Orbit calculations using the Runge-Kutta method

Date: 18/03/24; Student Nr: 217337

Runge-Kutta method implementation

Part a)

In the first part, the orbit of a rocket going around the Earth is calculated. Hereby the Earth is treated as stationary and as having a mass much larger than the rocket. According to Newton's law of gravitation, the rocket's equation of motion is following:

$$m\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{mM_eG}{|\boldsymbol{r}|^3}\boldsymbol{r}$$

Whereby m is its mass and M_e is the mass of the Earth.

The 4th order Runge-Kutta method produces following solutions for its position and velocity:

$$\boldsymbol{r}_n = \boldsymbol{r}_{n-1} + \frac{h}{6}(\boldsymbol{k}_{r1} + \boldsymbol{k}_{r2} + \boldsymbol{k}_{r3} + \boldsymbol{k}_{r4})$$

$$\boldsymbol{v}_n = \boldsymbol{v}_{n-1} + \frac{h}{6}(\boldsymbol{k}_{v1} + \boldsymbol{k}_{v2} + \boldsymbol{k}_{v3} + \boldsymbol{k}_{v4})$$

Whereby h is the timestep and the k's are calculated (in the order given) as:

$$\boldsymbol{k}_{r1} = f(v_{n-1})$$

$$\boldsymbol{k}_{v1} = f2(r_{n-1})$$

$$\boldsymbol{k}_{r2} = f1\left(\boldsymbol{v}_{n-1} + h\frac{\boldsymbol{k}_{v1}}{2}\right)$$

$$\boldsymbol{k}_{v2} = f2\left(\boldsymbol{r}_{n-1} + h\frac{\boldsymbol{k}_{r1}}{2}\right)$$

$$\boldsymbol{k}_{r3} = f1\left(\boldsymbol{v}_{n-1} + h\frac{\boldsymbol{k}_{v2}}{2}\right)$$

$$\boldsymbol{k}_{v3} = f2\left(\boldsymbol{r}_{n-1} + h\frac{\boldsymbol{k}_{r2}}{2}\right)$$

$$\boldsymbol{k}_{r4} = f1(\boldsymbol{v}_{n-1} + h\boldsymbol{k}_{v3})$$

$$\boldsymbol{k}_{v4} = f2(\boldsymbol{r}_{n-1} + h\,\boldsymbol{k}_{r3})$$

With:

$$f1(\boldsymbol{v}) = \boldsymbol{v}$$

$$f2(\boldsymbol{r}) = \frac{d\boldsymbol{v}}{dt} = -\frac{M_eG}{|\boldsymbol{r}|^3}\boldsymbol{r}$$

Part b)

In this part, the orbit of a rocket going around a stationary Earth *and* a stationary Moon is calculated.

The Runge-Kutta method is used just as in part a), except the rocket's equation of motion now includes the combined gravitational effects of both the Earth and the Moon, changing f2(**r**) to:

$$f2(\boldsymbol{r}) = \frac{d\boldsymbol{v}}{dt} = -\frac{M_e G}{|\boldsymbol{r}_e|^3}\boldsymbol{r}_e - \frac{M_m G}{|\boldsymbol{r}_m|^3}\boldsymbol{r}_m$$

Where M_m is the mass of the Moon, and:

$$\boldsymbol{r}_e = \boldsymbol{r} - \boldsymbol{R}_e$$

$$\boldsymbol{r}_m = \boldsymbol{r} - \boldsymbol{R}_m$$

With $\boldsymbol{R}_e$and $\boldsymbol{R}_m$ being the positions of the Earth and the Moon, respectively.

Part c)

Here the orbits of the Earth and the Moon resulting from their gravitational effects on one another are calculated.

This produces two Runge-Kutta "blocks" (each with eight equations for the k's) such as the ones in a) and b) - one for the Earth, and one for the Moon.

