b) to emphasize how different trajectories can be in a Lorenz 63 model even with small parameter changes. Note that particle filters will have to deal with both transition noise and observation noise.

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