Recall the definition of the Gamma function:

$$\Gamma(n) = \int_0^{+\infty} t^{n-1}e^{-t}dt = (n-1)!$$

Show that for a fixed integer $n \geq 1$ the function:

$$f(t) = \frac{1}{(n-1)!}t^{n-1}e^{-t} \quad for\ t > 0$$

is a probability density function.
Let X be a random variable with that density, compute the mean and variance of X.

1. Show that for a fixed integer $n \geq 1$, $f(x)$ is a p.d.f.

$f(t)$ is non-negative in its domain:

$$\forall\, t > 0 \quad f(x) \geq 0$$

Its integral from 0 to $+\infty$ should be 1:

$$\int_0^{+\infty} \frac{1}{(n-1)!}t^{n-1}e^{-t}dt = \frac{\Gamma(n)}{(n-1)!} = \frac{(n-1)!}{(n-1)!} = 1$$

$$\forall t > 0\; f(x) \geq 0\ \ and \int_0^{+\infty} f(t)dt = 1 \implies f(t)\ \ is\ \ a\ \ p.d.f.$$

2. Compute the mean and variance of X.

$$X \sim f(t)$$

$$\mathbb{E}[X] = \int_0^{+\infty} \frac{1}{(n-1)!}t^{n}e^{-t}dt = \frac{\Gamma(n+1)}{(n-1)!} = \frac{n!}{(n-1)!} = n$$

$$\mathbb{E}[X^2] = \int_0^{+\infty} \frac{1}{(n-1)!}t^{n+1}e^{-t}dt = \frac{\Gamma(n+2)}{(n-1)!} = \frac{(n+1)!}{(n-1)!} = n(n+1) = n^2+n$$

$$V(X) = E[X^2] - (E[X])^2 = n^2 + n - n^2 = n$$

The mean is n and the variance is n.