Earth's equation of motion is now a function of both the Earth's and the Moon's position vectors, turning the function f2 into:

$$f2_{earth(\boldsymbol{r}_e,\boldsymbol{r}_m)} = \frac{d\boldsymbol{v}_e}{dt} = -\frac{M_m G}{|\boldsymbol{r}_{\boldsymbol{em}}|^3}\boldsymbol{r}_{\boldsymbol{em}}$$

with

$$\boldsymbol{r}_{\boldsymbol{em}} = \boldsymbol{r}_e - \boldsymbol{r}_m$$

Equally, from the Moon's equation of motion we get:

$$f2_{moon(\boldsymbol{r}_e,\boldsymbol{r}_m)} = \frac{d\boldsymbol{v}_m}{dt} = -\frac{M_e G}{|\boldsymbol{r}_{\boldsymbol{me}}|^3}\boldsymbol{r}_{\boldsymbol{me}}$$

with

$$\boldsymbol{r}_{\boldsymbol{me}} = \boldsymbol{r}_m - \boldsymbol{r}_e$$

Which with the same implementation of the method as in a) solve for the positions of the Earth and Moon over time.

Part d)

In the last part, the gravitational effects of a stationary Sun are added into the Earth-Moon scenario of part c).

This changes $f2_{earth(\boldsymbol{r_e},\boldsymbol{r_m})}$ into:

$$f2_{earth(\boldsymbol{r_e},\boldsymbol{r_m})} = \frac{d\boldsymbol{v}_e}{dt} = -\frac{M_m G}{|\boldsymbol{r_1}|^3}\boldsymbol{r_1} - \frac{M_s G}{|\boldsymbol{r_{es}}|^3}\boldsymbol{r_{es}}$$

where

$$\boldsymbol{r_{em}} = \boldsymbol{r}_e - \boldsymbol{r}_m$$

$$\boldsymbol{r_{es}} = \boldsymbol{r}_e - \boldsymbol{R}_s$$

And $f2_{moon(\boldsymbol{r_e},\boldsymbol{r_m})}$ into:

$$f2_{moon(\boldsymbol{r_e},\boldsymbol{r_m})} = \frac{d\boldsymbol{v}_m}{dt} = -\frac{M_e G}{|\boldsymbol{r_{me}}|^3}\boldsymbol{r_{me}} - \frac{M_s G}{|\boldsymbol{r_{ms}}|^3}\boldsymbol{r_{ms}}$$

where

$$\boldsymbol{r_{em}} = \boldsymbol{r}_m - \boldsymbol{r}_e$$

$$\boldsymbol{r_{ms}} = \boldsymbol{r}_m - \boldsymbol{R}_s$$

Part a) – Rocket around Earth

Analysis Method

In this part both circular and elliptical orbits are calculated. The code (here and in the other parts) obtains circular orbits by implementing following equation for the velocity of the rocket, given an initial distance in x of x0:

$$\boldsymbol{v_{y\_circular}} = \sqrt{\frac{G * M_e}{|x0|}}$$

Additionally, the orbital plots were animated to visualise the speed of the rocket.

Their orbital period is calculated by indexing the troughs in the values of (Rocket's position – Rocket's initial position). This is done to also obtain a list of the difference values and an additional measure of accuracy, as in a completely accurate calculation of a circular orbit one would expect no deviation from the initial position values.

The accuracy of the Runge Kutta method is mainly calculated by taking into account the expected conservation of energy. Hereby the difference of all total energy values over time from the initial total energy value is calculated, and the last value additionally returned as it represents the accumulated error over time.

Finally, the effect of the timestep on the errors is analysed.

Results

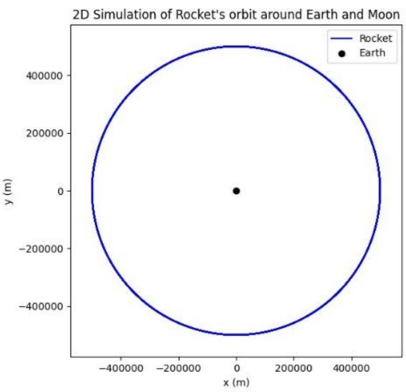
<u>Parameters:</u>

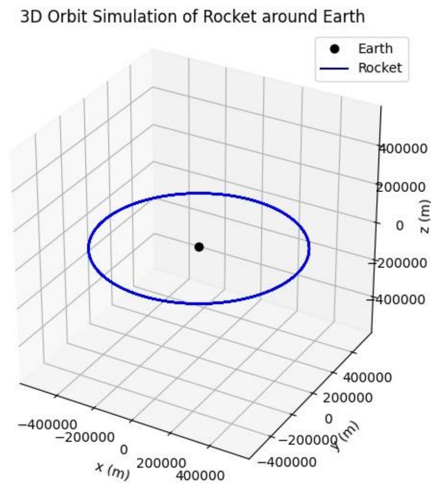
Initial rocket position (x, y, z) (m): (-5e5, 0 0)

