Suppose that T has density $f(x) = C(n,m)x^n(1-x)^m\mathbf{1}_{(0,1)}(x)$ for $n,m \in \mathbb{N}$.

i. Compute $C(n,m)$

We need $f(x)$ to be a probability density function:

$$f(x) \geq 0: \quad true \quad \forall\, x \in (0,1)$$

We now need to impose that:

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^1 C(n,m)x^n(1-x)^m dx$$

Using:

$$(1-x)^m = \sum_{k=0}^{m} \binom{m}{k}(-1)^k(x)^k$$

We write:

$$C(n,m)\int_0^1 \sum_{k=0}^{m} \binom{m}{k}(-1)^k x^{m+k}dx =$$

$$= C(n,m)\sum_{k=0}^{m} \binom{m}{k}(-1)^k \int_0^1 x^{n+k}dx =$$

$$= C(n,m)\sum_{k=0}^{m} \binom{m}{k}(-1)^k \left[\frac{x^{n+k+1}}{n+k+1}\right]_0^1 dx =$$

$$= C(n,m)\sum_{k=0}^{m} \binom{m}{k}\frac{(-1)^k}{n+k+1}$$

$$C(n,m)\sum_{k=0}^{m} \binom{m}{k}\frac{(-1)^k}{n+k+1} = 1 \implies C(n,m) = \left[\sum_{k=0}^{m} \binom{m}{k}\frac{(-1)^k}{n+k+1}\right]^{-1}$$

ii. Compute the mean value of T for every n,m.

$$\mathbb{E}(X) = \int_0^1 xf(x)dx$$

Using the computations above:

$$\int_0^1 xf(x)dx = C(n,m)\sum_{k=0}^{m} \binom{m}{k}(-1)^k \int_0^1 x^{n+k+1}dx =$$

$$= C(n,m)\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n+k+2} = \left[\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n+k+1}\right]^{-1} * \sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n+k+2}$$

$$\mathbb{E}(X) = \left[\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n+k+1}\right]^{-1} * \sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{n+k+2}$$