$$X = "\#heads\ when\ tossing\ two\ fair\ coins"$$

$$R_X = \{0, 1, 2\}$$

1. Construct the probability distribution of X.

$$p_X(k) = \begin{cases} 1/4 & k = 0 \\ 1/2 & k = 1 \\ 1/4 & k = 2 \end{cases}$$

2. Compute the average and the standard deviation of X.

$$\mu = \sum_{x \in R} x p_X(x) = \frac{1}{2} + \frac{1}{2} = 1$$

$$V(X) = \sum_{x \in r} (x - \mu)^2 p_X(x) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\sigma_X = \sqrt{V(X)} = \frac{\sqrt{2}}{2}$$

3. Now suppose you toss a fair coin repeatedly until you get a tail. Let Y be the number of heads obtained before the first tail. Construct the probability distribution of Y.

$$Y \sim Geo\left(\frac{1}{2}\right)$$

$$R_Y = \mathbf{N}$$

$$p_Y(k) = (1 - p)^k p$$

since we know $p = 1/2$:

$$p_Y(k) = \left(\frac{1}{2}\right)^{k+1}$$

0.0.1 Compute the average and the standard deviation of Y and compare them with those of X.

$$\mu_Y = \sum_{i=1}^{\infty} i * p_Y(i) = \sum_{i=1}^{\infty} i * \left(\frac{1}{2}\right)^{i+1} =$$

$$= \frac{1}{2} \sum_{i=1}^{\infty} \frac{i}{2^i} =$$

Since the power series $\sum_{i=1}^{\infty} \frac{i}{2^i} = 2$:

$$\mu_Y = \frac{1}{2} * 2 = 1$$

$$V(Y) = \sum_{i=0}^{\infty} (i - \mu)^2 p_Y(i) = \frac{1}{2} \sum_{i=0}^{\infty} \frac{i(i-1)^2}{2^i} =$$

$$= \frac{1}{2} \sum_{i=0}^{\infty} \frac{i^3 - 2i^2 + i}{2^i} =$$

$$= \frac{1}{2} \left(\sum_{i=0}^{\infty} \frac{i^3}{2^i} - 2 \sum_{i=0}^{\infty} \frac{i^2}{2^i} + \sum_{i=0}^{\infty} \frac{i}{2^i} \right)$$

$$\sum_{i=0}^{\infty} \frac{i^3}{2^i} = \frac{\frac{1}{2} \left(1 + 4 * \frac{1}{2} + \left(\frac{1}{2} \right)^2 \right)}{\left(1 - \frac{1}{2} \right)^4} = 26$$

$$\sum_{i=0}^{\infty} \frac{i^2}{2^i} = \frac{\frac{1}{2} \left(1 + \frac{1}{2} \right)}{\left(1 - \frac{1}{2} \right)^3} = 6$$

$$\sum_{i=0}^{\infty} \frac{i}{2^i} = 2$$

Then:

$$V(Y) = \frac{1}{2} (26 - 2 * 6 + 2) = 8$$

$$\sigma_Y = \sqrt{V(Y)} = \sqrt{8} = 2\sqrt{2}$$

The average of Y is equal to the average of X, while the standard deviation of Y is 4 times the one of X:

$$\mu_X = \mu_Y \qquad \sigma_Y = 4\sigma_X$$