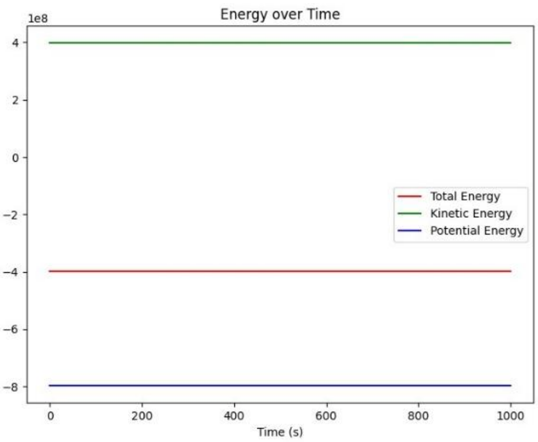
Earth's position (x, y, z) (m): (0, 0 0)

Time step h: 0.1

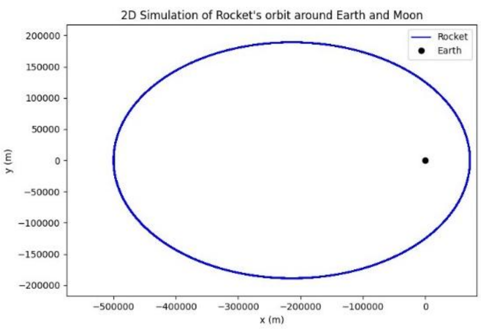
	Initial rocket velocity (vx, vy, vz) (m/s)
Circular orbit	(0, 28220.52, 0)
Elliptical orbit	(0, 0.5*vy-circular, 0)

