$$input \quad X \sim Ber(p)$$

$$output \quad Y \sim Ber(q)$$

$$\varepsilon \quad error \quad probability$$

$$q = p + \varepsilon - 2\varepsilon p$$

**1. Find, for a binary channel with error probability $\varepsilon$, the value of p that maximises the channel capacity $C = max_p(I(X,Y))$.**

$$H(Y) = -q\log_2 q - (1-q)\log_2(1-q)$$

$$H_0(Y) = H_1(Y) = -\varepsilon\log_2\varepsilon - (1-\varepsilon)\log_2(1-\varepsilon)$$

$$H_X(Y) = pH_0(Y) + (1-p)H_1(Y) = H_0(Y)(1-p+p) = H_0(Y) = -\varepsilon\log_2\varepsilon - (1-\varepsilon)\log_2(1-\varepsilon)$$

$$I(X,Y) = H(Y) - H_X(Y)$$

Since $H_X(Y)$ only depends on $\varepsilon$ and not on $p$, to maximise $I(X,Y)$ it is sufficient to maximise $H(Y)$.

The principle of maximum entropy states that the entropy is maximum when there is a uniform distribution, therefore we chose $q = 1/2$:

$$q = \frac{1}{2} \implies p + \varepsilon - 2\varepsilon p = \frac{1}{2}$$

$$p(1-2\varepsilon) + \varepsilon = \frac{1}{2}$$

$$p = \frac{1-2\varepsilon}{2} * \frac{1}{1-2\varepsilon} = \frac{1}{2} \; if \;\; \varepsilon \neq \frac{1}{2}$$

If $\varepsilon \neq \frac{1}{2}$, $p = \frac{1}{2}$ maximises the capacity of the binary channel:

$$H(Y) = \log_2 2 = 1$$

$$I(X,Y) = H(Y) - H_X(Y) = 1 + \varepsilon\log_2\varepsilon + (1-\varepsilon)\log_2(1-\varepsilon)$$

If $\varepsilon = \frac{1}{2}$:

$$q = p + \frac{1}{2} - p = \frac{1}{2}$$

$$H(Y) = 1$$

$$H_X(Y) = 1$$

$$I(X,Y) = H(Y) - H_X(Y) = 1 - 1 = 0$$

$X$ and $Y$ act like independent random variables, therefore their joint information is zero if $\varepsilon = \frac{1}{2}$.