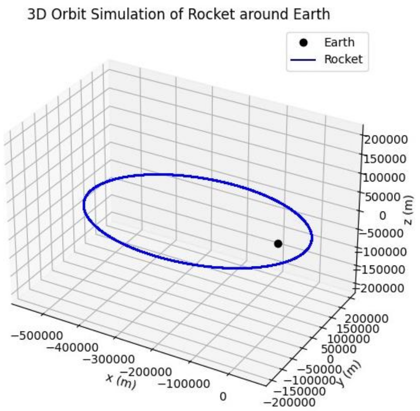


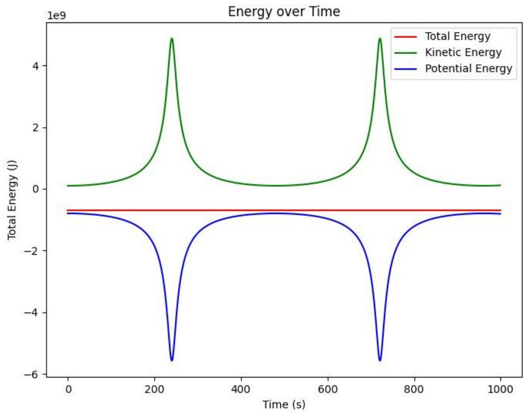




Figs. 1-3: Circular orbit





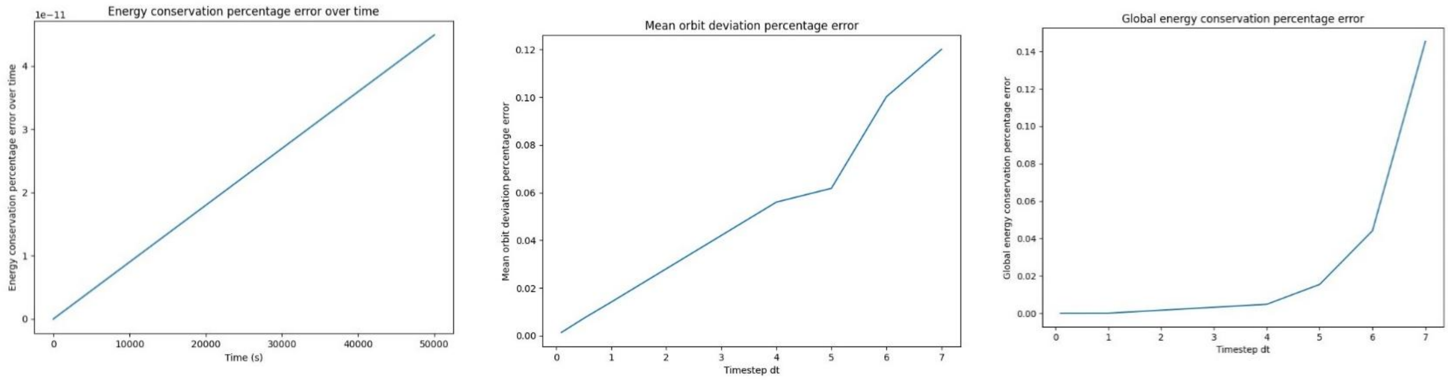


Figs. 4-6: Elliptical orbit

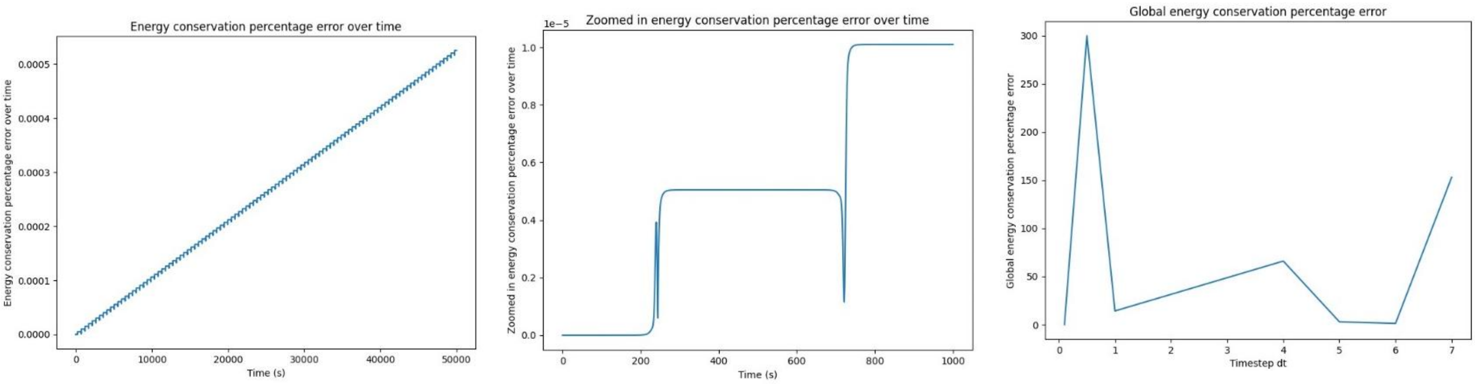
Orbits of a rocket around Earth in 2D, 3D, and its kinetic, potential and total energy over time.

Figs. 1-3: Circular orbit with an orbital period of T = 3.09 Hours.
Figs. 1-3: Elliptical orbit with an orbital period of T = 1.34 Hours.

Error Analysis



Figs. 5-7: Circular Orbit Errors



Figs. 9-11: Elliptical Orbit Errors

Figs. 5-7 (left to right): Circular orbit errors:
- Fig.5: Energy conservation percentage error over time
- Fig.6: Mean orbit deviation percentage error over time
- Fig.7: Accumulated energy conservation error as a function of timestep

Figs. 8-10: Elliptical orbit errors:
- Fig.8: Energy conservation percentage error over time
- Fig.9: Zoomed in energy conservation percentage error over time
- Fig.10: Accumulated energy conservation error as a function of timestep

Part b) – Moon slingshot

Method

In this part the initial position and velocity values for the rocket are chosen such that an external orbit of the rocket around the Earth and the Moon is produced or a figure-eight orbit, which is done by trial and error.

From the plots of the orbits it is clear when the rocket crashed into either body, as the rocket would suddenly deviates paths and shoots of into infinity most of the time. However I implemented an additional crash-test that takes into account the changes in angle of the position vectors and the distances from the Moon and the Earth.

The calculation is stopped (and simultaneously the orbital period is calculated) when the rocket completes its roundtrip – this is done by checking when the y-value changes signs (given that Earth's y-value is zero) and taking into consideration the distance from the Moon. This method is also used in parts c) and d).

To obtain the closest possible distance of the rocket from the Moon without it crashing, trial and error is again needed – the vy velocity is changed very slightly and various orbits plotted.

Finally I consider the conservation of energy and its error again to get a measure of the accuracy, and also analyse the effect of a changing timestep.

Results

Parameters:

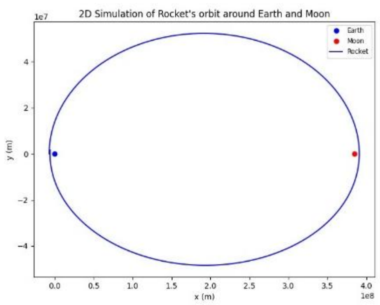
Earth's position (x, y, z) (m): (0, 0 0)

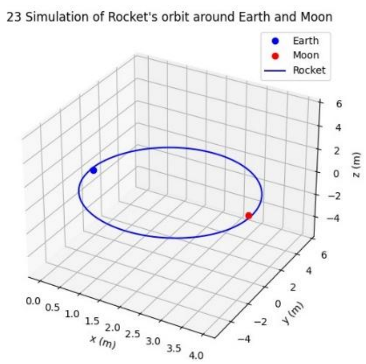
Moon's position (x, y, z) (m): (384.4e6, 0 0)

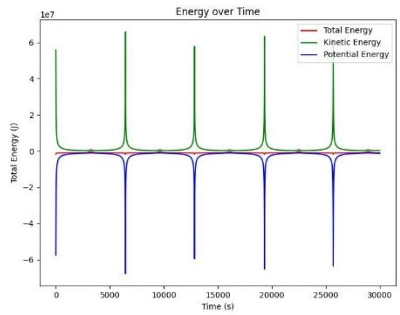
Initial rocket position (x, y, z) (m): (-7e6, 0 0)

Time step h: 100

	Initial rocket velocity (vx, vy, vz) (m/s)
External loop	(0, 10555.75, 0)
Eight figure loop	(0, 10572, 0)







Figs. 11-13: External Loop

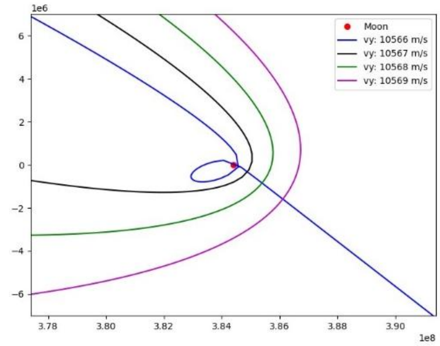


Fig. 14

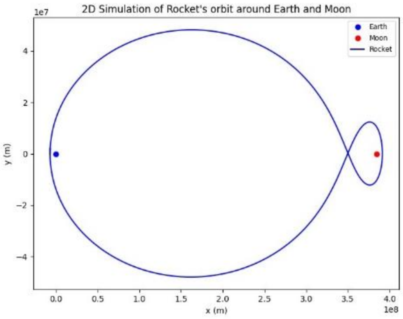


Fig. 15: Eight-figure Loop

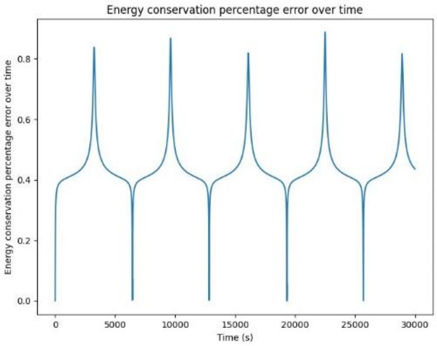


Fig.16

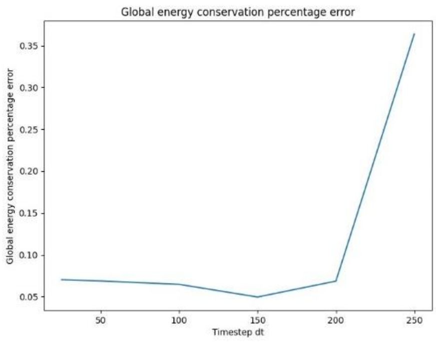


Fig.17

Figs. 11-13: External loop of Rocket around Earth and Moon in 2D (Fig. 11) and 3D (Fig.: 12)
Time to return: 179.06 Hours or 7.46 Days.

Fig.13: Energy of the rocket on the external orbit

Fig.14: Orbits for different initial vy values.
Closest orbit to Moon without crashing is with:
vy = 10567, with a distance from the Moon of 660306.95m.

Fig. 15: Eight figure loop around the Moon.
Time to return: 9.61 Days.

Fig.16: Energy conservation percentage error over time
Fig.17: Accumulated energy conservation error as a function of timestep

Discussion of parts a) and b)

The results for part a) and b) (with the given timesteps) obtained via the Runge Kutta Method exhibit the behaviour predicted by Newton's law of gravitation. As can be seen by the energy plots ,the total energy is always negative as is for bound orbits, and can be seen as a horizontal line, which proves that it is being conserved.

A look into the percentage error for the energy conservation (the global error at each step) shows that it follows a linear behaviour for the circular case, but is of the order e-11, and thus the change is negligible. For the elliptical case, this error also follows a linear path, however the local error is now visible at each timestep as it varies a periodically.

As predicted, the local Runge Kutta error is of the order h^5 for the elliptical case, and the accumulated error of the order h^4. The accumulated energy conservation percentage error increases exponentially with increasing time steps.

It can be seen that in order for the energy to be accurately conserved, a timestep of at least 1 is necessary here.

In part b) the accumulated error also increases exponentially with increasing time step, and here it can be seen that a time step of 150 produces the best accuracy. The energy conservation percentage error over time has a periodic behaviour.

Both an external loop and an 8 figure loop were produced, however the external loop leads to a faster return time and should thus be favoured.

Part c) and d)

Methods

In part c), the orbits of a moving Earth and Moon are calculated both for an Earth with zero initial velocity and a moving Earth. The code also lets you choose between a scenario where the mass of the Moon is equal to the mass of the Earth (to check whether it works correctly), and with their actual masses.

For the moving Earth, I use a relatively small initial vy velocity to better visualize the loops of the Moon around the Earth.

In part d) I include a stationary Sun and plot the orbits all three bodies under their combined gravitational influences, using the realistic parameters.

The orbital period of the Moon going around the Earth (part c) and the Earth going around the Sun (part d) is calculated in the same way as in b).

Results – Earth and Moon (c)

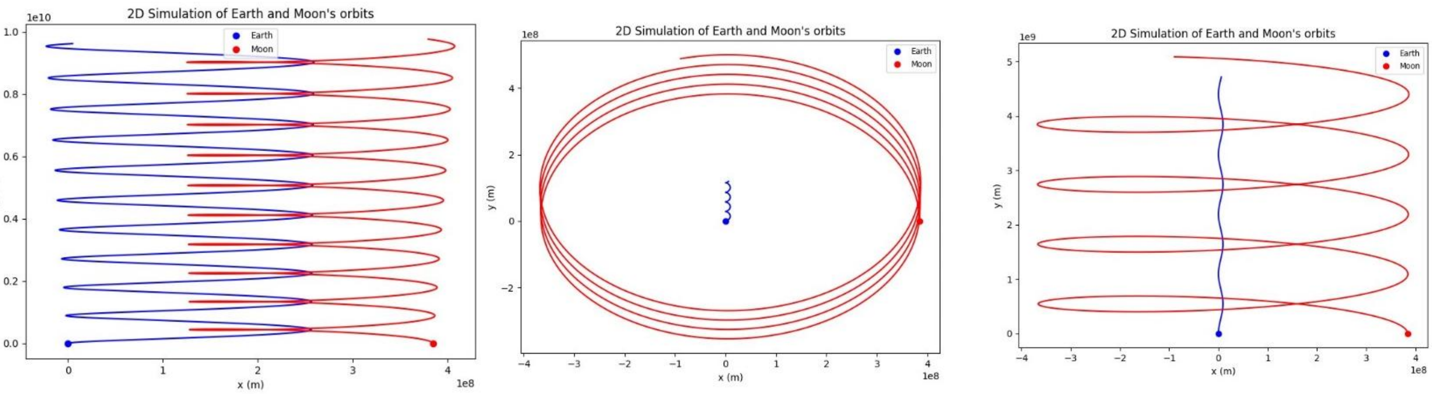
Parameters:

Initial Earth position (x, y, z) (m): (0, 0 0)

Initial Moon position (x, y, z) (m): (384.4e6, 0 0)

Time step h: 100

	Initial Earth velocity (vx, vy, vz) (m/s)	Initial Moon velocity (vx, vy, vz) (m/s)
Stationary Earth	(0,0, 0)	(0, 1017.79, 0)
Moving earth	(0, 460, 0)	(0, 1017.79 + 460, 0)



Figs. 18-20

Fig. 18: Orbits of the Earth and the Moon with Mm = Me (equal masses)

Fig.19: Orbits of an Earth with 0 initial velocity and an orbiting Moon. The orbital period of the Moon is calculated as being T = 26.51 days.

Fig. 20: Orbits of a moving Earth and moving Moon.

Results – Earth, Moon and Sun (d)

<u>Parameters:</u>

Initial Earth position (x, y, z) (m): (0, 0 0)

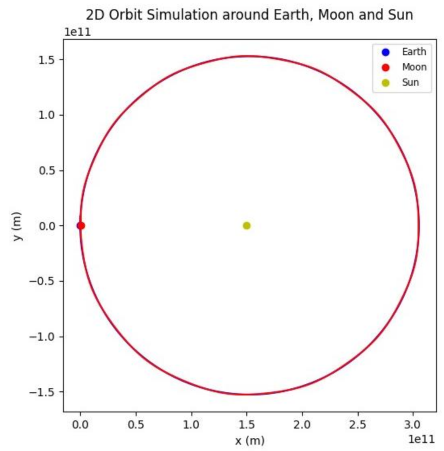
Initial Moon position (x, y, z) (m): (384.4e6, 0 0)

Position of the Sun (x, y, z) (m): (150e9, 0, 0=

Time step h: 1000

	Initial velocity (vx, vy, vz) (m/s)
Earth	(0, 30e3, 0)

Moon	(0, 1017.79 + 30e3, 0)



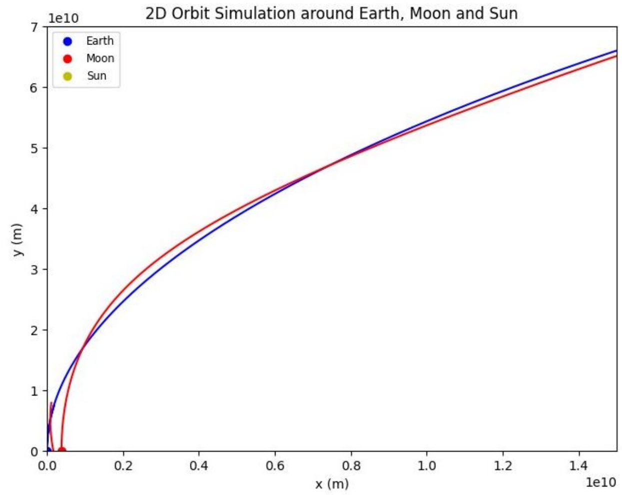


Fig.21(left): The orbits of the Earth, Moon and the Sun.
The orbital period of the Earth around the Sun is calculated as being 376.75 Days.

Fig.22 (right): A zoomed-in version

Discussion part c) and d)

The results for parts c) and d) show very accurate orbit calculations, giving values for the orbital period that are very close to the real values.

Conclusion

In conclusion, the Runge Kutta Method has been proven to be extremely accurate for the calculation of orbits and computationally efficient, giving an accurate representation of real-world orbits.